Stationary Navier-Stokes flow around a rotating obstacle

By
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Abstract

Consider a viscous incompressible fluid filling the whole 3-dimensional space exterior to a rotating body with constant angular velocity ω. By using a coordinate system attached to the body, the problem is reduced to an equivalent one in a fixed exterior domain. The reduced equation involves the crucial drift operator $(\omega \wedge x) \cdot \nabla$, which is not subordinate to the usual Stokes operator. This paper addresses stationary flows to the reduced problem with an external force $f = \mathrm{div}\, F$, that is, time-periodic flows to the original one. Generalizing previous results of G. P. Galdi [19] we show the existence of a unique solution $(\nabla u, p)$ in the class $L_{3/2,\infty}$ when both $F \in L_{3/2,\infty}$ and ω are small enough; here $L_{3/2,\infty}$ is the weak-$L_{3/2}$ space.

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1 Introduction

Let D be an exterior domain in $\mathbb{R}^3$ with smooth boundary ∂D. We consider the motion of a viscous incompressible fluid filling the domain D when the obstacle $\mathbb{R}^3 \setminus D$, which consists of a finite number of rigid bodies, is rotating about an axis with constant angular velocity ω. Without loss of generality, we may assume that $\omega = |\omega| e_3 = (0, 0, |\omega|)^T$. The fluid at space infinity is assumed to be at rest. Our aim is to solve the Navier-Stokes system in the time-dependent domain

$$D(t) = \{y = O(|\omega|t)x;\ x \in D\}, \quad \text{where} \quad O(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

subject to the non-slip boundary condition on the surface $\partial D(t)$, so that the fluid velocity attains $\omega \wedge y$ there. Here, $\wedge$ denotes the usual exterior product of three-dimensional vectors; thus,

$$\omega \wedge y = |\omega|(-y_2, y_1, 0)^T = \frac{d}{dt} O(|\omega|t)x$$

which describes the velocity at a point y of the rotating rigid body. By using a coordinate system attached to the obstacle (see [25]), one can reduce the problem to an equivalent one in the fixed exterior domain D; that is,

$$\begin{aligned} \frac{\partial u}{\partial t} + u \cdot \nabla u &= \Delta u + (\omega \wedge x) \cdot \nabla u - \omega \wedge u - \nabla p + f, \\ \text{div } u &= 0, \end{aligned}$$

in $D \times (0, \infty)$ subject to

$$u|_{\partial D} = \omega \wedge x, \qquad u \to 0 \text{ as } |x| \to \infty, \qquad u|_{t=0} = a,$$

where $u = (u_1, u_2, u_3)^T$ and p are the unknown velocity and pressure of the fluid, respectively, while the external force f and initial velocity a are prescribed.

A significant feature which distinguishes the problem above from the usual Navier-Stokes system ($\omega = 0$) is the presence of the first order term with unbounded coefficients, viz.

$$(\omega \wedge x) \cdot \nabla u = |\omega| \left(-x_2 \frac{\partial u}{\partial x_1} + x_1 \frac{\partial u}{\partial x_2} \right) = |\omega| \frac{\partial u}{\partial \theta},$$

where, in the final representation, $\theta = \tan^{-1}(x_2/x_1)$ is the angular variable in a cylindrical coordinate system. Even though $|\omega| > 0$ is small, this term is not subordinate to the viscous term Δu in the sense that some properties of the linear operator

$$L = -\Delta - (\omega \wedge x) \cdot \nabla + \omega \wedge$$

are worse than those of the Laplacian. In fact, by [25], we know that the generated semigroup e^{-tL} for the whole space problem does not map $L_2(\mathbb{R}^3)$ into the domain $D(L)$ for $t > 0$, and that

$$\sigma(-L) \supseteqq \sigma_{\text{ess}}(-L) = \{\lambda = \alpha + ik|\omega| \in \mathbb{C};\ \alpha \leq 0, k \in \mathbb{Z}\},$$

see [16] for the operator $-PL$ where P is the Helmholtz projection on $L_2(\Omega)$. Hence e^{-tL} is *not* an analytic semigroup. Moreover, by [14], the fundamental solution $\Gamma(x, y)$ of the operator L cannot be dominated by $|x - y|^{-1}$; more precisely, its component $\Gamma_{33}(x, y)$ satisfies

$$\Gamma_{33}(x, y) \geq C \, \frac{\log |x - y|}{|x - y|} \tag{1.1}$$

when, for example, $x = \rho e_1$ and $y = \rho e_2$ with large $\rho > 0$. These observations tell us that the operator L cannot be treated by means of any perturbation argument and could cause some mathematical difficulties.

In the last decade a lot of effort has gone into analyzing nonstationary as well as stationary problems for the nonlinear and linearized system in either an exterior domain or the whole space; moreover, L_2 theory and L_q theory were used for problems without as well as with translation of the obstacle. We refer to [3], [9], [10], [12], [13], [14], [15] [16] [18], [19], [20], [21], [23], [24], [25], [26], [27], [28], [32], [33] and [35]. Nevertheless, our mathematical understanding is still far from complete. There are also some other studies, to which we don't refer here, such as moving bodies in a bounded or unbounded fluid region, rotating fluids without bodies in the whole space, etc.

The present paper is devoted to the study of the nonlinear stationary problem in exterior domains:

$$Lu + \nabla p + u \cdot \nabla u = f, \qquad \text{div } u = 0 \qquad \text{in } D \tag{1.2}$$

subject to

$$u|_{\partial D} = \omega \wedge x, \qquad u \to 0 \;\text{ as } |x| \to \infty. \tag{1.3}$$

Note that the stationary motion in the frame attached to the obstacle corresponds to a time-periodic one in the original frame; in fact, given a solution (u, p) of (1.2), (1.3), the pair of $O(|\omega|t)\, u\big(O(|\omega|t)^T x\big)$ and $p\,\big(O(|\omega|t)^T x\big)$ provides a periodic solution of the original problem in the domain $D(t)$ with external force $O(|\omega|t)\, f\big(O(|\omega|t)^T x\big)$. It is possible to construct solutions of class $\nabla u \in L_2$ to (1.2), (1.3) by means of the Galerkin method in the L_2–framework for arbitrary ω and $f = \text{div } F, F \in L_2$, see Borchers [3], Galdi [18], Serre [33] and Silvestre [35]. When ω is small enough and $f = \text{div F}$ satisfies the decay estimates

$$|x|^2 |F(x)| + |x|^3 |f(x)| + |x|^4 |\text{div } f(x)| \leq c_0$$

for some small $c_0 > 0$, Galdi [19] derived the pointwise estimates

$$|x||u(x)| + |x|^2 (|\nabla u(x)| + |p(x)|) + |x|^3 |\nabla p(x)| \leq C \tag{1.4}$$

of a unique solution. These decay properties are important in the study of stability ([3], [20]), but, at first glance, rather surprising in view of (1.1). In [21] Galdi and Silvestre have recently extended a part of [19]; that is, they have shown $|x||u(x)| \leq C$ for a small force $f = \mathrm{div}\, F \in L_2$ with $|x|^2|F(x)| \leq c_0$ when the translation of the obstacle is also taken into account. We may expect an *anisotropic* decay structure of solutions similar to the Oseen case ([11], [17]), but as far as simple *isotropic* decay estimates are concerned, the result of Galdi [19] shows that the rate of decay of Navier-Stokes flow at infinity is the same as in the usual case $\omega = 0$ in spite of the slightly worse behavior (1.1) of the fundamental solution.

The purpose of the present paper is to provide another outlook on the pointwise estimate (1.4) in a different framework by use of special function spaces. To be more precise, we show in the class $L_{3/2,\infty}$ the existence of a unique solution $(\nabla u, p)$ to (1.2), (1.3) with force $f \in \dot{W}^{-1}_{3/2,\infty}$ when both f and ω are small enough; here $L_{3/2,\infty}$ is the weak-$L_{3/2}$ space, one of the Lorentz spaces. We note that $f \in \dot{W}^{-1}_{3/2,\infty}$ if and only if $f = \mathrm{div}\, F$ with $F \in L_{3/2,\infty}$, see Lemma 2.2 (i) below. For more precise definitions, in particular of weak solutions to (1.2), (1.3), see Section 2.

Now our main theorem reads as follows.

Theorem 1.1 *There is a constant* $\eta = \eta(D) > 0$ *such that for* $f \in \dot{W}^{-1}_{3/2,\infty}(D)$ *with*

$$|\omega| + \|f\|_{\dot{W}^{-1}_{3/2,\infty}(D)} \leq \eta,$$

problem (1.2), (1.3) possesses a unique weak solution (u, p) *with*

$$\nabla u, p \in L_{3/2,\infty}(D)$$

subject to the estimate

$$\|\nabla u\|_{3/2,\infty} + \|u\|_{3,\infty} + \|p\|_{3/2,\infty} \leq C\left(|\omega| + \|f\|_{\dot{W}^{-1}_{3/2,\infty}(D)}\right), \tag{1.5}$$

with some $C > 0$ *independent of* $|\omega|$ *and* f.

Our class of solutions is consistent with (1.4), and our class of external forces is larger than in [19], [21], though one cannot expect explicit pointwise estimates as in (1.4) for such external forces. For the case $\omega = 0$, a result analogous to Theorem 1.1 has been proved by Kozono and Yamazaki [31].

In [28], based on an idea from [14], one of the present authors has established the existence, uniqueness and L_q estimate

$$\|\nabla u\|_q + \|p\|_q \leq C\|f\|_{\dot{W}^{-1}_q(D)} \tag{1.6}$$

of weak solutions to the linearized problem

$$Lu + \nabla p = f, \qquad \mathrm{div}\, u = 0 \qquad \text{in } D; \qquad u|_{\partial D} = 0, \tag{1.7}$$

provided that $(n/(n-1) =)\, 3/2 < q < 3\, (= n)$. This result is regarded as a generalization of [4], [22], [29], [30] in the usual case $\omega = 0$, since the restriction on the exponent q is the same. Since the case $q = 3/2\, (= n/2)$ needed to estimate the nonlinearity $u \cdot \nabla u$ is missing in [28], an L_q theory does not help to solve the nonlinear problem (1.2), (1.3). Note that $L_{3/2}$ is too restrictive at infinity to expect $\nabla u \in L_{3/2}$ for (1.7) even if $f = \mathrm{div}\, F$ with $F \in C_0^\infty$. Therefore, we have to replace $L_{3/2}$ by a larger space. To do so, we follow Kozono and Yamazaki [31] who for the first time used Lorentz spaces in the case $\omega = 0$. This paper shows that the right class to find a solution $(\nabla u, p)$ to (1.2), (1.3) is $L_{3/2,\infty}$ as well.

An important step is to derive, instead of (1.6), the *a priori* estimate

$$\|\nabla u\|_{3/2,\infty} + \|p\|_{3/2,\infty} \le C\|f\|_{\dot{W}^{-1}_{3/2,\infty}(D)} \tag{1.8}$$

for the linearized problem (1.7); in fact, once this is established, a fixed point argument yields a unique solution $(\nabla u, p)$ of (1.2), (1.3) in the class $L_{3/2,\infty}$. In the proof of the solvability of (1.7) for all $f \in \dot{W}^{-1}_{3/2,\infty}$, a duality argument due to Kozono and Yamazaki [31] does not seem to be applicable to our problem because of lack of homogeneity of the equation, unlike the usual case $\omega = 0$. We thus follow, in principle, the argument of Shibata and Yamazaki [34], in which the solution is constructed without any duality argument for the Oseen problem. Note that one cannot use any continuity argument since C_0^∞ is not dense in $L_{q,\infty}$. So, as in [34], given $f \in \dot{W}^{-1}_{3/2,\infty}$, we try to construct directly the solution to (1.7). Though cut-off procedures were carried out twice in [34], we use such a procedure only once to obtain the solution; in this point, the proof of [34] is simplified in the present paper. In fact, a parametrix (v, π), an approximation of the solution, constructed by use of solutions in the whole space and in a bounded domain combined with the Bogovskiĭ operator [2] satisfies

$$Lv + \nabla \pi = f + Rf, \qquad \mathrm{div}\, v = 0 \qquad \text{in } D; \qquad v|_{\partial D} = 0,$$

where Rf is a remainder term with compact support. We show that the operator $1 + R$ has a bounded inverse in $\dot{W}^{-1}_{3/2,\infty}$. In the proof, the embedding $\dot{W}^1_{3,1} \hookrightarrow L_\infty$ ([31, Lemma 2.1]) and the fact that the dual space of $\dot{W}^1_{3,1}$ is $\dot{W}^{-1}_{3/2,\infty}$ play a fundamental role.

We would like to mention an advantage of the method of [34] (rather than the simplified one of this paper) from the viewpoint of linear theory. In the case $f \in \dot{W}^{-1}_{q,\infty}$, $1 < q < 3/2$, the argument above does not lead to any result. For such an external force, among other techniques, Proposition 3.3 (without the term $(\omega \wedge x) \cdot \nabla u$) for the whole space case, see Section 3 below, was important in [34].

This paper is organized as follows. In the next section we start with some preliminaries. To prove our main theorem on the stationary Navier-Stokes problem (1.2), (1.3) in Section 6, we carry out the analysis of the linearized equation (1.7) in the following order: the whole space problem in Section 3, the interior problem in Section 4 and finally the exterior problem in Section 5.

2 Preliminaries

Given any smooth domain $\Omega \subset \mathbb{R}^3$ such as $\Omega = D$, $\Omega = \mathbb{R}^3$ or a bounded domain we introduce the following function spaces. By $C_0^\infty(\Omega)$ we denote the class of smooth functions with compact support in Ω. For $1 \le q \le \infty$, the usual Lebesgue spaces are denoted by $L_q(\Omega)$ with norm $\|\cdot\|_{q,\Omega}$. We need the Lorentz spaces $L_{q,r}(\Omega)$, with norm $\|\cdot\|_{q,r,\Omega}$: for $1 < q < \infty$ and $1 \le r \le \infty$, the Lorentz spaces can be constructed via real interpolation

$$L_{q,r}(\Omega) = (L_1(\Omega), L_\infty(\Omega))_{1-1/q,r}\,;$$

for details, see Bergh and Löfström [1]. Note that

$$L_{q,r_0}(\Omega) \subset L_{q,r_1}(\Omega) \quad \text{if } r_0 \le r_1; \quad L_{q,q}(\Omega) = L_q(\Omega),$$

and, if Ω is bounded, that

$$L_{p,r}(\Omega) \subset L_{q,s}(\Omega) \quad \text{for all } 1 < q < p < \infty,\ r, s \in [1, \infty], \tag{2.1}$$

with continuous embeddings. For

$$1 < q < \infty, \quad 1 < r \le \infty, \quad \frac{1}{q'} + \frac{1}{q} = 1, \quad \frac{1}{r'} + \frac{1}{r} = 1, \tag{2.2}$$

we have the duality relation

$$L_{q,r}(\Omega) = L_{q',r'}(\Omega)^*.$$

In particular, $L_{q,\infty}(\Omega) = L_{q',1}(\Omega)^*$ is well known as the weak-L_q space, in which $C_0^\infty(\Omega)$ is *not* dense; moreover, $f \in L_{q,\infty}(\Omega)$ if and only if

$$\sup_{\sigma>0} \sigma \left|\{x \in \Omega; |f(x)| > \sigma\}\right|^{1/q} < \infty,$$

where $|\cdot|$ stands for the Lebesgue measure. In what follows, we adopt the same symbols for vector and scalar function spaces as long as there is no confusion, and we use the abbreviations $\|\cdot\|_q = \|\cdot\|_{q,D}$ and $\|\cdot\|_{q,r} = \|\cdot\|_{q,r,D}$ for the exterior domain D.

Furthermore, we need homogeneous Sobolev spaces. For $1 < q < \infty$, let $\dot{W}^1_q(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla(\cdot)\|_{q,\Omega}$, and let $\dot{W}^{-1}_q(\Omega)$ be the dual space of $\dot{W}^1_{q'}(\Omega)$ where $1/q' + 1/q = 1$. Let

$$1 < q_0 < q < q_1 < \infty, \quad 1/q = (1-\theta)/q_0 + \theta/q_1, \quad 1 \le r \le \infty. \tag{2.3}$$

We then define

$$\dot{W}^1_{q,r}(\Omega) = \left(\dot{W}^1_{q_0}(\Omega), \dot{W}^1_{q_1}(\Omega)\right)_{\theta,r},$$

which is independent of the choice of (q_0, q_1), with norm $\|\nabla(\cdot)\|_{q,r,\Omega}$. Note that $C_0^\infty(\Omega)$ is *not* dense in $\dot{W}^1_{q,\infty}(\Omega)$.

For (q, r) satisfying (2.2), the space $\dot{W}^{-1}_{q,r}(\Omega)$ is defined as the dual space of $\dot{W}^1_{q',r'}(\Omega)$; by the duality theorem for interpolation spaces ([1, 3.7.1]), we see that

$$\dot{W}^{-1}_{q,r}(\Omega) = \left(\dot{W}^{-1}_{q_0}(\Omega), \dot{W}^{-1}_{q_1}(\Omega)\right)_{\theta,r} \tag{2.4}$$

for q, q_0, q_1, r satisfying (2.3) but $r \ne 1$. For later use we cite the following results.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain.*

(i) The embedddings $\dot{W}^1_{q,r}(\Omega) \hookrightarrow L_{q,r}(\Omega)$, $1 < q < \infty$, $1 \le r \le \infty$, and $L_{q,r}(\Omega) \hookrightarrow \dot{W}^{-1}_{q,r}(\Omega)$, $1 < q < \infty$, $1 < r \le \infty$, are compact.

(ii) Let $1 < q < \infty$ and $1 \le r \le \infty$. Then

$$\dot{W}^1_{q,r}(\Omega) = \{u \in L_{q,r}(\Omega);\ \nabla u \in L_{q,r}(\Omega), u|_{\partial\Omega} = 0\},$$

and for all $u \in \dot{W}^1_{q,r}(\Omega)$ holds the Poincaré inequality

$$\|u\|_{q,r,\Omega} \le C\|\nabla u\|_{q,r,\Omega}. \tag{2.5}$$

Proof (i) follows from (2.1), classical embedding theorems, duality and interpolation, see [1, 3.14.8]. (ii) is based on the classical Poincaré inequality and interpolation. □

Lemma 2.2 *(i)* [31, Lemma 2.2] *Let $\Omega \subset \mathbb{R}^3$ be any domain and let $1 < q < \infty$, $1 < r \leq \infty$. Then there exists a constant $C > 0$ such that for every $f \in \dot{W}^{-1}_{q,r}(\Omega)$ there is $F \in L_{q,r}(\Omega)$ satisfying*

$$\operatorname{div} F = f, \qquad \|F\|_{q,r,\Omega} \leq C\|f\|_{\dot{W}^{-1}_{q,r}(\Omega)}.$$

(ii) [31, Lemma 2.1] *Let $D \subset \mathbb{R}^3$ be an exterior domain. Then for any $1 < q < 3$ and $r \in [1,\infty]$ we have, with $1/q_* := 1/q - 1/3$, the characterization*

$$\dot{W}^1_{q,r}(D) = \{u \in L_{q_*,r}(D);\ \nabla u \in L_{q,r}(D), u|_{\partial D} = 0\}, \tag{2.6}$$

together with the embedding estimate

$$\|u\|_{q_*,r} \leq C\|\nabla u\|_{q,r}. \tag{2.7}$$

(iii) [31, Lemma 2.1] *Let $D \subset \mathbb{R}^3$ be an exterior domain. Then for $q = 3$ we have the embedding $\dot{W}^1_{3,1}(D) \hookrightarrow L_\infty(D) \cap C(D)$ and the estimate*

$$\|u\|_\infty \leq \frac{1}{3}\|\nabla u\|_{3,1}. \tag{2.8}$$

(iv) [5, Theorem 5.9] *Let $D \subset \mathbb{R}^3$ be an exterior domain, let $1 < q < 3$, $r \in [1,\infty]$, and let $u \in L_{1,\mathrm{loc}}(\overline{D})$ satisfy $\nabla u \in L_{q,r}(D)$. Then there is a constant $k = k(u)$ such that $u + k \in L_{q_*,r}(D)$ and*

$$\|u + k\|_{q_*,r} \leq C\|\nabla u\|_{q,r},$$

with some $C > 0$ independent of u; here $1/q_* = 1/q - 1/3$.

For given $1 < q < \infty$, $1 < r \leq \infty$ and $f \in \dot{W}^{-1}_{q,r}(D)$ let us first consider the boundary value problem for the linearized equation

$$\left\{ \begin{array}{rcll} -\Delta u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u + \nabla p & = & f & \text{in } D, \\ \operatorname{div} u & = & 0 & \text{in } D, \\ u & = & 0 & \text{on } \partial D. \end{array} \right. \tag{2.9}$$

Then the pair of functions (u,p) is called (q,r)-*weak solution* (q-*weak solution* when $q = r$) of (2.9) if

1. $(u,p) \in \dot{W}^1_{q,r}(D) \times L_{q,r}(D)$;
2. $\operatorname{div} u = 0 \quad \text{in } L_{q,r}(D)$;
3. $(\omega \wedge x) \cdot \nabla u - \omega \wedge u \in \dot{W}^{-1}_{q,r}(D)$;

4. (u,p) satisfies $(2.9)_1$ in the sense of distributions, that is,

$$\langle \nabla u, \nabla\varphi\rangle - \langle (\omega\wedge x)\cdot\nabla u - \omega\wedge u, \varphi\rangle - \langle p, \operatorname{div}\varphi\rangle = \langle f,\varphi\rangle \qquad (2.10)$$

holds for all $\varphi\in C_0^\infty(D)$, where $\langle\cdot,\cdot\rangle$ stands for various duality pairings.

By continuity (note $r>1$), (u,p) satisfies (2.10) for all $\varphi\in \dot W^1_{q',r'}(D)$. When $1<q<3$, we have $u\in L_{q_*,r}(D)$ by (2.6), so that $u\to 0$ at infinity in this weak sense.

Choose a cut-off function $\zeta\in C_0^\infty(\mathbb{R}^3;[0,1])$ satisfying $\zeta=1$ near the boundary ∂D, and set

$$b(x) = -\frac{1}{2}\operatorname{rot}\left(\zeta(x)|x|^2\omega\right). \qquad (2.11)$$

Obviously $\operatorname{div} b = 0$ and $b|_{\partial D} = \omega\wedge x$. We thus intend to find the solution to (1.2), (1.3) as $u=v+b$, so that (v,p) should obey

$$\left\{\begin{array}{rcll} -\Delta v - (\omega\wedge x)\cdot\nabla v + \omega\wedge v + \nabla p &=& f-\Phi(v,b) & \text{in } D,\\ \operatorname{div} v &=& 0 & \text{in } D,\\ v &=& 0 & \text{on } \partial D,\\ v &\to& 0 & \text{at } \infty, \end{array}\right. \qquad (2.12)$$

with

$$\begin{aligned}\Phi(v,b) &= (v+b)\cdot\nabla(v+b) + Lb\\ &= \operatorname{div}\left[(v+b)\otimes(v+b) - \nabla b - (\omega\wedge x)\otimes b + b\otimes(\omega\wedge x)\right],\end{aligned} \qquad (2.13)$$

where $w\otimes\widetilde w = (w_j\widetilde w_k)$; here, note that

$$(\omega\wedge x)\cdot\nabla b = \operatorname{div}\left[(\omega\wedge x)\otimes b\right], \qquad \omega\wedge b = \operatorname{div}\left[b\otimes(\omega\wedge x)\right].$$

Let $f\in \dot W^{-1}_{3/2,\infty}(D)$. Since $v\in \dot W^1_{3/2,\infty}(D)$ implies $\Phi(v,b)\in \dot W^{-1}_{3/2,\infty}(D)$, see Section 6, one can define the *weak solution* $(v,p)\in \dot W^1_{3/2,\infty}(D)\times L_{3/2,\infty}(D)$ of (2.12) as the $(3/2,\infty)$-weak solution of (2.9) with f replaced by $f-\Phi(v,b)$. In this sense $(u=v+b,p)$ is the weak solution of the original problem (1.2), (1.3).

3 Linearized problem in the whole space

Let us consider the whole space problem

$$-\Delta u - (\omega\wedge x)\cdot\nabla u + \omega\wedge u + \nabla p = f, \quad \operatorname{div} u = g \qquad \text{in } \mathbb{R}^3. \qquad (3.1)$$

In the first half of this section we recall the L_q–theory developed in [28] and extend it to the case of Lorentz spaces. The second half is devoted to a proposition concerning the class of solutions when the support of f is compact.

We begin with the L_q–estimate of weak solutions.

Proposition 3.1 ([28]) *Let $1 < q < \infty$ and suppose that*

$$f \in \dot{W}_q^{-1}(\mathbb{R}^3), \quad g \in L_q(\mathbb{R}^3), \quad (\omega \wedge x)g \in \dot{W}_q^{-1}(\mathbb{R}^3).$$

Then problem (3.1) possesses a q-weak solution $(u,p) \in \dot{W}_q^1(\mathbb{R}^3) \times L_q(\mathbb{R}^3)$ subject to the estimate

$$\begin{aligned} &\|\nabla u\|_{q,\mathbb{R}^3} + \|p\|_{q,\mathbb{R}^3} + \|(\omega \wedge x)\cdot \nabla u - \omega \wedge u\|_{\dot{W}_q^{-1}(\mathbb{R}^3)} \\ &\quad \le C\Big(\|f\|_{\dot{W}_q^{-1}(\mathbb{R}^3)} + \|g\|_{q,\mathbb{R}^3} + \|(\omega \wedge x)g\|_{\dot{W}_q^{-1}(\mathbb{R}^3)}\Big), \end{aligned} \tag{3.2}$$

where $C > 0$ is independent of $|\omega|$, f and g. The solution is unique in the class above up to a constant multiple of ω for u.

Although this result was proved in [28] only for the case $|\omega| = 1$, a scaling argument implies that the constant $C > 0$ in (3.2) is independent of $|\omega|$. By real interpolation we obtain the estimate of weak solutions in Lorentz spaces.

Proposition 3.2 *Let $1 < q < \infty$ and suppose that*

$$f \in \dot{W}_{q,\infty}^{-1}(\mathbb{R}^3), \quad g \in L_{q,\infty}(\mathbb{R}^3), \quad (\omega \wedge x)g \in \dot{W}_{q,\infty}^{-1}(\mathbb{R}^3).$$

Then problem (3.1) possesses a (q,∞)-weak solution $(u,p) \in \dot{W}_{q,\infty}^1(\mathbb{R}^3) \times L_{q,\infty}(\mathbb{R}^3)$ subject to the estimate

$$\begin{aligned} &\|\nabla u\|_{q,\infty,\mathbb{R}^3} + \|p\|_{q,\infty,\mathbb{R}^3} + \|(\omega \wedge x)\cdot \nabla u - \omega \wedge u\|_{\dot{W}_{q,\infty}^{-1}(\mathbb{R}^3)} \\ &\quad \le C\left(\|f\|_{\dot{W}_{q,\infty}^{-1}(\mathbb{R}^3)} + \|g\|_{q,\infty,\mathbb{R}^3} + \|(\omega \wedge x)g\|_{\dot{W}_{q,\infty}^{-1}(\mathbb{R}^3)}\right), \end{aligned} \tag{3.3}$$

where $C > 0$ is independent of $|\omega|$. The solution is unique in the class above up to a constant multiple of ω for u.

Proof. We first obtain the pressure written formally by

$$p = -\mathrm{div}\,(-\Delta)^{-1}[f + \nabla g + (\omega \wedge x)g];$$

here, $(-\Delta)^{-1}$ can be considered as a bounded operator from $\dot{W}_q^{-1}(\mathbb{R}^3)$ to $\dot{W}_q^1(\mathbb{R}^3)$ ([22], [30]). By real interpolation it is bounded from $\dot{W}_{q,\infty}^{-1}(\mathbb{R}^3)$ to $\dot{W}_{q,\infty}^1(\mathbb{R}^3)$, see (2.4). Therefore,

$$\|p\|_{q,\infty,\mathbb{R}^3} \le C\|f + \nabla g + (\omega \wedge x)g\|_{\dot{W}_{q,\infty}^{-1}(\mathbb{R}^3)}. \tag{3.4}$$

We next consider the equation $Lu = f - \nabla p$. As shown in the proof of Proposition 3.1, the operator L^{-1} is bounded from $\dot{W}_q^{-1}(\mathbb{R}^3)$ to $\dot{W}_q^1(\mathbb{R}^3)$; thus it is bounded from $\dot{W}_{q,\infty}^{-1}(\mathbb{R}^3)$ to $\dot{W}_{q,\infty}^1(\mathbb{R}^3)$ by real interpolation. Hence we obtain a solution $u \in \dot{W}_{q,\infty}^1(\mathbb{R}^3)$ with

$$\|\nabla u\|_{q,\infty,\mathbb{R}^3} \le C\|f - \nabla p\|_{\dot{W}_{q,\infty}^{-1}(\mathbb{R}^3)},$$

where $C > 0$ is independent of $|\omega|$, which combined with (3.4) implies (3.3). Finally, applying div to the first equation of (3.1), we get $L(\text{div } u - g) = 0$ yielding div $u = g$, see below.

Let $(u, p) \in \dot{W}_{q,\infty}^1(\mathbb{R}^3) \times L_{q,\infty}(\mathbb{R}^3)$ satisfy $Lu + \nabla p = 0$, $\operatorname{div} u = 0$ in $\mathbb{R}^3$. We immediately obtain $p = 0$, since $\Delta p = 0$, so that $Lu = 0$. Since $\nabla u \in L_{q,\infty}(\mathbb{R}^3) \subset \mathcal{S}'$ and consequently also $u \in \mathcal{S}'$, see [8, Proposition 1.2.1], we get $\operatorname{supp} \widehat{u} \subset \{0\}$ as shown in [14], [28]. Hence u is a constant vector, which should be a constant multiple of ω because $\omega \wedge u = 0$. This completes the proof. □

The final proposition on the linear whole space problem gives a heuristic argument why in spite of the negative result (1.1) the pointwise estimate (1.4) may hold. We note that this proposition will not be used later on, but it would be essential when employing a cut-off procedure as in [34]. It extends a similar result from [34] where $\omega = 0$.

Proposition 3.3 *Let $1 < q < \infty$ and suppose that*

$$f \in L_q(\mathbb{R}^3), \qquad \operatorname{supp} f \subset B_R, \qquad g = 0.$$

Then a representative (u, p) of the strong solution to (3.1) obtained in [14] enjoys $(u, p) \in \dot{W}_{3/2,\infty}^1(\mathbb{R}^3) \times L_{3/2,\infty}(\mathbb{R}^3)$ and the estimates

$$\|\nabla u\|_{3/2,\infty,\mathbb{R}^3} + \|p\|_{3/2,\infty,\mathbb{R}^3} \le C_R\|f\|_{q,\mathbb{R}^3}, \tag{3.5}$$

$$\operatorname{ess\,sup}_{|x|\ge 4R} \left(|x|\,|u(x)| + |x|^2\left(|\nabla u(x)| + |p(x)|\right) + |x|^3|\nabla p(x)|\right) \le C_R\|f\|_{q,\mathbb{R}^3}, \tag{3.6}$$

with some $C_R > 0$ independent of $|\omega|$.

Proof. Since $\Delta p = \operatorname{div} f$, it follows from the Hausdorff-Young inequality that

$$\|p\|_{3/2,\infty,\mathbb{R}^3} \le C\|f\|_{1,\mathbb{R}^3}, \tag{3.7}$$

for $p(x) = -\frac{1}{4\pi}\nabla\frac{1}{|\cdot|} * f$ and $\frac{1}{|\cdot|^2} \in L_{3/2,\infty}(\mathbb{R}^3)$. We also find

$$\text{ess sup}_{|x|\ge 2R} \big(|x|^2|p(x)| + |x|^3|\nabla p(x)|\big) \le C\|f\|_{1,\mathbb{R}^3}, \tag{3.8}$$

because of $\operatorname{supp} f \subset B_R$.

By the relation $L_q(\mathbb{R}^3) \hookrightarrow \dot{W}^{-1}_{3q/(3-q)}(\mathbb{R}^3)$, where we may assume $q < 3$, Proposition 3.1 implies $\nabla u \in L_{3q/(3-q)}(\mathbb{R}^3)$. Since $3q/(3-q) > 3/2$, we obtain

$$\|\nabla u\|_{3/2,\infty,B_{4R}} \le C\|\nabla u\|_{3q/(3-q),\mathbb{R}^3} \le C\|f\|_{\dot{W}^{-1}_{3q/(3-q)}(\mathbb{R}^3)} \le C\|f\|_{q,\mathbb{R}^3}, \tag{3.9}$$

where $C = C_R > 0$ is independent of $|\omega|$. For the proof of (3.5) and (3.6), therefore, it suffices to show

$$\text{ess sup}_{|x|\ge 4R} |x|^2|\nabla u(x)| \le C_R\|f\|_{1,\mathbb{R}^3}, \tag{3.10}$$

with some $C_R > 0$ independent of $|\omega|$. To prove (3.10), we decompose $u = v + w$ where v, w are defined by $Lv = -\nabla p$ and $Lw = f$ in $\mathbb{R}^3$, respectively. Using the fundamental solution matrix

$$K(x,y) = \int_0^\infty (4\pi t)^{-3/2} \exp\left(-\frac{|O(|\omega|t)x - y|^2}{4t}\right) O(|\omega|t)^T dt$$

of the operator L, see [14], [28], we first consider

$$\begin{aligned} \nabla v(x) &= -\int_{\mathbb{R}^3} \nabla_x K(x,y)\nabla_y p(y)\, dy \\ &= \Big(-\int_{|y|<|x|/2} - \int_{|y|\ge|x|/2}\Big)(\cdots)\, dy =: I_0(x) + I_\infty(x). \end{aligned}$$

It is easily seen that

$$\begin{aligned} I_\infty(x) = \int_0^\infty (4\pi t)^{-3/2} \int_{|y|\ge|x|/2} &\exp\left(-\frac{|x-y|^2}{4t}\right) \\ &\times \big[O(|\omega|t)^T(\nabla p)(O(|\omega|t)y)\big] \otimes \frac{x-y}{2t}\, dy\, dt. \end{aligned}$$

Since $|y| \ge |x|/2 \ge 2R$ for $|x| \ge 4R$, one can use (3.8) to obtain

$$|I_\infty(x)| \le C\|f\|_{1,\mathbb{R}^3} \int_{|y|\ge|x|/2} \frac{|x-y|}{|y|^3} \int_0^\infty t^{-5/2} \exp\left(-\frac{|x-y|^2}{4t}\right) dt\, dy.$$

Recall

$$\int_0^\infty t^{-\alpha} \exp\left(-\frac{|x|^2}{ct}\right) dt = \frac{c^{\alpha-1}\Gamma(\alpha-1)}{|x|^{2(\alpha-1)}}, \qquad x \neq 0,\ \alpha > 1,\ c > 0, \quad (3.11)$$

where $\Gamma(\cdot)$ is the Gamma function, so that

$$|I_\infty(x)| \leq C\|f\|_{1,\mathbb{R}^3} \int_{|y|\geq|x|/2} \frac{1}{|y|^3|x-y|^2}\, dy.$$

Using the change of variable $y' = \frac{y}{|x|}$, it is easily seen that the integral above is bounded by $\frac{c}{|x|^2}$. We thus obtain

$$|I_\infty(x)| \leq \frac{C\|f\|_{1,\mathbb{R}^3}}{|x|^2}, \qquad |x| \geq 4R. \quad (3.12)$$

Integration by parts yields

$$I_0(x) = I_{01}(x) + I_{02}(x)$$

with

$$I_{01}(x) = \int_0^\infty (4\pi t)^{-3/2} \int_{|y|=|x|/2} \exp\left(-\frac{|x-y|^2}{4t}\right) \times \frac{\left(O(|\omega|t)^T \frac{y}{|y|}\right) \otimes (x-y)}{2t} p(O(|\omega|t)y)\, d\sigma_y\, dt,$$

and

$$I_{02}(x) = \int_0^\infty (4\pi t)^{-3/2} \int_{|y|<|x|/2} \exp\left(-\frac{|x-y|^2}{4t}\right) \times \left(\frac{(x-y)\otimes(x-y)}{-4t^2} + \frac{\mathbb{I}}{2t}\right) p(O(|\omega|t)y)\, dy\, dt,$$

where $\mathbb{I} = (\delta_{jk})$. For $|y| = |x|/2 \geq 2R$, we have $|p(O(t)y)| \leq C|x|^{-2}\|f\|_{1,\mathbb{R}^3}$ by (3.8). This together with (3.11) implies

$$|I_{01}(x)| \leq \frac{C\|f\|_{1,\mathbb{R}^3}}{|x|^2} \int_{|y|=|x|/2} \frac{d\sigma_y}{|x-y|^2} \leq \frac{C\|f\|_{1,\mathbb{R}^3}}{|x|^2}, \qquad |x| \geq 4R. \quad (3.13)$$

Since $|x|/2 < |x-y| < 3|x|/2$ for $|y| < |x|/2$, it follows from (3.11) and (3.8) that

$$\begin{aligned} |I_{02}(x)| &\leq C \int_0^\infty \left(t^{-7/2}|x|^2 + t^{-5/2}\right) \exp\left(-\frac{|x|^2}{16t}\right) dt \int_{|y|<|x|/2} |p(y)|\, dy \\ &\leq \frac{C}{|x|^3}\left(\|p\|_{1,B_{2R}} + C\|f\|_{1,\mathbb{R}^3} \int_{B_{|x|/2}} \frac{dy}{|y|^2}\right). \end{aligned}$$

By (3.7) we find $\|p\|_{1,B_{2R}} \le C_R \|p\|_{3/2,\infty,\mathbb{R}^3} \le C_R \|f\|_{1,\mathbb{R}^3}$. As a consequence,

$$|I_{02}(x)| \le \frac{C\|f\|_{1,\mathbb{R}^3}}{|x|^3}(C_R + |x|), \qquad |x| \ge 4R. \tag{3.14}$$

We collect (3.12), (3.13) and (3.14) to get

$$\text{ess sup}_{|x|\ge 4R}\, |x|^2 |\nabla v(x)| \le C_R \|f\|_{1,\mathbb{R}^3}, \tag{3.15}$$

with some $C_R > 0$ independent of $|\omega|$.

Finally, we see from supp $f \subset B_R$ that

$$|\nabla w(x)| \le C \int_0^\infty t^{-5/2} \int_{B_R} \exp\left(-\frac{|x-y|^2}{4t}\right) |x-y| |f(O(|\omega|t)y)|\, dy\, dt.$$

Since $|y| < R < |x|/2$, we have $|x|/2 < |x-y| < 3|x|/2$, so that

$$|\nabla w(x)| \le C|x| \int_0^\infty t^{-5/2} \exp\left(-\frac{|x|^2}{16t}\right) dt \int_{B_R} |f(y)|\, dy,$$

yielding with the help of (3.11) the estimate

$$\text{ess sup}_{|x|\ge 4R}\, |x|^2 |\nabla w(x)| \le C \|f\|_{1,\mathbb{R}^3}.$$

This inequality combined with (3.15) implies (3.10). By analogy, we get the estimate $|x|\,|u(x)| \le C\|f\|_{1,\mathbb{R}^3}$. Now we have completed the proof. □

4 Linearized problem in bounded domains

Given a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$ we consider weak solutions to the boundary value problem

$$\begin{cases} -\Delta u - (\omega \wedge x)\cdot \nabla u + \omega \wedge u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

For the usual Stokes problem, the case $\omega = 0$, the following results are known, see Cattabriga [7], Solonnikov [36], Kozono and Sohr [29] and Kozono and Yamazaki [31]. The second part (ii) follows from (i) by real interpolation.

Lemma 4.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, let $1 < q < \infty$ and consider problem (4.1) with $\omega = 0$.*

(i) ([7], [36], [29]) *Suppose that*

$$f \in \dot{W}_q^{-1}(\Omega), \quad g \in L_q(\Omega), \quad \int_\Omega g(x)dx = 0.$$

Then there exists a unique q-weak solution $(u,p) \in \dot{W}_q^1(\Omega) \times L_q(\Omega)$ *(up to an additive constant for p) subject to the estimate*

$$\|\nabla u\|_{q,\Omega} + \|u\|_{q,\Omega} + \|p - \overline{p}\|_{q,\Omega} \le C \left(\|f\|_{\dot{W}_q^{-1}(\Omega)} + \|g\|_{q,\Omega} \right), \tag{4.2}$$

where $\overline{p} = \frac{1}{|\Omega|} \int_\Omega p(x)dx$.

(ii) ([31, Lemma 2.7]) *Suppose that*

$$f \in \dot{W}_{q,\infty}^{-1}(\Omega), \quad g \in L_{q,\infty}(\Omega), \quad \int_\Omega g(x)dx = 0.$$

Then there exists a unique (q,∞)*-weak solution* $(u,p) \in \dot{W}_{q,\infty}^1(\Omega) \times L_{q,\infty}(\Omega)$ *(up to an additive constant for p) subject to the estimate*

$$\|\nabla u\|_{q,\infty,\Omega} + \|u\|_{q,\infty,\Omega} + \|p - \overline{p}\|_{q,\infty,\Omega} \le C \left(\|f\|_{\dot{W}_{q,\infty}^{-1}(\Omega)} + \|g\|_{q,\infty,\Omega} \right). \tag{4.3}$$

In bounded domains, the operator L can be treated as a perturbation to the Laplace operator. For the argument in the next section it suffices to consider the case $\operatorname{div} u = g = 0$.

Proposition 4.1 *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with smooth boundary, and let* $1 < q < \infty$*. Suppose that* $f \in \dot{W}_{q,\infty}^{-1}(\Omega)$ *and* $g = 0$*. Then problem (4.1) possesses a unique* (q,∞)*-weak solution* $(u,p) \in \dot{W}_{q,\infty}^1(\Omega) \times L_{q,\infty}(\Omega)$ *(up to an additive constant for p) subject to the estimate*

$$\|\nabla u\|_{q,\infty,\Omega} + \|u\|_{q,\infty,\Omega} + \|p - \overline{p}\|_{q,\infty,\Omega} \le C\|f\|_{\dot{W}_{q,\infty}^{-1}(\Omega)}, \tag{4.4}$$

with some $C = C(M) > 0$ *uniformly in* $|\omega| \in [0, M]$, $M > 0$*, where* $\overline{p}$ *is as in Lemma 4.1.*

Proof. Set

$$Du = (e_3 \wedge x) \cdot \nabla u - e_3 \wedge u = \operatorname{div}\,[(e_3 \wedge x) \otimes u - u \otimes (e_3 \wedge x)].$$

Then (4.1) can be rewritten as

$$-\Delta u + \nabla p = f + |\omega| Du, \ \operatorname{div} u = 0.$$

Let $T : \dot{W}^{-1}_{q,\infty}(\Omega) \to \dot{W}^1_{q,\infty}(\Omega), f \mapsto u$, be the solution operator defined by Lemma 4.1 (ii) with $g = 0$. Given $f \in \dot{W}^{-1}_{q,\infty}(\Omega)$, we intend to solve

$$(1 - |\omega| TD)u = Tf$$

in $\dot{W}^1_{q,\infty}(\Omega)$. Lemma 4.1 together with (2.5) yields

$$\|\nabla TDv\|_{q,\infty,\Omega} \le C\|Dv\|_{\dot{W}^{-1}_{q,\infty}(\Omega)} \le C\|v\|_{q,\infty,\Omega} \le c\|\nabla v\|_{q,\infty,\Omega}$$

for some $c = c(q, \Omega) > 0$ independent of $v \in \dot{W}^1_{q,\infty}(\Omega)$. Therefore, the operator $TD : L_{q,\infty}(\Omega) \to \dot{W}^1_{q,\infty}(\Omega)$ is bounded and, due to Lemma 2.1, even compact as an operator from $\dot{W}^1_{q,\infty}(\Omega)$ into itself. Moreover, we use the injectivity of the operator $1 - |\omega|TD : \dot{W}^1_{q,\infty}(\Omega) \to \dot{W}^1_{q,\infty}(\Omega)$: In fact, the equation $(1 - |\omega|TD)u = 0$ is equivalent to the Stokes system $-\Delta u + \nabla p = |\omega|Du$, $\operatorname{div} u = 0$ where $|\omega|Du \in L_{q,\infty}(\Omega)$. Then a bootstrapping argument yields $u \in \dot{W}^1_2(\Omega)$, and testing the equation with u itself shows that $u = 0$.

Now Fredholm theory proves that the operator $1 - |\omega|TD$ has a bounded inverse. Hence the unique solution u to (4.1) satisfies the estimate

$$\|\nabla u\|_{q,\infty,\Omega} \le C_\omega \|f\|_{\dot{W}^{-1}_{q,\infty}(\Omega)}.$$

We also obtain an associated pressure p from Lemma 4.1 satisfying

$$\|p - \overline{p}\|_{q,\infty,\Omega} \le C_\omega \|f + |\omega|Du\|_{\dot{W}^{-1}_{q,\infty}(\Omega)} \le C_\omega \|f\|_{\dot{W}^{-1}_{q,\infty}(\Omega)}.$$

So far the constant C_ω will depend somehow on ω. However, an argument by contradiction as at the end of Section 5 will easily prove that $C_\omega = C(M)$ uniformly in $|\omega| \in [0, M]$, $M > 0$, where (4.3) is used instead of (5.12). This completes the proof. □

Finally we mention a well-known result on the divergence problem.

Lemma 4.2 ([2], [6], [17]) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $1 < q < \infty$ and $k \ge 0$ integer. Then there is a linear operator $B : C_0^\infty(\Omega) \to C_0^\infty(\Omega)^n$ such that*

$$\|\nabla^{k+1} Bf\|_{q,\Omega} \le C\|\nabla^k f\|_{q,\Omega}$$

with some $C = C(\Omega, q, k) > 0$ and that

$$\operatorname{div}(Bf) = f \qquad \text{if} \quad \int_\Omega f(x)dx = 0.$$

By continuity, B is extended uniquely to a bounded operator from $\dot{W}^k_q(\Omega)$ to $\dot{W}^{k+1}_q(\Omega)^n$, where $\dot{W}^k_q(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla^k(\cdot)\|_{q,\Omega}$. Furthermore, by real interpolation, it is extended uniquely to a bounded operator from $\dot{W}^k_{q,\infty}(\Omega)$ to $\dot{W}^{k+1}_{q,\infty}(\Omega)^n$, where

$$\dot{W}^k_{q,\infty}(\Omega) = \left(\dot{W}^k_{q_0}(\Omega), \dot{W}^k_{q_1}(\Omega)\right)_{\theta,\infty}$$

with q_0, q_1 and θ satisfying (2.3).

5 Linearized problem in exterior domains

This section establishes Theorem 5.1 below on existence, uniqueness and $L_{3/2,\infty}$-estimate for solutions of the boundary value problem

$$Lu + \nabla p = f, \quad \operatorname{div} u = 0 \quad \text{in } D; \quad u|_{\partial D} = 0, \tag{5.1}$$

in an exterior domain D.

Theorem 5.1 *Let $f \in \dot{W}^{-1}_{3/2,\infty}(D)$. Then problem (5.1) possesses a unique $(3/2,\infty)$-weak solution*

$$(u, p) \in \dot{W}^1_{3/2,\infty}(D) \times L_{3/2,\infty}(D)$$

subject to the estimate

$$\|(\nabla u,\, p)\|_{3/2,\infty} + \|(\omega \wedge x)\cdot \nabla u - \omega \wedge u\|_{\dot{W}^{-1}_{3/2,\infty}(D)} \le C\|f\|_{\dot{W}^{-1}_{3/2,\infty}(D)}, \tag{5.2}$$

with some $C = C(M) > 0$ uniformly in $|\omega| \in [0, M]$, $M > 0$.

We begin with the question of uniqueness.

Proposition 5.1 *Let $f = 0$. Then the only $(3/2,\infty)$-weak solution $(u,p) \in \dot{W}^1_{3/2,\infty}(D) \times L_{3/2,\infty}(D)$ of (5.1) is the trivial one, that is, $(u,p) = (0,0)$.*

Proof. We use the cut-off procedure, as in [28, Lemma 5.2], to show the regularity (5.3) below. We fix $\rho > \rho_0 > 0$ so large that $\mathbb{R}^3 \setminus D \subset B_{\rho_0}$, and take a cut-off function $\phi \in C_0^\infty(B_\rho; [0,1])$ such that $\phi = 1$ on B_{ρ_0} and also supp $(\nabla\phi) \subset A = \{x \in \mathbb{R}^3; \rho_0 < |x| < \rho\}$. The solution (u,p) is decomposed as

$$\begin{cases} u = U + V, & U = (1-\phi)u, & V = \phi u, \\ p = \sigma + \tau, & \sigma = (1-\phi)p, & \tau = \phi p. \end{cases}$$

Then (U, σ) is a distribution solution to

$$LU + \nabla\sigma = F, \quad \operatorname{div} U = -u \cdot \nabla\phi \qquad \text{in } \mathbb{R}^3,$$

where

$$F = 2\nabla\phi \cdot \nabla u + [\Delta\phi + (\omega \wedge x) \cdot \nabla\phi]u - (\nabla\phi)p.$$

Similarly, (V, τ) is a distribution solution to the usual Stokes problem

$$-\Delta V + \nabla\tau = G, \quad \operatorname{div} V = u \cdot \nabla\phi \quad \text{in } D_\rho; \quad V|_{\partial D_\rho} = 0,$$

where $D_\rho = D \cap B_\rho$ and

$$G = \phi[(\omega \wedge x) \cdot \nabla u - \omega \wedge u] - 2\nabla\phi \cdot \nabla u - (\Delta\phi)u + (\nabla\phi)p.$$

By the embedding relation (2.6) and by duality we have

$$\dot{W}^1_{3/2,\infty}(D) \hookrightarrow L_{3,\infty}(D), \qquad L_{3/2,\infty}(D) \hookrightarrow \dot{W}^{-1}_{3,\infty}(D),$$

from which it follows that

$$u|_A \in L_q(A), \qquad p|_A \in \dot{W}^{-1}_q(A) \qquad \forall\, q \in (1, 3),$$

where A is the annulus above. Let $\psi \in C_0^\infty(\mathbb{R}^3)$. If $q > 3/2$, so that $q' = q/(q-1) < 3$, then we see

$$\|\psi\|_{q',A} \le C\|\psi\|_{r,A} \le C\|\psi\|_{r,\mathbb{R}^3} \le C\|\nabla\psi\|_{q',\mathbb{R}^3},$$

where $1/r = 1/q' - 1/3$. We thus assume $3/2 < q < 3$. Then

$$\begin{aligned} |\langle F, \psi\rangle| &= |\langle 2\nabla\phi \cdot \nabla u + [\Delta\phi + (\omega \wedge x) \cdot \nabla\phi]u - (\nabla\phi)p,\, \psi\rangle| \\ &\le C\left(\|u\|_{q,A} + \|p\|_{\dot{W}^{-1}_q(A)}\right)(\|\nabla\psi\|_{q',A} + \|\psi\|_{q',A}) \\ &\le C\left(\|u\|_{q,A} + \|p\|_{\dot{W}^{-1}_q(A)}\right)\|\nabla\psi\|_{q',\mathbb{R}^3}, \end{aligned}$$

which implies $F \in \dot{W}^{-1}_q(\mathbb{R}^3)$. Furthermore,

$$|\langle(\omega \wedge x)(u \cdot \nabla\phi), \psi\rangle| \le C\|u\|_{q,A}\|\psi\|_{q',A} \le C\|u\|_{q,A}\|\nabla\psi\|_{q',\mathbb{R}^3},$$

so that $(\omega \wedge x)(u \cdot \nabla\phi) \in \dot{W}^{-1}_q(\mathbb{R}^3)$ as well as $u \cdot \nabla\phi \in L_q(\mathbb{R}^3)$. In view of Proposition 3.1 we find $(\nabla U, \sigma) \in L_q(\mathbb{R}^3)$ for all $q \in (3/2, 3)$. In the same way, Lemma 4.1 implies $(\nabla V, \tau) \in L_q(D_\rho)$ for all $q \in (1, 3)$.

Therefore, $\nabla u \in L_q(D)$ for $3/2 < q < 3$, which yields $u - a \in L_{q_*}(D)$ for some $a \in \mathbb{R}^3$, where $1/q_* = 1/q - 1/3$ (see also Lemma 2.2 (iv)); but, since $u \in L_{3,\infty}(D)$, we find $a = 0$. As a consequence,

$$u \in \dot{W}^1_q(D), \qquad p \in L_q(D), \qquad \forall\, q \in (3/2, 3). \tag{5.3}$$

Due to Proposition 5.1 of [28], a q-weak solution to (5.1) is unique provided $1 < q < 3$. We thus obtain $(u, p) = (0, 0)$. $\square$

For given $f \in \dot{W}^{-1}_{3/2,\infty}(D)$ we will construct the solution of (5.1) by means of a cut-off technique, using the solutions in the whole space (Section 3) and in a bounded domain (Section 4).

Fix $\rho > 0$ so large that $\mathbb{R}^3 \setminus D \subset B_{\rho-5}$, and choose cut-off functions $\phi_j \in C^\infty(\mathbb{R}^3; [0,1])$, $j = 0, 1, 2$, satisfying

$$\phi_1(x) = \begin{cases} 0, & |x| \le \rho - 5, \\ 1, & |x| \ge \rho - 4, \end{cases} \qquad \phi_j(x) = \begin{cases} 1, & |x| \le \rho - 3 + j, \\ 0, & |x| \ge \rho - 2 + j, \end{cases} \quad j = 0, 2.$$

We set

$$D_\rho = D \cap B_\rho, \qquad A_\rho = \{x \in \mathbb{R}^3; \rho - 4 < |x| < \rho - 1\}.$$

Consider (3.1) with f replaced by $\phi_1 f$ and $g = 0$ in the whole space $\mathbb{R}^3$. We see that $\phi_1 f \in \dot{W}^{-1}_{3/2,\infty}(\mathbb{R}^3)$ with

$$\|\phi_1 f\|_{\dot{W}^{-1}_{3/2,\infty}(\mathbb{R}^3)} \le C\|f\|_{\dot{W}^{-1}_{3/2,\infty}(D)}. \tag{5.4}$$

This is observed by using $\dot{W}^1_{3,1}(\mathbb{R}^3) \hookrightarrow L_\infty(\mathbb{R}^3)$, see (2.8); in fact, for all $\psi \in C_0^\infty(\mathbb{R}^3)$, we find

$$\begin{aligned} |\langle \phi_1 f, \psi\rangle| &\le \|f\|_{\dot{W}^{-1}_{3/2,\infty}(D)} \|\nabla(\phi_1\psi)\|_{3,1} \\ &\le C\|f\|_{\dot{W}^{-1}_{3/2,\infty}} (\|\psi\|_{\infty, A_\rho} + \|\nabla\psi\|_{3,1}) \\ &\le C\|f\|_{\dot{W}^{-1}_{3/2,\infty}(D)} \|\nabla\psi\|_{3,1,\mathbb{R}^3}. \end{aligned}$$

Let (u_∞, p_∞) be the solenoidal solution obtained in Proposition 3.2 for the external force $\phi_1 f$, and let

$$(S_\infty, \Pi_\infty) : \dot{W}^{-1}_{3/2,\infty}(D) \ni f \mapsto (u_\infty, p_\infty) \in \dot{W}^1_{3/2,\infty}(\mathbb{R}^3) \times L_{3/2,\infty}(\mathbb{R}^3)$$

denote the solution operator. Here, u_∞ is uniquely chosen in such a way that $u_\infty \in L_{3,\infty}(\mathbb{R}^3)$.

We also consider (4.1) with f replaced by $\phi_2 f$ and $g = 0$ in the bounded domain $\Omega = D_\rho$. We easily see that $\phi_2 f \in \dot{W}^{-1}_{3/2,\infty}(D_\rho)$ with

$$\|\phi_2 f\|_{\dot{W}^{-1}_{3/2,\infty}(D_\rho)} \le C\|f\|_{\dot{W}^{-1}_{3/2,\infty}(D)}. \tag{5.5}$$

Let (u_0, p_0) be the solution obtained in Proposition 4.1 for the external force $\phi_2 f$, and let

$$(S_0, \Pi_0) : \dot{W}^{-1}_{3/2,\infty}(D) \ni f \mapsto (u_0, p_0) \in \dot{W}^1_{3/2,\infty}(D_\rho) \times L_{3/2,\infty}(D_\rho)$$

denote the solution operator. Here, p_0 is chosen such that $\int_{D_\rho} p_0(x)dx = 0$.

As a parametrix (an approximation of the solution) for the exterior problem, we take

$$\begin{cases} Sf = (1-\phi_0)S_\infty f + \phi_0 S_0 f + B[(S_\infty f - S_0 f)\cdot\nabla\phi_0], \\ \Pi f = (1-\phi_0)\Pi_\infty f + \phi_0 \Pi_0 f, \end{cases} \tag{5.6}$$

where B is the operator defined by Lemma 4.2 in the bounded domain A_ρ. Note that

$$\int_{A_\rho} (S_\infty f - S_0 f)\cdot\nabla\phi_0\, dx = 0,$$

which implies $\mathrm{div}\,(Sf) = 0$. Concerning the class of $(Sf, \Pi f)$, we have

Proposition 5.2 *Let $f \in \dot{W}^{-1}_{3/2,\infty}(D)$ be given. Then $(Sf, \Pi f) \in \dot{W}^1_{3/2,\infty}(D) \times L_{3/2,\infty}(D)$ and*

$$\|\nabla Sf\|_{3/2,\infty} + \|\Pi f\|_{3/2,\infty} \le C\|f\|_{\dot{W}^{-1}_{3/2,\infty}(D)} \tag{5.7}$$

for some $C = C(M) > 0$ uniformly in $|\omega| \in [0, M]$, $M > 0$.

Proof. By Lemma 4.2 we obtain

$$\begin{aligned} \|\nabla Sf\|_{3/2,\infty} &\le \|\nabla u_\infty\|_{3/2,\infty,\mathbb{R}^3} + \|\nabla u_0\|_{3/2,\infty,D_\rho} + C\|u_\infty - u_0\|_{3/2,\infty,A_\rho}, \\ \|\Pi f\|_{3/2,\infty} &\le \|p_\infty\|_{3/2,\infty,\mathbb{R}^3} + \|p_0\|_{3/2,\infty,D_\rho}, \end{aligned}$$

where $(u_\infty, p_\infty) := (S_\infty f, \Pi_\infty f)$ and $(u_0, p_0) := (S_0 f, \Pi_0 f)$. The Sobolev inequality (2.7) in $\mathbb{R}^3$ and the Poincaré inequality (2.5) in D_ρ lead us to the estimates

$$\|u_\infty\|_{3/2,\infty,A_\rho} \le C\|u_\infty\|_{3,\infty,\mathbb{R}^3} \le C\|\nabla u_\infty\|_{3/2,\infty,\mathbb{R}^3},$$

$$\|u_0\|_{3/2,\infty,A_\rho} \le \|u_0\|_{3/2,\infty,D_\rho} \le C\|\nabla u_0\|_{3/2,\infty,D_\rho}.$$

Thus Propositions 3.2 and 4.1 imply

$$\|\nabla Sf\|_{3/2,\infty} + \|\Pi f\|_{3/2,\infty} \le C\big(\|\phi_1 f\|_{\dot{W}^{-1}_{3/2,\infty}(\mathbb{R}^3)} + \|\phi_2 f\|_{\dot{W}^{-1}_{3/2,\infty}(D_\rho)}\big),$$

which together with (5.4) and (5.5) gives the estimate (5.7). We also have $Sf + a \in L_{3,\infty}(D)$ for some $a \in \mathbb{R}^3$, see Lemma 2.2. But since $Sf = S_\infty f$ for large $|x|$ and $S_\infty f \in \dot{W}^1_{3/2,\infty}(\mathbb{R}^3) \hookrightarrow L_{3,\infty}(\mathbb{R}^3)$, we find $a = 0$. We have proved $Sf \in \dot{W}^1_{3/2,\infty}(D)$. □

We see that $(v,\pi) = (Sf, \Pi f)$ is a distribution solution to

$$Lv + \nabla\pi = f + Rf, \quad \operatorname{div} v = 0 \quad \text{in } D; \quad v|_{\partial D} = 0, \tag{5.8}$$

where the remainder term Rf is defined as

$$\begin{aligned} Rf =& -2\nabla\phi_0 \cdot \nabla(S_\infty f - S_0 f) - [\Delta\phi_0 + (\omega \wedge x)\cdot\nabla\phi_0](S_\infty f - S_0 f) \\ & - LB[(S_\infty f - S_0 f)\cdot\nabla\phi_0] + (\nabla\phi_0)(\Pi_\infty f - \Pi_0 f). \end{aligned}$$

Lemma 5.1 *Suppose that $f \in \dot{W}^{-1}_{3/2,\infty}(D)$. Then we have $Rf \in L_{3/2,\infty}(A_\rho) \cap \dot{W}^{-1}_{3/2,\infty}(D)$ and*

$$\|Rf\|_{3/2,\infty,A_\rho} + \|Rf\|_{\dot{W}^{-1}_{3/2,\infty}(D)} \le C\|f\|_{\dot{W}^{-1}_{3/2,\infty}(D)}. \tag{5.9}$$

Proof. We start with the estimate of Rf in $\dot{W}^{-1}_{3/2,\infty}(D)$. Let $\psi \in C_0^\infty(D)$. By (2.8) we have, for any $r \in (1,\infty)$,

$$\|\psi\|_{r,1,A_\rho} \le C\|\psi\|_{\infty,A_\rho} \le C\|\psi\|_{\infty,D} \le C\|\nabla\psi\|_{3,1,D},$$

from which together with Lemma 4.2 it follows that

$$\begin{aligned} &|\langle 2\nabla\phi_0\cdot\nabla u_\infty + [\Delta\phi_0 + (\omega\wedge x)\cdot\nabla\phi_0]u_\infty - (\nabla\phi_0)p_\infty, \psi\rangle| \\ &\quad \le C\left(\|\nabla u_\infty\|_{3/2,\infty,\mathbb{R}^3} + \|u_\infty\|_{3,\infty,\mathbb{R}^3} + \|p_\infty\|_{3/2,\infty,\mathbb{R}^3}\right)\|\psi\|_{\infty,A_\rho} \\ &\quad \le C\left(\|\nabla u_\infty\|_{3/2,\infty,\mathbb{R}^3} + \|p_\infty\|_{3/2,\infty,\mathbb{R}^3}\right)\|\nabla\psi\|_{3,1,D}, \end{aligned}$$

and that

$$\begin{aligned} |\langle LB[u_\infty\cdot\nabla\phi_0],\psi\rangle| &\le C\|LB[u_\infty\cdot\nabla\phi_0]\|_{3/2,\infty,A_\rho}\|\psi\|_{3,1,A_\rho} \\ &\le C\left(\|\nabla u_\infty\|_{3/2,\infty,A_\rho} + \|u_\infty\|_{3/2,\infty,A_\rho}\right)\|\psi\|_{\infty,A_\rho} \\ &\le C(\|\nabla u_\infty\|_{3/2,\infty,\mathbb{R}^3} + \|u_\infty\|_{3,\infty,\mathbb{R}^3})\|\nabla\psi\|_{3,1,D} \\ &\le C\|\nabla u_\infty\|_{3/2,\infty,\mathbb{R}^3}\|\nabla\psi\|_{3,1,D}, \end{aligned}$$

where $(u_\infty, p_\infty) = (S_\infty f, \Pi_\infty f)$. The terms including $(S_0 f, \Pi_0 f)$ can easily be estimated. As a result, we obtain $Rf \in \dot{W}^{-1}_{3/2,\infty}(D)$ and, by Propositions 3.2 and 4.1,

$$\begin{aligned} &\|Rf\|_{\dot{W}^{-1}_{3/2,\infty}(D)} \\ &\le C\left(\|\nabla u_\infty\|_{3/2,\infty,\mathbb{R}^3} + \|p_\infty\|_{3/2,\infty,\mathbb{R}^3} + \|\nabla u_0\|_{3/2,\infty,D_\rho} + \|p_0\|_{3/2,\infty,D_\rho}\right) \\ &\le C\big(\|\phi_1 f\|_{\dot{W}^{-1}_{3/2,\infty}(\mathbb{R}^3)} + \|\phi_2 f\|_{\dot{W}^{-1}_{3/2,\infty}(D_\rho)}\big). \end{aligned}$$

This together with (5.4) and (5.5) implies the estimate of Rf in $\dot{W}^{-1}_{3/2,\infty}(D)$ in (5.9).

Since the support of Rf is contained in A_ρ we get as in the first part of the proof the estimate of Rf in $L_{3/2,\infty}(A_\rho)$. $\square$

By Lemma 5.1 and Proposition 5.2, we find that $(v,\pi)=(Sf,\Pi f)$ is a $(3/2,\infty)$-weak solution of (5.8).

Proposition 5.3 *The operator R is compact from $\dot W^{-1}_{3/2,\infty}(D)$ into itself, and $1+R$ has a bounded inverse in $\dot W^{-1}_{3/2,\infty}(D)$.*

Proof. By Lemma 2.1 and (5.9) the operator R is compact in $\dot W^{-1}_{3/2,\infty}(D)$. To complete the proof, owing to the Fredholm theorem, it suffices to show that $1+R$ is injective in $\dot W^{-1}_{3/2,\infty}(D)$. Suppose that $(1+R)f=0$. Then, in view of (5.8), the pair $(v,\pi)=(Sf,\Pi f)$ for such f is a $(3/2,\infty)$-weak solution to

$$Lv+\nabla\pi=0,\quad \operatorname{div} v=0\quad \text{in } D;\quad v|_{\partial D}=0.$$

From Proposition 5.1 it follows that $(v,\pi)=(0,0)$, which yields

$$(S_\infty f,\Pi_\infty f)=(0,0),\quad |x|\ge\rho-1;\qquad (S_0 f,\Pi_0 f)=(0,0),\quad x\in D_{\rho-4}.$$

Therefore, from the equations in $\mathbb{R}^3$ and in D_ρ we find $\operatorname{supp} f\subset \overline{A}_\rho$. Both $(S_\infty f,\Pi_\infty f)$ and $(S_0 f,\Pi_0 f)$ are of class $\dot W^1_{3/2,\infty}(B_\rho)\times L_{3/2,\infty}(B_\rho)$ and they are $(3/2,\infty)$-weak solutions to

$$Lu+\nabla p=f,\quad \operatorname{div} u=0\quad \text{in } B_\rho;\quad u|_{\partial B_\rho}=0,$$

where now $(S_0 f,\Pi_0 f)$ is understood as its extension by zero on the region $B_\rho\setminus D_\rho\equiv\mathbb{R}^3\setminus D$. By the uniqueness of solutions on B_ρ, see Proposition 4.1, we obtain $(S_\infty f,\Pi_\infty f)=(S_0 f,\Pi_0 f)$ in B_ρ, which implies

$$S_\infty f=Sf=v=0,\qquad \Pi_\infty f=\Pi f=\pi=0\quad \text{in } B_\rho.$$

Returning to the equation in the whole space $\mathbb{R}^3$, we obtain $f=0$ in D. This completes the proof. $\square$

Proof of Theorem 5.1. By Propositions 5.2 and 5.3 the pair of

$$u=S(1+R)^{-1}f,\qquad p=\Pi(1+R)^{-1}f,$$

provides a $(3/2,\infty)$-weak solution of (5.1) with $f\in\dot W^{-1}_{3/2,\infty}(D)$ and the estimate (5.2) holds. This shows the existence part. Uniqueness follows from Proposition 5.1.

Finally, we show that the constant $C > 0$ in (5.2) is independent of $|\omega| \in [0, M]$, $M > 0$. Suppose the contrary. Then there are sequences (ω_n) with $|\omega_n| \in [0, M]$ and $(f_n) \subset \dot{W}^{-1}_{3/2,\infty}(D)$ such that

$$\lim_{n\to\infty} \|f_n\|_{\dot{W}^{-1}_{3/2,\infty}(D)} = 0,$$

$$\|\nabla u_n\|_{3/2,\infty} + \|p_n\|_{3/2,\infty} + \|(\omega_n \wedge x) \cdot \nabla u_n - \omega_n \wedge u_n\|_{\dot{W}^{-1}_{3/2,\infty}(D)} = 1, \quad (5.10)$$

where $(u_n, p_n) \in \dot{W}^1_{3/2,\infty}(D) \times L_{3/2,\infty}(D)$ is the corresponding weak solution. Hence there are subsequences, which we denote by the same symbols, so that

$$w^* - \lim_{n\to\infty} (u_n, p_n) = (u, p) \quad \text{in } \dot{W}^1_{3/2,\infty}(D) \times L_{3/2,\infty}(D), \qquad \lim_{n\to\infty} \omega_n = \alpha e_3.$$

Let $\rho > 0$ be arbitrarily large and set

$$W^1_{3/2,\infty}(D_\rho) = \{u \in L_{3/2,\infty}(D_\rho);\ \nabla u \in L_{3/2,\infty}(D_\rho)\}, \quad D_\rho = D \cap B_\rho.$$

Since this space is compactly embedded into $L_{3/2,\infty}(D_\rho)$ and so is $L_{3/2,\infty}(D_\rho)$ in $\dot{W}^{-1}_{3/2,\infty}(D_\rho)$, we have

$$\lim_{n\to\infty} (u_n, p_n) = (u, p) \quad \text{in } L_{3/2,\infty}(D_\rho) \times \dot{W}^{-1}_{3/2,\infty}(D_\rho). \qquad (5.11)$$

On the other hand, it is possible to derive the *a priori* estimate

$$\begin{aligned} &\|\nabla u_n\|_{3/2,\infty} + \|p_n\|_{3/2,\infty} + \|(\omega_n \wedge x) \cdot \nabla u_n - \omega_n \wedge u_n\|_{\dot{W}^{-1}_{3/2,\infty}(D)} \\ &\quad \le C\Big\{\|f_n\|_{\dot{W}^{-1}_{3/2,\infty}(D)} + (1 + |\omega_n|)\|u_n\|_{3/2,\infty,D_\rho} \\ &\qquad + \|p_n\|_{\dot{W}^{-1}_{3/2,\infty}(D_\rho)} + \Big|\int_{D_\rho} \phi(x)p_n(x)dx\Big|\Big\} \end{aligned} \qquad (5.12)$$

for the weak solutions $(u_n, p_n) \in \dot{W}^1_{3/2,\infty}(D) \times L_{3/2,\infty}(D)$, where the cut-off function ϕ is as in the proof of Proposition 5.1 and $C > 0$ is independent of $|\omega_n|$. In fact, using Proposition 3.2, Lemma 4.1 and (2.8), we follow the cut-off argument in the proof of [28, Lemma 5.2] to obtain (5.12).

By (5.10), (5.11) and (5.12), we find, as $n \to \infty$,

$$1 \le C\Big\{(1 + \alpha)\|u\|_{3/2,\infty,D_\rho} + \|p\|_{\dot{W}^{-1}_{3/2,\infty}(D_\rho)} + \Big|\int_{D_\rho} \phi(x)p(x)dx\Big|\Big\}. \qquad (5.13)$$

However, the limit (u, p) is a weak solution to (5.1) with angular velocity $\omega = \alpha e_3$ and $f = 0$, so that Proposition 5.1 implies $(u, p) = (0, 0)$, contradicting (5.13). □

6 Navier-Stokes problem

This section is devoted to the proof of Theorem 1.1. We begin with the uniqueness.

Proposition 6.1 *For each $M > 0$ there is a constant $\widetilde{\eta} = \widetilde{\eta}(M) > 0$ such that for $|\omega| \in [0, M]$ the weak solution of (1.2), (1.3) in the class*

$$(\nabla u, p) \in L_{3/2,\infty}(D), \quad u \in L_{3,\infty}(D), \quad \|u\|_{3,\infty} \le \widetilde{\eta}, \tag{6.1}$$

is unique.

Proof. Let (u, p) and $(\widetilde{u}, \widetilde{p})$ be solutions of (1.2), (1.3) in the class (6.1). Set $(w, \pi) = (u - \widetilde{u}, p - \widetilde{p})$, which obeys

$$\begin{cases} Lw + \nabla\pi + w \cdot \nabla u + \widetilde{u} \cdot \nabla w &= 0 \qquad \text{in } D, \\ \operatorname{div} w &= 0 \qquad \text{in } D, \\ w &= 0 \qquad \text{on } \partial D, \\ w &\to 0 \qquad \text{as } |x| \to \infty. \end{cases}$$

Since $w = 0$ on ∂D, it follows from (2.6) that $w \in \dot{W}^1_{3/2,\infty}(D)$. By Theorem 5.1, the weak Hölder inequality ([5, Lemma 2.1]) and (2.7), we see that

$$\begin{aligned} \|\nabla w\|_{3/2,\infty} + \|\pi\|_{3/2,\infty} &\le C\|w \cdot \nabla u + \widetilde{u} \cdot \nabla w\|_{\dot{W}^{-1}_{3/2,\infty}(D)} \\ &\le C\left(\|u\|_{3,\infty} + \|\widetilde{u}\|_{3,\infty}\right)\|w\|_{3,\infty} \\ &\le \widetilde{c}\left(\|u\|_{3,\infty} + \|\widetilde{u}\|_{3,\infty}\right)\|\nabla w\|_{3/2,\infty}. \end{aligned}$$

One may take $\widetilde{\eta} < 1/2\widetilde{c}$, so that the condition (6.1) yields $w = 0$ in $\dot{W}^1_{3/2,\infty}(D)$, and thus $\pi = 0$ in $L_{3/2,\infty}(D)$. □

Let

$$T : \dot{W}^{-1}_{3/2,\infty}(D) \ni f \mapsto u \in \dot{W}^1_{3/2,\infty}(D)$$

be the solution operator defined by Theorem 5.1 for the linearized problem (5.1). By (5.2) we have

$$\|\nabla Tf\|_{3/2,\infty} \le C\|f\|_{\dot{W}^{-1}_{3/2,\infty}(D)}. \tag{6.2}$$

Given $v \in \dot{W}^1_{3/2,\infty}(D)$ and b defined by (2.11), we have

$$\|(v + b) \otimes (v + b)\|_{3/2,\infty} \le \|v + b\|^2_{3,\infty} \le C\left(\|\nabla v\|_{3/2,\infty} + |\omega|\right)^2$$

by (2.7). Since $b \in C_0^\infty(\mathbb{R}^3)$, we also have

$$\|\nabla b + (\omega \wedge x) \otimes b - b \otimes (\omega \wedge x)\|_{3/2,\infty} = C(|\omega| + |\omega|^2).$$

Thus, $\Phi(v, b) \in \dot{W}^{-1}_{3/2,\infty}(D)$ satisfying

$$\|\Phi(v, b)\|_{\dot{W}^{-1}_{3/2,\infty}(D)} \le C\|\nabla v\|^2_{3/2,\infty} + C(|\omega| + |\omega|^2). \tag{6.3}$$

Therefore, given $f \in \dot{W}^{-1}_{3/2,\infty}(D)$, we define the operator S from $\dot{W}^1_{3/2,\infty}(D)$ into itself by

$$Sv = T[f - \Phi(v, b)]$$

so that problem (2.12) is reduced to the fixed point problem

$$v = Sv \qquad \text{in } \dot{W}^1_{3/2,\infty}(D).$$

Once we find a fixed point v of the operator S, we obtain the associated pressure p as well by Theorem 5.1 with f replaced by $f - \Phi(v, b)$, and the pair (v, p) is actually a weak solution to (2.12).

Proof of Theorem 1.1. It follows from (6.2) and (6.3) that

$$\begin{aligned} \|\nabla Sv\|_{3/2,\infty} &= \|\nabla T[f - \Phi(v, b)]\|_{3/2,\infty} \\ &\le c_0 \left(\|f\|_{\dot{W}^{-1}_{3/2,\infty}(D)} + \|\nabla v\|^2_{3/2,\infty} + |\omega|^2 + |\omega| \right). \end{aligned} \tag{6.4}$$

Put $\rho = 2c_0\big(\|f\|_{\dot{W}^{-1}_{3/2,\infty}(D)} + |\omega|^2 + |\omega|\big)$ and define the closed ball

$$\mathcal{B}_\rho = \{v \in \dot{W}^1_{3/2,\infty}(D); \|\nabla v\|_{3/2,\infty} \le \rho\}$$

in $\dot{W}^1_{3/2,\infty}(D)$. Assume that

$$\|f\|_{\dot{W}^{-1}_{3/2,\infty}(D)} + |\omega|^2 + |\omega| \le \frac{1}{8c_0^2}, \tag{6.5}$$

or equivalently, $\rho \le 1/4c_0$. Then we see by (6.4) that $v \in \mathcal{B}_\rho$ implies

$$\|\nabla Sv\|_{3/2,\infty} \le c_0\rho^2 + \frac{\rho}{2} \le \rho.$$

Similarly, we have

$$\begin{aligned} \|\nabla(Sv - Sw)\|_{3/2,\infty} &= \|\nabla T[\Phi(v, b) - \Phi(w, b)]\|_{3/2,\infty} \\ &\le C\|\Phi(v, b) - \Phi(w, b)\|_{\dot{W}^{-1}_{3/2,\infty}(D)} \\ &\le C\|(v - w) \otimes (v + b) + (w + b) \otimes (v - w)\|_{3/2,\infty} \\ &\le c_0 \big(\|\nabla v\|_{3/2,\infty} + \|\nabla w\|_{3/2,\infty} + |\omega|\big) \|\nabla(v - w)\|_{3/2,\infty} \\ &\le c_0(2\rho + |\omega|)\|\nabla(v - w)\|_{3/2,\infty} \end{aligned}$$

for $v, w \in \mathcal{B}_\rho$, where $c_0 > 0$ is the same constant as in (6.4). By (6.5) we find $c_0(2\rho + |\omega|) < 1$, so that S is contractive from $\mathcal{B}_\rho$ into itself. We thus obtain a solution by Banach's fixed point theorem. By Proposition 6.1, the obtained solution $(u, p) = (v + b, p)$ provides a unique solution of (1.2), (1.3) in the class (6.1) as long as $\|v + b\|_{3,\infty} \leq \widetilde{\eta}$. $\square$

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