Equivalences of Smooth and Continuous Principal Bundles with Infinite-Dimensional Structure Group

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Abstract

This paper is on the equivalence of continuous and smooth principal bundles. Throughout the text, let K be a a Lie group, modeled on a locally convex space, and M be a finite-dimensional paracompact manifold with corners. We show that each continuous principal K- bundle over M is continuously equivalent to a smooth one and that two smooth principal K-bundles over M which are continuously equivalent are also smoothly equivalent. In the concluding section, we relate our results to neighboring topics.

Keywords: infinite-dimensional Lie groups, manifolds with corners, continuous principal bundles, smooth principal bundles, equivalences of continuous and smooth principal bundles ,smoothing of continuous principal bundles, smoothing of continuous bundle equivalences, non abelian Cech cohomology, twisted K-theory

MSC: 22E65, 55R10, 57R10

Introduction

This paper deals with the close interplay between continuous and smooth principal K-bundles over M, where K is a Lie group modeled on an arbitrary locally convex space (following [Mi84]) and M a finite-dimensional (paracompact) manifold with corners. The main point here is that there is no essential difference between the two concepts as long as one is only interested in equivalence classes of bundles (as one usually is).

We denote the set of equivalence classes of continuous K-principal bundles over M by $H^1_c(M,K)$ and the set of equivalence classes of smooth K-principal bundles over M by $H^1_s(M,K)$, which is only a nomenclature for now. Since each smooth principal bundle is in particular continuous, we have a canonic map $H^1_s(M,K) \to H^1_c(M,K)$. The question is whether this map is injective, surjective, or both.

One approach to this questions could be to introduce smooth structures on classifying spaces and to smooth classifying maps. As an example, the classifying space of $K = \mathrm{GL}_n(\mathbb{C})$ is isomorphic to the direct limit of the Grassmanians

$$B\,\mathrm{GL}_n(\mathbb{C}) \cong G_n(\infty) := \varinjlim G_n(k).$$

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Then [Gl05, Th. 3.1] provides a smooth manifold structure on $B\ \mathrm{GL}_n(\mathbb{C})$, and one can smooth classifying maps as in [Hi76, Th. 4.3.5] for the case of vector bundles. In the non-linear case, the classifying space of the diffeomorphism group $B\,\mathrm{Diff}(N)$ for a compact manifold N, which can be viewed as a nonlinear Grassmanian, can also be given a smooth structure [KM97, 44.21]. A general approach to this question has been taken in [Mo79], but with a focus on a de Rham cohomology, which has not been traced further. However, a general theory for differentiable structures on classifying spaces seems to be missing. On the other hand, there exist partial answers to the above question arising from the comparison of continuous and analytic fiber bundles (cf. [Gr58], [To67] and [Gu02]). Since these considerations use strong constraints on the structure group (e.g. its compactness), they can not be used in the great generality that we are aiming for.

The approach in this paper is to use an approximation result for Lie group-valued functions in order to smooth representatives of continuous bundles or bundle equivalences (cf. Proposition II.1 or [Wo05, Prop. III.8]). We use this in combination with the fact that there is a large freedom of choice in the description of principal bundles by locally trivial covers and transition functions. In this way, we construct new representatives of bundles and bundle equivalences that satisfy cocycle or compatibility conditions on probably finer locally trivial covers, but which describe equivalent objects. Since this technique uses heavily the local compactness of the base manifold, there seems to be no generalization of this method to infinite-dimensional base manifolds.

We now describe our results in some detail. In the first section we recall the basic facts of continuous and smooth principal bundles with a focus on the description of bundles and bundle equivalences in terms of locally trivial covers and transition functions. Furthermore, we recall briefly the concept of differential calculus and the concept of manifolds with corners that we work with in this text.

The second section is exclusively devoted to the proofs of our main results and to their technical prerequisites. As it is explained in the last section, the following theorems assert the surjectivity and injectivity, respectively, of the map $H^1_s(M,K) \to H^1_c(M,K)$.

Theorem (Smoothing Continuous Principal Bundles). *Let K be a Lie group modeled on a locally convex space, M be a connected manifold with corners and $\mathcal{P}$ be a continuous principal K-bundle over M. Then there exists a smooth principal K-bundle $\widetilde{\mathcal{P}}$ over M and a continuous bundle equivalence $\Omega : \mathcal{P} \to \widetilde{\mathcal{P}}$.*

Theorem (Smoothing Continuous Bundle Equivalences). *If $\mathcal{P}$ and $\mathcal{P}'$ are smooth principal K-bundles over the connected manifold with corners M and $\Omega : P \to P'$ is a continuous bundle equivalence, then there exists a smooth bundle equivalence $\widetilde{\Omega} : P \to P'$.*

In the third section we relate our results to some neighboring topics, in particular to non-abelian Čech cohomology and to twisted K-theory. This presentation is not meant to be exhaustive, but to give some ideas of the implications of our results.

I Principal Fiber Bundles

In this section we provide the basic material concerning manifolds with corners and smooth and continuous principal bundles.

Definition I.1 (Continuous Principal Bundle). Let K be a topological group and M be a topological space. Then a *continuous principal K-bundle over M* (or shortly a *continuous principal bundle*) is a topological space P together with a continuous action $P \times K \to P$, $(p,k) \mapsto p \cdot k$, and a map $\eta : P \to M$ such that there exists an open cover $(U_i)_{i \in I}$ of M, called a *locally trivial cover*, and homeomorphisms

$$\Omega_i : \eta^{-1}(U_i) \to U_i \times K,$$

called *local trivializations*, satisfying $\mathrm{pr}_1 \circ \Omega_i = \eta|_{\eta^{-1}(U_i)}$ and $\Omega(p \cdot k) = \Omega(p) \cdot k$. Here K acts on $U_i \times K$ by right multiplication in the second factor. We will use the calligraphic letter $\mathcal{P}$ for the tuple $(K, \eta : P \to M)$.

A *morphism of continuous bundles* or a *continuous bundle map* between two principal bundles $\mathcal{P}$ and $\mathcal{P}'$ over M is a continuous map $\Omega : P \to P'$ satisfying $\Omega(p \cdot k) = \Omega(p) \cdot k$. Since $P'/K \cong M \cong P/K$, it induces a map $\Omega^{\#} : M \to M$. We call Ω a *continuous bundle equivalence* if it is an isomorphism and $\Omega^{\#} = \mathrm{id}_M$.

Remark I.2 (Transition Functions). If $\mathcal{P}$ is a continuous principal K-bundle over M, then the local trivializations define continuous mappings $k_{ij} : U_i \cap U_j \to K$ by

$$\Omega_i^{-1}(x,e) \cdot k_{ij}(x) = \Omega_j^{-1}(x,e) \text{ for all } x \in U_i \cap U_j \tag{1}$$

called *transition functions*. The k_{ij} satisfy the *cocycle condition*

$$k_{ii}(x) = e \text{ for all } x \in U_i \quad \text{and} \quad k_{ij}(x) \cdot k_{jn}(x) \cdot k_{ni}(x) = e \text{ for all } x \in U_i \cap U_j \cap U_k. \tag{2}$$

On the other hand, if $(V_j)_{j \in J}$ is an open cover of a space N, and $k_{ij} : V_{ij} \to K$ are continuous that satisfy condition (2), then

$$P_{k_{ij}} = \bigcup_{j \in J} \{j\} \times U_j \times K/\sim \quad \text{with} \quad (j,x,k) \sim (j',x',k') \Leftrightarrow x = x' \text{ and } k_{j'j}(x) \cdot k = k'$$

defines a continuous principal K-bundle over M. Here η is given by $[i,x,k] \mapsto x$, the local trivializations by $[(i,x,k)] \mapsto (x,k)$ and the K-action by $([(i,x,k)],k') \mapsto [(i,x,kk')]$. We will call this bundle $\mathcal{P}_{k_{ij}}$.

If the k_{ij} arise from the local trivializations of a given bundle $\mathcal{P}$ as in (1), then

$$\Omega : P \to P_{k_{ij}}, \;\; p \mapsto [i, \Omega_i(p)] \text{ if } p \in \eta^{-1}(U_i)$$

defines a bundle equivalence between $\mathcal{P}$ and $\mathcal{P}_{k_{ij}}$ whose inverse is given by $[i,x,k] \mapsto \Omega_i^{-1}(x,k)$.

Definition I.3 (Differential Calculus on Locally Convex Spaces). Let E and F be a locally convex spaces and $U \subseteq E$ be open. Then $f : U \to F$ is called *continuously differentiable* or C^1 if it is continuous, for each $v \in E$ the differential quotient

$$df(x).v := \lim_{h \to 0} \frac{1}{h}(f(x+hv) - f(x))$$

exists and the map $df : U \times E \to F$ is continuous. For $n > 1$ we, recursively define

$$d^n f(x).(v_1, \ldots, v_n) := \lim_{h \to 0} \frac{1}{h} \left(d^{n-1} f(x+h).(v_1, \ldots, v_{n-1}) - d^{n-1} f(x).(v_1, \ldots, v_n) \right)$$

and say that f is C^n if $d^k f : U \times E^k \to F$ exists for all $k = 1, \ldots, n$ and is continuous. We say that f is C^∞ or *smooth* if it is C^n for all $n \in \mathbb{N}$.

Definition I.4 (Lie Group). From the definition above the notion of a *smooth Lie group* is clear. It is a group which is a smooth manifold modeled on a locally convex space such that the group operations are smooth.

Remark I.5 (Convenient Calculus). (cf. [Ne02, Rem. 3.2]) We briefly recall the basic definitions of the convenient calculus from [KM97]. Again, let E and F be locally convex spaces. A curve $f : \mathbb{R} \to E$ is called smooth if it is smooth in the sense of Definition I.3. Then the c^∞-topology on E is the final topology induced from all smooth curves $f \in C^\infty(\mathbb{R}, E)$. If E is a Fréchet space, then the c^∞-topology is again a locally convex vector topology which coincides with the original topology [KM97, Th. 4.11]. If $U \subseteq E$ is c^∞-open, then $f : U \to F$ is said to be of class C^∞ or smooth if

$$f_* \left(C^\infty(\mathbb{R}, U) \right) \subseteq C^\infty(\mathbb{R}, F),$$

i.e. if f maps smooth curves to smooth curves. The chain rule [Gl02, Prop. 1.15] implies that each smooth map in the sense of Definition I.3 is smooth in the convenient sense. On the other hand, [KM97, Th. 12.8] implies that on a Frèchet space a smooth map in the convenient sense is smooth in the sense of Definition I.3. Hence for Fréchet spaces, the two notions coincide.

Remark I.6 (Manifold with Corners). We refer to [Wo06] for an introduction to the concept of manifolds with corners, which may, in general, be modeled on a locally convex space. Roughly speaking, an n-dimensional manifold with corners (which we will assume to be paracompact throughout this paper) is a topological space such that each point has a neighborhood that is homeomorphic to an open subset of $[0,1]^n$ and that the corresponding coordinate changes are smooth. The crucial point here is the notion of smoothness for non-open domains. The usual notion is to define a map $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ to be smooth if for each $x \in A$, there exists a neighborhood U_x of x, which is open in $\mathbb{R}^n$, and a smooth map $f_x : U_x \to \mathbb{R}^m$ such that $f_x|_{A\cap U_x} = f|_{A\cap U_x}$. The notion in [Wo06] is slightly different, but is the appropriate one for a treatment of mapping spaces [Mi80]. However, it turns out in [Wo06] that it is equivalent to the usual notion.

Remark I.7 (Paracompact Spaces). We recall some basic facts from general topology. If X is a topological space, then a collection of subsets $(U_i)_{i\in I}$ of X is called *locally finite* if each $x \in X$ has a neighborhood that has non-empty intersection with only finitely many U_i, and X is called *paracompact* if each open cover has a locally finite refinement. If X is the union of countably many compact subsets, then it is called σ-compact, and if each open cover has a countable subcover, it is called Lindelöf.

Now let M be a manifold with corners, which is in particular locally compact and locally connected. For these spaces, [Du66, Th. XI.7.2+3] imply that M is paracompact if and only if each component is σ-compact, equivalently, Lindelöf. Furthermore, [Br93, Th. I.12.5] implies that M is normal in each of these cases.

Definition I.8 (Smooth Principal Bundle). If K is a smooth Lie group and M is a smooth manifold with corners (both modeled on locally convex vector spaces), then a continuous principal K-bundle over M is called a *smooth principal K-bundle over M* (or shortly a *smooth principal bundle*) if the transition functions from Remark I.2 are smooth for some choice of local trivializations.

Remark I.9 (Smooth Structure on Smooth Principal Bundles). If $\mathcal{P}$ is a smooth principal bundle, then we define a smooth structure on P by requiring the local trivializations

$$\Omega_i : \eta^{-1}(U_i) \to U_i \times K$$

that define the smooth transition functions from Definition I.8 to be diffeomorphisms. This actually defines a smooth structure on P since it is covered by $(\eta^{-1}(U_i))_{i\in I}$ and the coordinate changes

$$U_i \cap U_j \times K \to U_i \cap U_j \times K,\ \ (x,k) \mapsto \Omega_j(\Omega_i^{-1}(x,k)) = (x, k\cdot k_{ij}(x))$$

are smooth because the k_{ij} are assumed to be smooth. A continuous bundle map between smooth principal bundles is a *morphism of smooth principal bundles* or a *smooth bundle map* if it is smooth with respect to the the smooth structure on the bundles just described.

Remark I.10 (Bundle Equivalences). If $\mathcal{P}$ and $\mathcal{P}'$ are two principal K-bundles over M, then there exists an open cover $(U_i)_{i\in I}$ of M such that we have local trivializations

$$\Omega_i : \eta^{-1}(U_i) \to U_i \times K$$
$$\Omega_i' \colon \eta'^{-1}(U_i) \to U_i \times K$$

for $\mathcal{P}$ and $\mathcal{P}'$. In fact, if $(V_j)_{j\in J}$ and $(V'_{j'})_{j'\in J'}$ are locally trivial covers of M (for $\mathcal{P}$ and for $\mathcal{P}'$, respectively), then

$$(V_j \cap V_{j'})_{(j,j')\in J\times J'}$$

is simultaneously a locally trivial cover for both $\mathcal{P}$ and $\mathcal{P}'$, and the local trivializations are given by restricting the original ones.

If $\mathcal{P}_{k_{ij}}$ and $\mathcal{P}_{k'_{ij}}$ are given by transition functions k_{ij} and k'_{ij} with respect to the same open cover $(U_i)_{i\in I}$ (i.e., $k_{ij} : U_i \cap U_j \to K$ and $k'_{ij} : U_i \cap U_j \to K$), then a bundle equivalence $\Omega : P_{k_{ij}} \to P_{k'_{ij}}$ defines for each $i \in I$ a continuous map

$$\varphi_i : U_i \times K \to K \text{ by } \Omega([(i,x,k)]) = [(i,x,\varphi_i(x,k))]. \tag{3}$$

Furthermore, we have $\varphi_i(x,k) = \varphi_i(x,e)\cdot k$ since Ω is assumed to satisfy $\Omega(p\cdot k) = \Omega(p)\cdot k$. Setting $f_i(x) := \varphi_i(x,e)$, we thus obtain continuous maps $f_i : U_i \to K$ satisfying

$$k'_{ji}(x)\cdot f_i(x)\cdot k_{ji}(x) = f_j(x) \text{ for all } x \in U_i \cap U_j, \tag{4}$$

since $[(i,x,k)] = [(j,x,k_{ji}(x)k)]$ has to be mapped to the same element of $P_{k'_{ij}}$ by Ω. On the other hand, if for each $i \in I$ we have continuous maps $f_i : U_i \to K$ satisfying (4), then

$$P_{k_{ij}} \ni [(i,x,k)] \mapsto [(i,x,f_i(x)\cdot k)] \in P_{k'_{ij}}$$

defines a bundle equivalence between $\mathcal{P}_{k_{ij}}$ and $\mathcal{P}_{k'_{ij}}$ which covers the identity on M.

If $\mathcal{P}_{k_{ij}}$ and $\mathcal{P}_{k'_{ij}}$ are smooth and the maps k_{ij} and k'_{ij} are smooth, then it follows directly from (3) that a bundle equivalence described by continuous maps $f_i : U_i \to K$ is smooth if and only if these maps are smooth.

II Equivalences of Smooth and Continuous Bundles

In this section, we state and prove the two main results of this paper. The proofs use two important tools that we describe first: a proposition to smooth continuous maps and a lemma to fade out continuous functions.

Proposition II.1 (Smoothing). *Let M be a paracompact manifold with corners, K a Lie group modeled on a locally convex space and $f \in C(M,K)$. If $A \subseteq M$ is closed and $U \subseteq M$ is open such that f is smooth on a neighborhood of $A \setminus U$, then each open neighborhood O of f in $C(M,K)_{c.o.}$ contains a map g, which is smooth on a neighborhood of A and equals f on $M \setminus U$.*

Proof. This is [Wo05, Prop. III.8]. □

Remark II.2 (Centered Chart, Convex Subset). Let K be a Lie group modeled on a locally convex topological vector space E. A chart $\varphi : W \to \varphi(W) \subseteq E$ with $e \in W$ and $\varphi(e) = 0$ is called a *centered chart*. A subset L of W is called *φ-convex* (or just *convex*, if φ is obvious) if it is identified with a convex subset $\varphi(L)$ in E. If W itself is φ-convex, we speak of a *convex centered chart*.

It is clear that every open identity neighborhood in K contains a φ-convex open neighborhood for some centered chart φ, because we can pull back any convex open neighborhood that is small enough from the underlying locally convex vector space.

Lemma II.3 (Fading-Out). *Let M be a manifold with corners, A and B be closed subsets satisfying $B \subseteq A^0$, $\varphi : W \to \varphi(W)$ be a convex centered chart of a Lie group K modeled on a locally convex space, and $f : A \to W$ be a continuous function. Then there is a continuous function $F : M \to W \subseteq K$ with $F|_B = f$ and $F|_{M\setminus A^0} \equiv e$. Moreover, if $W' \subseteq W$ is another φ-convex set, then $f(x) \in W'$ always implies $F(x) \in W'$.*

Proof. Since M is paracompact, it is also normal (see Remark I.7). The closed sets $M \setminus A^0$ and B are disjoint by assumption, so the Urysohn's Lemma as in [Br93, Th. I.10.2] yields a continuous function $\lambda : M \to [0,1]$ such that $\lambda|_B \equiv 1$ and $\lambda|_{M\setminus A^0} \equiv 0$. Since $\varphi(W)$ is a convex zero neighborhood in E, we have $[0,1]\cdot\varphi(W) \subseteq \varphi(W)$. We use this to define the continuous function

$$f_\lambda : A \to W, \quad x \mapsto \varphi^{-1}\Big(\lambda(x)\cdot\varphi\big(f(x)\big)\Big),$$

that satisfies, by the choice of λ, $f_\lambda|_B = f|_B$ and $f_\lambda|_{\partial A} = e$ because $\partial A \subseteq M \setminus A^0$. So we may extend f_λ to the continuous function

$$F : M \to W, \quad x \mapsto \begin{cases} f_\lambda(x), & \text{if } x \in A \\ e, & \text{if } x \in M \setminus A^0 \end{cases}$$

that satisfies all requirements. □

Before we state and prove the main results, we need some technical data: suitable covers of a compact manifold with corners and identity neighborhoods in a Lie group that satisfy certain conditions.

Lemma II.4 (Squeezing-in Manifolds with Corners). *Let W be an open neighborhood of a point x in $\mathbb{R}^{d+} = \{(x_1,\dots,x_d) \in \mathbb{R}^d : x_i \geq 0$ for all $i = 1,\dots,n\}$ and $C \subseteq W$ be a compact set containing x. Then there exists an open set V satisfying $x \in C \subseteq V \subseteq \overline{V} \subseteq W$, whose closure $\overline{V}$ is a compact manifold with corners.*

Proof. For every $x = (x_1,\dots,x_d) \in C$, there is an $\varepsilon_x > 0$ such that

$$B(x,\varepsilon) := [x_1 - \varepsilon_x, x_1 + \varepsilon_x] \times \cdots \times [x_d - \varepsilon_x, x_d + \varepsilon_x] \cap \mathbb{R}^{d+} \tag{5}$$

is contained in W. The interiors $V_x := B(x,\varepsilon_x)^0$ in $\mathbb{R}^{d+}$ form an open cover of the compact set C, of which we may choose a finite subcollection $(V_{x_i})_{i=1,\dots,m}$ covering C. The union $V := \bigcup_{i=1}^m V_{x_i}$ satisfies all requirements. In particular, $\overline{V}$ is a compact manifold with corners, because it is a finite union of cubes. □

Proposition II.5 (Nested Covers). *Let M be a connected manifold with corners and $(U_j)_{j\in J}$ be an open cover of M. Then there exist countable open covers $\big(U_i^{[\infty]}\big)_{i\in\mathbb{N}}$ and $\big(U_i^{[0]}\big)_{i\in\mathbb{N}}$ of M such that $\overline{U}_i^{[\infty]} := \overline{U_i^{[\infty]}}$ and $\overline{U}_i^{[0]} := \overline{U_i^{[0]}}$ are compact manifolds with corners, $\overline{U}_i^{[\infty]} \subseteq U_i^{[0]}$ for all $i \in \mathbb{N}$, and such that even the cover $\big(\overline{U}_i^{[0]}\big)_{i\in\mathbb{N}}$ of M by compact sets is locally finite and subordinate to $(U_j)_{j\in J}$.*

In this situation, let L be any countable subset of the open interval $(0,\infty)$. Then for every $\lambda \in L$, there exists a countable, locally finite cover $\big(U_i^{[\lambda]}\big)_{i\in I}$ of M by open sets whose closures are compact manifolds with corners such that $\overline{U}_i^{[\lambda]} \subseteq U_i^{[\mu]}$ holds whenever $\infty \geq \lambda > \mu \geq 0$.

Proof. For every $x \in M$, we have $x \in U_{j(x)}$ for some $j(x) \in J$. Let (U_x, φ_x) be a chart of M around x such that $\overline{U_x} \subseteq U_{j(x)}$. We can even find an open neighborhood V_x of x whose closure $\overline{V_x}$ is compact and contained in U_x. Since M is paracompact, the open cover $\big(V_x\big)_{x\in M}$ has a locally finite subordinated cover $(V_i)_{i\in I}$, where $V_i \subseteq V_x$ and $\overline{V_i} \subseteq \overline{V_x} \subseteq U_x$ for suitable $x = x(i)$. Since M is also Lindelöf, we may assume that $I = \mathbb{N}$.

To find suitable covers $U_i^{[\infty]}$ and $U_i^{[0]}$, we are going to enlarge the sets V_i so carefully in two steps that the resulting covers remain locally finite. More precisely, $U_i^{[\infty]}$ and $U_i^{[0]}$ will be defined inductively so that even the family $(V_k^i)_{k\in\mathbb{N}}$ with

$$V_k^i := \begin{cases} \overline{U}_k^{[0]} & \text{for } k \leq i \\ V_k & \text{for } k > i \end{cases}$$

is still a locally finite cover of M for every $i \in \mathbb{N}$. We describe the construction for a fixed $i \geq 1$.

For every point $y \in \overline{V_i}$, there is an open neighborhood $V_{i,y}$ of y inside $U_{x(i)}$ whose intersection with just finitely many V_i^{i-1} is non-empty. Under the chart $\varphi_{x(i)}$, this neighborhood $V_{i,y}$ is mapped to an open neighborhood of $\varphi_{x(i)}(y)$ in the modeling space $\mathbb{R}^{d+}$ of M. There we can find real numbers $\varepsilon_0(y) > \varepsilon_\infty(y) > 0$ such that the cubes $B(y,\varepsilon_\infty(y))$ and $B(y,\varepsilon_0(y))$ introduced in (5) are compact neighborhoods of $\varphi_{x(i)}(y)$ contained in $\varphi_{x(i)}(V_{i,y})$. Since $\overline{V_i}$ is compact, it is already covered by finitely many sets $V_{i,y}$, say by $\big(V_{i,y}\big)_{y\in Y}$ for a finite subset Y of $\overline{V_i}$. We define the open sets

$$U_i^{[\infty]} := \bigcup_{y\in Y} \varphi_{x(i)}^{-1}\left(B(y,\varepsilon_\infty(y))^0\right) \text{ and } U_i^{[0]} := \bigcup_{y\in Y} \varphi_{x(i)}^{-1}\left(B(y,\varepsilon_0(y))^0\right),$$

whose closures are compact manifolds with corners, because they are a finite union of cubes under the chart $\varphi_{x(i)}$. On the one hand, the construction guarantees

$$V_i \subseteq U_i^{[\infty]} \subseteq \overline{U}_i^{[\infty]} \subseteq U_i^{[0]} \subseteq \overline{U}_i^{[0]} \subseteq \bigcup_{y\in Y} V_{i,y} \subseteq U_{x(i)}.$$

On the other hand, the cover $\big(V_k^i\big)_{k\in\mathbb{N}}$ is locally finite, because it differs from the locally finite cover $\big(V_k^{i-1}\big)_{k\in\mathbb{N}}$ in the single set $V_i^i = \overline{U}_i^{[0]}$, each point of which has a neighborhood $V_{i,y}$ intersecting just finitely many other sets of either cover.

For a proof of the second claim, let $\lambda_1, \lambda_2, \ldots$ be any enumeration of L. Then for any $n \geq 1$ and $i \in \mathbb{N}$, we apply Lemma II.4 to $C := \varphi_i\big(\overline{U}_i^{[\overline{\lambda}]}\big)$ and $W := \varphi_i\big(U_i^{[\underline{\lambda}]}\big)$, where $\overline{\lambda}$ (resp. $\underline{\lambda}$) is the smallest (resp. largest) element of $\lambda_1, \ldots, \lambda_{n-1}$ larger than (resp. smaller than) λ_n for $n > 1$ and ∞ (resp. 0) for $n = 1$. We get open sets $U_i^{[\lambda_n]}$ such that the condition $\overline{U}_i^{[\lambda]} \subseteq U_i^{[\mu]}$ holds whenever $\infty \geq \lambda > \mu \geq 0$ are elements in $\{\lambda_1, \ldots, \lambda_n\}$, and eventually in L. This completes the proof. □

Remark II.6 (Locally Finite Covers by Compact Sets). If $\big(\overline{U}_i\big)_{i\in I}$ is a locally finite cover of M by compact sets, then for fixed $i \in I$, the intersection $\overline{U}_i \cap \overline{U}_j$ is non-empty for only finitely many $j \in I$. Indeed, for every $x \in \overline{U}_i$, there is an open neighborhood U_x of x such that $I_x := \{j \in I : U_x \cap \overline{U}_j \neq \emptyset\}$ is finite. Since $\overline{U}_i$ is compact, it is covered by finitely many of these sets, say by $U_{x_1}, \ldots, U_{x_n}$. Then $J := I_{x_1} \cup \cdots \cup I_{x_n}$ is the finite set of indices $j \in J$ such that $\overline{U}_i \cap \overline{U}_j$ is non-empty, proving the claim.

Remark II.7 (Intersections). From now on, multiple lower indices on subsets always indicate intersection, i.e., $U_{1\cdots r} := U_1 \cap \ldots \cap U_r$.

Lemma II.8 (Suitable Identity Neighborhoods). *Let M be manifold with corners that is covered locally finitely by countably many compact sets $\big(\overline{U}_i\big)_{i\in\mathbb{N}}$. Moreover, let $k_{ij} : \overline{U}_{ij} \to K$ be continuous functions into a Lie group K modeled on a locally convex space so that $k_{ij} = k_{ji}^{-1}$ holds for all $i, j \in \mathbb{N}$. Then for any convex centered chart $\varphi : W \to \varphi(W)$, there are φ-convex open identity neighborhoods $W_{ij}^\alpha \subseteq W$ in K for indices $i < j$ and α in $\mathbb{N}$ that satisfy*

$$k_{ji}(x) \cdot (W_{ij}^\alpha)^{-1} \cdot W_{in}^\alpha \cdot k_{ij}(x) \subseteq W_{jn}^\alpha \text{ for all } x \in \overline{U}_{ijn\alpha},\ i < j < n \text{ and } \alpha \text{ in } \mathbb{N}, \tag{6}$$

$$k_{nj}(x) \cdot (W_{jn}^n)^{-1} \cdot k_{jn}(x) \subseteq W \text{ for all } x \in \overline{U}_{jn},\quad j < n \text{ in } \mathbb{N}. \tag{7}$$

Proof. Disregarding condition (6) initially, we set $W_{jn}^\alpha = W$ for $n \neq \alpha$ and choose $W_{jn}^n \subseteq W$ such that (7) is satisfied for all $j < n$ in $\mathbb{N}$. To do the latter, we observe that the function

$$\varphi_{jn} : \overline{U}_{jn} \times W \to K, \quad (x, k) \mapsto k_{nj}(x) \cdot k^{-1} \cdot k_{jn}(x)$$

is continuous and maps the point (x, e) to the identity e for every $x \in \overline{U}_{jn}$. So we may choose an open neighborhoods U_x of x and a convex (short for φ-convex) open identity neighborhood $W_x \subseteq W$ such that $\varphi_{jn}(U_x \times W_x) \subseteq W$. Since $\overline{U}_{jn}$ is compact, it is covered by finitely many U_x, say by $U_{x_1}, \ldots, U_{x_m}$. Then $W_{jn}^n := \bigcap_{i=1}^m W_{x_i}$ is a convex open identity neighborhood in W such that $\varphi_{jn}\big(\overline{U}_{jn} \times W\big) \subseteq W$, in other words, it satisfies (7).

To organize a step-by-step construction of the final open identity neighborhoods W_{jn}^α, which may become smaller than their initial values above, we define the following total order

$$(i, j) < (i', j') \quad :\Leftrightarrow \quad j < j' \text{ or } (j = j' \text{ and } i < i') \tag{8}$$

on pairs of real numbers, in particular on pairs of indices (i, j) in $\mathbb{N} \times \mathbb{N}$ with $i < j$. Note that this guarantees $(i, j) < (j, n)$ and $(i, n) < (j, n)$ whenever i, j, n are as in condition (6). We use this in the following construction that we perform for each fixed $\alpha \in \mathbb{N}$.

Since the cover $\big(\overline{U}_i\big)_{i\in\mathbb{N}}$ is locally finite, Remark II.6 says that there are only finitely many indices $j < n$ such that the intersection $\overline{U}_{jn\alpha}$ is non-empty. Starting at the largest such pair $j < n$ with respect to the total order (8), we adjust the identity neighborhoods W_{jn}^α one at a time in decreasing order down to and including W_{12}^α. For the index (j, n), we

- fixate the identity neighborhood W_{jn}^α (which already satisfies (7) if $n = \alpha$)

- and make sure that all conditions (6) with this W_{jn}^{α} on the right hand side are satisfied by making the corresponding sets W_{ij}^{α} and W_{in}^{α} on the left hand side smaller, if necessary. Here we use the same line of argument as above: Given such a condition, the compactness of $\overline{U}_{ijn\alpha}$ and the continuity of the function

$$\psi_{ijn\alpha} : \overline{U}_{ijn\alpha} \times W_{ij}^{\alpha} \times W_{in}^{\alpha} \to K, \quad (x, k, k') \mapsto k_{ji}(x) \cdot k^{-1} \cdot k' \cdot k_{ij}(x),$$

at the points (x, e, e) yield convex open identity neighborhoods inside W_{ij}^{α} and W_{in}^{α}, replacing these sets, so that condition (6) is satisfied. Note that we are still free to modify these sets with indices smaller than (j, n). Furthermore, making them smaller cannot violate any conditions that we guaranteed in previous steps of the construction, because they can only appear on the left hand side of such conditions.

This completes the proof. □

Theorem II.9 (Smoothing Continuous Principal Bundles). *Let K be a Lie group modeled on a locally convex space, M be a connected manifold with corners and $\mathcal{P}$ be a continuous principal K-bundle over M. Then there exists a smooth principal K-bundle $\widetilde{\mathcal{P}}$ over M and a continuous bundle equivalence $\Omega : \mathcal{P} \to \widetilde{\mathcal{P}}$.*

Proof. We may assume that the continuous bundle $\mathcal{P}$ is equivalent to a bundle $\mathcal{P}_{k_{ij}}$ as in Remark I.2, where $(U_j)_{j\in J}$ is a locally trivial cover of M and $k_{ij} : U_{ij} \to K$ are continuous transition functions that satisfy the cocycle condition $k_{ij} \cdot k_{jn} = k_{in}$ pointwise on U_{ijn}.

Proposition II.5 yields open covers $\big(U_i^{[\infty]}\big)_{i\in\mathbb{N}}$ and $\big(U_i^{[0]}\big)_{i\in\mathbb{N}}$ of M subordinate to $(U_j)_{j\in J}$. For every $i \in \mathbb{N}$, we denote by U_i an open set of the cover $(U_j)_{j\in J}$ that contains $U_i^{[0]}$ and observe that $(U_i)_{i\in\mathbb{N}}$ is still a locally trivial open cover of M. In our construction, we need open covers not only for pairs $(j, n) \in \mathbb{N}\times\mathbb{N}$ with $j < n$, but also for pairs $(j-1/3, n)$, $(j-2/3, n)$ in-between and (n, n) to enable continuous extensions and smoothing. The function

$$\lambda : \{(j, n) \in \{0, 1/3, 2/3, \dots\} \times \mathbb{N} : j \le n\} \to [0, \infty), \quad \lambda(j, n) = \frac{n(n-1)}{2} + j,$$

is tailored to map the pairs $(0, 1), (1, 1), (1, 2), (2, 2), (1, 3), (2, 3), (3, 3), (1, 4), \dots$ to the integers $0, 1, 2, \dots$, respectively, and the other pairs in-between. If we apply the second part of Proposition II.5 to the countable subset $L := (\operatorname{im}\lambda) \setminus \{0\}$ of $(0, \infty)$, we get open sets $U_i^{[jn]} := U_i^{[\lambda(j,n)]}$ for all pairs (j, n) in the domain of λ such that $\big(\overline{U}_i^{[jn]}\big)_{i\in\mathbb{N}}$ are again locally finite covers. We note that $(j, n) < (j', n')$ in the sense of (8) implies $\overline{U}_i^{[j'n']} \subseteq U_i^{[jn]}$.

Let $\varphi : W \to \varphi(W)$ be an arbitrary centered chart of K and consider the countable compact cover $\big(\overline{U}_i^{[0]}\big)_{i\in\mathbb{N}}$ of M and the restrictions $k_{ij}|_{\overline{U}_{ij}^{[0]}}$ of the continuous transition functions to the corresponding intersections. Then Lemma II.8 yields open φ-convex (convex, for short) identity neighborhoods W_{ij}^{α} for all $i < j$ and α in $\mathbb{N}$ that satisfy condition (6) stated there.

Our first goal is the construction of smooth maps $\widetilde{k}_{ij} : U_{ij} \to K$ that satisfy the cocycle condition on the open cover $\big(U_i^{[\infty]}\big)_{i\in\mathbb{N}}$ of M, which uniquely determines a smooth principal K-bundle $\widetilde{\mathcal{P}}$ by Remarks I.2 and I.9. These maps $\widetilde{k}_{ij}$ will be constructed step-by-step in increasing order with respect to (8), starting with the minimal index $(1, 2)$. At all times during the construction, the conditions

(a) $\widetilde{k}_{jn} = \widetilde{k}_{ji} \cdot \widetilde{k}_{in}$ pointwise on $\overline{U}_{ijn}^{[jn]}$ for $1 \le i < j < n \le N$ and

(b) $\big(\widetilde{k}_{jn} \cdot k_{nj}\big)\big(\overline{U}_{jn\alpha}^{[jn]}\big) \in W_{jn}^{\alpha}$ for all $j < n$ and α in $\mathbb{N}$,

will be satisfied whenever all $\widetilde{k}_{ij}$ involved have already been constructed. We are now going to construct the smooth maps $\widetilde{k}_{jn}$ for $1 \le j < n \le N$ (and implicitly $\widetilde{k}_{nj}$ as $\widetilde{k}_{nj}(x) := \widetilde{k}_{jn}(x)^{-1}$), assuming that this has already been done for pairs of indices smaller than (j, n).

- To satisfy all relevant cocycle conditions, we start with

$$\widetilde{k}'_{jn} : \bigcup_{i<j} \overline{U}_{ijn}^{[j-1,n]} \to K, \quad \widetilde{k}'_{jn}(x) := \widetilde{k}_{ji}(x) \cdot \widetilde{k}_{in}(x) \text{ for } x \in \overline{U}_{ijn}^{[j-1,n]}.$$

  This smooth function is well-defined, because the cocycle conditions (a) for lower indices assert that for any indices $i' < i < j$ and any point $x \in \overline{U}_{i'jn}^{[j-1,n]} \cap \overline{U}_{ijn}^{[j-1,n]}$, we have

$$\widetilde{k}_{ji'}(x) \cdot \widetilde{k}_{i'n}(x) = \widetilde{k}_{ji'}(x) \cdot \widetilde{k}_{i'i}(x) \cdot \widetilde{k}_{ii'}(x) \cdot \widetilde{k}_{in}(x) = \widetilde{k}_{ji}(x) \cdot \widetilde{k}_{in}(x),$$

  because $\overline{U}_{i'ijn}^{[j-1,n]}$ is contained in both $\overline{U}_{i'ij}^{[ij]}$ and $\overline{U}_{i'in}^{[in]}$.

- Next, we want to extend the smooth map $\widetilde{k}'_{jn}$ on $\bigcup_{i<j} \overline{U}_{ijn}^{[j-1,n]}$ to a continuous map k'_{jn} on U_{jn} without compromising the cocycle conditions too much. To do this, we consider the function $\varphi_{jn} := \widetilde{k}'_{jn} k_{nj} : \bigcup_{i<j} \overline{U}_{ijn}^{[j-1,n]} \to K$. For all $i < j$, $\alpha \in \mathbb{N}$ and $x \in \overline{U}_{ijn\alpha}^{[j-1,n]}$, conditions (b) above and (6) of Lemma II.8 imply

$$\begin{aligned} \varphi_{jn}(x) = (\widetilde{k}'_{jn} k_{nj})(x) &= k_{ji}(x) \cdot (\underbrace{(\widetilde{k}_{ij} \cdot k_{ji})(x)}_{\in W_{ij}^{\alpha}})^{-1} \cdot \underbrace{(\widetilde{k}_{in} \cdot k_{ni})(x)}_{\in W_{in}^{\alpha}} \cdot k_{ij}(x) \\ &\in k_{ji}(x) \cdot (W_{ij}^{\alpha})^{-1} \cdot W_{in}^{\alpha} \cdot k_{ij}(x) \subseteq W_{jn}^{\alpha}, \end{aligned}$$

  because $\overline{U}_{ijn\alpha}^{[j-1,n]}$ is contained in both $U_{ij\alpha}^{[ij]}$ and $U_{in\alpha}^{[in]}$. Since the values of φ_{jn} are contained in particular in the identity neighborhood W, we may apply Lemma II.3 to $M = U_{jn}$ and its subsets $A = \bigcup_{i<j} \overline{U}_{ijn}^{[j-1,n]}$ and $B = \bigcup_{i<j} \overline{U}_{ijn}^{[j-2/3,n]}$. It yields a continuous function $\Phi_{jn} : U_{jn} \to W$ that coincides with φ_{jn} on B, is the identity outside A, and satisfies $\Phi_{jn}(x) \in W_{jn}^{\alpha}$ for all $x \in \overline{U}_{jn\alpha}^{[j-1,n]}$. We define $k'_{jn} : U_{jn} \to K$ by $k'_{jn} := \Phi_{jn} k_{jn}$ and note that k'_{jn} coincides with the smooth function $\widetilde{k}'_{jn}$ on B and with k_{jn} outside A.

- We finally get the smooth map $\widetilde{k}_{jn} : U_{jn} \to K$ that we are looking for if we apply Proposition II.1 to the function k'_{jn} on $M = A = U_{jn}$, to the open complement U of $\bigcup_{i<j} \overline{U}_{ijn}^{[j-1/3,n]}$ in M, and to the open neighborhood

$$O_{jn} = \left(\bigcap_{\alpha \in \mathbb{N}} \left\lfloor \overline{U}_{jn\alpha}^{[jn]}, W_{jn}^{\alpha} \right\rfloor \right) \cdot k_{jn}$$

  of both k_{jn} and k'_{jn}, where $k'_{jn} \in O_{jn}$ follows from $\Phi_{jn}(x) \in W_{jn}^{\alpha}$ and $k'_{jn}(x) = \Phi_{jn}(x) \cdot k_{jn}(x) \in W_{jn} \cdot k_{jn}(x)$. Note that O_{jn} is really open, because Remark II.6 asserts that just finitely many of the sets $\overline{U}_{jn\alpha}^{[jn]}$ for $\alpha \in \mathbb{N}$ are non-empty and may influence the intersection.

  By the choice of U, the result $\widetilde{k}_{jn}$ coincides with both k'_{jn} and $\widetilde{k}'_{jn}$ on $\bigcup_{i<j} \overline{U}_{ijn}^{[jn]}$, so it satisfies the cocycle conditions (a). It also satisfies (b) by the choice of O_{jn}.

This concludes the construction of the smooth principal K-bundle $\widetilde{\mathcal{P}}$. We use the same covers of M and identity neighborhoods in K for the construction of continuous functions $f_i : \overline{U}_i^{[0]} \to K$ such that

(c) $f_n = \widetilde{k}_{nj} \cdot f_j \cdot k_{jn}$ pointwise on $\overline{U}_{jn}^{[nn]}$ for $j < n$ in $\mathbb{N}$,

(d) $f_n\big(\overline{U}_n^{[0]}\big) \subseteq W$ for $n \in \mathbb{N}$, and

(e) $f_n \equiv e$ outside $\bigcup_{j<n} \overline{U}_{jn}^{[jn]}$ for $n \in \mathbb{N}$,

and Remark I.10 tells us that the restriction of the maps f_i to the sets $U_i^{[\infty]}$ of the open cover is the local description of a bundle equivalence $\Omega : \mathcal{P} \to \widetilde{\mathcal{P}}$ of the bundles $\mathcal{P} \cong \mathcal{P}_{k_{ij}|U_{ij}^{[\infty]}}$ and $\widetilde{\mathcal{P}} \cong \mathcal{P}_{\widetilde{k}_{ij}|U_{ij}^{[\infty]}}$ that we are looking for. Indeed, all the sets $\overline{U}_{jn}^{[nn]}$ of condition (c) contain the corresponding sets $U_{jn}^{[\infty]}$ of the open cover.

We start with the constant function $f_1 \equiv e$, which clearly satisfies conditions (d) and (e). The construction of f_n for $n > 1$ is as follows:

- To satisfy condition (c), we start with

$$f_n' : \bigcup_{j<n} \overline{U}_{jn}^{[jn]} \to K, \quad f_n'(x) = \widetilde{k}_{nj}(x) \cdot f_j(x) \cdot k_{jn}(x) \text{ for } x \in \overline{U}_{jn}^{[jn]}.$$

  This continuous function is well-defined, because the conditions (c) for f_j and (a) for $j' < j < n$ guarantee that

$$\widetilde{k}_{nj}(x) \cdot f_j(x) \cdot k_{jn}(x) = \widetilde{k}_{nj}(x) \cdot \widetilde{k}_{jj'}(x) \cdot f_{j'}(x) \cdot k_{j'j}(x) \cdot k_{jn}(x) = \widetilde{k}_{nj'}(x) \cdot f_{j'}(x) \cdot k_{j'n}(x)$$

  holds for all $x \in \overline{U}_{j'jn}^{[jn]} = \overline{U}_{j'n}^{[jn]} \cap \overline{U}_{jn}^{[jn]}$.

- To apply Lemma II.3, we need to know something about the values of f_n'. Let x be an arbitrary point in $\bigcup_{j<n} \overline{U}_{jn}^{[jn]}$, and let $j < n$ be the smallest index such that $x \in \overline{U}_{jn}^{[jn]}$. Using condition (e) for f_j, we learn that $f_j(x) = e$. Then we get the estimate

$$\begin{aligned} f_n'(x) &= \widetilde{k}_{nj}(x) \cdot f_j(x) \cdot k_{jn}(x) = \widetilde{k}_{nj}(x) \cdot k_{jn}(x) \\ &= k_{nj}(x) \cdot \big(\widetilde{k}_{jn}(x) \cdot \widetilde{k}_{nj}(x)\big)^{-1} \cdot k_{jn}(x) \in k_{nj}(x) \cdot (W_{jn}^n)^{-1} \cdot k_{jn}(x) \subseteq W, \end{aligned}$$

  so that the values of f_n are, altogether, contained in the identity neighborhood W of K. If we apply Lemma II.3 to $M := \overline{U}_n^{[0]}$, to f_n' on $A = \bigcup_{j<n} \overline{U}_{jn}^{[jn]}$ and to the smaller set $B = \bigcup_{j<n} \overline{U}_{jn}^{[nn]}$, then we get a continuous function $f_n : \overline{U}_n^{[0]} \to W$ that coincides with f_n' on B and is e outside A. Accordingly, f_n satisfies all the conditions (c) to (e).

This concludes the construction of the bundle equivalence. □

Lemma II.10 (More Identity Neighborhoods). *If $\mathcal{P}_{k'_{ij}}$ is a continuous principal K-bundle over M with locally trivial cover $(U_i)_{i\in I}$, then there exist open unit neighborhoods W_j^α for $j, \alpha \in \mathbb{N}$ such that*

(9) $$k'_{ji}(x) \cdot W_i^\alpha \cdot k'_{ij}(x) \subseteq W_j^\alpha \quad \text{for all } i < j \text{ and } x \in \overline{U}_{ij\alpha}.$$

Proof. For any fixed $\alpha \in \mathbb{N}$, let $j < \infty$ be maximal with $\overline{U}_{j\alpha} \neq \emptyset$ (cf. Remark II.6) and set $W_j^\alpha = W$. With the same continuity argument as in Lemma II.8, we get W_i^α for any $i < j$ that satisfy (9), but which we do not fixate yet. In the next step, let $j' < j$ be maximal such that $\overline{U}_{j'\alpha} \neq \emptyset$. We now fixate the $W_{j'}^\alpha$ just constructed and compromise the remaining W_i^α for $i < j'$ by continuity as before. Proceeding in this way, we end up after finitely many steps with the construction of all W_i^α. □

Theorem II.11 (Smoothing Continuous Bundle Equivalences). *If $\mathcal{P}$ and $\mathcal{P}'$ are smooth principal K-bundles over the connected manifold with corners M and $\Omega : P \to P'$ is a continuous bundle equivalence, then there exists a smooth bundle equivalence $\widetilde{\Omega} : P \to P'$.*

Proof. Let $(V_l)_{l\in L}$ be a locally trivial locally finite open cover of M. First, we use Proposition II.5 to obtain for each $i, j \in \mathbb{N}$ open sets $U_i^{[\infty]}$, $U_i^{[j]}$, $U_i^{[j+1/3]}$, $U_i^{[j+2/3]}$, and $U_i^{[0]}$ such that their closures are compact manifolds with corners, that $\overline{U}_i^{[0]} \subseteq V_l$ for some $l \in L$, that $(U_i^{[\infty]})_{i\in\mathbb{N}}$ and $(\overline{U}_i^{[0]})_{i\in\mathbb{N}}$ are locally finite covers of M, and such that

(10) $$\overline{U}_i^{[\infty]} \subseteq U_i^{[j+1]} \subseteq \overline{U}_i^{[j+1]} \subseteq U_i^{[j+2/3]} \subseteq \overline{U}_i^{[j+2/3]} \subseteq U_i^{[j+1/3]} \subseteq \overline{U}_i^{[j+1/3]} \subseteq U_i^{[j]} \subseteq U_i^{[0]}$$

holds for all $i,j \in \mathbb{N}$.

Since the $U_i^{[\infty]}$ also cover M, we have a countable subcover $(V_i)_{i\in\mathbb{N}}$ of the locally finite cover $(V_l)_{l\in L}$ with $V_i := V_{l(i)}$. This defines smooth transition functions $k_{ij} : V_{ij} \to K$, $\kappa_{ij} : U_{ij}^{[\infty]} \to K$ and smooth bundle equivalences $\mathcal{P}_{k_{ij}} \cong \mathcal{P} \cong \mathcal{P}_{\kappa_{ij}}$. Performing the same construction for $\mathcal{P}'$ we get smooth transition functions $k'_{ij} : V_{ij} \to K$, $\kappa'_{ij} : U_{ij}^{[\infty]} \to K$ and smooth bundle equivalences $\mathcal{P}_{k'_{ij}} \cong \mathcal{P} \cong \mathcal{P}_{\kappa'_{ij}}$. According to Remark I.10, the bundle equivalence Ω defines continuous maps $f_i : V_{ij} \to K$ that satisfy

$$f_i(x) = k'_{ij}(x) \cdot f_j(x) \cdot k_{ji}(x) \quad \text{for all} \quad x \in V_{ij}. \tag{11}$$

We shall inductively construct smooth maps $\widetilde{f}_i : \overline{U}_i^{[0]} \to K$ such that

$$\widetilde{f}_i(x) \cdot f_i(x)^{-1} \in W_i^{\alpha} \quad \text{for all} \quad i < \alpha \text{ and } x \in \overline{U}_{i\alpha}^{[i]}, \tag{12}$$

$$\widetilde{f}_j(x) = k'_{ji}(x) \cdot \widetilde{f}_i(x) \cdot k_{ij}(x) \quad \text{for all} \quad i < j \quad \text{and } x \in \overline{U}_{ij}^{[j]}. \tag{13}$$

This will finish the proof, since $k_{ij}\big|_{U_{ij}^{[\infty]}} = \kappa_{ij}$, $k'_{ij}\big|_{U_{ij}^{[\infty]}} = \kappa'_{ij}$ and according to Remark I.10, the restriction of $\widetilde{f}_i$ to $U_i^{[\infty]}$ defines a smooth bundle equivalence $\widetilde{\Omega} : \mathcal{P}_{\kappa_{ij}} \to \mathcal{P}_{\kappa'_{ij}}$.

To construct $\widetilde{f}_1$, first note that

$$O_1 := \bigcap_{\alpha\in\mathbb{N}} \left\lfloor \overline{U}_{1\alpha}^{[0]}, W_1^{\alpha} \right\rfloor \cdot f_1 \tag{14}$$

is an open neighborhood of f_1, since only finitely many $\overline{U}_{1\alpha}^{[0]}$ are non-empty. We now obtain $\widetilde{f}_1$ if we apply Proposition II.1 to $f = f_1$, $M = A = U = \overline{U}_1^{[0]}$, and the open neighborhood O_1 from (14).

To construct $\widetilde{f}_j$ we set $\widetilde{f}'_j(x) := k'_{ji}(x) \cdot \widetilde{f}_i(x) \cdot k_{ij}(x)$ for $i < j$ and $x \in \overline{U}_{ij}^{[j-1]}$. This defines a continuous map on $\bigcup_{i<j} \overline{U}_{ij}^{[j-1]}$, since on each $\overline{U}_{ij}^{[j-1]}$ the compatibility condition (13) is satisfied by induction. Furthermore, we have

$$\varphi_j(x) := \widetilde{f}'_j(x) \cdot f_j(x)^{-1} = k'_{ji}(x) \cdot \widetilde{f}_i(x) \cdot k_{ij}(x) \cdot f_j(x)^{-1} = k'_{ji}(x) \cdot \underbrace{\widetilde{f}_i(x) \cdot f_i(x)^{-1}}_{\in W_i^{\alpha}} \cdot k'_{ij}(x) \in W_j^{\alpha}$$

for $x \in \overline{U}_{ij\alpha}^{[j-1]}$, $i < j < \alpha$ due to (9), (11) and (12). Since $W_j^{\alpha} \subseteq W$ is convex, we may apply Proposition II.3 to $A = \bigcup_{i<j} \overline{U}_{ij}^{[j-1]}$ and $B = \bigcup_{i<j} \overline{U}_{ij}^{[j-2/3]}$ to extend φ_j to a continuous map Φ_j on $\overline{U}_j^{[0]}$. Then Φ_j coincides with φ_i on B and maps $\overline{U}_{j\alpha}^{[j]}$ into W_j^{α} if $j < \alpha$. Accordingly, $\Phi_j \cdot f_j$ is an element of the open neighborhood

$$O_2 := \bigcap_{\alpha>j} \left\lfloor \overline{U}_{j\alpha}^{[j]}, W_j^{\alpha} \right\rfloor \cdot f_j \tag{15}$$

of f_j and is smooth on $\bigcup_{i<j} U_{ij}^{[j-2/3]}$. We then obtain $\widetilde{f}_j$ by applying Proposition II.1 to the map $f = \Phi_j \cdot f_j$, $M = A = \overline{U}_j^{[0]}$, $U = M \setminus \bigcup_{i<j} \overline{U}_{ij}^{[j-1/3]}$, and to the open neighborhood from (15). □

III Related Topics

In this section, we explain the relations of the results of the preceding section to non-abelian Čech cohomology and to twisted K-theory. While the first one is simply a reformulation of the previous setting in terms of sheaf theory, the latter shows how applications of the previous results may arise.

Remark III.1 (Abelian Čech Cohomology). Let M be a paracompact topological space with an open cover $\mathcal{U} = (U_i)_{i\in I}$ and A be an abelian topological group. Then, for $n \geq 0$ an n-cochain f is a collection of continuous functions $f_{i_1\ldots i_{n+1}} : U_{i_1\ldots i_{n+1}} \to A$, and we denote the set of n-cocycles by $C^n(\mathcal{U}, A)$ and set it to 0 if $n < 0$. We then define the boundary operator

$$\delta_n : C^n(\mathcal{U}, A) \to C^{n+1}(\mathcal{U}, A) \ \ \delta(f)_{i_0 i_1\ldots i_{n+1}} = \sum_{k=0}^{n} (-1)^k f_{i_0\ldots \widehat{i_k}\ldots i_{n+1}},$$

where $\widehat{i_k}$ means that we omit the index i_n. Then $\delta_{n+1} \circ \delta_n = 0$, and we define

$$H^n_c(\mathcal{U}, A) := \ker(\delta_n)/\operatorname{im}(\delta_{n-1}) \quad \text{and} \quad H^n_c(M, A) := \lim_{\rightarrow} H^n_c(\mathcal{U}, A), \tag{16}$$

where the order on covers is induced by being refinements of one another. The group $H^1(M, A)$ is the *n-th continuous Čech cohomology.* If, in addition, M is a smooth manifold with or without corners and A is a smooth Lie group, then the same construction with smooth instead of continuous functions leads to the corresponding *n-th smooth Čech cohomology.*

Remark III.2 (Non-Abelian Čech Cohomology). (cf. [De53, Sect. 12] and [GM99, 3.2.3]) If $n = 0, 1$, then we can perform a similar construction as in the previous remark in the case of a not necessarily commutative group K. The definition of an n-cochain is the same as in the commutative case, but we run into problems when writing down the boundary operator δ, since the computation of $\delta_{n+1} \circ \delta_n = 0$ uses heavily the commutativity of A. Even for $n = 0$, where an almost trivial computation shows that $\delta_1 \circ \delta_0$, vanishes, $\operatorname{im}(\delta_0)$ is *not* a normal subgroup of $\ker(\delta_1)$, whence we may not adopt the definition of $H^1_c(M, K)$ as in (16). However, we may define $\delta_0(f)_{ij} = f_i \cdot f_j^{-1}$, $\delta_1(k)_{ijl} = k_{ij} \cdot k_{jl} \cdot k_{li}$ and call the elements of $\ker(\delta_1)$ 2-cocycles (or cocycles for short).

The way to circumvent difficulties for $n = 1$ is the observation that even in the non-abelian case, $C^1_s(\mathcal{U}, K)$ acts on $\ker(\delta_1)$ by $(f_i, k_{ij}) \mapsto f_i \cdot k_{ij} \cdot f_j^{-1}$. Thus we define two cocycles k_{ij} and k'_{ij} to be equivalent if $k'_{ij} = f_i \cdot k_{ij} \cdot f_j^{-1}$ on U_{ij} for some $f_i \in C^1(\mathcal{U}, K)$ and by $H^1_c(\mathcal{U}, K)$, the equivalence classes (or the orbit space) of this action. Then $H^1_c(\mathcal{U}, K)$ is not a group, but we may nevertheless take the direct limit

$$H^1_c(M, K) := \lim_{\rightarrow} H^1_c(\mathcal{U}, K)$$

as sets and define it to be the 1^{st} (non-abelian) continuous Čech cohomology of M with coefficients in K. A representing space of $H^1_c(M, K)$ would then be the set of equivalence classes of continuous principal K-bundles over M.

Again, if M is a smooth manifold with corners and K is a smooth Lie, we can adopt this construction to define the 1^{st} (non-abelian) smooth Čech cohomology $H^1_s(M, K)$.

Theorem III.3 (Isomorphism for Non-Abelian Čech Cohomology). *If M is a finite-dimensional connected manifold with corners and K is a smooth Lie group modeled on a locally convex space, then the canonical injection*

$$\iota : H^1_s(M, K) \to H^1_c(M, K)$$

is a bijection.

Proof. We identify smooth and continuous principal bundles with Čech 1-cocycles and smooth and continuous bundle equivalences with Čech 0-cochains as in Remark I.10. For each open cover $\mathcal{U}$ of M, we have the canonical map $H^1_s(\mathcal{U}, K) \to H^1_c(\mathcal{U}, K)$. Now each cocycle $k_{ij} : U_{ij} \to K$ defines a principal bundle $\mathcal{P}$ with locally trivial covering $\mathcal{U}$. We may assume by Theorem II.9 that $\mathcal{P}$ is continuously equivalent to a smooth principal bundle $\widetilde{\mathcal{P}}$, and thus that $\mathcal{U}$ is also a locally trivial covering for $\widetilde{\mathcal{P}}$. This shows that the map is surjective and the injectivity follows from Theorem II.11 in the same way. Accordingly, the map induced on the direct limit is a bijection. □

Remark III.4 (The Projective Unitary Group). Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space and denote by $\mathrm{U}(\mathcal{H})$ the group of unitary operators. If we equip $\mathrm{U}(\mathcal{H})$ with the norm topology, then the exponential series, restricted to skew-self-adjoint operators $L(\mathrm{U}(\mathcal{H}))$, defines an exponential function and turns $\mathrm{U}(\mathcal{H})$ into an infinite-dimensional Banach–Lie group (cf. [Mi84, Ex. 1.1]). Then $\mathrm{U}(1)$ is a normal subgroup of $\mathrm{U}(\mathcal{H})$ and it can also be shown that $\mathrm{PU}(\mathcal{H}) := \mathrm{U}(\mathcal{H})/\mathrm{U}(1)$ is also a Lie group modeled on $L(\mathrm{U}(\mathcal{H}))/i\mathbb{R}$.

Remark III.5 (Eilenberg–MacLane Spaces). If X is a topological space with non-trivial n-th homotopy group $\pi_n(X)$ for all but one $n \in \mathbb{N}$, then it is called an *Eilenberg–MacLane space* $K(n, \pi_n(X))$. Since $\mathrm{U}(1)$ is a $K(1, \mathbb{Z})$, the long exact homotopy sequence [Br93, Th. VII.6.7] shows that $\mathrm{PU}(\mathcal{H})$ is a $K(2, \mathbb{Z})$, since $\mathrm{U}(\mathcal{H})$ is contractible [Ku65, Th. 3]. By the same argument, the classifying space $B\,\mathrm{PU}(\mathcal{H})$ [Hu94, Ch. 4.11] is a $K(3, \mathbb{Z})$ since its total space $P\,\mathrm{PU}(\mathcal{H})$ is contractible. Thus $[M, B\,\mathrm{PU}(\mathcal{H})] \cong H^3(M, \mathbb{Z})$ by [Br93, Cor. VII.13.16] classifies the equivalence classes of principal $\mathrm{PU}(\mathcal{H})$-bundles over M. The representing class $[\mathcal{P}]$ in $H^3(M, \mathbb{Z})$ is called the *Dixmier-Douady* class of $\mathcal{P}$ (cf. [CCM98], [DD63]). It describes the restriction of $\mathcal{P}$ to be the projectivization of an (automatically trivial) principal $\mathrm{U}(\mathcal{H})$-bundle.

Remark III.6 (Twisted K-theory). (cf. [Ro89, Sect. 2], [BCM+02]) The Dixmier-Douady class of a bundle $\mathcal{P}$ induces a twisting of K-theory in the following manner. For any paracompact space M, the K-theory $K^0(M)$ is defined to be the ring completion of the equivalence classes of finite-dimensional complex vector bundles over X, where addition and multiplication is defined by taking direct sums and tensor products of vector bundles [Hu94]. Furthermore, the space of Fredholm operators $\mathrm{Fred}(\mathcal{H})$ is a representing space for K-theory, i.e. $K^0(M) \cong [M, \mathrm{Fred}(\mathcal{H})]$, where $[\cdot, \cdot]$ denotes homotopy classes of continuous maps. Since $\mathrm{PU}(\mathcal{H})$ acts (continuously) on $\mathrm{Fred}(\mathcal{H})$ by conjugation, we can form the associated vector bundle $\mathcal{P}_{\mathrm{Fred}(\mathcal{H})} := \mathrm{Fred}(\mathcal{H}) \times_{\mathrm{PU}(\mathcal{H})} \mathcal{P}$. Then the homotopy classes of sections $[M, P_{\mathrm{Fred}(\mathcal{H})}]$ (or equivalently the equivariant homotopy classes of equivariant maps $[P_{\mathrm{Fred}(\mathcal{H})}, \mathrm{Fred}(\mathcal{H})]^{\mathrm{PU}(\mathcal{H})}$) define the *twisted K-theory* $K_{\mathcal{P}}(M)$. Since Theorem II.9 implies that we may assume $\mathcal{P}$ to be smooth and the action of $\mathrm{PU}(\mathcal{H})$ on $\mathrm{Fred}(\mathcal{H})$ is smooth, being given locally in terms of a continuous linear map, $\mathcal{P}_{\mathrm{Ad}}$ is also smooth. We may thus, in the computation of $K_{\mathcal{P}}(M)$, restrict our attention to smooth sections and smooth homotopies if M has a sufficiently nice triangularization.

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