

# Aspects of $p$ -Adic Non-Linear Functional Analysis

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## Abstract

The article provides an introduction to infinite-dimensional differential calculus over topological fields and surveys some of its applications, notably in the areas of infinite-dimensional Lie groups and dynamical systems.

## Introduction

We describe aspects of non-linear functional analysis over topological fields, considered as the study of non-linear mappings between topological vector spaces, their fixed points and differentiability properties. Our first aim is to give an introduction to the differential calculus of smooth and  $C^k$ -maps between topological vector spaces over topological fields developed in [6]. This approach generalizes Schikhof's single variable calculus over complete ultrametric fields (as in [52]) to multi- and infinite-dimensional situations. In the case of mappings between real locally convex spaces, it is equivalent to the usual locally convex calculus (Keller's  $C_c^k$ -theory, as in [39] or [47]). Our second aim is to survey various recent applications of differential calculus over topological fields, and the specific techniques underlying them. In particular, we discuss the following topics:

- The existence of fixed points and their  $C^k$ -dependence on parameters (as in [28]);
- An implicit function theorem for  $C^k$ -maps from arbitrary topological vector spaces over valued fields to Banach spaces (established in [25] and [28]);
- The construction of the main types of infinite-dimensional Lie groups (linear Lie groups, mapping groups, diffeomorphism groups, direct limit groups) over topological fields (carried out in [27], also [21] and [24]);
- The construction of invariant manifolds around hyperbolic fixed points for dynamical systems modelled on Banach spaces over valued fields (as in [29] and [30]), by an adaptation of Irwin's method (developed in [36] and [37], also [13] and [59]);
- Some special tools of calculus used in the preceding constructions (results ensuring differentiability properties of non-linear mappings between function spaces [27] and spaces of sequences [29]; exponential laws for function spaces [27]).

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While analytic mappings between Banach spaces over complete valued fields and the corresponding analytic Banach-Lie groups are classical objects of study (see [9], [11]; also [53] for the finite-dimensional case), we are interested just as well in mappings between general topological vector spaces, which have hardly been investigated so far. This is essential for infinite-dimensional Lie theory, since many interesting groups cannot be modelled on Banach spaces. It is also essential to work with smooth (and  $C^k$ -) maps, rather than analytic ones. In fact, already in the real case, typical examples of infinite-dimensional Lie groups (like diffeomorphism groups) are smooth Lie groups, but cannot be given an analytic Lie group structure. Even worse, over each local field of positive characteristic, finite-dimensional smooth Lie groups exist which do not admit an analytic Lie group structure compatible with their topological group structure [23]. For this reason, we shall use smooth maps as the basis of Lie theory.<sup>1</sup>

We mention that many important techniques usually applied in finite-dimensional non-archimedean analysis do not possess infinite-dimensional counterparts: Neither the techniques of algebraic geometry, which help to analyze the most prominent examples of finite-dimensional  $p$ -adic Lie groups (linear algebraic groups); nor the techniques of rigid analytic geometry (see [8], [17]), the non-archimedean analogue of complex geometry and function theory. However, the ideas of *real* differential calculus turn out to be extremely robust: surprisingly large parts of ordinary differential calculus carry over to mappings between open subsets of topological vector spaces over topological fields, once they are reformulated in an appropriate way. Schikhof's textbook [52] bears witness of this phenomenon in the case of functions of a single variable (see also [1], [3], [14], [15], [38], [41], [42] and [57], part of which include functions of several variables). The present article is based on the infinite-dimensional extension of Schikhof's ultrametric calculus developed in [6] (cf. [45] and [46] for a different, earlier approach to  $p$ -adic infinite-dimensional differential calculus, which will be discussed in Remark 1.9 (e) here). We hope to illustrate that the techniques of real differential calculus remain powerful also when dealing with mappings between topological vector spaces over topological fields.

Our illustrating examples are taken from Lie theory and dynamical systems. In the theory of non-archimedean dynamical systems, a wealth of results is available for one-dimensional polynomial or analytic systems, inspired by classical complex dynamics (see, e.g., [2], [4], [40], [44]). Motivated by Lie-theoretic applications, we here describe complementary results dealing with higher-dimensional (and infinite-dimensional) dynamical systems, which have hardly been investigated before. Also the study of infinite-dimensional Lie groups over topological fields is of recent origin. It started with discussions of irreducible representations and invariant measures for certain infinite-dimensional  $p$ -adic groups (see [45], [46] and the references therein). In the meantime, all major constructions of real infinite-dimensional Lie groups have been adapted to more general topological fields ([27], [24]).

Outside the realms of non-linear functional analysis, Lie theory and dynamical systems

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<sup>1</sup>Another problem is that a convincing notion of analytic map between (non-normable) locally convex spaces over ultrametric fields does not seem to exist yet, in contrast to the real and complex cases.

discussed in this article, the differential calculus over topological fields (and suitable commutative topological rings) has been used to give an essentially algebraic approach to differential geometry, which is entirely based on differentiation and does not involve integrals, nor flows [5]. Jordan theoretic applications have been explored in [7].

## 1 Differential calculus over topological fields

The basic idea of differential calculus over topological fields is to call a map  $C^1$  if one can pass from directional difference quotients to directional derivatives continuously. This idea, and its implications, will be described in more detail now. Throughout this section,  $\mathbb{K}$  denotes a topological field (i.e., a field, equipped with a non-discrete Hausdorff topology which turns the field operations into continuous mappings). Topological  $\mathbb{K}$ -vector spaces are defined as in the real case, and are assumed Hausdorff. As the default,  $E$ ,  $F$  and  $E_1, E_2, \dots$  denote topological  $\mathbb{K}$ -vector spaces, and  $U \subseteq E$  an open subset. To define continuous differentiability, let  $E$  and  $F$  be topological  $\mathbb{K}$ -vector spaces, and  $f: U \rightarrow F$  be a map on an open subset  $U \subseteq E$ . Then the directional difference quotient

$$f^{[1]}(x, y, t) := \frac{f(x + ty) - f(x)}{t}$$

makes sense for all  $(x, y, t)$  in the subset  $U^{[1]} := \{(x, y, t) \in U \times E \times \mathbb{K}^\times : x + ty \in U\}$  of  $E \times E \times \mathbb{K}$ . To incorporate directional derivatives, we must enlarge this set by allowing also the value  $t = 0$ . Hence, we consider now

$$U^{[1]} := \{(x, y, t) \in U \times E \times \mathbb{K} : x + ty \in U\}.$$

Thus  $U^{[1]} = U^{[1]} \cup (U \times E \times \{0\})$ , as a disjoint union.

**Definition 1.1**  $f: U \rightarrow F$  is called *continuously differentiable* (or  $C^1$ ) if  $f$  is continuous and there exists a continuous map  $f^{[1]}: U^{[1]} \rightarrow F$  which extends  $f^{[1]}: U^{[1]} \rightarrow F$ .

Thus, we assume the existence of a continuous map  $f^{[1]}: U^{[1]} \rightarrow F$  such that

$$f^{[1]}(x, y, t) = \frac{f(x + ty) - f(x)}{t} \quad \text{for all } (x, y, t) \in U^{[1]} \text{ such that } t \neq 0.$$

Given a  $C^1$ -map  $f: U \rightarrow F$  as before, its directional derivative

$$df(x, y) := D_y f(x) := \lim_{0 \neq t \rightarrow 0} \frac{1}{t} (f(x + ty) - f(x)) = \lim_{0 \neq t \rightarrow 0} f^{[1]}(x, y, t) = f^{[1]}(x, y, 0)$$

at  $x \in U$  in the direction  $y \in E$  exists, by continuity of  $f^{[1]}$ . The map  $df: U \times E \rightarrow F$  is continuous, being a partial map of  $f^{[1]}$ , and it can be shown that the ‘‘differential’’  $f'(x) := df(x, \cdot): E \rightarrow F$  of  $f$  at  $x \in U$  is a continuous  $\mathbb{K}$ -linear map [6, Proposition 2.2]. Since  $f^{[1]}(x, y, 0)$  is a limit of difference quotients,  $f^{[1]}$  is uniquely determined by  $f$ .

**Example 1.2** To prove that a given map is  $C^1$ , usually one first writes down directional difference quotients and then tries to guess a continuous extension. To illustrate this strategy, let us show that every continuous linear map  $\lambda: E \rightarrow F$  between topological  $\mathbb{K}$ -vector spaces is continuously differentiable. For  $x, y \in E$  and  $t \in \mathbb{K}^\times$ , we have

$$\frac{\lambda(x + ty) - \lambda(x)}{t} = \lambda(y), \quad (1)$$

exploiting the linearity of  $\lambda$ . We now note that the right hand side of (1) makes sense just as well for  $t = 0$ , and defines a continuous function

$$\lambda^{[1]}: E \times E \times \mathbb{K} \rightarrow F, \quad \lambda^{[1]}(x, y, t) := \lambda(y). \quad (2)$$

Thus  $\lambda$  is  $C^1$ , with  $\lambda^{[1]}$  as just described (which is again a continuous linear map). In particular,  $\lambda'(x) = \lambda$  for each  $x \in E$ .

The same idea can be used to prove the Chain Rule for  $C^1$ -maps.

**Proposition 1.3** *If  $f$  and  $g$  are composable  $C^1$ -maps, then also  $f \circ g$  is  $C^1$ , and*

$$(f \circ g)^{[1]}(x, y, t) = f^{[1]}(g(x), g^{[1]}(x, y, t), t).$$

*In particular,  $d(f \circ g)(x, y) = df(g(x), dg(x, y))$ .*

**Proof.** For  $t \neq 0$ , we have

$$\frac{f(g(x + ty)) - f(g(x))}{t} = \frac{f(g(x) + t \frac{g(x+ty) - g(x)}{t}) - f(g(x))}{t} = f^{[1]}(g(x), g^{[1]}(x, y, t), t).$$

Since the final expression makes sense just as well for  $t = 0$  and defines a continuous function, we see that  $f \circ g$  is  $C^1$ , with  $(f \circ g)^{[1]}$  as asserted.  $\square$

Since  $U^{[1]}$  is an open subset of the topological  $\mathbb{K}$ -vector space  $E \times E \times \mathbb{K}$ , it is possible to define  $C^k$ -maps by recursion.

**Definition 1.4** Given an integer  $k \geq 2$ , a map  $f: E \supseteq U \rightarrow F$  is called  $C^k$  if it is  $C^1$  and  $f^{[1]}: U^{[1]} \rightarrow F$  is  $C^{k-1}$ . As usual,  $f$  is called  $C^\infty$  (or *smooth*) if it is  $C^k$  for each  $k \in \mathbb{N}$ .

The present recursive definition of  $C^k$ -maps is particularly well-adapted to inductive arguments. In [45] and [46], a different approach to higher differentiability was used.

**Example 1.5** We have seen above that  $\lambda^{[1]}$  is again continuous linear if  $\lambda: E \rightarrow F$  is a continuous linear map. Hence a straightforward induction shows that  $\lambda$  is  $C^k$  for each  $k \in \mathbb{N}$  and thus smooth. A similar argument shows that every continuous  $n$ -linear map  $\beta: E_1 \times \cdots \times E_n \rightarrow F$  is smooth (use [6, §3.3] and induction).

**Remark 1.6**  $C^k$ -maps have many of the properties familiar from the real case.

- (a) Compositions of composable  $C^k$ -maps are  $C^k$  (see [6, Proposition 4.5]).
- (b) Higher differentials can be defined and have the usual properties: If  $f: E \subseteq U \rightarrow F$  is  $C^k$ , then the iterated directional derivative

$$d^j f(x, y_1, \dots, y_j) := (D_{y_j} \cdots D_{y_1} f)(x)$$

exists for all  $j \in \mathbb{N}$  such that  $j \leq k$  and all  $x \in U, y_1, \dots, y_j \in E$ . The mapping  $d^j f: U \times E^j \rightarrow F$  so defined is continuous, and  $d^j f(x, \bullet): E^j \rightarrow F$  is a continuous, symmetric,  $j$ -linear map, for each  $x \in U$  (see [6, Chapter 4]).

- (c) Finite order Taylor expansions are available in the following form: If  $f: E \supseteq U \rightarrow F$  is  $C^k$ , with  $k$  finite, then there are functions  $a_j: U \times E \rightarrow F$  for  $j \in \{0, 1, \dots, k\}$  and a continuous map  $R_k: U^{[1]} \rightarrow F$  with  $R_k(x, y, 0) = 0$  for all  $(x, y) \in U \times E$ , such that

$$f(x + ty) = \sum_{j=0}^k t^j a_j(x, y) + t^k R_k(x, y, t) \quad \text{for all } (x, y, t) \in U^{[1]}.$$

The functions  $a_0, \dots, a_k$  and  $R_k$  are uniquely determined. Furthermore,  $a_j$  is  $C^{k-j}$ , homogeneous of degree  $j$  in the second argument, and  $j! a_j(x, y) = d^j f(x, y, \dots, y)$ . If  $\mathbb{K}$  has characteristic 0, we can divide by  $j!$  and recover Taylor's formula in its familiar form with  $a_j(x, y) = \frac{1}{j!} d^j f(x, y, \dots, y)$  (see [6, Chapter 5] for details).

- (d) Despite its appearance, being  $C^k$  is a local property: If  $(U_i)_{i \in I}$  is an open cover of  $U$  and  $f|_{U_i}$  is  $C^k$  for each  $i \in I$ , then  $f: U \rightarrow F$  is  $C^k$  (see [6, Lemma 4.9]).

The preceding facts make it possible to define  $C^k$ -manifolds in the usual way.

**Definition 1.7** A  $C^k$ -manifold modelled on a topological  $\mathbb{K}$ -vector space  $E$  is a Hausdorff topological space  $M$ , equipped with a set  $\mathcal{A}$  of homeomorphisms  $\phi: U \rightarrow V$  from open subsets of  $M$  onto open subsets of  $E$ , such that the domains  $U$  cover  $M$  and the transition maps  $\phi \circ \psi^{-1}$  are  $C^k$  on their domain, for all  $\phi, \psi \in \mathcal{A}$ .

Now  $C^k$ -maps between  $C^k$ -manifolds can be defined as usual (such that the  $C^k$ -property can be tested in charts). Also the direct product of two manifolds can be defined as usual.

**Definition 1.8** A Lie group over a topological field  $\mathbb{K}$  is a group  $G$ , equipped with a smooth manifold structure modelled on a topological  $\mathbb{K}$ -vector space which turns the group inversion  $G \rightarrow G$  and the group multiplication  $G \times G \rightarrow G$  into smooth mappings.

$C^k$ -Lie groups for finite  $k$  are defined analogously.

**Remark 1.9** The  $C^k$ -maps considered here are related to traditional concepts as follows.

- (a) First of all, a map  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is  $C^k$  in our sense if and only if it is  $C^k$  in the usual sense. In fact, if  $f$  is an ordinary  $C^1$ -map, then

$$f^{[1]}(x, y, t) := \int_0^1 df(x + ty, y) dt$$

defines a map  $f^{[1]}: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^m$  which is continuous as a parameter-dependent integral, and produces the desired directional difference quotient for  $t \neq 0$ , by the Mean Value Theorem. Conversely, existence of  $f^{[1]}$  easily entails that  $f$  is  $C^1$  in the usual sense. For higher  $k$ , one argues by a simple induction.

- (b) More generally, a map  $f: E \supseteq U \rightarrow F$  to a real locally convex space  $F$  is  $C^k$  in our sense if and only if it is a “Keller  $C_c^k$ -map,” i.e.  $f$  is continuous, the iterated directional derivatives  $d^j f(x, y_1, \dots, y_j)$  (as in Remark 1.6 (b)) exist for all  $j \in \mathbb{N}$  such that  $j \leq k$ , and define continuous maps  $d^j f: U \times E^j \rightarrow F$  (see [6, Proposition 7.4]).
- (c) Every  $k$  times continuously Fréchet differentiable map ( $FC^k$ -map) between real Banach spaces is  $C^k$ , and conversely every  $C^{k+1}$ -map is  $FC^k$  (cf. (b) and [39]).
- (d) A map  $E \supseteq U \rightarrow F$  to a complex locally convex space is  $C^\infty$  if and only if it is complex analytic, i.e.,  $f$  is continuous and given locally around each point by a pointwise convergent series of continuous homogeneous polynomials [6, Proposition 7.7].
- (e) Finally, a map  $\mathbb{K} \supseteq U \rightarrow \mathbb{K}$  on an open subset of a complete ultrametric field  $\mathbb{K}$  is  $C^k$  in our sense if and only if it is a  $C^k$ -map in the sense of [52, Definition 29.1], as usually considered in non-archimedean analysis (see [6, Proposition 6.9]). More generally, our  $C^k$ -maps of several variables coincide with the usual ones, as in [52, §84] and [14] (see [32]). By contrast, being  $C(k)$  for  $k \geq 2$  in the sense considered in [45, §2.3] and [46] is a properly weaker property in general than being  $C^k$  in our sense, already for functions of one variable (which therefore do not coincide with Schikhof’s): see [32]. Examples show that such  $C(k)$ -maps need not admit  $k$ -th order Taylor expansions.

Of course, we are mainly interested in differential calculus over a valued field  $(\mathbb{K}, |\cdot|)$ ; thus  $\mathbb{K}$  is a field and  $|\cdot|: \mathbb{K} \rightarrow [0, \infty[$  an “absolute value” with properties analogous to the modulus of real or complex numbers. We shall always assume that  $|\cdot|$  is non-trivial in the sense that the metric  $d(x, y) := |x - y|$  defines a non-discrete (field) topology on  $\mathbb{K}$ . A valued field  $(\mathbb{K}, |\cdot|)$  is called an *ultrametric field* if its absolute value  $|\cdot|$  satisfies the ultrametric inequality, viz.  $|x + y| \leq \max\{|x|, |y|\}$  for all  $x, y \in \mathbb{K}$ . Besides  $\mathbb{R}$  and  $\mathbb{C}$ , the most prominent examples of valued fields are local fields (i.e., totally disconnected, locally compact topological fields). Any such is known to admit an ultrametric absolute value defining its topology. Up to isomorphism, every local field either is a finite extension of the field  $\mathbb{Q}_p$  of  $p$ -adic numbers, or a field  $\mathbb{F}((X))$  of formal Laurent series over a finite field  $\mathbb{F}$  (see [58]). If  $(\mathbb{K}, |\cdot|)$  is a valued field, then we can speak of norms and seminorms on  $\mathbb{K}$ -vector spaces, as in the real and complex cases. A normed space  $(E, \|\cdot\|)$  over  $\mathbb{K}$  which is complete in the metric determined by  $\|\cdot\|$  is called a *Banach space*. If  $(\mathbb{K}, |\cdot|)$  is

an ultrametric field here and  $\|\cdot\|$  satisfies the ultrametric inequality, then  $(E, \|\cdot\|)$  is called an *ultrametric* Banach space. See also [10], [48] and [51].

**Remark 1.10** We mention that a certain range of strengthened differentiability properties is available for mappings between topological vector spaces over a valued field  $(\mathbb{K}, |\cdot|)$ . In this case, also  $k$  times strictly differentiable mappings ( $SC^k$ -maps) and  $k$  times Lipschitz differentiable mappings ( $LC^k$ -maps) can be defined [28, §2 and §3]. Then

$$C^{k+1} \Rightarrow LC^k \Rightarrow SC^k \Rightarrow C^k$$

(see [28, Remark 3.17]). The strengthened differentiability properties are useful for some refined results. Strict differentiability at a point for mappings on normed spaces is a classical concept (see [9, 1.2.2]). Mappings between real Banach spaces are strictly differentiable if and only if they are once continuously Fréchet differentiable [9, 2.3.3]. If  $\mathbb{K}$  is locally compact and  $E$  a finite-dimensional  $\mathbb{K}$ -vector space, then a map  $f: U \rightarrow F$  on an open subset  $U \subseteq E$  is  $C^k$  if and only if it is  $SC^k$  (cf. [28, Lemma 3.11]).

## 2 Fixed points, inverse and implicit functions

In this section, we present an implicit function theorem for  $C^k$ -maps from arbitrary topological vector spaces over valued fields to Banach spaces. We also describe some of the tools used in its proof, notably a theorem ensuring the  $C^k$ -dependence of fixed points on parameters, which may be of independent interest. The results are taken from [28] (for slightly less general versions, see [25]). Similar implicit function theorems in the real and complex cases, in different settings of analysis, were also obtained in [34], [35] (for Keller's  $C_{\text{II}}^k$ -theory) and [54] (in the Convenient Setting of Analysis, [18], [43]).

While classical implicit function theorems are restricted to mappings between Banach spaces, the following result (a special case of [28, Theorem 5.2]) only requires that the range space be a Banach space (for the case  $k = 1$ , see Remark 2.7).

**Theorem 2.1 (Generalized Implicit Function Theorem)** *Let  $\mathbb{K}$  be a valued field,  $E$  be a topological  $\mathbb{K}$ -vector space,  $F$  be a Banach space over  $\mathbb{K}$ , and  $f: U \times V \rightarrow F$  be a  $C^k$ -map, where  $U \subseteq E$  and  $V \subseteq F$  are open subsets and  $k \in \mathbb{N} \cup \{\infty\}$ ,  $k \geq 2$ . Assume that  $f(x_0, y_0) = 0$  for some  $(x_0, y_0) \in U \times V$  and  $f'_{x_0}(y_0) \in \text{GL}(F) = \mathcal{L}(F)^\times$ , where  $f_x := f(x, \bullet): V \rightarrow F$  for  $x \in U$ . Then there exist open neighbourhoods  $U_0 \subseteq U$  of  $x_0$  and  $V_0 \subseteq V$  of  $y_0$  such that  $\{(x, y) \in U_0 \times V_0: f(x, y) = 0\}$  is the graph of a  $C^k$ -map  $\lambda: U_0 \rightarrow V_0$ .*

**Strategy of the proof.** The idea is to deduce the implicit function theorem from an “Inverse Function Theorem with Parameters.” Since  $f'_{x_0}(y_0) \in \text{GL}(F)$ , where  $\text{GL}(F)$  is open in  $\mathcal{L}(F)$  and the mapping  $U \rightarrow \mathcal{L}(F)$ ,  $x \mapsto f'_x(y_0)$  is continuous, we see that  $f'_x(y_0) \in \text{GL}(F)$  for  $x$  close to  $x_0$ . Then  $f_x^{-1}$  exists locally around  $f_x(y_0)$ , by a suitable

Inverse Function Theorem (e.g., [9, 1.5.1] or Proposition 2.2 below). Now the essential point is that the map

$$\psi : (x, z) \mapsto (f_x)^{-1}(z) \quad (3)$$

actually makes sense on a whole neighbourhood  $U_0 \times W$  of  $(x_0, 0)$  in  $U \times F$ , and is  $C^k$  there (further explanations will be given below). Once we have this, the rest is easy: The map  $\lambda: U_0 \rightarrow F$ ,  $\lambda(x) := (f_x)^{-1}(0) = \psi(x, 0)$  is  $C^k$ , and  $f(x, \lambda(x)) = 0$ .  $\square$

To see that the map  $\psi$  in (3) is defined on a whole neighbourhood, one exploits that suitable versions of the inverse function theorem provide *quantitative* information on the size of the domain of the local inverse. For example, one can use the following Lipschitz version of the inverse function theorem, [28, Theorem 5.3] (adapted from the real case in [59]). If  $E$  is a Banach space over  $\mathbb{K}$  and  $f: E \supseteq X \rightarrow E$  a map, we set  $\text{Lip}(f) := \sup\{\|f(y) - f(x)\| \cdot \|y - x\|^{-1} : x \neq y \in X\}$  and call  $f$  *Lipschitz* if  $\text{Lip}(f) < \infty$ .

**Proposition 2.2 (Lipschitz Inverse Function Theorem)** *Let  $(E, \|\cdot\|)$  be a Banach space over a valued field  $(\mathbb{K}, |\cdot|)$ . Let  $x \in E$ ,  $r > 0$  and  $f: B_r(x) \rightarrow E$  be a map on the ball  $B_r(x) := \{y \in E : \|y - x\| < r\}$  of the form*

$$f = A + \tilde{f},$$

where  $A \in \text{GL}(E)$  and  $\tilde{f}: B_r(x) \rightarrow E$  is a Lipschitz map with  $\text{Lip}(\tilde{f}) < \|A^{-1}\|^{-1}$ . Then  $f$  has open image and is a homeomorphism onto its image. Furthermore, the inverse map  $f^{-1}: f(B_r(x)) \rightarrow B_r(x)$  is Lipschitz with  $\text{Lip}(f^{-1}) \leq (\|A^{-1}\|^{-1} - \text{Lip}(\tilde{f}))^{-1}$ , and

$$B_{ar}(f(x)) \subseteq f(B_r(x)) \subseteq B_{br}(f(x))$$

with  $a := \|A^{-1}\|^{-1} - \text{Lip}(\tilde{f})$  and  $b := \|A\| + \text{Lip}(\tilde{f})$ .

The proof of Proposition 2.2 is based on a simple Newton-type iteration. Accordingly,  $\psi(x, z)$  in (3) is the fixed point of a contraction  $g_{x,z}$  of a (subset of a) Banach space. The following general result ensures that the fixed point  $\psi(x, z)$  of  $g_{x,z}$  is a  $C^k$ -function of  $(x, z)$ . It applies to so-called “uniform” families of contractions.

**Definition 2.3** Let  $F$  be a Banach space over a valued field  $\mathbb{K}$ , and  $U \subseteq F$  be a subset. A family  $(f_p)_{p \in P}$  of mappings  $f_p: U \rightarrow F$  is called a *uniform family of contractions* if there exists  $\theta \in [0, 1[$  such that  $\|f_p(x) - f_p(y)\| \leq \theta\|x - y\|$  for all  $x, y \in U$  and  $p \in P$ .

We now formulate the technical backbone of the generalized implicit function theorem (a special case of [28, Theorem 4.7]). It was stimulated by the discussion of fixed points in real Banach spaces and their dependence on parameters in *Banach spaces* in [37].

**Theorem 2.4 (on the Parameter-Dependence of Fixed Points)** *Let  $\mathbb{K}$  be a valued field,  $E$  be a topological  $\mathbb{K}$ -vector space,  $F$  be a Banach space over  $\mathbb{K}$  and  $f: P \times U \rightarrow F$  be a  $C^k$ -map, where  $P \subseteq E$  and  $U \subseteq F$  are open and  $k \in \mathbb{N}_0 \cup \{\infty\}$ . Assume that the maps  $f_p := f(p, \bullet): U \rightarrow F$  define a uniform family of contractions  $(f_p)_{p \in P}$ . Then*

$$Q := \{p \in P : f_p \text{ has a fixed point } x_p\}$$

is open in  $P$ , and  $\phi: Q \rightarrow F$ ,  $p \mapsto x_p$  is a  $C^k$ -map.



Exploiting the  $C^k$ -dependence of fixed points on parameters, one also obtains the following analogue of the classical Inverse Function Theorem (see [28, Theorem 5.1]):

**Theorem 2.5 (Inverse Function Theorem)** *Let  $E$  be a Banach space over a valued field  $\mathbb{K}$  and  $f: U \rightarrow E$  be a  $C^k$ -map on an open subset  $U \subseteq E$ , where  $k \in \mathbb{N} \cup \{\infty\}$  and  $k \geq 2$ . If  $f'(x) \in \text{GL}(E)$  for some  $x \in U$ , then there exists an open neighbourhood  $V \subseteq U$  of  $x$  such that  $f(V)$  is open in  $E$  and  $f|_V: V \rightarrow f(V)$  is a  $C^k$ -diffeomorphism.*

In the ultrametric case, closer inspection of the mapping  $\psi$  in (3) leads to the following useful result (which is a special case of [28, Theorem 5.17]).

**Theorem 2.6 (Ultrametric Inverse Function Theorem with Parameters)** *Let  $F$  be an ultrametric Banach space over an ultrametric field  $(\mathbb{K}, |\cdot|)$  and  $E$  be a topological  $\mathbb{K}$ -vector space. Let  $P_0 \subseteq E$  and  $U \subseteq F$  be open, and  $f: P_0 \times U \rightarrow F$  be a  $C^k$ -map, where  $k \in \mathbb{N} \cup \{\infty\}$  and  $k \geq 2$ . Let  $(p_0, x_0) \in P_0 \times U$  be given such that  $A := f'_{p_0}(x_0) \in \text{GL}(F)$ , where  $f_p := f(p, \cdot): U \rightarrow F$  for  $p \in P_0$ . Then there exists an open neighbourhood  $P \subseteq P_0$  of  $p_0$  and  $r > 0$  such that  $B := B_r(x_0) \subseteq U$  and the following holds:*

- (a)  $f_p(B) = f(p_0, x_0) + A \cdot B_r(0) =: V$ , for each  $p \in P$ , and  $\phi_p: B \rightarrow V$ ,  $\phi_p(y) := f(p, y)$  is a  $C^k$ -diffeomorphism;
- (b)  $f_p(B_s(y)) = f_p(y) + A \cdot B_s(0)$  for all  $p \in P$ ,  $y \in B$  and  $s \in ]0, r[$ ;
- (c) The map  $\psi: P \times V \rightarrow B$ ,  $\psi(p, v) := \phi_p^{-1}(v)$  is  $C^k$ ;
- (d)  $\xi: P \times B \rightarrow P \times V$ ,  $\xi(p, y) := (p, f(p, y))$  is a  $C^k$ -diffeomorphism, with inverse given by  $\xi^{-1}(p, v) = (p, \psi(p, v))$ .

A similar result is available for non-ultrametric Banach spaces [28, Theorem 5.13].

**Remark 2.7** Theorems 2.1 and 2.6 remain valid for  $k = 1$  if  $F$  is finite-dimensional, and Theorem 2.5 remains valid for finite-dimensional  $E$  (see [28]). In the infinite-dimensional case,  $C^1$ -maps need not be approximated good enough by their linearization around a given point to guarantee the hypotheses of Proposition 2.2. This problem disappears for  $SC^k$ -maps or  $LC^k$ -maps. All of the results presented so far in this section have analogues for  $SC^k$ -maps and  $LC^k$ -maps, valid for all  $k \in \mathbb{N} \cup \{\infty\}$  (see [28]).

**Applications.** We mention some applications of the preceding results in non-archimedean analysis. For applications in the real and complex cases, see [25] and [33].

- As described in more detail in Section 3, the implicit function theorem can be used to construct stable manifolds around hyperbolic fixed points.
- In [27], the ultrametric inverse function theorem with parameters in a Fréchet space is used to prove that the inversion map  $\text{Diff}(M) \rightarrow \text{Diff}(M)$ ,  $\gamma \mapsto \gamma^{-1}$  of the diffeomorphism group of a paracompact, finite-dimensional smooth manifold over a local field is smooth (see also Section 4 below).

- Varying an idea from [12],  $C^{k+n}$ -solutions to (systems of)  $p$ -adic differential equations of the form  $y^{(k)} = f(x, y, y', \dots, y^{(k-1)})$  for  $f$  an  $LC^n$ -map and  $k \in \mathbb{N}$ ,  $n \in \mathbb{N}_0$  can be constructed using our inverse function theorems (with and without parameters), which depend on initial conditions and parameters in a controlled way (work in progress; cf. [52, § 65] for  $C^1$ -solutions to scalar-valued first order equations). This is a first step towards the study of equations which are not accessible by the existing highly-developed techniques for linear or analytic equations (cf. [16], [55]).
- The ultrametric inverse function theorem with parameters is also used in the construction of a  $C^k$ -compatible analytic Lie group structure on each finite-dimensional  $p$ -adic  $C^k$ -Lie group in [26] (for  $k \in \mathbb{N} \cup \{\infty\}$ ).

### 3 Invariant manifolds around fixed points

We briefly discuss invariant manifolds over valued fields and an application.

**Definition 3.1** Let  $E$  be a Banach space over a valued field  $(\mathbb{K}, |\cdot|)$ , and  $a \in ]0, 1]$ . A (bicontinuous) linear automorphism  $\alpha \in \text{GL}(E)$  is called  $a$ -hyperbolic if  $E = E_1 \oplus E_2$  for certain  $\alpha$ -invariant closed vector subspaces  $E_1, E_2$ ;

$$\|x_1 + x_2\| = \max\{\|x_1\|, \|x_2\|\} \quad \text{for all } x_1 \in E_1 \text{ and } x_2 \in E_2$$

holds for a norm  $\|\cdot\|$  on  $E$  equivalent to the original one; and  $\alpha = \alpha_1 \oplus \alpha_2$  with  $\|\alpha_1\| < a$  and  $\|\alpha_2^{-1}\|^{-1} > a$ . The 1-hyperbolic automorphisms are simply called *hyperbolic*.

**Remark 3.2** If  $E$  is an ultrametric Banach space, then the norm  $\|\cdot\|$  described in the definition can be chosen ultrametric as well.

**Remark 3.3** If  $\mathbb{K}$  is a local field and  $\dim_{\mathbb{K}}(E) < \infty$ , then  $\alpha$  is  $a$ -hyperbolic if and only if  $a \neq |\lambda|$  for each eigenvalue  $\lambda \in \overline{\mathbb{K}}$  of  $\alpha \otimes_{\mathbb{K}} \text{id}_{\overline{\mathbb{K}}} \in \text{GL}(E \otimes_{\mathbb{K}} \overline{\mathbb{K}})$ , where  $\overline{\mathbb{K}}$  is an algebraic closure of  $\mathbb{K}$  ([29]; cf. also [19, proof of Lemma 3.3]).

We begin with the global Stable Manifold Theorem (see [29]).

**Theorem 3.4** Let  $M$  be a  $C^\infty$ -manifold modelled on a Banach space over a valued field,  $f: M \rightarrow M$  be a  $C^\infty$ -diffeomorphism, and  $p \in M$  be a hyperbolic fixed point of  $f$ , i.e.  $f(p) = p$  and the differential  $T_p f: T_p M \rightarrow T_p M$  is a hyperbolic automorphism of the tangent space  $T_p M$ . Then

$$W_s := \{x \in M : f^n(x) \rightarrow p \text{ as } n \rightarrow \infty\}$$

is an immersed  $C^\infty$ -submanifold of  $M$ .

**Remark 3.5** Analogous conclusions are available for manifolds and diffeomorphisms which are  $C^k$  (with  $k \geq 2$ ), resp.,  $SC^k$  and  $LC^k$  (with  $k \in \mathbb{N}$ ). The theorem also holds for  $\mathbb{K}$ -analytic diffeomorphism of  $\mathbb{K}$ -analytic manifolds, in the sense of [9] (see [29]).

Using standard arguments, the global stable manifold theorem follows from corresponding local results. Therefore, we now briefly discuss the construction of local  $a$ -stable manifolds, for  $a \in ]0, 1]$ . For simplicity, we only consider smooth maps, and concentrate on the case where  $E$  is an ultrametric Banach space over an ultrametric field  $(\mathbb{K}, |\cdot|)$ .

Let  $\alpha \in \text{GL}(E)$  be  $a$ -hyperbolic, say  $E = E_1 \oplus E_2$  with  $E_1$  and  $E_2$  as in Definition 3.1. Let  $r > 0$  and  $f: U \rightarrow E$  be a smooth map on the ball  $U := B_r^E(0) = U_1 \times U_2$ , where  $U_j := B_r^{E_j}(0)$ , such that  $f(0) = 0$  and  $f'(0) = \alpha$ . The  $a$ -stable set of  $f$  is defined as

$$W_{s,a} := \{z \in U : f^n(z) \text{ is defined for all } n \in \mathbb{N}_0, a^{-n} \|f^n(z)\| < r \text{ and } f^n(z) = o(a^n)\}.$$

Then  $f(W_{s,a}) \subseteq W_{s,a}$ . If  $z \in W_{s,a}$ , then the orbit  $\omega := (f^n(z))_{n \in \mathbb{N}_0}$  is an element of the Banach sequence space

$$\mathcal{S}_a(E) := \{z = (z_n)_{n \in \mathbb{N}_0} \in E^{\mathbb{N}_0} : z_n = o(a^n)\}$$

with norm  $\|z\|_a := \max\{a^{-n} \|z_n\| : n \in \mathbb{N}_0\}$ . After shrinking  $r$ , we may assume that  $\text{Lip}(f - \alpha) < \min\{1, \|\alpha_2^{-1}\|^{-1}\}$ . Then the following holds:

**Theorem 3.6**  *$W_{s,a}$  is the graph of a  $C^\infty$ -map  $\phi: U_1 \rightarrow U_2$  with  $\phi(0) = 0$  and  $\phi'(0) = 0$ . Thus  $W_{s,a}$  is a  $C^\infty$ -submanifold of  $E$ , which is tangent to the  $a$ -stable subspace  $E_1$  at 0.*

The proof given in [29] follows Irwin's method (see [36] and [59]). The idea is to construct not  $\phi(x)$  itself, but the orbit  $\omega$  of  $(x, \phi(x))$ , which is an element of the ball

$$\mathcal{U} := \{z \in \mathcal{S}_a(E) : \|z\|_a < r\}.$$

The orbit  $\omega$  turns out to solve an equation  $\Phi(x, \omega) = 0$ , for a suitable smooth map

$$\Phi : U_1 \times \mathcal{U} \rightarrow \mathcal{S}_a(E),$$

to which the implicit function theorem can be applied. This idea, used by Irwin in the real case, works just as well in the present situation. Essentially, one simply uses Theorem 2.1 instead of the implicit function theorem for real Banach spaces; the desired differentiability properties of  $\Phi$  are ensured by the next proposition (from [29]).

**Proposition 3.7** *Let  $E$  and  $F$  be Banach spaces over a valued field,  $f: B_r^E(0) \rightarrow F$  be a mapping such that  $f(0) = 0$ ,  $a \in ]0, 1]$  and  $\mathcal{U} := \{w \in \mathcal{S}_a(E) : \|w\|_a < r\}$ . If  $f$  is  $C^k$ ,  $SC^k$ ,  $LC^k$  with  $k \in \mathbb{N} \cup \{\infty\}$ , resp.,  $\mathbb{K}$ -analytic, then also the map*

$$\mathcal{S}_a(f) : \mathcal{U} \rightarrow \mathcal{S}_a(F), \quad (x_n)_{n \in \mathbb{N}_0} \mapsto (f(x_n))_{n \in \mathbb{N}_0}$$

*is  $C^k$ ,  $SC^k$ ,  $LC^k$ , resp.,  $\mathbb{K}$ -analytic.*

We mention that also Irwin's construction of pseudo-stable manifolds (see [37] and [13]) can be adapted to general valued fields [30]. Combining these constructions, all main types of invariant manifolds (stable and unstable manifolds, center-stable and center-unstable manifolds, as well as center manifolds) become available also over valued fields. Using these invariant manifolds, the following result (from [31]) can be obtained, which provided the original stimulus for the author's studies compiled in this section.

**Theorem 3.8** *Given  $k \in \mathbb{N} \cup \{\infty, \omega\}$ , let  $G$  be a finite-dimensional  $C^k$ -Lie group over a local field  $\mathbb{K}$  and  $\alpha: G \rightarrow G$  be a  $C^k$ -automorphism whose contraction group*

$$U_\alpha := \{x \in G : \alpha^n(x) \rightarrow 1 \text{ as } n \rightarrow \infty\}$$

*is closed in  $G$ . Then  $U_\alpha$ ,  $U_{\alpha^{-1}}$  and  $M_\alpha := \{x \in G : \alpha^{\mathbb{Z}}(x) \text{ is relatively compact}\}$  are closed Lie subgroups, and the product map  $U_\alpha \times M_\alpha \times U_{\alpha^{-1}} \rightarrow G$ ,  $(x, y, z) \mapsto xyz$  is a  $C^k$ -diffeomorphism onto an open,  $\alpha$ -stable identity neighbourhood of  $G$ .*

For  $\mathbb{K} = \mathbb{Q}_p$ , this is a classical result by Wang [56, Theorem 3.5]. In this case,  $U_\alpha$  is always closed, and  $\alpha$  simply looks like a linear automorphism in an exponential chart, which facilitates to reduce the proof to the case of linear automorphisms of vector spaces. By contrast, automorphisms of Lie groups over local fields of positive characteristic need not be linear in any chart (cf. [23]), whence non-linear analysis cannot be avoided.

## 4 Examples of infinite-dimensional Lie groups

In this section, we discuss the main examples of Lie groups over topological fields, in parallel with some of the specific techniques of non-linear functional analysis needed to construct their Lie group structures. In particular, we describe various results concerning non-linear mappings between function spaces.

### Linear Lie groups

Among the easiest examples of real or complex Lie groups are linear Lie groups, i.e., unit groups of unital Banach algebras (or other well-behaved topological algebras) and their Lie subgroups. If  $\mathbb{K}$  is a general topological field, then a good class of topological algebras to look at are the *continuous inverse algebras*, i.e., unital associative topological  $\mathbb{K}$ -algebras  $A$  such that the group of units  $A^\times$  is open and the inversion map  $\eta: A^\times \rightarrow A$ ,  $a \mapsto a^{-1}$  is continuous. An elementary argument shows (see [27, Proposition 2.2]):

**Proposition 4.1** *If  $A$  is a continuous inverse algebra, then the inversion map  $\eta: A^\times \rightarrow A$  is smooth and thus  $A^\times$  is a Lie group.*

For example,  $\mathbb{K}$  is a continuous inverse algebra, and more generally every finite-dimensional unital associative  $\mathbb{K}$ -algebra when equipped with the canonical vector topology ( $\cong \mathbb{K}^d$ ), see [27, Proposition 2.6]. If  $A$  is a continuous inverse algebra over  $\mathbb{K}$ , then so is the matrix algebra  $M_n(A)$ , for each  $n \in \mathbb{N}$  (see [27, Proposition 2.3]). Furthermore,  $A \otimes_{\mathbb{K}} \mathbb{L}$  is a continuous inverse algebra over  $\mathbb{L}$ , for each finite extension  $\mathbb{L}$  of  $\mathbb{K}$ , by [27, Corollary 2.8]. If  $A$  a continuous inverse algebra over a locally compact topological field  $\mathbb{K}$ , and  $K$  a compact  $C^r$ -manifold over  $\mathbb{K}$  (where  $r \in \mathbb{N}_0 \cup \{\infty\}$ ), then also the algebra  $C^r(K, A)$  of  $A$ -valued  $C^r$ -maps is a continuous inverse algebra, with respect to pointwise operations and the natural topology on this function space described below (see [27, Proposition 5.7]). For further examples over  $\mathbb{R}$  or  $\mathbb{C}$ , see [20].

Summing up, we always have a certain supply of continuous inverse algebras over each topological field, whose unit groups provide a certain supply of Lie groups.

## Mapping groups and related constructions

The second widely studied class of infinite-dimensional real Lie groups are the mapping groups, for example, loop groups  $C(\mathbb{S}^1, G)$  and  $C^\infty(\mathbb{S}^1, G)$ , where  $\mathbb{S}^1$  is the unit circle and  $G$  a finite-dimensional real Lie group ([47], [50]). The classical constructions of mapping groups can be generalized to a large extent to the case of Lie groups over topological fields. For example,  $C(K, G)$  can be made a Lie group for each compact topological space  $K$  and Lie group  $G$  over a topological field  $\mathbb{K}$  (see [27, Proposition 5.1]). We now discuss groups of differentiable maps in more detail. [27, Proposition 5.1] subsumes:

**Proposition 4.2** *Let  $K$  be a compact (and hence finite-dimensional)  $C^k$ -manifold over a locally compact topological field  $\mathbb{K}$ , where  $k \in \mathbb{N}_0 \cup \{\infty\}$ , and  $G$  a Lie group, modelled on a topological  $\mathbb{K}$ -vector space  $E$ . Then there is a uniquely determined smooth manifold structure on  $C^k(K, G)$  making it a Lie group modelled on  $C^k(K, E)$ , and such that*

$$\Phi : C^k(K, U) \rightarrow C^k(K, V), \quad \gamma \mapsto \phi \circ \gamma$$

*defines a chart of  $C^k(K, G)$  around 1, for some chart  $\phi : G \supseteq U \rightarrow V \subseteq E$  of  $G$ .*

Here  $C^k(K, U) = \{\gamma \in C^k(K, G) : \gamma(K) \subseteq U\}$  and  $C^k(K, V) = \{\gamma \in C^k(K, E) : \gamma(K) \subseteq V\}$ , which is an open subset of the topological  $\mathbb{K}$ -vector space  $C^k(K, E)$  of  $E$ -valued  $C^k$ -maps on  $K$ . The topology on the latter is defined as follows.

**Definition 4.3** Given topological  $\mathbb{K}$ -vector spaces  $X$  and  $E$  over a topological field  $\mathbb{K}$ , and a  $C^k$ -map  $\gamma : U \rightarrow E$  on an open subset  $U \subseteq X$ , where  $k \in \mathbb{N}_0 \cup \{\infty\}$ , we recursively define  $U^{[j]} := (U^{[1]})^{[j-1]}$  and  $\gamma^{[j]} := (\gamma^{[1]})^{[j-1]} : U^{[j]} \rightarrow E$  for each  $j \in \mathbb{N}$  such that  $j \leq k$  (with  $U^{[0]} := U$  and  $f^{[0]} := f$ ). We equip  $C^k(U, E)$  with the initial topology with respect to the family of mappings

$$C^k(U, E) \rightarrow C(U^{[j]}, E), \quad \gamma \mapsto \gamma^{[j]}$$

such that  $j \in \mathbb{N}_0$  and  $j \leq k$ , where  $C(U^{[j]}, E)$  carries the compact-open topology. If  $M$  is a  $C^k$ -manifold modelled on  $X$ , with  $C^k$ -atlas  $\mathcal{A}$ , we equip  $C^k(M, E)$  with the initial topology with respect to the family of maps  $C^k(M, E) \rightarrow C^k(V, E)$ ,  $\gamma \mapsto \gamma \circ \phi^{-1}$ , for  $\phi : M \supseteq U \rightarrow V \subseteq X$  ranging through  $\mathcal{A}$  (see [27, §4] for details).

This is a very natural definition, which provides spaces with the expected completeness, metrizability and local convexity properties in relevant situations, and which produces the conventional topologies in the real locally convex case [27, Proposition 4.19].

The proof of Proposition 4.2 is based on the following result (and variants for mappings between spaces of  $C^k$ -maps, [27, Proposition 4.20 and Corollary 4.21]). It ensures smoothness of the relevant non-linear mappings between spaces of smooth functions:

**Proposition 4.4** *Let  $K$  be a compact smooth manifold over a locally compact topological field  $\mathbb{K}$ ,  $E$  and  $F$  be topological  $\mathbb{K}$ -vector spaces, and  $U \subseteq E$  be open. Then*

$$C^\infty(K, f) : C^\infty(K, U) \rightarrow C^\infty(K, F), \quad \gamma \mapsto f \circ \gamma$$

is a smooth map, for each smooth map  $f: U \rightarrow F$ . More generally,

$$f_* : C^\infty(K, U) \rightarrow C^\infty(K, F), \quad \gamma \mapsto (x \mapsto f(x, \gamma(x)))$$

is smooth, for each smooth map  $f: K \times U \rightarrow F$ .

Now the  $C^\infty$ -version of Proposition 4.2 easily follows. For example, assuming  $U = U^{-1}$ , the inversion map of  $C^\infty(K, G)$  is smooth on  $C^\infty(K, U)$  (considered as a smooth manifold with global chart  $\Phi$ ), by the following argument: The inversion map of  $G$  restricts to a smooth map  $i: U \rightarrow U$ , corresponding to the smooth map  $j := \phi \circ i \circ \phi^{-1}: V \rightarrow V$  in the local chart  $\phi$ . The restriction of the inversion map of  $C^\infty(K, G)$  to  $C^\infty(K, U)$  is  $C^\infty(K, i)$ . This map is smooth, since  $\Phi \circ C^\infty(K, i) \circ \Phi^{-1} = C^\infty(K, j)$  is smooth by Proposition 4.4.

**Remark 4.5** We mention that an analogue of Proposition 4.4 is available for mappings between spaces of compactly supported vector-valued  $C^k$ -functions on a paracompact finite-dimensional manifold  $M$  over a locally compact topological field  $\mathbb{K}$  (see [27, Proposition 8.22 and Corollary 8.23]). For each Lie group  $G$  modelled on a topological  $\mathbb{K}$ -vector space  $E$ , this facilitates to turn the “test function group”

$$C_c^k(M, G) = \{\gamma \in C^k(M, G) : \gamma(x) = 1 \text{ for all } x \text{ outside some compact set}\}$$

of compactly supported  $G$ -valued  $C^k$ -maps into a Lie group modelled on  $C_c^k(M, E)$ , equipped with a suitable vector topology (see [27, Proposition 9.1]).

As a tool for the discussion of diffeomorphism groups, it is useful to know that the *weak direct product*

$$\prod_{i \in I}^* G_i := \{(x_i)_{i \in I} \in \prod_{i \in I} G_i : x_i = 1 \text{ for all but finitely many } i\}$$

of a family  $(G_i)_{i \in I}$  of Lie groups over a valued field  $\mathbb{K}$  can always be turned into a Lie group, modelled on the direct sum  $E := \bigoplus_{i \in I} E_i$  of the respective modelling spaces, equipped with the box topology (see [27, Proposition 7.1]). Thus, sets of the form  $\bigoplus_{i \in I} U_i := E \cap \prod_{i \in I} U_i$  are taken as a basis of open 0-neighbourhoods for  $E$ , where each  $U_i$  is an open 0-neighbourhood in  $E_i$ . The following result concerning typical non-linear mappings between direct sums ([27, Proposition 6.9]) is used to construct the Lie group structure on weak direct products.

**Proposition 4.6** *Let  $(E_i)_{i \in I}$  and  $(F_i)_{i \in I}$  be families of topological  $\mathbb{K}$ -vector spaces indexed by a set  $I$ . Let  $k \in \mathbb{N}_0 \cup \{\infty\}$ , and  $f_i: U_i \rightarrow F_i$  be a  $C^k$ -map, for  $i \in I$ , defined on an open 0-neighbourhood  $U_i \subseteq E_i$ . Then  $\bigoplus_{i \in I} f_i: \bigoplus_{i \in I} U_i \rightarrow \bigoplus_{i \in I} F_i$ ,  $(x_i)_{i \in I} \mapsto (f_i(x_i))_{i \in I}$  is a  $C^k$ -map on the open subset  $\bigoplus_{i \in I} U_i$  of  $\bigoplus_{i \in I} E_i$ .*

For  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ , analogous results can be obtained using locally convex direct sums [22].

## Diffeomorphism groups

Let  $M$  be a finite-dimensional, paracompact smooth manifold over a local field  $\mathbb{K}$ . We explain some ideas used in [27, §13] to define a Lie group structure on  $\text{Diff}(M)$ , the group of all  $C^\infty$ -diffeomorphisms of  $M$ :

**Proposition 4.7**  *$\text{Diff}(M)$  is a Lie group modelled on the space  $C_c^\infty(M, TM)$  of compactly supported smooth vector fields on  $M$ .*

To construct the Lie group structure, one exploits that  $M$  is a disjoint union  $M = \bigcup_{i \in I} B_i$  of balls, i.e., open subsets  $B_i \subseteq M$  diffeomorphic to  $\mathbb{O}^d$ , where  $\mathbb{O}$  is the maximal compact subring of  $\mathbb{K}$  and  $d$  the dimension of the modelling space of  $M$ . The main step (explained more closely below) is to make each  $\text{Diff}(B_i)$  a Lie group. Then the weak direct product  $\prod_{i \in I}^* \text{Diff}(B_i)$  can be made a Lie group, as described above. Here  $\prod_{i \in I}^* \text{Diff}(B_i)$  can be identified with a subgroup of  $\text{Diff}(M)$  in an apparent way. In a third step, one verifies that  $\text{Diff}(M)$  can be given a Lie group structure making  $\prod_{i \in I}^* \text{Diff}(B_i)$  an open subgroup.

The following “exponential law” (covered by [27, Lemma 12.1 and Proposition 12.2]) is essential (cf. also [27, Proposition 12.6] for a variant for metrizable manifolds).

**Proposition 4.8** *Let  $\mathbb{K}$  be a topological field,  $M$  and  $N$  be smooth manifolds modelled on topological  $\mathbb{K}$ -vector spaces, and  $E$  be a topological  $\mathbb{K}$ -vector space.*

- (a) *For every smooth mapping  $f: M \times N \rightarrow E$ , also the mapping  $f^\vee: M \rightarrow C^\infty(N, E)$ ,  $f^\vee(x) := f(x, \bullet)$  is smooth.*
- (b) *If  $\mathbb{K}$  is locally compact and  $N$  is finite-dimensional, then a map  $g: M \rightarrow C^\infty(N, E)$  is smooth if and only if  $g^\wedge: M \times N \rightarrow E$ ,  $g^\wedge(x, y) := g(x)(y)$  is smooth.*

Furthermore, the map  $C^\infty(M \times N, E) \rightarrow C^\infty(M, C^\infty(N, E))$ ,  $f \mapsto f^\vee$  is an isomorphism of topological vector spaces in the situation of (b), with inverse  $g \mapsto g^\wedge$ .

We now explain the essential first step of the above construction of the Lie group structure on diffeomorphism groups in more detail, namely the construction of the Lie group structure on  $\text{Diff}(M)$  for  $M = \mathbb{O}^d$  a ball. Then  $P := \text{Diff}(M)$  is an open subset of the topological vector space  $C^\infty(M, \mathbb{K}^d)$  (as above).

*Smoothness of inversion.* The inclusion map  $i: P \rightarrow C^\infty(M, \mathbb{K}^d)$ ,  $\gamma \mapsto \gamma$  being smooth, the second half of the exponential law (Proposition 4.8 (b)) ensures that also

$$f: P \times M \rightarrow \mathbb{K}^d, \quad f(\gamma, x) := i^\wedge(\gamma, x) := i(\gamma)(x) = \gamma(x)$$

is smooth, using that  $M$  is finite-dimensional. Then  $f_\gamma := f(\gamma, \bullet) = \gamma$  for each  $\gamma \in P$ , whence  $f'_\gamma(x) = \gamma'(x) \in \text{GL}(\mathbb{K}^d)$  for each  $x \in M$ . Applying the inverse function with parameters (Theorem 2.6) with the diffeomorphism  $\gamma$  as the parameter, we see that

$$g: P \times M \rightarrow M, \quad g(\gamma, x) := (f_\gamma)^{-1}(x) = \gamma^{-1}(x)$$

is smooth. Now the first half of the exponential law (Proposition 4.8(a)) shows that  $\text{Diff}(M) \rightarrow C^\infty(M, \mathbb{K}^d)$ ,  $\gamma \mapsto g^\vee(\gamma) := g(\gamma, \bullet) = \gamma^{-1}$  is smooth. But this is the inversion map of the group  $\text{Diff}(M)$ .

*Smoothness of composition.* Since  $\text{Diff}(M)$  is open in  $C^\infty(M, \mathbb{K}^d)$ , we only need to show that  $\Gamma: C^\infty(M, \mathbb{K}^d \times \mathbb{K}^d) \cong C^\infty(M, \mathbb{K}^d) \times C^\infty(M, \mathbb{K}^d) \rightarrow C^\infty(M, \mathbb{K}^d)$ ,  $\Gamma(\gamma, \eta) := \gamma \circ \eta$  is smooth. Because the evaluation map  $\text{ev}: C^\infty(M, \mathbb{K}^d) \times M \rightarrow \mathbb{K}^d$ ,  $\text{ev}(\gamma, x) := \gamma(x)$  is smooth by [27, Proposition 11.1], we deduce from the formula

$$\Gamma^\wedge((\gamma, \eta), x) = \gamma(\eta(x)) = \text{ev}(\gamma, \text{ev}(\eta, x)) \quad \text{for } \gamma, \eta \in C^\infty(M, \mathbb{K}^d), x \in M$$

that  $\Gamma^\wedge$  is smooth and hence also  $\Gamma$ , by Proposition 4.8(b).

**Remark 4.9** For differentiability properties of the composition map between spaces of  $C^k$ -functions, see [27, §11]. Diffeomorphism groups can also be found in [46].

## Direct limit groups

Consider an ascending sequence  $G_1 \subseteq G_2 \subseteq \dots$  of finite-dimensional Lie groups over a locally compact topological field  $\mathbb{K}$ , such that each inclusion map  $G_n \rightarrow G_{n+1}$  is a smooth immersion. Then  $G := \bigcup_{n \in \mathbb{N}} G_n$  can be given a Lie group structure modelled on the direct limit topological  $\mathbb{K}$ -vector space  $\varinjlim E_n$  of the respective finite-dimensional modelling spaces, which makes  $G$  the direct limit of the given directed sequence in the category of  $\mathbb{K}$ -Lie groups and smooth homomorphisms [24] (cf. also [49] and [21]).

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