Existence and Exponential Stability in L^r-spaces of Stationary Navier-Stokes Flows with Prescribed Flux in Infinite Cylindrical Domains

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Abstract

We prove existence, uniqueness and exponential stability of stationary Navier-Stokes flows with prescribed flux in an unbounded cylinder of $\mathbb{R}^n, n \geq 3$, with several exits to infinity provided the total flux and external force are sufficiently small. The proofs are based on analytic semigroup theory, perturbation theory and $L^r - L^q$-estimates of a perturbation of the Stokes operator in L^q-spaces.

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1 Introduction

Let $\Omega = \bigcup_{i=0}^{m} \Omega_i$ be a cylindrical domain of $C^{1,1}$-class where Ω_0 is a bounded domain and $\Omega_i, i = 1, \dots, m$, are disjoint semi-infinite straight cylinders, that is, in possibly different coordinates,

$$\Omega_i = \{x^i = (x_1^i, \dots, x_n^i) \in \mathbb{R}^n : \, x_n^i > 0, x'^i = (x_1^i, \dots, x_{n-1}^i) \in \Sigma^i\},$$

where $\Sigma^i \subset \mathbb{R}^{n-1}, n \geq 3, \; i = 1, \dots, m$, is a bounded domain and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. Without loss of generality, we assume for each $i = 1, \dots, m$ that the coordinate system which is fixed in Ω_i is such that x'^i, x_n^i denote the variables with respect to the cross section Σ^i and the axial direction of Ω_i, respectively. Then we consider the existence, uniqueness and stability of the stationary Navier-Stokes

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system

$$
\text{(SNS)} \qquad \begin{aligned} -\Delta w + (w\cdot\nabla)w + \nabla q &= f && \text{in } \Omega \\ \operatorname{div} w &= 0 && \text{in } \Omega \\ w &= 0 && \text{on } \partial\Omega \\ w &= u_\infty && \text{at } \infty, \end{aligned} \tag{1.1}
$$

where u_∞ is a function depending on the variables x'^i in the cross section Σ^i of $\Omega_i, i = 1,\dots,m$. It is well known that for the existence of a unique solution to (1.1) some additional conditions, e.g. a *flux condition* in each exit, must be given, i.e.,

$$
\Phi_i = \int_{\Sigma_i} u\cdot \mathbf{n}^i\, ds, \quad i = 1,\dots,m, \tag{1.2}
$$

where $\mathbf{n}^i$ is the unit vector along the positive axial direction of Ω_i, is prescribed. Note that, due to the solenoidalness of the fluid, $\Phi_i \equiv$ const in x_n^i, $i = 1\dots,m$, and

$$
\sum_{i=1}^m \Phi_i = 0. \tag{1.3}
$$

Moreover, it is natural to assume that the velocity at infinity u_∞ in each $\Omega_i, i = 1,\dots,m$, equals the *Poiseuille flow* $\mathbf{v}_i$ corresponding to the flux Φ_i.

The Poiseuille flow $(\mathbf{v}_0, p_0)$ corresponding to a given flux Φ_0 in an infinite straight cylinder $\Sigma\times\mathbb{R}$ with bounded cross section $\Sigma\subset\mathbb{R}^{n-1}$ is the solution to the stationary Stokes system in $\Sigma\times\mathbb{R}$ such that $\mathbf{v}_0 = v_0(x')\mathbf{n}, \nabla p_0 = -k\mathbf{n}$ with constant $k = k(\Phi_0)$ and

$$
\int_\Sigma \mathbf{v}_0\cdot\mathbf{n}\, ds = \Phi_0;
$$

here $\mathbf{n}$ is the unit vector along the positive direction of the cylinder $\Sigma\times\mathbb{R}$. Then it is easily seen that

$$
-\Delta v_0 = k, \quad v_0|_{\partial\Sigma} = 0;
$$

in particular, if Σ is a Lipschitz domain, one gets the explicit representation $v_0 = \frac{\Phi_0}{k_0} w_0(x'),\ k = \frac{\Phi_0}{k_0}$, where w_0 is the unique solution to the Dirichlet problem $-\Delta' w_0 = 1,\ w_0|_{\partial\Sigma} = 0$ and $k_0 = \int_\Sigma |\nabla' w_0|^2\, dx'$. Moreover, if Σ is of $C^{1,1}$-class, then

$$
v_0 \in H^{2,s}(\Sigma)\cap H_0^{1,s}(\Sigma), \quad \|v_0\|_{H^{2,s}(\Sigma)} \le c(s,\Sigma)|\Phi_0| \quad \forall s\in(1,\infty), \tag{1.4}
$$

in particular, $v_0, \nabla v_0 \in L^\infty(\Sigma)$ and $\|v_0,\, \nabla v_0\|_{L^\infty(\Sigma)} \le c(\Sigma)|\Phi_0|$ due to the Sobolev embedding theorem. Note that the Poiseuille solution $(\mathbf{v}_0, p_0)$ also solves the stationary Navier-Stokes system in $\Sigma\times\mathbb{R}$.

There is a number of papers dealing with stability of stationary Navier-Stokes flows on various domains, see e.g. [21] for the whole space, [19] for the half space, [16] for bounded domains and [14], [15], [27] for exterior domains and the references therein.

The existence of stationary Navier-Stokes flows in domains with noncompact boundaries has been studied in many papers, see e.g. [17], [18], [24], [25], [28] and references cited in [25]. The existence of solutions with *infinite* Dirichlet integral to stationary Navier-Stokes systems in unbounded domains with exits to infinity for which the cross section of each exit is a ball of $\mathbb{R}^{n-1}, n = 2, 3$, is considered for arbitrary fluxes in [18] for weak solution and in [24] for strong solution; it should be noted that the existence for large data is obtained without imposing *a priori* that the flow at infinity equals a Poiseuille flow, and it is not known whether the solutions will tend to a Poiseuille flow as $|x| \to \infty$, see [25], §2.6 or [11], Ch. XI, Remark 3.1. Moreover, in [24], for stationary Navier-Stokes systems in infinite cylindrical domains the existence of a strong solution which behaves at infinity like Poiseuille flows corresponding to given fluxes $\Phi_i, i = 1, \dots, m$, was shown under some smallness condition on the *total flux*

$$\Phi := \sum_{i=1}^{m} |\Phi_i|.$$

It is not clear whether the method used there will be applicable to the case of the unbounded cylinder Ω with arbitrary cross section. We refer to [24], [28] and [10], Ch. VI, for more details of solvability of stationary Navier-Stokes systems in domains with noncompact boundaries. In the case of our cylindrical domain Ω the existence of a weak solution to (SNS) was shown first in [2] under a smallness condition on the flux, see also [11], Ch. XI. Recently, in [26] the instationary Navier-Stokes system in Ω with time-dependent prescribed flux has been considered in Hilbert spaces using Galerkin approximation. For cylindrical domains with several exits to infinity and with bounded varying cross sections the stationary Stokes system is considered in [22].

In this paper we consider the existence, uniqueness and stability of a strong solution to the stationary Navier-Stokes system (SNS) with prescribed flux via an L^r-space approach.

In order to prove existence and uniqueness to (SNS), first, a *carrier* $\mathbf{a}$ on Ω of the Poiseuille flows $\mathbf{v}_i$, corresponding to the given fluxes $\Phi_i, i = 1, \dots, m$, in each exit of the domain Ω is constructed. The original system (SNS) is reduced to a modified stationary Navier-Stokes system with respect to the new unknown $v = w - \mathbf{a}$, see the system (SNS$'$) in (2.13), with zero flux. Our first main result gives the existence of a stationary solution w to system (SNS) using Banach's fixed point theorem, cf. Theorem 2.4:

Theorem 1.1 *Let $\frac{n}{3} < r < \infty$ and $f \in L^r(\Omega)$. Furthermore, let the velocity u_∞ at infinity for each exit $\Omega_i, i = 1, \dots, m$, be the Poiseuille flow corresponding to the given flux $\Phi_i, i = 1, \dots, m$, satisfying* (1.3). *Then for sufficiently small $\|f\|_r$ and total flux Φ system (SNS) has a unique solution $w = \mathbf{a} + v$ satisfying $v \in H^{2,r}(\Omega)$ and*

$$\|v\|_{H^{2,r}(\Omega)} \le c(r, \Omega)(\|f\|_r + \Phi^2).$$

Next we consider the stability of stationary Navier Stokes flows in Ω. If the stationary solution $\{w, \nabla q\}$ is perturbed by a velocity field u_0 at time $t = 0$, then the corresponding perturbed instationary flow $\{u(t) + w, \nabla(p(t) + q)\}$ is governed by the system

$$\begin{aligned} u_t - \Delta u + (u \cdot \nabla)w + (w \cdot \nabla)u + (u \cdot \nabla)u + \nabla p &= 0 \quad \text{in} \quad \Omega \times (0,T) \\ \operatorname{div} u &= 0 \quad \text{in} \quad \Omega \times (0,T) \\ u &= 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ u(0) &= u_0 \quad \text{in} \quad \Omega. \end{aligned} \tag{1.5}$$

Hence the study of stability for (SNS) is reduced to the investigation of the behavior of solutions to (1.5) for $t \to \infty$. To this end, we consider an abstract formulation of (1.5), namely,

$$u_t + S_r u + P_r(u \cdot \nabla)u = 0, \quad t \in (0,T), \quad u(0) = u_0,$$

in $L^r_\sigma(\Omega)$, where $S_r u = A_r u + P_r((u \cdot \nabla)w + (w \cdot \nabla)u)$ with w the solution to (SNS), A_r is the Stokes operator in $L^r_\sigma(\Omega)$ and P_r is the Helmholtz projection in $L^r(\Omega)$. Note that the Stokes operator in the cylindrical domain Ω generates a bounded and exponentially decaying analytic semigroup in $L^r_\sigma(\Omega)$, see [9].

Using perturbation techniques we show that, if $\|f\|_r$ and the total flux Φ are sufficiently small, then the operator S_r (depending on w) generates a bounded and exponentially decaying analytic semigroup in $L^r_\sigma(\Omega)$ and, moreover, admits a bounded H^∞-calculus in $L^r_\sigma(\Omega)$ for $r > \frac{n}{3}$ (Theorem 3.5). Then, based on $L^r - L^q$ estimates for the semigroups $\{e^{-tS_r}\}_{t\geq 0}$ and $\{e^{-tS^*_{r'}}\}_{t\geq 0}$ (Lemma 3.9) and a standard fixed point argument, we get our main result on exponential stability:

Theorem 1.2 *Let $n \leq r < \infty$, and let $f \in L^r(\Omega)$ and the total flux Φ be sufficiently small. Then for all $u_0 \in L^r_\sigma(\Omega)$ with sufficiently small norm $\|u_0\|_r$ the system* (1.5) *– with the unique solution w to (SNS) corresponding to $f, \Phi_1, \dots, \Phi_m$ given by Theorem 1.1 – has a global strong solution u satisfying for certain $\alpha > 0$*

$$\lim_{t\to\infty} e^{\alpha t}\|u(t)\|_q = 0 \quad \textit{for all} \quad q \geq r.$$

In particular, the stationary solution w of (SNS) is exponentially stable.

For the proof of this theorem we first prove the existence of a global mild solution to (1.5) which decays exponentially as $t \to \infty$ (Theorem 4.6). Furthermore, $L^r - L^q$ estimates imply that the global mild solution has certain regularity properties depending on $r \geq n$. Then sharp estimates for the nonlinear term $(u \cdot \nabla)u$, see Lemma 2.1, combined with the theory of abstract parabolic equations imply that this global mild solution is actually a strong solution to (1.5) in the sense of Definition 3.12, see Theorem 4.8. Finally, in Theorem 4.9, we consider the uniqueness of strong solutions to (1.5). We note that, when $w = 0$, these results yield the existence and uniqueness of a global in time strong solution with zero flux to the instationary Navier-Stokes system in $L^r(\Omega)$ for $r \geq n$.

We use the following notation. Not distinguishing between spaces of vector functions and scalar functions, $L^r(\Omega)$, $1 < r < \infty$, may denote Lebesgue spaces of vector or scalar functions depending on the context. We denote by $H^{s,p}(\Omega)$ and $B^s_{q,r}(\Omega)$, for $s \geq 0$, $p \in (1,\infty)$, $1 \leq q,r \leq \infty$ Bessel potential and Besov spaces, respectively. For $1 < r < \infty$ let $L^r_\sigma(\Omega)$ be the completion in L^r-norm $\|\cdot\|_r$ of the set

$$C^\infty_{0,\sigma}(\Omega) = \{\varphi \in C^\infty_0(\Omega)^n : \operatorname{div}\varphi = 0\}.$$

Given a Banach space X and an interval $J \subset \mathbb{R}$ let $BC(J,X)$ denote the space of all uniformly bounded and continuous X-valued functions defined on J endowed with norm

$$\|u\|_{BC(J,X)} = \sup_{s\in J} \|u(s)\|_X.$$

For linear normed spaces X, Y the notation $X \hookrightarrow Y$ means that X is continuously embedded into Y. For $0 < \theta < 1, 1 \leq p \leq \infty$ we denote by $(\cdot,\cdot)_{\theta,p}$, $[\cdot,\cdot]_\theta$ the real and complex interpolation functor, respectively. Throughout this paper we put

$$\bar{\alpha} = \min\{\alpha^{(i)} : \ i = 0,\dots,m\}, \tag{1.6}$$

where $\alpha^{(0)} > 0$ and $\alpha^{(i)} > 0$, $i = 1,\dots,m$, are the smallest eigenvalues of Dirichlet Laplacians in Ω_0 and in Σ^i, $i = 1,\dots,m$, respectively.

We use the short notation $\|u,v\|_X$ for $\|u\|_X + \|v\|_X$, even if u and v are tensors of different order and, as long as no confusion arises, denote various constants in estimates by the same symbol, say c, C etc..

This paper is organized as follows. In Section 2, existence and uniqueness of solutions to the stationary Navier-Stokes system (SNS) with prescribed flux in Ω is shown, cf. Theorem 1.1 or Theorem 2.4. Section 3 is devoted to preliminaries on modified Stokes operators, and Section 4 discusses the exponential stability of (SNS), cf. Theorem 1.2 – or in more details – Theorems 4.6, 4.8 and 4.9.

2 Stationary Navier-Stokes Flows

Let the cylindrical domain Ω be given as in the Introduction. We consider the system (SNS), see (1.1), with

$$u_\infty = \mathbf{v}_i,$$

in each exit $\Omega_i, i = 1,\dots,m$, where $\mathbf{v}_i$ is the Poiseuille flow corresponding to the flux Φ_i through the cross section Σ^i of Ω_i and (1.3) is assumed. Note that $\mathbf{v}_i, i = 1,\dots,m$, depends only on the variable $x'^i \in \Sigma^i$.

First of all, we construct a *carrier* $\mathbf{a}$ of the Poiseuille flows $\mathbf{v}_i, i = 1,\dots,m$. Let $1 < r < \infty$. A carrier $\mathbf{a}$ is defined as a function on Ω such that

$$\mathbf{a} \in H^{2,r}_{\mathrm{loc}}(\overline{\Omega}), \quad \operatorname{div}\mathbf{a} = 0 \text{ in } \Omega, \quad \mathbf{a} = 0 \text{ on } \partial\Omega, \quad \mathbf{a} = \mathbf{v}_i \text{ in } \Omega_i \setminus \Omega_0, i = 1,\dots,m.$$

In [10], Ch. 6, §1, a carrier $\mathbf{a}$ for the case $r = 2$ is constructed. The idea used there can be applied to the general case $r \in (1,\infty)$. Without loss of generality we may assume that there exist cut-off functions $\{\varphi_i\}_{i=0}^m$ such that

$$\begin{aligned} &\textstyle\sum_{i=0}^m \varphi_i(x) = 1, \quad 0 \leq \varphi_i(x) \leq 1 \quad \text{for } x \in \Omega, \\ &\varphi_i \in C^\infty(\bar{\Omega}_i), \quad \operatorname{dist}(\operatorname{supp}\varphi_i,\, \partial\Omega_i \cap \Omega) \geq d > 0, \ i = 0,\dots,m. \end{aligned}$$

For $i = 1, \dots, m$ let $\tilde{\mathbf{v}}_i = \chi_i \mathbf{v}_i$, where χ_i is the characteristic function of Ω_i, and set

$$\mathbf{v}(x) := \sum_{i=1}^{m} \varphi_i(x) \tilde{\mathbf{v}}_i(x) \quad \text{for } x \in \Omega.$$

Then from the construction of $\{\varphi_i\}$ and (1.4) we get

$$\mathbf{v}|_{\Omega_0} \in H^{2,r}(\Omega_0), \quad \|\mathbf{v}|_{\Omega_0}\|_{H^{2,r}(\Omega_0)} \le c(r,\Omega)\Phi \quad \forall r \in (1,\infty) \tag{2.1}$$

where and in what follows we use the notation

$$\Phi := \sum_{i=1}^{m} |\Phi_i|$$

for the total flux. Note that $\operatorname{div} \mathbf{v}|_{\Omega_0} \in H_0^{1,r}(\Omega_0)$ for all $r \in (1,\infty)$ and by (1.3)

$$\int_{\Omega_0} \operatorname{div} \mathbf{v}\, dx = \sum_{i=1}^{m} \int_{\Sigma_i} \mathbf{v}_i(x'^i) \cdot \mathbf{n}^i \, dx'^i = \sum_{i=1}^{m} \Phi_i = 0,$$

where $\mathbf{n}^i, i = 1, \dots, m$, is the unit vector towards the positive direction of the axis of $\Omega_i, i = 1, \dots, m$. Then, by [10], Ch. III, Theorem 3.2, Remark 3.6, (cf. [5], Theorem 2.4) there is a vector field $\mathbf{z}$ such that

$$\mathbf{z} \in H_0^{2,r}(\Omega_0) \quad \text{and} \quad \operatorname{div} \mathbf{z} = -\operatorname{div} \mathbf{v}|_{\Omega_0} \quad \text{for all } r \in (1,\infty)$$

and

$$\|\mathbf{z}\|_{H_0^{2,r}(\Omega_0)} \le c(r,\Omega_0) \|\operatorname{div} \mathbf{v}|_{\Omega_0}\|_{H_0^{1,r}(\Omega_0)} \le c(r,\Omega)\Phi \quad \forall r \in (1,\infty), \tag{2.2}$$

where we used (2.1). Now extend the function $\mathbf{z}$ from Ω_0 to Ω by 0 and denote it again by $\mathbf{z}$. Then

$$\mathbf{a} := \mathbf{z} + \mathbf{v} \tag{2.3}$$

is a carrier of the Poiseuille flows $\mathbf{v}_i, i = 1, \dots, m$, and, by (2.1), (2.2) satisfies the estimate

$$\|\mathbf{a}\|_{H^{2,r}(\Omega_0)} \le c(r,\Omega)\Phi \quad \forall r \in (1,\infty). \tag{2.4}$$

In particular, we get $\mathbf{a}$, $\nabla \mathbf{a} \in L^\infty(\Omega)$ and

$$\|\mathbf{a}, \nabla \mathbf{a}\|_{L^\infty(\Omega)} \le c(\Omega)\Phi. \tag{2.5}$$

Lemma 2.1 *Let $n \ge 3, 1 < r < \infty$ and let*

$$\delta = \begin{cases} \frac{n}{r} - 2 & \text{for } 1 < r < \frac{n}{2} \\ \delta' & \text{for } r = \frac{n}{2} \\ 0 & \text{for } r > \frac{n}{2}, \end{cases} \tag{2.6}$$

with $\delta' > 0$ arbitrarily small.

(1) For all $u \in H^{1+\delta,r}(\Omega)$ *and* $v \in H^{2,r}(\Omega)$ *we have* $(u\cdot\nabla)v, (v\cdot\nabla)u \in L^r(\Omega)$ *and*

$$\|(u\cdot\nabla)v,\ (v\cdot\nabla)u\|_r \le c\|u\|_{H^{1+\delta,r}(\Omega)}\|v\|_{H^{2,r}(\Omega)} \tag{2.7}$$

where $c = c(r,\Omega) > 0$ *is independent of* δ *unless* $r = \frac{n}{2}$*.*

(2) Let $r \in (1,\infty)$ *and* $r \ge \frac{n}{3}$*. Then for all* $u, v \in H^{2,r}(\Omega)$ *we have* $(u\cdot\nabla)v \in H^{1-\delta,r}(\Omega)$ *and*

$$\|(v\cdot\nabla)u\|_{H^{1-\delta,r}(\Omega)} \le c\|u\|_{H^{2,r}(\Omega)}\|v\|_{H^{2,r}(\Omega)}. \tag{2.8}$$

(3) Let

$$\eta = \begin{cases} \frac{n+r}{2r}, & r < n \\ 1+\delta', & r = n \\ 1, & r > n \end{cases} \tag{2.9}$$

with $\delta' > 0$ *arbitrarily small. Then for* $r \in (1,\infty), r \ge \frac{n}{3}$*, and* $\xi \in [\eta, 2]$

$$\|(v\cdot\nabla)u\|_{H^{(1-\delta)\frac{\xi-\eta}{2-\eta},r}(\Omega)} \le c\|u\|_{H^{\xi,r}(\Omega)}\|v\|_{H^{\xi,r}(\Omega)}, \tag{2.10}$$

where $c = c(r,\xi,\Omega) > 0$ *is independent of* δ *(*δ'*) unless* $r = \frac{n}{2}$ *(*$r = n$*).*

Proof: First of all, we note that for the unbounded domain Ω the usual Sobolev embedding theorems hold since Ω has a *minimally smooth boundary* and hence, extension theorems for Sobolev spaces hold for Ω, cf. [1], Ch. V, Theorem 2.4.5 (cf. [29], Theorem 3.21). In the proof we shall write shortly $H^{s,r}$, L^q in place of $H^{s,r}(\Omega)$, $L^q(\Omega)$, respectively.

(1) First let $1 < r < \frac{n}{2}$. Observe that for $\delta = \frac{n}{r} - 2$ the Sobolev embeddings $H^{1+\delta,r} \hookrightarrow L^n$ and $H^{1,r} \hookrightarrow L^{nr/(n-r)}$ hold. Hence we get for all $u \in H^{1+\delta,r}, v \in H^{2,r}$ that

$$\|(u\cdot\nabla)v\|_r \le \|u\|_n\|\nabla v\|_{\frac{nr}{n-r}} \le c\|u\|_{H^{1+\delta,r}}\|v\|_{H^{2,r}}$$

with $c = c(r,\Omega) > 0$. Moreover, by the embeddings $H^{2,r} \hookrightarrow L^{nr/(n-2r)}$, $H^{\delta,r} = H^{\frac{n}{r}-2,r} \hookrightarrow L^{n/2}$, we get

$$\|(v\cdot\nabla)u\|_r \le \|v\|_{\frac{nr}{n-2r}}\|\nabla u\|_{n/2} \le c(r,\Omega)\|v\|_{H^{2,r}}\|u\|_{H^{1+\delta,r}}.$$

Now let $\frac{n}{2} < r < \infty$. Then

$$\|(u\cdot\nabla)v\|_r \le \|u\|_{2r}\|\nabla v\|_{2r} \le c\|u\|_{H^{1+\delta,r}}\|v\|_{H^{2,r}}$$

with $\delta = 0$. Note that the embedding $H^{2,r} \hookrightarrow L^\infty$ holds for $r > \frac{n}{2}$. Hence,

$$\|(v\cdot\nabla)u\|_r \le c\|v\|_\infty\|\nabla u\|_r \le c\|v\|_{H^{2,r}}\|u\|_{H^{1+\delta,r}}$$

with $\delta = 0$.

In the limit case $r = \frac{n}{2}$ note that $H^{2,r} \hookrightarrow L^p$ for all $p \in [r,\infty)$ and that for all $\delta \in (0,1)$ there exists an $\varepsilon = \varepsilon(r,\delta,\Omega) > 0$ such that $H^{\delta,r} \hookrightarrow L^{r+\varepsilon}$. Hence for $u \in H^{1+\delta,r}, v \in H^{2,r}$

$$\|(u\cdot\nabla)v\|_r \le c\|u\|_{2r}\|v\|_{2r} \le c\|u\|_{H^{1,r}}\|v\|_{H^{2,r}},$$

and there exists $p_\varepsilon > r$ such that

$$\|(v\cdot\nabla)u\|_r \le \|\nabla u\|_{r+\varepsilon}\|v\|_{p_\varepsilon} \le c_\delta \|u\|_{H^{1+\delta,r}}\|v\|_{H^{2,r}}.$$

(2) First observe that for all $u \in H^{2+\delta,r}, v \in H^{2,r}$

$$\|(v\cdot\nabla)u\|_{H^{1,r}} \le c\|u\|_{H^{2+\delta,r}}\|v\|_{H^{2,r}}. \tag{2.11}$$

Actually $D(v\cdot\nabla)u = (Dv\cdot\nabla)u + (v\cdot\nabla)Du$, where D is any first order derivative. By (2.7) we get that

$$\|(Dv\cdot\nabla)u\|_r \le c\|\nabla u\|_{H^{1+\delta,r}}\|v\|_{H^{2,r}} \le c\|u\|_{H^{2+\delta,r}}\|v\|_{H^{2,r}},$$
$$\|(v\cdot\nabla)Du\|_r \le c\|\nabla u\|_{H^{1+\delta,r}}\|v\|_{H^{2,r}} \le c\|u\|_{H^{2+\delta,r}}\|v\|_{H^{2,r}},$$

proving (2.11). Note that $1-\delta \in (0,1]$ for $r \ge \frac{n}{3}$ and that by complex interpolation $[H^{1+\delta,r}, H^{2+\delta,r}]_{1-\delta} = H^{2,r}$ and $[L^r, H^{1,r}]_{1-\delta} = H^{1-\delta,r}$, cf. [3], [30]. Therefore, by complex interpolation of (2.7), (2.11) with the index $1-\delta$, we get for all $u, v \in H^{2,r}$ that $(v\cdot\nabla)u \in H^{1-\delta,r}$ and

$$\|(v\cdot\nabla)Du\|_{H^{1-\delta,r}} \le c\|u\|_{H^{2,r}}\|v\|_{H^{2,r}},$$

where $c = c(r,\delta,\Omega)$ for $r = \frac{n}{2}$ and arbitrarily small δ. Thus (2.8) is proved.

(3) First let us prove for η given by (2.9) and for $u, v \in H^{\eta,r}$ that

$$\|(v\cdot\nabla)u\|_r \le c\|u\|_{H^{\eta,r}}\|v\|_{H^{\eta,r}}, \tag{2.12}$$

with $c = c(r,\Omega) > 0$ ($c = c(r,\delta',\Omega) > 0$ for $r = n$). Actually, for $1 < r < n$ we get with $\alpha = \frac{1}{2}(1-\frac{r}{n}) \in (0,1)$ that

$$\|(v\cdot\nabla)u\|_r \le \|v\|_{\frac{r}{\alpha}}\|\nabla u\|_{\frac{r}{1-\alpha}} \le c\|v\|_{H^{\eta,r}}\|u\|_{H^{\eta,r}},$$

where we used that $H^{\eta,r} \hookrightarrow L^{r/\alpha}$ and $H^{\eta-1,r} \hookrightarrow L^{r/(1-\alpha)}$. For $r = n$

$$\|(v\cdot\nabla)u\|_r \le \|v\|_\infty\|\nabla u\|_r \le c\|v\|_{H^{1+\delta',r}}\|u\|_{H^{1,r}},$$

and finally, for $r > \frac{n}{2}$ we get

$$\|(v\cdot\nabla)u\|_r \le \|v\|_\infty\|\nabla u\|_r \le c\|v\|_{H^{1,r}}\|u\|_{H^{1,r}},$$

thus proving (2.12). Now bilinear complex interpolation of (2.8) and (2.12) (see [30], 1.19.5) yields

$$\|(v\cdot\nabla)u\|_{H^{(1-\delta)\theta,r}} \le c\|u\|_{H^{\eta(1-\theta)+2\theta,r}}\|v\|_{H^{\eta(1-\theta)+2\theta,r}}, \quad \theta \in [0,1],$$

which coincides with (2.10) for $\theta = \frac{\xi-\eta}{2-\eta}$.

The proof of the lemma is complete. ■

Lemma 2.2 *Let $1 < r < \infty$, let the constant δ be given as in Lemma 2.1, and let* $\mathbf{a}$ *be defined by* (2.3).

(1) For all $u \in H^{1+\delta,r}(\Omega)$ we have $(\mathbf{a}\cdot\nabla)u, (u\cdot\nabla)\mathbf{a} \in L^r(\Omega)$ and

$$\|(\mathbf{a}\cdot\nabla)u, (u\cdot\nabla)\mathbf{a}\|_r \le c(r,\Omega)\Phi\|u\|_{H^{1+\delta,r}}.$$

(2) Let $1 < r < \infty, r \ge \frac{n}{3}$. For all $u \in H^{2,r}(\Omega)$ we have $(\mathbf{a}\cdot\nabla)u, (u\cdot\nabla)\mathbf{a} \in H^{1-\delta,r}(\Omega)$ and

$$\|(\mathbf{a}\cdot\nabla)u, (u\cdot\nabla)\mathbf{a}\|_{H^{1-\delta,r}} \le c(r,\Omega)\Phi\|u\|_{H^{2,r}}.$$

Proof: Since Lemma 2.1 (1) holds for Ω_0 as well in place of Ω, we get by (2.4) that

$$(u\cdot\nabla)\mathbf{a} \in L^r(\Omega_0), (\mathbf{a}\cdot\nabla)u \in L^r(\Omega_0)$$

and

$$\begin{aligned}\|(u\cdot\nabla)\mathbf{a}, (\mathbf{a}\cdot\nabla)u\|_{L^r(\Omega_0)} &\le c(r,\Omega_0)\|u\|_{H^{1+\delta,r}(\Omega)}\|\mathbf{a}\|_{H^{2,r}(\Omega_0)} \\ &\le c(r,\Omega)\Phi\|u\|_{H^{1+\delta,r}(\Omega)}.\end{aligned}$$

Now it remains to show that $(\mathbf{a}\cdot\nabla)u, (u\cdot\nabla)\mathbf{a} \in L^r(\Omega\setminus\Omega_0)$ and

$$\|(\mathbf{a}\cdot\nabla)u, (u\cdot\nabla)\mathbf{a}\|_{L^r(\Omega\setminus\Omega_0)} \le c(r,\Omega)\Phi\|u\|_{H^{1+\delta,r}(\Omega)},$$

which is obvious since $\mathbf{a}|_{\Omega_i\setminus\Omega_0} = \mathbf{v}_i, i = 1,\dots,m$, due to the construction of $\mathbf{a}$ and $\mathbf{v}_i|_{\Omega_i\setminus\Omega_0}, \nabla\mathbf{v}_i|_{\Omega_i\setminus\Omega_0} \in L^\infty(\Omega_i\setminus\Omega_0), i = 1,\dots,m$ due to the Sobolev embedding $H^{1,s}(\Sigma^i) \hookrightarrow L^\infty(\Sigma^i)$ for $s > n-1$. Hence (1) is proved.

The proof of (2) is analogous to the proof of Lemma 2.1 (2) using complex interpolation and will be omitted. ■

Now we consider the system (SNS). Let $1 < r < \infty$. By the transform $v := w - \mathbf{a}$ the system (SNS) is reduced to

$$(\mathrm{SNS}')\qquad \begin{aligned} -\Delta v + (v\cdot\nabla)\mathbf{a} + (\mathbf{a}\cdot\nabla)v + (v\cdot\nabla)v + \nabla q &= F && \text{in } \Omega \\ \operatorname{div} v &= 0 && \text{in } \Omega \\ v &= 0 && \text{on } \partial\Omega \\ v(x) &= 0 && \text{at infinity,} \end{aligned} \qquad (2.13)$$

where $F = f - (\mathbf{a}\cdot\nabla)\mathbf{a}$.

It is easily seen that the reduced system (SNS′) is equivalent to

$$G_r v + P_r(v\cdot\nabla)v = P_r F; \qquad (2.14)$$

where P_r is the Helmholtz projection and the operator G_r is defined by

$$\begin{aligned} &D(G_r) = D(A_r) = H^{2,r}(\Omega) \cap H_0^{1,r}(\Omega) \cap L^r_\sigma(\Omega), \\ &G_r v := A_r v + P_r\big((v\cdot\nabla)\mathbf{a} + (\mathbf{a}\cdot\nabla)v\big), \end{aligned}$$

with the Stokes operator $A_r = -P_r\Delta$ in $L^r_\sigma(\Omega)$.

First we consider the linearization of (2.14):

$$G_r v + P_r(y\cdot\nabla)v = P_r F, \qquad (2.15)$$

for fixed $y \in D(A_r)$.

Lemma 2.3 *Let* $1 < r < \infty$, $r \geq \frac{n}{3}$. *There exists a constant* $K_0 = K_0(r, \Omega) > 0$ *such that, if* $\Phi \leq K_0$ *and* $\|y\|_{H^{2,r}(\Omega)} \leq K_0$, *then problem* (2.15) *has a unique solution* $v_y \in H^{2,r}(\Omega)$ *satisfying the estimate*

$$\|v_y\|_{H^{2,r}(\Omega)} \leq M\big(\|f\|_r + \Phi^2\big) \tag{2.16}$$

with a constant $M = M(r, \Omega) > 0$.

Proof: For $v \in H^{2,r}(\Omega)$ let

$$E_y v := P_r\big((v \cdot \nabla)\mathbf{a} + (\mathbf{a} \cdot \nabla)v + (y \cdot \nabla)v\big)$$

so that (2.15) is equivalent to $(A_r + E_y)v = P_r F$. By Lemma 2.1 and Lemma 2.2

$$\begin{aligned} \|E_y v\|_{L^r_\sigma(\Omega)} &\leq C_1(r, \Omega)\big(\Phi + \|y\|_{H^{2,r}(\Omega)}\big)\,\|v\|_{H^{2,r}(\Omega)} \\ &\leq C_2(r, \Omega)\big(\Phi + \|y\|_{H^{2,r}(\Omega)}\big)\,\|A_r v\|_r; \end{aligned}$$

here note that by [9], Theorem 1.1, $A_r^{-1} \in \mathcal{L}(L^r_\sigma(\Omega), H^{2,r}(\Omega))$. Therefore, if

$$\Phi \leq K_0 := \frac{1}{4C_2}, \quad \|y\|_{H^{2,r}(\Omega)} \leq K_0, \tag{2.17}$$

then $\|E_y A_r^{-1}\|_{\mathcal{L}(L^r_\sigma, L^r_\sigma)} \leq \frac{1}{2}$ yielding the invertibility of $I + E_y A_r^{-1}$ on $L^r_\sigma(\Omega)$. Hence

$$(A_r + E_y)^{-1} \in \mathcal{L}(L^r_\sigma(\Omega), H^{2,r}(\Omega)) \quad \text{and} \quad \|(A_r + E_y)^{-1}\|_{\mathcal{L}(L^r_\sigma, H^{2,r})} \leq M_0 \tag{2.18}$$

with some $M_0 = M_0(r, \Omega)$. Thus (2.15) has a unique solution $v_y = (A_r + E_y)^{-1}F \in H^{2,r}(\Omega)$ satisfying

$$\begin{aligned} \|v_y\|_{H^{2,r}(\Omega))} &\leq c\|F\|_r & &\leq c\big(\|f\|_r + \|(\mathbf{a} \cdot \nabla)\mathbf{a}\|_{L^r(\Omega_0)}\big) \\ &\leq c\big(\|f\|_r + \|\mathbf{a}\|^2_{H^{2,r}(\Omega_0)}\big) & &\leq c(\|f\|_r + \Phi^2) \end{aligned}$$

with $c = c(r, \Omega) > 0$, where we used $(\mathbf{a} \cdot \nabla)\mathbf{a} = 0$ in $\Omega \setminus \Omega_0$, Lemma 2.2 (1) for Ω_0 and (2.4). ∎

Now we state the theorem on the existence of solutions for (SNS).

Theorem 2.4 *Let* $1 < r < \infty$, $r \geq \frac{n}{3}$, *and let* $f \in L^r(\Omega)$. *Furthermore, let the velocity* u_∞ *at infinity for each exit* $\Omega_i, i = 1, \ldots, m$, *be the Poiseuille flow corresponding to the given flux* $\Phi_i, i = 1, \ldots, m$, *satisfying* (1.3). *Then there is a constant* $K_1 = K_1(r, \Omega) > 0$ *such that, if* $\|f\|_r + \Phi^2 < K_1$, *then (SNS) has a unique solution* $w = \mathbf{a} + v$ *satisfying* $v \in H^{2,r}(\Omega)$ *and*

$$\|v\|_{H^{2,r}(\Omega)} \leq c(r, \Omega)(\|f\|_r + \Phi^2).$$

Proof: It is enough to show the unique solvability of (SNS′) in a ball of $H^{2,r}(\Omega)$. Let K_0 be the number given by Lemma 2.3 and let

$$U_{K_0} = \big\{v \in H^{2,r}(\Omega) : \ \|v\|_{H^{2,r}} \leq K_0\big\}.$$

Assuming $\Phi < K_0$, let us define the mapping

$$\Psi : U_{K_0} \to H^{2,r}(\Omega), \quad \Psi y = v_y,$$

where v_y is the unique solution to the linearized problem (2.15). Then for $y_1, y_2 \in U_{K_0}$

$$G_r v_{y_j} + P_r(y_j \cdot \nabla)v_{y_j} = P_r(f - (\mathbf{a} \cdot \nabla)\mathbf{a}), \quad j = 1, 2,$$

which, by subtraction, yields

$$G_r(v_{y_1} - v_{y_2}) + P_r(y_1 \cdot \nabla)(v_{y_1} - v_{y_2}) = -P_r((y_1 - y_2) \cdot \nabla)v_{y_2},$$

i.e.,

$$(A_r + E_{y_1})(v_{y_1} - v_{y_2}) = -P_r((y_1 - y_2) \cdot \nabla)v_{y_2}.$$

Hence, (2.18), Lemma 2.1 (1) and (2.16) yield

$$\begin{aligned} \|v_{y_1} - v_{y_2}\|_{H^{2,r}} &\leq M_0 \|P_r((y_1 - y_2) \cdot \nabla)v_{y_2}\|_r \\ &\leq M_0 \tilde{C} \|v_{y_2}\|_{H^{2,r}} \|y_1 - y_2\|_{H^{2,r}} \\ &\leq M_0 M \tilde{C} \left(\|f\|_r + \Phi^2\right) \|y_1 - y_2\|_{H^{2,r}} \end{aligned}$$

where $\tilde{C} = \tilde{C}(r, \Omega) > 0$. Therefore, if

$$\|f\|_r + \Phi^2 < K_1 := \min\left\{\frac{1}{M_0 M \tilde{C}}, \frac{K_0}{M}, K_0^2\right\}, \tag{2.19}$$

then $\Psi(U_{K_0}) \subset U_{K_0}$ due to Lemma 2.3 and $\Psi : U_{K_0} \to U_{K_0}$ is a contraction mapping. Thus by Banach's fixed point theorem there is a unique fixed point $\tilde{y} \in U_{K_0}$ of Ψ, which implies that, if (2.19) is satisfied, (SNS$'$) has a unique solution $v = v_{\tilde{y}} \in H^{2,r}(\Omega)$. Moreover, this solution satisfies

$$\|v\|_{H^{2,r}(\Omega)} \leq M(\|f\|_r + \Phi^2) \ \ (< K_0)$$

by Lemma 2.3. ■

3 Preliminaries on Linearized Instationary Problems

In this section we discuss some preliminaries on modified instationary Stokes problems which will be used in Section 4. Let us introduce the operator

$$S_r := A_r + B_r \tag{3.1}$$

with

$$B_r u := P_r((u \cdot \nabla)w + (w \cdot \nabla)u), \tag{3.2}$$

where w is the unique solution to (SNS) given by Theorem 2.4. It is easily seen that B_r with domain

$$D(B_r) = \{u \in L^r_\sigma(\Omega) : \ (u \cdot \nabla)w + (w \cdot \nabla)u \in L^r(\Omega)\}$$

is closed. Note that (1.5) is equivalent to

$$\begin{aligned} u_t + S_r u + P_r(u\cdot\nabla)u &= 0 \\ u(0) &= u_0. \end{aligned} \tag{3.3}$$

where P_r is the L^r-Helmholtz projection.

Lemma 3.1 *Let $1 < r < \infty$ and*

$$D(\Delta_r) = H^{2,r}(\Omega)\cap H_0^{1,r}(\Omega), \quad \Delta_r u = \Delta u.$$

Then there is a continuous projection Q_r such that

$$Q_r \in \mathcal{L}(D(\Delta_r), D(A_r)) \cap \mathcal{L}(L^r(\Omega), L^r_\sigma(\Omega)).$$

Proof: This lemma can be proved in the same way as [12], Lemma 6, using that $P_r^* = P_{r'}$, $A_r^* = A_{r'}$ (see e.g. the proof of Theorem 1.1 of [9]) and $\Delta_r^* = \Delta_{r'}, r' = r/(r-1)$, for all $r \in (1,\infty)$. ■

Corollary 3.2 *Let $1 < r < \infty$, $0 < \theta < 1$. Then*

$$[L^r_\sigma(\Omega), D(A_r)]_\theta = [L^r(\Omega), H^{2,r}(\Omega)\cap H_0^{1,r}(\Omega)]_\theta \cap L^r_\sigma(\Omega).$$

In particular, if $\theta < \frac{1}{2r}$, then

$$[L^r_\sigma(\Omega), D(A_r)]_\theta = H^{2\theta,r}(\Omega)\cap L^r_\sigma(\Omega). \tag{3.4}$$

Proof: Due to Lemma 3.1 we can apply [30], Theorem 1.17.1/1, that is,

$$\begin{aligned} [L^r_\sigma(\Omega), D(A_r)]_\theta &= [L^r(\Omega)\cap L^r_\sigma(\Omega), H^{2,r}(\Omega)\cap H_0^{1,r}(\Omega)\cap L^r_\sigma(\Omega)]_\theta \\ &= [L^r(\Omega), H^{2,r}(\Omega)\cap H_0^{1,r}(\Omega)]_\theta \cap L^r_\sigma(\Omega). \end{aligned}$$

It is well known that, if $\theta < \frac{1}{2r}$, then

$$H^{2\theta,r}(\Omega) = [L^r(\Omega), H_0^{2,r}(\Omega)]_\theta = [L^r(\Omega), H^{2,r}(\Omega)]_\theta$$

yielding $H^{2\theta,r}(\Omega) = [L^r(\Omega), H^{2,r}(\Omega)\cap H_0^{1,r}(\Omega)]_\theta$, cf. [30], 4.3.2. ■

In [9], it is shown that the Stokes operator A_r for all $r \in (1,\infty)$ admits a bounded H^∞-calculus in $L^r_\sigma(\Omega)$ with H^∞-angle $\phi^\infty_{A_r} = 0$. It is known that, if a sectorial operator $\mathcal{A}$ admits a bounded H^∞-calculus in a Banach space X, then the operator $\mathcal{A}$ has bounded imaginary powers and

$$D(\mathcal{A}^\theta) = [X, D(\mathcal{A})]_\theta \quad \forall\theta \in (0,1). \tag{3.5}$$

Moreover, if the H^∞-angle $\phi^\infty_{\mathcal{A}}$ of $\mathcal{A}$ is less than $\pi/2$, then it has maximal regularity. For definitions and further results we refer to [6], [7]. In particular, the following perturbation result holds.

Theorem 3.3 ([7], *Theorem* 3.2) *Let X be a UMD space and let $\mathcal{A}$ admit a bounded H^∞-calculus in X. Let $\mathcal{B}$ be a linear operator such that $D(\mathcal{B}) \supset D(\mathcal{A})$.*

(i) Assume that there exists $\kappa > 0$ such that

$$\|\mathcal{B}u\|_X \le \kappa \|\mathcal{A}u\|_X, \quad u \in D(\mathcal{A}).$$

(ii) Suppose that there exist $\gamma \in (0,1)$ and $C > 0$ such that

$$\mathcal{B}(D(\mathcal{A}^{1+\gamma}) \subset D(\mathcal{A}^\gamma) \quad \text{and} \quad \|\mathcal{A}^\gamma \mathcal{B} u\|_X \le C\|\mathcal{A}^{1+\gamma}u\|_X \quad \forall u \in D(\mathcal{A}^{1+\gamma}).$$

Then $\mathcal{A}+\mathcal{B}$ admits a bounded H^∞-calculus provided κ is sufficiently small. Moreover, for each $\phi > \phi^\infty_{\mathcal{A}}$ there is $\kappa_0(\phi) > 0$ such that $\phi^\infty_{\mathcal{A}+\mathcal{B}} \le \phi$ if $\kappa < \kappa_0(\phi)$.

To apply Theorem 3.3 to S_r we need the following lemma.

Lemma 3.4 *Let $1 < r < \infty$ and let the assumption of Theorem 2.4 be satisfied.*

(1) For all $u \in H^{1+\delta,r}(\Omega) \cap L^r_\sigma(\Omega)$

$$\|B_r u\|_r \le c(r,\Omega)(\|f\|_r + \Phi + \Phi^2)\|u\|_{H^{1+\delta,r}(\Omega)}. \tag{3.6}$$

(2) Let $r \ge \frac{n}{3}$. For $\alpha \in (0,\bar{\alpha})$ we have $-\alpha + \Sigma_\varepsilon \subset \rho(-S_r)$ and

$$\|(\lambda + S_r)^{-1}\|_{\mathcal{L}(L^r_\sigma(\Omega))} \le \frac{C}{|\lambda+\alpha|} \tag{3.7}$$

with some constant $C = C(r,\Omega,\alpha,\varepsilon) > 0$. In particular, $-S_r$ generates an analytic semigroup $\{e^{-tS_r}\}_{t\ge 0}$ satisfying the estimate

$$\|e^{-tS_r}\|_{\mathcal{L}(L^r_\sigma(\Omega))} \le Ce^{-\alpha t} \quad \forall t > 0 \tag{3.8}$$

with some constant $C = C(r,\Omega,\alpha) > 0$.

Proof: (1) Since $w = v + \mathbf{a}$, Lemma 2.1 (1), Lemma 2.2 (1) and Theorem 2.4 yield that

$$\begin{aligned} \|B_r u\|_{L^r_\sigma} &\le c(r,\Omega)\big(\|(v\cdot\nabla)u\|_r + \|(u\cdot\nabla)v\|_r + \|(\mathbf{a}\cdot\nabla)u\|_r + \|(u\cdot\nabla)\mathbf{a}\|_r\big) \\ &\le c(r,\Omega)(\|v\|_{H^{2,r}(\Omega)} + \Phi)\|u\|_{H^{1+\delta,r}(\Omega)} \\ &\le c(r,\Omega)(\|f\|_r + \Phi + \Phi^2)\|u\|_{H^{1+\delta,r}(\Omega)} \end{aligned}$$

for all $u \in H^{1+\delta,r}(\Omega)$.

(2) In [9], Theorem 1.1, it was shown that for any $r \in (1,\infty)$, $\alpha \in (0,\bar{\alpha})$ (see (1.6) for $\bar{\alpha}$) and $\varepsilon \in (\pi/2,\pi)$

$$\|u\|_{H^{2,r}(\Omega)} \le c(r,\Omega,\alpha,\varepsilon)\|(\lambda + A_r)u\|_{L^r_\sigma(\Omega)} \quad \forall u \in D(A_r)\ \forall \lambda \in -\alpha + \Sigma_\varepsilon.$$

This inequality together with (3.6) where $\delta \le 1$ since $r \ge \frac{n}{3}$ yields the assertions, if $\|f\|_r + \Phi + \Phi^2$ is small enough. ■

In the next theorem we shall show that the operator S_r, $r \in (1,\infty)$, $r > \frac{n}{3}$, $n \ge 3$, admits a bounded H^∞-calculus in $L^r_\sigma(\Omega)$ under smallness conditions on f and Φ. Note that $L^r(\Omega)$ and $L^r_\sigma(\Omega)$ are UMD spaces, see e.g. [1].

Theorem 3.5 *Let $r > \frac{n}{3}$ and let $w = v + \mathbf{a}$ be the solution to (SNS) given by Theorem 2.4. There is a constant $K_2 = K_2(r, \Omega) > 0$ such that if $\|f\|_r + \Phi + \Phi^2 < K_2$, then the operator S_r admits a bounded H^∞-calculus with H^∞-angle less than $\pi/2$ in $L^r_\sigma(\Omega)$. Moreover, the adjoint operator $S^*_{r'}$ of S_r in $L^{r'}_\sigma(\Omega)$ has a bounded H^∞-calculus with H^∞-angle less than $\pi/2$ as well.*

Proof: Based on the fact that the Stokes operator A_r admits a bounded H^∞-calculus with H^∞-angle 0 in $L^r_\sigma(\Omega)$, see [9], Theorem 1.2, we shall use the perturbation theorem 3.3. Hence, let us show that the operator B_r given by (3.2) satisfies the assumptions (i), (ii) of Theorem 3.3 with $\mathcal{A} = A_r$, $\mathcal{B} = B_r$. By Lemma 3.4 for all $u \in D(A_r), r \in (1, \infty)$,

$$\|B_r u\|_{L^r_\sigma(\Omega)} \leq c(r, \Omega)(\|f\|_r + \Phi + \Phi^2)\|u\|_{H^{2,r}(\Omega)}$$

proving (i) of Theorem 3.3.

In view of $w = v + \mathbf{a}$, Lemma 2.1 (2), Lemma 2.2 (2) and Theorem 2.4 yield

$$\begin{aligned} \|B_r u\|_{H^{1-\delta,r}(\Omega)} &\leq c(r, \Omega)\|(u \cdot \nabla)w + (w \cdot \nabla)u\|_{H^{1-\delta,r}(\Omega)} \\ &\leq c(r, \Omega)(\|v\|_{H^{2,r}(\Omega)} + \Phi)\|u\|_{H^{2,r}(\Omega)} \\ &\leq c(r, \Omega)(\|f\|_r + \Phi + \Phi^2)\|u\|_{H^{2,r}(\Omega)} \end{aligned} \tag{3.9}$$

for all $u \in D(A_r)$. Note that for $\gamma \in (0, 1)$ the complex interpolation space $[L^r_\sigma(\Omega), D(A_r)]_\gamma$ coincides with the domain $D(A^\gamma_r)$ of A^γ_r since A_r has bounded imaginary powers, cf. [30], Theorem 1.15.3. Therefore, by (3.4), (3.9) we get that if $0 < \gamma < \min\{\frac{1-\delta}{2}, \frac{1}{2r}\}$, then $B_r u \in D(A^\gamma_r)$ for all $u \in D(A_r)$ and

$$\|A^\gamma_r B_r u\|_{L^r_\sigma(\Omega)} \leq c(\delta, \gamma, r, \Omega)\|B_r u\|_{H^{1-\delta,r}(\Omega)} \leq c(\delta, \gamma, r, \Omega)(\|f\|_r + \Phi + \Phi^2)\|A_r u\|_{L^r_\sigma(\Omega)}$$

which is an even stronger estimate than needed in Theorem 3.3 (ii). Now fix suitable δ, γ depending on r, n. Thus Theorem 3.3 implies that there is a sufficiently small number K_2 depending only on r, Ω such that, if $\|f\|_r + \Phi + \Phi^2 < K_2$, then $S_r = A_r + B_r$ admits a bounded H^∞-calculus in $L^r_\sigma(\Omega)$ with H^∞-angle less than $\pi/2$.

Finally [6], Proposition 2.11, proves the assertion on the adjoint operator $S^*_{r'}$. ■

As important consequences of Theorem 3.5 we characterize domains of fractional powers of S_r and get maximal regularity for the linearization of (1.5), cf. [6].

Proposition 3.6 *Let $r > \frac{n}{3}$. If $\|f\|_r + \Phi + \Phi^2$ is small enough depending on r, δ, Ω, for $\theta \in (0, 1)$ we have $D(S^\theta_r) = D(A^\theta_r)$. In particular,*

$$D(S^\theta_r) = [L^r(\Omega), H^{2,r}(\Omega) \cap H^{1,r}_0(\Omega)]_\theta \cap L^r_\sigma(\Omega) \tag{3.10}$$

with equivalent norms, and

$$\|u\|_{H^{2\theta,r}(\Omega)} \leq C\|S^\theta_r u\|_{L^r_\sigma(\Omega)} \quad \forall u \in D(S^\theta_r) \tag{3.11}$$

with $C = C(r, \theta, \Omega) > 0$. Moreover, for $\theta < \frac{1}{2r}$, the norms $\|\cdot\|_{D(S^\theta_r)} = \|\cdot\|_{H^{2\theta,r}(\Omega)}$ are equivalent.

Proof: By Theorem 3.5 S_r has bounded imaginary powers so that (3.5) applies. Hence by Corollary 3.2

$$D(S_\theta^r) = [L_r^\sigma(\Omega), D(S_r)]_\theta = [L_r^\sigma(\Omega), D(A_r)]_\theta = [L^r(\Omega), H^{2,r}(\Omega) \cap H_0^{1,r}(\Omega)]_\theta \cap L_\sigma^r(\Omega).$$

For $\theta < \frac{1}{2r}$ we additionally use (3.4). ■

Proposition 3.7 *Let $1 < p < \infty$, $\frac{n}{3} < r < \infty$, and let the smallness assumptions of Theorem 3.5 be satisfied. Furthermore, let $h \in L^p(0,\infty; L^r(\Omega))$ and $u_0 \in D(A_r)$. Then the linear system*

$$\begin{aligned} u_t - \Delta u + (u\cdot\nabla)w + (w\cdot\nabla)u + \nabla p &= h \quad \text{in } \Omega\times(0,\infty) \\ \operatorname{div} u &= 0 \quad \text{in } \Omega\times(0,\infty) \\ u &= 0 \quad \text{on } \partial\Omega\times(0,\infty) \\ u(0) &= u_0 \quad \text{in } \Omega, \end{aligned}$$

where $w \in H^{2,r}(\Omega)$ is the solution to (SNS) given by Theorem 2.4, has a unique solution

$$u \in L^p(0,\infty; H^{2,r}(\Omega)), u_t \in L^p(0,\infty; L_\sigma^r(\Omega))$$

satisfying

$$\|u\|_{L^p(0,\infty;H^{2,r}(\Omega))} + \|u_t\|_{L^p(0,\infty;L_\sigma^r(\Omega))} \le c(\|h\|_{L^p(0,\infty;L_\sigma^r(\Omega))} + \|u_0\|_{D(A_r)}).$$

Let us have a closer look at the adjoint operator $S_{r'}^*$ of S_r in $L_\sigma^r(\Omega)$ and characterize the domains of its fractional powers. Note that for all $\varphi \in C_{0,\sigma}^\infty(\Omega)$

$$\begin{aligned} (B_r u, \varphi)_{L^r, L^{r'}} &= \int_\Omega \big((w\cdot\nabla)u + (u\cdot\nabla)w\big)\cdot\varphi\,dx \\ &= -\int_\Omega \Big((w\cdot\nabla)\varphi + \sum_{j=1}^n w_j\nabla\varphi_j\Big)\cdot u\,dx, \end{aligned}$$

where we used that $\operatorname{div} w = \operatorname{div} v + \operatorname{div}\mathbf{a} = 0$. Let us prove that, if $r > \max\{\frac{n}{3}, \frac{2n}{n+2}\}$, then $(w\cdot\nabla)\varphi + \sum_{j=1}^n w_j\nabla\varphi_j \in L^{r'}(\Omega)$ and

$$\|(w\cdot\nabla)\varphi + \textstyle\sum_{j=1}^n w_j\nabla\varphi_j\|_{r'} \le c\big(\|v\|_{H^{2,r}(\Omega)} + \|\mathbf{a}\|_{L^\infty(\Omega)}\big)\|\varphi\|_{H^{1+\delta,r'}(\Omega)}, \qquad (3.12)$$

where $\delta \in [0,1)$ is given by (2.6). In fact, if $\max\{\frac{n}{3}, \frac{2n}{n+2}\} < r < \frac{n}{2}$, then $H^{2,r}(\Omega) \hookrightarrow L^{\frac{nr}{n-2r}}(\Omega)$, $\frac{nr}{n-2r} > r'$ and there exists $s > 1$ such that $\frac{n-2r}{nr} + \frac{1}{s} = \frac{1}{r'}$, $H^{\delta,r'}\Omega) = H^{\frac{n}{r}-2,r'}(\Omega) \hookrightarrow L^s(\Omega)$. Hence

$$\|(v\cdot\nabla)\varphi\|_{r'} \le \|v\|_{\frac{nr}{n-2r}}\|\nabla\varphi\|_s \le c\|v\|_{H^{2,r}}\|\nabla\varphi\|_{H^{\delta,r'}};$$

in the case $r \ge \frac{n}{2}$ the inequality $\|(v\cdot\nabla)\varphi\|_{r'} \le c\|v\|_{H^{2,r}}\|\nabla\varphi\|_{H^{\delta,r'}}$ can be proved in a similar way as in the proof Lemma 2.1. The remaining estimate for $(\mathbf{a}\cdot\nabla)\varphi$ is trivial since $\mathbf{a} \in L^\infty(\Omega)$ (see (2.5)).

Let $B^*_{r'}$ denote the adjoint of the (closed) operator B_r in $L^{r'}_\sigma(\Omega)$. Then (3.12) and the embedding $D(A^{(1+\delta)/2}_{r'}) \subset H^{1+\delta,r'}(\Omega)$ imply that

$$D(A^{\frac{1+\delta}{2}}_{r'}) \subset D(B^*_{r'}) \tag{3.13}$$

with $B^*_{r'}\varphi = -(w\cdot\nabla)\varphi - \sum_{j=1}^n w_j \nabla\varphi_j$ for $\varphi \in D(A^{\frac{1+\delta}{2}}_{r'})$ and

$$\begin{aligned} \|B^*_{r'}\varphi\|_{L^{r'}_\sigma(\Omega)} &\le c(r,\delta,\Omega)\big(\|v\|_{H^{2,r}(\Omega)} + \|\mathbf{a}\|_{L^\infty(\Omega)}\big)\|\varphi\|_{H^{1+\delta,r'}(\Omega)} \\ &\le c(r,\delta,\Omega)\big(\|f\|_r + \Phi + \Phi^2\big)\|\varphi\|_{D(A^{(1+\delta)/2}_{r'})}. \end{aligned} \tag{3.14}$$

Since $L^r_\sigma(\Omega)$ is reflexive, also $S^*_{r'} = A_{r'} + B^*_{r'}$ generates a bounded analytic semigroup in $L^{r'}_\sigma(\Omega)$, see [23], Ch. 1, Corollary 10.6. Note that (3.14) and an interpolation inequality ([23], Ch. 2, Theorem 6.10) imply the $A_{r'}$-boundedness of $B^*_{r'}$ with $A_{r'}$-bound less than 1. Hence $A_{r'}+B^*_{r'}$ is closed and $D(A_{r'}+B^*_{r'}) = D(A_{r'})$, see [13], Ch. IV, Theorem 1.1. Moreover, (3.14) shows that $A_{r'}+B^*_{r'}$ is invertible if $\|f\|_r$ and Φ are sufficiently small. Since it is easily seen that $A_{r'}+B^*_{r'} \subset S^*_{r'}$ and since both operators $A_{r'}+B^*_{r'}$ and $S^*_{r'}$ are invertible, we conclude that $D(A_{r'}) = D(S^*_{r'})$. Now Theorem 3.5 and (3.5) imply for all $\theta\in[0,1]$ that

$$D((S^*_{r'})^\theta) = [L^{r'}_\sigma(\Omega), D(S^*_{r'})]_\theta = [L^{r'}_\sigma(\Omega), D(A_{r'})]_\theta = D(A^\theta_{r'}). \tag{3.15}$$

In particular, for all $r > \max\left\{\frac{n}{3}, \frac{2n}{n+2}\right\}$ and $\theta \in (0,1)$

$$\|u\|_{H^{2\theta,r'}(\Omega)} \le c(r,\theta,\Omega)\|(S^*_{r'})^\theta u\|_{L^{r'}_\sigma(\Omega)} \quad \forall u \in D((S^*_{r'})^\theta) = D(A^\theta_{r'}). \tag{3.16}$$

In the remainder of this paper we shall assume that the constant K_2 in Theorem 3.5 is so small that (3.15), (3.16) hold as well.

Remark 3.8 If $\max\left\{\frac{n}{3}, \frac{2n}{n+2}\right\} < r < q < \infty$, then obviously for all $t > 0$

$$e^{-tS^*_{r'}}\varphi = e^{-tS^*_{q'}}\varphi \quad \forall\varphi \in C^\infty_{0,\sigma}(\Omega).$$

Therefore, we shall write $e^{-tS^*}\varphi$ for $e^{-tS^*_{r'}}\varphi$ in the following.

Lemma 3.9 *(L^r-L^q estimates) Let $\frac{n}{2} < r < q < \infty, n \ge 3$ and let $\alpha \in (0,\bar\alpha)$ be fixed, where $\bar\alpha$ is given by (1.6). Then the following estimates hold for all $u \in L^r_\sigma(\Omega)$ and $t > 0$:*

(1) $\|e^{-tS_r}u\|_q \le c(r,q,\alpha,\Omega)\, t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\, e^{-\alpha t}\|u\|_r.$

(2) $\|\nabla S^\beta_r e^{-tS_r}u\|_q \le c(r,q,\alpha,\beta,\Omega)\, t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}-\beta}\, e^{-\alpha t}\|u\|_r \quad \forall\beta\in(0,\frac{1}{2}).$

For all $\varphi \in L^{r'}_\sigma(\Omega)$ and $\xi > r'$ the following estimates hold:

(1′) $\|e^{-tS^*_{r'}}\varphi\|_\xi \le c(r,\xi,\alpha,\Omega)\, t^{-\frac{n}{2}(\frac{1}{r'}-\frac{1}{\xi})}\, e^{-\alpha t}\|\varphi\|_{r'}.$

(2′) $\|\nabla(S^*_{r'})^\beta e^{-tS^*_{r'}}\varphi\|_\xi \le c(r,\xi,\alpha,\beta,\Omega)\, t^{-\frac{n}{2}(\frac{1}{r'}-\frac{1}{\xi})-\frac{1}{2}-\beta}\, e^{-\alpha t}\|\varphi\|_{r'} \quad \forall\beta\in(0,\frac{1}{2}).$

Proof: First let us prove (1). Let $\gamma = \frac{n}{2}(\frac{1}{r} - \frac{1}{q})$. Obviously, $\gamma \in (0,1)$. By the embedding $H^{2\gamma,r}(\Omega) \hookrightarrow L^q(\Omega)$ and (3.11) we get for all $u \in L^r_\sigma(\Omega)$ that

$$\begin{aligned}
\|e^{-tS_r}u\|_q &\leq c_1(r,q,\Omega)\|e^{-tS_r}u\|_{H^{2\gamma,r}} \\
&\leq c_2(r,q,\Omega)\|S_r^\gamma e^{-tS_r}u\|_{L^r_\sigma} \\
&= c_2(r,q,\Omega)\|S_r^\gamma e^{-\frac{\bar\alpha-\alpha}{\bar\alpha+\alpha}tS_r}\, e^{-\frac{2\alpha}{\bar\alpha+\alpha}tS_r}u\|_{L^r} \\
&\leq c_3(r,\alpha,q,\Omega)\, t^{-\gamma}\, \|e^{-\frac{2\alpha}{\bar\alpha+\alpha}tS_r}u\|_{L^r},
\end{aligned}$$

where we used the well-known estimate $\|S_r^\theta e^{-tS_r}\|_{\mathcal{L}(L^r_\sigma(\Omega))} \leq c(r,\theta,\Omega)t^{-\theta}$ for $\theta \in (0,1)$, $t>0$, for analytic semigroups. Thus by (3.8) with α replaced by $\frac{\bar\alpha+\alpha}{2}$ we get (1).

The assertion (2) can be proved in a similar way as (1) using additionally that

$$\|\nabla u\|_q \leq c\|u\|_{H^{1,q}} \leq c\|S_r^{1/2}u\|_q \quad \text{for all} \quad u \in C^\infty_{0,\sigma}(\Omega).$$

Using (3.16) the proofs of (1′) and (2′) are similar and are omitted. ■

4 Stability of the Stationary Navier Stokes Flows

In this section we fix $r \in [n,\infty)$ and an initial value $u_0 \in L^r_\sigma(\Omega)$.

Definition 4.1 *A function u is called a strong solution to* (1.5) *on* $[0,T)$, $0 < T \leq \infty$, *if*

$$u \in BC([0,T), L^r_\sigma(\Omega)) \cap C^1((0,T), L^r_\sigma(\Omega)) \cap C((0,T), D(A_r)) \tag{4.1}$$

and u satisfies (1.5) *pointwise in* $t \in (0,T)$.

Remark 4.2 *Due to the Sobolev embedding $H^{2,r}(\Omega) \hookrightarrow L^q(\Omega)$ for $q \geq r$, any strong solution to* (1.5) *on* $(0,T)$ *belongs to* $C((0,T), L^q(\Omega))$ *for any* $q \geq r$.

If u is a strong solution to (1.5), then u satisfies the integral equation

$$u(t) = e^{-tS_r}u_0 - \int_0^t e^{-(t-s)S_r}P_r(u\cdot\nabla)u(s)\,ds,\ \ t \in (0,T), \tag{4.2}$$

hence, in consideration of Remark 3.8, for all $\varphi \in C^\infty_{0,\sigma}(\Omega)$ and $t \in (0,T)$

$$(u(t),\varphi) = (e^{-tS_r}u_0,\varphi) + \int_0^t \big((u\cdot\nabla)e^{-(t-s)S^*}\varphi, u(s)\big)\,ds. \tag{4.3}$$

For $r < q < \infty$ and $\alpha \in [\frac{\bar\alpha}{2},\bar\alpha)$ let

$$\begin{aligned}
X_q(\alpha) := \{u : {}& e^{\alpha t}u \in BC([0,\infty), L^r_\sigma(\Omega)), \\
& t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}e^{\alpha t}u \in BC((0,\infty), L^q_\sigma(\Omega)),\ \lim_{t\to+0} t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|u(t)\|_q = 0\},
\end{aligned}$$

$$\|u\|_{X_q(\alpha)} = \|e^{\alpha t}u\|_{BC([0,\infty),L^r_\sigma(\Omega))} + \|t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}e^{\alpha t}u\|_{BC((0,\infty),L^q_\sigma(\Omega))}.$$

Obviously, $X_q(\alpha)$ is a Banach space. Moreover, $X_{q_1}(\alpha) \hookrightarrow X_{q_2}(\alpha)$ for $q_1 \geq q_2$ and

$$X_q(\alpha_1) \hookrightarrow X_q(\alpha_2) \quad \text{for } \alpha_1 > \alpha_2. \tag{4.4}$$

Definition 4.3 *A function u belonging to $X_q(\alpha)$ for any $q > r, \alpha \in [\frac{\bar\alpha}{2}, \bar\alpha)$ and satisfying* (4.3) *for all $t \in (0,\infty)$ is called a global mild solution to* (1.5)*.*

For each $u, z \in X_q$ define the functional $F(u,z)(t), t \geq 0$, on $C^\infty_{0,\sigma}(\Omega)$ by

$$\langle F(u,z)(t), \varphi\rangle = \int_0^t \big((u(s)\cdot\nabla)e^{-(t-s)S^*}\varphi, z(s)\big)\,ds. \tag{4.5}$$

Then (4.3) can be rewritten formally as

$$u(t) = e^{-tS_r}u_0 + F(u,u)(t), \quad t > 0. \tag{4.6}$$

Lemma 4.4 *Let $n \leq r < q < \infty$ and $\alpha \in [\bar\alpha/2, \bar\alpha)$.*

(1) The operator $F(\cdot,\cdot)$ is a bilinear continuous mapping from $X_q(\alpha)\times X_q(\alpha)$ to $X_q(\alpha)$, i.e.,

$$\|F(u,z)\|_{X_q(\alpha)} \leq c\|u\|_{X_q(\alpha)}\|z\|_{X_q(\alpha)} \quad \forall u, z \in X_q(\alpha)$$

with $c = c(r,q,\alpha,\Omega) > 0$.

(2) For all $q \in (r,\infty)$ the operator $F(\cdot,\cdot)$ is a bilinear continuous mapping from $X_{2r}(\alpha)\times X_{2r}(\alpha)$ to $X_q(\alpha)$, i.e.,

$$\|F(u,z)\|_{X_q(\alpha)} \leq c\|u\|_{X_{2r}(\alpha)}\|z\|_{X_{2r}(\alpha)}$$

with $c = c(r,q,\alpha,\Omega) > 0$ for all $u, z \in X_{2r}(\alpha)$.

Proof: (1) For simplicity we write $X_q = X_q(\alpha)$ and $\gamma = \frac{n}{2}\big(\frac1r - \frac1q\big) \in \big(0,\frac12\big)$. For $u, z \in X_q$ and $\varphi \in C^\infty_{0,\sigma}(\Omega)$

$$\begin{aligned} |\langle F(u,z)(t),\varphi\rangle| &\leq \int_0^t \|u(s)\|_r \|\nabla e^{-(t-s)S^*}\varphi\|_\xi \|z(s)\|_q\,ds \\ &\leq \sup_{0<s<t}\big\{e^{\alpha s}\|u(s)\|_r\big\}\cdot\sup_{0<s<t}\big\{s^\gamma e^{\alpha s}\|z(s)\|_q\big\} \\ &\quad\times \int_0^t s^{-\gamma}e^{-2\alpha s}\|\nabla e^{-(t-s)S^*}\varphi\|_\xi\,ds, \end{aligned} \tag{4.7}$$

for all $t > 0$, where $\frac1\xi = 1 - \frac1r - \frac1q$. By Lemma 3.9 ($2'$) with α replaced by $\frac{\bar\alpha+\alpha}{2}$

$$\|\nabla e^{-(t-s)S^*}\varphi\|_\xi \leq c(t-s)^{-\frac n2(\frac{1}{r'}-\frac1\xi)-\frac12}e^{-\frac{\bar\alpha+\alpha}{2}(t-s)}\|\varphi\|_{r'} = c(t-s)^{-\frac{n}{2q}-\frac12}e^{-\frac{\bar\alpha+\alpha}{2}(t-s)}\|\varphi\|_{r'}$$

with $c = c(r,q,\alpha,\Omega) > 0$. Hence (4.7) yields for all $t > 0$ that

$$|\langle F(u,z)(t),\varphi\rangle| \leq c\sup_{0<s<t}\big\{e^{\alpha s}\|u(s)\|_r\big\}\cdot\sup_{0<s<t}\big\{s^\gamma e^{\alpha s}\|z(s)\|_q\big\}e^{-\alpha t}\cdot I_1(t)\|\varphi\|_{r'},$$

where

$$\begin{aligned} I_1(t) &= e^{-\frac{\bar\alpha-\alpha}{2}t}\int_0^t s^{-\gamma}(t-s)^{-\frac{n}{2q}-\frac12}e^{-2\alpha+\frac{\bar\alpha+\alpha}{2}s}\,ds \\ &= e^{-\frac{\bar\alpha-\alpha}{2}t}t^{\frac12-\frac{n}{2r}}\int_0^1 \tau^{-\gamma}(1-\tau)^{-\frac{n}{2q}-\frac12}\,d\tau \\ &\leq cB(1-\gamma, \tfrac12 - \tfrac{n}{2q}) \end{aligned}$$

and $B(\cdot,\cdot)$ denotes the Beta function; note here that $-2\alpha + \frac{\bar\alpha+\alpha}{2} < 0$ for all $\alpha \in [\bar\alpha/2, \bar\alpha)$. Therefore, for $u, z \in X_q$ we have $F(u,z)(t) \in L^r_\sigma(\Omega)$ for all $t > 0$ and

$$e^{\alpha t}\|F(u,z)(t)\|_r \le c \sup_{0\le s\le t}\big\{e^{\alpha s}\|u(s)\|_r\big\} \cdot \sup_{0<s\le t}\big\{s^\gamma e^{\alpha s}\|z(s)\|_q\big\}, \tag{4.8}$$

where $c = c(r, q, \alpha, \Omega) > 0$.

Furthermore, for $u, z \in X_q$ we have

$$F(u,z) \in BC([0,\infty), L^r_\sigma(\Omega)), \tag{4.9}$$

since $t \to F(u,z)(t)$ is continuous from $[0,\infty)$ to $L^r_\sigma(\Omega)$. In fact, $t \mapsto F(u,z)(t)$ is continuous at $t = 0$ in $L^r_\sigma(\Omega)$ due to (4.8). Moreover, for $t_1, t_2 \in (0,\infty), t_1 > t_2$,

$$\begin{aligned} |\langle F(u,z)(t_1) - F(u,z)(t_2), \varphi\rangle| = \Big| &\int_{t_2}^{t_1} \big((u(s)\cdot\nabla)e^{-(t_1-s)S^*}\varphi,\, z(s)\big)\,ds \\ &+ \int_0^{t_2} \big((u(s)\cdot\nabla)(e^{-(t_1-t_2)S^*} - I)e^{-(t_2-s)S^*})\varphi,\, z(s)\big)\,ds\Big| \end{aligned} \tag{4.10}$$

Then, by the same technique as in the proof of (4.8),

$$\begin{aligned} &\Big|\int_{t_2}^{t_1} \big((u(s)\cdot\nabla)e^{-(t_1-s)S^*}\varphi,\, z(s)\big)\,ds\Big| \\ &\qquad \le c\|u\|_{X_q}\|z\|_{X_q}\int_{t_2}^{t_1} s^{-\gamma}(t_1-s)^{-\frac{n}{2q}-\frac12}\,ds\,\|\varphi\|_{r'} \end{aligned} \tag{4.11}$$

where

$$\int_{t_2}^{t_1} s^{-\gamma}(t_1-s)^{-\frac{n}{2q}-\frac12}\,ds \le c\,t_2^{-\gamma}(t_1-t_2)^{\frac12-\frac{n}{2q}} \to 0$$

as $t_1 \to t_2$ or $t_2 \to t_1$. Moreover, we have

$$\begin{aligned} &\Big|\int_0^{t_2} \big((u(s)\cdot\nabla)(e^{-(t_1-t_2)S^*} - I)e^{-(t_2-s)S^*}\varphi,\, z(s)\big)\,ds\Big| \\ &\qquad \le \|u\|_{X_q}\|z\|_{X_q}\int_0^{t_2} s^{-\gamma}\|\nabla(e^{-(t_1-t_2)S^*} - I)e^{-(t_2-s)S^*}\varphi\|_\xi\,ds \end{aligned} \tag{4.12}$$

where $\frac{1}{\xi} = 1 - \frac1r - \frac1q$. Note that $0 \in \rho(S^*_{r'})$, and by [23], Ch. 2, Theorem 6.13 (d), and Lemma 3.9 ($2'$)

$$\begin{aligned} \|\nabla(e^{-(t_1-t_2)S^*} - I)e^{-(t_2-s)S^*}\varphi\|_\xi &= \|\nabla e^{-\frac{t_2-s}{2}S^*}(e^{-(t_1-t_2)S^*} - I)e^{-\frac{t_2-s}{2}S^*}\varphi\|_\xi \\ &\le c(t_2-s)^{-\frac{n}{2q}-\frac12}\|(e^{-(t_1-t_2)S^*} - I)e^{-\frac{t_2-s}{2}S^*}\varphi\|_{r'} \\ &\le c_\zeta(t_2-s)^{-\frac{n}{2q}-\frac12}(t_1-t_2)^\zeta\|(S^*)^\zeta e^{-\frac{t_2-s}{2}S^*}\varphi\|_{r'} \\ &\le c_\zeta(t_2-s)^{-\frac{n}{2q}-\frac12-\zeta}(t_1-t_2)^\zeta\|\varphi\|_{r'}, \end{aligned}$$

where ζ is arbitrarily fixed in $(0, \frac12 - \frac{n}{2q})$. Thus, from (4.12) we get

$$\begin{aligned} &\Big|\int_0^{t_2} \big((u(s)\cdot\nabla)(e^{-(t_1-t_2)S^*} - I)e^{-(t_2-s)S^*}\varphi,\, z(s)\big)\,ds\Big| \\ &\qquad \le c_\zeta(t_1-t_2)^\zeta\|u\|_{X_q}\|z\|_{X_q}\int_0^{t_2} s^{-\gamma}(t_2-s)^{-\frac{n}{2q}-\frac12-\zeta}\,ds\,\|\varphi\|_{r'} \\ &\qquad \le \tilde c_\zeta(t_2)(t_1-t_2)^\zeta\|u\|_{X_q}\|z\|_{X_q}\|\varphi\|_{r'}, \end{aligned}$$

which together with (4.10), (4.11) and (4.12) implies that the function $t \mapsto F(u,z)(t)$ is continuous from $(0,\infty)$ to $L^r_\sigma(\Omega)$.

By a similar technique as in the proof of (4.8) we get for all $t>0$ that

$$\begin{aligned} |\langle F(u,z)(t),\varphi\rangle| &\leq \int_0^t \|u(s)\|_q \|\nabla e^{-(t-s)S^*}\varphi\|_{q/(q-2)} \|z(s)\|_q \, ds \\ &\leq c \sup_{0<s\leq t}\left\{s^\gamma e^{\alpha s}\|u(s)\|_q\right\} \cdot \sup_{0<s\leq t}\left\{s^\gamma e^{\alpha s}\|z(s)\|_q\right\} \\ &\qquad \cdot e^{-\frac{\bar\alpha+\alpha}{2}t} \int_0^t s^{-2\gamma}(t-s)^{-\frac{n}{2q}-\frac12}\, ds \|\varphi\|_{q'}. \end{aligned}$$

Hence for all $t>0$ we have $F(u,z)(t)\in L^q_\sigma(\Omega)$ and

$$\begin{aligned} \|F(u,z)(t)\|_q \leq c \sup_{0<s\leq t}\left\{s^\gamma e^{\alpha s}\|u(s)\|_q\right\} \cdot \sup_{0<s\leq t}\left\{s^\gamma e^{\alpha s}\|z(s)\|_q\right\} \\ \cdot t^{-\frac{n}{r}+\frac{n}{2q}+\frac12} e^{-\frac{\bar\alpha+\alpha}{2}t} \int_0^1 \tau^{-2\gamma}(1-\tau)^{-\frac{n}{2q}-\frac12}\, d\tau, \end{aligned}$$

yielding

$$\begin{aligned} t^\gamma e^{\alpha t}\|F(u,z)(t)\|_q &\leq c_1 \sup_{0<s\leq t}\left\{s^\gamma e^{\alpha s}\|u(s)\|_q\right\} \cdot \sup_{0<s\leq t}\left\{s^\gamma e^{\alpha s}\|z(s)\|_q\right\} t^{\frac12-\frac{n}{2r}} e^{-\frac{\bar\alpha-\alpha}{2}t} \\ &\leq c_2 \sup_{0<s\leq t}\left\{s^\gamma e^{\alpha s}\|u(s)\|_q\right\} \cdot \sup_{0<s\leq t}\left\{s^\gamma e^{\alpha s}\|z(s)\|_q\right\}, \end{aligned} \tag{4.13}$$

where $c_i = c_i(r,q,\alpha,\Omega)>0, i=1,2$. In particular, (4.13) implies that

$$\lim_{t\to 0} t^\gamma \|F(u,z)(t)\|_q = 0.$$

Moreover, by a similar argument as in the proof of (4.9) we get that

$$t^\gamma e^{\alpha t} F(u,z)(t) \in BC((0,\infty), L^q(\Omega)).$$

Thus we proved (1).

(2) For $u,z\in X_{2r}$ and $\varphi\in C^\infty_{0,\sigma}(\Omega)$ we get

$$|\langle F(u,z)(t),\varphi\rangle| \leq \int_0^t \|u(s)\|_{2r}\|z(s)\|_{2r}\|\nabla e^{-(t-s)S^*}\varphi\|_{\frac{r}{r-1}}\, ds \quad \forall t>0. \tag{4.14}$$

By Lemma 3.9 (2′)

$$\|\nabla e^{-(t-s)S^*}\varphi\|_{\frac{r}{r-1}} \leq c(r,q,\alpha,\Omega)\,(t-s)^{-\gamma-\frac12}\, e^{-\frac{\bar\alpha+\alpha}{2}(t-s)}\|\varphi\|_{q'}.$$

Hence (4.14) yields for all $t>0$ that

$$\begin{aligned} |\langle F(u,z)(t),\varphi\rangle| \leq c \sup_{0<s\leq t}\left\{s^{\frac{n}{4r}} e^{\alpha s}\|u(s)\|_{2r}\right\} \cdot \sup_{0<s\leq t}\left\{s^{\frac{n}{4r}} e^{\alpha s}\|z(s)\|_{2r}\right\} \\ \cdot e^{-\frac{\bar\alpha+\alpha}{2}t} \int_0^t s^{-\frac{n}{2r}}(t-s)^{-\gamma-\frac12}\, ds\, \|\varphi\|_{q'} \\ \leq c \sup_{0<s\leq t}\left\{s^{\frac{n}{4r}} e^{\alpha s}\|u(s)\|_{2r}\right\} \cdot \sup_{0<s\leq t}\left\{s^{\frac{n}{4r}} e^{\alpha s}\|z(s)\|_{2r}\right\} t^{-\gamma} e^{-\alpha t} I_2(t)\|\varphi\|_{q'}, \end{aligned}$$

where

$$I_2(t) = t^{\frac{1}{2}-\frac{n}{2r}} e^{-\frac{\bar\alpha-\alpha}{2}t} \int_0^1 \tau^{-\frac{n}{2r}}(1-\tau)^{-\gamma-\frac{1}{2}}\, d\tau \le cB\big(1-\frac{n}{2r}, \frac{1}{2}-\gamma\big)$$

for all $t > 0$. Therefore, for all $t > 0$ we have $F(u,z)(t) \in L^q_\sigma(\Omega)$ and

$$t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} e^{\alpha t} \|F(u,z)(t)\|_q \le c \sup_{0<s\le t} \big\{s^{\frac{n}{4r}} e^{\alpha s}\|u(s)\|_{2r}\big\} \cdot \sup_{0<s\le t} \big\{s^{\frac{n}{4r}} e^{\alpha s}\|z(s)\|_{2r}\big\}, \quad (4.15)$$

where $c = c(r,q,\alpha,\Omega) > 0$. It follows directly from (4.15) that

$$\lim_{t\to+0} t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \|F(u,z)(t)\|_q = 0.$$

Moreover, as in the proof of (4.9), it is easily seen that the mapping $t \mapsto t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}F(u,z)(t)$ is continuous from $(0,\infty)$ to $L^q_\sigma(\Omega)$. Therefore, from (4.8) with $q = 2r$ and (4.15) we get $F(u,z) \in X_{2r}$ and the inequality in (2).

The proof of this lemma is complete. ■

Remark 4.5 Due to Lemma 4.4 (2) and Lemma 3.9 (1), it follows that, if a function u satisfying $u \in X_{2r}(\alpha)$ for all $\alpha \in [\bar\alpha/2, \bar\alpha)$ solves the equation (4.6), then it is a global mild solution to the system (1.5).

Theorem 4.6 *(Existence of Global Mild Solutions) Let $n \le r < \infty$, $f \in L_r(\Omega)$ and let the fluxes $\Phi_1, \dots, \Phi_m \in \mathbb{R}$ satisfy*

$$\|f\|_r + \Phi + \Phi^2 < \min\{K_1, K_2\},$$

where $\Phi = \sum_{i=1}^m |\Phi_i|$ and $K_i = K_i(r,\Omega), i = 1,2,$ are the constants in Theorem 2.4, Theorem 3.5, respectively. Then there exists a constant $\delta_0 = \delta_0(r,\Omega) > 0$ such that for all $u_0 \in L^r_\sigma(\Omega)$ satisfying $\|u_0\|_r < \delta_0$ the system (1.5) – with the unique solution w to (SNS) corresponding to $f, \Phi_1, \dots, \Phi_m$ given by Theorem 2.4 – has a global mild solution u which is unique in a small ball of $X_{2r}(\bar\alpha/2)$. This solution u has the following properties for all $\alpha \in (0,\bar\alpha)$ and $\theta \in (0, \frac{1}{2}+\frac{n}{2r})$:

$$\lim_{t\to\infty} e^{\alpha t}\|u(t)\|_q = 0 \qquad \textit{for all} \quad q \ge r, \quad (4.16)$$

$$\lim_{t\to+0} t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|u(t)\|_q = 0 \qquad \textit{for all} \quad q > r, \quad (4.17)$$

$$t^\theta e^{\alpha t} u \in BC((0,\infty), D(S_r^\theta)), \quad (4.18)$$

$$\lim_{t\to\infty} e^{\alpha t}\|u(t)\|_{D(S_r^\theta)} = 0, \quad (4.19)$$

$$\lim_{t\to+0} t^\theta \|u(t)\|_{D(S_r^\theta)} = 0. \quad (4.20)$$

In particular, the stationary solution $w \in L^r_\sigma(\Omega)$ of (SNS) is exponentially stable.

Remark 4.7 It follows from (4.18) that the global mild solution given by Theorem 4.6 solves the integral equation (4.2).

Proof of Theorem 4.6: First we note that

$$\lim_{t\to+0} t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|e^{-tS_r}u_0\|_q = 0 \quad \text{for all } q > r. \tag{4.21}$$

In fact, for $\gamma = \frac{n}{2}(\frac{1}{r} - \frac{1}{q}) \in \left(0, \frac{1}{2}\right)$ and with the embedding $H^{2\gamma,r}(\Omega) \hookrightarrow L^q(\Omega)$, Proposition 3.6 and [23], Ch. 2, Theorem 6.10, yield

$$\begin{aligned} t^{\gamma}\|e^{-tS_r}u_0\|_q &\leq ct^{\gamma}\|e^{-tS_r}u_0\|_{H^{2\gamma,r}(\Omega)} \\ &\leq ct^{\gamma}\|S_r^{\gamma}e^{-tS_r}u_0\|_r \\ &\leq c\|e^{-tS_r}u_0\|_r^{1-\gamma}\|tS_r\, e^{-tS_r}u_0\|_r^{\gamma}, \end{aligned} \tag{4.22}$$

where $c = c(r, q, \Omega) > 0$. Since $\|tS_r e^{-tS_r}u_0\|_r \to 0$ as $t \to 0$ for $u_0 \in D(S_r)$, the denseness of $D(S_r)$ in $L^r_\sigma(\Omega)$ and the boundedness of the operator family $\{tS_r\, e^{-tS_r}\}_{t\geq 0}$ in $\mathcal{L}(L^r_\sigma(\Omega))$ imply (4.21).

By [9], Theorem 1.1, Lemma 3.9 (1) and (4.21) we get for all $\alpha \in [\bar{\alpha}/2, \bar{\alpha})$ that

$$e^{-tS_r}u_0 \in X_q(\alpha), \quad \forall q > r \tag{4.23}$$

and, in particular,

$$\|e^{-tS_r}u_0\|_{X_{2r}(\alpha)} \leq \sup_{t>0} e^{\alpha t}\|e^{-tS_r}u_0\|_r + \sup_{t>0} t^{\frac{n}{4r}}e^{\alpha t}\|e^{-tS_r}u_0\|_{2r} < C_*\|u_0\|_r \tag{4.24}$$

with some constant $C_* = C_*(r, \alpha, \Omega) > 0$.

Now let us define the mapping $\Psi_{\alpha,u_0} : X_{2r}(\alpha) \to X_{2r}(\alpha)$ by

$$\Psi_{\alpha,u_0}u = e^{-tS_r}u_0 + F(u, u)$$

for a fixed $\alpha \in [\bar{\alpha}/2, \bar{\alpha})$. Let $C_{**} = C_{**}(r, \alpha, \Omega)$ denote the constant in the inequality of Lemma 4.4 (1) with $q = 2r$. Then

$$\|\Psi_{\alpha,u_0}u\|_{X_{2r}(\alpha)} \leq \|e^{-tS_r}u_0\|_{X_{2r}(\alpha)} + \|F(u, u)\|_{X_{2r}(\alpha)} \leq C_*\|u_0\|_r + C_{**}\|u\|^2_{X_{2r}(\alpha)}. \tag{4.25}$$

Note that, if

$$\|u_0\|_r < C_0(r, \alpha, \Omega) := \frac{1}{8C_*C_{**}}, \tag{4.26}$$

then

$$K = K(\alpha, \|u_0\|_r) := \frac{1 - \sqrt{1 - 4C_*C_{**}\|u_0\|_r}}{2C_{**}} < \frac{1}{2C_{**}} \tag{4.27}$$

and the inequality $C_*\|u_0\|_r + C_{**}K^2 \leq K$ holds. Therefore, we get from (4.25) that

$$\Psi_{\alpha,u_0}(U_{K,\alpha}) \subset U_{K,\alpha} := \{u \in X_{2r}(\alpha) : \|u\|_{X_{2r}(\alpha)} \leq K\}.$$

For any $u, z \in U_{K,\alpha}$

$$\begin{aligned} \|\Psi_{\alpha,u_0}u - \Psi_{\alpha,u_0}z\|_{X_{2r}(\alpha)} &= \|F(u, u - z) - F(u - z, z)\|_{X_{2r}(\alpha)} \\ &\leq C_{**}(\|u\|_{X_{2r}(\alpha)} + \|z\|_{X_{2r}(\alpha)})\|u - z\|_{X_{2r}(\alpha)} \\ &\leq 2C_{**}K\|u - z\|_{X_{2r}(\alpha)}. \end{aligned}$$

Hence, in view of $2C_{**}K < 1$, see (4.27), $\Psi_\alpha : U_{K,\alpha} \to U_{K,\alpha}$ is a contraction mapping, and by the Banach fixed point theorem it has a unique fixed point u in $U_{K,\alpha}$.

Now let $u \in X_{2r}(\bar\alpha/2)$ be the unique fixed point of $\Psi_{\bar\alpha/2}$ in $U_{K(\bar\alpha/2,\|u_0\|_r),\bar\alpha/2}$. We shall show that $u \in X_{2r}(\alpha)$ for all $\alpha \in [\bar\alpha/2, \bar\alpha)$. Since $\|u(t)\|_r$ decays as time tends to infinity, for any $\alpha \in (\bar\alpha/2, \bar\alpha)$ there is a (sufficiently large) $t_1(\alpha) > 0$ such that

$$\|u(t_1)\|_r \le \min\{C_0(r, \bar\alpha/2, \Omega), C_0(r, \alpha, \Omega)\}, \tag{4.28}$$

see (4.26), and

$$U_{K(\alpha,\|u(t_1)\|_r),\alpha} \subset U_{K(\bar\alpha/2,\|u_0\|_r),\bar\alpha/2} \tag{4.29}$$

due to (4.4) and the fact that $K(\alpha, \|u_0\|_r) \to 0$ as $\|u_0\|_r \to 0$, see (4.27). Then by (4.28) there is a fixed point $\tilde u \in U_{K(\alpha,\|u(t_1)\|_r),\alpha} \subset X_{2r}(\alpha)$ of $\Psi_{\alpha,u(t_1)}$. Note that $\tilde u$ is also a fixed point of $\Psi_{\bar\alpha/2,u(t_1)}$ in $U_{K(\bar\alpha/2,\|u_0\|_r),\bar\alpha/2}$ due to (4.29). We shall show that $\tilde u(t)$ coincides with $u(t + t_1), t \ge 0$. Obviously, $u(\cdot + t_1) \in X_{2r}(\bar\alpha/2)$ and $\|u(\cdot + t_1)\|_{X_{2r}(\bar\alpha/2)} \le K(\bar\alpha/2, \|u_0\|_r)$. Moreover, we can check that $u(\cdot + t_1)$ solves (4.3), hence (4.6), since for all $t > t_1$ and $\varphi \in C^\infty_{0,\sigma}(\Omega)$

$$\begin{aligned}(u(t), \varphi) &= (e^{-tH_r} u_0, \varphi) + \int_0^t \big((u(s) \cdot \nabla)e^{-(t-s)H^*}\varphi, u(s)\big)\, ds \\ &= (e^{-(t-t_1)H_r} u(t_1), \varphi) + \int_{t_1}^t \big((u(s) \cdot \nabla)e^{-(t-s)H^*}\varphi, u(s)\big)\, ds.\end{aligned}$$

and $\lim_{t\to+t_1}(t-t_1)^{\frac{n}{4r}}\|u(t)\|_{2r} = 0$. Therefore, in view of (4.28), $u(\cdot+t_1)$ is the unique fixed point of $\Psi_{\bar\alpha/2,u(t_1)}$ in $U_{K(\bar\alpha/2,\|u_0\|_r),\bar\alpha/2}$. Consequently, we get $\tilde u(\cdot) = u(\cdot + t_1)$ yielding $u \in X_{2r}(\alpha)$.

Formulae (4.16) and (4.17) are direct consequences of $u \in X_q(\alpha)$ for all $q \in (r, \infty)$ and $\alpha \in [\bar\alpha/2, \bar\alpha)$, see Lemma 4.4 (2).

Now let $\theta \in (0, \frac12 + \frac{n}{2r})$ and fix $p \in (r, \infty)$ such that

$$\frac{n}{p} < \frac{n}{2r} + \frac12 - \theta.$$

It is enough to prove (4.18)-(4.20) for $\alpha \in [\bar\alpha/2, \bar\alpha)$. By Lemma 3.9 (2′) with α replaced by $\frac{\bar\alpha+\alpha}{2}$ we get for all $\varphi \in D(S^{*\theta}_{r'})$ that

$$\begin{aligned}&\big|\langle F(u,u)(t), (S^*_{r'})^\theta\varphi\rangle_{L^r,L^{r'}}\big| = \Big|\int_0^t \big((u(s)\cdot\nabla)(S^*)^\theta e^{-(t-s)S^*_{r'}}\varphi, u(s)\big)\, ds\Big| \\ &\quad\le \int_0^t \|u(s)\|_p^2 \|\nabla(S^*_{r'})^\theta e^{-(t-s)S^*_{r'}}\varphi\|_{p/(p-2)}\, ds \\ &\quad\le c \sup_{0<s\le t}\big\{s^\gamma e^{\alpha s}\|u(s)\|_p\big\}^2 \int_0^t s^{-2\gamma}(t-s)^{-\frac{n}{p}+\frac{n}{2r}-\frac12-\theta}\, e^{-2\alpha s}\, e^{-\frac{\alpha+\bar\alpha}{2}(t-s)}\, ds\, \|\varphi\|_{r'} \\ &\quad\le c \sup_{0<s\le t}\big\{s^\gamma e^{\alpha s}\|u(s)\|_p\big\}^2 t^{-\theta} e^{-\alpha t} I_3(t)\|\varphi\|_{r'}\end{aligned}$$

for all $t > 0$, where $\gamma = \frac{n}{2}(\frac1r - \frac1p)$

$$I_3(t) \equiv t^{\frac12-\frac{n}{2r}} e^{-\frac{\bar\alpha-\alpha}{2}t} \int_0^1 s^{-2\gamma}(1-s)^{-\frac{n}{p}+\frac{n}{2r}-\frac12-\theta}\, ds \le c \quad \forall t > 0.$$

Therefore, in view of $(S^*_{r'})^\theta = (S^\theta_r)^*$, see [1], Ch. V, Lemma 1.4.11, we get $F(u,u)(t) \in D(S^\theta_r)$ for all $t > 0$ and

$$t^\theta e^{\alpha t}\|S^\theta_r F(u,u)(t)\|_r \le c \sup_{0<s\le t}\left\{s^{\frac{n}{2}(\frac{1}{r}-\frac{1}{p})} e^{\alpha s}\|u(s)\|_p\right\}^2. \tag{4.30}$$

On the other hand, by the same technique as in the proof of (4.9) we see that the function $t \mapsto S^\theta_r F(u,u)(t)$ is continuous from $(0,\infty)$ to $L^r_\sigma(\Omega)$, which together with (4.30) yields (4.18), (4.19). Moreover, (4.30) implies (4.20) due to $u \in X_p(\alpha)$.

Finally let us prove that this fixed point is unique in the whole space $X_{2r}(\alpha)$ rather than only in $U_{K(\alpha,\|u_0\|_r),\alpha}$. Given fixed points $u_1, u_2 \in X_{2r}(\alpha)$ of Ψ_{α,u_0} we get from (4.15) with $q = 2r$ that for all $t > 0$

$$\begin{aligned}&\sup_{0<s\le t}\left\{s^{\frac{n}{4r}} e^{\alpha s}\|u_1(s)-u_2(s)\|_{2r}\right\}\\ &\quad\le c \sup_{0<s\le t}\left\{s^{\frac{n}{4r}} e^{\alpha s}\big(\|u_1(s)\|_{2r}+\|u_2(s)\|_{2r}\big)\right\}\cdot \sup_{0<s\le t}\left\{s^{\frac{n}{4r}} e^{\alpha s}\|u_1(s)-u_2(s)\|_{2r}\right\}.\end{aligned}$$

Since $s^{\frac{n}{4r}}\big(\|u_1(s)\|_{2r}+\|u_2(s)\|_{2r}\big) \to 0$ as $s \to 0$, there exists $t_1 = t_1(u_1,u_2) > 0$ such that $u_1 \equiv u_2$ in $[0,t_1]$. Defining $T = \sup\{t_1 > 0 : u_1 \equiv u_2 \text{ on } [0,t_1]\}$, a continuity argument yields $u_1 \equiv u_2$ on $[0,T]$. If $T < \infty$, we repeat the above argument by starting at T and conclude that $u_1 \equiv u_2$ on $[0, T+t_2]$ for some $t_2 = t_2(u_1,u_2) > 0$ in contradiction to the definition of T.

The proof of this theorem is complete. ■

Theorem 4.8 *(Existence of Global Strong Solution) The global weak solution given by Theorem 4.6 is a strong solution to* (1.5).

Proof: Let u be the global mild solution to (1.5) given by Theorem 4.6. We shall prove that for all $\varepsilon > 0$ and $T > \varepsilon$

$$P_r(u\cdot\nabla)u \in C([\varepsilon,T], D(S^\zeta_r)) \tag{4.31}$$

with some $\zeta \in (0,1)$. Then by well-known results on analytic semigroups (see e.g. [23], Ch. 4, Theorem 3.6 or [1], Ch. II, Theorem 1.2.2)

$$u(t) = e^{-tS_r}u_0 - \int_0^t e^{-(t-s)S_r}P_r(u\cdot\nabla)u(s)\,ds$$

is a strong solution on $(\varepsilon, T]$ to (1.5) for any $0 < \varepsilon < T < \infty$, i.e.,

$$u \in C([\varepsilon,T], L^r_\sigma(\Omega)) \cap C^1((\varepsilon,T], L^r_\sigma(\Omega)) \cap C((\varepsilon,T], D(S_r)) \quad \forall t \in (\varepsilon, T].$$

Note $D(S_r) = D(A_r)$, see (3.1). Therefore, $u \in C^1((0,\infty), L^r_\sigma(\Omega)) \cap C((0,T], D(A_r))$ and consequently, u is a global strong solution to (1.5) since u belongs to $BC([0,\infty), L^r_\sigma(\Omega))$ as a global mild solution.

Fix $\theta \in (\frac{1}{2}, \frac{1}{2}+\frac{1}{2r})$, and let $\xi = 2\theta$ and $\zeta = \frac{1}{2}\frac{\xi-\eta}{2-\eta}$ where $\eta \ge 1$ is defined by (2.9) (with $\delta' > 0$ arbitrarily small when $r = n$) so that $2\zeta \le \xi - 1 < \frac{1}{r}$. Then by Lemma 2.1 (3) (with $\delta = 0$) and Proposition 3.6

$$\|P_r(u\cdot\nabla)v\|_{H^{2\zeta,r}(\Omega)} \le c\|u\|_{H^{\xi,r}(\Omega)}\|v\|_{H^{\xi,r}(\Omega)} \le c\|u\|_{D(S^\theta_r)}\|v\|_{D(S^\theta_r)}.$$

Since $\|\cdot\|_{D(S_r^\zeta)}$ is equivalent to $\|\cdot\|_{H^{2\zeta,r}(\Omega)}$, see Proposition 3.6, we conclude that $P_r(u(t)\cdot\nabla)u(t)\in D(S_r^\zeta)$ for all $t\in[\varepsilon,T]$ and

$$\begin{aligned}&\|P_r((u(t_1)\cdot\nabla)u(t_1)-(u(t_2)\cdot\nabla)u(t_2))\|_{D(S_r^\zeta)}\\ &\qquad\le c\big(\|u(t_1)\|_{D(S_r^\theta)}+\|u(t_2)\|_{D(S_r^\theta)}\big)\|u(t_1)-u(t_2)\|_{D(S_r^\theta)}.\end{aligned}$$

Hence (4.18) yields (4.31). ■

Theorem 4.9 *(Uniqueness of Strong Solution)*

(1) Let $r\in(n,\infty)$. If $u_0\in L^r_\sigma(\Omega)$, then the strong solution to (1.5) *is unique.*

(2) If $u_0\in H^{s,n}(\Omega)\cap L^n_\sigma(\Omega)$ for some $s>0$, then the strong solution to (1.5) *is unique.*

(3) Let $u_0\in L^n_\sigma(\Omega)$ and let u_1,u_2 be strong solutions to (1.5) *on* $[0,T)$ *satisfying*

$$\lim_{t\to+0} t^{\frac12-\frac{n}{2q}}u_i(t)=0\quad in\quad L^q(\Omega),\ i=1,2, \tag{4.32}$$

for some $q>n$. Then $u_1\equiv u_2$.

Proof: (1) Let $r>n$. If u_1,u_2 are strong solutions on $[0,T)$, $0<T<\infty$, to (1.5), we have by Lemma 3.9 (2′) for all $\varphi\in C^\infty_{0,\sigma}(\Omega)$ and $t\in(0,T)$ that

$$\begin{aligned}\big|\langle u_1(t)-u_2(t),\varphi\rangle\big| &\le \Big|\int_0^t \big((u_1(s)-u_2(s)\cdot\nabla)e^{-(t-s)S^*}\varphi,u_1(s)\big)\,ds\Big|\\ &\qquad+\Big|\int_0^t \big((u_2(s)\cdot\nabla)e^{-(t-s)S^*}\varphi,u_1(s)-u_2(s)\big)\,ds\Big|\\ &\le \|u_1,u_2\|_{BC([0,T),L^r_\sigma(\Omega))}\int_0^t \|\nabla e^{-(t-s)S^*}\varphi\|_{\frac{r}{r-2}}\|u_1(s)-u_2(s)\|_r\,ds\\ &\le c\|u_1,u_2\|_{BC([0,T),L^r_\sigma(\Omega))}\int_0^t (t-s)^{-\frac{n}{2r}-\frac12}\|u_1(s)-u_2(s)\|_r\,ds\|\varphi\|_{r'}.\end{aligned}$$

Hence we get

$$\|u_1(t)-u_2(t)\|_r\le c\int_0^t (t-s)^{-\frac{n}{2r}-\frac12}\|u_1(s)-u_2(s)\|_r\,ds\quad \forall t\in(0,T),$$

which implies after a finite number of integrations and due to Gronwall's lemma that $u_1(t)=u_2(t)$ on $(0,T)$.

(2) Due to the Sobolev embedding $H^{s,n}(\Omega)\hookrightarrow L^r(\Omega)$ for some $r>n$, the assertion (2) follows from (1).

(3) Let $u_0\in L^n_\sigma(\Omega)$ and let u_1,u_2 be strong solutions to (1.5) satisfying (4.32). In view of the uniqueness results (1), (2) proved above and the fact that any strong solution belongs to $C([\varepsilon,T],L^r(\Omega))$ for any $0<\varepsilon<T$ and some $r>n$, it is enough to show $u_1(t)=u_2(t)$ for some $\delta>0$; for a similar proof we refer to [15], Lemma 3.2. By the same technique as above, see also (4.7), (4.8), we get for all $t>0$ and $\varphi\in C^\infty_{0,\sigma}(\Omega)$ that

$$\begin{aligned}|\langle u_1(t)-u_2(t),\varphi\rangle| &\le \int_0^t \|\nabla e^{-(t-s)S^*}\varphi\|_\xi\|u_1(s)-u_2(s)\|_n(\|u_1(s)\|_q+\|u_2(s)\|_q)\,ds\\ &\le D(t)K(t)\int_0^t s^{-\frac n2(\frac1n-\frac1q)}\|\nabla e^{-(t-s)S^*}\varphi\|_\xi\,ds\\ &\le cD(t)K(t)\int_0^t s^{-\frac n2(\frac1n-\frac1q)}(t-s)^{-\frac{n}{2q}-\frac12}\|\varphi\|_{n'},\end{aligned}$$

where $\frac{1}{\xi} = 1 - \frac{1}{n} - \frac{1}{q}$ and

$$D(t) = \sup_{0<s\leq t} \|u_1(s) - u_2(s)\|_n, \quad K(t) = \sum_{i=1}^{2} \sup_{0<s\leq t} s^{\frac{n}{2}(\frac{1}{n}-\frac{1}{q})} \|u_i(s)\|_q.$$

Therefore, we get $\|u_1(t) - u_2(t)\|_n \leq C_0 K(t) D(t)$ with some $C_0 > 0$ and even

$$D(t) \leq C_0 K(t) D(t)$$

for all $t > 0$. By assumption, $\lim_{t\to 0} K(t) = 0$; hence, there is some $\delta > 0$ such that $C_0 K(t) < 1$ for all $t \in (0, \delta)$. Thus $D(t) = 0$, i.e., $u_1(t) = u_2(t)$ for $t \in (0, \delta)$. ■

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