Odd-dimensional orthogonal groups as amalgams of unitary groups.
Part 2: machine computations

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Abstract

In the first part [PART1], a characterization of central quotients of the group $\mathrm{Spin}(2n+1, q)$ is given for $n \geq 3$ and all odd prime powers q, with the exception of the cases $n = 3$, $q \in \{3, 5, 7, 9\}$. The present article treats these cases computationally, thus completing the Phan-type theorem for the group $\mathrm{Spin}(2n+1, q)$.

1 Introduction

The goal of the paper is to complete the proof of a Phan-type theorem that characterizes the group $\mathrm{Spin}(2n+1, q)$ for $n \geq 3$ and an odd prime power q. The characterization relies on the following theorem on a certain geometry Γ defined in Section 2.

Theorem *Let $q \geq 3$ be odd and let $n \geq 2$. Then the following hold.*

(i) *Γ is a rank n geometry, on which $G_\sigma \cong \mathrm{SO}(2n+1, q)$ and its index two subgroup $G'_\sigma \cong \mathrm{S}\Omega(2n+1, q)$ act as flag-transitive groups of automorphisms.*

(ii) *Γ is residually connected for $q \geq 5$.*

(iii) *Γ is simply connected for $n \geq 4$ and for $n = 3$ and $q \geq 5$.*

Part 1 [PART1] of this work proves this theorem except for the cases $n = 3$ and $5 \leq q \leq 9$. For $n = q = 3$ the geometry Γ is not simply connected but rather admits a 2187-fold universal cover.

This work is a case study of how theoretic arguments, covering all but a few small cases, and machine calculations for those cases complement each other to obtain the complete statement of a theorem. It is not uncommon in combinatorial situations that the general arguments do not apply to small parameters. As it happens in this work, these cases can be too involved to be accessible to hand calculations. Here, sophisticated algebra software provides a useful tool for dealing with these cases. Furthermore, machine-based computations help to decide whether the theorem under consideration can be extended. For the above theorem, we showed with the help of the computer that Γ is not simply connected if $n = q = 3$.

Before we turn to the problem setting described above, we first want to state the Phan-type theorem which motivates this work. Full details can be found in [PART1]. For convenience we restate some relevant definitions from [PART1], starting with the concept of *standard pairs* in the groups $\mathrm{SU}(3, q^2)$ and $\mathrm{Spin}(5, q)$.

Definition 1.1 *Let $G \cong \mathrm{SU}(3, q^2)$ and V be the natural G-module over $\mathbb{F}_{q^2}$. Subgroups U_1 and U_2 of $G \cong \mathrm{SU}(3, q^2)$ form a standard pair whenever each U_i is the stabilizer in G of a nonsingular vector $v_i \in V$ and, moreover, v_1 and v_2 are perpendicular. If U_1 and U_2 form a standard pair in G, and π describes a quotient map whose kernel is a subgroup of the center of G, then $\pi(U_1)$ and $\pi(U_2)$ are called a standard pair in $\pi(G)$.*

Definition 1.2 *Let $G \cong \mathrm{S}\Omega(5,q)$ and V be its natural module, where the invariant form is chosen to be of discriminant one. Subgroups U_1 and U_2 of $G \cong \mathrm{S}\Omega(5,q)$ form a standard pair if there is an orthogonal decomposition $V = V_2 \oplus V_2' \oplus V_1$, where*

(i) *V_2 is 2-dimensional of minus type, and U_1 is the vector-wise stabilizer of V_2;*

(ii) *$V_2 \oplus V_2'$ is 4-dimensional of plus type, and U_2 is one of the two direct factors in the vector-wise stabilizer of V_1.*

We remark that here $U_1 \cong \mathrm{S}\Omega(3,q) \cong \mathrm{PSL}(2,q)$ and $U_2 \cong \mathrm{SL}(2,q)$.

For $\widehat{G} \cong \mathrm{Spin}(5,q)$ and the natural homomorphism from $\pi : \widehat{G} \to G$, subgroups $U_1, U_2 \cong \mathrm{SL}(2,q)$ of $\widehat{G}$ form a standard pair if $\pi(U_1)$ and $\pi(U_2)$ form a standard pair in G.

We use diagrams to describe configurations involving standard pairs. In such a diagram an edge $\underset{i}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{j}{\circ}$ represents the fact that a suitable group G contains subgroups U_i and U_j such that $U_{ij} := \langle U_i, U_j \rangle$ is isomorphic to $\mathrm{SU}(3,q^2)$ (or its central quotient $\mathrm{PSU}(3,q^2)$) and that U_i and U_j form a standard pair in U_{ij}. Similarly, the edge $\underset{i}{\circ}\!\overset{>}{=\!=\!=}\!\underset{j}{\circ}$ requires that U_{ij} be isomorphic to $\mathrm{Spin}(5,q)$ (or its central quotient $\mathrm{S}\Omega(5,q)$) and that U_i and U_j again form a standard pair in U_{ij}. In addition to the above two types of edges we will need the "empty" edge $\underset{i}{\circ}\qquad\underset{j}{\circ}$ which means that U_{ij} is a central product of U_i and U_j.

Definition 1.3 *Let $n \geq 2$. A group G contains a* weak Phan system of type B_n over $\mathbb{F}_{q^2}$ *if G is generated by a family of subgroups U_i, $i \in I = \{1, \ldots, n\}$, so that, for $1 \leq i < j \leq n$, the subgroups U_i and U_j form a standard pair in $U_{ij} := \langle U_i, U_j \rangle$ according to the Dynkin diagram B_n:*

$$\underset{1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{2}{\circ}\ \cdots\ \underset{n-2}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{n-1}{\circ}\!\overset{>}{=\!=\!=}\!\underset{n}{\circ}.$$

The main result of [Part1] consists of the following two theorems.

Main Theorem A *For $n \geq 3$ and $q \geq 5$ an odd prime power, let G be a group containing a weak Phan system of type B_n over $\mathbb{F}_{q^2}$. Then G is isomorphic to $\mathrm{Spin}(2n+1,q)$ or a central quotient thereof.*

Main Theorem B *For $n \geq 4$, let G be a group containing a weak Phan system of type B_n over $\mathbb{F}_9$. In addition, assume that $\langle U_{i-1}, U_i, U_{i+1} \rangle$ is isomorphic to a central quotient of $\mathrm{SU}(4,9)$ (if $2 \leq i \leq n-2$) or $\mathrm{Spin}(7,3)$ (if $i = n-1$). Then G is isomorphic to $\mathrm{Spin}(2n+1,3)$ or a central quotient thereof.*

By the argument given in the introduction of [Part1], Main Theorems A and B are an immediate consequence of the Theorem.

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2 Geometrical setting

In this section we introduce the geometry Γ dealt with in the Theorem. For a brief introduction to synthetic geometry, refer to [Part1] or, for a comprehensive introduction, to [BC].

Fix a dimension n (in this article we actually only deal with $n = 3$) and an odd prime power q. Let B be the $(2n+1) \times (2n+1)$ matrix

$$\begin{pmatrix} & \mathrm{id}_{n\times n} & \\ \mathrm{id}_{n\times n} & & \\ & & 1 \end{pmatrix}$$

over $\mathbb{F}_{q^2}$ and $(\cdot,\cdot)$ be the bilinear form defined via $(x,y) := x^T By$. As representation of $\mathrm{SO}(2n+1,q^2)$ we choose the set of all invertible $(2n+1)\times(2n+1)$-matrices A over $\mathbb{F}_{q^2}$ which preserve $(\cdot,\cdot)$, that is, $A^T BA = B$ holds. With $G = \mathrm{S}\Omega(2n+1,q^2)$ we denote the commutator subgroup of $\mathrm{SO}(2n+1,q^2)$.

Let V be the vector space $\mathbb{F}_{q^2}^{2n+1}$ endowed with the form $(\cdot,\cdot)$ and let $\{e_1,\dots,e_n,f_1,\dots,f_n,x\}$ be a hyperbolic basis, i.e. a basis such that the Gram matrix of $(\cdot,\cdot)$ with respect to that basis equals B. We denote by $^-$ the unique non-trivial involutory field automorphism $x\mapsto x^q$ of $\mathbb{F}_{q^2}$. Consider the $^-$-semi-linear map $\sigma : V\to V$ defined by $e_i\mapsto f_i, f_i\mapsto e_i, x\mapsto x$ and $\sigma(c\cdot v) = \overline{c}\sigma(v)$ for $c\in\mathbb{F}_{q^2}, v\in V$. The basis $\{e_1,\dots,e_n,f_1,\dots,f_n,x\}$ is called *canonical* for σ. Note that $\sigma(v) = \overline{Bv} = B\overline{v}$. The centralizer $G_\sigma := \{g\in G \mid \forall v\in V : g\sigma(v) = \sigma(gv)\}$ of σ in $\mathrm{S}\Omega(2n+1,q^2)$ is isomorphic to $\mathrm{SO}(2n+1,q)$ (see [Part1], Proposition 2.10). For our computations, we take G_σ as our representation of $\mathrm{SO}(2n+1,q)$. Note that for a matrix $A\in\mathrm{SO}(2n+1,q^2)$, centralizing σ is equivalent to the condition $A^{-1} = \overline{A}^T$.

We now define the flipflop geometry Γ which we are studying in this article. (For an introduction to flipflop geometries, see [BGHS] or [G2].) To this end, we define a $^-$-hermitian form $((\cdot,\cdot))$ by $((u,v)) := (u,\sigma(v))$, cf. [Part1], Definition 2.4 and Lemma 2.5. To denote orthogonality with respect to the form $(\cdot,\cdot)$, we use the symbol $\perp$. To denote orthogonality with respect to the form $((\cdot,\cdot))$, we use the symbol $\perp\!\!\!\perp$.

Definition 2.1 (Cf. Proposition 3.1 in [Part1]**)** *The objects of the geometry Γ are all nontrivial subspaces of V that are totally isotropic with respect to $(\cdot,\cdot)$ and nondegenerate with respect to $((\cdot,\cdot))$. Incidence is defined by symmetrized containment.*

3 Simple connectedness

In this section we prove the simple connectedness of the flipflop geometry Γ for $n = 3$ and $q\in\{5,7,9\}$. The concept of simple connectedness or homotopies of a geometry are based on said concepts found in algebraic topology, especially the theory of simplicial complexes. For a brief overview, refer to Section 3.1 in [Part1]. For details see [ST].

In Proposition 3.7 of [Part1], it was shown that for $q\geq 5$, the collinearity graph of the geometry Γ has diameter two. Furthermore it was shown — cf. Lemma 4.1 of [Part1] and the subsequent discussion — that in order to prove the simple connectedness of Γ, it suffices to deal with triangles, quadrangles and pentagons in the collinearity graph of Γ. To decompose them, we will make use of the fact that all geometric triangles — that is, triangles in the collinearity graph for which an element of the geometry exists that is incident to all points and lines in the triangle — are trivially null-homotopic and thus can be used to decompose all other cycles. The pentagons was already covered by Lemma 4.12 in [Part1].

3.1 Triangles

As a first step, we want to decompose triangles. In [Part1], Lemmas 4.6 – 4.8, this was already done for $q\geq 7$. Hence we need to show that this is possible for $q = 5$. Lemma 4.6 in [Part1] states that any nongeometric triangle can be decomposed into triangles that have two g-orthogonal vertices. We can therefore restrict our attention to such triangles. As was shown in [Part1], we can further restrict ourselves to consider triangles (e_1,e_2,v_3) where $v_3 = \alpha e_1+\beta e_2+r$, where r is in the g-radical. Assume that $r = \gamma e_3+\delta f_3+\varepsilon x$. Then the condition on r gives $0 = (r,r) = 2\gamma\delta+\varepsilon^2$ and $0 = ((r,r)) = \gamma\overline{\gamma}+\delta\overline{\delta}+\varepsilon\overline{\varepsilon}$. If $\delta = 0$, then also $\varepsilon = 0 = \gamma$, which would mean our triangle was really a line; hence $\delta\neq 0$. Thus $\gamma = -\frac{\varepsilon^2}{2\delta}$ and a short calculation shows $\gamma\overline{\gamma} = \delta\overline{\delta} = -\frac{\varepsilon\overline{\varepsilon}}{2}$ (in particular we also see that $\gamma\neq 0\neq\delta$).

To decompose a triangle (e_1,e_2,v_3) we construct an octahedron with that triangle as one face, and a suitably chosen geometric triangle (p_1,p_2,p_3) as the opposite face (see Figure 1) such that all other faces of the octahedron are geometric.

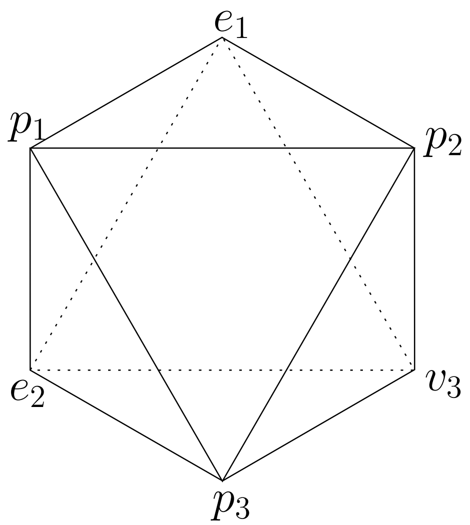

Figure 1: Octahedron construction used in Lemma 3.1

In Lemma 4.7 of [Part1], this construction is used with the triangle $(p_1, p_2, p_3) = (f_3, f_3 - \frac{\gamma}{\beta} f_2, f_3 - \frac{\gamma}{\alpha} f_1)$ to show that for $q \geq 7$ all triangles can be decomposed into geometric triangles and hence are null-homotopic. This result is sharp in the sense that for $q = 5$ there are triangles which cannot be decomposed by the octahedron construction using that triangle. For an example, let z denote a primitive element in $\mathbb{F}_{25}$ over $\mathbb{F}_5$ with minimal polynomial $x^2 - x + 2$. Then

$$a := \langle e_1 \rangle, \quad b := \langle e_2 \rangle, \quad c := \langle z e_1 + e_2 + z^5 e_3 + z f_3 + x \rangle$$

form a triangle that cannot be decomposed using the octahedron construction. Hence a different triangle (p_1, p_2, p_3) or even a completely different approach is required. Fortunately, it turns out that the former suffices.

Lemma 3.1 *Let $q = 5$. Then all triangles can be decomposed into geometric triangles.*

Proof. We assume as above that the triangle is (e_1, e_2, v_3) where $v_3 = \alpha e_1 + \beta e_2 + r$ where $r = \gamma e_3 + \delta f_3 + \varepsilon x$ is in the g-radical. We will now construct an octahedron using the initial vertices and the points

$$\begin{aligned} p_1 &= f_3, \\ p_2 &= \theta_1 \theta_2 \alpha \gamma e_1 + \frac{1}{2} \theta_1^2 \beta \gamma e_2 - \frac{\gamma}{\beta} f_2 + (1 - \theta_1 \varepsilon) f_3 + \theta_1 \gamma x, & (3.1) \\ p_3 &= \frac{1}{2} \theta_2^2 \alpha \gamma e_1 - \frac{\gamma}{\alpha} f_1 + (1 - \theta_2 \varepsilon) f_3 + \theta_2 \gamma x. & (3.2) \end{aligned}$$

where $\theta_1, \theta_2 \in \mathbb{F}_{q^2}$. For $\theta_1 = 0 = \theta_2$ this is identical to the triangle used in Lemma 4.7 of [Part1].

We now have to show that for each triangle (e_1, e_2, v_3) there exist values for θ_1 and θ_2 such that the octahedron construction with the triangle (p_1, p_2, p_3) decomposes our starting triangle, i.e. in which all faces except for the starting triangle are geometric triangles. Since the stabilizer of the pair e_1, e_2 acts transitively on the f- and g-singular one-dimensional subspaces of $\langle e_3, f_3, x \rangle$, we can fix one admissable value for r, and only have to vary α and β. Thus only about $|\mathbb{F}_{q^2}|^2 = q^4 = 625$ triangles have to be considered. We do this by an exhaustive search through all values for α and β. For each pair we search through admissible values for θ_1 and θ_2 where a suitable geometrical triangle (p_1, p_2, p_3) is formed. This search was successful in all cases. A simple GAP program (see Appendix A) is sufficient to verify the claim in all these cases. $\square$

3.2 Quadrangles

In this section we show that all quadrangles a, b, c, d can be decomposed into triangles for $5 \leq q \leq 9$. Consider a generic quadrangle:

$$\begin{aligned} a &= (a_1, a_2, a_3, a_4, a_5, a_6, a_7), \\ b &= (b_1, b_2, b_3, b_4, b_5, b_6, b_7), \\ c &= (c_1, c_2, c_3, c_4, c_5, c_6, c_7), \\ d &= (d_1, d_2, d_3, d_4, d_5, d_6, d_7). \end{aligned}$$

We will present an algorithm that verifies for all quadrangles that they are decomposable. To this end, we first explain how to reduce the number of conjugacy classes to be considered as much as possible. A naive check would have to check approximately $|\mathbb{F}_{q^2}|^{4\cdot 6} = q^{48}$ quadrangles, a number which exceeds the capacities of commonly available hardware and even of many super computers.

Finally, we need a way to efficiently decompose a representative of each quadrangle conjugacy class into triangles.

3.2.1 Determining quadrangle conjugacy classes

The group $G_\sigma = \mathrm{SO}(7, q)$ acts transitively on the flags of the geometry Γ_σ. Since a quadrangle is decomposable if and only if all its conjugates are decomposable, we can use the action of G_σ to reduce the number of cases we have to consider.

The flag $\langle a\rangle, \langle a, b\rangle$ is conjugated to the flag $\langle e_1\rangle, \langle e_1, e_2\rangle$. Hence we can assume that

$$a = e_1, \quad b = \lambda e_1 + e_2, \quad \lambda\bar{\lambda} \neq -1.$$

Let U be the natural module of G. Let M_2 be the stabilizer, called *line stabilizer*, of the subspace $U_2 := \langle e_1, e_2\rangle$ of U. Since M_2 preserves both our forms, it also stabilizes $U_2^\sigma = \langle f_1, f_2\rangle$ and $W := U_2^\perp \cap U_2^{\perp\!\!\perp} = \langle e_3, f_3, x\rangle$, i.e. it stabilizes the direct decomposition

$$U_2 \oplus U_2^\sigma \oplus \langle U_2, U_2^\sigma\rangle^\perp = \langle e_1, e_2\rangle \oplus \langle f_1, f_2\rangle \oplus \langle e_3, f_3, x\rangle.$$

By sharp observation, we determine that the action of M_2 on U induces $\mathrm{GU}(2, q^2)$ on U_2 (and U_2^σ) and $\mathrm{SO}(3, q)$ on W. In particular, the group M_2 is isomorphic to

$$\mathrm{GU}(2, q^2) \times \mathrm{SO}(3, q).$$

We use the action of $\mathrm{GU}(2, q^2)$ on U_2 to reduce the number of values for λ we have to regard. For this, we take the stabilizer S of $\langle e_1\rangle$ in $\mathrm{GU}(2, q^2)$, which is generated by diagonal matrices with respect to our chosen basis. We observe that the orbit of $\langle b\rangle$ under the action of S consists of all elements $\langle \lambda' e_1 + e_2\rangle$ for which $\lambda'\bar{\lambda}' = \lambda\bar{\lambda}$. Hence for λ we only need to consider one representative for each of the $q - 1$ values $\lambda\bar{\lambda} \in \mathbb{F}_q \setminus \{-1\}$.

Next take the action of $\mathrm{SO}(3, q)$ on W and notice that again it is sufficient to consider one representative for each orbit of c_3, c_6, c_7. A computation in GAP shows that there are $q^3 + q^2 + q$ orbits for $q \in \{3, 5, 7, 9\}$ (we do not know whether this holds for larger values of q).

Combining all this (ignoring for the moment that we are in a projective space) leaves eleven free parameters (c_1, c_2, c_4, c_5 and d_1 to d_7), which gives a quadrangle conjugacy class count of at most $|\mathbb{F}_{q^2}|^{11} \cdot (q - 1) \cdot (q^3 + q^2 + q) = q^{26} - q^{23}$.

3.2.2 Reducing the conjugacy classes using geometric arguments

It is sufficient for us to consider quadrangles for which $\langle a, c\rangle$ and $\langle b, d\rangle$ are not f-singular, as the other cases are already covered in Lemmas 4.8 and 4.10 of [Part1]. So $(a, c) = c_4 \neq 0$ and $(b, d) = d_5 + d_4\lambda \neq 0$.

Since we are working in a projective space, we can scale c such that

$$c_4 = 1.$$

Since $\langle c\rangle$ is a point, we have that $(c,c)=0$. This gives

$$0=(c,c)=2c_1+2c_2c_5+2c_3c_6+c_7^2,\quad \text{whence}\quad c_1=-(c_2c_5+c_3c_6+\frac{c_7^2}{2}).$$

Next, we use that the sides of the quadrangle must be lines of the geometry and hence:

$$\begin{aligned}
0&=(a,b),\\
0&=(b,c)=c_5+\lambda c_4=c_5+\lambda,\quad \text{whence}\quad c_5=-\lambda,\\
0&=(d,a)=d_4.
\end{aligned}$$

Since $(b,d)=d_5+d_4\lambda\neq 0$, and $d_4=0$, we conclude that $d_5\neq 0$ and hence we can scale d such that

$$d_5=1.$$

Finally we consider the line $\langle c,d\rangle$ and the point $\langle d\rangle$:

$$0=(c,d)=c_2+d_1+c_6d_3+c_3d_6+c_7d_7-d_2\lambda,\quad \text{whence}\quad d_1=-(c_2+c_6d_3+c_3d_6+c_7d_7-d_2\lambda),$$
$$0=(d,d)=2d_2+2d_3d_6+d_7^2,\quad \text{whence}\quad d_2=-d_3d_6-\frac{d_7^2}{2}.$$

Collecting all we know, we have

$$\begin{aligned}
a&=(1,0,0,0,0,0,0),\\
b&=(\lambda,1,0,0,0,0,0),\\
c&=(\lambda c_2-c_3c_6-\frac{c_7^2}{2},\ c_2,\ c_3,\ 1,\ -\lambda,\ c_6,\ c_7), &(3.3)\\
d&=(-(c_2+c_6d_3+c_3d_6+c_7d_7+\lambda(d_3d_6+\frac{d_7^2}{2})),\ -d_3d_6-\frac{d_7^2}{2},\ d_3,\ 0,\ 1,\ d_6,\ d_7). &(3.4)
\end{aligned}$$

Combining this with the results from the previous section we have at most

$$|\mathbb{F}_{q^2}|^4\cdot(q-1)\cdot(q^3+q^2+q)=q^{12}-q^9$$

quadrangle conjugacy classes to consider. Not all of these will be quadrangles of the geometry Γ, since the conditions we have used so far are necessary but not sufficient to ensure that the one- and two-dimensional spaces occurring here are elements of the geometry. We have not yet ensured that they must be g-nondegenerate. That condition yields inequalities, which makes it difficult to use it to narrow down the conjugacy classes. Instead, our program will filter out those conjugacy class representatives which are not quadrangles of the geometry before trying to decompose them.

3.2.3 Decomposing quadrangles, the simple way

The simplest way to decompose a quadrangle is to find a point p which is collinear to all of its corners, thus decomposing the quadrangle into four triangles. We call this pyramid construction (see Figure 2). Such a point must form an f-singular space with each of a,b,c,d, yielding the following conditions in our situation:

$$\begin{aligned}
(p,a)=0 &\implies p_4=0,\\
(p,b)=0 &\implies \lambda p_4+p_5=0\implies p_5=0,\\
(p,c)=0 &\implies p_1+c_6p_3+c_3p_6+c_7p_7-p_2\lambda=0\implies p_1=-c_6p_3-c_3p_6-c_7p_7+p_2\lambda,\\
(p,d)=0 &\implies p_2+d_6p_3+d_3p_6+d_7p_7=0\implies p_2=-d_6p_3-d_3p_6-d_7p_7.
\end{aligned}$$

Thus p looks as follows:

$$p=(-c_6p_3-c_3p_6-c_7p_7+p_2\lambda,-d_6p_3-d_3p_6-d_7p_7,p_3,0,0,p_6,p_7). \qquad (3.5)$$

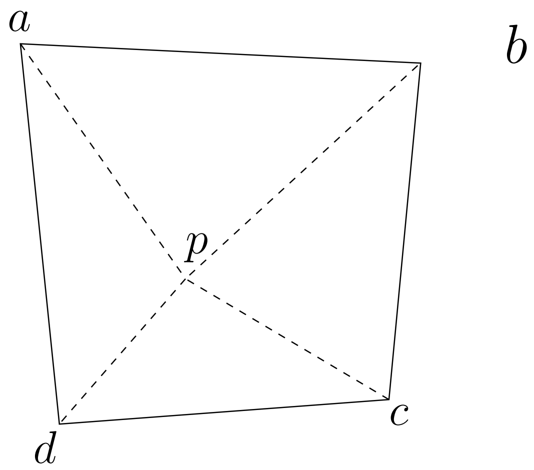

Figure 2: Basic quadrangle decomposition: Pyramid construction

Since p should be a point of the geometry, i.e. $(p,p)=0$, we get as a further restriction the equation $2p_3p_6+p_7^2=0$. If $p_3=0$, then also $p_7=0$ and we can assume that $p_6=1$. If on the other hand $p_3\neq 0$, we can assume that $p_3=1$, p_7 is arbitrary and $p_6=-\frac{p_7^2}{2}$. Thus we get q^2+1 candidates for p. Not all of these are points or generate lines with the corners of the quadrangle.

In fact, it turns out that there are some quadrangles for which such a point p does not exist. For an example, let z denote a primitive element in $\mathbb{F}_{81}$ over $\mathbb{F}_3$ with minimal polynomial x^4-x^3-1. Then

$$a:=\langle e_1\rangle,\quad b:=\langle e_2\rangle,\quad c:=\langle e_2+e_3+f_1+f_3+x\rangle,\quad d:=\langle 2e_1+z^2e_3+f_2+z^2f_3+z^2x\rangle$$

form a quadrangle that cannot be decomposed using the pyramid construction. For such quadrangles a more sophisticated approach is needed.

3.2.4 Decomposing quadrangles, the hard way

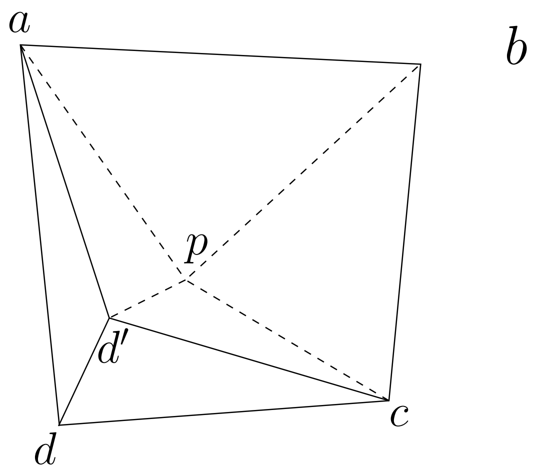

Figure 3: Enhanced quadrangle decomposition

When we encounter a quadrangle that fails to decompose using the pyramid construction described in the previous section, we use the following refined technique (see Figure 3): We search for a point d' which is collinear to a, c, d auch that the pyramid construction is applicable to the quadrangle formed by a, b, c, d'. In particular, $\langle b, d' \rangle$ must be f-singular. We conclude that $0 = (a, d') = d'_4$ and $0 \neq (b, d') = d'_5 + \lambda d'_4 = d'_5$, and hence we can scale d' such that $d'_5 = 1$. Furthermore, d' must be a point and collinear to c, thus

$$0 = (c, d') = c_2 + d'_1 + c_6 d'_3 + c_3 d'_6 + c_7 d'_7 - d'_2 \lambda, \quad \text{whence} \quad d'_1 = -(c_2 + c_6 d'_3 + c_3 d'_6 + c_7 d'_7 - d'_2 \lambda),$$

$$0 = (d', d') = 2d'_2 + 2d'_3 d'_6 + {d'_7}^2, \quad \text{whence} \quad d'_2 = -d'_3 d'_6 - \frac{{d'_7}^2}{2}.$$

With this d' has the following form:

$$d' = (-(c_2 + c_6 d'_3 + c_3 d'_6 + c_7 d'_7 - d'_2 \lambda),\ -d'_3 d'_6 - \frac{{d'_7}^2}{2},\ d'_3,\ 0,\ 1,\ d'_6,\ d'_7). \tag{3.6}$$

Note that d' has the same form as d as given in Equation 3.4. Hence we can then reuse the computations we made in the previous section and apply the pyramid decomposition strategy to the new quadrangle a, b, c, d'.

We have not yet made use of the collinearity of d and d':

$$0 = (d, d') = -d_3 d_6 - \frac{d_7^2}{2} + d_6 d'_3 + d_3 d'_6 - 2d'_3 d'_6 + d_7 d'_7 - {d'_7}^2.$$

However, we chose not to directly use this condition to further improve our algorithm (the condition is still being verified before accepting any solution), since there seems to be no easy way to directly exploit it, and empirically we determined that there are comparatively few quadrangles that need to be treated by this advanced decomposition strategy, hence further optimizations at this point yield little benefit for the overall run time.

The advantage of this improved strategy is that it is simple to implement and reuses what we already know. It turned out that it managed to decompose all (remaining) quadrangles.

3.2.5 Putting it all together

Assembling everything we have described above yields the algorithm described in Appendix A.

The approach described in the previous sections works because the geometry turns out to be connected, and for each quadrangle, we can relatively quickly determine a decomposition. Disproving simple connectedness this way would be much harder: To show simple connectedness, it sufficies to find a decomposition of our quadrangle, but to disprove it would require finding a quadrangle for which no decomposition is possible, in any conceivable way.

In Sections 3.2.1 and 3.2.2 we saw that we need to check about q^{12} quadrangle conjugacy classes. In Section 3.2.3 we saw that decomposing a quadrangle with the pyramid construction works in $O(q^2)$ (if such a decomposition exists), while the more complicated scheme in 3.2.4 runs in $O(q^6)$. This would suggest a total run time complexity of $O(q^{19})$.

When we run the algorithm, it quickly became apparent that the advanced decomposition algorithm usually succeeds in far less time. In addition, only a tiny fraction of all quadrangles fail to decompose via the pyramid construction. Hence the actual complexity is empirically $O(q^{14})$.

3.3 Summary

Since the collinearity graph of Γ has diameter 2, we have thus proven the following:

Proposition 3.2 *Let $n = 3$ and $q \in \{5, 7, 9\}$. Then the flipflop geometry Γ is simply connected.*

Together with Propositions 4.5 and 4.14 of [Part1] this completes the proof of (iii) of the Theorem. Main Theorems A and B now hold by the argument given in the introduction of [Part1].

4 The case $q = 3$

So far we were able to deal with the open cases using geometric methods. We now turn to the last open case, namely $n = 3 = q$. We will prove that the geometry Γ is *not* simply connected, but for this proof we will use group-theoretic methods. In particular, Tits' Lemma plays a central role:

Proposition 4.1 (Tits' Lemma) *Suppose a group H acts flag-transitively on a geometry Γ and let $\mathcal{A}$ be the amalgam of maximal parabolics associated with some maximal flag F. Then H is the universal completion of the amalgam $\mathcal{A}$ if and only if Γ is simply connected.*

Proof. This follows from [T], *Corollaire* 1, applied to the flag complex of Γ. See also Corollary 1.4.6 of [IS], or, in more general form, Corollary 3.2 of [GVM]. $\square$

4.1 A representation of $\mathrm{S\Omega}(7,3)$

We start by giving a representation of $\mathrm{S\Omega}(7,3)$ matching the definitions and constructions in Section 2. Namely, we construct it as a perfect subgroup of $\mathrm{SO}(7,9)$ preserving the forms $(\cdot,\cdot)$ and $((\cdot,\cdot))$, that has the correct size.

Let z denote a primitive element in $\mathbb{F}_9$ over $\mathbb{F}_3$ with minimal polynomial $x^2 - x - 1$. We define the following matrices:

$$U : \left(\begin{array}{ccc|ccc|c} z^7 & z^5 & & & & & \\ z^3 & z^5 & & & & & \\ & & 1 & & & & \\ \hline & & & z^5 & z^7 & & \\ & & & z & z^7 & & \\ & & & & & 1 & \\ \hline & & & & & & 1 \end{array}\right) \qquad V : \left(\begin{array}{ccc|ccc|c} 1 & & & & & & \\ & z^5 & z^7 & & & & \\ & z & z^7 & & & & \\ \hline & & & 1 & & & \\ & & & & z^7 & z^5 & \\ & & & & z^3 & z^5 & \\ \hline & & & & & & 1 \end{array}\right)$$

$$W : \left(\begin{array}{ccc|ccc|c} 1 & & & & & & \\ & 1 & & & & & \\ & & z^6 & & & 1 & z^7 \\ \hline & & & 1 & & & \\ & & & & 1 & & \\ & & 1 & & & z^2 & z^5 \\ \hline & & z^3 & & & z & 0 \end{array}\right)$$

In addition to these elements we use the diagonal matrices $\widehat{D}_1 := \mathrm{diag}(z^2,1,1,z^6,1,1,1)$, $\widehat{D}_2 := \mathrm{diag}(1,z^2,1,1,z^6,1,1)$, $\widehat{D}_3 := \mathrm{diag}(1,1,z^2,1,1,z^6,1)$, that generate the stabilizer of the flag F in $\mathrm{SO}(7,3)$, a half-split torus isomorphic to C_4^3, as well as $D_1 := \widehat{D}_1\widehat{D}_2$, $D_2 := \widehat{D}_2\widehat{D}_3$, $D_3 := \widehat{D}_3\widehat{D}_1$. which generate the stabilizer of the flag F in $\mathrm{S\Omega}(7,3)$.

Lemma 4.2 *The matrices U, V, W and $\widehat{D}_i, 1 \leq i \leq 3$ generate* $\mathrm{SO}(7,3)$. *The matrices U, V, W and $D_i, 1 \leq i \leq 3$ generate* $\mathrm{S\Omega}(7,3)$.

Proof. Using GAP or another computer algebra system one verifies that the matrices are all invertible and preserve $(\cdot,\cdot)$ and $((\cdot,\cdot))$, hence are in $\mathrm{SO}(7,3)$, and generate groups of the correct sizes. In the second case, one then verifies (again with the help of a computer) that the generated group is perfect. $\square$

We have seen earlier what the isomorphism type of the line stabilizer M_2 is. One can similarly deduce the isomorphism types of the other maximal parabolics in $\mathrm{SO}(7,3)$ and arrives at the following table (the size and index of each group is readily computed using GAP or well-known formulas for the sizes of orthogonal and unitary groups, see e.g. [Ta]):

stabilizer	stabilized element(s)	isomorphism type	size	index
$\widehat{M}_{123}$	$\langle e_1\rangle, \langle e_1, e_2\rangle, \langle e_1, e_2, e_3\rangle$	C_4^3	64	~ 143M
$\widehat{M}_1$	$\langle e_1\rangle$	$\mathrm{GU}(1,9) \times \mathrm{SO}(5,3)$	207360	44226
$\widehat{M}_2$	$\langle e_1, e_2\rangle$	$\mathrm{GU}(2,9) \times \mathrm{SO}(3,3)$	2304	3980340
$\widehat{M}_3$	$\langle e_1, e_2, e_3\rangle$	$\mathrm{GU}(3,9)$	24192	379080

Lemma 4.3 *Each maximal parabolic in* $\mathrm{S}\Omega(7,3)$ *is generated by the matrices, and has the size and index, as specified in the following table.*

stabilizer	*stabilized element(s)*	*generators*	*size*	*index*
M_{123}	$\langle e_1\rangle, \langle e_1, e_2\rangle, \langle e_1, e_2, e_3\rangle$	D_1, D_2, D_3	32	$\sim 143M$
M_1	$\langle e_1\rangle$	D_1, D_2, D_3, V, W	103680	44226
M_2	$\langle e_1, e_2\rangle$	D_1, D_2, D_3, U, W	1152	3980340
M_3	$\langle e_1, e_2, e_3\rangle$	D_1, D_2, D_3, U, V	12096	379080

Furthermore, the pairwise intersection of the stabilizers is generated by the intersection of the generating sets given above.

Proof. The claimed generators of each M_i obviously each stabilize the corresponding flag in the table. Hence they generate subgroups of the stabilizers. Clearly, the intersection of the generators of any two M_i generates a subgroup of their intersection. Since M_{123} must be a subgroup of the flag stabilizer, and since $\widehat{D}_1$ is not in $\mathrm{S}\Omega(7,3)$, we know that M_{123} has to be smaller than $\widehat{M}_{123}$ which has 64 elements. We compute its size to be 32, hence it already is the full flag stabilizer in $\mathrm{S}\Omega(7,3)$.

The stabilizers in $\mathrm{S}\Omega(7,3)$ are obtained by intersecting the stabilizers $\widehat{M}_i$ with $\mathrm{S}\Omega(7,3)$. Since $\mathrm{S}\Omega(7,3)$ has index two in $\mathrm{SO}(7,3)$, its stabilizers are either the same or also have index two in their counterparts. But clearly the flag stabilizer in $\mathrm{S}\Omega(7,3)$ is already smaller, and it is contained in all other parabolics. Hence all stabilizers M_i are strictly smaller than the corresponding $\widehat{M}_i$. This justifies the table.

We compute using GAP the sizes of the M_i and verify that these match the sizes of the maximal parabolics given in the table. Finally, again using GAP we compute the sizes of the pairwise intersections of the M_i and verify that these are generated as claimed. $\square$

4.2 Amalgam

Based on the above table, we determine with the help of GAP a finite presentation of the maximal parabolics on the generators d_1, d_2, d_3, u, v, w. Taking these elements as generators, and the union of the relators of said presentations yields a finite presentation of the universal completion H of the amalgam of the maximal parabolics (by virtue of the intersection property shown in the preceeding lemma). We obtained the following relators:

$$d_1^4,\ d_2^4,\ u^3,\ v^3,\ w^3,\ [d_1,d_2],\ [d_1,d_3],\ [d_1,u],\ [d_1,w],\ [d_2,d_3],\ [d_2,v],\ [u,w],\ (d_1d_2d_3)^2,$$
$$d_1^{-1}wd_3wd_2,\ (d_3uw)^2d_1,\ (vd_1d_3^{-1})^3,\ d_3vd_2d_1^{-1}vd_2,\ d_3^{-1}d_2ud_2ud_2,\ uvu(vuv)^{-1},\ wvwv(vwvw)^{-1},$$
$$d_2v^{-1}d_1d_2v^{-1}d_3^{-1},\ d_3^{-1}u^{-1}d_2d_3^{-1}u^{-1}d_2u,\ d_3v^{-1}d_1^{-1}w^{-1}vd_2w^{-1}d_3v^{-1}w^{-1}d_2d_1^{-1}vw^{-1},$$
$$v^{-1}d_1^{-1}w^{-1}vd_2wd_2vd_3w^{-1}d_2d_1^{-1}vwd_1d_2^{-1},\ d_3d_2vwv^{-1}d_2wvd_3^{-1}d_1vwvd_3^{-1}d_2wd_2^{-1}w^{-1},$$
$$d_2wvd_3v^{-1}d_1^{-1}w^{-1}d_2vd_2vwvd_3v^{-1}d_1^{-1}w^{-1}d_2v^{-1}.$$

Note that $\widetilde{G} := \mathrm{S}\Omega(7,3)$, generated as above, is a quotient of H. In the following we will determine the exact structure of H.

Lemma 4.4 *Let $\phi : H \to \widetilde{G}$ be the canonical group epimorphism which maps the generators of H onto the generators of $\widetilde{G}$. Then $|\ker\phi| = 3^7 = 2187$, or equivalently $|H| = 2187 \cdot |\widetilde{G}|$.*

Proof. Let $M_1' := \langle d_1, d_2, d_3, v, w\rangle$, and note that (by construction) $\phi\mid_{M_1'}$ is an isomorphism between M_1' and M_1. We use a two stage approach to determine the size of $\ker\phi$. Consider the following subgroup R of H and its intersection S with M_1':

$$R := \langle d_1^2, d_2, d_3 d_1^{-1}, v, w, uv(uvw)^{-1} d_3^{-1} v(uvw)^{-1}\rangle$$
$$S := \langle d_1^2, d_2, d_3, v, w\rangle \subseteq W \cap M_1'$$

But ϕ induces an isomorphism on M_1' and also on S, thus we see

$$|H| = |H : R| \cdot |R : S| \cdot |S| = |H : R| \cdot |R : S| \cdot 51840.$$

Computing $|H : R| = 378$ using coset enumeration is easy. To compute $|R : S|$, some more effort is required: One has to compute a suitable finite presentation of R first. Fortunately, GAP does that for us, and we find that $|R : S| = 511758$. Hence

$$|H| = 378 \cdot 511758 \cdot 51840 = 10028164124160 = 2187 \cdot |\mathrm{S}\Omega(7, 3)|.$$

□

As a small remark, notice that one could try to directly compute the index of a suitable subgroup, e.g. make use of the fact that

$$|\ker\phi| = |H : M_1'|/|\widetilde{G} : M_1| = |H : M_1'|/44226$$

and use coset enumeration to compute $|H : M_1'|$. If the geometry was indeed simply connected, this would require only about 44026 cosets. Unfortunately, it is not, as we have seen above. Rather we would actually need at least $44026 \cdot 2187 \simeq 96$ million cosets (which translates, when using [ACE], to about 5 GB of RAM).

We can now proceed to determine the nature of the kernel.

Lemma 4.5 *The kernel of ϕ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^7$.*

Proof. Let $k_7 := (uvw)^9$. Consider the subgroup

$$V := \langle d_1, d_2, d_3, u, v, w, d_2^{-1} k_7 d_2\rangle$$

of H. A coset enumeration yields that $|H : V| = 398034$. The resulting coset table can be used to derive a permutation representation of (a subgroup of) H of degree 398034. Computing the size of this permutation group, we find that it is indeed a faithful representation of H.

Computing the kernel of ϕ is a relatively simple computation. Taking the normal closure of k_7, we indeed obtain a (normal) subgroup of size 2187 of H, which we verify to be the kernel of ϕ. Computing its isomorphism type finally proves the claim. □

Proposition 4.6 *The group H is isomorphic to a non-split extension of* $\mathrm{S}\Omega(7, 3)$ *by $K := (\mathbb{Z}/3\mathbb{Z})^7$, i.e. the following sequence is exact and non-split:*

$$1 \to K \to H \xrightarrow{\phi} \mathrm{S}\Omega(7, 3) \to 1.$$

Proof. We observe that $K = \ker\phi$ corresponds to the 7-dimensional vector space $\mathbb{F}_3^7$. We compute the action of H on $\ker\phi$ to compute a 7-dimensional representation of H which is not faithful, but rather is isomorphic to $\mathrm{S}\Omega(7, 3)$. It is generated by the following two matrices:

$$A: \begin{pmatrix} & & 1 & & & & \\ 1 & 1 & 1 & & 2 & 1 & 1 \\ 1 & 1 & & 1 & 2 & & 2 \\ & 1 & 2 & 1 & & 2 & 2 \\ & 1 & & 1 & & & 2 \\ 2 & 2 & & 2 & & 2 & 2 \\ 2 & & 2 & 2 & & & 2 \end{pmatrix} \qquad B: \begin{pmatrix} 1 & & & & & & \\ & & & 1 & & & \\ & 1 & & & & & \\ 1 & 2 & & 2 & 1 & & \\ 2 & 2 & & 1 & 1 & 2 & \\ 2 & & 1 & 2 & 2 & 2 & 1 \\ & 2 & 2 & 2 & 2 & & 1 \end{pmatrix}$$

which preserve the bilinear form induced by the matrix

$$\begin{pmatrix} 1 & & 1 & 1 & 2 & & 2 \\ & 2 & & 1 & 2 & 1 & 1 \\ 1 & & & & 2 & 2 & \\ 1 & 1 & & 2 & 2 & & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ & 1 & 2 & & 2 & 1 & 1 \\ 2 & 1 & & 2 & 1 & 1 & 1 \end{pmatrix}.$$

We now apply the `OCOneCocycles` command in GAP to our extension (making use of the finite presentation, the permutation presentation, and the action on the module described above), which, after a few computations, confirms that we are indeed dealing with a non-split extension here.

In [Ku] it is proved that $S\Omega(7,3)$ admits precisely one non-split extension by its natural 7-dimensional module. Hence H is uniquely determined. □

We also used the cohomolo package [Cohom] to compute the the second cohomology group of $S\Omega(7,3)$ with respect to the module given above. This yielded dimension one, confirming the results presented in [F] and [Ku].

We finally conclude the following:

Corollary 4.7 *For $(n,q) = (3,3)$, the geometry Γ admits a 2187-fold covering. In particular it is not simply connected.*

Proof. This is a direct consequence from Proposition 4.6 and Tits' Lemma. □

A Algorithm description

The following pseudocode demonstrates how the observations regarding triangle and quadrangle decomposition from Section 3 can be turned into a working computer program. We implemented the triangle decomposition in GAP, and the quadrangle decomposition in C++. The full source for our implementation of these algorithms can be downloaded from:

`http://www.mathematik.tu-darmstadt.de/~mhorn/papers/quad-scan-20060204.tar.gz`.

The triangle code, for $q = 5$, when run on a 1.5Ghz PowerBook G4 under Mac OS X 10.4 and GAP 4.4.5, takes about 5 seconds to successfully decompose all relevant triangle conjugacy classes.

The quadrangle code had to be run for $q \in \{5, 7, 9\}$. In the biggest case, i.e. for $q = 9$, a total of about 150 billion quadrangles had to be split. This would have taken about a week of computation time, but by letting the code run on multiple computers in parallel, we were able to get the result after about 20 hours. Note that only 2430 of the quadrangles required the more advanced decomposition strategy described in Section 3.2.4.

Algorithm 1 Triangle decomposition

function TryToDecomposeTri($\alpha, \beta, \gamma, \delta, \varepsilon$) ▷ Uses the method described in Lemma 3.1
$a \leftarrow (1, 0, 0, 0, 0, 0, 0)$
$b \leftarrow (0, 1, 0, 0, 0, 0, 0)$
$c \leftarrow (\alpha, \beta, \gamma, 0, 0, \delta, \varepsilon)$
$p_1 \leftarrow (0, 0, 0, 0, 0, 1, 0)$
for $\theta_2 \in \mathbb{F}_{q^2}$ **do**
$p_3 \leftarrow (\frac{1}{2}\theta_2^2\alpha\gamma, 0, 0, -\frac{\gamma}{\alpha}, 0, (1 - \theta_2\varepsilon), \theta_2\gamma)$ ▷ Equation 3.2
if p is a point collinear to each of p_1, c, b **and** (p_3, b, c), (p_3, b, p_1) are triangles **then**
for $\theta_1 \in \mathbb{F}_{q^2}$ **do**
$p_2 \leftarrow (\theta_1\theta_2\alpha\gamma, \frac{1}{2}\theta_1^2\beta\gamma, 0, 0, -\frac{\gamma}{\beta}, (1 - \theta_1\varepsilon), \theta_1\gamma)$ ▷ Equation 3.1
if (p_1, p_2, p_3) forms an octahedron with (a, b, c) **then**
return success
end if
end for
end if
end for
return failure
end function

Algorithm 2 Triangle main program

$\varepsilon \leftarrow 1$ ▷ We can fix an arbitrary value for ε
for $\delta \in \mathbb{F}_{q^2}$ **do** ▷ Find suitable values for δ, γ
$\gamma \leftarrow -\frac{\varepsilon^2}{2\delta}$
if $\gamma\bar{\gamma} + \delta\bar{\delta} + \varepsilon\bar{\varepsilon} = 0$ **then**
`end loop`
end if
end for
for $(\alpha, \beta) \in \mathbb{F}_{q^2}^2$ **do**
if TryToDecomposeTri($\alpha, \beta, \gamma, \delta, \varepsilon$) fails **then**
`abort and print FAILURE`
end if
end for

Algorithm 3 Hard quadrangle decomposition

function TryToSplitQuadHard(a, b, c, d) ▷ Uses the method described in Section 3.2.4
for $(d'_3, d'_6, d'_7) \in \mathbb{F}_{q^2}^3$ **do**
$d'_2 \leftarrow -(d'_3 d'_6 + {d'_7}^2/2)$
$d'_1 \leftarrow -(c_2 + c_3 d'_6 + c_6 d'_3 + c_7 d'_7) + b_1 d'_2$
$d' \leftarrow (d'_1, d'_2, d'_3, 0, 1, d'_6, d'_7)$ ▷ See Equation 3.6
if d' is a point collinear to each of a, c, d **then**
if TryToSplitQuad(a, b, c, d') succeeds **then**
return success
end if
end if
end for
return failure
end function

Algorithm 4 Simple quadrangle decomposition

```
function IsPointSplitting($a, b, c, d, x, y, z$)
    $p \leftarrow (-(c_6 + d_6 b_1)x - (c_3 + d_3 b_1)y - (c_7 + d_7 b_1)z, -(d_6 x + d_3 y + d_7 z), x, 0, 0, y, z)$
                                                                  ▷ See Equation 3.5
    if $p$ is a point collinear to each of $a, b, c, d$ then
        return success
    else
        return failure
    end if
end function

function TryToSplitQuad($a, b, c, d$)          ▷ Uses the method described in Section 3.2.3
    if IsPointSplitting($0, 1, 0$) then
        return success
    end if
    for $x \in \mathbb{F}_{q^2}$ do
        if IsPointSplitting($1, -x^2/2, x$) then
            return success
        end if
    end for
    return failure
end function
```

Algorithm 5 Quadrangle main program

```
ORB ← orbit representatives for the action of $\mathrm{SO}(3, q)$ on $\mathbb{F}_{q^2}^3$      ▷ See Section 3.2.1
$a \leftarrow (1, 0, 0, 0, 0, 0, 0)$
for $\lambda' \in \mathbb{F}_q \setminus \{-1\}$ do
    $\lambda \leftarrow$ value from $\mathbb{F}_{q^2}$ such that $\lambda\overline{\lambda} = \lambda'$
    $b \leftarrow (\lambda, 1, 0, 0, 0, 0, 0)$
    for $c_2 \in \mathbb{F}_{q^2}$ do
        for $(c_3, c_6, c_7) \in$ ORB do
            $c \leftarrow (c_2 b_1 - c_3 c_6 - c_7^2/2, c_2, c_3, 1, -\lambda, c_6, c_7)$      ▷ Equation 3.3
            if $c$ is a point collinear to $a$ and $b$ and $c \notin a^\perp$ then
                for $(d_3, d_6, d_7) \in \mathbb{F}_{q^2}^3$ do
                    $d_2 \leftarrow -(d_3 d_6 + d_7^2/2)$
                    $d_1 \leftarrow -(c_2 + c_3 d_6 + c_6 d_3 + c_7 d_7) + \lambda d_2$
                    $d \leftarrow (d_1, d_2, d_3, 0, 1, d_6, d_7)$                ▷ Equation 3.4
                    if $d$ is a point collinear to each of $a, b, c$ and $d \notin b^\perp$ then
                        if TryToSplitQuad($a, b, c, d$) fails then
                            if TryToSplitQuadHard($a, b, c, d$) fails then
                                abort and print FAILURE
                            end if
                        end if
                    end if
                end for
            end if
        end for
    end for
end for
```

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