The complete Phan-type theorem for $\mathrm{Sp}(2n, q)$

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Abstract

The articles [8] and [9] give a characterization of central quotients of the group $\mathrm{Sp}(2n, q)$ for $n \geq 3$ and all prime powers q up to some small cases that are left open. The present article fills in this gap, thus providing the definitive version of the Phan-type theorem for $\mathrm{Sp}(2n, q)$.

1 Introduction

The modern approach to Phan-type theorems, i.e., characterizations of finite Chevalley groups in the spirit of [18] and [19], falls into two parts, as outlined in [1]. On one hand one has to prove the simple connectedness of some suitable geometry, on the other hand one has to classify related amalgams. Hoffman, Shpectorov and the first author [8] gave a Phan-type characterization of the group $\mathrm{Sp}(2n, q)$ by studying the so-called flipflop geometry Γ of type C_n over $\mathbb{F}_{q^2}$. The Main Theorem of that paper states that this flipflop geometry is simply connected for $n \geq 3$ and $q \geq 8$, for $n \geq 4$ and $q \geq 3$, and for $n \geq 5$. By Tits' Lemma (Corollaire 1 of [23], see also Lemma 3.1 of the present article) this implies that the amalgam consisting of the rank 1 and rank 2 parabolics of the flag-transitive group $\mathrm{Sp}(2n, q)$ of automorphisms of the flipflop geometry Γ admits $\mathrm{Sp}(2n, q)$ as its universal completion. We refer to [8] for details. The second part, the classification of amalgams was dealt with in [9].

The purpose of the present paper is to prove the Phan-type theorem, see Subsection 2.3 for the precise statements, in the remaining open case over the fields $\mathbb{F}_3$, $\mathbb{F}_4$, $\mathbb{F}_5$, $\mathbb{F}_7$.

In Section 2 we remind the reader of the setting and state the main results. In Section 3 we give a geometric argument why our main result is true over the field $\mathbb{F}_7$, the other cases being dealt with by a coset enumeration with computer, cf. Appendix C, that can be checked by the interested reader on a standard desktop machine. In Section 4 we correct a claim made in [11]. Appendices A and B remind the reader of some notions from amalgam theory and incidence geometry.

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2 The Phan-type theorem for $\mathrm{Sp}(2n, q)$

2.1 Geometrical setting

Let B_{2n} be the matrix

$$\left(\begin{array}{c|c} 0 & \mathrm{id}_{n\times n} \\ \hline -\mathrm{id}_{n\times n} & 0 \end{array}\right)$$

over $\mathbb{F}_{q^2}$. Let $(\cdot,\cdot)$ be the bilinear form defined by B_{2n} via $(x,y) := x^T B_{2n} y$. We represent $G := \mathrm{Sp}(2n,q^2)$ by the set of all invertible $(2n)\times(2n)$-matrices A over $\mathbb{F}_{q^2}$ which preserve $(\cdot,\cdot)$, that is, $A^T B_{2n} A = B_{2n}$ holds.

Let V be the vector space $\mathbb{F}_{q^2}^{2n}$ and let $\{e_1,\ldots,e_n,f_1,\ldots,f_n\}$ be the standard basis. We denote by $^-$ the unique non-trivial involutory field automorphism $x\mapsto x^q$ of $\mathbb{F}_{q^2}$. Consider the $^-$-semi-linear map $\sigma : V\to V$ defined by $e_i\mapsto f_i, f_i\mapsto -e_i$ and $\sigma(c\cdot v) = \overline{c}\sigma(v)$ for $c\in\mathbb{F}_{q^2}, v\in V$. Note that $\sigma(v) = \overline{B_{2n}v} = B_{2n}\overline{v}$. Then the centralizer $G_\sigma := \{g\in G \mid \forall v\in V : g\sigma(v) = \sigma(gv)\}$ of σ in $\mathrm{Sp}(2n,q^2)$ is isomorphic to $\mathrm{Sp}(2n,q)$ (see [8], Proposition 3.8). For our computations in the later sections, we take G_σ as our representation of $\mathrm{Sp}(2n,q)$. Note that for a matrix $A\in\mathrm{Sp}(2n,q^2)$, centralizing σ is equivalent to the condition $A^{-1} = \overline{A}^T$.

We now define the (so-called flipflop) geometry Γ which we are studying in this article. (For an introduction to flipflop geometries, see [1] or [10].) To this end, we define a $^-$-hermitian form $((\cdot,\cdot))$ by $((u,v)) := (u,\sigma(v))$, cf. [8], Lemma 3.2. To denote orthogonality with respect to the form $(\cdot,\cdot)$, we use the symbol $\perp$. To denote orthogonality with respect to the form $((\cdot,\cdot))$, we use the symbol $\perp\!\!\!\perp$.

Definition 2.1 The objects of the geometry Γ are all non-trivial subspaces of V which are totally isotropic with respect to $(\cdot,\cdot)$ and nondegenerate with respect to $((\cdot,\cdot))$; incidence is defined by symmetrized containment.

As $G_\sigma\cong\mathrm{Sp}(2n,q)$ respects both forms, it acts on the geometry. This action is in fact flag-transitive (see [8], Proposition 4.2).

For our computations we choose the maximal flag F equal to $\langle e_1\rangle\subset\langle e_1,e_2\rangle\subset\ldots\subset\langle e_1,e_2,\ldots,e_n\rangle$. When computing stabilizers, we will refer to the stabilizers of each of these subspaces as the point stabilizer M_1, the line stabilizer M_2, the plane stabilizer M_3 and (for $n=4$) the space stabilizer M_4, respectively. The M_i are the maximal parabolics of G_σ.

Definition 2.2 For $\lambda\in\mathbb{F}_{q^2}$, let $V_\lambda := \{u\in V \mid \sigma(u) = \lambda u\}$.

Lemma 2.3
The following hold.

(i) V_λ *is* G_σ*-invariant.*

(ii) V_λ *is an* $\mathbb{F}_q$*-subspace of* V*.*

(iii) $V_\lambda\neq 0$ *if and only if* $\lambda\bar{\lambda} = -1$

(iv) *If* $V_\lambda\neq 0$ *then* V_λ *contains a basis of* V*.*

Proof. See Lemma 3.6 of [8] or Lemma 2.4 of [9]. □

Let $\lambda\in\mathbb{F}_{q^2}$ such that $\lambda\overline{\lambda} = -1$. Since V_λ contains a basis for V, it has the same dimension as V. As it is a $\mathbb{F}_q$-subspace, we deduce that $|V_\lambda| = q^{2n}$. Let $v_n := \overline{\lambda}e_n + f_n\in V_\lambda$. We observe that $G_\sigma v_n\subset V_\lambda$, i.e. the orbit of v_n, is a subset of V_λ, and hence $|v_n^{G_\sigma}| < q^{2n}$.

Thus we have found a vector with an orbit that is short enough to be suitable for our purposes. For we can now use this to effectively compute lower bounds on the size of G_σ and its subgroups: All these groups induce a permutation action on the orbit $G_\sigma v_n$. Hence we can compute an homomorphic image into a permutation group. There are good algorithms (and implementations of them) for determining the size of such a permutation group. (They work better the smaller the set is upon which the group acts, which is why we went to some effort to find vectors with relatively small orbit.) Thus, we can efficiently compute a lower bound on the size of a factor group of any subgroup H of G_σ. If the action induced by the group on the orbit is faithful, as is the case here, then we actually obtain the exact size of the group.

2.2 Phan systems

Definition 2.4 (cf. [2]) Subgroups U_1 and U_2 of $\mathrm{SU}(3, q^2)$ form a **standard pair** whenever each $U_i \cong \mathrm{SU}(2, q^2)$ is the stabilizer in $\mathrm{SU}(3, q^2)$ of a nonsingular vector v_i and, furthermore, v_1 and v_2 are perpendicular. Standard pairs in central quotients of $\mathrm{SU}(3, q^2)$ are defined as the images under the natural homomorphism of the standard pairs from $\mathrm{SU}(3, q^2)$. We denote a standard pair U_1, U_2 of a central quotient of $\mathrm{SU}(3, q^2)$ by $\underset{U_1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\underset{U_2}{\circ}$.

For an element U of Γ, i.e. a $(\cdot,\cdot)$-totally singular, $((\cdot,\cdot))$-nondegenerate subspace of V, let $\mathrm{GU}(U)$ denote the subgroup of G_σ that preserves the form $((\cdot,\cdot))_{|U\times U}$ and acts trivially on $U^\perp \cap U^{\perp\!\perp}$. For a nondegenerate σ-invariant subspace W of V denote by $\mathrm{Sp}(W)$ the subgroup of G_σ that preserves the form $(\cdot,\cdot)_{|W_\lambda \times W_\lambda}$ (see Definition 2.2 for the definition of W_λ) and acts trivially on $U^\perp \cap U^{\perp\!\perp}$.

Definition 2.5 (cf. [9]) In case $n = 2$, we have $V = \langle e_1, e_1^\sigma, e_2, e_2^\sigma \rangle$, $G \cong \mathrm{Sp}(4, q^2)$, and $G_\sigma \cong \mathrm{Sp}(4, q)$. Subgroups $U_1 \cong \mathrm{Sp}(2, q)$ and $U_2 \cong \mathrm{SU}(2, q^2)$ are called a **standard pair** in G_σ if there exists a $(\cdot,\cdot)$-isotropic and $((\cdot,\cdot))$-non-isotropic vector v of V and a two-dimensional $(\cdot,\cdot)$-totally isotropic and $((\cdot,\cdot))$-nondegenerate subspace $U \ni v$ of V such that the group U_1 coincides with $\mathrm{Sp}(v^\perp \cap v^{\perp\!\perp})$ and the group U_2 coincides with $\mathrm{SU}(U)$. Standard pairs in central quotients of $\mathrm{Sp}(4, q)$ are defined as the images under the natural homomorphism of the standard pairs from $\mathrm{Sp}(4, q)$. We denote the standard pair U_1, U_2 of $\mathrm{Sp}(4, q)$ by $\underset{U_1}{\circ}\!\overset{>}{=\!=\!=}\!\underset{U_2}{\circ}$ or by $\underset{U_2}{\circ}\!\overset{<}{=\!=\!=}\!\underset{U_1}{\circ}$.

Definition 2.6 Let $n \geq 2$, let Δ be a Dynkin diagram with rank two subdiagrams isomorphic to $\circ \qquad \circ$ or $\circ\!-\!\!\!-\!\!\!-\!\circ$ or $\circ\!\overset{>}{=\!=\!=}\!\circ$, and let $I = \{1, \ldots, n\}$. A group G admits a **weak Phan system of type Δ over $\mathbb{F}_{q^2}$** if G contains subgroups $U_i \cong \mathrm{SL}(2, q) \cong \mathrm{Sp}(2, q) \cong \mathrm{SU}(2, q^2)$, for $i \in I$, and subgroups $U_{i,j}$, for $i \neq j \in I$, so that the following hold:

(i) If (i, j) is not an edge in Δ, then $U_{i,j}$ is a central product of U_i and U_j;

(ii) if (i, j) is an edge in Δ, then $U_{i,j}$ is isomorphic to a central quotient of $\mathrm{SU}(3, q^2)$, if (i, j) is a single edge, and isomorphic to a central quotient of $\mathrm{Sp}(4, q)$, if (i, j) is a double edge; moreover, U_i and U_j form a standard pair in $U_{i,j}$ according to the diagram $\underset{U_i}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\underset{U_j}{\circ}$ or $\underset{U_i}{\circ}\!\overset{>}{=\!=\!=}\!\underset{U_j}{\circ}$; and

(iii) the subgroups $(U_{i,j})_{i,j \in I}$, generate G.

2.3 The main results

Main Theorem 1
Let $q \geq 3$, let $n \geq 3$, and let G be a group that contains a weak Phan system of type C_n over $\mathbb{F}_{q^2}$. Then G is isomorphic to a central quotient of $\mathrm{Sp}(2n, q)$.

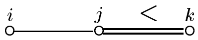
Main Theorem 2
Let $n \geq 4$ and let G be a group that contains a weak Phan system of type C_n over $\mathbb{F}_4$. Suppose further that

(i) *for any triple i, j, k of nodes of the Dynkin diagram C_n that form a subdiagram*

$$\overset{i}{\circ}\!-\!\!\!-\!\!\!-\!\!\!-\!\overset{j}{\circ}\!-\!\!\!-\!\!\!-\!\!\!-\!\overset{k}{\circ}$$

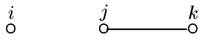
of type A_3, the subgroup $\langle U_{i,j}, U_{j,k} \rangle$ is isomorphic to a central quotient of $\mathrm{SU}(4, 2^2)$;

(ii) *for any triple i, j, k of nodes of the Dynkin diagram C_n that form a subdiagram*



of type C_3, the subgroup $\langle U_{i,j}, U_{j,k}\rangle$ is isomorphic to a central quotient of $\mathrm{Sp}(6, 2)$*;*

(iii) (a) *for any triple i, j, k of nodes of the Dynkin diagram C_n that form a subdiagram*

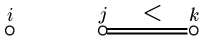


of type $A_1 \oplus A_2$, the groups U_i and $U_{j,k}$ commute elementwise; and

(b) *for any quadruple of nodes of the Dynkin diagram C_n that form a subdiagram*

of type $A_2 \oplus A_2$, the groups $U_{i,j}$ and $U_{k,l}$ commute elementwise; and

(c) *for any triple i, j, k of nodes of the Dynkin diagram C_n that form a subdiagram*



of type $A_1 \oplus C_2$, the groups U_i and $U_{j,k}$ commute elementwise; and

(d) *for any quadruple of nodes of the Dynkin diagram C_n that form a subdiagram*

of type $A_2 \oplus C_2$, the groups $U_{i,j}$ and $U_{k,l}$ commute elementwise.

Then G is isomorphic to a central quotient of $\mathrm{Sp}(2n, 2)$.

3 Simple connectedness of the geometry

In this section we will prove that for $n = 3, q \geq 7$, the geometry Γ is simply connected. By the following lemma, this implies that $\mathrm{Sp}(6, q)$ is the universal completion of the amalgam of its maximal parabolics, as desired. This extends the proof from [8] to include the field $\mathbb{F}_7$.

Lemma 3.1 (Tits' Lemma)
Suppose a group G acts flag-transitively on a geometry $\mathcal{G}$, and let $\mathcal{A}$ be the amalgam of parabolics associated with some maximal flag F of $\mathcal{G}$. Then G is the universal completion of the amalgam $\mathcal{A}$ if and only if $\mathcal{G}$ is simply connected.

Proof. See [5], [13], [16] or [23]. $\square$

3.1 Simple connectedness

Using Tits' Lemma, we have transformed our group-theoretic problem (analyzing the universal completion of an amalgam) into a geometric one (showing that a certain geometry is simply connected). Now we have to consider how to solve the latter problem. We need some mathematical tools and facts in order to tackle it successfully.

Being simply connected means the following for our geometry: All cycles in its incidence graph have to be null-homotopic, i.e., for every cycle there exists a triangulization (for details, see for example [20]). If $q \geq 3$, every cycle in the incidence graph of Γ is homotopic to a cycle passing exclusively through points and lines (Lemma 5.1 in [8]). Since Γ is a partially linear geometry, i.e., distinct points have at most one line joining them, the points of such a cycle uniquely determine the lines of the cycle. Hence it suffices to study cycles of the collinearity graph of Γ. Since the diameter of the collinearity graph is two (see Lemma 4.5 in [8]), every cycle of length at least six always decomposes into smaller cycles (i.e. it is the sum of these smaller cycles), and hence it suffices to study triangles, quadrangles and pentagons of the collinearity graph in order to prove simple connectedness.

3.2 Some tools

The following lemma will prove to be very useful throughout the whole section. Recall the terminology and definitions introduced in Section 2.1. Notice that if l is a two-dimensional subspace of V of $((\cdot,\cdot))$-rank at least one, then it contains at least q^2-q points of Γ. Indeed, if the $((\cdot,\cdot))$-rank of l is one then the radical is the only non-trivial isotropic subspace of l and if the $((\cdot,\cdot))$-rank of l is two then l contains $q+1$ distinct non-trivial isotropic subspaces. Since any point of l is $(\cdot,\cdot)$-singular, it contains q^2 (respectively, q^2-q) points of Γ, if it has $((\cdot,\cdot))$-rank one (respectively, two).

Lemma 3.2
Let p be a point of Γ and $\Pi \supset p$ be a three-dimensional subspace of V of $((\cdot,\cdot))$-rank at least two such that p is in the $(\cdot,\cdot)$-radical of Π. Then for any $((\cdot,\cdot))$-nondegenerate two-dimensional subspace l of Π, all points of Γ incident with l are collinear to p, with the exception of at most $q+1$ points.

Proof. This is a reformulation of Lemma 4.3 of [8]. □

A direct consequence of this is that if l has $((\cdot,\cdot))$-rank one (respectively, two) it contains at least q^2-q-1 (respectively, q^2-2q-1) points collinear to p. Furthermore, the following is true:

Lemma 3.3
Let p be a point of Γ and $\Pi \supset p$ be a three-dimensional subspace of V of $((\cdot,\cdot))$-rank at least two. Then any two-dimensional subspace l of Π not containing p is incident with at least q^2-q-1 (respectively, q^2-2q-1) points of Γ that generate a $((\cdot,\cdot))$-nondegenerate two space with p if l has $((\cdot,\cdot))$-rank one (respectively, two).

Proof. See Corollary 4.4 of [8]. □

3.3 Triangles

The first step is the analysis of triangles of the collinearity graph. We will call a triangle (a,b,c) a **good triangle** if a, b and c are incident to a common plane of the geometry. A triangle that is not good is called **bad**. Note that a good triangle is null-homotopic, so we only have to deal with the bad ones.

Lemma 3.4
Let (a, b, c) be a bad triangle. Then we can decompose this triangle into bad triangles, in such a way that for each new triangle T_i we can find a canonical basis e_1, e_2, e_3, f_1, f_2, f_3 of V such that each T_i equals $\langle e_1\rangle$, $\langle e_2\rangle$, $\langle x_i e_1 + y_i e_2 + (k_i e_3 + f_3)\rangle$ with $k_i\bar{k}_i = -1$ and $x_i y_i \neq 0$ and $x_i\bar{x}_i + y_i\bar{y}_i \neq 0$.

Proof. This is a consequence of the Lemmas 5.3, 5.4 and 6.1 in [8]. □

By the preceding Lemma we only have to show for a very limited class of (bad) triangles that they can be decomposed. To do this, we start with a triangle (a, b, c) and construct an octahedron with the triangle forming one face, and a suitably chosen null-homotopic triangle (p, p', p'') forming the opposite face. With suitably chosen we mean that all triangles except for the starting triangle shall be good. In the following we will prove that this is possible for $q \geq 4$.

Before we do that, we need some more tools.

Lemma 3.5
Let $k, l \in \mathbb{F}_{q^2}$ such that $k\bar{k} = -1$, $l \neq 0$ Then there exists a matrix of the form

$$A := \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & x & & & -\bar{y} \\ & & & 1 & & \\ & & & & 1 & \\ & & y & & & \bar{x} \end{pmatrix} \in G_\sigma$$

such that $(ke_3 + f_3)A = (k\bar{l}e_3 + lf_3)$.

Proof. It is easy to verify that $A \in G_\sigma$ if and only if $x\bar{x} + y\bar{y} = 1$. Furthermore $(ke_3 + f_3)A = (kx + y)e_3 + (\bar{x} - k\bar{y})f_3 = k\overline{(\bar{x} - k\bar{y})}e_3 + (\bar{x} - k\bar{y})f_3$. So the claim is equivalent to showing that the following system of equations has a solution: $x\bar{x} + y\bar{y} = 1$ and $\bar{x} - k\bar{y} = l$. Finding such a solution is easily achieved via straight forward computation: use the second equation to replace the variable x in the first equation:

$$\begin{aligned} & \overline{(l + k\bar{y})}(l + k\bar{y}) + y\bar{y} &&= 1 \\ \iff\quad & l\bar{l} + l\bar{k}y + \overline{(l\bar{k}y)} &&= 1 \\ \iff\quad & z + \bar{z} &&= 1 - l\bar{l} \in \mathbb{F}_q \end{aligned}$$

where $z := l\bar{k}y$. Now if r is a primitive root of $\mathbb{F}_{q^2}$, then $r + \bar{r} \neq 0$ and hence $z = \frac{r(1-l\bar{l})}{r+\bar{r}}$ is a solution to this last equation. Backward substitution yields the desired values for x and y. □

Lemma 3.6
For $4 \leq q \leq 11$, any bad triangle can be decomposed into good triangles.

Proof. Let a, b, c be a bad triangle. By Lemma 3.4, we can assume $(a, b, c) = (\langle e_1\rangle, \langle e_2\rangle, \langle xe_1 + ye_2 + (ke_3 + f_3)\rangle)$ satisfying $k\bar{k} = -1$ and $xy \neq 0$ and $x\bar{x} + y\bar{y} \neq 0$. Since $x \neq 0$, by Lemma 3.5 we can find $g \in G_\sigma$ such that $(g(a), g(b), g(c)) = (\langle e_1\rangle, \langle e_2\rangle, \langle xe_1 + ye_2 + (k\bar{x}e_3 + xf_3)\rangle) = (\langle e_1\rangle, \langle e_2\rangle, \langle e_1 + y'e_2 + (k'e_3 + f_3)\rangle)$ with $y' := \frac{y}{x}$ and $k' := \frac{k\bar{x}}{x}$. So every bad triangle is conjugate to such a triangle. Note that $k'\bar{k}' = \frac{k\bar{k}x\bar{x}}{x\bar{x}} = -1$, so k' can take at most $q + 1$ different values. Since $y' \neq 0$, it can take at most $q^2 - 1$ different values. Hence there are at most $(q + 1)(q^2 - 1)$ different conjugacy classes of bad triangles to consider.

It is now a simple matter of combinatorics to determine all the possible conjugacy classes of bad triangles for a given q, and then testing for each whether the triangle defined this way is decomposable. We now claim that for $4 \leq q \leq 11$ this is possible by using the octahedron construction described above, and setting $(p,p',p'') = (\langle f_3 \rangle, \langle se_1 + kf_1 - xf_3 \rangle, \langle te_2 + kf_1 - yf_3 \rangle)$ where $s,t \in \mathbb{F}_{q^2} \setminus \{0\}$ are chosen suitably.

Verifying that this is possible requires at most $(q+1)(q^2-1)^3$ checks. This can readily be done using a simple GAP program (see Appendix D of [12]). In particular we successfully performed these checks for $4 \leq q \leq 11$. (This upper bound could easily be increased, but of course we had to stop at some point. We picked it so that it complements the previous proof presented in [8] which works without computational help for $q \geq 13$. So stopping at $q = 11$ is arbitrary, and the code in Appendix D of [12] should work for larger values of q, too.) □

3.4 Quadrangles

Now we will shift our attention to quadrangles. By the preceding results, it is enough to decompose quadrangles into triangles, regardless whether they are good or bad. Notice that if in a quadrangle a, b, c, d we have that a and c (or b and d) are collinear then this quadrangle is immediately decomposed into two triangles.

Definition 3.7 We call a quadrangle a, b, c, d **half-special** if $\langle a,c \rangle$ or $\langle b,d \rangle$ is nondegenerate with respect to both forms $(\cdot,\cdot)$ and $((\cdot,\cdot))$. We call it **special** if both $\langle a,c \rangle$ and $\langle b,d \rangle$ are nondegenerate with respect to both forms.

Lemma 3.8
Let $q \geq 5$. Then any quadrangle can be decomposed into triangles and half-special quadrangles.

Proof. Consider an arbitrary quadrangle a, b, c, d. Without loss of generality we may assume that b and d are noncollinear. Pick an arbitrary point $s \in X = a^{\perp\!\perp} \cap b^{\perp} \cap d^{\perp}$. The point s exists because X is not totally isotropic with respect to $((\cdot,\cdot))$, being a three-dimensional space contained in the nondegenerate five-dimensional space $a^{\perp\!\perp}$. The line $l = \langle a,s \rangle$ has $((\cdot,\cdot))$-rank two. Using Lemma 3.3, the line l contains at least q^2-2q-1 points of Γ that are collinear with b, respectively d, and at least q^2-2q-1 points of Γ that generate a nondegenerate two-dimensional space with c. Since $q \geq 5$ and since l contains q^2-q points of Γ, the space l has to contain a point p of Γ that generates a nondegenerate two-dimensional space with c and that is collinear to both b and d. Clearly a, b, c, d decomposes into a, b, p, d and c, b, p, d. If $(a,p) = 0$ then $\langle a,p \rangle$ is a line, implying that a, b, p, d decomposes into triangles. Otherwise, a, b, p, d is half-special with respect to $\langle a,p \rangle$. Similarly for c, b, p, d. □

Lemma 3.9
Let $q \geq 5$. Then any quadrangle can be decomposed into triangles and special quadrangles.

Proof. Apply Lemma 3.8 once to obtain triangles and half-special quadrangles. Then apply Lemma 3.8 again, after suitably renaming the vertices of the quadrangles, to obtain special quadrangles. □

Proposition 3.10
Let $q \geq 7$. Then any quadrangle can be decomposed into triangles.

Proof. Denote the quadrangle by (a,b,c,d). By the preceding lemma, we can assume without loss of generality that it is special, so $(a,c) \neq 0 \neq (b,d)$ and both $\langle a,c \rangle$ and $\langle b,d \rangle$ are $((\cdot,\cdot))$-nondegenerate. We try to find a point p collinear to all of a,b,c,d, which means we can decompose the quadrangle into triangles.

Set $W := a^{\perp} \cap c^{\perp}$ and $U_1 := W \cap b^{\perp}$ and $U_2 := W \cap d^{\perp}$ and $l := U_1 \cap U_2$. Note that $\dim W = 4$, $\dim U_1 = \dim U_2 = 3$, $\dim l = 2$. Also, W is $((\cdot,\cdot))$-nondegenerate since $\langle a,c \rangle$ is $((\cdot,\cdot))$-nondegenerate and

$W = a^\perp \cap c^\perp = (a^\sigma)^{\perp\!\!\perp} \cap (c^\sigma)^{\perp\!\!\perp} = \langle a^\sigma, c^\sigma \rangle^{\perp\!\!\perp} = (\langle a,c\rangle^\sigma)^{\perp\!\!\perp}$. Similar arguments hold for $a^\perp \cap b^\perp$, $b^\perp \cap c^\perp$ and so on.

We now distinguish three cases:

(i) If l is of $((\cdot,\cdot))$-rank two, then we can apply Lemma 3.2 to the planes $\langle a,l\rangle$, $\langle b,l\rangle$, $\langle c,l\rangle$, and $\langle d,l\rangle$ to obtain q^2-5q-4 points of Γ on l collinear to all of a, b, c, d. Notice that this is a positive number for $q \geq 7$.

(ii) Suppose now that l is of $((\cdot,\cdot))$-rank one. Then the plane $\Pi := \langle b,l\rangle$ has $((\cdot,\cdot))$-rank at least one. It lies inside the four-dimensional $((\cdot,\cdot))$-nondegenerate space W. Assume Π had $((\cdot,\cdot))$-rank one. Then it has a two-dimensional $((\cdot,\cdot))$-radical R, which would be maximal totally isotropic in W, since $\dim(R) + \dim(R^{\perp\!\!\perp} \cap W) = \dim(W)$ and $R \subseteq R^{\perp\!\!\perp}$. Similarly, R can not have a polar of dimension three, which Π would be. Contradiction, thus Π has $((\cdot,\cdot))$-rank two. Similar arguments hold for the points a, c, d instead of b. Applying Lemma 3.2 gives us q^2-4q-4 points of Γ collinear to all of a, b, c, d. Notice that this is a positive number for $q \geq 5$.

(iii) Suppose now l is totally isotropic with respect to $((\cdot,\cdot))$. Then the planes U_1 and U_2 are $((\cdot,\cdot))$-degenerate. They must have $((\cdot,\cdot))$-rank two (this can be shown with similar arguments as used in case (ii) for Π).

Let R_1 and R_2 be the one-dimensional $((\cdot,\cdot))$-radical s of U_1 and U_2. They are contained in l. For assume that $R_1 \not\subseteq l$; then $U_1 = \langle R_1, l\rangle$. But then U_1 would be totally isotropic (since l is totally isotropic, and also orthogonal to R_1, the radical of U_1), a contradiction. We argue likewise for R_2.

Furthermore, the radicals cannot coincide as otherwise we would obtain a radical for the $((\cdot,\cdot))$-nondegenerate space $a^\perp \cap c^\perp$. So we have $l = \langle R_1, R_2\rangle$. Notice that $b \not\subset l$, since $(b,d) \neq 0$. Hence b is different from R_1 and R_2.

Choose a line t of Γ through b inside U_1. This line exists since the $((\cdot,\cdot))$-rank of U_1 is two, and b is not in the $((\cdot,\cdot))$-radical R_1 of U_1. Applying first Lemma 3.3 to $\langle d,t\rangle$ and then Lemma 3.2 to $\langle a,t\rangle$ and $\langle c,t\rangle$ yields the existence of $(q^2-2q-1)-2(q+1) = q^2-4q-3 > 0$ points on t collinear to a, b, c and which span a $((\cdot,\cdot))$-nondegenerate space with d. Choose one of these points not equal to b and call it b'. Then $(b',d) \neq 0$, for otherwise, $b' \in l$, contradicting that l is totally isotropic with respect to $((\cdot,\cdot))$. Hence a, b', c, d form a special quadrangle.

Let $U_1' := b'^\perp \cap W$. We claim that U_1' intersects U_2 in a line l' that does not contain R_2, implying the $((\cdot,\cdot))$-rank of l' is two (since it is contained in U_2 which has $((\cdot,\cdot))$-rank two and doesn't intersect its radical) and so we have reduced to case (i) of this proof.

It remains to verify our last claim. Assume $R_2 \subset U_1' \cap U_2 = l'$. Then $R_2 \subseteq l \cap l' = (b^\perp \cap b'^\perp) \cap U_2 = (\langle b,b'\rangle^\perp) \cap U_2 \subset \langle b,b'\rangle^\perp = t^\perp$, thus $t \subseteq R_2^\perp \cap U_1$. Notice that $R_2^\perp \cap U_1 = \langle b, R_2\rangle$: Clearly $\langle b,R_2\rangle \subseteq R_2^\perp \cap U_1$, since $R_2 \subset R_2^\perp$, $R_2 \subset l \subset U_1$, $b \subset U_1$ and $b \subset l^\perp \subset R_2^\perp$. Equality holds since R_2 is one-dimensional, and R_2 is not the $(\cdot,\cdot)$-radical of U_1 (which is b, and we already know that $b \neq R_2$), and thus both sides of the equation have the same dimension. But then also $t = \langle b, R_2\rangle$, implying that t has $((\cdot,\cdot))$-rank one, a contradiction since t is a line of Γ.

□

Note that the 'pyramid' construction used in the preceding proposition is *not* sufficient for $q \leq 5$, so a different approach would be needed to cover it. For a specific example, let z denote a primitive element in $\mathbb{F}_{25}$ over $\mathbb{F}_5$ with minimal polynomial x^2-x+2. Then let $a := \langle e_1\rangle$, $b := \langle e_2\rangle$, $c := \langle e_2 + z^{-1}e_3 + z^{-1}f_1\rangle$, $d := \langle e_1 + z^5 e_2 + z^6 f_2 + z^9 f_3\rangle$. This is a special quadrangle, and using the definitions from Proposition 3.10, $l := \langle u,v\rangle$ with $u := \langle e_1 + f_3\rangle$, $v := \langle e_2 + z^9 e_3\rangle$. Now l has $((\cdot,\cdot))$-rank two, but contains no point p collinear to all of a, b, c, d.

3.5 Pentagons

Proposition 3.11
Let $q \geq 5$. Then any pentagon can be decomposed into triangles and quadrangles.

Proof. Let (a, b, c, d, e) be a pentagon. Consider the space $U := \langle a, b, d\rangle^\perp$ of dimension three. Its $((\cdot,\cdot))$-rank has to be at least two, as the $((\cdot,\cdot))$-rank of $\langle a, b\rangle$ is two. Choosing a $((\cdot,\cdot))$-nondegenerate two-dimensional subspace l of U and applying Lemma 3.2 on the planes $\langle a, l\rangle$, $\langle b, l\rangle$, $\langle d, l\rangle$, we will find $(q^2 - q) - 3(q + 1) = q^2 - 4q - 3 > 0$ points on l collinear to all of a, b, d, decomposing the pentagon. □

3.6 Proof of the Main Results

Combining the results from the preceding sections yields this proposition:

Proposition 3.12
If $n = 3$ and $7 \leq q \leq 11$, the geometry Γ is simply connected. □

By $A_{(k)}$ we denote the amalgam of rank k parabolics with respect of the flag-transitive action of G_σ on Γ.

Proposition 3.13
For $(n, q) \in \{(3, 3), (3, 4), (3, 5), (3, 7), (4, 2)\}$, the group G_σ is the universal completion of $\mathcal{A}_{(n-1)}$.

Proof. This proposition is proved by the computations described in Appendix C.1. □

Using the results from [8] and the work done in the present article we can prove the following theorem:

Theorem 3.14
G_σ is the universal completion of $\mathcal{A}_{(n-1)}$ if and only if $n \geq 3$ and $(n, q) \neq (3, 2)$.

Proof. We show that the flipflop geometry Γ is simply connected if and only if $n \geq 3$ and $(n, q) \neq (3, 2)$. From this follows the claim via Tits' Lemma (Lemma 3.1). Simple connectedness for $n \geq 3$ and $(n, q) \neq (3, 2)$ is proved conjointly by Proposition 3.12, by combining Proposition 3.13 with Tits' Lemma, and finally by Theorem 6.8 from [8].

If $(n, q) = (3, 2)$, then the geometry is not simply connected, as shown in [8], right after Theorem 6.8. Finally, if $n = 2$, the simplicial complex is one dimensional, and hence only simply connected if it contains no cycles (i.e. if it is a tree). But the points $\langle e_1\rangle$, $\langle e_2\rangle$, $\langle f_1\rangle$, $\langle f_2\rangle$ form a quadrangle, and hence there exists a non-trivial cycle in the simplicial complex, thus the geometry is not simply connected. □

Proof of Main Result 1. See Sections 3 and 4 of [9] or Chapter 8 of [10]. □

Proof of Main Result 2. See [2] or Chapter 8 of [10]. □

4 The case $(n, q) = (3, 3)$ reviewed

In this article, we prove that for $(n, q) = (3, 3)$, the geometry Γ corresponding to $\mathrm{Sp}(6, 3)$ and C_3 is simply connected, and the group itself is the universal completion of the amalgam corresponding to the weak Phan system.

In case of A_3 or equivalently D_3, however, one can show that either a three-fold or a nine-fold cover of the geometry exists. (Richard Lyons gave a simple argument for this, see page 86 of [10]). However, nothing was known so far about the universal cover (contrary to the claim made on page 144 in [11]). Recently the authors successfully applied the techniques used in this article to the case A_3 and determined the universal cover, which turned out to be nine-fold.

Theorem 4.1
The flipflop geometry $\Gamma(3,3)$ *studied in [2] and [11] admits a nine-fold universal covering.*

Finally, for B_3 nothing was known so far. The authors applied the techniques described in Appendix C here as well. The unexpected result was that the coset enumeration (used to determine the size of the universal completion of the amalgam of parabolics) did not terminate. However, by adding additional relations we succeeded in proving that there has to exist a three-fold covering of the geometry, an observation that has been confirmed by Sergey Shpectorov, who informed us that he constructed a three-fold cover of $\mathrm{Spin}(7,3)$ containing a weak Phan system.

A Appendix: Amalgams

In this section, we introduce the notion of group amalgams. Note that we only need a special kind of amalgams; for a more general definition, see for example [21].

Definition A.1 An **amalgam of groups** is a set $\mathcal{A}$ endowed with a partial multiplication and a finite family of subsets $(G_i)_{i\in I}$ such that the following holds:

(i) $\mathcal{A} = \cup_{i\in I} G_i$,

(ii) the restriction of the multiplication to G_i turns G_i into a group for $i \in I$,

(iii) $G_i \cap G_j$ is a subgroup both in G_i and G_j for all $i, j \in I$.

Definition A.2 A group G is called a **completion** of an amalgam $\mathcal{A}$ if there exists a map $\pi : \mathcal{A} \to G$ (called the **completion map**) such that

(i) for all $i \in I$ the restriction of π to G_i is a homomorphism

(ii) $\pi(\mathcal{A})$ generates G.

Among all completions of $\mathcal{A}$ there is a largest one which, if $\mathcal{A}$ is finite, can be defined as the group having the following finite presentation:

$$\mathcal{U}(\mathcal{A}) = \langle t_h \mid h \in \mathcal{A}, t_x t_y = t_{xy} \text{ if } xy \text{ is defined } \rangle \,.$$

$\mathcal{U}(\mathcal{A})$ is called the **universal completion**. Its completion map is given by

$$\psi : \mathcal{A} \to \mathcal{U}(\mathcal{A}) : g \mapsto t_g .$$

We call this completion universal since it has the universal property that for any other completion G with completion map π, there exists a unique group epimorphism $\widehat{\pi}$ from $\mathcal{U}(\mathcal{A})$ onto G such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\psi"] \arrow[dr, "\pi"'] & \mathcal{U}(\mathcal{A}) \arrow[d, "\widehat{\pi}"] \\ & G \end{tikzcd}$$

We define the map $\widehat{\pi}$ by first defining it on the the generators $\psi(\mathcal{A})$ of $\mathcal{U}(\mathcal{A})$ only, via $\widehat{\pi}_{|\psi(\mathcal{A})} : t_x \mapsto \pi(x)$. This can be extended to a group epimorphism because

$$\widehat{\pi}(t_x t_y) = \widehat{\pi}(t_x)\widehat{\pi}(t_y) = \pi(x)\pi(y) = \pi(xy) = \widehat{\pi}(t_{xy})$$

whenever xy and, thus, t_{xy} are defined.

We now consider the amalgam formed by subgroups of a given group G.

Lemma A.3
Let $(G_i)_{i\in I}$ be a finite family of subgroups of a finite group G which generates G, let $\mathcal{A} := \cup_{i\in I} G_i$ be the associated amalgam of groups, and let $\psi : \mathcal{A} \to \mathcal{U}(\mathcal{A})$ be the completion map. Then for each $i \in I$ the restriction $\psi_{|G_i} : G_i \to \mathcal{U}(\mathcal{A})$ is injective. Furthermore $\mathcal{U}(\mathcal{A}) \cong G$ if and only if there exists an $i \in I$ such that the index of $\psi(G_i)$ in $\mathcal{U}(\mathcal{A})$ equals the index of G_i in G.

Proof. Note that G is a completion of $\mathcal{A}$, for which the inclusion map ι is a completion map. By the universal nature of $\mathcal{U}(\mathcal{A})$, there exists an epimorphism π from $\mathcal{U}(\mathcal{A})$ onto G such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\psi"] \arrow[dr, "\iota"'] & \mathcal{U}(\mathcal{A}) \arrow[d, "\pi"] \\ & G \end{tikzcd}$$

The map $\psi_{|G_i}$ must be injective, as $\iota_{|G_i}$ is injective. Let $\widetilde{G}_i := \psi(G_i)$. If $\mathcal{U}(\mathcal{A}) \cong G$, then the indices $[G : G_i]$ and $\left[\mathcal{U}(\mathcal{A}) : \widetilde{G}_i\right]$ coincide, whence this implication is immediate. Conversely, assume that for some i, the group G_i has the same index in G as $\widetilde{G}_i$ has in $\mathcal{U}(\mathcal{A})$. The group $\widetilde{G}_i$ intersects the kernel of π trivially, since by the above $\pi\left(\widetilde{G}_i\right) = G_i \cong \psi(G_i) = \widetilde{G}_i$. But by hypothesis $\left[\mathcal{U}(\mathcal{A}) : \widetilde{G}_i\right] = [G : G_i]$, so

$$\begin{aligned} |G| : |G_i| &= [G : G_i] \\ &= \left[\mathcal{U}(\mathcal{A}) : \widetilde{G}_i\right] \\ &= |\mathcal{U}(\mathcal{A})| : \left|\widetilde{G}_i\right| \\ &= |G| \cdot |\ker(\pi)| : |G_i|, \end{aligned}$$

whence $|\ker(\pi)| = 1$ and π is an isomorphism between $\mathcal{U}(\mathcal{A})$ and G. □

B Appendix: Geometries

In the following, we give a quick run-down on the basics of synthetic geometry. For a more complete introduction to the subject, refer for example to [3] or [17].

Definition B.1 A **pregeometry over a set** I is a triple $\mathcal{G} = (X, *, typ)$ where X is a set (its elements are called the **elements of** $\mathcal{G}$), $*$ is a symmetric and reflexive relation defined on X which is called the **incidence relation of** $\mathcal{G}$, and typ is a map from X to I (the set I is called the **type set of** $\mathcal{G}$) such that $typ(x) = typ(y)$ and $x * y$ imply $x = y$. The pregeometry $\mathcal{G}$ is called **connected** if the graph $(X, *)$ is connected.

Definition B.2 If $A \subseteq X$, then A is of the **type** $typ(A)$, of **rank** $|typ(A)|$, and of **corank** $|I \setminus typ(A)|$. The cardinality $|I|$ of I is called the **rank of** $\mathcal{G}$. A **flag** of $\mathcal{G}$ is a set of mutually incident elements of $\mathcal{G}$. Flags of type I are called **chambers**.

Definition B.3 If F is a flag of $\mathcal{G}$, then the **residue** of F in $\mathcal{G}$ is the pregeometry $\mathcal{G}_F = (X_F, *_F, typ_F)$, where X_F is the set of elements of X that are incident with but distinct from all elements of F, and $*_F$ and typ_F are the restrictions of $*$ and typ to $X_F \times X_F$ respectively X_F. The pregeometry $\mathcal{G}$ is called **residually connected** if $(X_F, *_F)$ is a connected graph for each flag F of $\mathcal{G}$ of corank greater or equal two, and non-empty for each flag F of corank one.

Definition B.4 A **geometry over** I is a pregeometry $\mathcal{G}$ over I in which every maximal flag is a chamber.

Definition B.5 Let G be a group of automorphisms of a geometry $\mathcal{G}$ over I. We say G acts **flag-transitively** on $\mathcal{G}$ if for each $J \subseteq I$, the group G acts transitively on the set of flags of type J. In other words, if F_1 and F_2 are flags in $\mathcal{G}$ of equal type, then there exists $g \in G$ such that $g(F_1) = F_2$.

Definition B.6 Let $\mathcal{G}$ be a geometry of rank n, let $\phi : G \to \mathrm{Aut}\mathcal{G}$ be a group homomorphism such that $\phi(G)$ acts incidence-transitively on $\mathcal{G}$. A **rank** k **parabolic** is the stabilizer of a flag of corank k from $\mathcal{G}$ with respect to the action given by $gF := \phi(g)F$. Parabolics of rank $n-1$ are called **maximal parabolics**. They are exactly the stabilizers in G of single elements of $\mathcal{G}$.

Definition B.7 Let $\mathcal{G}$ be a geometry which admits points and lines as two if its types. The **collinearity graph** is an undirected graph which has as its vertices the points of $\mathcal{G}$, and in which two vertices v_1, v_2 corresponding to points p_1, p_2 are connected by an edge if and only if there exists a line l incident to both p_1 and p_2.

C Appendix: Determining universal completions

C.1 General approach for computing the amalgams: GAP

In order to compute the universal completion of the amalgams of parabolics which we are studying here, we do the following: First, we determine generators for each parabolic. They will be chosen such that the intersection of the parabolics is generated by the intersection of their respective generating set. Specifically, in the case $n = 3$ the maximal parabolics we consider are the point, line and plane stabilizers of our flag F, with suitably chosen generators u, v, w. These stabilizers all intersect in the flag stabilizer, and so generators of the flag stabilizer together with u, v, w generate the desired parabolics as well as their intersections (which are also parabolics).

To prove that the parabolics and their intersections are generated by the matrices for which we claim this, we first show that they generate a subgroup U of the desired group H; then we compute a lower bound of the size for U. If this bound equals the size of the full group H, we have thus established that $H \cong U$.

We proceed by using GAP [7] to compute finite presentations of the parabolics in terms of these generators: We first find a permutation group isomorphic to our group, then from that determine the corresponding relators (to learn more about the algorithms involved, which GAP implements, refer to [4] and [14]). Due to our choice of generators, the universal completion of the amalgam is obtained by forming the union of all the generators and relators of the parabolics.

Finally, we have to prove that this universal completion is identical to $\mathrm{Sp}(2n, q)$. For this it would be sufficient to compute the size of the group. Doing that directly via a coset enumeration over the trivial group is not feasible due to the size of this finitely presented group. Instead we compute the index of one of the parabolics inside the amalgam, which also establishes the desired isomorphism (see Lemma A.3).

All computations (except those for $(n, q) = (3, 7)$) were performed on an Apple PowerBook G4 1.5GHz with 1 GB RAM using GAP 4.4.5. Computation times ranged from a few seconds up to about half an hour (for $(n, q) = (3, 5)$); memory requirements ranged up to 350 MB (again for $(n, q) = (3, 5)$). Details regarding $(n, q) = (3, 7)$ are mentioned in Section C.6.

Before we proceed with the details of this, we present some auxiliary results which are useful for computing lower bounds on the group sizes.

C.2 Subgroups and their sizes

The maximal parabolics of M_i, with respect to our maximal flag F are subgroups of $G_\sigma \cong \mathrm{Sp}(2n, q)$ with the following isomorphism type (see [8]):

$$M_i \cong \begin{cases} \mathrm{Sp}(2n-2i, q) \times \mathrm{GU}(i, q^2) & \text{for } 1 \le i \le n-1 \\ \mathrm{GU}(n, q^2) & \text{for } i = n \end{cases}.$$

Note that by $\mathrm{GU}(n, q^2)$ we denote the general unitary group of dimension n over the field $\mathbb{F}_{q^2}$ (sometimes in the literature this is referred to as $\mathrm{GU}(n, q)$, which is also the notation used by GAP).

So we can compute the size of the M_i, since it is well known (see e.g. [22]) that $|\mathrm{Sp}(2n, q)| = q^{n^2} \prod_{i=1}^{n} (q^{2i} - 1)$ and $|\mathrm{GU}(n, q^2)| = q^{n(n-1)/2} \prod_{i=1}^{n} (q^i + (-1)^{i+1})$.

These size formulas are important in the following sections, where we use them to prove that the groups generated by certain matrices are precisely the groups we are looking for.

C.3 The case $n = 3, q = 3$

In this section z denotes a primitive element in $\mathbb{F}_9$ over $\mathbb{F}_3$ with minimal polynomial $x^2 - x - 1$. We define the following matrices:

$$U := \left(\begin{array}{ccc|ccc} z^7 & z^1 & & & & \\ z^7 & z^5 & & & & \\ & & 1 & & & \\ \hline & & & z^5 & z^3 & \\ & & & z^5 & z^7 & \\ & & & & & 1 \end{array}\right) \qquad V := \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & z^7 & z^1 & & & \\ & z^7 & z^5 & & & \\ \hline & & & 1 & & \\ & & & & z^5 & z^3 \\ & & & & z^5 & z^7 \end{array}\right)$$

$$W := \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & & & & \\ & & z^7 & & & z^1 \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & z^7 & & & z^5 \end{array}\right)$$

In addition to these elements we use diagonal matrices $D_i, 1 \le i \le 3$, that generate the stabilizer of the flag F, a half-split torus isomorphic to C_4^3.

Lemma C.1
Each maximal parabolic in $\mathrm{Sp}(6, 3)$ *is generated by the matrices specified in the following table together with generators of the flag stabilizer.*

stabilizer	*element*	*generators*	*isomorphism type*	*index*
M_1	$\langle e_1 \rangle$	V, W	$\mathrm{Sp}(4, 3) \times \mathrm{GU}(1, 9)$	44226
M_2	$\langle e_1, e_2 \rangle$	U, W	$\mathrm{Sp}(2, 3) \times \mathrm{GU}(2, 9)$	3980340
M_3	$\langle e_1, e_2, e_3 \rangle$	U, V	$\mathrm{GU}(3, 9)$	379080

Furthermore, the pairwise intersection of the stabilizers is generated by the intersection of their generators as given above.

Proof. The claimed generators of each M_i obviously each stabilize the corresponding element in the table. Hence they generate subgroups of the stabilizers. Also, the intersection of the generators of any two M_i forms a subgroup of the intersection of the two groups. To complete the proof, we compute, using GAP and Lemma 2.3 and the subsequent discussion, lower bounds of the group sizes. We then verify that they are equal to the sizes of the maximal parabolics respectively the double stabilizers. □

Based on the above table, we find presentations of the maximal parabolics on the generators d_1, d_2, d_3, u, v, w. These, together with the union of the relators of said presentations give a presentation for the universal completion of the amalgam of the maximal parabolics.

Coset enumeration over the subgroup generated by d_1, d_2, d_3, v, w gives an index of 44226 which corresponds to the index of M_1 in $\mathrm{Sp}(6,3)$. By Lemma A.3 this shows that $\mathrm{Sp}(6,3)$ is the universal completion of the amalgam of maximal parabolics.

We give here presentations of the maximal parabolics on the generators d_1, d_2, d_3, u, v, w. To each presentation the relators d_i^4 for $1 \leq i \leq 3$ and $[d_i, d_j]$ for $1 \leq i < j \leq 3$ need to be added.

Generators for M_1: d_1, d_2, d_3, v, w.

Relators for M_1:

$$\begin{aligned}
&v^3, w^3, [v, d_1], [w, d_1], [w, d_2], d_2v^{-1}d_3v^{-1}d_3^{-2}, wd_3wd_3wd_3^{-1},\\
&vwv^{-1}wd_3^{-1}vw^{-1}v^{-1}d_3w^{-1}, d_3vwv^{-1}w^{-1}vw^{-1}v^{-1}d_3^{-1}vwv^{-1},\\
&d_2vwv^{-1}w^{-1}v^{-1}d_3^{-1}d_2^{-1}wvwv^{-1}w^{-1}d_3w
\end{aligned}$$

Generators for M_2: d_1, d_2, d_3, u, w.

Relators for M_2:

$$u^3, w^3, [u, w], [u, d_3], [w, d_1], [w, d_2], d_1u^{-1}d_2u^{-1}d_2^{-2}, wd_3^2w^{-1}d_3^{-2}, d_3wd_3^{-1}wd_3w$$

Generators for M_3: d_1, d_2, d_3, u, v.

Relators for M_3:

$$u^3, v^3, [v, d_1], [u, d_3], d_2^2vd_2vd_3, ud_1^2d_2ud_1, u^{-1}vuv^{-1}uv$$

C.4 The case $n = 3, q = 4$

In this section z denotes a primitive element in $\mathbb{F}_{16}$ over $\mathbb{F}_2$ with minimal polynomial $x^4 + x + 1$. We define the following matrices:

$$U := \left(\begin{array}{ccc|ccc}
z^5 & z^1 & & & & \\
z^4 & z^5 & & & & \\
 & & 1 & & & \\
\hline
 & & & z^5 & z^4 & \\
 & & & z^1 & z^5 & \\
 & & & & & 1
\end{array}\right)
\qquad
V := \left(\begin{array}{ccc|ccc}
1 & & & & & \\
 & z^5 & z^1 & & & \\
 & z^4 & z^5 & & & \\
\hline
 & & & 1 & & \\
 & & & & z^5 & z^4 \\
 & & & & z^1 & z^5
\end{array}\right)$$

$$W := \left(\begin{array}{ccc|ccc}
1 & & & & & \\
 & 1 & & & & \\
 & & z^5 & & & z^1 \\
\hline
 & & & 1 & & \\
 & & & & 1 & \\
 & & z^4 & & & z^5
\end{array}\right)$$

In addition to these elements we use diagonal matrices $D_i, 1 \leq i \leq 3$, that generate the stabilizer of the flag F, a half-split torus isomorphic to C_5^3.

Lemma C.2
Each maximal parabolic in $\mathrm{Sp}(6,4)$ *is generated by the matrices specified in the following table together with generators of the flag stabilizer.*

stabilizer	*element*	*generators*	*isomorphism type*	*index*
M_1	$\langle e_1 \rangle$	V, W	$\mathrm{Sp}(4,4) \times \mathrm{GU}(1,16)$	838656
M_2	$\langle e_1, e_2 \rangle$	U, W	$\mathrm{Sp}(2,4) \times \mathrm{GU}(2,16)$	228114432
M_3	$\langle e_1, e_2, e_3 \rangle$	U, V	$\mathrm{GU}(3,16)$	13160448

Proof. See the proof of Lemma C.1. □

Based on the above table, we find presentations of the maximal parabolics on the generators d_1, d_2, d_3, u, v, w. These, together with the union of the relators of said presentations give a presentation for the universal completion of the amalgam of the maximal parabolics.

Coset enumeration over the subgroup generated by d_1, d_2, d_3, v, w gives an index of 838656 which corresponds to the index of M_1 in $\mathrm{Sp}(6,4)$. By Lemma A.3 this shows that $\mathrm{Sp}(6,4)$ is the universal completion of the amalgam of maximal parabolics.

We give here presentations of the maximal parabolics on the generators d_1, d_2, d_3, u, v, w. To each presentation the relators d_i^5 for $1 \leq i \leq 3$ and $[d_i, d_j]$ for $1 \leq i < j \leq 3$ need to be added.

Generators for M_1: d_1, d_2, d_3, v, w.

Relators for M_1:

$$\begin{gathered}v^2, w^2, [v, d_1], [w, d_1], [w, d_2], (wd_3)^3, (vw)^4, vd_2d_3vd_2^{-1}d_3^{-1}, (vd_3d_2^{-1})^3,\\ (wvwvd_3^2)^4, wd_1vd_3wd_3d_2vd_3d_2^2vwd_3vd_1^{-1}d_2wvd_3wvwd_3^{-1}vd_2^{-1}d_3wd_3^{-1}wd_3,\\ (wd_3^{-2}wvwvd_3^{-2}wd_3^2)^2, d_3d_1d_3vwvwd_1d_3vwvd_3^{-1}vwd_1^{-2}vd_3^{-2}wd_3^{-2}w,\\ d_3^{-2}wvd_2^{-1}d_3^2wd_3^{-1}vwvd_3wd_3^{-1}vwvd_3wd_3^{-2}wd_3^2wvd_3wv,\\ d_2vd_3wvwd_3^{-1}vwd_2^{-1}d_3vwvd_3wd_3^{-1}wd_3wd_3^{-1}wd_3^2wd_3^{-2}vwvd_3^{-2}\end{gathered}$$

Generators for M_2: d_1, d_2, d_3, u, w.

Relators for M_2:

$$\begin{gathered}u^2, w^2, [u, w], [u, d_3], [w, d_1], [w, d_2], d_3^{-1}wd_3^{-1}wd_3^{-1}w, ud_2^2ud_1^{-2}ud_1^{-1}d_2,\\ ud_2d_1ud_1^{-1}d_2^{-1}, ud_1ud_1^{-1}ud_1^{-1}ud_1ud_1^{-1}ud_2, wd_3^{-2}wd_3^2wd_3^{-2}wd_3^2wd_3^{-2}wd_3^2\end{gathered}$$

Generators for M_3: d_1, d_2, d_3, u, v.

Relators for M_3:

$$\begin{gathered}u^2, v^2, [u, d_3], [v, d_1], d_2d_3^{-1}vd_3^{-2}vd_2^2v, ud_1vd_3vd_3^{-1}d_2vd_3vd_1^{-1}ud_3^{-2},\\ vd_3^2ud_1vd_3^2uvd_3^{-2}uvd_1^{-1}ud_3^{-2}, d_2d_3^{-1}uvd_1^{-1}ud_3^{-2}vd_1^{-2}d_2d_3^{-1}uvd_1^{-1}ud_3^{-2}d_1^{-2},\\ d_2d_3vd_3^{-1}d_2^{-1}v, d_2uvud_3vud_2^2d_3^{-1}uvd_1^{-1}ud_3^{-2}d_1^{-1}d_2d_3^{-1}uvd_1^{-1}ud_3^{-2}d_1^{-1},\\ vd_1^{-1}ud_3^{-2}d_2d_3^{-1}uvd_1^{-1}ud_3^{-2}d_1^2d_2d_3^{-1}uvd_1^{-1}ud_3^{-2}d_2^{-1}uvd_3^{-1}uvd_3^{-1},\\ ud_2d_1ud_2^{-1}d_1^{-1}, d_2^{-1}vd_3^{-1}vd_3^{-1}d_2^{-2}d_3^2ud_1vud_3d_2^{-1}uvd_2^{-1}vud_1ud_3^2d_1vud_3d_2^{-1}vd_1^{-1},\\ d_2d_3^{-1}uvd_1^{-1}uvd_2^{-2}vuvd_1^{-1}d_3^{-2}ud_2d_3^{-1}uvd_1^{-1}d_3^{-2}ud_2d_3^{-1}uvd_1^{-1}d_3^{-2}ud_2d_3^{-1}uvd_1^{-1}d_3^{-2}u\end{gathered}$$

C.5 The case $n = 3, q = 5$

In this section z denotes a primitive element in $\mathbb{F}_{25}$ over $\mathbb{F}_5$ with minimal polynomial $x^2 - x + 2$. We define the following matrices:

$$U := \left(\begin{array}{ccc|ccc} z^{18} & z^{1} & & & & \\ z^{17} & z^{18} & & & & \\ & & 1 & & & \\ \hline & & & z^{18} & z^{5} & \\ & & & z^{13} & z^{18} & \\ & & & & & 1 \end{array}\right) \qquad V := \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & z^{18} & z^{1} & & & \\ & z^{17} & z^{18} & & & \\ \hline & & & 1 & & \\ & & & & z^{18} & z^{5} \\ & & & & z^{13} & z^{18} \end{array}\right)$$

$$W := \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & & & & \\ & & z^{18} & & & z^{1} \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & z^{17} & & & z^{18} \end{array}\right)$$

In addition to these elements we use diagonal matrices $D_i, 1 \leq i \leq 3$, that generate the stabilizer of the flag F, a half-split torus isomorphic to C_6^3.

Lemma C.3
Each maximal parabolic in $\mathrm{Sp}(6,5)$ *is generated by the matrices specified in the following table together with generators of the flag stabilizer.*

stabilizer	*element*	*generators*	*isomorphism type*	*index*
M_1	$\langle e_1 \rangle$	V, W	$\mathrm{Sp}(4,5) \times \mathrm{GU}(1,25)$	8137500
M_2	$\langle e_1, e_2 \rangle$	U, W	$\mathrm{Sp}(2,5) \times \mathrm{GU}(2,25)$	5289375000
M_3	$\langle e_1, e_2, e_3 \rangle$	U, V	$\mathrm{GU}(3,25)$	201500000

Proof. See the proof of Lemma C.1. □

Based on the above table, we find presentations of the maximal parabolics on the generators d_1, d_2, d_3, u, v, w. These, together with the union of the relators of said presentations give a presentation for the universal completion of the amalgam of the maximal parabolics.

Coset enumeration over the subgroup generated by d_1, d_2, d_3, v, w gives an index of 8137500 which corresponds to the index of M_1 in $\mathrm{Sp}(6,5)$. By Lemma A.3 this shows that $\mathrm{Sp}(6,5)$ is the universal completion of the amalgam of maximal parabolics.

We give here presentations of the maximal parabolics on the generators d_1, d_2, d_3, u, v, w. To each presentation the relators d_i^6 for $1 \leq i \leq 3$ and $[d_i, d_j]$ for $1 \leq i < j \leq 3$ need to be added.

Generators for M_1: d_1, d_2, d_3, v, w.

Relators for M_1:

$$\begin{aligned}
&[v,d_1],[w,d_2],[w,d_1],[v,d_2d_3],[w,d_3^3],v^{-3}(d_2d_3)^3,v^{-1}d_2^{-1}d_3^2v^{-1}d_3^{-2}d_2,\\
&w^{-1}d_3^2w^{-1}d_3w^{-1}d_3^{-1}w^{-1}d_3,d_3w^{-1}d_3wd_3w^2d_3w,d_3^{-1}vd_2^{-2}d_3^{-1}v^{-1}d_3^{-1}vd_2^{-1}v^{-1},\\
&vd_2^2v^{-1}d_3v^{-1}d_3^{-2}vd_2^{-1},d_3vwd_2vd_2^{-1}w^{-1}vd_3^{-2}d_2w^{-1}d_2^{-1}v^{-1}wd_3,\\
&vd_3^{-2}w^{-1}v^{-1}d_3^{-1}wd_3^{-1}w^{-1}vd_2wvd_2^{-1}d_3w^{-1}d_3^{-1}v^{-1}d_3w^{-1}v^{-1}d_3^{-1}w^{-1}d_3,\\
&d_2v^{-1}wv^{-1}d_3d_2wv^{-1}d_3^{-1}d_2v^{-1}w^{-1}d_2^{-1}d_3^{-2}w^{-1}v^{-1}d_3^{-1}wd_3w^3d_3w,\\
&d_3vwd_3^{-1}vd_3^{-1}v^{-1}wd_3^{-1}v^{-1}d_3vwd_3^2v^{-1}d_3^{-1}vd_2wvd_2^{-1}d_3^{-1}w^2d_3w^{-1}d_3
\end{aligned}$$

Generators for M_2: d_1, d_2, d_3, u, w.
Relators for M_2:

$$w^6, [u,w], [w,d_2], [w,d_1], [u,d_3], d_3^3w^3, [u,d_1d_2], (u^{-1}d_1^3)^2, d_3^2wd_3^{-1}w^{-1}d_3^{-1}wd_3^{-1}w^{-1},$$
$$d_1^3u^2d_2^{-3}u^{-1}, wd_3wd_3^{-1}wd_3^{-2}wd_3^{-1}, ud_2^{-1}u^{-1}d_1^{-1}u^{-2}d_1u^{-1}d_2, d_1^2ud_1^{-1}ud_1d_2^{-1}u^{-1}d_1^{-1}u^{-1}$$

Generators for M_3: d_1, d_2, d_3, u, v.
Relators for M_3:

$$[v,d_1], [u,d_3], [v,d_2d_3], [u,d_1d_2], (uv)^{-3}, (v^{-1}d_2^3)^2, (d_1^3u)^{-2}, d_1^3u^2d_2^{-3}u^{-1}, d_2^3v^2d_3^{-3}v^{-1},$$
$$ud_2^{-1}u^{-1}d_1^{-1}u^{-2}d_1u^{-1}d_2, vd_3^{-1}v^{-1}d_2^{-1}v^{-2}d_2v^{-1}d_3, d_2^2vd_2^{-1}vd_2d_3^{-1}v^{-1}d_2^{-1}v^{-1},$$
$$ud_1^{-1}ud_2ud_2^{-2}u^{-1}d_1^2, vuv^{-1}d_3^{-1}vu^{-1}v^{-1}ud_1u^{-1}$$

C.6 The case $n = 3, q = 7$

This is the biggest of the open cases, with an index of 247163742. Using our standard representation of the involved groups, that amounts to a memory requirement of about 12 GB when using ACE [6] to perform the coset enumeration. This means that one has to use a 64bit machine with sufficient memory in order to perform the enumeration. George Havas, one of the authors of ACE, performed these computations for us on both a Sparc and an Itanium system with sufficient memory.

In this section z denotes a primitive element in $\mathbb{F}_{49}$ over $\mathbb{F}_7$ with minimal polynomial $x^2 - x + 3$. We define the following matrices:

$$U := \left(\begin{array}{ccc|ccc} z^{11} & z^{1} & & & & \\ z^{31} & z^{29} & & & & \\ & & 1 & & & \\ \hline & & & z^{29} & z^{7} & \\ & & & z^{25} & z^{11} & \\ & & & & & 1 \end{array}\right) \qquad V := \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & z^{11} & z^{1} & & & \\ & z^{31} & z^{29} & & & \\ \hline & & & 1 & & \\ & & & & z^{29} & z^{7} \\ & & & & z^{25} & z^{11} \end{array}\right)$$

$$W := \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & & & & \\ t & & z^{11} & & & z^{1} \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & z^{31} & & & z^{29} \end{array}\right)$$

In addition to these elements we use diagonal matrices $D_i, 1 \leq i \leq 3$, that generate the stabilizer of the flag F, a half-split torus isomorphic to C_8^3.

Lemma C.4
Each maximal parabolic in $\mathrm{Sp}(6,7)$ *is generated by the matrices specified in the following table together with generators of the flag stabilizer.*

stabilizer	*element*	*generators*	*isomorphism type*	*index*
M_1	$\langle e_1 \rangle$	V, W	$\mathrm{Sp}(4,7) \times \mathrm{GU}(1,49)$	247163742
M_2	$\langle e_1, e_2 \rangle$	U, W	$\mathrm{Sp}(2,7) \times \mathrm{GU}(2,49)$	605551167900
M_3	$\langle e_1, e_2, e_3 \rangle$	U, V	$\mathrm{GU}(3,49)$	12070787400

Proof. See the proof of Lemma C.1. □

Based on the above table, we find presentations of the maximal parabolics on the generators d_1, d_2, d_3, u, v, w. These, together with the union of the relators of said presentations give a presentation for the universal completion of the amalgam of the maximal parabolics.

Coset enumeration over the subgroup generated by d_1, d_2, d_3, v, w gives an index of 247163742 which corresponds to the index of M_1 in $\mathrm{Sp}(6,7)$. By Lemma A.3 this shows that $\mathrm{Sp}(6,7)$ is the universal completion of the amalgam of maximal parabolics.

We give here presentations of the maximal parabolics on the generators d_1, d_2, d_3, u, v, w. To each presentation the relators d_i^8 for $1 \le i \le 3$ and $[d_i, d_j]$ for $1 \le i < j \le 3$ need to be added.

Generators for M_1: d_1, d_2, d_3, v, w.

Relators for M_1:

$$\begin{aligned}
&[v,d_1],[w,d_1],[w,d_2],[v,d_2d_2],v^3d_2v^{-1}d_3^{-1},w^2d_3^{-1}wd_3wd_3wd_3^{-1},d_2^2d_3d_2d_3^3vd_3v,\\
&d_3wd_3^{-1}w^{-1}d_3^{-1}w^{-1}d_3^{-1}wd_3,v^{-1}d_3wvd_3vw^{-1}d_3^{-1}v^{-1}d_3^{-1},d_3w^2d_3^{-2}w^{-1}d_3^{-2}w^{-2},\\
&vd_3w^{-1}vw^{-1}v^{-1}w^3d_3^{-1}wd_3^{-2}w^{-1}d_3v^2d_3wv^{-2}w^{-1}d_3^{-1}w^{-1},\\
&wv^{-1}d_3^{-1}v^{-1}d_2^{-1}d_3^2d_2^{-1}w^2d_3d_2^{-1}d_1^{-1}v^{-1}d_3^{-1}d_1v^{-1}d_3^2d_2^{-2}w^2d_3^2d_2^{-1},\\
&d_2v^{-1}d_3v^{-1}w^{-1}d_1v^{-1}d_3v^{-1}d_3^{-2}w^{-1}vd_1d_2d_3^{-1}w^{-2}d_3^{-2}d_2^2vd_1^{-2}d_3^{-3}wd_3,\\
&d_1^2v^{-1}d_2^{-2}d_3^2w^2d_3d_2^{-1}d_1^{-1}v^{-1}d_1^{-1}d_3wd_3d_1vd_1d_2d_3^{-1}w^{-2}d_3^{-2}d_2^2vd_1^{-2}w^{-1}d_3^{-2},\\
&d_1vd_1d_2d_3^{-1}w^{-2}d_3^{-2}d_2d_3^{-1}v^{-2}d_3d_2^{-2}d_3^2w^2d_3d_2^{-1}v^{-2}d_2^{-2}d_3^2w^2d_3d_2^{-1}d_1^{-1}v^{-1}d_1^{-1}wd_3,\\
&d_1v^2d_3wv^{-2}w^{-1}d_1v^{-1}d_2^{-2}d_3^2w^2d_3d_2^{-1}d_1^{-1}v^{-1}w^{-1}d_1v^{-1}d_2^{-2}d_3^2w^2d_3d_2^{-1}d_1^{-1}v^{-1}d_1^{-1}d_3^{-2}w^{-1}d_3
\end{aligned}$$

Generators for M_2: d_1, d_2, d_3, u, w.

Relators for M_2:

$$\begin{aligned}
&[u,w],[u,d_3],[w,d_1],[w,d_2],ud_1d_2u^{-1}d_2^{-1}d_1^{-1},d_2^{-1}u^3d_1u^{-1},ud_2u^5d_1^{-1},\\
&wd_3^2wd_3^{-1}w^{-1}d_3^{-1}w^{-1}d_3^{-1},d_3^{-1}w^2d_3^{-1}wd_3wd_3w,d_1d_2d_1u^{-1}d_2^2d_1^3u^{-1},\\
&w^2d_3^{-4}w^2,d_2^{-1}d_1^{-1}u^{-1}d_1^{-1}u^{-1}d_1^{-2}u^{-1}d_1^{-1}u^{-1}d_1^{-2},ud_2^{-1}d_1ud_1^{-1}u^{-1}d_2^{-1}ud_2u^{-1}d_1^2d_2^{-1}
\end{aligned}$$

Generators for M_3: d_1, d_2, d_3, u, v.

Relators for M_3:

$$\begin{aligned}
&[v,d_1],[u,d_3],vd_2d_3v^{-1}d_3^{-1}d_2^{-1},ud_2d_1u^{-1}d_1^{-1}d_2^{-1},d_3^{-1}v^3d_2v^{-1},\\
&u^3d_1u^{-1}d_2^{-1},vd_3v^5d_2^{-1},d_2ud_1^{-1}ud_1^{-1}ud_2u,d_2d_1^2d_2d_1d_2u^{-1}d_1u^{-1}d_1,\\
&d_2^2d_3v^{-1}d_3^2d_2^3v^{-1},v^{-1}u^{-1}v^{-1}d_3v^{-1}u^{-1}v^{-1}u^{-1}d_1^{-1}u^{-1},\\
&d_3^{-1}d_2^{-1}v^{-1}d_2^{-1}v^{-1}d_2^{-2}v^{-1}d_2^{-1}v^{-1}d_2^{-2},ud_2^{-1}ud_2^{-1}d_1^{-1}d_2^{-2}ud_2^{-1}ud_2^{-2},\\
&ud_1d_2^{-1}ud_2^2u^{-2}d_1^{-1}u^{-1}d_1d_2^{-2},v^{-2}u^{-1}v^{-1}u^{-1}v^{-2}u^2d_1ud_2^{-2}d_1
\end{aligned}$$

C.7 The case $n = 4, q = 2$

In this section z denotes a primitive element in $\mathbb{F}_4$ over $\mathbb{F}_2$ with minimal polynomial $x^2 + x + 1$. We define the following matrices:

$$P_1 := \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & z & z^2 \\ & & & 1 & & & z^2 & 1 \\ \hline & & & & 1 & & & \\ & & & & & 1 & & \\ & & z^2 & z & & & 1 & \\ & & z & 1 & & & & 1 \end{array}\right) \qquad P_2 := \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & & 1 & & & & & \\ & 1 & & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & & & 1 & \\ & & & & & 1 & & \\ & & & & & & & 1 \end{array}\right)$$

$$P_3 := \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & z^2 & 1 & 1 & & & & \\ & 1 & z^2 & 1 & & & & \\ & 1 & 1 & z^2 & & & & \\ \hline & & & & 1 & & & \\ & & & & & z & 1 & 1 \\ & & & & & 1 & z & 1 \\ & & & & & 1 & 1 & z \end{array}\right) \qquad P_4 := \left(\begin{array}{cccc|cccc} & 1 & & & & & & \\ 1 & & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ \hline & & & & & 1 & & \\ & & & & 1 & & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{array}\right)$$

$$P_5 := \left(\begin{array}{cccc|cccc} z & 1 & 1 & & & & & \\ 1 & z & 1 & & & & & \\ 1 & 1 & z & & & & & \\ & & & 1 & & & & \\ \hline & & & & z^2 & 1 & 1 & \\ & & & & 1 & z^2 & 1 & \\ & & & & 1 & 1 & z^2 & \\ & & & & & & & 1 \end{array}\right) \qquad P_6 := \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & 1 & & & & & & \\ & & & 1 & & & & \\ & & 1 & & & & & \\ \hline & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & & 1 \\ & & & & & & 1 & \end{array}\right)$$

$$P_7 := \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & & & & & 1 \\ \hline & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & 1 & & & & \end{array}\right)$$

In addition to these elements we use diagonal matrices $D_i, 1 \le i \le 4$, that generate the stabilizer of the flag F, a half-split torus isomorphic to C_3^4.

Lemma C.5
Each maximal parabolic in $\mathrm{Sp}(8,2)$ *is generated by the matrices specified in the following table together with generators of the flag stabilizer.*

stabilizer	*element*	*generators*	*isomorphism type*	*index*
M_1	$\langle e_1 \rangle$	P_1, P_2, P_3, P_6, P_7	$\mathrm{Sp}(6,2) \times \mathrm{GU}(1,4)$	10880
M_2	$\langle e_1, e_2 \rangle$	P_1, P_4, P_6, P_7	$\mathrm{Sp}(4,2) \times \mathrm{GU}(2,4)$	3655680
M_3	$\langle e_1, e_2, e_3 \rangle$	P_2, P_4, P_5, P_7	$\mathrm{Sp}(2,2) \times \mathrm{GU}(3,4)$	12185600
M_4	$\langle e_1, e_2, e_3, e_4 \rangle$	P_2, P_3, P_4, P_5, P_6	$\mathrm{GU}(4,4)$	609280

Proof. See the proof of Lemma C.1. □

Based on the above table, we find presentations of the maximal parabolics on the generators d_1, d_2, d_3, d_4, p_1, p_2, p_3, p_4, p_5, p_6, p_7. These, together with the union of the relators of said presentations give a presentation for the universal completion of the amalgam of the maximal parabolics. The following presentations define each maximal parabolic.

Coset enumeration over the subgroup generated by $d_1, d_2, d_3, d_4, p_1, p_2, p_3, p_6, p_7$ gives an index of 10880 which corresponds to the index of M_1 in $\mathrm{Sp}(8,2)$. By Lemma A.3 this shows that $\mathrm{Sp}(8,2)$ is the universal completion of the amalgam of maximal parabolics.

We give here presentations of the maximal parabolics on the generators d_1 till d_4 and p_1 till p_7. To each presentation the relators d_i^3 for $1 \leq i \leq 4$ and $[d_i, d_j]$ for $1 \leq i < j \leq 4$ need to be added.

Generators for M_1: $d_1, d_2, d_3, d_4, p_1, p_2, p_3, p_6, p_7$.

Relators for M_1:

$$\begin{gathered}
p_1^2, p_2^2, p_3^3, p_6^2, p_7^2, [p_1, d_1], [p_1, d_2], [p_2, d_1], [p_2, d_4], [p_2, p_3], [p_2, p_7], [p_3, d_1], [p_3, p_6], \\
[p_6, d_1], [p_6, d_2], [p_7, d_1], [p_7, d_2], [p_7, d_3], [p_7, d_4], (p_7 p_1)^4, p_2 d_2^{-1} p_2 d_3, p_6 d_3 p_6 d_4^{-1}, \\
p_3^{-1} p_2 d_4 p_3^{-1} d_2^{-1} d_3^{-1}, p_6 p_3^{-1} d_2 p_3^{-1} d_4^{-1} d_3^{-1}, p_1 p_7 p_1 p_6 d_4 p_7 p_6, d_4^{-1} p_1 d_3 p_1 d_3^{-1} p_6 p_1 p_7, \\
d_4 p_6 p_1 d_4 p_6 d_4^{-1} p_1 d_3^{-1}, d_4 p_1 p_7 p_1 d_4^{-1} p_1 p_7 p_1, p_1 p_6 d_4 p_1 d_3^{-1} d_4^{-1} p_1 p_6 d_4, \\
d_4 p_3 p_7 p_3^{-1} p_7 d_4^{-1} p_3 p_7 p_3^{-1} d_4^{-1} p_7, p_2 p_1 d_3^{-1} p_3^{-1} p_1 p_7 p_6 p_7 p_6 p_3^{-1} d_3 p_1 d_3, \\
p_2 p_6 p_1 p_2 d_2^{-1} d_4 p_1 d_4 p_3 p_7 d_2^{-1} p_3 d_4 p_6 p_1, p_3 p_7 d_2^{-1} p_3 d_3 p_3^{-1} d_2 p_7 p_3 d_2 p_7 p_6 p_3^{-1} p_7 d_3^{-1}, \\
p_3 p_7 d_2^{-1} p_3 p_1 p_3^{-1} d_2 p_7 p_3^{-1} p_7 d_4 p_6 p_1 d_4^{-1} p_6 p_7 p_1 d_3^{-1}
\end{gathered}$$

Generators for M_2: $d_1, d_2, d_3, d_4, p_1, p_4, p_6, p_7$.

Relators for M_2:

$$\begin{gathered}
p_1^2, p_4^2, p_6^2, p_7^2, [p_1, d_1], [p_1, d_2], [p_1, p_4], [p_4, d_3], [p_4, d_4], [p_4, p_7], [p_6, d_2], [p_7, d_1], \\
[p_7, d_2], [p_7, d_3], [p_7, d_4], (p_1 p_7)^4, p_4 d_1^{-1} p_4 d_2, p_6 d_4^{-1} p_6 d_3, p_1 p_7 p_1 d_3 p_6 p_7 p_6, \\
p_1 p_6 d_4^{-1} p_1 p_7 d_4^{-1} p_1 d_3, d_4 p_1 p_6 p_1 d_3 d_4^{-1} p_7 p_1 d_3^{-1} p_7 p_6
\end{gathered}$$

Generators for M_3: $d_1, d_2, d_3, d_4, p_2, p_4, p_5, p_7$.

Relators for M_3:

$$\begin{gathered}
p_2^2, p_4^2, p_5^3, p_7^2, [p_2, d_1], [p_2, d_4], [p_2, p_5], [p_2, p_7], [p_4, d_3], [p_4, d_4], [p_4, p_5], [p_4, p_7], \\
[p_5, d_4], [p_5, p_7], [p_7, d_1], [p_7, d_2], [p_7, d_3], [p_7, d_4], d_3 p_2 d_2^{-1} p_2, p_4 d_1^{-1} p_4 d_2, \\
p_4 p_5 d_3 p_5 d_1^{-1} d_2^{-1}, p_5^{-1} d_1^{-1} p_5^{-1} d_2 d_3 p_2
\end{gathered}$$

Generators for M_4: $d_1, d_2, d_3, d_4, p_2, p_3, p_4, p_5, p_6$.

Relators for M_4:

$$\begin{aligned}
&p_2^2, p_4^2, p_6^2, [p_2, d_1], [p_2, d_4], [p_3, p_6], [p_4, d_3], [p_4, d_4], [p_4, p_5], [p_4, p_6], [p_6, d_1], [p_6, d_2],\\
&(p_5p_3)^3, p_6d_3^{-1}p_6d_4, d_2p_4d_1^{-1}p_4, d_3p_2d_2^{-1}p_2, p_3p_5d_1p_5p_3d_4^{-1}, p_2p_5p_3p_2p_3^{-1}p_5^{-1},\\
&(p_5p_3d_4d_1^{-1})^2, d_3p_6p_3d_3d_1^{-1}d_2^{-1}d_3^{-1}p_3d_1d_4, d_1^{-1}d_4^{-1}p_3p_5p_3^{-1}p_5^{-1}d_1^{-1}p_3^{-1}p_5^{-1}p_2,\\
&p_4d_1p_5^{-1}d_2p_5p_3d_1d_4^{-1}p_3p_5, p_5p_3^2p_5d_1d_4d_2d_3p_3^{-1}, d_4d_1d_4^{-1}p_5p_3d_1d_4^{-1}d_1^{-1}p_3p_5,\\
&p_5d_2d_3d_4p_5p_3d_4p_5p_3, d_2d_1p_3p_4p_3^{-1}d_4^{-1}p_5p_6p_5^{-1}d_2^{-1}d_1^{-1}d_4p_5p_3,\\
&p_5^{-1}p_3p_4p_3^{-1}d_1^{-1}d_2^{-1}p_5p_3p_5^{-1}p_3^{-1}d_2d_1p_5p_6,\\
&p_3p_4p_3^{-1}d_1^{-1}d_2^{-1}p_5p_3d_4p_5^{-1}p_3^{-1}d_1d_2p_5p_6p_5^{-1}d_1^{-1},\\
&p_3^{-1}p_5^{-1}p_2d_2d_1p_3p_4p_3^{-1}d_2^{-1}p_2d_2p_5p_6p_5^{-1}d_2^{-1}d_1^{-1}p_3p_5,\\
&p_3^{-1}p_5^{-1}d_2p_5p_6p_5^{-1}d_1^{-1}d_2^{-1}d_4p_3^{-1}p_5^{-1}d_2p_5p_6p_5^{-1}d_4d_2^{-1}d_1^{-1}
\end{aligned}$$

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