Stationary Stokes- and Navier-Stokes Equations with Low Regularity Data in Weighted Bessel-Potential Spaces

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Abstract

We investigate the stationary Navier-Stokes equations in Spaces with Muckenhoupt weights. The aim is to find a class of solutions as large as possible. We join the notation of very weak solutions in [1] and [10]. When estimating the nonlinear term the weighted context causes difficulties. For this reason we consider solutions in weighted Bessel-potential spaces.

Thus using complex interpolation we establish a theory of solutions to the Stokes equations in weighted Bessel-potential spaces.

Keywords: Stokes and Navier-Stokes equations, Muckenhoupt weights, very weak solutions, Bessel Potential spaces, nonhomgeneous data

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1 Introduction

We consider the stationary Navier-Stokes equations with inhomogeneous data and viscosity 1

$$-\Delta u + u \cdot \nabla u + \nabla p = F, \quad \text{in} \quad \Omega \tag{1.1}$$

$$\operatorname{div} u = K, \quad \text{in} \quad \Omega \tag{1.2}$$

$$u|_{\partial\Omega} = g. \tag{1.3}$$

If one multiplies (1.1) with a test function ϕ vanishing on the boundary and (1.2) with a test function ψ then formal integration by parts yields

$$-\langle u, \Delta\phi\rangle + \langle u \cdot \nabla u, \phi\rangle = \langle F, \phi\rangle - \langle g, N \cdot \nabla\phi\rangle \quad \text{and} \quad -\langle u, \nabla\psi\rangle = \langle K, \psi\rangle - \langle g, N\psi\rangle.$$

Following [1], [3], [4], [10] we will use these equations for the definition of very weak solutions.

Our aim is to consider the stationary Navier-Stokes equations requiring the least possible regularity of the data. We investigate this problem in the context of function spaces with Muckenhoupt weights. This is a large class of locally integrable weight-functions defined by (2.1). Their good properties concerning harmonic analysis [15], [11] where the base to treat to solvability of the Stokes and Navier Stokes equations [5], [6], [7], [8].

When it comes to estimate the nonlinear term $u \cdot \nabla u$, one needs a weighted analogue to the Sobolev embedding theorems. Such embeddings can be proved as in [9] using the continuity of weakly singular integral operators established in [14]. However, these estimates require strong assumptions to the weight function. As a rule the more general data and solutions are the more restrictions we have to impose on the weight function. This is the reason why we study this problem in weighted Bessel potential spaces. Depending on the weight function w we find a class of indices β such that the class of solutions is contained in the Bessel-potential space $H_w^{\beta,q}(\Omega)$, presumed the data is chosen appropriately. The classical weak and strong solutions are contained in the presented theory for $\beta = 1$ and $\beta = 2$, respectively.

Our main result concerning very weak solutions to the stationary Navier Stokes problem is the following:

Theorem 1.1. *Let* $0 \le \beta < 1$ *and* $\beta \ge \frac{ns}{q} - 1$ *if* $n \ge 3$ *and* $\beta > -\frac{1}{2} + \frac{2s}{q}$ *if* $n = 2$. *Moreover, let* $F \in W_w^{-1,t}(\Omega)$, $K \in L_w^t(\Omega)$ *with*

$$\frac{1-\beta}{ns} + \frac{1}{q} - \frac{1}{t} = 0$$

and $g \in T_w^{\beta,q}(\partial\Omega)$ *with* $\langle K, 1\rangle = \langle g, N\rangle_{\partial\Omega}$. *Then there exists a constant* $\rho > 0$ *such that, if*

$$\|F\|_{-1,t,w} + \|K\|_{t,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)} \le \rho,$$

then there exists a very weak solution $u \in H_w^{\beta,q}(\Omega)$ *to the stationary Navier-Stokes equations. This solution fulfills* $u|_{\partial\Omega} = g$ *in the sense of (2.12) and (2.11) and satisfies the estimate*

$$\|u\|_{\beta,q,w} \le c(\|F\|_{-1,t,w} + \|K\|_{t,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)})$$

with $c = c(\beta, q, w, \Omega)$.

If $1 \le \beta \le 2$ we are in the situation between weak and strong solutions. In this case we have the following existence theorem

Theorem 1.2. *Let* $1 \le \beta \le 2$ *and* $\beta \ge \frac{ns}{q} - 1$ *if* $n \ge 3$ *and* $\beta > \frac{2s}{q}$ *if* $n = 2$. *Moreover let* $F \in H_w^{\beta-2,q}(\Omega)$, $K \in H_w^{\beta-1,q}(\Omega)$ *and* $g \in T_w^{\beta,q}(\partial\Omega)$ *with* $\int K\,dx = \int_{\partial\Omega} gN\,dS$. *Then there exists a constant* $\rho > 0$ *such that, if*

$$\|F\|_{\beta-2,q,w} + \|K\|_{\beta-1,q,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)} \le \rho,$$

then there exists a weak solution $u \in H_w^{\beta,q}(\Omega)$ *to the stationary Navier-Stokes equations. This solution satisfies the estimate*

$$\|u\|_{\beta,q,w} \le c(\|F\|_{\beta-2,q,w} + \|K\|_{\beta-1,q,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)})$$

with $c = c(\beta, q, w, \Omega)$.

The solution in Theorem 1.1 and 1.2 is unique with an additional smallness assumption to u and K. It is shown in Theorem 4.8 using the uniqueness in the unweighted case established in [4].

The difference between the case $0 \le \beta \le 1$ and $1 \le \beta \le 2$ lies in the way of treating the boundary conditions and the inhomogeneous divergence. If $1 \le \beta \le 2$ then the solution $u \in H_w^{\beta,q}(\Omega)$ is smooth enough to ensure that the restriction $u|_{\partial\Omega}$ to the boundary is well-defined. Moreover the divergence $K \in H_w^{\beta-1,q}(\Omega) \hookrightarrow L_w^q(\Omega)$ and we are dealing with products of functions. If $0 \le \beta \le 1$ one has to demand more regularity of f and k to ensure this.

As a difficult step in the proof of the mentioned theorems we need the solvability of the stationary Stokes equations in weighted Bessel potential spaces. To establish this we use complex interpolation between the strong and the very weak solution. These very weak solutions can be obtained by dualization of the strong solutions as done in [13]. This in turn requires interpolation theorems of spaces with 0 boundary values. See Section 3 for details.

2 Preliminaries

2.1 Weighted Function Spaces

Let A_q, $1 < q < \infty$ be the set of Muckenhoupt weights which is given by all $0 \le w \in L^1_{loc}(\mathbb{R}^n)$ for which

$$A_q(w) := \sup_Q \left(\frac{1}{|Q|} \int_Q w \, dx \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{q-1}} dx \right)^{q-1} < \infty \tag{2.1}$$

The supremum is taken over all cubes Q in $\mathbb{R}^n$ and $|Q|$ stands for the Lebesgue measure of Q. To shorten the notation we write $w(U) = \int_U w \, dx$ for every measurable set $U \subset \mathbb{R}^n$.

For $w \in A_q$ and an open set Ω we define

$$L_w^q(\Omega) := \{ f \in L^1_{loc}(\overline{\Omega}) \mid \|f\|_{q,w} := \left(\int_\Omega |f|^q w \, dx \right)^{\frac{1}{q}} < \infty \}.$$

It is easily seen that $(L_w^q(\Omega))' = L_{w'}^{q'}(\Omega)$ with $\frac{1}{q} + \frac{1}{q'} = 1$ and $w' = w^{-\frac{1}{q-1}} \in A_{q'}$.

Moreover, for $k \ge 0$ we introduce the weighted Sobolev spaces

$$W_w^{k,q}(\Omega) := \Big\{ u \in L_w^q(\Omega), \mid \|u\|_{k,q,w} := \sum_{|\alpha| \le k} \|D^\alpha u\|_{q,w} < \infty \Big\}$$

and $W_{w,0}^{k,q}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{k,q,w}}$ as well as the dual space $W_w^{-k,q}(\Omega) := (W_{w',0}^{k,q'}(\Omega))'$. With this notation one has $W_w^{0,q}(\Omega) = W_{w,0}^{0,q}(\Omega) = L_w^q(\Omega)$.

Since for $k \ge 1$ one has $W_w^{k,q}(\Omega) \subset W_{loc}^{k,1}(\overline{\Omega})$, the restriction $u \mapsto u|_{\partial\Omega}$ is well-defined. Thus we may define $T_w^{k,q}(\partial\Omega) := (W_w^{k,q}(\Omega))|_{\partial\Omega}$ equipped with the norm of the factor space

$$\|g\|_{T_w^{k,q}(\partial\Omega)} := \inf\{ u \in W_w^{k,q}(\Omega) \mid u|_{\partial\Omega} = g \}.$$

Furthermore, we set $T_w^{0,q}(\partial\Omega) := (T_{w'}^{1,q'}(\partial\Omega))'$. Then $L_w^q(\Omega)$, $W_w^{k,q}(\Omega)$, $W_{w,0}^{k,q}(\Omega)$ and $T_w^{k,q}(\partial\Omega)$ are reflexive Banach spaces in which $C^\infty(\overline{\Omega})$ ($C^\infty(\overline{\Omega})|_{\partial\Omega}$, resp.) is dense.

We also use the divergence-free version of the spaces

$$W^{k,q}_{w,0,\sigma}(\Omega) := \{u \in W^{k,q}_{w,0}(\Omega) \mid \operatorname{div} u = 0\}$$

and $C^\infty_{0,\sigma}(\Omega)$, the space of smooth and compactly supported divergence-free functions.

By [6] change of variables is continuous between weighted Sobolev spaces. More precisely if $w \in A_q$ and $\alpha \in C^{1,1}(\mathbb{R}^n)$, is a diffeomorphism then $w \circ \alpha \in A_q$ and

$$u \mapsto u \circ \alpha \quad : W^{k,q}_{w}(\mathbb{R}^n) \to W^{k,q}_{w\circ\alpha}(\mathbb{R}^n) \;\text{ is continuous for }\; k = 0,1,2. \tag{2.2}$$

For $w \in A_q$ let

$$\tilde{w}(x', x_n) = \begin{cases} w(x', x_n) & \text{on } \mathbb{R}^n_+ \\ w(x', -x_n) & \text{on } \mathbb{R}^n_- \end{cases} \tag{2.3}$$

then an elementary proof (see [8]) shows $\tilde{w} \in A_q$.

2.2 Very Weak Solutions to the Stokes Problem

Before dealing with very weak solutions to the Stokes equations in weighted Bessel-potential spaces we treat them in special weighted spaces of functionals. In this lowest regularity context the data is given by functionals which are in general no distributions on the domain Ω. More precisely, the force is contained in $Y'_{q',w'}$, the dual space of

$$Y_{q',w'} =: Y_{q',w'}(\Omega) := \{u \in W^{2,q'}_{w'}(\Omega) \mid u|_{\partial\Omega} = 0\},$$

and the divergence k is contained in $W^{-1,q}_{w,0}(\Omega) := (W^{1,q'}_{w'}(\Omega))'$. As a space of test functions we use

$$Y_\sigma := Y_{q',w',\sigma} := \{\phi \in Y_{q',w'} \mid \operatorname{div} \phi = 0\}.$$

Then the definition of very weak solutions reads as follows.

Definition 2.1. *Let $f \in Y_{w',q'}(\Omega)'$ and $k \in W^{-1,q}_{w,0}(\Omega)$. A function $u \in L^q_w(\Omega)$ is called a very weak solution to the Stokes problem with respect to the data f and k if*

$$\langle f, \phi\rangle \;= -\langle u, \Delta\phi\rangle, \qquad \textit{for all } \phi \in Y_\sigma \;\textit{ and} \tag{2.4}$$

$$\langle k, \psi\rangle \;= -\langle u, \nabla\psi\rangle, \qquad \textit{for all } \psi \in W^{1,q'}_{w'}(\Omega). \tag{2.5}$$

The following two theorems guaranty existence, uniqueness and regularity of very weak solutions. They are special cases of theorems in [13]

Theorem 2.2. *Let $f \in Y'_{q',w'}$, $k \in W^{-1,q}_{w,0}(\Omega)$ with $\langle k, 1\rangle = 0$. Then there exists a unique very weak solution $u \in L^q_w(\Omega)$ to the Stokes problem in the sense of Definition 2.1.*

Moreover there exists a unique pressure functional $p \in W^{-1,q}_{0,w}(\Omega)$ (unique modulo constants) such that (u,p) solves

$$-\langle u, \Delta\phi\rangle - \langle p, \operatorname{div}\phi\rangle = \langle F, \phi\rangle \quad \textit{for all } \; \phi \in Y_{q',w'}.$$

In particular $-\Delta u + \nabla(p|_{C^\infty_0(\Omega)}) = f|_{C^\infty_0(\Omega)}$ in the sense of distributions. The functions (u,p) fulfill the inequality

$$\|u\|_{q,w} + \|p\|_{-1,q,w,0} \le c(\|f\|_{Y'_{q',w'}} + \|k\|_{W^{-1,q}_{w,0}(\Omega)}), \tag{2.6}$$

with $c = c(\Omega, q, w)$.

Theorem 2.3. *Assume that* $f \in Y'_{q',w'}$ *and* $k \in W^{-1,q}_{w,0}(\Omega)$ *allow a decomposition into*

$$\begin{aligned} \langle f, \phi\rangle &= \langle F, \phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega} \qquad && \text{for all } \phi \in Y_{q',w'}, \\ \langle k, \psi\rangle &= \langle K, \psi\rangle - \langle g, N\cdot\psi\rangle_{\partial\Omega} && \text{for all } \psi \in W^{1,q'}_{w'}(\Omega) \end{aligned} \tag{2.7}$$

with $g \in T^{2,q}_w(\partial\Omega)$, $F \in L^q_w(\Omega)$ *and* $K \in W^{1,q}_w(\Omega)$. *Then the very weak solution* u *to the Stokes problem with respect to* f *and* k *is a strong solution with respect to* F, K *and* g. *In particular* $u \in W^{2,q}_w(\Omega)$ *and*

$$\|u\|_{2,q,w} \le c(\|F\|_{q,w} + \|K\|_{1,q,w} + \|g\|_{T^{2,q}_w(\partial\Omega)}). \tag{2.8}$$

One consequence of this theorem is the following: If u is a very weak solution to the Stokes problem with sufficiently regular f and k (It suffices to assume that f and k are contained in some space of distributions embedded into $Y'_{q',w'}$, $W^{-1,q}_{w,0}(\Omega)$, resp.) then $u|_{\partial\Omega} = 0$.

However, we need to define boundary conditions in a far more general context. Thus we choose $1 < r < \infty$ and $\tilde{w} \in A_r$ such that

$$W^{-1,r}_{\tilde{w}}(\Omega) \hookrightarrow Y'_{q',w'} \quad \text{and} \quad L^r_{\tilde{w}}(\Omega) \hookrightarrow W^{-1,q}_{w,0}(\Omega) \tag{2.9}$$

and define the space

$$\tilde{W}^{q,r}_{w,\tilde{w}}(\Omega) := \{u \in L^q_w(\Omega) \,|\, \Delta u|_{C^\infty_{0,\sigma}(\Omega)} \text{ can be extended to an element of } (W^{1,r'}_{0,\tilde{w}',\sigma}(\Omega))'\}. \tag{2.10}$$

For $u \in \tilde{W}^{q,r}_{w,\tilde{w}}(\Omega)$ the tangential trace u_T is given by

$$\langle u_T, N\cdot\nabla\phi\rangle_{\partial\Omega} = \langle u, \Delta\phi\rangle - \langle \Delta u, \phi\rangle \tag{2.11}$$

for every $\phi \in Y_{q',w',\sigma}$.

For the normal component of the boundary condition we set

$$E^{q,r}_{w,\tilde{w}} := \{v \in L^q_w(\Omega) \mid \operatorname{div} v \in L^r_{\tilde{w}}(\Omega)\}$$

and define for $u \in E^{q,r}_{w,\tilde{w}}$ as in the classical case

$$\langle u_N, Nv\rangle_{\partial\Omega} := \langle \operatorname{div} u, v\rangle + \langle u, \nabla v\rangle \quad \text{for all} \quad v \in W^{1,q'}_{w'}(\Omega). \tag{2.12}$$

For $u \in \tilde{W}^{q,r}_{w,\tilde{w}}(\Omega) \cap E^{q,r}_{w,\tilde{w}}$ we write $u|_{\partial\Omega} = g$, if $\langle u_T, h\rangle_{\partial\Omega} = \langle g, h\rangle_{\partial\Omega}$ for every purely tangential h and $\langle u_N, h\rangle_{\partial\Omega} = \langle g, h\rangle_{\partial\Omega}$ for every purely normal h.

See [13] for the well-definedness and the continuity of this restriction. Finally a very weak solution assumes a given boundary condition in the above sense. (See [13] for the proof.)

Proposition 2.4. *Let* u *be a very weak solution to the Stokes problem corresponding to the data* $\langle f, \phi\rangle = \langle F, \phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}$ *and* $\langle k, \psi\rangle = \langle K, \psi\rangle - \langle g, N\cdot\psi\rangle_{\partial\Omega}$ *with* $F \in W^{-1,r}_{\tilde{w}}(\Omega)$, $K \in L^r_{\tilde{w}}(\Omega)$ *where* r *and* $\tilde{w} \in A_r$ *are chosen according to (2.9) and* $g \in T^{0,q}_w(\partial\Omega)$.
Then $u \in \tilde{W}^{q,r}_{w,\tilde{w}} \cap E^{q,r}_{w,\tilde{w}}$ *and* $u|_{\partial\Omega} = g$.

2.3 Weighted Bessel Potential Spaces and Complex Interpolation

Let $\{X_1, X_2\}$ be an interpolation couple and $D = \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1\}$.

We denote by $F(X_1, X_2)$ the space of all holomorphic functions from D to $X_1 + X_2$ which are extendable to continuous functions on $\overline{D}$ such that

$$\|f\|_{F(X_1,X_2)} := \max\{\sup_{y\in\mathbb{R}} \|f(iy)\|_{X_1}, \sup_{y\in\mathbb{R}} \|f(iy+1)\|_{X_2}\} < \infty.$$

For $0 < \theta < 1$ we write $[X_1, X_2]_\theta$ for the complex interpolation in the usual sense, see e.g. [17].

On the space $\mathcal{S}'(\mathbb{R}^n)$ of temperate distributions we define for all $\beta \in \mathbb{C}$ the operator

$$\Lambda^\beta f = \mathcal{F}^{-1}(1 + |\xi|^2)^{\frac{\beta}{2}} \mathcal{F} f \quad \text{for all } f \in S'(\mathbb{R}^n),$$

where $\mathcal{F}$ stands for the Fourier transformation on $\mathcal{S}'(\mathbb{R}^n)$. Then for $1 < q < \infty$, $w \in A_q$ and $\beta \in \mathbb{R}$ the weighted Bessel potential space is given by

$$H_w^{\beta,q}(\mathbb{R}^n) = \big\{f \in \mathcal{S}' \mid \|f\|_{H_w^{\beta,q}(\mathbb{R}^n)} := \|\Lambda^\beta f\|_{q,w,\mathbb{R}^n} < \infty\big\}.$$

Let Ω be an extension domain, i.e. Ω admits a continuous extension operator $E : W_w^{k,q}(\Omega) \to W_w^{k,q}(\mathbb{R}^n)$ which is universal for all $k \le m$. In particular Lipschitz domains are extension domains (see Chua [2] and Jones [12]). Then the weighted Bessel potential space on Ω is defined by

$$H_w^{\beta,q}(\Omega) = \{u|_\Omega \mid u \in H_w^{\beta,q}(\mathbb{R}^n)\}$$

equipped with the norm

$$\|u\|_{\beta,q,w} := \|u\|_{H_w^{\beta,q}(\Omega)} := \inf\{\|U\|_{H_w^{\beta,q}(\mathbb{R}^n)} \mid U \in H_w^{\beta,q}(\mathbb{R}^n),\ U|_\Omega = u\}.$$

Theorem 2.5. *Let Ω be an extension domain, $1 < q < \infty$, $w \in A_q$.*

1. *For $k \in \mathbb{N}_0$ one has $H_w^{k,q}(\Omega) = W_w^{k,q}(\Omega)$ with equivalent norms.*
2. *For $k \in \mathbb{N}$, $0 \le \beta \le k$ one has*

$$H_w^{\beta,q}(\Omega) = [L_w^q(\Omega), W_w^{k,q}(\Omega)]_{\frac{\beta}{k}}.$$

3. *The spaces $H_w^{\beta,q}(\Omega)$, $\beta \in \mathbb{R}$ are independent of the values of the weight function $w \in A_q$ outside Ω, i.e., if $w_1, w_2 \in A_q$, $w_1|_\Omega = w_2|_\Omega$ then $H_{w_1}^{\beta,q}(\Omega) = H_{w_2}^{\beta,q}(\Omega)$ with equivalent norms.*

Proof. [6, 8.2.2] and [9] □

3 Stokes Equations in Weighted Bessel Potential Spaces

3.1 Interpolation of Weighted Bessel Potential Spaces with Zero Boundary Values

For an extension domain $\Omega \subset \mathbb{R}^n$, $1 < q < \infty$, $w \in A_q$ and $0 \le \beta \le 2$ we define the space

$$\tilde{H}_w^{\beta,q}(\Omega) := \begin{cases} \overline{Y_{q,w}(\Omega)}^{H_w^{\beta,q}(\mathbb{R}^n)}, & \text{if } \ 0 \le \beta \le 1 \ \text{ equipped with } \|\cdot\|_{H_w^{\beta,q}(\mathbb{R}^n)}, \\ \overline{Y_{q,w}(\Omega)}^{H_w^{\beta,q}(\Omega)}, & \text{if } \ 1 < \beta \le 2 \ \text{ equipped with } \|\cdot\|_{H_w^{\beta,q}(\Omega)}, \end{cases}$$

where in the case $0 \le \beta \le 1$ the functions of $Y_{q,w}(\Omega)$ are assumed to be extended by 0 to functions defined on the whole space $\mathbb{R}^n$. This is possible, since $C_0^\infty(\Omega)$ is dense in $W_{w,0}^{1,q}(\Omega) \supset Y_{q,w}(\Omega)$ and $W_{w,0}^{1,q}(\Omega) \hookrightarrow W_w^{1,q}(\mathbb{R}^n) \hookrightarrow H_w^{\beta,q}(\mathbb{R}^n)$.

Moreover, for such β it follows immediately from the definition of $\tilde{H}_w^{\beta,q}(\Omega)$ that the extension of functions $u \in \tilde{H}_w^{\beta,q}(\Omega)$ by 0 to functions on $\mathbb{R}^n$ is a continuous linear map to $H_w^{\beta,q}(\mathbb{R}^n)$.

Finally, for $\beta = 1$ the two definitions are equivalent, i.e. $\tilde{H}_w^{1,q}(\Omega) = W_{w,0}^{1,q}(\Omega) = \overline{Y_{q,w}(\Omega)}^{H_w^{1,q}(\Omega)}$, equipped with $\|\cdot\|_{H_w^{1,q}(\Omega)}$. The reason is that for $u \in Y_{q,w}$ one has by Theorem 2.5

$$\begin{aligned} \|u\|_{H_w^{1,q}(\Omega)} &\le c_1\|u\|_{W_w^{1,q}(\Omega)} = c_1\|\tilde{u}\|_{W_w^{1,q}(\mathbb{R}^n)} \le c_2\|\tilde{u}\|_{H_w^{1,q}(\mathbb{R}^n)} \\ &\le c_3\|\tilde{u}\|_{W_w^{1,q}(\mathbb{R}^n)} \le c_4\|u\|_{H_w^{1,q}(\Omega)}, \end{aligned}$$

where $\tilde{u}$ denotes the extension of u by 0 to the whole space $\mathbb{R}^n$.

For symmetrie reasons the question arises whether $\tilde{H}_w^{\beta,q}(\Omega) = \overline{Y_{q,w}(\Omega)}^{H_w^{\beta,q}(\Omega)}$ for all $0 \le \beta \le 2$. However this is not the case, not even in the unweighted case. Indeed by Triebel [18, I.6.5.23] one has

$$\overline{Y_{q,1}(\Omega)}^{H^{1-\frac{1}{q},q}(\Omega)} = \overline{C_0^\infty(\Omega)}^{H^{1-\frac{1}{q},q}(\Omega)} \ne \{u \in H^{1-\frac{1}{q},q}(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\} = \tilde{H}^{1-\frac{1}{q},q}(\Omega). \tag{3.1}$$

We choose the spaces $\tilde{H}_w^{\beta,q}(\Omega)$ because of their good properties with respect to interpolation.

Theorem 3.1. *Let $1 < q < \infty$, $w \in A_q$ and $0 \le \beta \le 2$. Then*

$$\left[L_w^q(\mathbb{R}_+^n), Y_{q,w}(\mathbb{R}_+^n)\right]_\theta = \tilde{H}_w^{\beta,q}(\mathbb{R}_+^n),$$

where $\theta = \frac{\beta}{2}$ with equivalent norms.

Proof. By Theorem 2.5 we may assume that $w = \tilde{w}$ (given by (2.3)), i.e. w is even in x_n.

Step 1:

$$\left[L_w^q(\mathbb{R}_+^n), Y_{q,w}(\mathbb{R}_+^n)\right]_\theta \hookrightarrow \tilde{H}_w^{\beta,q}(\mathbb{R}_+^n).$$

To see this let $u \in \left[L_w^q(\mathbb{R}_+^n), Y_{q,w}(\mathbb{R}_+^n)\right]_\theta$.

We begin with the case $1 \le \beta \le 2$. Then there is a function $U \in F(L_w^q(\mathbb{R}_+^n), Y_{q,w}(\mathbb{R}_+^n))$

such that $U(\theta)=u$ and $\|U\|_{F(L^q_w(\mathbb{R}^n_+),Y_{q,w}(\mathbb{R}^n_+))}\le c\|u\|_{[L^q_w(\mathbb{R}^n_+),Y_{q,w}(\mathbb{R}^n_+)]_\theta}$.
Since $F(L^q_w(\mathbb{R}^n_+),Y_{q,w}(\mathbb{R}^n_+))\subset F(L^q_w(\mathbb{R}^n_+),H^{2,q}_w(\mathbb{R}^n_+))$ we obtain

$$u=U(\theta)\in[L^q_w(\mathbb{R}^n_+),H^{2,q}_w(\mathbb{R}^n_+)]_\theta$$

and

$$\begin{aligned}\|u\|_{H^{\beta,q}_w(\mathbb{R}^n_+)} &\le \inf\left\{\|V\|_{F(L^q_w(\mathbb{R}^n_+),H^{2,q}_w(\mathbb{R}^n_+))}\mid V\in F(L^q_w(\mathbb{R}^n_+),H^{2,q}_w(\mathbb{R}^n_+)),\ V(\theta)=u\right\}\\ &\le \|U\|_{F(L^q_w(\mathbb{R}^n_+),Y_{q,w}(\mathbb{R}^n_+))}\le c\|u\|_{[L^q_w(\mathbb{R}^n_+),Y_{q,w}(\mathbb{R}^n_+)]_\theta}.\end{aligned}$$

Moreover, by [17, Theorem 1.9.3] we know that $Y_{q,w}(\mathbb{R}^n_+)$ is dense in $[L^q_w(\mathbb{R}^n_+),Y_{q,w}(\mathbb{R}^n_+)]_\theta$ which yields the assertion of Step 1 in the case $s\ge 1$.
In the case $0\le s\le 1$ we assume that we already know

$$\left[L^q_w(\mathbb{R}^n_+),Y_{q,w}(\mathbb{R}^n_+)\right]_{\frac12}=\tilde H^{1,q}_w(\mathbb{R}^n_+)=\overline{C_0^\infty(\mathbb{R}^n_+)}^{W^{1,q}_w(\mathbb{R}^n)}=W^{1,q}_{0,w}(\mathbb{R}^n_+),$$

which will be proved later on in this proof without using the present fact. Then, as $0\le\theta\le\frac12$, the reiteration property implies

$$\left[L^q_w(\mathbb{R}^n_+),Y_{q,w}(\mathbb{R}^n_+)\right]_\theta=\left[L^q_w(\mathbb{R}^n_+),W^{1,q}_{w,0}(\mathbb{R}^n_+)\right]_{2\theta}.$$

Since the extension

$$Tu(x)=\begin{cases}u(x) & \text{for } x\in\mathbb{R}^n_+\\ 0 & \text{for } x\in\mathbb{R}^n_-\end{cases}$$

of functions defined on the half space is continuous from $W^{1,q}_{w,0}(\mathbb{R}^n_+)$ to $W^{1,q}_w(\mathbb{R}^n)$ and from $L^q_w(\mathbb{R}^n_+)$ to $L^q_w(\mathbb{R}^n)$, we find by interpolation

$$T:\left[L^q_w(\mathbb{R}^n_+),W^{1,q}_{w,0}(\mathbb{R}^n_+)\right]_{2\theta}\to H^{\beta,q}_w(\mathbb{R}^n)$$

is continuous. Thus for every $u\in C_0^\infty(\mathbb{R}^n_+)$ we obtain

$$\|u\|_{\tilde H^{\beta,q}_w(\mathbb{R}^n_+)}=\|Tu\|_{\beta,q,w,\mathbb{R}^n}\le c\|u\|_{[L^q_w(\mathbb{R}^n_+),W^{1,q}_{w,0}(\mathbb{R}^n_+)]_{2\theta}}.$$

Then the density of the embedding $C_0^\infty(\mathbb{R}^n_+)\hookrightarrow\left[L^q_w(\mathbb{R}^n_+),W^{1,q}_{w,0}(\mathbb{R}^n_+)\right]_{2\theta}$ finishes the proof of Step 1.
Step 2: We show the following:
If the odd extension

$$E:\tilde H^{\beta,q}_w(\mathbb{R}^n_+)\to H^{\beta,q}_w(\mathbb{R}^n),\quad Eu(x)=\begin{cases}u(x) & \text{if } x\in\mathbb{R}^n_+\\ -u(x',-x_n) & \text{if } x\in\mathbb{R}^n_-\end{cases}$$

where $x=(x',x_n)$ is continuous, then the assertion is true for β.
Let $u\in\tilde H^{\beta,q}_w(\mathbb{R}^n_+)$ and set

$$U(z)=e^{z^2}\Lambda^{(\theta-z)2}Eu.$$

Then one has $U\in F(L^q_w(\mathbb{R}^n),W^{2,q}_w(\mathbb{R}^n))$ with $U(\theta)=e^{\theta^2}Eu$. Moreover, since for every $\mu\in\mathbb{C}$ the operator Λ^μ maps odd functions to odd functions, one has $U(iy+1)|_{\mathbb{R}^{n-1}}=0$

which implies $U(iy+1)|_{\mathbb{R}^n_+} \in Y_{q,w}(\mathbb{R}^n_+)$ for every y. Thus $U|_{\mathbb{R}^n_+} \in F(L^q_w(\mathbb{R}^n_+), Y_{q,w}(\mathbb{R}^n_+))$ and we obtain $u \in \left[L^q_w(\mathbb{R}^n_+), Y_{q,w}(\mathbb{R}^n_+)\right]_\theta$.
Step 3: The assertion is true for $\beta < 1$:
By the definition of $\tilde{H}^{\beta,q}_w(\mathbb{R}^n_+)$ for $\beta < 1$ we know that the extension $\tilde{u}$ of u by 0 on $\mathbb{R}^n$ is continuous from $\tilde{H}^{\beta,q}_w(\mathbb{R}^n_+)$ to $H^{\beta,q}_w(\mathbb{R}^n)$. Thus, the odd extension of u, which is equal to

$$Eu(x) = \tilde{u}(x) - \tilde{u}(x', -x_n),$$

is also continuous. Step 2 completes the argument.
Step 4: The assertion is true for $1 \le \beta \le 2$.
For $g \in T^{2,q}_w(\mathbb{R}^{n-1})$ there exists an extension $S(g)$ with the following properties:

- $S(g)|_{\mathbb{R}^{n-1}} = g$.
- S is a continuous linear mapping

$$: T^{2,q}_w(\mathbb{R}^{n-1}) \to W^{2,q}_w(\mathbb{R}^n) \quad \text{and} \quad : T^{1,q}_w(\mathbb{R}^{n-1}) \to W^{1,q}_w(\mathbb{R}^n).$$

To see this we define $S(g)|_{\mathbb{R}^n_+}$ to be the solution of

$$(1-\Delta)S(g) = 0 \ \text{ on } \mathbb{R}^n_+ \ \text{ and } \ S(g) = g \ \text{ on } \mathbb{R}^{n-1}.$$

Then by [6, Lemma 3.14, Satz 3.7] we know that $S(u)|_{\mathbb{R}^n_+}$ is well-defined and has the two properties on $\mathbb{R}^n_+$. By [2, Theorem 1.5] there exists an extension operator, continuous from $W^{2,q}_w(\mathbb{R}^n_+)$ to $W^{2,q}_w(\mathbb{R}^n)$ as well as from $W^{1,q}_w(\mathbb{R}^n_+)$ to $W^{1,q}_w(\mathbb{R}^n)$. Thus the existence of such an S is proved.
Now we consider the operator

$$B : H^{2,q}_w(\mathbb{R}^n_+) \to H^{2,q}_w(\mathbb{R}^n), \quad u \mapsto S(u|_{\mathbb{R}^{n-1}}) + E(u - S(u|_{\mathbb{R}^{n-1}}))$$

where E is the odd extension operator from Step 2. Since $w = \tilde{w}$, the operator E is continuous from $Y_{q,w}(\mathbb{R}^n_+)$ to $W^{2,q}_w(\mathbb{R}^n)$ and from $W^{1,q}_{w,0}(\mathbb{R}^n_+)$ to $W^{1,q}_w(\mathbb{R}^n)$. Thus, we have constructed an operator B continuous from $W^{2,q}_w(\mathbb{R}^n_+)$ to $W^{2,q}_w(\mathbb{R}^n)$ as well as from $W^{1,q}_w(\mathbb{R}^n_+)$ to $W^{1,q}_w(\mathbb{R}^n)$ and which coincides with E on $\tilde{H}^{\beta,q}_w(\Omega)$, $\beta = 1, 2$. By interpolation we find that

$$B : H^{\beta,q}_w(\mathbb{R}^n_+) \to H^{\beta,q}_w(\mathbb{R}^n)$$

is continuous for every $1 \le \beta \le 2$. Thus for every $u \in \tilde{H}^{\beta,q}_w(\mathbb{R}^n_+) \subset \tilde{H}^{1,q}_w(\mathbb{R}^n_+)$ one has

$$\|Eu\|_{H^{\beta,q}_w(\mathbb{R}^n)} = \|Bu\|_{H^{\beta,q}_w(\mathbb{R}^n)} \le c\|u\|_{H^{\beta,q}_w(\mathbb{R}^n_+)} = c\|u\|_{\tilde{H}^{\beta,q}_w(\mathbb{R}^n_+)}.$$

Thus Step 2 finishes the proof. □

Theorem 3.2. *The assertion of Theorem 3.1 holds true, when replacing $\mathbb{R}^n_+$ by a bounded $C^{1,1}$-domain Ω, i.e.,*

$$\left[L^q_w(\Omega), Y_{q,w}(\Omega)\right]_\theta = \tilde{H}^{\beta,q}_w(\Omega),$$

where $\theta = \frac{\beta}{2}$ with equivalent norms.

Proof. Let α_j, $j = 1, ..., m$, be a collection of $C^{1,1}$-charts and ψ_j a decomposition of the unity subordinate to the domains of the charts. We assume that every ψ_j is extended to an element of $C_0^\infty(\mathbb{R}^n)$ and that every α_i is extended to an element of $C^{1,1}(\mathbb{R}^n)$ such that it has an inverse $\alpha_i^{-1} \in C^{1,1}(\mathbb{R}^n)$.

Then we fix j, write $\psi = \psi_j$ and $\alpha = \alpha_j$ and define the mapping

$$B : \tilde{H}^{\beta,q}_{w\circ\alpha}(\mathbb{R}^n_+) \to \tilde{H}^{\beta,q}_w(\Omega), \quad u \mapsto (u(\psi \circ \alpha)) \circ \alpha^{-1}.$$

We have to show, that B is a continuous mapping into the asserted image space.
Case 1 $(0 \leq \beta \leq 1)$: In this case the extension $\tilde{u}$ of a function $u \in \tilde{H}^{\beta,q}_{w\circ\alpha}(\mathbb{R}^n_+)$ by 0 is a continuous operation into the space $H^{\beta,q}_{w\circ\alpha}(\mathbb{R}^n)$. By interpolation the operator

$$\overline{B} : H^{\beta,q}_{w\circ\alpha}(\mathbb{R}^n) \to H^{\beta,q}_w(\mathbb{R}^n), \quad u \mapsto (u(\psi \circ \alpha)) \circ \alpha^{-1}$$

is continuous and we obtain for $u \in \tilde{H}^{\beta,q}_{w\circ\alpha}(\mathbb{R}^n_+)$

$$\|Bu\|_{\tilde{H}^{\beta,q}_w(\Omega)} = \|\widetilde{Bu}\|_{\beta,q,w,\mathbb{R}^n} = \|\overline{B}\tilde{u}\|_{\beta,q,w,\mathbb{R}^n} \leq c\|\tilde{u}\|_{\beta,q,w\circ\alpha,\mathbb{R}^n} = c\|u\|_{\tilde{H}^{\beta,q}_{w\circ\alpha}(\mathbb{R}^n_+)}.$$

The assertion for $0 \leq \beta \leq 1$ is proved.
Case 2 $(1 \leq \beta \leq 2)$: Interpolation shows that B, extended in the canonic way, maps $H^{\beta,q}_{w\circ\alpha}(\mathbb{R}^n_+)$ continuously to $H^{\beta,q}_w(\Omega)$. Since

$$B(\{u \in W^{2,q}_{w\circ\alpha}(\mathbb{R}^n) \mid u|_{\mathbb{R}^{n-1}} = 0\}) \subset \{u \in W^{2,q}_w(\Omega) \mid u|_{\partial\Omega} = 0\}$$

the operator $B : \tilde{H}^{\beta,q}_{w\circ\alpha}(\mathbb{R}^n_+) \to \tilde{H}^{\beta,q}_w(\Omega)$ is continuous by the density of $Y_{q,w}$ in $\tilde{H}^{\beta,q}_w(\Omega)$. Now setting $B_j u = (u(\psi_j \circ \alpha_j)) \circ \alpha_j^{-1}$ we define the operator

$$B_\Omega : \prod_{i=1}^m \tilde{H}^{\beta,q}_{w\circ\alpha_i}(\mathbb{R}^n_+) \to \tilde{H}^{\beta,q}_w(\Omega), \quad (u_1, ..., u_m) \mapsto \sum_{i=1}^m B_i u_i$$

which is continuous and surjective. (Surjectivity follows considering the operator $H^{\beta,q}_w(\Omega) \ni u \mapsto (u\phi_j) \circ \alpha_j \in H^{\beta,q}_{w\circ\alpha_j}(\mathbb{R}^n_+)$, where ϕ_j is an appropriate cut-off function, $\equiv 1$ on $\operatorname{supp}\psi_j$.)

Moreover, by interpolation it follows that

$$B_\Omega : \prod_{i=1}^m \tilde{H}^{\beta,q}_{w\circ\alpha_i}(\mathbb{R}^n_+) \to [L^q_w(\Omega), Y_{q,w}(\Omega)]_{\frac{\beta}{2}}$$

is continuous. Thus we obtain $[L^q_w(\Omega), Y_{q,w}(\Omega)]_{\frac{\beta}{2}} \supset \tilde{H}^{\beta,q}_w(\Omega)$.
The inclusion $'' \subset''$ is proved in the same way as in the proof of Theorem 3.1, Step 1. $\square$

3.2 Interpolation of Bessel Potential Spaces of Negative Order

It is an easy consequence of the definition of Bessel potential spaces that $H^{-\beta,q}_w(\mathbb{R}^n) = \left(H^{\beta,q'}_{w'}(\mathbb{R}^n)\right)'$ isometrically for every $\beta > 0$.

Theorem 3.3. *If $1 < q < \infty$, $w \in A_q$, $l, k \in \mathbb{N}$ and $-l < \beta < k$ then*

$$\left[H_w^{-l,q}(\mathbb{R}^n), H_w^{k,q}(\mathbb{R}^n)\right]_\theta = H_w^{\beta,q}(\mathbb{R}^n),$$

where $\theta = \frac{\beta+l}{k+l}$.

Proof. The proof is analogous to the one of [16, Proposition 13.6.2] using the weighted multiplier theorem [11, IV Theorem 3.9]. □

For $\beta > 0$ the weighted Bessel potential space of negative order on an extension domain Ω is defined by $H_w^{-\beta,q}(\Omega) = \{u|_{C_0^\infty(\Omega)} \mid u \in H_w^{-\beta,q}(\mathbb{R}^n)\}$, equipped with the norm

$$\|u\|_{-\beta,q,w,\Omega} = \inf\{\|v\|_{-\beta,q,w,\mathbb{R}^n} \mid v \in H_w^{-\beta,q}(\mathbb{R}^n),\ v|_{C_0^\infty(\Omega)} = u\}.$$

Moreover, we set

$$H_{w,0}^{\beta,q}(\Omega) = \overline{(C_0^\infty(\Omega))}^{H_w^{\beta,q}(\mathbb{R}^n)},$$

assuming that $\phi \in C_0^\infty(\Omega)$ is extended by 0 to a function $\tilde{\phi}$ on $\mathbb{R}^n$. The space $H_{w,0}^{\beta,q}(\Omega)$ is equipped with the norm $\|\cdot\|_{\beta,q,w,0,\Omega} := \|\cdot\|_{\beta,q,w,\mathbb{R}^n}$. Note that by (3.1) this norm is in general not equivalent to $\|\cdot\|_{s,q,w,\Omega}$. With this definition one obtains a good behavior of the dual spaces and interpolation properties. It holds

$$H_w^{-\beta,q}(\Omega) = \left(H_{w',0}^{\beta,q'}(\Omega)\right)', \tag{3.2}$$

with equivalent norms.

To see this, let $u \in H_w^{-\beta,q}(\Omega)$. Then by definition there exists $U \in H_w^{-\beta,q}(\mathbb{R}^n)$ such that $U|_{C_0^\infty(\Omega)} = u$ with

$$\begin{aligned} 2\|u\|_{-\beta,q,w,\Omega} &\geq \|U\|_{-\beta,q,w,\mathbb{R}^n} = \sup_{\phi\in\mathcal{S}(\mathbb{R}^n),\|\phi\|_{\beta,q',w',\mathbb{R}^n}\leq 1} \langle U, \phi\rangle \\ &\geq \sup_{\phi\in C_0^\infty(\Omega),\|\phi\|_{\beta,q',w',\mathbb{R}^n}\leq 1} \langle u, \phi\rangle = \|u\|_{(H_{w',0}^{\beta,q'}(\Omega))'}. \end{aligned}$$

Thus $u \in (H_{w',0}^{\beta,q'}(\Omega))'$.

Vice versa, by Hahn-Banach's theorem every $u \in \left(H_{w',0}^{\beta,q'}(\Omega)\right)'$ can be extended to an element $U \in H_w^{-\beta,q}(\mathbb{R}^n)$ with $\|U\|_{-\beta,q,w,\mathbb{R}^n} = \|u\|_{(H_{w',0}^{\beta,q'}(\Omega))'}$. Then a similar calculation as above yields $u \in H_w^{-\beta,q}(\Omega)$ with $\|u\|_{-\beta,q,w,\Omega} \leq \|u\|_{(H_{w',0}^{\beta,q'}(\Omega))'}$.

(3.2) also yields the completeness of $H_w^{-\beta,q}(\Omega)$.

Lemma 3.4. *There exists a continuous linear extension operator*

$$E : H_w^{-1,q}(\Omega) \to H_w^{-1,q}(\mathbb{R}^n)$$

such that $Eu|_{C_0^\infty(\Omega)} = u$ which is also continuous as a mapping $: H_w^{1,q}(\Omega) \to H_w^{1,q}(\mathbb{R}^n)$.

Proof. We begin with showing the assertion for the half space $\Omega = \mathbb{R}^n_+$.

By [8] for every $f \in W^{-1,q}_w(\mathbb{R}^n_+)$ there exists a unique $u \in W^{1,q}_{w,0}(\mathbb{R}^n_+)$ solving $(1-\Delta)u = f$ which depends linearly on f and fulfills the estimate $\|u\|_{1,q} \le c\|f\|_{-1,q}$. We write $u = (1-\Delta)^{-1} f$.

To construct E we remind that by [2] there exists a linear continuous extension operator

$$\tilde{E} : W^{1,q}_w(\mathbb{R}^n_+) \to W^{1,q}_w(\mathbb{R}^n) \quad \text{and} \quad : W^{3,q}_w(\mathbb{R}^n_+) \to W^{3,q}_w(\mathbb{R}^n) \text{ with } \tilde{E}u|_{\mathbb{R}^n_+} = u.$$

For $u \in W^{-1,q}_w(\mathbb{R}^n_+)$ let $v = (1-\Delta)^{-1}u \in W^{1,q}_w(\mathbb{R}^n_+)$. Then $\|v\|_{1,q,w} \le c\|u\|_{-1,q,w}$ and by [13] from $u \in H^{1,q}_w(\mathbb{R}^n_+)$ it follows $v \in H^{3,q}_w(\mathbb{R}^n_+)$ with $\|v\|_{3,q,w} \le c\|u\|_{1,q,w}$. Now we set

$$Eu = (1-\Delta)\tilde{E}v.$$

Thus E has the asserted properties on the half space $\mathbb{R}^n_+$.

For a bounded $C^{1,1}$-domain Ω we take a collection of charts $(\alpha_j)_{j=1}^m$ and a decomposition of the unity $(\psi_j)_{j=1}^m$ subordinate to the domains of the charts. Then for $u \in W^{1,q}_w(\Omega)$ we set

$$E_\Omega u = \sum_{j=1}^m E_{\mathbb{R}^n_+}((u\psi_j) \circ \alpha_j) \circ \alpha_j^{-1},$$

where $E_{\mathbb{R}^n_+} : W^{1,q}_{w\circ\alpha_j}(\mathbb{R}^n_+) \to W^{1,q}_{w\circ\alpha_j}(\mathbb{R}^n)$ is the operator just constructed. Obviously $E_\Omega : W^{1,q}_w(\Omega) \to W^{1,q}_w(\mathbb{R}^n)$ is continuous. Moreover, it follows from (2.2) that $u \mapsto u \circ \alpha_j$ is a continuous operation from $W^{-1,q}_{w\circ\alpha_j}(\Omega) \to W^{-1,q}_w(\Omega)$. This shows the continuity of $E_\Omega : W^{-1,q}_w(\Omega) \to W^{-1,q}_w(\mathbb{R}^n)$ and the proof is complete. ☐

Theorem 3.5. *Let $1 < q < \infty$, $w \in A_q$, $-1 \le \beta \le 1$ and Ω a bounded $C^{1,1}$-domain. Then*

1. $[H^{-1,q}_w(\Omega), H^{1,q}_w(\Omega)]_\theta = H^{\beta,q}_w(\Omega)$, *where* $\theta = \frac{1+\beta}{2}$.

2. $$\left[H^{-1,q}_{w,0}(\Omega), H^{1,q}_w(\Omega)\right]_\theta = \begin{cases} H^{\beta,q}_{w,0}(\Omega) := (H^{-\beta,q'}_{w'}(\Omega))', & \text{if } \theta < \frac{1}{2} \\ H^{\beta,q}_w(\Omega), & \text{if } \theta \ge \frac{1}{2} \end{cases},$$

   where $\theta = \frac{1+\beta}{2}$.

Proof. 1. "$\subset$" By complex interpolation we obtain that

$$E : \left[H^{-1,w}_w(\Omega), H^{1,q}_w(\Omega)\right]_\theta \to \left[H^{-1,w}_w(\mathbb{R}^n), H^{1,q}_w(\mathbb{R}^n)\right]_\theta = H^{\beta,q}_w(\mathbb{R}^n)$$

is linear and continuous, where E is the extension operator from Lemma 3.4. Thus every $u \in [H^{-1,w}_w(\Omega), H^{1,q}_w(\Omega)]_\theta$ is the restriction of $Eu \in H^{\beta,q}_w(\mathbb{R}^n)$ to Ω. By definition, this implies $u \in H^{\beta,q}_w(\Omega)$.
"$\supset$" Follows from the same arguments as in the proof of "$\subset$", when replacing the extension operator E by the restriction operator

$$R : W^{\pm1,q}_w(\mathbb{R}^n) \to W^{\pm1,q}_w(\Omega), \quad u \mapsto u|_{C_0^\infty(\Omega)}.$$

2. An application of the duality theorem in [17] to 1. yields

$$\left[H_{w,0}^{-1,q}(\Omega), H_{w,0}^{1,q}(\Omega)\right]_\theta = H_{w,0}^{\beta,q}(\Omega). \tag{3.3}$$

Since

$$F(H_{w,0}^{-1,q}(\Omega), H_{w,0}^{1,q}(\Omega)) \subset F(H_{w,0}^{-1,q}(\Omega), H_{w}^{1,q}(\Omega))$$

and the same is true when replacing q by q' and w by w', we have

$$L_w^q(\Omega) = \left[H_{w,0}^{-1,q}(\Omega), H_{w,0}^{1,q}(\Omega)\right]_{\frac{1}{2}} \hookrightarrow \left[H_{w,0}^{-1,q}(\Omega), H_{w}^{1,q}(\Omega)\right]_{\frac{1}{2}} \tag{3.4}$$

and

$$L_{w'}^{q'}(\Omega) \hookrightarrow \left[H_{w',0}^{-1,q'}(\Omega), H_{w'}^{1,q'}(\Omega)\right]_{\frac{1}{2}} = \left[H_{w,0}^{-1,q}(\Omega), H_{w}^{1,q}(\Omega)\right]_{\frac{1}{2}}' . \tag{3.5}$$

By the density of the embedding $H_{w'}^{1,q'}(\Omega) \hookrightarrow \left[H_{0,w'}^{-1,q'}(\Omega), H_{w'}^{1,q'}(\Omega)\right]_{\frac{1}{2}}$ we obtain that the embedding (3.5) is dense. Thus we obtain, dualizing (3.5) and combining it with (3.4),

$$\left[H_{0,w}^{-1,w}(\Omega), H_{w}^{1,q}(\Omega)\right]_{\frac{1}{2}} = L_w^q(\Omega).$$

Now the assertion follows by the reiteration property in [17], using (3.3) for $\beta < \frac{1}{2}$ and the assertion of 1. for $\beta \geq \frac{1}{2}$ $\square$

3.3 Stokes Equations in Weighted Bessel Potential Spaces

Theorem 3.6. *Let $1 < q < \infty$, $w \in A_q$, $0 \leq \beta \leq 2$ and let Ω be a bounded $C^{1,1}$-domain. Moreover let*

$$f \in \tilde{H}_w^{\beta-2,q}(\Omega) := \left(\tilde{H}_{w'}^{2-\beta,q'}(\Omega)\right)' \quad \text{and} \quad k \in H_{w,*}^{\beta-1,q}(\Omega) := \begin{cases} H_w^{\beta-1,q}(\Omega), & \text{if } \beta \geq 1 \\ H_{0,w}^{\beta-1,q}(\Omega), & \text{if } \beta < 1 \end{cases}$$

with $\langle k, 1\rangle = 0$. Then there exists a unique very weak solution $u \in \tilde{H}_w^{\beta,q}(\Omega)$ to the Stokes problem with respect to the data f, k, i.e.,

$$\begin{aligned} \langle f, \varphi\rangle &= -\langle u, \Delta\varphi\rangle, && \text{for all } \varphi \in Y_\sigma \text{ and} \\ \langle k, \psi\rangle &= -\langle u, \nabla\psi\rangle, && \text{for all } \psi \in W_{w'}^{1,q'}(\Omega). \end{aligned} \tag{3.6}$$

This u fulfills the estimate

$$\|u\|_{\tilde{H}_w^{\beta,q}(\Omega)} \leq c\left(\|f\|_{\tilde{H}_w^{\beta-2,q}(\Omega)} + \|k\|_{\beta-1,q,w,*,\Omega}\right),$$

where $\|\cdot\|_{\beta-1,q,w,*,\Omega} = \|\cdot\|_{\beta-1,q,w,0,\Omega}$ if $\beta < 1$ and $\|\cdot\|_{\beta-1,q,w,*,\Omega} = \|\cdot\|_{\beta-1,q,w,\Omega}$ if $\beta \geq 1$. Moreover there exists a pressure functional $p \in H_w^{\beta-1,q}(\Omega)$, unique modulo constants, such that the Stokes equations

$$-\Delta u + \nabla p = f|_{C_0^\infty(\Omega)}, \quad \text{on } \Omega$$

are fulfilled in the sense of distributions.

Proof. In Sections 3.1 and 3.2 we have shown

$$\left[(Y_{q',w'})' \times H^{-1,q}_{w,0}(\Omega), L^q_w(\Omega) \times H^{1,q}_w(\Omega)\right]_\theta = \tilde{H}^{\beta-2,q}_w(\Omega) \times H^{\beta-1,q}_{w,*}(\Omega),$$

where $\theta = \frac{\beta}{2}$. It is immediate that

$$k \mapsto K := k - \langle k, 1\rangle \in \mathcal{L}(W^{-1,q}_{w,0}(\Omega)) \cap \mathcal{L}(W^{1,q}_w(\Omega)).$$

By Theorem 2.2 the mapping

$$\mathcal{S} : (Y_{q',w'})' \times H^{-1,q}_{w,0}(\Omega) \ni (f, k) \mapsto u \in L^q_w(\Omega),$$

is continuous, where $u \in L^q_w(\Omega)$ is the very weak solution to the Stokes problem with respect to the data f and $K = k - \langle k, 1\rangle$.

If u is a solution to (3.6) with sufficiently regular data f and k then by Theorem 2.3 we find that u is a strong solution with 0 boundary values. In particular $\mathcal{S}$ is also continuous from $L^q_w(\Omega) \times H^{1,q}_w(\Omega)$ to $Y_{q,w}$.
Now we obtain from interpolation that

$$\mathcal{S} : \tilde{H}^{\beta-2,q}_w(\Omega) \times H^{\beta-1,q}_{w,*}(\Omega) \to \tilde{H}^{\beta,q}_w(\Omega)$$

is continuous, which finishes the proof of existence and estimates of u. Uniqueness follows from the uniqueness of the very weak solutions in $L^q_w(\Omega)$ (Theorem 2.2).
To show the existence of p we use the interpolation Theorem 3.5.1. and the existence and uniqueness (modulo constants) of the pressure in the case of strong [8] and of very weak solutions (Theorem 2.2). Then we obtain a functional $\tilde{p} \in H^{\beta-1,q}_{w,*}(\Omega)$ such that

$$-\langle u, \Delta\phi\rangle - \langle \tilde{p}, \operatorname{div}\phi\rangle = \langle F, \phi\rangle \quad \text{for all} \quad \phi \in Y_{q',w'}.$$

The restriction $p := \tilde{p}|_{C^\infty_0(\Omega)}$ solves the problem. □

By the definition of $\tilde{H}^{\beta,q}_w(\Omega)$ it follows, that whenever a restriction operator

$$\operatorname{tr} : H^{\beta,q}_w(\Omega) \to T(D)$$

for a boundary portion $D \subset \partial\Omega$ is well-defined (as a continuous linear operator into some boundary space $T(D)$, which coincides with the usual trace on $W^{2,q}_w(\Omega)$), then for the solution $u \in \tilde{H}^{\beta,q}_w(\Omega)$ one has $\operatorname{tr} u = 0$.

In the case, where data and solutions are regular enough (including the case $\beta = 1$ of weak solutions) we want to deal with inhomogeneous boundary values.

If $\beta \geq 1$ then $H^{\beta,q}_w(\Omega) \hookrightarrow W^{1,q}_w(\Omega)$ which implies the existence of a continuous restriction operator

$$\operatorname{tr} : H^{\beta,q}_w(\Omega) \to T^{1,q}_w(\partial\Omega), \quad \operatorname{tr} u = u|_{\partial\Omega} \text{ if } u \in C^\infty(\overline{\Omega}).$$

As in the case of weighted Sobolev spaces we define the associated boundary space by

$$T^{\beta,q}_w(\partial\Omega) = \operatorname{tr}\left(H^{\beta,q}_w(\Omega)\right)$$

equipped with the norm of the factor space

$$\|g\|_{T^{\beta,q}_w(\partial\Omega)} = \inf\{\|u\|_{\beta,q,w,\Omega} \mid u \in H^{\beta,q}_w(\Omega),\ \operatorname{tr} u = g\}.$$

Lemma 3.7.

$$[T_w^{1,q}(\partial\Omega), T_w^{2,q}(\partial\Omega)]_{\beta-1} = T_w^{\beta,q}(\partial\Omega)$$

and there exists a continuous linear extension operator ext $: T_w^{\beta,q}(\partial\Omega) \to H_w^{\beta,q}(\Omega)$.

Proof. By [6] or [13] there exists a continuous linear extension operator

$$\text{ext } : T_w^{1,q}(\partial\Omega) \to W_w^{1,q}(\Omega) \ \text{ and } \ : T_w^{2,q}(\partial\Omega) \to W_w^{2,q}(\Omega),$$

with $(\text{ext}\, g)|_{\partial\Omega} = g$ for every $g \in T_w^{k,q}(\partial\Omega)$, $k = 1, 2$. Thus by interpolation

$$\text{ext } : [T_w^{1,q}(\partial\Omega), T_w^{2,q}(\partial\Omega)]_{\beta-1} \to H_w^{\beta,q}(\Omega)$$

is continuous and we obtain $[T_w^{1,q}(\partial\Omega), T_w^{2,q}(\partial\Omega)]_{\beta-1} \subset T_w^{\beta,q}(\partial\Omega)$.

Vice versa the restriction operator

$$\text{tr } : H_w^{\beta,q} = [W_w^{1,q}(\Omega), W_w^{2,q}(\Omega)]_{\beta-1} \to [T_w^{1,q}(\partial\Omega), T_w^{2,q}(\partial\Omega)]_{\beta-1}$$

is continuous which implies $[T_w^{1,q}(\partial\Omega), T_w^{2,q}(\partial\Omega)]_{\beta-1} \supset T_w^{\beta,q}(\partial\Omega)$.

Thus the first assertion is proved. The second assertion follows from the first assertion when applying complex interpolation to ext . ☐

Theorem 3.8. *Let* $1 < q < \infty$*,* $w \in A_q$ *and* $1 \le \beta \le 2$*. Moreover let* $F \in H_w^{\beta-2,q}(\Omega)$*,* $K \in H_w^{\beta-1,q}(\Omega)$ *and* $g \in T_w^{\beta,q}(\partial\Omega)$*. Then there exists a unique weak solution* $u \in H_w^{\beta,q}(\Omega)$*, i.e.*

$$\langle \nabla u, \nabla \phi \rangle = \langle F, \phi \rangle, \quad \textit{for all} \quad \phi \in W_{w,0,\sigma}^{1,q}(\Omega)$$

fulfilling $u|_{\partial\Omega} = g$ *and* $\operatorname{div} u = K$ *in the sense of distributions. This solution fulfills the estimate*

$$\|u\|_{\beta,q,w} \le c(\|F\|_{\beta-2,q,w} + \|K\|_{\beta-1,q,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)}).$$

Moreover there exists a pressure function $p \in H_w^{\beta-1,q}(\Omega)$*, unique modulo constants, such that the Stokes equations are fulfilled distributionally.*

Proof. Existence: For $g \in T_w^{\beta,q}(\partial\Omega)$ there exists $v \in H_w^{\beta,q}(\Omega)$ such that $\operatorname{tr} v = g$ and $\|v\|_{\beta,q,w,\Omega} \le 2\|g\|_{T_w^{\beta,q}(\partial\Omega)}$. Since there exists an extension V of v to the whole space $\mathbb{R}^n$ with $\|V\|_{\beta,q,w,\mathbb{R}^n} \le c\|v\|_{\beta,q,w,\Omega}$, one has $\Delta v = (\Delta V)|_{C_0^\infty(\Omega)} \in H_w^{\beta-2,q}(\Omega) = \tilde{H}_w^{\beta-2,q}(\Omega)$.

Hence by Theorem 3.6 there exists $U \in H_w^{\beta,q}(\Omega)$ solving

$$\begin{aligned} \langle F + \Delta v, \varphi \rangle &= -\langle U, \Delta\varphi \rangle, && \text{for all } \varphi \in Y_\sigma \text{ and} \\ \langle K - \operatorname{div} v, \psi \rangle &= -\langle U, \nabla\psi \rangle, && \text{for all } \psi \in W_{w'}^{1,q'}(\Omega). \end{aligned}$$

Since $U \in \tilde{H}_w^{\beta,q}(\Omega) \subset W_{w,0}^{1,q}(\Omega)$, we obtain by integration by parts for $\phi \in Y_{q',w'}$, which is dense in $W_{w,0}^{1,q}(\Omega)$, that

$$\langle \nabla(U + v), \nabla\phi \rangle = -\langle U, \Delta\phi \rangle - \langle \Delta v, \phi \rangle = \langle F, \phi \rangle.$$

Setting now $u := U + v$ we clearly obtain $\operatorname{div} u = K$ distributionally and $\operatorname{tr} u = \operatorname{tr} v + \operatorname{tr} U = \operatorname{tr} v = g$. Moreover

$$\|u\|_{\beta,q,w,\Omega} \le \|v\|_{\beta,q,w,\Omega} + \|U\|_{\beta,q,w,\Omega} \le c(\|g\|_{T_w^{\beta,q}(\partial\Omega)} + \|F\|_{\beta-2,q,w,\Omega} + \|K\|_{\beta-1,q,w,\ Omega}).$$

Uniqueness: Let u be a weak solution to the Stokes problem w.r.t the data F, K and g. Then integration by parts yields

$$\langle u, \Delta\phi\rangle = -\langle \nabla u, \nabla\phi\rangle + \langle u, N\cdot\nabla\phi\rangle_{\partial\Omega} = -\langle F, \phi\rangle + \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}. \tag{3.7}$$

Since $|\langle u, N\cdot\nabla\phi\rangle_{\partial\Omega}| \le \|g\|_{T_w^{0,q}(\partial\Omega)}\|\phi\|_{2,q',w',\Omega}$ we find that the right hand side of (3.7), considered as a map in ϕ, is contained in $(Y_{q',w'})'$. Thus u is a very weak solution. By the uniqueness of very weak solutions in Theorem 2.2, we obtain the uniqueness of u.
Pressure: To show the existence of p we use that by de Rahm's Theorem there exists $p \in (C_0^\infty(\Omega))'$ such that the Stokes equations are fulfilled distributionally. From the equation we obtain $\nabla p \in H_w^{\beta-2,q}(\Omega)$. It remains to show $p \in H_w^{\beta-1,q}(\Omega)$. For $\beta = 1$ and $\beta = 2$ this is clear by the Poicarrée inequality [7, Corollary 2.1]. If we assume in addition that $\int p = 0$ we obtain that p depends linearly and continuously on f, k and g. Thus interpolation shows $p \in H_w^{\beta-1,q}(\Omega)$. □

Now we turn to the case $0 \le \beta \le 1$. Here we define boundary spaces by

$$T_w^{\beta,q}(\partial\Omega) = \left[T_w^{0,q}(\partial\Omega), T_w^{1,q}(\partial\Omega)\right]_\beta,$$

equipped with the norm of the interpolation space.

To ensure the well-definedness of the boundary conditions we need to demand that the force F and the divergence K is contained in some space of distributions on Ω. Since Sobolev embeddings require strong assumptions to the weight function w we assume (3.9). See Lemma 4.3 for sufficient conditions such that (3.9) is fulfilled.

Theorem 3.9. *Let $1 < q < \infty$, $w \in A_q$ and $0 \le \beta \le 1$. Assume that $f \in \tilde{H}_w^{\beta-2,q}(\Omega)$ and $k \in H_{w,0}^{\beta-1,q}(\Omega)$ allow decompositions into*

$$\begin{aligned} \langle f, \phi\rangle &= \langle F, \phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}, \quad &&\text{for every } \phi \in Y_{q',w'} \\ \langle k, \psi\rangle &= \langle K, \psi\rangle - \langle g, N\psi\rangle_{\partial\Omega}, &&\text{for every } \psi \in W_{w'}^{1,q'}(\Omega) \end{aligned} \tag{3.8}$$

with $F \in W_{\tilde{w}}^{-1,r}(\Omega)$, $K \in L_{\tilde{w}}^r(\Omega)$ and $g \in T_w^{\beta,q}(\partial\Omega)$, where r and $\tilde{w} \in A_r$ are chosen such that

$$W_{\tilde{w}}^{-1,r}(\Omega) \hookrightarrow \tilde{H}_w^{\beta-2,q}(\Omega) \quad \text{and} \quad L_{\tilde{w}}^r(\Omega) \hookrightarrow H_{w,0}^{\beta-1,q}(\Omega). \tag{3.9}$$

Then the very weak solution $u \in L_w^q(\Omega)$ with respect to f and k which exists according to Theorem 2.2 is contained in $H_w^{\beta,q}(\Omega)$, assumes the boundary value g in the sense of (2.11) and (2.12) and fulfills the estimate

$$\|u\|_{\beta,q,w} \le c(\|F\|_{\tilde{H}_w^{\beta-2,q}(\Omega)} + \|K\|_{H_{w,0}^{\beta-1,q}(\Omega)} + \|g\|_{T_w^{\beta,q}(\partial\Omega)}). \tag{3.10}$$

Proof. Step 1: We consider the operator

$$B : T_w^{0,q}(\partial\Omega) \to L_w^q(\Omega), \quad g \mapsto u,$$

where u is the very weak solution to the Stokes problem with data

$$f = [\phi \mapsto \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}] \quad \text{and} \quad k = [\psi \mapsto \langle g, N\psi\rangle_{\partial\Omega}].$$

B is linear and continuous, also considered as an operator $B : T^{1,q}_w(\partial\Omega) \to W^{1,q}_w(\Omega)$. This follows from Theorem 3.8 in the case $\beta = 1$ since the very weak solution with respect to f and k coincides with the weak solution with 0 force and divergence and boundary condition g. Thus, interpolation yields that

$$B : T^{\beta,q}_w(\partial\Omega) \to H^{\beta,q}_w(\Omega)$$

is continuous which implies the existence and estimates of the solution. Uniqueness follows from the uniqueness of the very weak solution in $L^q_w(\Omega)$ which is known from Theorem 2.2.

Step 2: Let $U = Bg \in H^{\beta,q}_w(\Omega)$ given by Step 1. Moreover let $v \in W^{1,r}_{\tilde{w}}(\Omega)$ be the weak solution to the Stokes Problem w.r.t the data F, K and zero boundary values which exists according to Theorem 3.8.

By the embeddings (3.9) and Theorem 3.6 there also exists a very weak solution in $H^{\beta,q}_w(\Omega)$. Since both, $H^{\beta,q}_w(\Omega)$ and $W^{1,r}_{\tilde{w}}(\Omega)$ are embedded into some common space $L^t(\Omega)$ for some $t > 1$ we obtain by the uniqueness of the very weak solution in $L^t(\Omega)$ (Theorem 3.9) that these solutions coincide. This yields the estimate

$$\|v\|_{\beta,q,w} \le c(\|F\|_{\tilde{H}^{\beta-2,q}_w(\Omega)} + \|K\|_{H^{\beta-1,q}_{0,w}(\Omega)})$$

Now we set $u := U + v$. Then u is a very weak solution with respect to f and k and the estimate (3.10). Moreover, by the definition of the trace in (2.11) and (2.12) we obtain $u|_{\partial\Omega} = g$. $\square$

The proof of the above theorem works in the same way, if one chooses $F \in \tilde{H}^{\beta-2,q}_w(\Omega)$ and $K \in H^{\beta-1,q}_{w,0}(\Omega)$. This is also visible in the a priori estimate (3.10). However, with such data it is not clear if the outcoming solution is regular enough to ensure that the boundary value $u|_{\partial\Omega}$ is well defined.

Corollary 3.10. *Let Ω be a bounded $C^{1,1}$-domain. Moreover, let $1 < q, r < \infty$, $w \in A_q$, $v \in A_r$ and $0 \le \beta \le 2$ be given such that $H^{\beta,q}_w(\Omega) \hookrightarrow L^r_v(\Omega)$. Then*

$$T^{\beta,q}_w(\partial\Omega) \hookrightarrow T^{0,r}_v(\partial\Omega).$$

Proof. Let $g \in T^{\beta,q}_w(\partial\Omega)$. Then the very weak solution $u \in H^{\beta,q}_w(\Omega)$ to

$$\begin{aligned} -\langle u, \Delta\phi\rangle &= \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega} \quad \text{for all } \phi \in Y_{q',w',\sigma}(\Omega) \\ -\langle u, \nabla\psi\rangle &= \langle g, N\psi\rangle_{\partial\Omega} \qquad \text{for all } \psi \in W^{1,q'}_{w'}(\Omega) \end{aligned}$$

fulfills $\|u\|_{\beta,q,w} \le c\|g\|_{T^{\beta,q}_w(\partial\Omega)}$ and $u \in \tilde{W}^{r,r}_{v,v}$ (defined in (2.10)) with $\|u\|_{\tilde{W}^{r,r}_{v,v}} = \|u\|_{r,v}$ and $\operatorname{div} u = 0$. Thus tangential and normal trace are well-defined for u and since $u|_{\partial\Omega} = g$ we obtain

$$\|g\|_{T^{0,r}_v(\partial\Omega)} \le c\|u\|_{r,v} \le c\|u\|_{\beta,q,w} \le c\|g\|_{T^{\beta,q}_w(\partial\Omega)}.$$

$\square$

4 Stationary Navier Stokes Equations with Irregular Data

4.1 Estimates of the Nonlinear Term

We prepare some embedding theorems. For $0 < \beta < n$ we define the weakly singular integral operator

$$I_\beta g(x) = \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n-\beta}} dy.$$

Theorem 4.1. *Let $0 < \beta < n$ and $1 < p < q < \infty$, $v \in A_p$ and $w \in A_q$. Moreover assume that v and w fulfill the condition*

$$|Q|^{\frac{\beta}{n}-1} \left(\int_Q w\right)^{\frac{1}{q}} \left(\int_Q v^{-\frac{1}{p-1}}\right)^{\frac{1}{p'}} < c \quad \textit{for every cube} \quad Q \subset \mathbb{R}^n.$$

Then

$$\|I_\beta f\|_{q,w} \leq c\|f\|_{p,v} \quad \textit{for every} \quad f \in L^q_v(\mathbb{R}^n).$$

Proof. This is a special case of [14, Theorem 1 (B)] □

Lemma 4.2. *Let $1 < q < \infty$, $w \in A_q$, $1 < s < q$ and $\Omega \subset \mathbb{R}^n$ be bounded and open. Moreover we assume that*

$$|Q|^s \leq cw(Q) \quad \textit{for every cube} \quad Q \subset \Omega_\delta = \{x \in \mathbb{R}^n,\ \operatorname{dist}(x,\Omega) \leq \delta\}.$$

Then there exists a weight function $W \in A_q$ with $w|_\Omega = W|_\Omega$ and

$$|Q|^s \leq cW(Q) \quad \textit{for every cube} \quad Q \subset \mathbb{R}^n.$$

Proof. [6, Lemma A.2] □

Lemma 4.3. *Let Ω be a bounded extension domain. Moreover, let $1 \leq s < r < q < \infty$ and assume $0 < \beta < n$ such that*

$$\frac{1}{q} \geq \frac{1}{r} - \frac{\beta}{ns}.$$

Then for every $w \in A_s$ the following embeddings are true:

1. $H^{\beta,r}_w(\Omega) \hookrightarrow L^q_w(\Omega)$.
2. $H^{\beta,q'}_{w_q}(\Omega) \hookrightarrow L^{r'}_{w_r}(\Omega)$, *where* $w_q = w^{-\frac{1}{q-1}}$ *and* $w_r = w^{-\frac{1}{r-1}}$.
3. $L^r_w(\Omega) \hookrightarrow H^{-\beta,q}_w(\Omega)$ *and* $L^r_w(\Omega) \hookrightarrow H^{-\beta,q}_{w,0}(\Omega)$.

Proof. 1. By [9, Corollary 3.2] the asserted embedding holds, if $|Q|^{\frac{\beta}{n}} w(Q)^{\frac{1}{q}-\frac{1}{r}} < C$ for all $Q \subset U$ for some open set $U \supset \overline{\Omega}$. By [15] we know that for every $Q \subset U$ and $w \in A_s$ it holds $|Q|^s \leq \frac{|U|^s}{w(U)} w(Q)$. Thus

$$|Q|^{\frac{\beta}{n}} w(Q)^{\frac{1}{q}-\frac{1}{r}} \leq cw(Q)^{\frac{\beta}{sn}+\frac{1}{q}-\frac{1}{r}} \leq cw(U)^{\frac{\beta}{sn}+\frac{1}{q}-\frac{1}{r}} =: C,$$

since $\frac{\beta}{sn}+\frac{1}{q}-\frac{1}{r}\geq 0$.
2. In the same way as in [9, Theorem 3.2] one shows using Theorem 4.1 that

$$|Q|^{\frac{\alpha}{n}-1}\left(\int_Q w_r\right)^{\frac{1}{r'}}\left(\int_Q (w_q)^{-\frac{1}{q'-1}}\right)^{\frac{1}{q}} < c \ \text{ for every cube } \ Q\subset\mathbb{R}^n \tag{4.1}$$

implies

$$H^{\gamma,q'}_{w_q}(\mathbb{R}^n)\hookrightarrow L^{r'}_{w_r}(\mathbb{R}^n) \ \text{ for every } \ \gamma\geq\alpha.$$

Thus we have to show (4.1). As above $w\in A_s$ implies $w(Q)\geq c(U)|Q|^s$ for every $Q\subset U$. Thus by Lemma 4.2 we can assume without loss of generality $w(Q)\geq c(U)|Q|^s$ for every cube $Q\subset\mathbb{R}^n$. Since

$$w_r^{-\frac{1}{r'-1}} = w^{\frac{1}{r'-1}\frac{1}{r-1}} = w = (w_q)^{-\frac{1}{q'-1}},$$

we can calculate using the definition of Muckenhoupt weights, $w\in A_r$ and $\frac{1}{q}-\frac{1}{r}<0$

$$\begin{aligned}|Q|^{\frac{\alpha}{n}-1}\left(\int_Q w_r\right)^{\frac{1}{r'}}\left(\int_Q (w_q)^{-\frac{1}{q'-1}}\right)^{\frac{1}{q}} &= |Q|^{\frac{\alpha}{n}-1}w_r(Q)^{\frac{1}{r'}}w(Q)^{\frac{1}{q}}\\ &\leq c|Q|^{\frac{\alpha}{n}}w(Q)^{(\frac{1}{q}-\frac{1}{r})}\\ &\leq c|Q|^{\frac{\alpha}{n}+s(\frac{1}{q}-\frac{1}{r})}.\end{aligned}$$

The last term is bounded if $\frac{\alpha}{n}+s(\frac{1}{q}-\frac{1}{r})=0$. There exists $0\leq\alpha\leq\beta$ so that this is true, because $s\left(\frac{1}{q}-\frac{1}{r}\right)<0$ and for $\alpha=\beta$ one has $\frac{\beta}{n}+s(\frac{1}{q}-\frac{1}{r})\geq\frac{\beta}{n}-s\frac{\beta}{sn}=0$.

This finishes the proof of 2.
3. Follows when considering the dual spaces in 2 and using $H^{\beta,q'}_{w',0}(\Omega)\hookrightarrow H^{\beta,q'}_{w'}(\Omega)$. $\square$

Lemma 4.4. *Let $\Omega\subset\mathbb{R}^n$ be a bounded $C^{1,1}$-domain. Assume $w\in A_s$ for some $s<q$ and $\beta\geq\left(\frac{ns}{q}-1\right)$.*

1. *Let in addition $\beta\leq 1$ and $1<t<\infty$ with*

$$\frac{1-\beta}{ns}+\frac{1}{q}-\frac{1}{t}=0. \tag{4.2}$$

If $n=2$ assume in addition $\beta>-\frac{1}{2}+\frac{2s}{q}$. Then $w\in A_t$,

$$L^t_w(\Omega)\hookrightarrow H^{\beta-1,q}_{w,0}(\Omega)$$

and

a)

$$\left|\int uv\psi\,dx\right|\leq c\|u\|_{\beta,q,w}\|v\|_{\beta,q,w}\|\psi\|_{t',w'}$$

for every $u,v\in H^{\beta,q}_w(\Omega)$ and $\psi\in H^{1-\beta,q'}_{w'}(\Omega)$.

b)

$$\left|\int ku\phi\,dx\right| \le c\|k\|_{t,w}\|u\|_{\beta,q,w}\|\phi\|_{1,t',w'}$$

for every $k \in L^t_w(\Omega)$, $u \in H^{\beta,q}_w(\Omega)$ *and* $\phi \in H^{2-\beta,q'}_{w'}(\Omega)$.

2. *If* $\beta \ge 1$ *then*

$$\|u\nabla v\|_{2-\beta,q,w} \le c\|u\|_{\beta,q,w}\|v\|_{\beta,q,w} \quad \textit{for every} \quad u,v \in H^{\beta,q}_w(\Omega),$$

if in the case $n=2$ *the additional condition* $\beta > \frac{2s}{q}$ *is satisfied.*

Proof. One has

$$t = \frac{nsq}{q(1-\beta)+ns} \ge \frac{nsq}{q(2-\frac{ns}{q})+ns} = \frac{ns}{2} > s$$

if $n \ge 3$. If $n=2$ then $t > s$ is guaranteed by the supplementary condition below (4.2). Thus, by Lemma 4.3 one has $L^t_w(\Omega) \hookrightarrow H^{\beta-1,q}_{w,0}(\Omega)$ and $H^{1-\beta,q'}_{w_q}(\Omega) \hookrightarrow L^{t'}_{w_t}(\Omega)$.
1. a) Let $r = \frac{nsq}{-q\beta+ns}$. Then by Lemma 4.3 one has $u \in H^{\beta,q}_w(\Omega) \hookrightarrow L^r_w(\Omega)$. Moreover we set $\eta = (1-\frac{2}{r})^{-1} = \frac{nsq}{nsq+2\beta q-2ns}$. Then

$$\frac{1}{\eta'} - \frac{1}{t} = \frac{-2\beta q+2ns}{nsq} - \frac{q-\beta q+ns}{nsq} = \frac{-q+ns-\beta q}{nsq} \le 0$$

Thus $\eta' \ge t$ which implies $L^{\eta'}_w(\Omega) \hookrightarrow L^t_w(\Omega)$ and hence $L^{t'}_{w_t}(\Omega) \hookrightarrow L^{\eta}_{w_{\eta'}}(\Omega)$.

Since $\frac{1}{r}+\frac{1}{r}+\frac{1}{\eta} = 1$ and $-\frac{1}{(\eta'-1)\eta}+\frac{1}{r}+\frac{1}{r} = 0$ we can calculate

$$\begin{aligned} \left|\int uv\phi\,dx\right| &= \left|\int uw^{\frac{1}{r}}vw^{\frac{1}{r}}\psi w^{\frac{1}{\eta}}_{\eta'}\,dx\right| \\ &\le \|u\|_{r,w}\|v\|_{r,w}\|\psi\|_{\eta,w_{\eta'}} \le c\|u\|_{\beta,q,w}\|v\|_{\beta,q,w}\|\psi\|_{t',w_t}. \end{aligned}$$

1. b) We set $\eta = \left(1-\frac{1}{r}-\frac{1}{t}\right)^{-1} = \frac{rt}{rt-t-r}$ and $r = \frac{nsq}{-q\beta+ns}$ as above. Then

$$\eta' = \frac{rt}{r+t} = \frac{nsq}{q+2ns-2q\beta} \ge \frac{nqs}{3q} \ge s \quad \text{if} \quad n \ge 3.$$

If $n=2$ one needs the condition below (4.2) to make sure $\eta' \ge s$. Using this and the fact that

$$-\frac{1}{\eta'}+\frac{1}{t}+\frac{1}{ns} = -\frac{1}{r}+\frac{1}{ns} = \frac{1+\beta-\frac{ns}{q}}{ns} \ge 0$$

we obtain $H^{1,t'}_{w_t}(\Omega) \hookrightarrow L^{\eta}_{w_{\eta'}}(\Omega)$. Since $\frac{1}{t}+\frac{1}{q}+\frac{1}{\eta} = 1$ and $-\frac{1}{(\eta'-1)\eta}+\frac{1}{t}+\frac{1}{r} = 0$ we can estimate

$$\begin{aligned} \left|\int ku\phi\,dx\right| &= \left|\int kw^{\frac{1}{t}}uw^{\frac{1}{r}}\phi w^{\frac{1}{\eta}}_{\eta'}\,dx\right| \\ &\le \|k\|_{t,w}\|u\|_{r,w}\|\phi\|_{\eta,w_{\eta'}} \le c\|k\|_{t,w}\|u\|_{\beta,q,w}\|\phi\|_{1,t',w_t}. \end{aligned}$$

2. Let $r := \frac{nsq}{ns+2q-q\beta}$, $\eta := \frac{nsq}{ns-q\beta}$ and $\mu := \frac{ns}{2}$. Then one has

- $\frac{1}{r} = \frac{1}{\eta} + \frac{1}{\mu}$.
- $L^r_w(\Omega) \hookrightarrow H^{\beta-2,q}_w(\Omega)$. If $n = 2$ we need the supplementary condition to ensure $r > s$.
- $H^{\beta,q}_w(\Omega) \hookrightarrow L^\eta_w(\Omega)$.
- $H^{\beta-1,q}_w(\Omega) \hookrightarrow L^\mu_w(\Omega)$ (using $\beta \geq \frac{ns}{q} - 1$).

Thus it follows

$$\begin{aligned}\|u\nabla v\|_{\beta-2,q,w} &\leq c\|u\nabla v\|_{r,w} = c\left(\int |u|^r w^{\frac{r}{\eta}} |\nabla v|^r w^{1-\frac{r}{\eta}}\right)^{\frac{1}{r}} \\ &\leq c\|u\|_{\eta,w}\|\nabla v\|_{\mu,w} \leq c\|u\|_{\beta,q,w}\|\nabla v\|_{\beta-1,q,w} \leq c\|u\|_{\beta,q,w}\|v\|_{\beta,q,w}.\end{aligned}$$

□

4.2 Stationary Navier Stokes Equations in Bessel Potential Spaces

In this section we always assume

- $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,1}$-domain,
- $1 < q < \infty$ and $w \in A_s$ for some $1 \leq s < q$,
- $\frac{ns}{q} - 1 \leq \beta \leq 2$ and $\beta \geq 0$. (If $n \leq 3$ this is always possible, since for $s = q$ one has $\frac{ns}{q} - 1 = n - 1 \leq 2$).

Definition 4.5. *Let $0 \leq \beta \leq 1$, $1 < q < \infty$ and $w \in A_q$. Moreover, let $g \in T^{\beta,q}_w(\partial\Omega)$, $F \in W^{-1,t}_w(\Omega)$ and $K \in L^t_w(\Omega)$. Then $u \in H^{\beta,q}_w(\Omega)$ is called a very weak solution to the stationary Navier-Stokes equations, if*

$$-\langle u, \Delta\phi\rangle + \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega} - \langle uu, \nabla\phi\rangle - \langle Ku, \phi\rangle = \langle F, \phi\rangle \quad \textit{for every} \quad \phi \in Y_{q',w',\sigma}(\Omega),$$

$\operatorname{div} u = K$ *is fulfilled in the sense of distributions and* $u \cdot N|_{\partial\Omega} = g \cdot N$ *in the sense of (2.12).*

Proof of Theorem 1.1. For $u \in H^{\beta,q}_w(\Omega)$ let $W(u) \in (C^\infty_0(\Omega))'$ be given by

$$\langle W(u), \phi\rangle = \langle uu, \nabla\phi\rangle + \langle Ku, \phi\rangle \quad \text{for all} \quad \phi \in C^\infty_0(\Omega).$$

By Lemma 4.4.1 one has for $\phi \in C^\infty_0(\Omega)$

$$\begin{aligned}|\langle W(u), \phi\rangle| &\leq c\|u\|^2_{\beta,q,w}\|\nabla\phi\|_{t',w'} + c\|K\|_{t,w}\|u\|_{\beta,q,w}\|\phi\|_{1,t',w'} \\ &\leq c(\|u\|^2_{\beta,q,w} + \|K\|_{t,w}\|u\|_{\beta,q,w})\|\phi\|_{1,t',w'}\end{aligned}$$

and hence $W(u) \in W^{-1,t}_w(\Omega)$ with

$$\|W(u)\|_{-1,t,w} \leq c(\|u\|^2_{\beta,q,w} + \|K\|_{t,w}\|u\|_{\beta,q,w}). \tag{4.3}$$

We define the mapping $S : H_w^{\beta,q}(\Omega) \to H_w^{\beta,q}(\Omega)$ by

$$\begin{aligned} -\langle Su, \Delta\phi\rangle &= \langle F,\phi\rangle + \langle W(u),\phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega} && \text{for every } \ \phi \in Y_{q',w',\sigma} \\ -\langle Su, \nabla\psi\rangle &= \langle K,\psi\rangle - \langle g, N\psi\rangle_{\partial\Omega} && \text{for every } \ \psi \in W_{w'}^{1,q'}(\Omega). \end{aligned}$$

This well-defined by Theorem 3.9.

We want to use Banach's Fixed Point Theorem to show that S has a fixed point, presumed the data is small enough.

By the a priori estimate in Theorem 3.9 we know that

$$\|v\|_{\beta,q,w} \le D(\|F\|_{-1,t,w} + \|K\|_{t,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)}), \tag{4.4}$$

if v is a very weak solution to the Stokes problem with respect to the data $F \in H_w^{-1,t}(\Omega)$, $K \in L_w^t(\Omega)$ and $g \in T_w^{\beta,q}(\partial\Omega)$.

We assume that the data F, K and g are chosen small enough such that the right hand side of (4.4) is $\le \rho := \frac{1}{6cD}$, where c is the constant in the estimate (4.3).

Next we show that for such data and $\delta = \frac{2}{6cD}$ the ball $B_\delta(0)$ is mapped by S into itself. By (4.4) and (4.3) one has for $u \in B_\delta(0)$

$$\begin{aligned} \|Su\|_{\beta,q,w} &\le D(\|F\|_{-1,t,w} + c(\|u\|_{\beta,q,w}^2 + \|K\|_{t,w}\|u\|_{\beta,q,w}) + \|K\|_{t,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)}) \\ &\le \rho + cD\rho\delta + cD\delta^2 = \frac{6cD + 2cD + 4cD}{(6cD)^2} = \frac{2}{6cD} = \delta. \end{aligned}$$

The next step is to show that S is a contraction on $B_\delta(0)$. Take $u, v \in B_\delta(0)$. Then $Su - Sv$ is a solution of

$$\begin{aligned} -\langle Su - Sv, \Delta\phi\rangle &= \langle W(u) - W(v), \phi\rangle && \text{for every } \ \phi \in Y_{q',w',\sigma} \\ \langle Su - Sv, \nabla\psi\rangle &= 0 && \text{for every } \ \psi \in W_{w'}^{1,q'}(\Omega). \end{aligned}$$

Moreover from Lemma 4.4.1 we obtain

$$\begin{aligned} |\langle W(u) - W(v), \phi\rangle| &\le |\langle (u-v)u, \nabla\phi\rangle| + |\langle v(u-v), \nabla\phi\rangle| + |\langle K(u-v), \phi\rangle| \\ &\le c(\|u\|_{\beta,q,w} + \|v\|_{\beta,q,w} + \|K\|_{t,w})\|u-v\|_{\beta,q,w}\|\phi\|_{1,t',w_t} \\ &\le (2c\delta + c\rho)\|u-v\|_{\beta,q,w}\|\phi\|_{1,t',w_t} = \frac{5}{6D}\|u-v\|_{\beta,q,w}\|\phi\|_{1,t',w_t}. \end{aligned}$$

Thus we again obtain from the a priori estimate that

$$\|Su - Sv\|_{\beta,q,w} \le D\|W(u) - W(v)\|_{-1,t,w} \le \frac{5}{6}\|u-v\|_{\beta,q,w}.$$

We have shown that there exists a unique fixed point of S in $B_\delta(0)$ and hence a solution $u \in H_w^{\beta,q}(\Omega)$ to the stationary Navier-Stokes system.

The a priori estimate follows from

$$\|u\|_{\beta,q,w} \le D(\|F\|_{-1,t,w} + \|K\|_{t,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)} + c(\|u\|_{\beta,q,w}^2 + \|K\|_{t,w}\|u\|_{\beta,q,w})).$$

Since $Dc(\|u\|_{\beta,q,w} + \|K\|_{t,w}) \le \frac{3}{6}$, one obtains the a priori estimate subtracting $\frac{3}{6}\|u\|_{\beta,q,w}$ from both sides of the above equation.

To show $u|_{\partial\Omega} = g$ one uses the fact that u is a very weak solution to the Stokes equations with respect to the data

$$\begin{aligned} f &= [\phi \mapsto \langle F, \phi\rangle + \langle W(u), \phi\rangle - \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega}] \\ k &= [\psi \mapsto \langle K, \psi\rangle - \langle g, N\psi\rangle_{\partial\Omega}], \end{aligned}$$

where $f = [\phi \mapsto \langle F, \phi\rangle + \langle W(u), \phi\rangle] \in W_w^{-1,t}(\Omega)$. Then the assertion follows from Proposition 3.9. □

Definition 4.6. *Let $1 \le \beta \le 2$. Moreover, let $F \in H_w^{\beta-2,q}(\Omega)$, $K \in H_w^{\beta-1,q}(\Omega)$ and $g \in T_w^{\beta,q}(\partial\Omega)$. Then $u \in H_w^{\beta,q}(\Omega)$ is called a weak solution to the stationary Navier-Stokes equations, if*

$$\langle \nabla u, \nabla\phi\rangle + \langle u \cdot \nabla u, \phi\rangle = \langle F, \phi\rangle \quad \textit{for every} \quad \phi \in C_{0,\sigma}^{\infty}(\Omega),$$

$\operatorname{div} u = K$ *and* $u|_{\partial\Omega} = g$.

Proof of Theorem 1.2.
This can be proved in the same way as Theorem 1.1 using Lemma 4.4.2. instead of Lemma 4.4.1. and Theorem 3.8 instead of Theorem 3.9. □

The very weak solution is unique even without the assumption of the smallness of the exterior force f and the boundary condition g. To see this we need the following embedding theorem.

Lemma 4.7. *If $1 \le s$, $w \in A(s)$ and $1 \le p < \infty$ then for $q \ge sp$ one has*

$$L_w^q(\Omega) \hookrightarrow L^p(\Omega).$$

Proof. First we assume that $s > 1$. Since $\frac{q}{p} \ge s$ one has $w \in A_{\frac{q}{p}}$. Thus $w^{-\frac{1}{\frac{q}{p}-1}} \in A_{(\frac{q}{p})'} \subset L^1_{loc}(\Omega)$. Together with the Hölder inequality this yields

$$\int_\Omega |f|^p\,dx = \int_\Omega |f|^p w^{\frac{p}{q}} w^{-\frac{p}{q}}\,dx \le \|f\|_{q,w}^p \left(\int_\Omega w^{-\frac{1}{\frac{q}{p}-1}}\,dx\right)^{\frac{q-p}{q}} = c\|f\|_{q,w}^p$$

for every $f \in L_w^q(\Omega)$.
If $s = 1$, then by [15] one can assume that w is bounded from below on Ω. This implies $L_w^p(\Omega) \hookrightarrow L^p(\Omega)$. □

Theorem 4.8. *Let the data F, K and g be given as in Theorem 1.1 and let u be a very weak solution to the stationary Navier Stokes system with respect to the data F, K and g.*

Then there exists a constant $\rho > 0$ such that under the condition

$$\|u\|_{\beta,q,w} + \|K\|_{t,w} \le \rho,$$

there exists at most one very weak solution to the stationary Navier Stokes equations according to Definition 4.5.

Proof. By Lemma 4.3 and Lemma 4.7 one has

$$H_w^{\beta,q}(\Omega) \hookrightarrow L_w^{\frac{nsq}{-q\beta+ns}}(\Omega) \hookrightarrow L^{\frac{nq}{-q\beta+ns}}(\Omega) \hookrightarrow L^n(\Omega),$$

by the assumptions on β. Analogously we find $F \in W^{-1,\frac{n}{2}}(\Omega)$ and $K \in L^{\frac{n}{2}}(\Omega)$. From Corollary 3.10 we thus obtain $g \in W^{-\frac{1}{n},n}(\partial\Omega) := T_1^{0,n}(\partial\Omega)$. Hence the assumptions of [4, Theorem 1.5] are fulfilled and we obtain the uniqueness. □

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