Odd-dimensional orthogonal groups as amalgams of unitary groups.
Part 1: general simple connectedness

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Abstract

We extend the Phan theory described in [BGHS] to the last remaining infinite series of classical Chevalley groups over finite fields. Namely, we prove that the twin buildings for the group $\mathrm{Spin}(2n+1,q^2)$, q odd, admit a unique unitary flip and that the corresponding flipflop geometry is simply connected for almost all finite fields $\mathbb{F}_{q^2}$. Applying standard methods from amalgam theory, this results in a characterization of central quotients of the group $\mathrm{Spin}(2n+1,q)$ by a Phan system of rank one and rank two subgroups. In the present first part of a series of two articles we present simple connectedness results for sufficiently large fields or sufficiently large rank. To be precise, the result stated in the present paper is proved for all cases but $n=3$ and $q \in \{3,5,7,9\}$, the remaining cases are dealt with in the sequel [PART2] computationally.

1 Introduction

The purpose of this paper is to establish the analog of Phan's theorems (cf. [P1] and [P2]) for the groups $\mathrm{Spin}(2n+1,q)$, q odd. To state these results we need some definitions, starting with the concept of *standard pairs* in the groups $\mathrm{SU}(3,q^2)$ and $\mathrm{Spin}(5,q)$.

Definition 1.1 Let $G \cong \mathrm{SU}(3,q^2)$ and V be the natural G-module over $\mathbb{F}_{q^2}$. Subgroups U_1 and U_2 isomorphic to $\mathrm{SU}_2(q^2)$ of $G \cong \mathrm{SU}(3,q^2)$ form a standard pair whenever each U_i is the stabilizer in G of a nonsingular vector $v_i \in V$ and, moreover, v_1 and v_2 are perpendicular. If U_1 and U_2 form a standard pair in G, and π describes a quotient map whose kernel is a subgroup of the center of G, then $\pi(U_1)$ and $\pi(U_2)$ are called a standard pair in $\pi(G)$.

Definition 1.2 Let $G \cong \mathrm{S\Omega}(5,q)$ and V be its natural module, where the invariant form is chosen to be of discriminant one. Subgroups U_1 and U_2 of $G \cong \mathrm{S\Omega}(5,q)$ form a standard pair if there is an orthogonal decomposition $V = V_2 \oplus V_2' \oplus V_1$,

(i) with V_2 being 2-dimensional of minus type, such that U_1 is the vector-wise stabilizer of V_2;

(ii) where $V_2 \oplus V_2'$ is 4-dimensional of plus type, such that U_2 is one of the two direct factors in the vector-wise stabilizer of V_1.

We remark that here $U_1 \cong \mathrm{S\Omega}(3,q) \cong \mathrm{PSL}(2,q)$ and $U_2 \cong \mathrm{SL}(2,q)$. For $\widehat{G} \cong \mathrm{Spin}(5,q)$ and the natural homomorphism from $\pi : \widehat{G} \to G$, subgroups $U_1, U_2 \cong \mathrm{SL}(2,q)$ of $\widehat{G}$ form a standard pair if $\pi(U_1)$ and $\pi(U_2)$ form a standard pair in G.

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Definition 1.1 repeats the definition in the introduction of [BS], while Definition 1.2 can be shown equivalent to the definition in [G1], Section 3, of the standard pairs in $\mathrm{Sp}(4,q) \cong \mathrm{Spin}(5,q)$. We use diagrams to describe configurations involving standard pairs. In such a diagram an edge $\underset{i}{\circ}\!\!\frac{\quad\quad}{}\!\!\underset{j}{\circ}$ represents the fact that a suitable group G contains subgroups U_i and U_j such that $U_{ij} := \langle U_i, U_j \rangle$ is isomorphic to $\mathrm{SU}(3,q^2)$ (or its central quotient $\mathrm{PSU}(3,q^2)$) and that U_i and U_j form a standard pair in U_{ij}. Similarly, the edge $\underset{i}{\circ}\!\overset{>}{=\!=\!=}\!\underset{j}{\circ}$ requires that U_{ij} be isomorphic to $\mathrm{Spin}(5,q)$ (or its central quotient $\mathrm{SO}(5,q)$) and that U_i and U_j again form a standard pair in U_{ij}. Note that Definition 1.1 is symmetric with respect to U_1 and U_2, whereas in Definition 1.2 the order of U_1 and U_2 matters, and so the diagram in this case is asymmetric. Notice also that our definition works well for all values of q, except $q = 2$, where the standard pair does not generate the entire $\mathrm{SU}(3,q^2)$ or $\mathrm{Spin}(5,q)$; recall that in this paper we are only interested in the case q odd, and so this complication does not arise. In addition to the above two types of edges we will need the "empty" edge $\underset{i}{\circ}\qquad\quad\underset{j}{\circ}$ which means that U_{ij} is a central product of U_i and U_j.

With this notation in place we can now give our main definition.

Definition 1.3 Let $n \geq 2$. A group G contains a *weak Phan system of type B_n over $\mathbb{F}_{q^2}$* if G is generated by a family of subgroups U_i, $i \in I = \{1, \ldots, n\}$, so that, for $1 \leq i < j \leq n$, the subgroups U_i and U_j form a standard pair in $U_{ij} := \langle U_i, U_j \rangle$ according to the Dynkin diagram B_n:

$$\underset{1}{\circ}\!\frac{\qquad}{}\!\underset{2}{\circ}\ \cdots\ \underset{n-2}{\circ}\!\frac{\qquad}{}\!\underset{n-1}{\circ}\!\overset{>}{=\!=\!=\!=}\!\underset{n}{\circ}.$$

Our main result consists of the following two theorems.

Main Theorem A *For $n \geq 3$ and $q \geq 5$ an odd prime power, let G be a group containing a weak Phan system of type B_n over $\mathbb{F}_{q^2}$. Then G is isomorphic to* $\mathrm{Spin}(2n+1,q)$ *or a central quotient thereof.*

There exists a counterexample to the statement of Main Theorem A for $n = 3$, $q = 3$, cf. Section 4 of [PART2]. However, the following statement is true in case $q = 3$.

Main Theorem B *For $n \geq 4$, let G be a group containing a weak Phan system of type B_n over $\mathbb{F}_9$. In addition, assume that $\langle U_{i-1}, U_i, U_{i+1} \rangle$ is isomorphic to a central quotient of* $\mathrm{SU}(4,9)$ *(if $2 \leq i \leq n-2$) or* $\mathrm{Spin}(7,3)$ *(if $i = n-1$). Then G is isomorphic to* $\mathrm{Spin}(2n+1,3)$ *or a central quotient thereof.*

We remark that the cases of diagrams A_n, C_n, and D_n have been dealt with previously (see [P1], [P2], and also [BS], [GHS], [G1], [H], [GHN], [GHNS]). So our present result completes the last series of Phan-type results for classical groups, see details below. We turn now to the motivation and history of the field and also outline how we approach the proof of Main Theorems A and B.

In 1977 Phan published two papers, [P1] and [P2], in which he stated and proved theorems giving presentations for some Chevalley groups, that were similar in spirit to the Curtis-Tits presentations for the groups with simply laced diagrams A_n, D_n, and E_n. Instead of the subgroups $\mathrm{SL}(2,q)$ and $\mathrm{SL}(3,q)$, as in the Curtis-Tits presentation, he used subgroups isomorphic to $\mathrm{SU}(2,q^2)$ and $\mathrm{SU}(3,q^2)$. Phan's results, along with the Curtis-Tits theorem, proved to be fundamental for the original classification of the finite simple groups announced in 1981, especially for Aschbacher's paper [A]. The current revision of the classification, lead by Lyons and Solomon, also requires a revision of Phan's results. Such a revision was started by Bennett and Shpectorov in [BS]; see [BGHS] for a survey.

It was soon discovered that Phan's results are not just similar to the Curtis-Tits theorem, but rather these theorems are much more closely related to each other (see [BGHS]). It turned out that the Curtis-Tits theorem is equivalent, via a certain reduction, to the simple connectedness of the so-called opposites geometry of the spherical twin buildings associated with the corresponding Chevalley group, cf. [M], also [AM]. When the Chevalley group is of untwisted type and is defined over a field $\mathbb{F}_{q^2}$, the twin buildings have a class of automorphisms that we call *unitary flips.* The subgeometry of the opposites geometry, consisting of all objects fixed by the flip, is called the *flipflop geometry.* It turned out that Phan's theorems, in essence (that is, again modulo some reduction), are the statements that the flipflop geometries of rank at least three for the simply laced diagrams A_n (Phan's first paper [P1]), D_n and E_n (Phan's second paper [P2]) are simply connected with some exceptions when q equals to 2 or 3. Thus, Phan's theorems can be viewed as twisted versions of particular cases of the Curtis-Tits theorem.

As we have already mentioned, unitary flips exist for all untwisted Chevalley groups over $\mathbb{F}_{q^2}$. On the other hand, Phan treated only the simply laced diagrams. So, naturally, it is interesting to ask whether the flipflop geometries for the diagrams B_n, C_n, $n \geq 3$, and F_4 are simply connected for sufficiently large q. The positive answer for the diagram C_n was obtained in [GHS] and refined in [H], see also [GHN]. Main Theorems A and B finish the last infinite series of Dynkin diagrams, B_n. Notice that in this case we only need to consider the case where q is odd, because B_n and C_n are the same when q is even.

We now outline how Main Theorems A and B are proved. The proof consists of two stages. At stage one let X be a group with a weak Phan system. Define the amalgam $\mathcal{A} = \bigcup_{1\leq i<j\leq n} U_{ij}$, as found in X. For the general concept of a group amalgam see [S]; we deal with a more restricted notion, as described, e.g., in [BS]. The goal of the first stage is to establish the uniqueness of the amalgam $\mathcal{A}$, that is, that it is essentially the same for all groups X with a weak Phan system of type B_n. This step is proved uniformly for all Dynkin diagrams. The first occurrence of this proof was in [BS], where the case of weak Phan systems of type A_n was dealt with. That original proof applies to all simply-laced diagrams. The proof was modified in [G1] to include the double bonds, and in this modified form it applies also to the diagrams B_n, C_n, and F_4. There is also an even more general treatment in [D]. Because of all of this, we do not include details of the first stage in the present paper.

Once the uniqueness of $\mathcal{A}$ is known, it must be the amalgam found in the known example, $\mathrm{Spin}(2n+1,q)$. We observe that an arbitrary group X, having a weak Phan system of type B_n, contains a copy of $\mathcal{A}$, and so X must be isomorphic to a factor group of the universal completion $\mathcal{U}(\mathcal{A})$ of $\mathcal{A}$. Thus, the main result follows if we prove that $\mathcal{A}$ contains enough relations to define $\mathrm{Spin}(2n+1,q)$. More precisely, it needs to be shown that the universal completion $\mathcal{U}(\mathcal{A})$ of $\mathcal{A}$ coincides with $\mathrm{Spin}(2n+1,q)$; for the definition of the universal completion see, for example, [BS]. This is the second stage, and the proof here consists of two steps. First we define $D_i = N_{U_i}(U_j)$, where j is a neighbour of i in the diagram (it turns out that D_i is independent of the neighbour j). Let $D = D_1 D_2 \cdots D_n$ (e.g., as subgroups of $\mathrm{Spin}(2n+1,q)$, where this product is direct). Let also $\widehat{U}_i = U_i D$ and $\widehat{U}_{ij} = U_{ij}D$ for all $1 \leq i < j \leq n$. It turns out that the amalgam $\widehat{\mathcal{A}} = \bigcup_{1\leq i<j\leq n} \widehat{U}_{ij}$ has exactly the same universal completion as $\mathcal{A}$. The proof of this step is again identical for all diagrams; in fact, it is a very general statement, cf. Lemma 29.3 of [GLS]. So we again skip the details of this step in the present paper. It remains to show that the universal completion of $\widehat{\mathcal{A}}$ coincides with $\mathrm{Spin}(2n+1,q)$, which is the second step of the second stage. For this, we observe that $\widehat{\mathcal{A}}$ is the amalgam of rank one and two parabolics for $\mathrm{Spin}(2n+1,q)$ acting flag-transitively on the corresponding flipflop geometry Γ. (For the geometric terminology see Section 3; for an overview over the topic of flipflop geometries see [BGHS] or [G2].) By Tits' Lemma (see [T], also Corollary 1.4.6 of [IS], or, in a more general form, Corollary 3.2 of [GVM]) the group $\mathrm{Spin}(2n+1,q)$ is the universal completion of $\widehat{\mathcal{A}}$ if and only if Γ and all its residues of rank at least three are simply connected. Thus, the proof is achieved via the study of Γ. This part is the only part that is individual for each diagram; and this is exactly what we do in the present paper for the case of the diagram B_n. (We would like to mention that recently a building-theoretic method has been found to treat all spherical diagrams simultaneously, including the exceptional ones, over sufficiently large finite fields, see [DM] and [GHMS].)

We now define Γ and state the result that we prove about Γ. Since Γ is the flipflop geometry related to a unitary flip, the initial setup involves the field $\mathbb{F}_{q^2}$. Let V be the natural module of the group $G \cong \mathrm{S}\Omega(2n+1,q^2)$, q odd, with the nondegenerate symmetric bilinear form $(\cdot,\cdot)$ and the corresponding quadratic form f. Let σ be an involutory semilinear transformation of V, satisfying $(\sigma(u),\sigma(v)) = \overline{(u,v)} = (u,v)^q$. We will show, see Proposition 2.10, that $G_\sigma = C_G(\sigma)$ is isomorphic to $\mathrm{SO}(2n+1,q)$. The *flipflop geometry* Γ consists of those singular subspaces of V that trivially intersect the polar of their images under σ; see Proposition 3.1 for an alternative description of Γ. Clearly G_σ acts on Γ and this also leads to the action of $\mathrm{Spin}(2n+1,q)$ on Γ, since the latter group is a double cover of the index two subgroup $G'_\sigma \cong \mathrm{S\Omega}(2n+1,q)$ of G_σ.

Theorem *Let $q \geq 3$ be odd and let $n \geq 2$. Then the following hold.*

(i) Γ *is a rank n geometry, on which $G_\sigma \cong \mathrm{SO}(2n+1,q)$ and its index two subgroup $G'_\sigma \cong \mathrm{S\Omega}(2n+1,q)$ act as flag-transitive groups of automorphisms.*

(ii) Γ *is residually connected for $q \geq 5$.*

(iii) Γ *is simply connected for $n \geq 4$ and for $n = 3$ and $q \geq 5$.*

This theorem, together with the results of [BS], implies that all residues of Γ of rank at least three are simply connected, provided that $q \geq 5$. For $q = 3$, it implies that all residues of rank at least four are simply connected, leading to Main Theorem B. The cases $n = 3$ and $5 \leq q \leq 9$ are dealt with in the second part [PART2] computationally. In the present paper we thus assume that $q \geq 11$ if $n = 3$. For $n = 3$ and $q = 3$ there exists a counterexample to the conclusion of part (iii) of the Theorem. Namely, the universal cover of Γ is finite of degree 3^7, see [PART2] for the details.

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2 Unitary flips

Let $G = \mathrm{S\Omega}(2n+1,q^2)$, q odd, let V be its natural module, and let $(\cdot,\cdot)$ be the symmetric bilinear form on V. Let $^-\colon a \mapsto \bar{a} = a^q$ be the involutory automorphism of $\mathbb{F}_{q^2}$. By a *unitary flip* we mean a semilinear transformation σ of V satisfying the following conditions:

(F1) $\sigma(av) = \bar{a}\sigma(v)$ for all $a \in \mathbb{F}_{q^2}$ and $v \in V$;

(F2) σ semi-preserves $(\cdot,\cdot)$ up to a scalar; that is, $(\sigma(u),\sigma(v)) = a\overline{(u,v)}$ for some $a \in \mathbb{F}_{q^2}$ and all $u, v \in V$;

(F3) σ^2 is a scalar transformation; that is, $\sigma^2(v) = bv$ for some $b \in \mathbb{F}_{q^2}$ and all $v \in V$.

The following is the main result of this section.

Proposition 2.1 *Up to conjugation with an element of $\Gamma\mathrm{O}(2n+1,q^2)$ and multiplication with a scalar, there exists a unique unitary flip σ of V. This σ has the additional property that $\sigma(U)\cap U = \{0\}$ for at least one maximal totally singular subspace U of V.*

Let σ be a semilinear transformation of V satisfying (F1), (F2) and (F3). Clearly, every scalar multiple of σ also satisfies these conditions.

Lemma 2.2 *There exists $c \in \mathbb{F}_{q^2}$ such that $(c\sigma)^2 = \mathrm{Id}$.*

Proof. By (F3) we have that $\sigma^2 = b \cdot \mathrm{Id}$ for some $b \in \mathbb{F}_{q^2}$. We claim that, in fact, $b \in \mathbb{F}_q$. Indeed, on one hand, $\sigma^3(v) = \sigma^2(\sigma(v)) = b\sigma(v)$, where $v \in V\backslash\{0\}$. On the other hand, $\sigma^3(v) = \sigma(\sigma^2(v)) = \sigma(bv) = \bar{b}\sigma(v)$. Since $\sigma(v) \neq 0$, we conclude that $b \in \mathbb{F}_q$. By surjectivity of the norm map $\mathbb{F}_{q^2} \to \mathbb{F}_q$, there exists $c \in \mathbb{F}_{q^2}$, such that $b^{-1} = c^{q+1} = c\bar{c}$. Let $\sigma' = c\sigma$. Then $(\sigma')^2(v) = c\sigma(c\sigma(v)) = c\bar{c}\sigma^2(v) = b^{-1}bv = v$ for all $v \in V$. $\square$

Because of this lemma we assume from now on that

(F3') $\sigma^2 = \mathrm{Id}$.

Notice that this condition does not specify σ uniquely among its scalar multiples. More precisely, $(c\sigma)^2 = \mathrm{Id}$ if and only if $c\bar{c} = 1$. Since $c\bar{c} = c^{q+1}$, the field $\mathbb{F}_{q^2}$ contains exactly $q+1$ such scalars c. Next, we choose among the scalar multiples of the form $(\cdot,\cdot)$, one that fits our σ best. For $d \in \mathbb{F}_{q^2}\backslash\{0\}$ let $(\cdot,\cdot)' := d(\cdot,\cdot)$. Clearly, $(\cdot,\cdot)'$ can be used in place of $(\cdot,\cdot)$, as it defines the same building geometry and the same orthogonal group. Furthermore, σ semi-preserves $(\cdot,\cdot)'$, up to a scalar.

Lemma 2.3 *There exists $d \in \mathbb{F}_{q^2}$ such that the corresponding form $(\cdot,\cdot)' = d(\cdot,\cdot)$ is semi-preserved by σ; that is, $(\sigma(u),\sigma(v))' = \overline{(u,v)'}$ for all $u, v \in V$.*

Proof. Since σ semi-preserves $(\cdot,\cdot)$ up to a scalar, there exists $a \in \mathbb{F}_{q^2}$ such that $(\sigma(u),\sigma(v)) = a\overline{(u,v)}$ for $u,v \in V$. Notice that $(u,v) = (\sigma^2(u),\sigma^2(v)) = a\overline{(\sigma(u),\sigma(v))} = a\bar{a}\overline{\overline{(u,v)}} = a\bar{a}(u,v)$. Hence $a\bar{a} = 1$, that is, $a^{q+1} = 1$. This means that there exists $d \in \mathbb{F}_{q^2}$ such that $a = d^{q-1} = \frac{\bar{d}}{d}$. For the corresponding form $(\cdot,\cdot)'$ we have $(\sigma(u),\sigma(v))' = d(\sigma(u),\sigma(v)) = (da)\overline{(u,v)} = \bar{d}\overline{(u,v)} = \overline{(u,v)'}$. □

In view of this lemma we assume in what follows that

(F2') $(\sigma(u),\sigma(v)) = \overline{(u,v)}$ for all $u, v \in V$.

Notice that a multiple $c\sigma$ of σ semi-preserves this $(\cdot,\cdot)$ if and only if $c^2 = 1$. Thus, $-\sigma$ is the only other multiple of σ that semi-preserves $(\cdot,\cdot)$. Notice also that $-\sigma$ squares to the identity, just like σ. Finally, σ semi-preserves a nonzero multiple $d(\cdot,\cdot)$ of $(\cdot,\cdot)$ if and only if $1 = \frac{\bar{d}}{d} = d^{q-1}$, that is, if and only if d is a nonzero element of $\mathbb{F}_q$.

Definition 2.4 For $u, v \in V$, let

$$((u,v)) := (u, \sigma(v)).$$

Moreover, let $f(v) := (v,v)$ and $g(v) := ((v,v))$.

Clearly, $((\cdot,\cdot))$ is a nondegenerate sesquilinear form on V. Perpendicularity with respect to $(\cdot,\cdot)$ will be denoted by $\perp$, while perpendicularity with respect to $((\cdot,\cdot))$ will be denoted by $\perp\!\!\!\perp$. A subspace of V degenerate (respectively, nondegenerate) with respect to $(\cdot,\cdot)$ and f will be called f-degenerate (respectively, f-nondegenerate), and similarly for $((\cdot,\cdot))$ and g. Notice that a σ-invariant subspace is f-degenerate if and only if it is g-degenerate, so for such a subspace we can speak simply of degeneracy or nondegeneracy.

Lemma 2.5 *The form $((\cdot,\cdot))$ is Hermitian. It is semi-preserved by σ in the sense that $((\sigma(u),\sigma(v))) = \overline{((u,v))}$.*

Proof. Indeed, $((v,u)) = (v,\sigma(u)) = \overline{(\sigma(v),\sigma^2(u))} = \overline{(\sigma(v),u)} = \overline{(u,\sigma(v))} = \overline{((u,v))}$. Thus $((\cdot,\cdot))$ is Hermitian. Since $((\sigma(u),\sigma(v))) = (\sigma(u),\sigma^2(v)) = \overline{(u,\sigma(v))} = \overline{((u,v))}$, we also see that σ semi-preserves $((\cdot,\cdot))$. □

Let $V_\sigma = C_V(\sigma) = \{v \in V \mid \sigma(v) = v\}$. We call V_σ the *model space*. If U is a σ-invariant subspace of V then $U_\sigma = \{u \in U \mid \sigma(u) = u\} = U \cap V_\sigma$ will be called the *model* of U.

Lemma 2.6 *The map $U \mapsto U_\sigma$ is a dimension-preserving bijection between all σ-invariant subspaces of V and all subspaces of the $\mathbb{F}_q$-space V_σ.*

Proof. Since σ is $\mathbb{F}_q$-linear, V_σ is a vector space over $\mathbb{F}_q$. Suppose $u_1, \ldots, u_k$ is the smallest linearly independent subset of V_σ that is linearly dependent over $\mathbb{F}_{q^2}$, and let $a_1u_1 + \cdots + a_ku_k = 0$ be a nontrivial linear dependence. Notice that we can assume $a_1 = 1$ and, furthermore, that at least one coefficient a_i is not contained in $\mathbb{F}_q$. Applying σ, we get a second relation $\bar{a}_1u_1 + \cdots + \bar{a}_ku_k = 0$, which is not a scalar multiple of the first relation. Using the second relation we can exclude at least one vector from the first relation, yielding a contradiction with the minimality of the set $u_1, \dots, u_k$. Thus, every linearly independent subset of V_σ is also independent over $\mathbb{F}_{q^2}$. To complete the proof of the second claim, it remains to show that U_σ spans U. Let m be the dimension of U. Consider U as a vector space over $\mathbb{F}_q$ of dimension $2m$ and σ as an $\mathbb{F}_q$-linear endomorphism of U. Since q is odd and $\sigma^2 = \mathrm{Id}$, the subspace U is the direct sum of the eigenspaces of σ corresponding to the eigenvalues 1 and -1. The first eigenspace is U_σ, the second one is $U_{-\sigma}$. By the above, the dimension of U_σ is at most m. Since $-\sigma$ semi-preserves $(\cdot,\cdot)$ and since $(-\sigma)^2 = \mathrm{Id}$, we also have that the dimension of $U_{-\sigma}$ is at most m. It follows that the dimensions of both eigenspaces are m. Thus, every basis of U_σ is a basis of U over $\mathbb{F}_{q^2}$. □

In particular, the dimension of V_σ is $2n+1$ and every basis of V_σ is a basis of V over $\mathbb{F}_{q^2}$. It follows from the definition of $((\cdot,\cdot))$ that its restriction to V_σ coincides with the restriction of $(\cdot,\cdot)$. Hence also the forms f and g agree on V_σ. Because of this, we can speak of singular vectors and subspaces instead of f- or g-singular, and similarly for all other properties of vectors and subspaces of V_σ.

The next two lemmas are consequences of Lemma 2.6.

Lemma 2.7 *The restrictions of $(\cdot,\cdot)$ on V_σ is a nondegenerate bilinear form over $\mathbb{F}_q$.*

Proof. Clearly, $(\cdot,\cdot)$ is $\mathbb{F}_q$-bilinear. For $u, v \in V_\sigma$, we have that $(u,v) = (\sigma(u), \sigma(v)) = \overline{(u,v)}$. Hence the values of $(\cdot,\cdot)$ on V_σ belong to $\mathbb{F}_q$. Finally, the restriction of $(\cdot,\cdot)$ to V_σ is nondegenerate because V_σ contains a basis of V. □

Lemma 2.8 *If U_1 and U_2 are σ-invariant subspaces of V with $U_1 \subseteq U_2$ then U_2 contains a σ-invariant complement to U_1.*

Proof. Indeed, W can be chosen so that W_σ is a complement to $(U_2)_\sigma$ in $(U_1)_\sigma$. □

Proof of Proposition 2.1. Suppose σ and σ' are two semi-linear transformations of V, satisfying (F1), (F2'), and (F3') with respect to $(\cdot,\cdot)$ and $(\cdot,\cdot)'$, respectively. Suppose further that $(\cdot,\cdot)'$ is a scalar multiple of $(\cdot,\cdot)$. Since in odd dimension all nondegenerate symmetric bilinear forms are isometric, up to a possible non-square factor, cf. [C], [K], we can assume that we have a bijective linear map $\psi : V_\sigma \to V_{\sigma'}$, such that $(\psi(u), \psi(v))' = c(u,v)$ for all $u, v \in V_\sigma$ and a fixed $c \in \mathbb{F}_q$. Extend ψ by $\mathbb{F}_{q^2}$-linearity to the entire space V. The resulting mapping is a bijective endomorphism of V and it preserves $(\cdot,\cdot)$, up to a scalar. It also conjugates σ to σ'.

Thus, all such σ's are conjugate. It remains to show that σ takes some maximal totally singular subspace U to an opposite maximal totally singular subspace, that is, $\sigma(U) \cap U = 0$. Since the dimension of V is odd, we may assume without loss of generality that the determinant of the Gram matrix of $(\cdot,\cdot)$ is a square. Then a basis $e_1, f_1, \dots, e_n, f_n, x$ exists such that, for all $1 \le i, j \le n$, we have $(e_i, e_j) = 0 = (f_i, f_j) = (e_i, x) = (f_i, x)$, $(e_i, f_j) = \delta_{i,j}$, and $(x, x) = 1$. Set $\sigma'(e_i) = f_i$, $\sigma'(f_i) = e_i$, $\sigma'(x) = x$, and extend by semi-linearity to the entire V. This σ' satisfies (F1), (F2'), and (F3'). By the above, our initial σ is conjugate to σ' up to a scalar factor. Since σ' manifestly takes $U = \langle e_1, \dots, e_n \rangle$ to $\sigma'(U) = \langle f_1, \dots, f_n \rangle$, the last claim of Proposition 2.1 follows as well. □

A *σ-point* in V is a 1-dimensional subspace $U = \langle u \rangle$ such that u is f-singular and g-nonsingular.

Lemma 2.9 *Suppose U is a σ-invariant nondegenerate subspace of V of dimension at least two. Then U contains no σ-point if and only if $\dim U = 2$ and U_σ is of plus type in V_σ.*

Proof. Suppose first that U is 2-dimensional. Since the restriction of $(\cdot,\cdot)$ to U_σ is a bilinear form over $\mathbb{F}_q$, we have that U is of plus type. In particular, U contains exactly two 1-dimensional f-singular subspaces, and they are not perpendicular to each other. Since σ takes an f-singular vector again to an f-singular vector, U contains a σ-point if and only if σ interchanges the two f-singular subspaces. Equivalently, U contains no σ-point if and only if both f-singular subspaces of U are σ-invariant. The latter condition means that the intersections of the two f-singular subspaces with U_σ are nontrivial, that is, the restriction of $(\cdot,\cdot)$ to U_σ is of plus type. Thus, the claim of the lemma holds when $\dim U = 2$.

If $\dim U \geq 3$ then U_σ contains a subspace X that is nondegenerate of minus type. By Lemma 2.6, we have $X = W_\sigma$ for some σ-invariant subspace $W \leq U$. By the above, W (and hence also U) contains a σ-point. $\square$

Let us now return to the proof of Proposition 2.1. A basis $e_1, \ldots, e_n, f_1, \ldots, f_n, x$ as in this proof is called a *standard basis* of V with respect to $(\cdot,\cdot)$ and σ. That is, a standard basis satisfies the following conditions: for all $1 \leq i, j \leq n$, we have $(e_i, e_j) = 0 = (f_i, f_j) = (e_i, x) = (f_i, x)$, $(e_i, f_j) = \delta_{i,j}$, and $(x, x) = 1$. Furthermore, $\sigma(e_i) = f_i$ (and hence $\sigma(f_i) = e_i$ for all i), and $\sigma(x) = x$. Suppose $(\cdot,\cdot)$ and σ satisfy (F1), (F2'), and (F3'). Does it mean that a standard basis exists in V? Not necessarily. Indeed, given a standard basis, set $E_i = \langle e_i, f_i \rangle$, $i = 1, \ldots, f_i$, and $X = \langle x \rangle$. Then V_σ decomposes as the orthogonal direct sum of all $(E_i)_\sigma$ and X_σ. Since E_i contains σ-points (namely, $\langle e_i \rangle$ and $\langle f_i \rangle$), each summand $(E_i)_\sigma$ is of minus type in V_σ. Notice that $x \in X_\sigma$. This means that the discriminant of $(\cdot,\cdot)$ on V_σ (determinant of the Gram matrix of $(\cdot,\cdot)$ with respect to an arbitrary basis of V_σ, taken modulo the squares in $\mathbb{F}_q^*$) is congruent to $(-\xi)^n$. Here ξ is an arbitrary non-square in $\mathbb{F}_q$. It is easy to reverse this argument and establish that *standard bases exist if and only if the discriminant of* $(\cdot,\cdot)$ *on* V_σ *is congruent to* $(-\xi)^n$. Since the dimension of V is odd, by taking, if necessary, $(\cdot,\cdot)' = \xi(\cdot,\cdot)$ in place of $(\cdot,\cdot)$, we may assume without loss of generality that the congruence condition is satisfied for $(\cdot,\cdot)$ and σ, and hence standard bases exist. This assumption stays throughout the remainder of the paper.

Conjugation by σ is an automorphism of $G = \mathrm{S\Omega}(2n+1, q^2)$. Let G_σ be the centralizer of σ in G. The above setup gives us means to identify G_σ. Let H be the group of linear transformations of V_σ of determinant one, preserving (the restriction of) the form $(\cdot,\cdot)$. By Lemma 2.6, the group H is isomorphic to $\mathrm{SO}(2n+1, q)$. Since V_σ contains a basis of V, we can use $\mathbb{F}_{q^2}$-linearity to extend the action of the elements of H to the entire V. This allows us to identify H with a subgroup of $\mathrm{SO}(V, f)$. Notice that under this identification H is contained in $\mathrm{S\Omega}(V, f) = G$. Indeed, every element of H can be written as a product of reflections in the nonsingular vectors $v \in V_\sigma$. Since $f(v) \in \mathbb{F}_q$ is a square in $\mathbb{F}_{q^2}$, every element of H lies in $\mathrm{S\Omega}(V, f)$.

Proposition 2.10 $G_\sigma = H \cong \mathrm{SO}(2n+1, q)$.

Proof. Choose a basis $\{w_1, \ldots, w_{2n+1}\}$ in V_σ. Then this set is also a basis of V. Let $h \in H$. If $u = \sum_{i=1}^{2n+1} x_i w_i \in V$ then $h\sigma(u) = h\left(\sum_{i=1}^{2n+1} \bar{x}_i w_i\right) = \sum_{i=1}^{2n+1} \bar{x}_i h(w_i)$. On the other hand, $\sigma h(u) = \sigma\left(\sum_{i=1}^{2n+1} x_i h(w_i)\right) = \sum_{i=1}^{2n+1} \bar{x}_i h(w_i)$. Therefore, $H \leq G_\sigma$. Now take $h \in G_\sigma$. If $u \in V_\sigma$ then $\sigma h(u) = h\sigma(u) = h(u)$. This proves that h leaves V_σ invariant. Hence h induces on V_σ an $\mathbb{F}_q$-linear transformation of determinant one, that preserves the restriction of $(\cdot,\cdot)$. That is, $h \in H$. $\square$

In what follows, G'_σ denotes the index two subgroup of G_σ isomorphic to $\mathrm{S\Omega}(2n+1, q)$.

3 The flipflop geometry

Geometries

In this section we give a brief rundown of the basic terminology of synthetic geometry. For a comprehensive introduction into the subject, refer to [BC].

Let I be a finite set, called the *set of types*. Its elements as well as its subsets are called *types*. Let $\Gamma = (X, *, \mathrm{typ})$ be a triple where X is a set, $* \subseteq X \times X$ is a symmetric and reflexive relation and $\mathrm{typ} : X \to I$ is a map, such that, for $x, y \in X$ we have $x = y$ if and only if $x * y$ and $\mathrm{typ}(x) = \mathrm{typ}(y)$. Then Γ is called a *pregeometry over* I. The elements of X are called the *elements of* Γ, the relation $*$ is called the *incidence relation of* Γ, the map typ is called the *type function of* Γ.

Let $\Gamma = (X, *, \mathrm{typ})$ be a pregeometry over I. If $A \subseteq X$, then A is of *type* $\mathrm{typ}(A) \subseteq I$, of *cotype* $I \backslash \mathrm{typ}(A)$, of *rank* $|\mathrm{typ}(A)|$, and of *corank* $|I \backslash \mathrm{typ}(A)|$. The rank of A is also denoted by $\mathrm{rk}(A)$. The cardinality $|I|$ of I is called the *rank of* Γ.

A *flag* F of a pregeometry Γ is a set of mutually incident elements of Γ. Notice that $\mathrm{typ}_{|F} : F \to I$ is a injection. A *maximal flag* of Γ is a flag that is maximal with respect to inclusion. Flags of type I are called *chambers*. A *geometry over* I is a pregeometry Γ over I in which every maximal flag is a chamber.

Let F be a flag of Γ, say of type $J \subseteq I$. Then the *residue* Γ_F *of* F is the geometry $(X', *_{|X' \times X'}, \mathrm{typ}_{|X})$ over $I \backslash J$, with $X' := \{x \in X \mid F \cup \{x\}$ is a flag of Γ and $\mathrm{typ}(x) \notin \mathrm{typ}(F)\}$.

The geometry Γ is *connected* if the graph $(X, *)$ is connected. The geometry Γ is *residually connected* if for any flag F of corank at least two the residue Γ_F is connected.

If $\Gamma = (X, *, \mathrm{typ})$ and $\Gamma' = (X', *', \mathrm{typ}')$ are two geometries, over I and I', respectively, with $I \cap I' = \emptyset$, then the *direct sum* $\Gamma \oplus \Gamma'$ is the geometry $(X \cup X', *'', \mathrm{typ} \cup \mathrm{typ}')$ over $I \sqcup I'$, with $*''_{|X} = *$, $*''_{|X'} = *'$ and $(X \times X') \subseteq *''$.

A group G of automorphisms of some pregeometry Δ is called *flag-transitive* if for each pair F_1, F_2 of flags of the same type there exists a $g \in G$ such that $g(F_1) = F_2$. Notice that for a geometry Δ this condition is equivalent to the condition that G is transitive on the set of chambers.

The flipflop geometry of type B_n

We will use the notation from Section 2. In particular, V is a nondegenerate orthogonal space over $\mathbb{F}_{q^2}$, of dimension $2n + 1$, with the bilinear form $(\cdot, \cdot)$ and quadratic form f. The semilinear map σ is a (unitary) flip with the corresponding Hermitian form $((\cdot, \cdot))$ and unitary form g. Also, G is isomorphic to $\mathrm{S}\Omega(2n+1, q^2)$. Furthermore, G_σ is the centralizer $C_G(\sigma)$ of σ in G. The group G_σ is isomorphic to $\mathrm{SO}(2n+1, q)$ and G'_σ is the index two subgroup of G_σ, isomorphic to $\mathrm{S}\Omega(2n+1, q)$.

Throughout this section, we assume $n \geq 2$. Let $\mathcal{B}$ be the building geometry associated with G. The elements of $\mathcal{B}$ of type $i = 1, 2, \ldots, n$ are the f-singular subspaces of V of dimension i. Incidence is given by symmetrized containment. We will use the customary geometric terminology. In particular, *points*, *lines*, and *planes* are subspaces of a vector space of dimension 1, 2, and 3, respectively.

Let Γ be the pregeometry consisting of those nontrivial f-singular subspaces of V that do not intersect the polar of their image under σ. (See [BGHS] for an explanation why this is a natural object to consider.) The pregeometry Γ is called the *flipflop geometry* of $\mathcal{B}$ associated with σ. Alternatively, we can describe the flipflop geometry Γ as follows.

Proposition 3.1 *The elements of* Γ *are all subspaces* $\{0\} \neq U \subsetneq V$*, which are* f*-singular and* g*-nondegenerate.*

Proof. We have $U^{\perp\!\!\perp} = \sigma(U^\perp)$. Hence, if X is the g-radical of U, we have $X = U \cap U^{\perp\!\!\perp} = U \cap \sigma(U^\perp)$. Therefore $X = \{0\}$ if and only if $U \cap \sigma(U^\perp) = \{0\}$. □

We remark that the σ-points, as defined in the preceding section, are just the points of Γ. We now establish that Γ is, in fact, a geometry.

Proposition 3.2 *The pregeometry* Γ *is a geometry of rank* n*. Moreover,* G_σ *and* G'_σ *act flag-transitively on* Γ*.*

Proof. For the first claim we need to show that a maximal flag F in Γ contains elements of all types. If F contains an element of type i then, clearly, it also contains elements of all types less than i. Suppose m is the highest type present in F, and let U be the element of type m in F. Let $W = \langle U, \sigma(U) \rangle$ and $T = W^\perp$. Since W is nondegenerate, so is T, and hence σ is a flip of T. Therefore, by Proposition 2.1 there exists a maximal f-singular subspace X in T, such that $\sigma(X) \cap X = \{0\}$. The space X has dimension $n - m$ and thus $\langle U, X \rangle$ is an element of Γ of type n incident to each element of F. This shows that Γ is a geometry.

For the second claim, let $V_1, V_2, \ldots, V_n$ and $V_1', V_2', \ldots, V_n'$ be two chambers ordered by types. Choose a base $e_1, \ldots, e_n$ in V_n that is orthonormal with respect to $((\cdot,\cdot))$ and such that $V_i = \langle e_1, \ldots, e_i \rangle$. Set $f_i := \sigma(e_i)$, for $i = 1, \ldots, n$, and let x be chosen in $\langle V_n, \sigma(V_n) \rangle^\perp$ so that $g(x) = 1$ and $\sigma(x) = x$. Such an x exists since the discriminant of $(\cdot,\cdot)$ on V_σ is congruent to $(-\xi)^n$, ξ a non-square in $\mathbb{F}_q$ (cf. the discussion after Lemma 2.9). Indeed, the discriminant of $(\cdot,\cdot)$ on $\langle V_n, \sigma(V_n) \rangle_\sigma$ is also congruent to $(-\xi)^n$, which yields that the discriminant of $(\cdot,\cdot)$ on the 1-dimensional space $(\langle V_n, \sigma(V_n) \rangle^\perp)_\sigma$ is congruent to one. Hence x can be chosen as claimed.

Now, $e_1, \ldots, e_n, f_1, \ldots, f_n, x$ is a standard basis for σ. Choose a similar standard basis $e_1', \ldots, e_n'$, $f_1', \ldots, f_n', x'$ for the second chamber. Let h be a linear transformation of V that sends every e_i to e_i', every f_i to f_i', and x to x'. Clearly, h preserves $(\cdot,\cdot)$ and hence it is an orthogonal transformation. Substituting x' with $-x'$, if necessary, we may assume that h has determinant one. Now observe that $h\sigma$ and σh are both semilinear and their actions on the basis $e_1, \ldots, e_n, f_1, \ldots, f_n, x$ coincide. This means that h commutes with σ. In particular, h acts on the model space V_σ as an orthogonal transformation of determinant one, and we conclude that $h \in H = G_\sigma$. Manifestly, h takes the first chamber to the second one. Hence G_σ is flag-transitive on Γ.

In order to show that G_σ' is also flag-transitive on Γ it suffices to show that the stabilizer in G_σ of a chamber F_1 is not contained in G_σ'. Let $e_1, \ldots, e_n, f_1, \ldots, f_n, x$ be as above. Let $U = \langle e_1, \ldots, e_{n-1}, f_1, \ldots, f_{n-1} \rangle$. The centralizer L of U in G_σ is isomorphic to $\mathrm{SO}(3,q)$ (acting on $U^\perp = \langle e_n, f_n, x \rangle$), while $L \cap G_\sigma'$ is the index two subgroup isomorphic to $\mathrm{S}\Omega(3,q)$. Let D consist of all linear transformations d_λ, $\lambda \in \mathbb{F}_{q^2}$, $\lambda\bar{\lambda} = 1$, centralizing U and acting on $U^\perp$ as follows: $d_\lambda(e_n) = \lambda e_n$, $d_\lambda(f_n) = \bar{\lambda} f_n$, and $d_\lambda(x) = x$. Then D is a cyclic group of order $q+1$ that stabilizes the chamber F_1. Clearly, $D \leq L$, but $D \not\leq L \cap G_\sigma' \cong \mathrm{S}\Omega(3,q) \cong \mathrm{PSL}(2,q)$. $\square$

We now collect some useful lemmas to be applied later.

Lemma 3.3 *Let p be a point of Γ and $W \supset p$ be a 3-dimensional f-singular subspace of V of g-rank at least two. Let U be a 2-dimensional subspace of W that contains at least one point of Γ and does not contain p. Then U contains at least $q^2 - 2q - 1$ (respectively, $q^2 - q - 1$) points of Γ that are collinear with p if it is (respectively, is not) a line.*

Proof. This is Lemma 4.4 of [GHNS]. $\square$

Lemma 3.4 *Let W be a 3-dimensional f-nondegenerate subspace of V that is not g-singular. Then W contains at least $q^2 - 2q - 1$ points of Γ.*

Proof. Since $(\cdot,\cdot)$ is nondegenerate on W, we can choose a basis $\{e_1, f_1, x\}$ in W for which either $(\cdot,\cdot)$ or $(\cdot,\cdot)' = \xi(\cdot,\cdot)$ (ξ a non-square in $\mathbb{F}_{q^2}$) has the following Gram matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A simple calculation shows that $e_1 + af_1 + bx$ is f-singular if and only if $a = \frac{-b^2}{2}$. The vector f_1 and the q^2 vectors $e_1 - \frac{b^2}{2} f_1 + bx$ represent all f-singular 1-dimensional subspaces of W. Now, $g(e_1 - \frac{b^2}{2} f_1 + bx) = 0$ if and only if

$$((e_1,e_1)) - \frac{\bar{b}^2}{2}((e_1,f_1)) + \bar{b}((e_1,x)) - \frac{b^2}{2}((f_1,e_1)) + \frac{b^2\bar{b}^2}{4}((f_1,f_1))$$
$$+\frac{b^2\bar{b}}{2}((f_1,x)) + b((x,e_1) + \frac{b\bar{b}^2}{2}((x,f_1)) + b\bar{b}((x,x)) = 0.$$

Since $\bar{b} = b^q$ and not all the inner products above can be 0 by hypothesis, the above yields that $g(e_1 + -\frac{b^2}{2} f_1 + bx) = 0$ if and only if b satisfies a polynomial of degree $2q + 2$ in b. Hence the number of 1-dimensional subspaces of W that are simultaneously f- and g-singular is at most $2q + 2$. Consequently, there are at least $q^2 + 1 - (2q + 2) = q^2 - 2q - 1$ points of Γ in W. $\square$

Lemma 3.5 *Let W be a subspace of V containing a vector u such that $g(u) \neq 0$. Define a unitary form g_1 on W via $g_1(w) = g(\mathrm{pr}_{u^{\perp\!\!\perp}}(w))$, where $\mathrm{pr}_{u^{\perp\!\!\perp}}$ denotes the orthogonal (with respect to $((\cdot,\cdot))$) projection onto $u^{\perp\!\!\perp}$. Then for, $w \in W \setminus \langle u \rangle$, the 2-dimensional subspace $\langle u, w \rangle$ is g-nondegenerate if and only if $g_1(w) \neq 0$.*

Proof. Indeed, u and $\mathrm{pr}_{u^{\perp\!\!\perp}}(w)$ form an orthogonal basis of $\langle u, w \rangle$, and the Gram matrix of $((\cdot,\cdot))$ with respect to this basis has determinant $g(u)g(\mathrm{pr}_{u^{\perp\!\!\perp}}(w)) = g(u)g_1(w)$. □

Now we turn to the question of connectedness of Γ. In the case $n = 2$ we have $B_2 = C_2$, so we can apply the results from [GHS] and obtain that Γ is connected.

Lemma 3.6 *Let $q \geq 5$ and $n = 3$. Then the collinearity graph of Γ has diameter two.*

Proof. Let p_1 and p_2 be points of Γ. If $p_1 \not\perp p_2$, then the 5-dimensional space $\langle p_1, p_2 \rangle^\perp$ is f-nondegenerate and has g-rank at least four. Hence $\langle p_1, p_2 \rangle^\perp$ contains an f-isotropic 2-dimensional subspace that has g-rank at least one, cf. Lemma 6.2 of [GHNS]. By Lemma 3.3 this subspace contains $q^2 - 3q - 2$ points of Γ collinear to p_1 and p_2. If $p_1 \perp p_2$, then $\langle p_1, p_2 \rangle$ is contained in an f-totally isotropic 3-dimensional subspace that has g-rank at least two, and again Lemma 3.3 applies. □

Proposition 3.7 *Let $q \geq 5$ and $n = 3$, or let $n \geq 4$. Then the collinearity graph of Γ has diameter two.*

Proof. The first case was covered by the preceding lemma. If $n \geq 4$ and p_1, p_2 are two points of Γ then $\langle p_1, p_2, p_1^\sigma, p_2^\sigma \rangle^\perp$ is at least 5-dimensional of rank at least three. Its radical is σ-invariant, so one can find a 3-dimensional subspace as in Lemma 3.4. That subspace contains a common neighbour of p_1 and p_2. □

4 Simple connectedness

Homotopies

Considering the flag complex of a geometry of rank n as an n-dimensional simplicial complex, we can use notions from combinatorial topology, cf. [ST].

Let $\mathcal{G}$ be a connected geometry. A path of length k in the geometry is a sequence of elements $(x_0, \ldots, x_k)$ such that x_i and x_{i+1} are incident, $0 \leq i \leq k-1$. A *cycle* based at an element x is a path in which $x_0 = x_k = x$. Two paths are *homotopically equivalent* if one can be obtained from the other via the following operations (called *elementary homotopies*): inserting or deleting a repetition (*i.e.*, replacing x by xx or vice versa), a return (*i.e.*, replacing x by xyx or vice versa), or a triangle (*i.e.*, replacing x by $xyzx$ or vice versa). The equivalence classes of cycles based at an element x form a group under the operation induced by concatenation of cycles. This group is called the *fundamental group* of $\mathcal{G}$ and denoted by $\pi_1(\mathcal{G}, x)$. A cycle based at x that is homotopically equivalent to the trivial cycle (x) is called *null-homotopic*. Every cycle of length two or three is null-homotopic.

Suppose $\mathcal{G}$ and $\widehat{\mathcal{G}}$ are geometries over the same type set and suppose $\phi : \widehat{\mathcal{G}} \to \mathcal{G}$ is a *homomorphism* of geometries, *i.e.*, ϕ preserves the types and sends incident elements to incident elements. A surjective homomorphism ϕ between connected geometries $\widehat{\mathcal{G}}$ and $\mathcal{G}$ is called a *covering* if and only if for every non-empty flag $\widehat{F}$ in $\widehat{\mathcal{G}}$ the mapping ϕ induces an isomorphism between the residue of $\widehat{F}$ in $\widehat{\mathcal{G}}$ and the residue of $F = \phi(\widehat{F})$ in $\mathcal{G}$. Coverings of a geometry correspond to the usual topological coverings of the flag complex. It is well-known that a surjective homomorphism ϕ between connected geometries $\widehat{\mathcal{G}}$ and $\mathcal{G}$ is a covering if and only if for every element $\widehat{x}$ in $\widehat{\mathcal{G}}$ the map ϕ induces an isomorphism between the residue of $\widehat{x}$ in $\widehat{\mathcal{G}}$ and the residue of $x = \phi(\widehat{x})$ in $\mathcal{G}$. If ϕ is an isomorphism, then the covering is said to be *trivial*.

Recall the well-known fact (see, e.g., Chapter 8 of [ST]) that if $\mathcal{G}$ is a connected geometry and x an element of $\mathcal{G}$, then every covering of the geometry $\mathcal{G}$ is trivial if and only if $\pi_1(\mathcal{G}, x)$ is trivial. A geometry satisfying the above equivalent conditions is called *simply connected*. A *geometric cycle* in the geometry $\mathcal{G}$ is a cycle each element of which is incident with a common element x. A geometric cycle γ is null-homotopic, because γ and x form a cone.

Simple connectedness of the flipflop geometry

Retain the notation from Section 3. In particular, Γ denotes the flipflop geometry. By the following lemma, it suffices to study the collinearity graph of Γ instead of the incidence graph when proving simple connectedness.

Lemma 4.1 *Let $q \geq 5$ and $n = 3$, or let $n \geq 4$. Then every cycle in the incidence graph of Γ is homotopically equivalent to a cycle in the incidence graph of Γ passing only through points and lines.*

Proof. Identical to the proof of Lemma 5.1 in [GHS], which essentially requires a residually connected geometry with a string diagram. See also Lemma 5.4 in [GHNS]. □

Therefore, in order to prove simple connectedness of the geometry, it suffices to analyze the point-line incidence graph of Γ, and, thus, the collinearity graph of Γ. Since by Lemma 3.7 the collinearity graph has diameter two, we only have to study triangles, quadrangles and pentagons in it.

Let us first consider $n \geq 4$. Recall that q is odd. Note that the space generated by the three points of a triangle is f-singular and of g-rank at least two. If the g-rank is three then the triangle is geometric.

Lemma 4.2 *Any triangle can be decomposed into geometric triangles.*

Proof. Let p_1, p_2, p_3 be a triangle. If $\langle p_1, p_2, p_3\rangle$ is nondegenerate then the triangle is geometric and there is nothing to prove. So we can assume that $\langle p_1, p_2, p_3\rangle$ is degenerate. Since $\langle p_1, p_2\rangle$ is a line, the g-radical r of $\langle p_1, p_2, p_3\rangle$ can only be 1-dimensional. We need to consider two separate cases.

If $\langle r\rangle^\sigma = \langle r\rangle$ then $W = \langle p_1, p_2, p_3, p_1^\sigma, p_2^\sigma, p_3^\sigma\rangle$ is 5-dimensional and $W^\perp$ is $(2n-4)$-dimensional of g-rank $2n-5 \geq 3$. Therefore, every σ-invariant complement $W^\perp$ to its radical satisfies the assumptions of Lemma 2.9. Hence, $W^\perp$ contains a point p of Γ. The geometric triangles p, p_1, p_2 and p, p_1, p_3 and p, p_2, p_3 decompose p_1, p_2, p_3.

If $\langle r\rangle^\sigma \neq \langle r\rangle$ then let $W = \langle p_1, p_2, p_1^\sigma, p_2^\sigma\rangle$, which is σ-invariant and nondegenerate. Both r and r^σ are in $W^\perp$, and the latter is a $(2n-3)$-dimensional nondegenerate subspace. Consider $U = W^\perp \cap r^{\perp\!\!\perp}$, which is a space of dimension $2n-4$. Pick a a vector $s \in U$ such that s is f-singular and $\langle s, r\rangle$ is f-nondegenerate. Then $V' = \langle p_1, p_2, p_1^\sigma, p_2^\sigma, r, r^\sigma, s, s^\sigma\rangle$ is an 8-dimensional σ-invariant nondegenerate subspace, on which σ acts as a flip (because $\langle p_1, p_2, r^\sigma, s\rangle$ satisfies condition (F4b) of Corollary 3.7 in [GHNS]) and so we can use the result in [GHNS] to see that the triangle can be decomposed in V', hence also in V. □

Lemma 4.3 *All quadrangles are null-homotopic.*

Proof. The proofs of Lemmas 6.4 – 6.6 of [GHNS] work in this case. □

Lemma 4.4 *All pentagons are null-homotopic.*

Proof. Let a, b, c, d, e be a pentagon. As in Lemma 6.7 from [GHNS], if $a \perp c$ and $a \perp d$, then the line $\langle c, d\rangle$ contains $q^2 - 2q - 1 > 0$ points that are collinear to a.

We can therefore assume that $a \not\perp d$ and $c \not\perp e$ and conclude that the space $\langle a, b, c, d, e \rangle$ has f-rank at least four and g-rank at least two. Therefore the space $W = \langle a, b, c, d, e \rangle^{\perp}$ is a $(2n-4)$-dimensional space of f-rank at least $2n-6$ and not g-singular. It will then contain a point w of Γ, and w will be collinear with at least one point on each of $\langle a, b \rangle$, $\langle b, c \rangle$, $\langle c, d \rangle$, $\langle d, e \rangle$, and $\langle e, a \rangle$, decomposing the pentagon into quadrangles. $\square$

This completes the proof of the following

Proposition 4.5 *Let $n \geq 4$. Then the flipflop geometry Γ is simply connected.*

Now consider the case $n = 3$. Again, we will first decompose the triangles.

Lemma 4.6 *Every nongeometric triangle can be decomposed as a sum of triangles that have two g-orthogonal vertices.*

Proof. Let p_1, p_2, p_3 be a triangle and let r be the g-radical of $\langle p_1, p_2, p_3 \rangle$. Consider $v \in p_1^{\perp\!\!\perp} \cap \langle p_2, p_3 \rangle$. If v is g-singular then the 2-dimensional subspace $\langle p_1, v \rangle$ is g-degenerate, so it must contain r, which implies that $\langle v \rangle = \langle r \rangle$, contradicting the fact that $\langle p_2, p_3 \rangle$ is a line of Γ. Therefore v must be g-nonsingular. As before, $\langle p_1, v \rangle$ is degenerate only if it contains r. However, that would mean that p_1 is in the g-radical of $\langle r, v \rangle$, a contradiction. Therefore, the initial triangle can be decomposed into the two triangles p_1, v, p_2 and p_1, v, p_3. $\square$

Lemma 4.7 *Let $q \geq 7$. Then all triangles can be decomposed into geometric triangles.*

Proof. By the preceding lemma we can restrict our attention to the triangles having g-orthogonal vertices. Since the group G_σ acts transitively on pairs of points that are orthogonal with respect to both forms, we may assume that our triangle has e_1 and e_2 as two of its vertices. It then follows that its g-radical $\langle r \rangle$ lies in the space $\langle e_3, f_3, x \rangle$. Let the third vertex of the triangle be $\langle v \rangle$, where $v = \alpha e_1 + \beta e_2 + r$ with $\alpha, \beta \neq 0$, and assume that $r = \gamma e_3 + \delta f_3 + \varepsilon x$. The conditions on r give $0 = (r, r) = 2\gamma\delta + \varepsilon^2$ and $0 = ((r, r)) = \gamma\bar{\gamma} + \delta\bar{\delta} + \varepsilon\bar{\varepsilon}$. Recall that q is odd, whence 2 is invertible, and notice that $\delta = 0$ would imply $\varepsilon = 0$ which is nonsense in view of the above choice for the radical r. So the first condition gives $\gamma = -\frac{\varepsilon^2}{2\delta}$. Multiplying γ with $\bar{\gamma}$ we get $\gamma\bar{\gamma} = \frac{(\varepsilon\bar{\varepsilon})^2}{4\delta\bar{\delta}}$, and hence the second condition gives $0 = \frac{(\varepsilon\bar{\varepsilon})^2}{4\delta\bar{\delta}} + \delta\bar{\delta} + \varepsilon\bar{\varepsilon} = \frac{1}{4\delta\bar{\delta}}(\varepsilon\bar{\varepsilon} + 2\delta\bar{\delta})^2$, thus $\gamma\bar{\gamma} = \delta\bar{\delta} = -\frac{\varepsilon\bar{\varepsilon}}{2}$. Note that the centralizer of the pair e_1, e_2 acts transitively on the f- and g-singular 1-dimensional subspaces of $\langle e_3, f_3, x \rangle$ and so we may assume that $\gamma\bar{\gamma} = \delta\bar{\delta}$ is any fixed element of $\mathbb{F}_q$ while α and β remain constant.

We will decompose the triangle (e_1, e_2, v) into a sum of seven geometric triangles by constructing an octahedron whose vertices are $\langle e_1 \rangle$, $\langle e_2 \rangle$, $\langle v \rangle$, $\langle f_3 \rangle$, $\langle f_3 - \frac{\gamma}{\beta} f_2 \rangle$, and $\langle f_3 - \frac{\gamma}{\alpha} f_1 \rangle$ and in which all sides except (e_1, e_2, v) are geometric triangles.

The space $\langle f_1, f_2, f_3 \rangle = \langle f_3, f_3 - \frac{\gamma}{\beta} f_2, f_3 - \frac{\gamma}{\alpha} f_1 \rangle$ is obviously an element of our geometry. Thus, $(f_3, f_3 - \frac{\gamma}{\beta} f_2, f_3 - \frac{\gamma}{\alpha} f_1)$ is a geometric triangle, once it is a triangle. For that we need $f_3 - \frac{\gamma}{\beta} f_2$ and $f_3 - \frac{\gamma}{\alpha} f_1$ to be points and $\langle f_3 - \frac{\gamma}{\beta} f_2, f_3 - \frac{\gamma}{\alpha} f_1 \rangle$ a line. These three conditions are equivalent to:

$$\begin{aligned} \alpha\bar{\alpha} + \gamma\bar{\gamma} &\neq 0, \\ \beta\bar{\beta} + \gamma\bar{\gamma} &\neq 0, \quad \text{and} \\ \alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} &\neq 0. \end{aligned}$$

Notice that (f_3, e_1, e_2), $(e_1, f_3, f_3 - \frac{\gamma}{\beta} f_2)$, and $(e_2, f_3, f_3 - \frac{\gamma}{\alpha} f_1)$ are geometric triangles, if $f_3 - \frac{\gamma}{\alpha} f_1$ and $f_3 - \frac{\gamma}{\beta} f_2$ are points. The vectors $f_3 - \frac{\gamma}{\beta} f_2$, e_1, and $\alpha e_1 + \beta e_2 + r$ generate a totally $(\cdot, \cdot)$-isotropic subspace, and the Gram matrix with respect to $((\cdot, \cdot))$ on it is

$$\begin{pmatrix} 1 + \frac{\gamma\bar{\gamma}}{\beta\bar{\beta}} & 0 & \bar{\delta} \\ 0 & 1 & \bar{\alpha} \\ \delta & \alpha & \alpha\bar{\alpha} + \beta\bar{\beta} \end{pmatrix},$$

whose determinant equals $\alpha\bar{\alpha}+\beta\bar{\beta}+\frac{\gamma\bar{\gamma}\alpha\bar{\alpha}}{\beta\bar{\beta}}+\gamma\bar{\gamma}-\delta\bar{\delta}-\alpha\bar{\alpha}-\frac{\gamma\bar{\gamma}\alpha\bar{\alpha}}{\beta\bar{\beta}}=\beta\bar{\beta}\neq 0$ and so we only need to verify that the sides are lines of Γ. The only nontrivial condition comes from $\langle f_3-\frac{\gamma}{\beta}f_2, \alpha e_1+\beta e_2+r\rangle$. Here the Gram matrix with respect to $((\cdot,\cdot))$ is

$$\begin{pmatrix} 1+\frac{\gamma\bar{\gamma}}{\beta\bar{\beta}} & \bar{\delta} \\ \delta & \alpha\bar{\alpha}+\beta\bar{\beta} \end{pmatrix}$$

and so the condition is $\alpha\bar{\alpha}+\beta\bar{\beta}+\frac{\alpha\bar{\alpha}}{\beta\bar{\beta}}\gamma\bar{\gamma}+\gamma\bar{\gamma}-\delta\bar{\delta}=\alpha\bar{\alpha}+\beta\bar{\beta}+\frac{\alpha\bar{\alpha}}{\beta\bar{\beta}}\gamma\bar{\gamma}\neq 0$.

Similarly the triangle $(f_3-\frac{\gamma}{\alpha}f_1, e_2, \alpha e_1+\beta e_2+r)$ gives the condition $\alpha\bar{\alpha}+\beta\bar{\beta}+\frac{\beta\bar{\beta}}{\alpha\bar{\alpha}}\gamma\bar{\gamma}\neq 0$.

The final triangle is $(f_3-\frac{\gamma}{\alpha}f_1, f_3-\frac{\gamma}{\beta}f_2, \alpha e_1+\beta e_2+r)$. It is clear that under the above conditions the sides are lines of Γ, so we only need to verify that the whole subspace is $((\cdot,\cdot))$-nondegenerate. The corresponding Gram matrix is

$$\begin{pmatrix} 1+\frac{\gamma\bar{\gamma}}{\beta\bar{\beta}} & 1 & \bar{\delta} \\ 1 & 1+\frac{\gamma\bar{\gamma}}{\alpha\bar{\alpha}} & \bar{\delta} \\ \delta & \delta & \alpha\bar{\alpha}+\beta\bar{\beta} \end{pmatrix}.$$

Computing its determinant we obtain $2\delta\bar{\delta}+\alpha\bar{\alpha}+\beta\bar{\beta}+\frac{\gamma\bar{\gamma}\alpha\bar{\alpha}}{\beta\bar{\beta}}+2\gamma\bar{\gamma}+\frac{\gamma\bar{\gamma}\beta\bar{\beta}}{\alpha\bar{\alpha}}+\frac{(\gamma\bar{\gamma})^2}{\beta\bar{\beta}}+\frac{(\gamma\bar{\gamma})^2}{\alpha\bar{\alpha}}-\alpha\bar{\alpha}-\beta\bar{\beta}-\frac{\delta\bar{\delta}\gamma\bar{\gamma}}{\beta\bar{\beta}}-\frac{\gamma\bar{\gamma}\delta\bar{\delta}}{\alpha\bar{\alpha}}=\frac{\gamma\bar{\gamma}}{\alpha\bar{\alpha}\beta\bar{\beta}}(\alpha\bar{\alpha}+\beta\bar{\beta})^2$, as $\gamma\bar{\gamma}=\delta\bar{\delta}$, and so the condition for this to be a geometric triangle is that $\alpha\bar{\alpha}+\beta\bar{\beta}\neq 0$ which is not a new condition.

To summarize, we can decompose the initial triangle into seven geometric triangles if there exists a $\gamma\bar{\gamma}\in\mathbb{F}_q$ such that:

$$\begin{aligned} \gamma\bar{\gamma} &\neq 0; \\ \alpha\bar{\alpha}+\gamma\bar{\gamma} &\neq 0; \\ \beta\bar{\beta}+\gamma\bar{\gamma} &\neq 0; \\ \alpha\bar{\alpha}+\beta\bar{\beta}+\gamma\bar{\gamma} &\neq 0; \\ \alpha\bar{\alpha}+\beta\bar{\beta}+\frac{\beta\bar{\beta}}{\alpha\bar{\alpha}}\gamma\bar{\gamma} &\neq 0; \\ \alpha\bar{\alpha}+\beta\bar{\beta}+\frac{\alpha\bar{\alpha}}{\beta\bar{\beta}}\gamma\bar{\gamma} &\neq 0. \end{aligned}$$

If $q\geq 7$, such a $\gamma\bar{\gamma}$ can be found. □

We now deal with quadrangles.

Lemma 4.8 *Let $q\geq 5$ and let (a,b,c,d) be a quadrangle with $a\perp c$ and $b\perp c$. Then a, b, c, d can be decomposed into triangles.*

Proof. The span $\langle a,b,c,d\rangle$ is f-totally isotropic, hence it is three-dimensional. By Lemma 3.3 there exist at least $q^2-3q-2>0$ points on $\langle a,b\rangle$ collinear to c and d, decomposing the quadrangle. □

Lemma 4.9 *Let $q\geq 11$ and let (a,b,c,d) be a quadrangle with $a\not\perp c$ and $b\not\perp d$. Then there exists a common neighbour of a, b, c, d.*

Proof. Consider the space $U=\langle a,b,c,d\rangle^{\perp}$. Then U is a 3-dimensional space which is nondegenerate with respect to f. If R denotes the g-radical of U, then $R=\mathrm{Rad}_g(U)=\mathrm{Rad}_g(U^{\perp\!\!\perp})=\mathrm{Rad}_g((U^{\perp})^{\sigma})=\mathrm{Rad}_g(\langle a,b,c,d\rangle^{\sigma})$. The latter is at most 2-dimensional, so the g-rank of U is at least one. Consider $\langle U,a\rangle$ and define for $u\in U$, as in Lemma 3.5, the unitary form $g_a(u)=g(\mathrm{pr}_{a^{\perp\!\!\perp}}(u))$, where $\mathrm{pr}_{a^{\perp\!\!\perp}}(u)$ denotes the projection of u onto $a^{\perp\!\!\perp}$ via the direct sum decomposition $V=\langle a\rangle\oplus a^{\perp\!\!\perp}$. Note that $U\cap a^{\perp\!\!\perp}$ is at least 2-dimensional and cannot be equal to R. Indeed, $\dim R=2$ and $a\perp\!\!\perp R$ imply $a\perp R^{\sigma}$, whence $R^{\sigma}\subseteq a^{\perp}\cap U^{\perp}=a^{\perp}\cap\langle a,b,c,d\rangle=\langle a,b,d\rangle$, contradicting the fact that the g-rank of $\langle a,b,d\rangle$ is at least two, as it contains lines of Γ. Therefore g_a is nontrivial on U. Similarly, g_b, g_c, g_d are nontrivial on U, so using Lemma 3.4, there are at least $q^2-10q-9>0$ points that are non-isotropic with respect to g, g_a, g_b, g_c, g_d and, thus, collinear to a, b, c, and d. Hence we are done. □

Lemma 4.10 *Let $q\geq 5$ and let (a,b,c,d) be a quadrangle with $a\not\perp c$ and $b\perp d$. Then there exists a point p collinear to a, c, such that $b\not\perp p$ and $d\not\perp p$.*

Proof. Consider the space $W=\langle a,c\rangle^\perp$ which is an f-nondegenerate 5-dimensional space of g-rank at least four. Pick a point $t\in\Gamma$ that is collinear with both a and c, but is different from b,d. Now pick $s\in W$ such that s is f-singular, $t\perp s$, but $b\not\perp s$ and $d\not\perp s$. Indeed, this is possible, because $\langle t\rangle^\perp\neq\langle b\rangle^\perp$ and $\langle t\rangle^\perp\neq\langle d\rangle^\perp$. Now Lemma 3.3 implies that the space $\langle s,t\rangle$ contains at least q^2-3q-2 points of Γ, that are collinear to a and c. Moreover, since $\langle st\rangle\not\subset\langle b\rangle^\perp$ and $\langle st\rangle\not\subset\langle d\rangle^\perp$, it follows that $\langle s,t\rangle$ contains at least q^2-3q-4 points satisfying the conclusion of the lemma. Since $q\geq 5$, the conclusion follows. □

We have proved the following.

Lemma 4.11 *Let $q\geq 11$. Then any quadrangle can be decomposed into triangles.* □

Finally we need to consider pentagons.

Lemma 4.12 *Any pentagon (a,b,c,d,e) with $a\perp c$ and $a\perp d$ can be decomposed into triangles and quadrangles.*

Proof. By Lemma 3.3, the line $\langle c,d\rangle$ contains q^2-2q-1 points of Γ collinear to a, decomposing the pentagon. □

Lemma 4.13 *Let $q\geq 5$. Then any pentagon can be decomposed into triangles and quadrangles.*

Proof. In view of Lemma 4.12, we will assume that the pentagon is a, b, c, d, e with $a\not\perp d$. The idea is to reduce to the case in Lemma 4.12. We construct a point d' collinear to both c and e and such that $d'\perp a$, decomposing the pentagon into the sum of the pentagon a, b, c, d', e and the quadrangle c, d, e, d'.

Note that if X is the f-radical of $\langle a,c,d\rangle$ then $X\in\langle c,d\rangle$ and X is also the radical of $\langle a,c,d\rangle^\perp$. If X is also the g-radical of $\langle a,c,d\rangle^\perp$ then it would be the f-radical of $\langle a,c,d\rangle^\sigma$, which contradicts the fact that $\langle c,d\rangle^\sigma$ is nondegenerate with respect to g.

We now want to construct a line of Γ that lies in $\langle a,c,d\rangle^\perp$ and contains X. If X is a point of Γ then $X^{\perp\!\!\perp}\cap\langle a,c,d\rangle^\perp$ is a complement to X and so it is an f-nondegenerate three-dimensional space. It is not totally isotropic for g, because it lies in $\langle a,c,d\rangle$, which has rank at least three. Lemma 3.4 gives a point of Γ in this space, hence the required line of Γ. If X is not a point of Γ and if p is f-singular 1-dimensional subspace of $\langle a,c,d\rangle^\perp\setminus X^{\perp\!\!\perp}$ then $\langle X,p\rangle$ is a line of Γ.

Finally, if l is a line of Γ as above, Lemma 3.3 implies that l has at least q^2-3q-2 points of Γ, that are collinear to both a and c, and if $q>3$, there exists a point b' collinear to both a and c and such that $d\perp b'$. We decompose the pentagon a, b, c, d, e as the sum of the quadrangle a, b, c, b' and the pentagon a, b', c, d, e, in which $b'\perp d$. If $b'\perp e$, we are done by Lemma 4.12. If $b'\not\perp e$, then we can repeat the argument above for b',e,d to get a point a' collinear to both e and b' and such that $a'\perp d$. □

Proposition 4.14 *Let $n=3$ and $q\geq 11$ be odd. Then the flipflop geometry Γ is simply connected.*

Proof of the Theorem. Part (i) follows from Proposition 3.2. Part (ii) follows from Proposition 3.7. Part (iii) follows from Propositions 4.5 and 4.14 plus [Part2], Proposition 3.2. □

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