Szegö limit theorems for operators with almost periodic diagonals

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Dedicated to Vladimir S. Rabinovich on the occasion of his 65th birthday

Abstract

The classical Szegö theorems study the asymptotic behaviour of the determinants of the finite sections $P_nT(a)P_n$ of Toeplitz operators, i.e., of operators which have constant entries along each diagonal. We generalize these results to operators which have almost periodic functions on their diagonals.

1 Introduction

This paper deals mainly with operators which are constituted by Laurent or Toeplitz operators and by band-dominated operators. So we start with introducing some notations and with recalling some facts about Toeplitz and band-dominated operators and their finite sections.

Spaces and projections. Given a non-empty subset $\mathbb{I}$ of the set $\mathbb{Z}$ of the integers, let $l^2(\mathbb{I})$ stand for the Hilbert space of all sequences $(x_n)_{n\in\mathbb{I}}$ of complex numbers with $\sum_{n\in\mathbb{I}} |x_n|^2 < \infty$. We identify $l^2(\mathbb{I})$ with a closed subspace of $l^2(\mathbb{Z})$ in the natural way, and we write $P_\mathbb{I}$ for the orthogonal projection from $l^2(\mathbb{Z})$ onto $l^2(\mathbb{I})$.

The set of the non-negative integers will be denoted by $\mathbb{Z}^+$, and we write P in place of $P_\mathbb{Z}^+$ and Q in place of the complementary projection $I - P$. Thus, $Q = P_{\mathbb{Z}^-}$ where $\mathbb{Z}^-$ refers to the set of all negative integers.

Further, for each positive integer n, set

$$P_n := P_{\{0,\,1,\,\ldots,\,n-1\}} \quad \text{and} \quad R_n := P_{\{-n,\,-n+1,\,\ldots,\,n-1\}}.$$

The projections R_n converge strongly to the identity operator on $l^2(\mathbb{Z})$, and the projections P_n converge strongly to the identity operator on $l^2(\mathbb{Z}^+)$ when considered as acting on $l^2(\mathbb{Z}^+)$ and to the projection P when considered as acting on $l^2(\mathbb{Z})$.

The C^*-algebra of all bounded linear operators on a Hilbert space H will be denoted by $L(H)$.

Functions and operators. Let $a \in L^\infty(\mathbb{T})$, the C^*-algebra of all essentially bounded measurable functions on the complex unit circle $\mathbb{T}$, and let

$$a_j := \frac{1}{2\pi} \int_0^{2\pi} a(e^{it}) e^{-ijt}\, dt.$$

refer to the jth Fourier coefficient of a. Then the operator on $l^2(\mathbb{Z})$ given by the matrix representation $(a_{i-j})_{i,j\in\mathbb{Z}}$ with respect to the standard basis of $l^2(\mathbb{Z})$ induces a bounded linear operator $L(a)$ on $l^2(\mathbb{Z})$, the so-called *Laurent operator with generating function a*. The operator $T(a) := PL(a)P$ acting on $l^2(\mathbb{Z}^+)$ is called the *Toeplitz operator with generating function a*.

Laurent operators are distinguished by their shift invariance. For $k \in \mathbb{Z}$, define the *shift operator*

$$U_k : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}), \quad (x_n) \mapsto (y_n) \text{ with } y_n = x_{n-k}.$$

Then $A \in L(l^2(\mathbb{Z}))$ is a Laurent operator if and only if $U_{-k}AU_k = A$ for each $k \in \mathbb{Z}$.

Further, each function $b \in l^\infty(\mathbb{Z})$, the C^*-algebra of all bounded sequences on $\mathbb{Z}$, induces a *multiplication operator*

$$aI : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}), \quad (x_n) \mapsto (a_n x_n).$$

Let X be a C^*-subalgebra of $L^\infty(\mathbb{T})$ and Y be a shift invariant C^*-subalgebra of $l^\infty(\mathbb{Z})$. The latter means that $U_{-k}aU_k \in Y$ whenever $a \in Y$. We let $\mathcal{A}_{X,Y}(\mathbb{Z})$ stand for the smallest closed C^*-subalgebra of $L(l^2(\mathbb{Z}))$ which contains all Laurent operators $L(a)$ with $a \in X$ and all multiplication operators bI with $b \in Y$. Similarly, we write $\mathcal{A}_{X,Y}(\mathbb{Z}^+)$ for the smallest closed C^*-subalgebra of $L(l^2(\mathbb{Z}^+))$ which contains all Toeplitz operators $T(a)$ with $a \in X$ and all operators PbP with $b \in Y$. So $\mathcal{A}_{L^\infty(\mathbb{T}),\mathbb{C}}(\mathbb{Z})$ is the C^*-algebra of all Laurent operators, which is *-isomorphic to the algebra $L^\infty(\mathbb{T})$, and $\mathcal{A}_{L^\infty(\mathbb{T}),\mathbb{C}}(\mathbb{Z}^+)$ is the smallest closed subalgebra of $L(l^2(\mathbb{Z}^+))$ which contains all bounded Toeplitz operators.

Of particular interest are the algebra $X = C(\mathbb{T})$ of the continuous functions on $\mathbb{T}$ and the algebra $Y = AP(\mathbb{Z})$ of the almost periodic functions. A function $a \in l^\infty(\mathbb{Z})$ is called *almost periodic* if the set of all multiplication operators $U_{-k}aU_k$ with $k \in \mathbb{Z}$ is relatively compact in the norm topology of $L(l^2(\mathbb{Z}))$ or, equivalently, in the norm topology of $l^\infty(\mathbb{Z})$.

The operators in $\mathcal{A}_{C(\mathbb{T}),l^\infty(\mathbb{Z})}(\mathbb{Z})$ are usually referred to as *band-dominated operators*, and the operators in $\mathcal{A}_{C(\mathbb{T}),Y}(\mathbb{Z})$ are called *band-dominated operators* *with coefficients in* Y. To operators in $\mathcal{A}_{C(\mathbb{T}),l^\infty(\mathbb{Z})}(\mathbb{Z}^+)$, we also refer as band-dominated operators over $\mathbb{Z}^+$. The reason for the notion *band-dominated* is that continuous functions on $\mathbb{T}$ can be uniformly approximated by trigonometric polynomials, hence, operators in $\mathcal{A}_{C(\mathbb{T}),l^\infty(\mathbb{Z})}$ can be approximated by band operators in the norm of $L(l^2(\mathbb{Z}))$. We will usually write $\mathcal{A}_Y(\mathbb{Z})$ and $\mathcal{A}_Y(\mathbb{Z}^+)$ in place of $\mathcal{A}_{C(\mathbb{T}),Y}(\mathbb{Z})$ and $\mathcal{A}_{C(\mathbb{T}),Y}(\mathbb{Z}^+)$, respectively, which is consistent with the notations in [32, 35]. It is easy to see that $PAP \in \mathcal{A}_Y(\mathbb{Z}^+)$ whenever $A \in \mathcal{A}_Y(\mathbb{Z})$.

Szegö theorems for Toeplitz matrices. There are several ways to express the so-called first Szegö limit theorem, and there are several kinds of hypotheses under which the theorem holds. A version which is convenient for us is via stability of the finite sections method. The nth *finite section* of the operator A is the operator P_nAP_n. Unless otherwise stated, we will consider this operator as acting on $\operatorname{im} P_n$. Thus, P_nAP_n can be represented by an $n \times n$ matrix. Instead of $P_nT(a)P_n$ we will also write $T_n(a)$.

The sequence $(P_nAP_n)_{n\in\mathbb{N}}$ of the finite sections of an operator $A \in L(l^2(\mathbb{Z}^+))$ is said to be stable if the matrices P_nAP_n are invertible for sufficiently large n and if the norms of their inverses are uniformly bounded.

Theorem 1.1 (First Szegö limit theorem) *Let $a \in L^\infty(\mathbb{T})$ and suppose that the finite sections sequence $(T_n(a))_{n\in\mathbb{N}}$ is stable. Then $T(a)$ is invertible and*

$$\lim_{n\to\infty} \frac{\det T_n(a)}{\det T_{n-1}(a)} = G[a] \tag{1}$$

where

$$G[a] := 1/(P_1T(a)^{-1}P_1)$$

and, of course, $P_1T(a)^{-1}P_1$ stands for the 00th entry of $T(a)^{-1}$.

If $a \in L^\infty(\mathbb{T})$ is real-valued and $T(a)$ is invertible, then the (compact) essential range of a is contained in the open interval $(0, \infty)$ by the Hartman-Wintner theorem (see 2.36 in [12] or Theorem 1.27 in [13]). Thus, the function a has a real-valued logarithm $\log a \in L^\infty(\mathbb{T})$, and it is not hard to show that

$$G[a] = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} (\log a)(e^{it})\,dt\right) = \exp(\log a)_0. \tag{2}$$

Szegö [39] proved (1) under the assumptions that $a \in L^1(\mathbb{T})$, $a \geq 0$ and $\log a \in L^1(\mathbb{T})$. The following theorems provide statements about the eigenvalue distribution of Toeplitz matrices. One has to distinguish between real-valued generating functions a, in which case the function f has to be merely continuous, whereas in case of arbitrary bounded functions a, one needs holomorphy of f.

They can be derived from Szegö's first limit theorem (compare the proofs of Theorems 5.9 and 5.10 in [13]). Although this derivation is not without effort, they are also referred to as First Szegö limit theorems. In the present paper we will call them the distributive versions of Theorem 1.1.

For each $n \times n$-matrix B, let $\lambda_i(B)$ with $i = 1, \ldots, n$ refer to the eigenvalues of B. The order of enumeration is not of importance.

Theorem 1.2 (First Szegö limit theorem, distributive version I) *Let $a \in L^\infty(\mathbb{T})$ be a real-valued function, and let g be any continuous function on the convex hull of the essential range of a. Then*

$$\lim_{n\to\infty} \frac{g(\lambda_1(T_n(a)) + \ldots + g(\lambda_n(T_n(a))}{n} = \frac{1}{2\pi}\int_0^{2\pi} g(a(e^{it}))\,dt. \tag{3}$$

Theorem 1.3 (First Szegö limit theorem, distributive version II) *Let a be an arbitrary function in $L^\infty(\mathbb{T})$, and let g be analytic on an open neighborhood of the convex hull of the essential range of a. Then* (3) *holds again.*

It is one thing to settle the convergence (1) and another one to describe the precise asymptotic behaviour of the determinants $\det T_n(a)$. The latter is the contents of the so-called strong Szegö limit theorem, proved by Szegö [40] for positive generating functions with Hölder continuous derivative. In the formulation below, there occurs an algebra, $W^{0,0} \cap B_{2,2}^{1/2,1/2}$, of continuous functions on $\mathbb{T}$ which is defined in [12], 10.21.

Theorem 1.4 (Strong Szegö limit theorem) *Let $a \in W_{0,0} \cap B^{2,2}_{1/2,1/2}$ have no zeros on $\mathbb{T}$ and winding number 0 with respect to the origin. Then*

$$\lim_{n\to\infty} \frac{\det T_n(a)}{G[a]^n} = E[a] \tag{4}$$

where

$$E[a] = \exp \sum_{k=1}^{\infty} k (\log a)_k \, (\log a)_{-k}. \tag{5}$$

We will not go into the long and rich history of the Szegö limit theorems here and refer to [12, 13] and to Chapter 2 of [37] instead. Let us only mention that E. Basor, G. Baxter, A. Böttcher, A. Devinatz, T. Ehrhardt, I. Gohberg, I. Feldman, I. I. Hirschman, M. Kac, M. G. Krein and H. Widom are among the main contributors and that [6, 17, 18, 19, 21, 24, 43] mark some milestones in this field.

About this paper. This paper is devoted to generalizations of the classical Szegö limit theorems to several classes of operators with variable coefficients (whereas Toeplitz and Laurent operators are considered as operators with constant coefficients). Particular attention is paid to operators with almost periodic coefficients for which we will obtain the most satisfying generalizations of Theorems 1.1 – 1.3. These results will be discussed in Sections 3 and 4 below. On the other hand, we have to report that the precise asymptotic behaviour of the determinants of an operator with almost periodic diagonals still remains mysterious for us. Thus, the question of a possible generalization of the strong Szegö limit theorem is still open (although Torsten Ehrhardt's wonderful paper [19] seems to offer a comfortable way to attack this problem).

We prepare our discussion in Section 2 by recalling some facts about algebras generated be sequences of finite sections and about band-dominated operators and their finite sections. The results cited in this section can be found in [33, 35]. The concluding fifth section is devoted to some applications of our general Szegö limit theorems.

The reader should have in mind that most of the results presented below can be formulated in two different settings: for operators acting on the two-sided infinite sequences (with the Laurent operators as an example) and for operators on one-sided infinite sequences (for instance, the Toeplitz operators). In order to make our results comparable with the classical Szegö theorems, we will state them for operators on $l^2(\mathbb{Z}^+)$.

2 Preliminaries

2.1 Algebras related with finite sections

Let $\mathcal{P} := (P_n)_{n\in\mathbb{N}}$ where the projections P_n are defined as in the introduction. Write $\mathcal{F}^{\mathcal{P}}$ for the set of all sequences (A_n) of operators $A_n : \operatorname{im} P_n \to \operatorname{im} P_n$ for which the strong limits

$$\operatorname{s-lim} A_n P_n \quad \text{and} \quad \operatorname{s-lim} A_n^* P_n$$

exist, and $\mathcal{G}$ for the subset of $\mathcal{F}^{\mathcal{P}}$ consisting of all sequences (G_n) with $\|G_n\| \to 0$. Provided with the operations

$$(A_n) + (B_n) := (A_n + B_n), \quad \lambda(A_n) := (\lambda A_n), \quad (A_n)\,(B_n) := (A_n B_n), \tag{6}$$

the involution $(A_n)^* := (A_n^*)$ and with the norm

$$\|(A_n)\| := \sup_{n\in\mathbb{N}} \|A_n\|,$$

the set $\mathcal{F}^{\mathcal{P}}$ becomes a C^*-algebra, and $\mathcal{G}$ is a closed ideal of $\mathcal{F}^{\mathcal{P}}$. We will often use boldface letters to refer to elements of $\mathcal{F}^{\mathcal{P}}$. For $\mathbf{A} := (A_n) \in \mathcal{F}^{\mathcal{P}}$, we denote the strong limit $\operatorname{s-lim} A_n P_n$ by $W(\mathbf{A})$. Thus, W is a *-homomorphism from $\mathcal{F}^{\mathcal{P}}$ onto $L(l^2(\mathbb{Z}^+))$.

A sequence $(A_n) \in \mathcal{F}^{\mathcal{P}}$ is called stable if the operators $A_n : \operatorname{im} P_n \to \operatorname{im} P_n$ are invertible for sufficiently large n and if the norms of their inverses are uniformly bounded. The following simple result is the basis for the algebraization of several problems from numerical analysis.

Proposition 2.1 (Kozak) *A sequence $\mathbf{A} \in \mathcal{F}^{\mathcal{P}}$ is stable if and only if the coset $\mathbf{A} + \mathcal{G}$ is invertible in the quotient algebra $\mathcal{F}^{\mathcal{P}}/\mathcal{G}$.*

The spectrum of the coset $\mathbf{A} + \mathcal{G}$ in $\mathcal{F}^{\mathcal{P}}/\mathcal{G}$ will be denoted by $\sigma_{\mathcal{F}^{\mathcal{P}}/\mathcal{G}}(\mathbf{A} + \mathcal{G})$ or simply by $\sigma(\mathbf{A}+\mathcal{G})$. It is also called the *stability spectrum* of the sequence $\mathbf{A}$ and will occur in the formulation of several results below. Here we only mention the following fact.

Proposition 2.2 *Let $\mathbf{A} = (A_n) \in \mathcal{F}^{\mathcal{P}}$. Then*

$$\sigma_{L(l^2(\mathbb{Z}^+))}(W(\mathbf{A})) \subseteq \sigma_{\mathcal{F}^{\mathcal{P}}/\mathcal{G}}(\mathbf{A} + \mathcal{G}),$$

and for each open neighborhood U of $\sigma_{\mathcal{F}^{\mathcal{P}}/\mathcal{G}}(\mathbf{A} + \mathcal{G})$ one has

$$\sigma_{L(im\, P_n)}(A_n) \subseteq U$$

for all sufficiently large n.

The proof of the first assertion is a consequence of Polski's theorem (Theorem 1.4 in [22]), and the second one follows easily from the inclusion

$$\limsup \sigma(A_n) \subseteq \sigma_{\mathcal{F}^{\mathcal{P}}/\mathcal{G}}(\mathbf{A} + \mathcal{G}) \tag{7}$$

stated in Theorem 3.19 in [22], where lim sup is the set-theoretical limes superior. Indeed, suppose there are an open neighborhood U of $\sigma_{\mathcal{F}^{\mathcal{P}}/\mathcal{G}}(\mathbf{A} + \mathcal{G})$, a strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$, and points $\lambda_n \in \sigma(A_{\eta(n)})$ with $\lambda_n \notin U$. Since (A_n) is a bounded sequence, the sequence (λ_n) is bounded, too. Hence, it possesses a partial limit λ^* which belongs to $\limsup \sigma(A_n)$ (by definition) but not to U (since U is open). This contradicts (7). ■

It what follows we will have to consider several subalgebras of $\mathcal{F}^{\mathcal{P}}$. For X and Y as in the introduction, let $\mathcal{S}_{X,Y}(\mathbb{Z}^+)$ stand for the smallest closed C^*-subalgebra of $\mathcal{F}^{\mathcal{P}}$ which contains all sequences $(P_n A P_n)$ of finite sections of operators $A \in \mathcal{A}_{X,Y}(\mathbb{Z}^+)$. Further we will often write $\mathcal{A}_Y(\mathbb{Z}^+)$ in place of $\mathcal{A}_{C(\mathbb{T}),Y}(\mathbb{Z}^+)$.

2.2 Band-dominated operators, their Fredholmness and finite sections

Here is a summary of the results from [30] needed in what follows. A comprehensive treatment of this topic is in [32]; see also the references mentioned there.

Fredholmness of band-dominated operators. An operator A on a Hilbert space H is called Fredholm if both its kernel $\ker A := \{x \in H : Ax = 0\}$ and its cokernel $\operatorname{coker} A := H/(AH)$ are finite dimensional linear spaces. There is a Fredholm criterion for a general band-dominated operator A which expresses the Fredholm property in terms of the limit operators of A. To state this result, we will need a few notations.

Let $\mathcal{H}$ stand for the set of all sequences $h : \mathbb{N} \to \mathbb{Z}$ which tend to infinity in the sense that given $C > 0$, there is an n_0 such that $|h(n)| > C$ for all $n \geq n_0$. An operator $A_h \in L(l^2(\mathbb{Z}))$ is called the *limit operator* of $A \in L(l^2(\mathbb{Z}))$ with respect to the sequence $h \in \mathcal{H}$ if $U_{-h(n)} A U_{h(n)}$ tends *-strongly to A_h as $n \to \infty$. Notice that every operator can possess at most one limit operator with respect to a given sequence $h \in \mathcal{H}$. The set $\sigma_{op}(A)$ of all limit operators of a given operator A is the *operator spectrum* of A.

We write $L^\$(l^2(\mathbb{Z}))$ for the set of all operators $A \in L(l^2(\mathbb{Z}))$ which own the following compactness property: Every sequence $h \in \mathcal{H}$ possesses a subsequence g for which the limit operator A_g exists. Thus, operators in $L^\$(l^2(\mathbb{Z}))$ possess, in a sense, *many* limit operators. They are also called operators with *rich* operator spectrum (therefore the notation).

Proposition 2.3 *(a) $L^\$(l^2(\mathbb{Z}))$ is a C^*-subalgebra of $L(l^2(\mathbb{Z}))$.*
(b) $\mathcal{A}_{L^\infty(\mathbb{T}), l^\infty(\mathbb{Z})}(\mathbb{Z}) \subseteq L^\$(l^2(\mathbb{Z}))$.

Assertion (a) is Proposition 1.2.6 (a) in [32]. Since $L^\$(l^2(\mathbb{Z}))$ is a closed algebra, assertion (b) will follow once it has been shown that all bounded Laurent operators and all bounded multiplication operators belong to $L^\$(l^2(\mathbb{Z}))$. The first inclusion is evident due to the shift invariance of Laurent operators, and the second one is Theorem 2.1.16 in [32]. ∎

It is not hard to see that every limit operator of a compact operator is 0 and that every limit operator of a Fredholm operator is invertible. A basic result of [30] (see also Theorems 2.2.1 and 2.5.7 in [32]) claims that the operator spectrum of a *band-dominated operator* is rich enough in order to guarantee the reverse implications.

Theorem 2.4 *Let $A \in L(l^2(\mathbb{Z}))$ be a band-dominated operator. Then the operator A is Fredholm if and only if each of its limit operators is invertible and if the norms of their inverses are uniformly bounded. If A is a band operator, then A is Fredholm if and only if each of its limit operators is invertible.*

An analogous result holds for band-dominated operators on $\mathbb{Z}^+$ in which case one has to take into account all limit operators with respect to sequences h tending to $+\infty$. (Simply apply Theorem 2.4 to the operator $PAP + Q$, now acting on all of $\mathbb{Z}$.) We let $\sigma_\pm(A)$ collect the set of all limit operators of A which are taken with respect to a sequence tending to $\pm\infty$.

Finite sections of band-dominated operators. One way to attack stability problems is based on the following observation. Associate to the sequence $\mathbf{A} = (A_n) \in \mathcal{F}^{\mathcal{P}}$ the block diagonal operator

$$\mathrm{Op}\,(\mathbf{A}) := \mathrm{diag}\,(A_1,\, A_2,\, A_3,\, \ldots) \tag{8}$$

considered as acting on $l^2(\mathbb{Z}^+)$. It is easy to check that the sequence $\mathbf{A}$ is stable if and only if the associated operator $\mathrm{Op}\,(\mathbf{A})$ is Fredholm. In general, this stability criterion seems to be of less use. But if one starts with the sequence $\mathbf{A} = (P_nAP_n)$ of the finite sections method of a band-dominated operator A, then one ends up with a band-dominated operator $\mathrm{Op}\,(\mathbf{A})$ on $l^2(\mathbb{Z}^+)$, and Theorem 2.4 applies to study the Fredholmness of $\mathrm{Op}\,(\mathbf{A})$. Basically, one has to compute the limit operators of $\mathrm{Op}\,(\mathbf{A})$, which leads to the following result (which is Theorem 3 in [31]). See also Chapter 6 in [32] and the detailed account on the finite sections method of band-dominated operators given in [35].

Theorem 2.5 *Let $A \in L(l^2(\mathbb{Z}))$ be a band-dominated operator. Then the finite sections method $(R_nAR_n)_{n\ge 1}$ is stable if and only if the operator A, all operators*

$$QA_hQ + P \quad \textit{with} \quad A_h \in \sigma_+(A)$$

and all operators

$$PA_hP + Q \quad \textit{with} \quad A_h \in \sigma_-(A)$$

are invertible on $l^2(\mathbb{Z})$, and if the norms of their inverses are uniformly bounded. The condition of the uniform boundedness of the inverses is redundant if A is a band operator.

Specifying this theorem to the case of band operators on $l^2(\mathbb{Z}^+)$ we get the following, where J refers to the unitary operator

$$l^2(\mathbb{Z}) \to l^2(\mathbb{Z}), \quad (Jx)_m := x_{-m-1},$$

and where we define $\sigma_+(A)$ as $\sigma_+(PAP + Q)$.

Theorem 2.6 *Let $A \in L(l^2(\mathbb{Z}^+))$ be a band-dominated operator. Then the finite sections method $(P_nAP_n)_{n\ge 1}$ is stable if and only if the operator A and all operators*

$$JQA_hQJ \quad \textit{with} \quad A_h \in \sigma_+(A)$$

are invertible on $l^2(\mathbb{Z}^+)$ and if the norms of their inverses are uniformly bounded. The condition of the uniform boundedness of the inverses is redundant if A is a band operator.

There are generalizations of Theorems 2.5 and 2.6 which can be verified in the same vein as their predecessors. We mention the result for the finite sections (P_nAP_n) only.

Theorem 2.7 *Let $A \in L(l^2(\mathbb{Z}^+))$ be a band-dominated operator, and let $\eta : \mathbb{N} \to \mathbb{N}$ be a strongly monotonically increasing sequence. Then the sequence $(P_{\eta(n)} A P_{\eta(n)})_{n \geq 1}$ is stable if and only if the operator A and all operators JQA_hQJ where A_h is a limit operator of A with respect to a subsequence h of η are invertible on $l^2(\mathbb{Z}^+)$ and if the norms of their inverses are uniformly bounded. The condition of the uniform boundedness of the inverses is redundant if A is a band operator.*

Thus, instead of taking all limit operators of A with respect to monotonically increasing sequences h, one has consider only those with respect to subsequences of η.

Corollary 2.8 *Let $A \in L(l^2(\mathbb{Z}^+))$ be a band-dominated operator, and let $h : \mathbb{N} \to \mathbb{N}$ be a strongly monotonically increasing sequence for which the limit operator A_h exists. Then the sequence $(P_{h(n)} A P_{h(n)})_{n \geq 1}$ is stable if and only if the operators A and JQA_hQJ are invertible.*

2.3 Band-dominated operators with almost periodic coefficients

Here we collect some basic facts from [33] which show that the conclusion of Corollary 2.8 can be essentially simplified if the sequence h is chosen appropriately. These results will only be needed in Subsection 5.1 (after Theorem 5.4) below.

It is one peculiarity of band-dominated operators $A \in \mathcal{A}_{AP}(\mathbb{Z})$ that there is a strongly monotonically increasing sequence $h : \mathbb{N} \to \mathbb{N}$ such that

$$\|U_{-h(n)} A U_{h(n)} - A\| \to 0 \quad \text{as } n \to \infty. \tag{9}$$

Thus, A is its own limit operator with respect to h, and it is a limit operator in the sense of *norm* convergence. We shall prove this fact in Section 5.3 in a more general context. Each sequence h with the properties mentioned above is called a *distinguished sequence for A*. If h is a distinguished sequence for A, then we call $(P_{h(n)} PAP P_{h(n)})$ the associated *distinguished finite sections method for PAP* and $(R_{h(n)} A R_{h(n)})$ the associated *distinguished finite sections method for A*.

Theorem 2.9 *Let $A \in \mathcal{A}_{AP}(\mathbb{Z})$ and let h be a distinguished sequence for A. Then the sequence $(P_{h(n)} PAP P_{h(n)})$ is stable if and only if the operators PAP and $JQAQJ$ are invertible.*

Of course, this follows immediately from Corollary 2.8. But there is also an elementary proof based on (9) which mimics the proof of the stability of the finite sections method for invertible Toeplitz operators with continuous generating function (see [10], Theorem 4.102 in [28] and Section 1.4.2 in [22] for the proof in the Toeplitz setting and [33] for band-dominated operators with almost periodic coefficients).

It is not always easy to find a distinguished sequence for a given operator in $\mathcal{A}_{AP}(\mathbb{Z})$. But sometimes it is, and here are two examples taken from [33].

Example 2.10 (Multiplication operators) For each real number $\alpha \in [0, 1)$, the function

$$a : \mathbb{Z} \to \mathbb{C}, \quad n \mapsto e^{2\pi i \alpha n} \tag{10}$$

is almost periodic. Indeed, for every integer k, $U_{-k} a U_k$ is the operator of multiplication by the function a_k with $a_k(n) = a(n+k) = e^{2\pi i \alpha k} a(n)$, i.e.,

$$U_{-k} a U_k = e^{2\pi i \alpha k} a. \tag{11}$$

Let $(U_{-k(n)} a U_{k(n)})$ by any sequence in $\{U_{-k} a U_k : k \in \mathbb{Z}\}$. Due to the compactness of $\mathbb{T}$, there are a subsequence $(e^{2\pi i \alpha k(n(r))})_{r \geq 1}$ of $(e^{2\pi i \alpha k(n)})_{n \geq 1}$ and a real number β such that

$$e^{2\pi i \alpha k(n(r))} \to e^{2\pi i \beta} \qquad \text{as } r \to \infty.$$

Thus, the functions $a_{k(n(r))} = e^{2\pi i \alpha k(n(r))} a$ converge in the norm of $l^\infty(\mathbb{Z})$ to $e^{2\pi i \beta} a$, whence the almost periodicity of a. Thus, every function as in (10) belongs to $AP(\mathbb{Z})$. Conversely, $AP(\mathbb{Z})$ is the closure in $l^\infty(\mathbb{Z})$ of the span of all functions of the form (10) with $\alpha \in [0, 1)$ ([16], Theorems 1.9 – 1.11 and Theorem 1.27).

For the operator spectrum of the operator aI one finds

$$\sigma_{op,\,s}(aI) = \sigma_{op,\,n}(aI) = \begin{cases} \{e^{2\pi i l/q}\, a : l = 1, 2, \ldots, q\} & \text{if} \quad \alpha = 2p/q \in \mathbb{Q}, \\ \{e^{it}\, a : t \in \mathbb{R}\} & \text{if} \quad \alpha \notin \mathbb{Q}, \end{cases}$$

Here, p and q are relatively prime integers with $q > 0$. Indeed, the inclusion $\subseteq$ follows immediately from (11). The reverse inclusion is evident in case $\alpha \in \mathbb{Q}$. If $\alpha \notin \mathbb{Q}$, then it follows from a theorem by Kronecker which states that the set of all numbers $e^{2\pi i \alpha k}$ with integer k lies dense in the unit circle $\mathbb{T}$.

Next we are looking for distinguished sequences for the operator of multiplication by the sequence 10. From (11) we infer that a sequence h is distinguished for aI if and only if

$$\lim_{n \to \infty} e^{2\pi i \alpha h(n)} = 1$$

In case $\alpha = p/q \in \mathbb{Q}$, the sequence a is q-periodic. Thus, $h(n) := qn$ is a distinguished sequence for aI. For non-rational $\alpha \in (0, 1)$, expand α into its continued fraction

$$\alpha = \lim_{n \to \infty} \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots \, b_{n-1} + \cfrac{1}{b_n}}}}$$

with uniquely determined positive integers b_i. Write this continued fraction as p_n/q_n with positive and relatively prime integers p_n, q_n. These integers satisfy the recursions

$$p_n = a_n p_{n-1} + p_{n-2}, \qquad q_n = a_n q_{n-1} + q_{n-2} \tag{12}$$

with $p_0 = 0$, $p_1 = 1$, $q_0 = 1$ and $q_1 = a_1$, and one has for all $n \geq 1$

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}. \tag{13}$$

Thus,

$$|\alpha q_n - p_n| \leq q_n \left| \alpha - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n} \to 0,$$

whence

$$e^{2\pi i \alpha q_n} = e^{2\pi i(\alpha q_n - p_n)} \to 1.$$

Since moreover $q_1 < q_2 < \ldots$ due to the recursion (12), this shows that the sequence $h(n) := q_n$ belongs to $\mathcal{H}_{A,n}$ and that $A_h = A$, i.e. h is a distinguished sequence for the operator aI with a as in (10). ■

Example 2.11 (Almost Mathieu operators) These are the operators $H_{\alpha,\lambda,\theta}$ on $l^2(\mathbb{Z})$ given by

$$(H_{\alpha,\lambda,\theta}x)_n := x_{n+1} + x_{n-1} + \lambda x_n \cos 2\pi(n\alpha + \theta)$$

with real parameters α, λ and θ. Thus, $H_{\alpha,\lambda,\theta}$ is a band operator with almost periodic coefficients, and

$$H_{\alpha,\lambda,\theta} = U_{-1} + U_1 + aI \qquad \text{with} \qquad a(n) = \lambda \cos 2\pi(n\alpha + \theta).$$

For a treatment of the spectral theory of almost Mathieu operators see [9] and the recently published papers [4, 29] where the long-standing *Ten Martini problem* is solved.

As in Example 2.10 one gets

$$U_{-k} H_{\alpha,\lambda,\theta} U_k = U_{-1} + U_1 + a_k I$$

with

$$\begin{aligned} a_k(n) &= a(n+k) = \lambda \cos 2\pi((n+k)\alpha + \theta) \\ &= \lambda(\cos 2\pi(n\alpha + \theta)\cos 2\pi k\alpha - \sin 2\pi(n\alpha+\theta)\sin 2\pi k\alpha). \end{aligned} \tag{14}$$

We will only consider the non-periodic case, i.e., we let $\alpha \in (0, 1)$ be irrational. As in the previous example, we write α as a continued fraction with nth approximant p_n/q_n such that (13) holds. Then

$$\cos 2\pi\alpha q_n = \cos 2\pi(\alpha q_n - p_n) = \cos 2\pi q_n(\alpha - p_n/q_n) \to \cos 0 = 1$$

and, similarly, $\sin 2\pi\alpha q_n \to 0$. Further we infer from (14) that

$$|(a_{q_n}) - a)(n)| \le |\lambda|\,|1 - \cos 2\pi\alpha q_n| + |\lambda|\,|\sin \pi\alpha q_n|.$$

Hence, $a_{q_n} \to a$ uniformly. Thus, $h(n) := q_n$ defines a distinguished sequence for the Almost Mathieu operator $H_{\alpha,\lambda,\theta}$. Notice that this sequence depends on the parameter α only. ■

Theorem 2.9 implies the following.

Corollary 2.12 *Let $A := H_{\alpha,\lambda,\theta}$ be an Almost Mathieu operator and h a distinguished sequence for A. Then the following conditions are equivalent:*

(a) the distinguished finite sections method $(P_{h(n)}PAPP_{h(n)})$ for PAP is stable;

(b) the distinguished finite sections method $(R_{h(n)}AR_{h(n)})$ for A is stable;

(c) the operators PAP and QAQ are invertible.

If $\theta = 0$, then the Almost Mathieu operator $A = H_{\alpha,\lambda,0}$ is flip invariant, i.e., $JAJ = A$. So we observe in this case that the third condition in Corollary 2.12 is equivalent to the invertibility of PAP alone.

For a different numerical treatment of Almost Mathieu and other operators in irrational rotation algebras consult [15].

3 The first Szegö limit theorem

3.1 Operators with rich spectrum

Let A be an operator on $l^2(\mathbb{N})$ for which the finite sections sequence (P_nAP_n) is stable. Then the matrices P_nAP_n are invertible for n large enough, and it makes sense to consider the sequence

$$n \mapsto \frac{\det(P_nAP_n)}{\det(P_{n-1}AP_{n-1})}. \tag{15}$$

In case $A = T(a)$ is an invertible Toeplitz operator with continuous generating function, the sequence (15) converges, and its limit is equal to

$$G[a] := 1/(P_1T(a)^{-1}P_1) \tag{16}$$

by the first Szegö limit theorem 1.1. For general A, one cannot expect convergence of (15) as already the band operator

$$A := \operatorname{diag}\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \ldots\right)$$

shows. In this case we denote by $\omega(A)$ the set of all partial limits of the sequence (15). It turns out that this set can be described via limit operators in case A is an operator with rich operator spectrum for which the finite sections method is stable. In this case, the operators JQA_hQJ are invertible on $l^2(\mathbb{Z}^+)$ by Theorem 2.6. In analogy to (16), set

$$G[A_h] := 1/(P_1(JQA_hQJ)^{-1}P_1) \tag{17}$$

which has to be read as follows: $P_1(JQA_hQJ)^{-1}P_1$ can be understood as an 1×1-matrix, and we identify this matrix with its only entry, which is a complex number.

Theorem 3.1 *Let $A \in L^\$(l^2(\mathbb{Z}^+))$ be an operator for which the finite sections sequence (P_nAP_n) is stable. Then*

$$\omega(A) = \{G[A_h] : A_h \in \sigma_+(A)\}. \tag{18}$$

Proof. For n a positive integer, let

$$W_n : l^2(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+), \quad (x_0, x_1, \ldots) \mapsto (x_{n-1}, x_{n-2}, \ldots, x_0, 0, 0, \ldots). \tag{19}$$

If the finite sections method (P_nAP_n) is stable, then the operators W_nAW_n, considered as acting on $\operatorname{im} W_n = \operatorname{im} P_n$, are invertible for large n, and

$$\frac{\det(P_{n-1}AP_{n-1})}{\det(P_nAP_n)} = \frac{\det(W_{n-1}AW_{n-1})}{\det(W_nAW_n)} =: \beta_n.$$

By Cramer's rule, β_n equals the first component of the solution $x^{(n)}$ to the equation

$$W_nAW_nx^{(n)} = (1, 0, 0, \ldots, 0)^T.$$

Let now $\alpha \in \omega(A)$, and let $h : \mathbb{N} \to \mathbb{N}$ be a sequence tending to infinity such that $\alpha^{-1} = \lim \beta_{h(n)}$. Since A has a rich operator spectrum, there is a subsequence g of h such that the limit operator

$$A_g = \text{s-lim}\, U_{-g(n)}AU_{g(n)} \in L(l^2(\mathbb{Z}))$$

exists. Then also the strong limit

$$\text{s-lim}\, JU_{-g(n)}P_{g(n)}AP_{g(n)}U_{g(n)}J \in L(l^2(\mathbb{N}))$$

exists and is equal to JQA_gQJ. Since $JU_{-n}P_n = W_n$ and $P_nU_nJ = W_n$, this shows that the strong limit $\text{s-lim}\, W_{g(n)}AW_{g(n)}$ exists and that this limit is equal to $JQA_gQJ \in L(l^2(\mathbb{N}))$. So one can consider $(W_{g(n)}AW_{g(n)})_{n\in\mathbb{N}}$ as a stable and convergent approximation sequence for the operator JQA_gQJ. In particular, the solutions $x^{(n)}$ to the equation

$$W_{g(n)}AW_{g(n)}x^{(n)} = (1, 0, 0, \ldots, 0)^T \tag{20}$$

converge in the norm of $l^2(\mathbb{N})$ to the solution x to the equation

$$JQA_gQJx = (1, 0, 0, \ldots)^T. \tag{21}$$

Thus, the first component $\beta_{g(n)}$ of the solution $x^{(n)}$ to equation (20) converges to the first component of the solution x to equation (21). Since the latter one is equal to

$$P_1x = P_1(JQA_gQJ)^{-1}P_1,$$

we arrive at $\alpha = (P_1(JQA_gQJ)^{-1}P_1)^{-1} = G[A_g]$. This settles the inclusion $\subseteq$ in (18). The reverse inclusion can be proved by similar arguments. ∎

3.2 Operators in the Toeplitz algebra

By Proposition 2.3 (b), the assertion of Theorem 3.1 holds in particular for operators in the algebra $\mathcal{A}_{L^\infty(\mathbb{T}),l^\infty(\mathbb{Z})}(\mathbb{Z}^+)$ and, thus, for all band-dominated operators $A \in \mathcal{A}_{l^\infty(\mathbb{Z})}(\mathbb{Z}^+)$ and for all operators A in the Toeplitz algebra $\mathcal{A}_{L^\infty(\mathbb{T}),\mathbb{C}}(\mathbb{Z}^+)$. The statement for band-operators has been already proved in [35], Theorem 7.23, whereas the Toeplitz case was the subject of Section 7.2.3 in [22]. In the Toeplitz case, one can complete the assertion of Theorem 3.1 essentially. The point is the following observation.

Proposition 3.2 *Let* $A \in \mathcal{A}_{L^\infty(\mathbb{T}),\mathbb{C}}(\mathbb{Z}^+)$.
(a) *Consider* A *as an operator on* $l^2(\mathbb{Z})$ *which acts as the zero operator on* l^2 *over the negative integers. Then the sequence* $(U_{-n}AU_n)_{n\in\mathbb{N}}$ *converges* **-strongly on* $l^2(\mathbb{Z})$. *Its limit is a bounded Laurent operator, i.e., it is of the form* $L(a)$ *with* $a \in L^\infty(\mathbb{T})$.
(b) *The sequence* $(W_nAW_n)_{n\in\mathbb{N}}$ *converges* **-strongly on* $l^2(\mathbb{Z}^+)$. *Its limit is a bounded Toeplitz operator, i.e., it is of the form* $T(b)$ *with* $b \in L^\infty(\mathbb{T})$.
Moreover, $b(t) = a(1/t)$ *a.e. on* $\mathbb{T}$.

The function a is also called the *symbol* of the operator $A \in \mathcal{A}_{L^\infty(\mathbb{T}),\mathbb{C}}(\mathbb{Z}^+)$. We denote it by s_A.

For a proof of assertion (a), write $T(a)$ as $PL(a)P$. Clearly, $U_{-n}L(a)U_n = L(a)$, and one easily checks that $U_{-n}PU_n \to I$ strongly. Thus,

$$U_{-n}T(a)U_n \to L(a) \quad \text{as } n \to \infty.$$

Assertion (b) follows from (a) since

$$W_nAW_n = JQU_{-n}AU_nQJ.$$

For another proof of (b) (and some facts around it) see Sections 4.3.3 and 7.2.3 in [22]. ∎

It follows from Proposition 3.2 that the only limit operator at $+\infty$ of $A \in \mathcal{A}_{L^\infty(\mathbb{T}),\mathbb{C}}(\mathbb{Z}^+)$ is the Laurent operator $L(s_A)$. Hence, the set $\omega(T(a))$ is the singleton $\{G[T(\widetilde{s_A})]\}$ in this case, whence the convergence of the sequence (15) to this value.

Corollary 3.3 *Let $A \in \mathcal{A}_{L^\infty(\mathbb{T}),\mathbb{C}}(\mathbb{Z}^+)$ be an operator for which the finite sections sequence (P_nAP_n) is stable. Then the sequence* (15) *converges, and its limit is*

$$G[T(\widetilde{s_A})] = 1/(P_1(T(\widetilde{s_A}))^{-1}P_1).$$

Corollary 3.4 *Let $a \in L^\infty(\mathbb{T})$ be such that the finite sections sequence (P_nAP_n) for the Toeplitz operator $A = T(a)$ is stable. Then the sequence* (15) *converges, and its limit is*

$$G[T(\tilde{a})] = 1/(P_1(T(\tilde{a}))^{-1}P_1).$$

In order to show that this corollary indeed reproduces the first Szegö limit theorem 1.1 we have to verify that

$$P_1T(a)^{-1}P_1 = P_1(T(\tilde{a}))^{-1}P_1. \tag{22}$$

Let $C : l^2(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+)$ denote the operator of conjugation $(x_n) \mapsto (\overline{x_n})$ (which is linear over the field of the real numbers only). One easily checks that

$$T(\tilde{a}) = CT(a)^*C \quad \text{for each function } a \in L^\infty(\mathbb{T}).$$

Hence, $T(a)$ is invertible if and only if $T(\tilde{a})$ is invertible, and if B is the inverse of $T(a)$, then CB^*C is the inverse of $T(\tilde{a})$. The 00th entries of B and CB^*C coincide obviously, whence (22). ■

There are two obstacles for the application of Corollary 3.4. The first one concerns the stability of the finite sections sequence $(P_nT(a)P_n)$ for which there is no general criterion known. But there are at least special classes of generating functions $a \in L^\infty(\mathbb{T})$ (e.g., piecewise continuous or piecewise quasicontinuous functions) for which one knows that the finite sections sequence for the Toeplitz operator $T(a)$ is stable if and only the operator $T(a)$ is invertible, and for which effective criteria for the invertibility of $T(a)$ are available. Details can be found in Section IV.3 in [21], Section 4.2 in [22] and Section 2.4 in [13] for Toeplitz operators with piecewise continuous generating functions and in Chapter 7 in [12] where a heavy machinery is developed to attack stability problems.

The second point concerns the constant $G[a] = (P_1T(a)^{-1}P_1)^{-1}$ for which one wants to have an effective way of computation. Under suitable assumptions for the generating function a (e.g., belonging to the Wiener algebra or being locally sectorial) one can identify the number $G[a]$ with $1/\exp(\log a)_0$ with b_0 referring to the 0th Fourier coefficient of the function b (details can be found in Section 5.4 of [13], for example).

The latter observation offers a also way to determine the constant $G[A_h]$ in some further instances. Let $A \in L(l^2(\mathbb{Z}^+))$ be a band-dominated operator with slowly oscillating coefficients where a function $b \in l^\infty(\mathbb{Z})$ is called *slowly oscillating* if the difference $b(n+1) - b(n)$ tends to zero as $n \to \pm\infty$. Then all limit operators of A are shift invariant (Proposition 2.4.1 in [32]); hence, all partial limits in $\omega(A)$ are of the form $P_1 T(\tilde{a_h})^{-1} P_1$ with a certain continuous function a_h. If, moreover, $A = \sum a_k V_k$ satisfies the Wiener condition $\sum \|a_k\|_\infty < \infty$, then all functions a_h belong to the Wiener algebra, and one has

$$P_1 T(\tilde{a_h})^{-1} P_1 = P_1 T(a_h)^{-1} P_1 = 1/\exp(\log a_h)_0.$$

4 Distributive versions of the first Szegö limit theorem

The goal of this section is to prove versions of Theorems 1.2 and 1.3 for operators in $\mathcal{A}_{L^\infty(\mathbb{T}), AP(\mathbb{Z})}(\mathbb{Z}^+)$. For their formulation, we need some preparations.

It will be convenient to put the proof into some algebraic framework which has been developed by Arveson, Bédos, and SeLegue [1, 2, 7, 8, 36] (see also Section 7.2.1 in [22]) and which we are going to recall first. For the reader's convenience, we include the proofs.

4.1 The Følner algebra

For each operator $A \in L(l^2(\mathbb{Z}^+))$, let $|A|$ denote its absolute value, i.e., the non-negative square root of A^*A. Let further tr refer to the canonical trace on $L(l^2(\mathbb{Z}^+))$, and abbreviate the sequence (P_n) to $\mathcal{P}$. Evidently, $\operatorname{tr} P_n = n$.

Proposition 4.1 *The set $\mathfrak{F}(\mathcal{P})$ of all operators $A \in L(l^2(\mathbb{Z}^+))$ with*

$$\lim_{n\to\infty} \frac{\operatorname{tr}(|P_n A - A P_n|)}{\operatorname{tr} P_n} = 0 \tag{23}$$

is a C^*-subalgebra of $L(l^2(\mathbb{Z}^+))$.

We refer to $\mathfrak{F}(\mathcal{P})$ as the *Følner algebra* associated with $\mathcal{P}$.

Proof. Recall that the set $\mathcal{N}_1 := \{A \in L(l^2(\mathbb{Z}^+)) : \operatorname{tr}(|A|) < \infty\}$ of the trace class operators is a two-sided (non-closed) ideal of $L(l^2(\mathbb{Z}^+))$, that the mapping $A \mapsto \operatorname{tr}(|A|)$ defines a norm on $\mathcal{N}_1$ which makes this set to a Banach space, and that

$$|\operatorname{tr}(A)| \le \operatorname{tr}(|A|), \tag{24}$$

$$\operatorname{tr}(|A+B|) \le \operatorname{tr}(|A|) + \operatorname{tr}(|B|), \tag{25}$$

$$\max\{\operatorname{tr}(|AC|), \operatorname{tr}(|CA|)\} \le \|C\| \operatorname{tr}(|A|), \tag{26}$$

$$\operatorname{tr}(|A|) = \operatorname{tr}(|A^*|) \tag{27}$$

for arbitrary operators $A, B \in \mathcal{N}_1$ and $C \in L(l^2(\mathbb{Z}^+))$. For details see [34], Section VI.6. Let now $A, B \in \mathfrak{F}(\mathcal{P})$. Then

$$\operatorname{tr}(|P_n(A+B) - (A+B)P_n|) \leq \operatorname{tr}(|P_nA - AP_n|) + \operatorname{tr}(|P_nB - BP_n|)$$

and

$$\begin{aligned} \operatorname{tr}(|P_n(AB) - (AB)P_n|) &= \operatorname{tr}(|(P_nA - AP_n)B + A(P_nB - BP_n)|) \\ &\leq \|B\| \operatorname{tr}(|P_nA - AP_n|) + \|A\| \operatorname{tr}(|P_nB - BP_n|) \end{aligned}$$

by (25) and (26), which implies that $A + B$ and AB are in $\mathfrak{F}(\mathcal{P})$ again. Further, if $A_m \in \mathfrak{F}(\mathcal{P})$ and $A_m \to A$ in the norm of $L(l^2(\mathbb{Z}^+))$, then

$$\begin{aligned} \operatorname{tr}(|P_nA - AP_n|) &\leq \operatorname{tr}(|P_n(A - A_m) - (A - A_m)P_n|) + \operatorname{tr}(|P_nA_m - A_mP_n|) \\ &\leq 2 \operatorname{tr} P_n \|A - A_m\| + \operatorname{tr}(|P_nA_m - A_mP_n|), \end{aligned}$$

which gives the closedness of $\mathfrak{F}(\mathcal{P})$ in $L(l^2(\mathbb{Z}^+))$. The symmetry of $\mathfrak{F}(\mathcal{P})$ is a consequence of (27). ■

Let $\mathcal{S}(\mathfrak{F}(\mathcal{P}))$ stand for the smallest closed subalgebra of $\mathcal{F}^{\mathcal{P}}$ which contains all finite sections sequences (P_nAP_n) where A is in $\mathfrak{F}(\mathcal{P})$. The following result is the key to several generalizations of the first Szegö limit theorem.

Theorem 4.2 *Let* $\mathbf{A} := (A_n) \in \mathcal{S}(\mathfrak{F}(\mathcal{P}))$. *Then*

$$\frac{1}{n} \operatorname{tr}(|A_n - P_nW(\mathbf{A})P_n|) \to 0 \tag{28}$$

as $n \to \infty$.

Proof. By (26), the functionals

$$L(\operatorname{im} P_n) \to \mathbb{C}, \quad A_n \mapsto \frac{1}{n} \operatorname{tr}(|A_n|)$$

are uniformly bounded with respect to n (by 1). Hence, it is sufficient to prove (28) for sequences $\mathbf{A}$ in a dense subalgebra of $\mathcal{S}(\mathfrak{F}(\mathcal{P}))$.

Every sequence in $\mathcal{S}(\mathfrak{F}(\mathcal{P}))$ can be approximated as closely as desired (with respect to the norm in $\mathcal{F}^{\mathcal{P}}$) by sequences of the form

$$\mathbf{B} := \sum_j \prod_i (P_nB_{ij}P_n) \quad \text{where} \quad B_{ij} \in \mathfrak{F}(\mathcal{P}).$$

Clearly,

$$W(\mathbf{B}) = \sum_i \prod_j B_{ij}.$$

Thus, and by (25), it is sufficient to prove (28) for sequences of the form $\mathbf{B} := \prod_i (P_n B_i P_n)$ where $B_i \in \mathfrak{F}(\mathcal{P})$, i.e., to verify that

$$\frac{1}{n}\mathrm{tr}(|P_n B_1 P_n B_2 P_n \dots P_n B_k P_n - P_n B_1 B_2 \dots B_k P_n|) \to 0 \tag{29}$$

as $n \to \infty$. We prove (29) in case $k = 2$ from which the case of general k follows by induction. Assertion (29) for $k = 2$ will follow as soon as we have shown that

$$\begin{aligned} &\mathrm{tr}\,(|P_n B_1 P_n B_2 P_n - P_n B_1 B_2 P_n|) \\ &\quad \le \ \max\{\|B_2\|\,\mathrm{tr}\,(|P_n B_1 - B_1 P_n|),\ \|B_1\|\,\mathrm{tr}\,(|P_n B_2 - B_2 P_n|)\} \end{aligned}$$

for arbitrary operators $B_1,\, B_2 \in L(l^2(\mathbb{Z}^+))$. This estimate is a consequence of

$$\begin{aligned} \mathrm{tr}\,(|P_n B_1 P_n B_2 P_n - P_n B_1 B_2 P_n|) &= \mathrm{tr}\,(|P_n B_1 (I - P_n) B_2 P_n|) \\ &\le \|B_1\|\,\mathrm{tr}\,((I - P_n) B_2 P_n|) \end{aligned}$$

and of

$$\begin{aligned} \mathrm{tr}\,(|(I - P_n) B_2 P_n|) &= \mathrm{tr}\,(|(I - P_n)(B_2 P_n - P_n B_2)|) \\ &\le \|I - P_n\|\,\mathrm{tr}\,(|P_n B_2 - B_2 P_n|) \end{aligned}$$

where we used (26). ■

From (24) and (28) we conclude that

$$\frac{1}{n}|\mathrm{tr}\,(A_n - P_n W(\mathbf{A}) P_n)| \to 0.$$

Thus, if $(w_{ij})_{i,j=0}^{\infty}$ refers to the matrix representation of $W(\mathbf{A})$ with respect to the standard basis of $l^2(\mathbb{Z}^+)$, then (28) implies

$$\left(\frac{\lambda_1(A_n) + \ldots + \lambda_n(A_n)}{n} - \frac{w_{00} + \ldots + w_{n-1,\,n-1}}{n}\right) \to 0 \tag{30}$$

as $n \to \infty$ for every sequence $\mathbf{A} := (A_n) \in \mathcal{S}(\mathfrak{F}(\mathcal{P}))$.

Remark 4.3 It is evident that the notion of a Følner algebra is not restricted to the context considered in this section. Indeed, for every sequence $\mathcal{P} = (P_n)$ of orthogonal projections of finite rank acting on a certain Hilbert space and tending strongly to the identity operator, there is an associated Følner algebra. This observation allows one to derive distributive versions of the first Szegö limit theorem also in the higher dimensional context, by employing exactly the same ideas which will be pointed out in the following sections. In this way, the results of [26, 38] can be both easily obtained and generalized.

4.2 Operators and their diagonals

A further utilization of (28) and (30) requires to examine the trace tr $(P_n W(\mathbf{A}) P_n)$ which clearly depends on the main diagonal of the operator $W(\mathbf{A})$ only. In this section we show that the main diagonal of operators in $\mathcal{A}_{L^\infty(\mathbb{T}), AP(\mathbb{Z})}(\mathbb{Z}^+)$ behaves quite well.

Let $A \in L(l^2(\mathbb{Z}))$ be an operator with matrix representation $(a_{ij})_{i,j\in\mathbb{Z}}$ with respect to the standard basis of $l^2(\mathbb{Z})$. Since

$$|a_{ii}| = \|P_{\{i\}} A P_{\{i\}}\| \leq \|A\|,$$

the sequence $(a_{ii})_{i\in\mathbb{Z}}$ belongs to $l^\infty(\mathbb{Z})$. Hence, it defines a multiplication operator on $l^2(\mathbb{Z})$ which we call the *main diagonal of* A and which we denote by $D(A)$. Similarly, the main diagonal of an operator $B \in L(l^2(\mathbb{Z}^+))$ is defined. It acts as a multiplication operator on $l^2(\mathbb{Z}^+)$, and we denote it also by $D(B)$ (which will not rise confusion if one takes into account where A and B live). In each case, $\|D(A)\| \leq \|A\|$.

Theorem 4.4 *If $A \in \mathcal{A}_{L^\infty(\mathbb{T}), AP(\mathbb{Z})}(\mathbb{Z})$, then $D(A) \in AP(\mathbb{Z})$.*

Of course, then *every* diagonal which is parallel to the main diagonal is almost periodic, too.

Proof. Since $D : L(l^2(\mathbb{Z})) \to l^\infty(\mathbb{Z})$ is a continuous linear mapping, and since $AP(\mathbb{Z})$ is a closed subalgebra of $l^\infty(\mathbb{Z})$, it is sufficient to prove the assertion for the case when A is a finite product of Laurent operators with generating functions in $L^\infty(\mathbb{T})$ and of operators of multiplication by almost periodic functions. Thus, we can assume that

$$A = L(a_1)\, b_1\, L(a_2)\, b_2\, \ldots\, L(a_k)\, b_k I$$

with $a_i \in L^\infty(\mathbb{T})$ and $b_i \in AP(\mathbb{Z})$. Consider the diagonal $D(A)$ and let $h : \mathbb{N} \to \mathbb{Z}$ be an arbitrary sequence. We have to show that $(U_{-h(n)} D(A) U_{h(n)})_{n\in\mathbb{N}}$ has a norm convergent subsequence. Since

$$U_{-h(n)} D(A) U_{h(n)} = D(U_{-h(n)} A U_{h(n)})$$

it is sufficient to show that $(U_{-h(n)} A U_{h(n)})_{n\in\mathbb{N}}$ has a convergent subsequence. Now one has

$$\begin{aligned} &U_{-h(n)} A U_{h(n)} \\ &\quad = L(a_1)\, (U_{-h(n)} b_1 U_{h(n)})\, L(a_2)\, (U_{-h(n)} b_2 U_{h(n)})\, \ldots\, L(a_k)\, (U_{-h(n)} b_k U_{h(n)}). \end{aligned}$$

Since b_1 is almost periodic, there is a subsequence h_1 of h such that the sequence $(U_{-h_1(n)} b_1 U_{h_1(n)})_{n\in\mathbb{N}}$ converges. Analogously, there is a subsequence h_2 of h_1 such that the sequence $(U_{-h_2(n)} b_2 U_{h_2(n)})_{n\in\mathbb{N}}$ converges. We proceed in this way. After k steps we arrive at a subsequence g of h for which each of the sequences $(U_{-g(n)} b_i U_{g(n)})_{n\in\mathbb{N}}$ and, thus, the sequence $(U_{-g(n)} A U_{g(n)})_{n\in\mathbb{N}}$ converges. ■

Let $c_0(\mathbb{Z}^+)$ stand for the set of all sequences $a : \mathbb{Z}^+ \to \mathbb{C}$ with $a(n) \to 0$ as $n \to \infty$, and write $AP(\mathbb{Z}^+)$ for the set of all functions PaP where $a \in AP(\mathbb{Z})$, considered as functions on $\mathbb{Z}^+$. Evidently, both $c_0(\mathbb{Z}^+)$ and $AP(\mathbb{Z}^+)$ are closed subalgebras of $l^\infty(\mathbb{Z}^+)$.

Theorem 4.5 *If $A \in \mathcal{A}_{L^\infty(\mathbb{T}),AP(\mathbb{Z})}(\mathbb{Z}^+)$, then $D(A) \in AP(\mathbb{Z}^+) + c_0(\mathbb{Z}^+)$.*

Proof. As is the proof of the previous theorem, it is sufficient to verify the assertion for operators of the form

$$\begin{aligned} A &= T(a_1)\, b_1\, T(a_2)\, b_2 \, \ldots \, T(a_k)\, b_k I \\ &= PL(a_1)Pb_1PL(a_2)Pb_2 \, \ldots \, PL(a_k)Pb_kP \end{aligned}$$

with $a_i \in L^\infty(\mathbb{T})$ and $b_i \in AP(\mathbb{Z}^+)$. We replace all inner projections P by $I - Q$ and factor out to get

$$A = PBP + R \quad \text{where} \quad B \in \mathcal{A}_{L^\infty(\mathbb{T}),AP(\mathbb{Z})}(\mathbb{Z}^+) \tag{31}$$

and where R is a finite sum, with each item in this sum being a product of Laurent operators, multiplication operators, projections P and *at least one* projection Q. Evidently, the projections P and Q have a rich operator spectrum, and $\sigma_+(Q) = \{0\}$. Since the set $L^\$(l^2(\mathbb{Z}))$ forms an algebra we conclude that the operator R has a rich operator spectrum, too, and the algebraic properties of limit operators stated in Proposition 1.2.2 in [32] yield that also $\sigma_+(R) = \{0\}$.

We claim that the main diagonal $D(R) =: \operatorname{diag}(r_{nn})$ of R is in $c_0(\mathbb{Z}^+)$. Suppose it is not. Then there is a $C > 0$ and a strongly monotonically increasing sequence $h : \mathbb{N} \to \mathbb{N}$ such that $|r_{h(n),h(n)}| \geq C$ for all $n \in \mathbb{N}$. Since $R \in L^\$(l^2(\mathbb{Z}))$ there is a subsequence g of h for which the limit operator R_g exists. Since h (thus, g) tends to $+\infty$, one has $R_g \in \sigma_+(R)$, whence $R_g = 0$. This implies in particular that

$$r_{g(n),g(n)} = P_1 U_{-g(n)} R U_{g(n)} P_1 \to 0,$$

a contradiction. Thus, $D(R) \in c_0(\mathbb{Z}^+)$, and passing to the main diagonals in (31) yields

$$D(A) = PD(B)P + D(R) \in AP(\mathbb{Z}^+) + c_0(\mathbb{Z}^+)$$

due to Theorem 4.4. ■

Proposition 4.6 *Each function $a \in AP(\mathbb{Z}^+) + c_0(\mathbb{Z}^+)$ has a unique representation in the form $a = PfP + c$ where $f \in AP(\mathbb{Z})$ and $c \in c_0(\mathbb{Z}^+)$.*

Proof. Let $f_1,\, f_2 \in AP(\mathbb{Z})$ and $c_1,\, c_2 \in c_0(\mathbb{Z}^+)$ be such that $Pf_1P + c_1 = Pf_2P + c_2$. Then $Pf_1P - Pf_2P = c_2 - c_1$. One easily checks that $c_2 - c_1$ is the restriction of an almost periodic function only if $c_2 - c_1 = 0$. Thus, $c_1 = c_2$ and $Pf_1P = Pf_2P$. The latter identity implies $f_1 = f_2$ by Corollary 3.3 in [33]. ■

Thus, for each operator $A \in \mathcal{A}_{L^\infty(\mathbb{T}),AP(\mathbb{Z})}(\mathbb{Z}^+)$, there is a uniquely determined function $f \in AP(\mathbb{Z})$ such that $D(A) - PfP \in c_0(\mathbb{Z}^+)$. We call this function the *almost periodic part of the main diagonal of* A and denote it by $D_{ap}(A)$. Note that $D_{ap}(PAP) = D(A)$ for each operator $A \in \mathcal{A}_{L^\infty(\mathbb{T}),AP(\mathbb{Z})}(\mathbb{Z})$.

4.3 The first Szegö limit theorem

We are now going to formulate a general version of the first Szegö limit theorem which will imply all other versions of Szegö limit theorems as particular instances. This version is based on a fundamental property of every almost periodic function a, namely that the arithmetic means

$$\frac{1}{n}\sum_{r=0}^{n-1} a(r) \tag{32}$$

tend to some value $M(a)$ called the *mean value of* a (see [16], Theorem 1.28 or [23], Example (b) in Section (18.15)).

Theorem 4.7 *Let* $\mathbf{A} = (A_n) \in \mathcal{S}_{L^\infty(\mathbb{T}),AP(\mathbb{Z})}(\mathbb{Z}^+)$. *Then*

$$\lim_{n\to\infty} \frac{\lambda_1(A_n) + \ldots + \lambda_n(A_n)}{n} = M(D_{ap}(W(\mathbf{A}))). \tag{33}$$

Proof. It is shown in Corollary 1 in [36] and in Section 7.2.1 of [22] that the Følner algebra $\mathfrak{F}(\mathcal{P})$ contains all Laurent operators and all band-dominated operators. Hence, $\mathcal{A}_{L^\infty(\mathbb{T}),AP(\mathbb{Z})}(\mathbb{Z}^+)$ is a subalgebra of the Følner algebra, and (28) and (24) imply

$$\frac{1}{n}|\mathrm{tr}\,(A_n - P_nW(\mathbf{A})P_n)| = \frac{1}{n}|\mathrm{tr}\,(A_n) - \mathrm{tr}\,(P_nW(\mathbf{A})P_n)| \to 0. \tag{34}$$

Evidently, $\mathrm{tr}\,(A_n) = \lambda_1(A_n) + \ldots + \lambda_n(A_n)$, and it remains to show that

$$\frac{1}{n}\mathrm{tr}\,(P_nW(\mathbf{A})P_n) \to M(D_{ap}(W(\mathbf{A}))). \tag{35}$$

Since $\mathbf{A} \in \mathcal{S}_{L^\infty(\mathbb{T}),AP(\mathbb{Z})}(\mathbb{Z}^+)$, one has $W(\mathbf{A}) \in \mathcal{A}_{L^\infty(\mathbb{T}),AP(\mathbb{Z})}(\mathbb{Z}^+)$. Then, by Proposition 4.6,

$$\frac{1}{n}\mathrm{tr}\,(P_nW(\mathbf{A})P_n) = \frac{1}{n}\mathrm{tr}\,(P_nD(W(\mathbf{A}))P_n) = \frac{1}{n}\left(\sum_{k=1}^{n} D_{ap}(W(\mathbf{A}))(k) + \sum_{k=1}^{n} c(k)\right)$$

with a certain function $c \in c_0(\mathbb{Z}^+)$. Since $\frac{1}{n}\sum_{k=1}^n c(k) \to 0$, and by what has been said before Theorem 4.7, the convergence (35) follows. ■

Note that it is exactly the mean value property of the almost periodic functions which allows us to prove the existence of the limit in (33).

Remark 4.8 For Toeplitz operators, the block case is considered as being of particular interest. In order to see how the block case follows from Theorem 4.7 we mention an obvious generalization of that theorem. Let $\eta : \mathbb{N} \to \mathbb{N}$ be a strongly monotonically increasing sequence. In place of the sequence $\mathbf{A} = (A_n) \in \mathcal{S}_{L^\infty(\mathbb{T}),AP(\mathbb{Z})}(\mathbb{Z}^+)$ we consider its subsequence $(A_{\eta(n)})$. Then the limit

$$\lim_{n\to\infty} \frac{\operatorname{tr}(A_{\eta(n)})}{\operatorname{tr}(P_{\eta(n)})} = \lim_{n\to\infty} \frac{\lambda_1(A_{\eta(n)}) + \ldots + \lambda_{\eta(n)}(A_{\eta(n)})}{\eta(n)}$$

exists and is equal to $M(D_{ap}(W(\mathbf{A})))$. The block case follows if one allows for d-periodic coefficients only and if one chooses $\eta(n) := dn$.

5 Special cases

5.1 Szegö-type theorems

Continuous functions of sequences. Here we are going to derive versions of Theorem 4.7 which hold for functions of sequences in $\mathcal{S}_{L^\infty(\mathbb{T}),AP(\mathbb{Z})}(\mathbb{Z}^+)$. Of course, they cannot yield anything which is substantially new since continuous functions of normal elements of this algebra belong to $\mathcal{S}_{L^\infty(\mathbb{T}),AP(\mathbb{Z})}(\mathbb{Z}^+)$ again. But they will bring us closer to the formulation of the classical Szegö limit theorems.

Theorem 5.1 *Let $\mathbf{A} = (A_n)$ be a normal sequence in $\mathcal{S}_{L^\infty(\mathbb{T}),AP(\mathbb{Z})}(\mathbb{Z}^+)$, and let g be any function which is continuous on a neighborhood in $\mathbb{R}$ of the stability spectrum $\sigma(\mathbf{A} + \mathcal{G})$. Then*

$$\lim_{n\to\infty} \frac{g(\lambda_1(A_n)) + \ldots + g(\lambda_n(A_n))}{n} = M(D_{ap}(g(W(\mathbf{A})))). \tag{36}$$

Proof. Let U be a neighborhood of $\sigma(\mathbf{A} + \mathcal{G})$ in $\mathbb{R}$ and g continuous on U. By Proposition 2.2,

$$\sigma(A_n) \subseteq U \quad \text{and} \quad \sigma(W(\mathbf{A})) \subseteq U,$$

and A_n and $W(\mathbf{A})$ are normal. Thus, $g(A_n)$ and $g(W(\mathbf{A}))$ are well-defined via the continuous functional calculus for normal elements of a C^*-algebra (Theorem 6.2.7 in [3]). Without loss we can also assume that $\sigma_{\mathcal{F}^\mathcal{P}}(\mathbf{A}) \subseteq U$ such that $g(\mathbf{A})$ is well-defined. Indeed, the spectrum of $\mathbf{A}$ in $\mathcal{F}^\mathcal{P}$ is the union of all spectra $\sigma(A_n)$ with the stability spectrum of $\mathbf{A}$. Thus, there is a finitely supported sequence $\mathbf{G}$ such that the spectrum of $(B_n) = \mathbf{B} := \mathbf{A} + \mathbf{G}$ lies in U. Since $B_n = A_n$ for sufficiently large n and since $W(\mathbf{B}) = W(\mathbf{A})$, one can replace $\mathbf{A}$ by $\mathbf{B}$ without loss. Clearly, one also has $B_n = g(A_n)$ for sufficiently large n.

Applying (33) to the sequence $g(A)$ yields

$$\lim_{n\to\infty} \frac{\lambda_1(g(A_n)) + \ldots + \lambda_n(g(A_n))}{n} = M(D_{ap}(W(g(\mathbf{A})))). \tag{37}$$

The continuous functional calculus for normal elements (or the Gelfand-Naimark theory for commutative C^*-algebras) further tells us that

$$\sigma(g(A_n)) = g(\sigma(A_n)) \tag{38}$$

for all n with $\sigma(A_n) \subseteq U$. Thus,

$$\lambda_1(g(A_n)) + \ldots + \lambda_n(g(A_n)) = g(\lambda_1(A_n)) + \ldots + g(\lambda_n(A_n)). \tag{39}$$

Finally one has

$$W(g(\mathbf{A})) = g(W(\mathbf{A})). \tag{40}$$

This equality is evident when $g(\lambda) = p(\lambda, \overline{\lambda})$ where p is a polynomial in two variables, in which case one has

$$g(W(\mathbf{A})) = p(W(\mathbf{A}), W(\mathbf{A})^*),$$

and it follows for general g since every compactly supported continuous function can be uniformly approximated by polynomials of the form $\lambda \mapsto p(\lambda, \overline{\lambda})$ due to the Stone-Weierstraß theorem (Theorem IV.10 in [34]). The equalities (37), (39) and (40) imply the assertion. ∎

Holomorphic functions of sequences. Next we will discuss a version for non-normal elements which has to be based on the holomorphic functional calculus. Recall that, for each element b of a Banach algebra $\mathcal{B}$ with identity e and for each function g which is holomorphic in a neighborhood U of $\sigma_{\mathcal{B}}(b)$, the element $g(b)$ is defined by

$$g(b) := \frac{1}{2\pi i} \int_\Gamma g(\zeta)(\zeta e - b)^{-1}\, d\zeta \tag{41}$$

where Γ is a smooth oriented Jordan curve in $U \setminus \sigma_{\mathcal{B}}(b)$ which surrounds $\sigma_{\mathcal{B}}(b)$. This definition is independent of the choice of Γ, and it settles a homomorphism from the algebra of the holomorphic function s on U into $\mathcal{B}$ which is continuous in the sense that if a sequence (g_n) converges to g uniformly on compact subsets of U, then $g(b) = \lim g_n(b)$ in the norm of $\mathcal{B}$. Moreover,

$$\sigma_{\mathcal{B}}(g(b)) = g(\sigma_{\mathcal{B}}(b)). \tag{42}$$

For details see [3], Section III.3.

Theorem 5.2 *Let $\mathbf{A} = (A_n)$ be a sequence in $\mathcal{S}_{L^\infty(\mathbb{T}),AP(\mathbb{Z})}(\mathbb{Z}^+)$, and let g be any function which is holomorphic on a neighborhood U in $\mathbb{C}$ of the stability spectrum $\sigma(\mathbf{A} + \mathcal{G})$. Then*

$$\lim_{n\to\infty} \frac{g(\lambda_1(A_n)) + \ldots + g(\lambda_n(A_n))}{n} = M(D_{ap}(g(W(\mathbf{A})))). \tag{43}$$

Proof. The proof runs completely parallel to that of Theorem 5.1. As there one checks that all occurring terms as well as the sequence $g(\mathbf{A})$ are well defined (the latter after modification by a finitely supported sequence if necessary). Thus, the analogue of (37) holds.

Further, the equality (42) implies the analogue of (38) which, on its hand, yields the analogue of (39). Finally, the analogue of (40) follows by applying the (continuous and unital) homomorphism W to the contour integral (41): approximate this integral by a sequence of Riemann sums $r_n(\mathbf{A})$ and use that $W(r_n(\mathbf{A})) = r_n(W(\mathbf{A}))$. ■

Another approach to this theorem employs Runge's approximation theorem ([20], Theorem 2 in Section III.1) in place of the holomorphic functional calculus. Runge's theorem yields approximations of $g(b)$ by linear combinations of $(\zeta_i e - b)^{-1}$ with simple poles ζ_i in $U \setminus \sigma(b)$. (Note that the Riemann sums for (41) also yield such approximations.)

Finite sections sequences. Next we specify these results to finite sections sequences (P_nAP_n) where A is a normal operator in $\mathcal{A}_{L^\infty(\mathbb{T}),AP(\mathbb{Z})}(\mathbb{Z}^+)$.

Theorem 5.3 *Let A be a normal operator in $\mathcal{A}_{L^\infty(\mathbb{T}),AP(\mathbb{Z})}(\mathbb{Z}^+)$ and let g be any continuous function on the convex hull of the spectrum of A. Then*

$$\lim_{n\to\infty} \frac{g(\lambda_1(P_nAP_n)) + \ldots + g(\lambda_n(P_nAP_n))}{n} = M(D_{ap}(g(A))). \tag{44}$$

Proof. The interesting new point is that g is merely assumed to be continuous on the convex hull I of the spectrum of the operator A. Of course, the operator $g(A)$ is still well defined. Further one knows that all eigenvalues of P_nAP_n belong to I, too. This can be most easily seen by introducing the numerical range

$$N(B) := \{\langle Bx, x\rangle : x \in l^2(\mathbb{Z}^+),\ \|x\| = 1\}$$

of an operator $B \in L(l^2(\mathbb{Z}^+))$. It is well known that

$$\operatorname{conv} \sigma(A) \subseteq \operatorname{clos} N(A)$$

for each operator $A \in L(l^2(\mathbb{Z}^+))$ and that equality holds in this inclusion if A is normal (see [14] or Section 3.4.1 in [22]). Here, $\operatorname{conv} M$ stands for the convex hull of the set $M \subset \mathbb{C}$. Consequently, for each normal operator A,

$$\sigma(P_nAP_n) \subseteq \operatorname{clos} N(P_nAP_n) \subseteq \operatorname{clos} N(A) = \operatorname{conv} \sigma(A)$$

where the second inclusion holds since each unit vector in $\operatorname{im} P_n$ is also a unit vector in $l^2(\mathbb{Z}^+)$. Thus, $g(P_nAP_n)$ is also well-defined. The inclusions $\sigma(P_nAP_n) \subseteq I$ holding for every $n \in \mathbb{N}$ together with the property of being normal further imply that the stability spectrum of the finite sections sequence (P_nAP_n) is in I, too. ■

In a similar way, one derives the following special case of Theorem 5.2.

Theorem 5.4 *Let $A \in \mathcal{A}_{L^\infty(\mathbb{T}),AP(\mathbb{Z})}(\mathbb{Z}^+)$ and $\mathbf{A} = (P_nAP_n)$. Further, let g be any function which is holomorphic on a neighborhood U in $\mathbb{C}$ of the stability spectrum $\sigma(\mathbf{A}+\mathcal{G})$. Then*

$$\lim_{n\to\infty} \frac{g(\lambda_1(P_nAP_n)) + \ldots + g(\lambda_n(P_nAP_n))}{n} = M(D_{ap}(g(A))). \tag{45}$$

Let now $A \in \mathcal{A}_{AP(\mathbb{Z})}(\mathbb{Z}^+)$ be a band-dominated operator with almost periodic coefficients. Then we can determine the stability spectrum of the finite sections sequence (P_nAP_n) by means of Theorem 2.6. If we pass from (P_nAP_n) to a subsequence $(P_{h(n)}AP_{h(n)})$ then the stability spectrum will decrease in accordance with Theorem 2.7 and, thus, the set of the holomorphic functions g for which (45) holds will become larger. The minimal possible stability spectrum (thus, the maximal set of holomorphic functions g for which (45) holds) is obtained if we choose h as a distinguished sequence of A. In this case, the stability spectrum of the sequence $(P_{h(n)}AP_{h(n)})$ is equal to

$$\sigma(PAP) \cup \sigma(JQaQJ)$$

by Theorem 2.9.

Operators in the Toeplitz algebra. Let now A be a normal operator in the Toeplitz algebra $\mathcal{A}_{L^\infty(\mathbb{T}),\mathbb{C}}(\mathbb{Z}^+)$ and let g be continuous. Then $D_{ap}(g(A))$ coincides with the constant function $g(s_A)_0$ where the symbol s_A of A is defined after Proposition 3.2. This equality follows by a similar reasoning as in the proofs of Theorems 4.4 and 4.5. Since $D_{ap}(g(A))$ is a constant function, one clearly has $M(D_{ap}(g(A))) = g(s_A)_0$. Thus, specifying Theorem 5.3 to operators in the Toeplitz algebra yields the following version of Szegö's first limit theorem which is due to SeLegue [36].

Corollary 5.5 (SeLegue) *Let A be a normal operator in $\mathcal{A}_{L^\infty(\mathbb{T}),\mathbb{C}}(\mathbb{Z}^+)$ and let g be any continuous function on the convex hull of the spectrum of A. Then*

$$\lim_{n\to\infty} \frac{g(\lambda_1(P_nAP_n)) + \ldots + g(\lambda_n(P_nAP_n))}{n} = g(s_A)_0 = \frac{1}{2\pi}\int_0^{2\pi} g(s_A(e^{it}))\, dt. \tag{46}$$

In particular, if $A = T(a)$ is a Toeplitz operator with a generating function $a \in L^\infty(\mathbb{T})$, then $s_A = a$. Thus, a further specification of Corollary 5.5 to the case of normal Toeplitz operators yields the following.

Corollary 5.6 *Let $a \in L^\infty(\mathbb{T})$ be such that the Toeplitz operator $T(a)$ is normal, and let g be any continuous function on the convex hull of the essential range of a. Then*

$$\lim_{n\to\infty} \frac{g(\lambda_1(T_n(a)) + \ldots + g(\lambda_n(T_n(a))}{n} = \frac{1}{2\pi}\int_0^{2\pi} g(a(e^{it}))\,dt. \qquad (47)$$

In this form, one finds the first Szegö theorem in [13], Theorem 5.10, for instance. Note that a Toeplitz operator $T(a)$ is normal if and only if it is a complex linear combination of a self-adjoint Toeplitz operator and the identity and, thus, if and only if the essential range of a is contained in a line segment (the Brown-Halmos theorem, see Section 3.3 in [13]. Thus, for Toeplitz operators, there is no basic difference between the normal and the self-adjoint case. Note also that the finite sections $P_nT(a)P_n$ are normal for a normal Toeplitz operator.

A final specification of Corollary 5.6 to self-adjoint Toeplitz operators yields precisely Theorem 1.2. Its holomorphic version Theorem 1.3 follows by a similar specification of Theorem 5.2.

Operators in algebras with unique tracial state. We finish this section with a few remarks on subalgebras $\mathcal{B}$ of the Følner algebra which own a unique tracial state, i.e., a state τ with $\tau(AB) = \tau(BA)$ for each pair of operators $A, B \in \mathcal{B}$. Their importance for generalized Szegö theorems rests on the following result. For its proof and all further facts cited here see [1, 7] or Sections 7.2.1 and 7.2.4 in [22].

Theorem 5.7 (Arveson, Bédos) *Let $\mathcal{B}$ be a unital C^*-subalgebra of the Følner algebra $\mathfrak{F}(\mathcal{P})$. For every $n \geq 1$, let ρ_n be the state of $\mathcal{B}$ defined by*

$$\rho_n(A) := \frac{1}{n}\mathrm{tr}\,(P_nAP_n),$$

and let $\mathcal{R}_n$ be the *-weak-closed convex hull of the set $\{\rho_n, \rho_{n+1}, \rho_{n+2}, \ldots\}$. Then $\mathcal{R}_\infty := \cap_{n\geq 1}\mathcal{R}_n$ is a non-empty set of tracial states of $\mathcal{B}$.

Thus, if $\mathcal{B}$ has a *unique* tracial state τ then the ρ_n converge *-weakly to τ. In particular,

$$\lim_{n\to\infty} \rho_n(g(A)) = \tau(g(A))$$

for each self-adjoint operator $A \in \mathcal{B}$ and each continuous function g. This implies easily the following version of the first Szegö limit theorem.

Theorem 5.8 (Arveson, Bédos) *Let $\mathcal{B}$ be a unital C^*-subalgebra of the Følner algebra $\mathfrak{F}(\mathcal{P})$ which possesses a unique tracial state τ. Let further $A \in \mathcal{B}$ be a self-adjoint operator. Then, for every compactly supported continuous function $g : \mathbb{R} \to \mathbb{R}$,*

$$\lim_{n\to\infty} \frac{g(\lambda_1(P_nAP_n)) + \ldots + g(\lambda_n(P_nAP_n))}{n} = \tau(g(A)).$$

Note that, for each self-adjoint operator $A \in \mathcal{B}$, the state τ gives rise to a natural probability measure μ_A on $\mathbb{R}$ via

$$\int_{-\infty}^{\infty} g(x)\, d\mu_A(x) := \tau(g(A)). \tag{48}$$

A particular example of a C^*-subalgebra of the Følner algebra with a unique tracial state is the irrational rotation algebra. The operators in this algebra can be also considered as band-dominated operators with almost periodic coefficients. Thus, they are subject both to the Arveson-Bédos theorem 5.8 and to our Theorem 5.3. This observation allows one to identify the tracial state τ of the irrational rotation algebra as well as the measures associated with τ by (48) via

$$\int_{-\infty}^{\infty} g(x)\, d\mu_A(x) = \tau(g(A)) = M(f_{g(A)}),$$

which holds for each compactly supported continuous function g.

5.2 Avram-Parter-type theorems

The Avram-Parter theorem establishes a formula for the trace of

$$g(P_n T(\overline{a}) P_n T(a) P_n) \quad \text{with } a \in L^\infty(\mathbb{T})$$

and is, thus, immediately related with products of finite sections sequences and with algebras generated by them. Indeed, we will see that this theorem can be considered as another simple special case of Theorem 4.7. For each $n \times n$-matrix B, let $\sigma_i(B)$ with $i = 1, \ldots, n$ refer to the singular values of B, i.e., to the non-negative square roots of the eigenvalues of B^*B. The order of enumeration is again not of importance.

Let $\mathbf{A} = (A_n) \in \mathcal{F}^{\mathcal{P}}$. Then the entries of the sequence $\mathbf{B} := (\mathbf{A}^*\mathbf{A})^{1/2}$ are the matrices $B_n := (A_n^* A_n)^{1/2}$, and

$$\sigma_j(A_n) = \lambda_j(B_n) \quad \text{for } j = 1, \ldots, n$$

under suitable enumeration. Thus, application of Theorem 5.1 to the sequence $\mathbf{B}$ yields the following.

Theorem 5.9 *Let $\mathbf{A} = (A_n)$ be a sequence in $\mathcal{S}_{L^\infty(\mathbb{T}), AP(\mathbb{Z})}(\mathbb{Z}^+)$, and let g be any function which is continuous on a neighborhood in $\mathbb{R}$ of the stability spectrum $\sigma(\mathbf{B} + \mathcal{G})$ with $\mathbf{B} := (\mathbf{A}^*\mathbf{A})^{1/2}$. Then*

$$\lim_{n \to \infty} \frac{g(\sigma_1(A_n)) + \ldots + g(\sigma_n(A_n))}{n} = M(D_{ap}(g(W(\mathbf{B})))). \tag{49}$$

Corollary 5.10 *Let* $\mathbf{A} := (P_n A P_n)$ *with* $A \in \mathcal{A}_{L^\infty(\mathbb{T}),AP(\mathbb{Z})}(\mathbb{Z}^+)$*, and let* g *be any function which is continuous on a neighborhood in* $\mathbb{R}$ *of the stability spectrum* $\sigma(\mathbf{B} + \mathcal{G})$ *with* $\mathbf{B} := (\mathbf{A}^*\mathbf{A})^{1/2}$*. Then*

$$\lim_{n\to\infty} \frac{g(\sigma_1(A_n)) + \ldots + g(\sigma_n(A_n))}{n} = M(D_{ap}(g(B))) \tag{50}$$

with $B := (A^*A)^{1/2}$*.*

Further specification to the case of operators in the Toeplitz algebra yields the following version of SeLegue's result (Corollary 5.5).

Corollary 5.11 *Let* $\mathbf{A} := (P_n A P_n)$ *with* $A \in \mathcal{A}_{L^\infty(\mathbb{T}),\mathbb{C}}(\mathbb{Z}^+)$*, and let* g *be any function which is continuous on a neighborhood in* $\mathbb{R}$ *of the stability spectrum* $\sigma(\mathbf{B} + \mathcal{G})$ *with* $\mathbf{B} := (\mathbf{A}^*\mathbf{A})^{1/2}$*. Then*

$$\lim_{n\to\infty} \frac{g(\sigma_1(A_n)) + \ldots + g(\sigma_n(A_n))}{n} = g(s_B)_0 = \frac{1}{2\pi}\int_0^{2\pi} g(s_B(e^{it}))\, dt \tag{51}$$

with $B := (A^*A)^{1/2}$*.*

Finally, if $A = T(a)$ is a Toeplitz operator with generating function $a \in L^\infty(\mathbb{T})$, then

$$s_B = s_{(A^*A)^{1/2}} = (\overline{a}a)^{1/2} = |a|.$$

Corollary 5.12 (Avram/Parter) *Let* $\mathbf{A} := (P_n T(a) P_n)$ *with* $a \in L^\infty(\mathbb{T})$*, and let* g *be any function which is continuous on a neighborhood in* $\mathbb{R}$ *of the stability spectrum* $\sigma(\mathbf{B} + \mathcal{G})$ *with* $\mathbf{B} := (\mathbf{A}^*\mathbf{A})^{1/2}$*. Then*

$$\lim_{n\to\infty} \frac{g(\sigma_1(A_n)) + \ldots + g(\sigma_n(A_n))}{n} = \frac{1}{2\pi}\int_0^{2\pi} g(|a(e^{it})|)\, dt. \tag{52}$$

This result was established by Parter [27] for locally self-adjoint (= products of continuous and real-valued) generating functions a, and Avram [5] proved it for general $L^\infty(\mathbb{T})$-functions. The algebraic approach to the Avram/Parter theorem goes back to Böttcher and one of the authors (Section 5.6 in [13]). There (Section 4.5) one also finds a short illustrated history of the Avram/Parter theorems which were aimed to explain Moler's phenomenon concerning the singular value distribution of Toeplitz matrices.

We would also like to mention that Tyrtyshnikov [41, 42] was able to show that Corollary 5.12 remains valid for arbitrary functions $a \in L^2(\mathbb{T})$ (in which case the Toeplitz operator $T(a)$ is no longer bounded and our techniques do not seem to apply).

5.3 Böttcher-Otte-type theorems

The continuous and holomorphic functional calculus can also be applied to the sequences considered in Theorem 4.2 and in (30). It seems that Böttcher and Otte [11] were interested in results of that type for the first time. The following two corollaries to Theorem 4.2 follow by a straightforward application of the functional calculus as in Subsection 5.1.

Corollary 5.13 *Let* $\mathbf{A} = (A_n)$ *be a normal sequence in* $\mathcal{S}(\mathfrak{F}(\mathcal{P}))$*, and let* g *be any function which is continuous on a neighborhood in* $\mathbb{R}$ *of the stability spectrum* $\sigma(\mathbf{A} + \mathcal{G})$*. Then*

$$\lim_{n\to\infty} \frac{1}{n} \left(\operatorname{tr} g(A_n) - \operatorname{tr}\left(P_n g(W(\mathbf{A}))P_n\right)\right) = 0. \tag{53}$$

Corollary 5.14 *Let* $\mathbf{A} = (A_n)$ *be a sequence in* $\mathcal{S}(\mathfrak{F}(\mathcal{P}))$*, and let* g *be any function which is holomorphic on a neighborhood* U *in* $\mathbb{C}$ *of the stability spectrum* $\sigma(\mathbf{A} + \mathcal{G})$*. Then* (53) *holds.*

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