Finite order differentiability properties, fixed points and implicit functions over valued fields

Helge Glöckner

Abstract

We prove an implicit function theorem for C^k-maps from arbitrary topological vector spaces over valued fields to Banach spaces (for $k \geq 2$). As a tool, we show the C^k-dependence of fixed points on parameters for suitable families of contractions of a Banach space. Similar results are obtained for k times strictly differentiable maps, and for k times Lipschitz differentiable maps. In the real case, our results subsume an implicit function theorem for Keller C^k_c-maps from arbitrary topological vector spaces to Banach spaces.

Introduction

Generalizations of the implicit function theorem for mappings from suitable real or complex topological vector spaces to Banach spaces have been obtained in various settings of analysis. Hiltunen ([18],[19]) studied implicit functions in the framework of Keller's C^k_Π-theory [23] (cf. also [20] for recent generalizations to non-Banach range spaces). Teichmann [32] proved an implicit function theorem in the "convenient setting" of analysis, under more restrictive conditions. In the framework of Keller's C^k_c-theory, implicit functions from real and complex topological vector spaces to Banach spaces were discussed in [12].

The notion of a Keller C^k_c-map can be generalized to a notion of C^k-map between open subsets of topological vector spaces over an arbitrary (non-discrete) topological field [2]. For mappings between open subsets of an ultrametric field, these C^k-maps coincide with those usually considered in Non-Archimedian Analysis (as in [31]). On the basis of [2], the paper [12] also provided an implicit function theorem for C^k-maps from metrizable topological vector spaces over complete valued fields to Banach spaces, with a possible loss of one order of differentiability in the case of an infinite-dimensional range space.

In the present paper, we discuss implicit functions from topological vector spaces over valued fields to Banach spaces by a different method, which enables us to remove the metrizability condition. We can also avoid the former loss of one order of differentiability. Thus, local solutions to C^k-equations are always C^k, as they should be (if $k \geq 2$). Stimulated by discussions of mappings between real Banach spaces in [22], our new strategy of proof now is to discuss, in a first step, the existence of fixed points for families of contractions of a Banach space and their differentiable dependence on parameters. Next, we prove a suitable Lipschitz Inverse Function Theorem (which varies a result from [35]). The desired implicit function theorem is then an immediate consequence. In fact, the Lipschitz Inverse Function Theorem entails that, locally, an implicit function λ exists and that its value $\lambda(x)$ at x can be constructed as a fixed point of a suitable contraction g_x (if $k \geq 2$, say). By the C^k-dependence of fixed points on parameters, $\lambda(x)$ is a C^k-function of x.

It useful to work with several variants of C^k-maps, because the mere C^k-property (notably, the C^1-property) is slightly too weak for some of our purposes, and envisaged applications. We therefore introduce so-called *k times strictly differentiable mappings* (SC^k-maps) between open subsets of topological $\mathbb{K}$-vector spaces (and subsets with dense interior), as well as *k times Lipschitz differentiable maps* (LC^k-maps). Our definition of SC^k-maps generalizes an earlier definition for mappings from normed spaces to polynormed vector spaces from [12]. Furthermore, a mapping between open subsets of Banach spaces is once strictly differentiable in our sense if and only if it is strictly differentiable at each point in the sense of Bourbaki [3, 1.2.2] (cf. also [25] and [5]). We also mention the "r-Lipschitz maps" $\mathbb{O} \to \mathbb{K}$ studied by Barsky [1], where $\mathbb{K}$ is a local field and $\mathbb{O}$ its maximal compact subring. The differentiability properties just described are related as follows:

$$C^{k+1} \implies LC^k \implies SC^k \implies C^k\,.$$

Thus C^∞-maps, SC^∞-maps and LC^∞-maps all coincide, but we have a certain range of finite-order differentiability properties.

Among our main results is the following generalization of the Implicit Function Theorem (Theorem 5.2).

Generalized Implicit Function Theorem. *Let $\mathbb{K}$ be a valued field, E be a topological $\mathbb{K}$-vector space, F be a Banach space over $\mathbb{K}$, and $f\colon U \times V \to F$ be a map, where $U \subseteq E$ is a subset with dense interior and $V \subseteq F$ is open. Given $x \in U$, abbreviate $f_x := f(x, \bullet)\colon V \to F$. Assume that $\mathbb{K}$, F, $k \in \mathbb{N} \cup \{\infty\}$ and f have properties as shown in the following table. Furthermore, assume that $f(x_0, y_0) = 0$ for some $(x_0, y_0) \in U \times V$ and $f'_{x_0}(y_0) \in \mathrm{GL}(F)$. Then there exists an open neighborhood $U_0 \subseteq U$ of x_0, an open neighborhood $V_0 \subseteq V$ of y_0, and a map $\lambda\colon U_0 \to V_0$ such that*

$$\{(x,y) \in U_0 \times V_0 \colon f(x,y) = 0\} \;=\; \text{graph}\,\lambda\,,$$

where λ has the differentiability property shown in the table:

$\mathbb{K}$	F	k	f	λ
arbitrary	arbitrary	arbitrary	LC^k	LC^k
arb.	arb.	arb.	SC^k	SC^k
arb.	arb.	≥ 2	C^k	C^k
locally compact	$\dim F < \infty$	arb.	C^k	C^k

The generalized implicit function theorem is deduced from a suitable "Inverse Function Theorem with Parameters" (Theorem 5.13), dealing with families of local diffeomorphisms. This theorem is our actual main result. As a technical tool, in Theorem 4.7 we prove differentiable dependence of fixed points on parameters, for uniform families of contractions in the sense of Definition 4.4:

Theorem (on the Parameter-Dependence of Fixed Points). *Let E be a topological $\mathbb{K}$-vector space, F be a Banach space over $\mathbb{K}$ and $f\colon P \times U \to F$ be a map, where $P \subseteq E$ is a subset with dense interior and $U \subseteq F$ is open. Given $p \in P$, abbreviate $f_p := f(p, \bullet)\colon U \to F$. Assume that $(f_p)_{p\in P}$ is a uniform family of contractions and assume that f is C^k, SC^k, resp., LC^k for some $k \in \mathbb{N}_0 \cup \{\infty\}$. Let Q be the set of all $p \in P$ such that f_p has a fixed point x_p. Then the following holds:*

(a) *Q is open in P;*

(b) *The map $\phi\colon Q \to F$, $p \mapsto x_p$ is C^k, SC^k, resp., LC^k.*

Applications. It is clear that generalizations of such basic and central results as the inverse- and implicit function theorems have immediate implications.

- In [13], the ultrametric inverse function theorem with parameters in a Fréchet space is used to prove that the inversion map $\mathrm{Diff}(M) \to \mathrm{Diff}(M)$, $\gamma \mapsto \gamma^{-1}$ of the diffeomorphism group of a paracompact, finite-dimensional smooth manifold over a local field is smooth. Also composition being smooth, $\mathrm{Diff}(M)$ is a Lie group (see also [12] for an outline of the proof).[1]

- Irwin [21] explained how to construct stable manifolds around hyperbolic fixed points for discrete dynamical systems modeled on real Banach spaces, with the help of the implicit function theorem (see also [35]). In [14], the C^k-, SC^k- and LC^k-versions of our inverse function theorem are used to adapt Irwin's method to dynamical systems modeled on Banach spaces over valued fields. Stable manifolds around hyperbolic fixed points of the respective differentiability class are constructed (and also analytic ones). Furthermore, the LC^k-inverse function theorem with parameters is used in [14] to study the dependence of the stable manifolds on the non-linearity (cf. also [12] for an outline of the smooth case).

[1] Cf. also [26], [27] for certain diffeomorphism groups for char $\mathbb{K} = 0$.

- Adapting Irwin's discussion of pseudo-stable manifolds ([22]; see also [9]), one can also construct pseudo-stable manifolds around hyperbolic fixed points for dynamical systems modeled on Banach spaces over valued fields (see [15]). At the heart of these studies is a refined analysis of the dependence of fixed points on parameters (in specialized situations).

- Varying a classical idea by Chow and Hale [6], C^{k+n}-solutions to (systems of) p-adic differential equations of the form $y^{(k)} = f(x, y, y', \ldots, y^{(k-1)})$ for f an LC^n-map and $k \in \mathbb{N}$, $n \in \mathbb{N}_0$ can be constructed using our inverse function theorems (with and without parameters), which depend on initial conditions and parameters in a controlled way (work in progress; cf. [31, § 65] for C^1-solutions to scalar-valued first order equations).

- Various applications of the Inverse Function Theorem with Parameters in the structure theory of infinite-dimensional real Lie groups can be found in [17]. For example, as shown by Neeb, it can be used to prove that locally compact subgroups of locally exponential Lie groups are Lie subgroups (as in the classical case of Banach-Lie groups).[2]

The stable, unstable and center-stable manifolds available through the preceding constructions are useful for the theory of finite-dimensional Lie groups over local fields. In [16], they are used to generalize structure theorems for p-adic Lie groups and their automorphisms (as in [33] and [10]) to the case of Lie groups over local fields of positive characteristic (under suitable hypotheses). Further applications of the generalized implicit function theorems provided here (and their precursors from [12]) are summarized in the introduction of [12].

Structure of the article. In Section 1 (as well as Appendix A), we compile definitions, notation and basic facts concerning the differential calculus of C^k-maps over topological fields. We also introduce a certain concept of a "gauge" as a convenient substitute for seminorms when dealing with arbitrary (not necessarily polynormed) topological vector spaces over valued fields. In Section 2, we discuss Lipschitz continuous maps between subsets of topological vector spaces over valued fields and Lipschitz differentiable maps between subsets with dense interior. Next, we define and discuss strictly differentiable maps and clarify their relations to C^k-maps and LC^k-maps (Section 3). Fixed points of contractions and their dependence on parameters are discussed in Section 4, and the results obtained are then used in Section 5 to deduce implicit function theorems as well as inverse function theorems with and without parameters. Appendix B records a variant of a technical lemma, which is not needed in the main text.

[2]A smooth Lie group G modeled on a real locally convex space is called *locally exponential* if it has an exponential function $\exp_G \colon L(G) \to G$ which induces a local C^∞-diffeomorphism from an open 0-neighborhood in $L(G)$ onto an open identity neighborhood in G.

1 Preliminaries and basic facts

The general framework for our studies is the differential calculus of C^k-maps between open subsets of topological vector spaces over non-discrete topological fields developed in [2]. In this section, we first recall basic definitions and facts from this theory, which applies to arbitrary topological ground fields, and slightly extend them by replacing open sets by sets with dense interior. We then introduce various concepts which are important when dealing with *valued* fields. In particular, we shall define certain generalizations of seminorms, which we call "gauges". With the help of these gauges, we can treat general topological vector spaces as though they were locally convex. Many definitions, results and proofs will look exactly as in the locally convex (or polynormed) case, except that continuous seminorms have been replaced with gauges.[3]

All topological fields are assumed Hausdorff and non-discrete; all topological vector spaces are assumed Hausdorff. In Definition 1.1–Lemma 1.13, we shall assume that $\mathbb{K}$ is a topological field; in Definition 1.14–Lemma 1.32, we assume that $(\mathbb{K}, |.|)$ is a (non-trivial) valued field.

C^k-maps over topological fields

Let E and F be topological $\mathbb{K}$-vector spaces, and $f\colon U \to F$ be a mapping, defined on a subset $U \subseteq E$ with dense interior. Then the directional difference quotient

$$f^{]1[}(x, y, t) \;:=\; \frac{f(x + ty) - f(x)}{t}$$

makes sense for all (x, y, t) in the subset

$$U^{]1[} \;:=\; \{(x, y, t) \in U \times E \times \mathbb{K}^\times : x + ty \in U\}$$

of $E \times E \times \mathbb{K}$. To define directional derivatives, we must enlarge this set by allowing also the value $t = 0$. Hence, we consider now

$$U^{[1]} \;:=\; \{(x, y, t) \in U \times E \times \mathbb{K} \colon x + ty \in U\}\,.$$

Thus $U^{[1]} = U^{]1[} \cup (U \times E \times \{0\})$, as a disjoint union.

Definition 1.1 $f\colon U \to F$ is called *continuously differentiable* (or C^1) if f is continuous (C^0) and there exists a continuous map $f^{[1]}\colon U^{[1]} \to F$ which extends $f^{]1[}\colon U^{]1[} \to F$.

Thus, we assume the existence of a continuous map $f^{[1]}\colon U^{[1]} \to F$ such that

$$f^{[1]}(x, y, t) \;=\; \frac{f(x + ty) - f(x)}{t} \quad \text{for all } (x, y, t) \in U^{[1]} \text{ such that } t \neq 0.$$

[3] A similar idea is also implicit in Lang's definition of total differentiability in general real topological vector spaces [24, I, § 3].

If it exists, then $f^{[1]}$ is unique, by part (a) of the following lemma:

Lemma 1.2 *Let E be a topological $\mathbb{K}$-vector space, and $U \subseteq E$ be a subset with dense interior U^0. Then the following holds:*

(a) $U^{]1[}$ *is a dense open subset of* $U^{[1]}$*;*

(b) $U^{[1]}$ *is a subset of* $E \times E \times \mathbb{K}$ *with dense interior.*

(c) $W^{]1[}$ *is open in* $E \times E \times \mathbb{K}$ *and dense in* $U^{[1]}$*, for each dense subset* $W \subseteq U$ *which is open in* E *(e.g., for* $W := U^0$*).*

Proof. Clearly $U^{]1[} = U^{[1]} \cap (E \times E \times \mathbb{K}^\times)$ is open in $U^{[1]}$. Furthermore, $W^{]1[}$ (and hence $(U^0)^{]1[}$) is open in $E \times E \times \mathbb{K}$, as a consequence of the continuity of the maps $(x, y, t) \mapsto x$ and $(x, y, t) \mapsto x + ty$ on $E \times E \times \mathbb{K}$. All other assertions will follow readily if we can show that $W^{]1[}$ is dense in $U^{[1]}$. To this end, let $(x, y, t) \in U^{[1]}$ and $X \subseteq E$, $Y \subseteq E$ and $S \subseteq \mathbb{K}$ be open neighborhoods of x, y and t, respectively.

If $t \neq 0$, then $x+tY$ is a neighborhood of $x+ty \in U \subseteq \overline{W}$, whence $(x+tY) \cap W \neq \emptyset$. We therefore find $y' \in Y$ such that $x + ty' \in W$. Then $(W - ty') \cap X$ is a neighborhood of x and hence has non-empty intersection with W. We pick $x' \in W \cap (W - ty') \cap X$. Then $x' \in W$ and $x' + ty' \in W$ and thus $(x', y', t) \in W^{]1[}$. Furthermore, $(x', y', t) \in X \times Y \times S$.

If $t = 0$, we pick an element $x' \in X \cap W$. By openness of W and continuity of scalar multiplication, after shrinking Y and S we may assume that $x' + SY \subseteq W$. Thus $\{x'\} \times Y \times S \subseteq W^{]1[}$. We choose $y' \in Y$ and $s \in S \setminus \{0\}$. Then $(x', y', s) \in W^{]1[} \cap (X \times Y \times S)$.

We have shown that $W^{]1[}$ is dense in $U^{[1]}$, which completes the proof. □

Lemma 1.2 (b) facilitates to define C^k-maps by recursion.

Definition 1.3 Let E and F be topological $\mathbb{K}$-vector spaces and $f \colon U \to F$ be a map, defined on a subset $U \subseteq E$ with dense interior. Given $k \in \mathbb{N}$ with $k \geq 2$, we say that f is *k times continuously differentiable* (or C^k) if f is C^1 and $f^{[1]} \colon U^{[1]} \to F$ is C^{k-1}. The map f is called C^∞ (or *smooth*) if it is C^k for all $k \in \mathbb{N}_0$.

1.4 For example, every continuous linear map $\lambda \colon E \to F$ is smooth [2], with $\lambda^{[1]}(x, y, t) = \lambda(y)$ for all $(x, y, t) \in E \times E \times \mathbb{K}$ (thus $\lambda^{[1]}$ is again continuous linear).

1.5 The domain of the mapping $f^{[2]} := (f^{[1]})^{[1]}$ is the set

$$U^{[2]} := (U^{[1]})^{[1]} \subseteq E \times E \times \mathbb{K} \times E \times E \times \mathbb{K} \times \mathbb{K}\,.$$

We write $U^{[k]} := (U^{[1]})^{[k-1]}$ for the domain of $f^{[k]}$, and set $U^{]k[} := (U^{]1[})^{]k-1[}$.

Remark 1.6 Applying Lemma 1.2 (c) to $U^{[1]}$ and its dense open subset $W := (U^0)^{]1[}$, we find that $(U^0)^{]2[}$ is dense in $U^{[2]}$, and inductively that $(U^0)^{]k[}$ is dense in $U^{[k]}$, for each $k \in \mathbb{N}$. Hence $(f|_{U^0})^{[k]}$ uniquely determines $f^{[k]}$ in particular (if it exists). This enables us to extend all relevant results and proofs from [2] (where only open domains were considered) to the case of non-open domains, by trivial arguments based on continuous extension. We shall therefore cite (and apply) results from [2] freely also for mappings on non-open domains. We remark that, if one is working exclusively with Hausdorff topological vector spaces over Hausdorff topological fields (in contrast to [2], where the scope was wider), it can be convenient to replace the somewhat technical axioms from [2] by simpler ones. Such variants are suggested in Appendix A.

Remark 1.7 A trivial induction shows that $U^{[k]} = (U^{[k-1]})^{[1]}$, and that f is C^k if and only if f is C^{k-1} and $f^{[k-1]}$ is C^1 (cf. [2, Rem. 4.2]). In this article, we shall avoid to use $f^{[k]}$ for higher k as far as possible. Usually, we only need $f^{[1]}$.

1.8 Given a C^1-map $f\colon U \to F$ as before, we define its *directional derivative* at $x \in U$ in the direction $y \in E$ via

$$(D_y f)(x) \;:=\; df(x,y) \;:=\; f^{[1]}(x,y,0)\,.$$

If $x \in U^0$, then

$$df(x,y) \;=\; f^{[1]}(x,y,0) \;=\; \lim_{0\neq t\to 0} f^{[1]}(x,y,t) \;=\; \lim_{0\neq t\to 0} \tfrac{1}{t}(f(x+ty)-f(x))\,,$$

and thus $df(x,y)$ can be interpreted as a limit of difference quotients. The map $df\colon U \times E \to F$ is continuous, being a partial map of $f^{[1]}$, and it can be shown that the "differential" $f'(x) := df(x,\bullet)\colon E \to F$ of f at x is a continuous $\mathbb{K}$-linear map, for each $x \in U$ (cf. [2, Proposition 2.2]). If f is C^2, we define a continuous map $d^2 f\colon U \times E^2 \to F$ via $d_2 f(x,y_1,y_2) := (D_{y_2}(D_{y_1} f))(x)$. Thus

$$d^2 f(x,y_1,y_2) := \lim_{t\to 0} \tfrac{1}{t}(df(x+ty_2,y_1) - df(x,y_1)) = f^{[2]}((x,y_1,0),\,(y_2,0,0),\,0)\,.$$

Similarly, if f is C^k, we obtain continuous maps $d^j f\colon U \times E^j \to F$ for all $j \in \mathbb{N}_0$ such that $j \leq k$ via $d^j f(x,y_1,\ldots,y_j) := (D_{y_j}\cdots D_{y_1} f)(x)$. It can be shown that $d^j f(x,\bullet)\colon E^j \to F$ is a symmetric j-linear map (cf. [2, Lemma 4.8]).

Remark 1.9 We remark that, in the real locally convex case, our C^k-maps coincide with Keller's C^k_c-maps (as used, e.g., in [11], [17], [28]). More precisely, let E be a real topological vector space, F be a locally convex real topological vector space, $U \subseteq E$ be open, $f\colon U \to F$ be a map, and $k \in \mathbb{N}_0 \cup \{\infty\}$ Then f is C^k if and only if f is continuous, the limits $d^j f(x,y_1,\ldots,y_j) := (D_{y_j}\cdots D_{y_1} f)(x)$ described in **1.8** exist for all $j \in \mathbb{N}$ with $j \leq k$, and the maps $d^j f\colon U \times E`j \to F$ so obtained are continuous (see [2, Proposition 7.4]). We mention that if also E is locally convex, then this characterization remains valid for mappings on locally convex subsets $U \subseteq E$ with dense interior (as considered in [17]).

1.10 (Chain Rule). If E, F, and H are topological $\mathbb{K}$-vector spaces, $U \subseteq E$ and $V \subseteq F$ are subsets with dense interior, and $f\colon U \to V \subseteq F$, $g\colon V \to H$ are C^k-maps, then also the composition $g \circ f\colon U \to H$ is C^k. If $k \geq 1$, we have $(f(x), f^{[1]}(x,y,t), t) \in V^{[1]}$ for all $(x,y,t) \in U^{[1]}$, and

$$(g \circ f)^{[1]}(x,y,t) = g^{[1]}(f(x), f^{[1]}(x,y,t), t)\,. \tag{1}$$

In particular, $d(g \circ f)(x,y) = dg(f(x), df(x,y))$ for all $(x,y) \in U \times E$ (cf. [2, Prop. 3.1 and 4.5]).

We recall from [2, La. 4.9] that being C^k is a local property.

Lemma 1.11 *Let E and F be topological $\mathbb{K}$-vector spaces, and $f\colon U \to F$ be a map, defined on an open subset U of E. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. If there is an open cover $(U_i)_{i \in I}$ of U such that $f|_{U_i}\colon U_i \to F$ is C^k for each $i \in I$, then f is C^k.* □

Symmetry properties of $f^{[1]}$

The map $f^{[1]}$ and also the higher different quotient maps satisfy various identities, which we need to exploit occasionally. We now describe some properties of $f^{[1]}$. A symmetry property of $f^{[2]}$ is described in Appendix B. Slightly less explicit (and more complicated) results concerning $f^{[k]}$ for arbitrary k can be found in [13, La. 6.8]. They are also essential for the construction of invariant manifolds in [14] and [15].

Lemma 1.12 *Let E, F be topological $\mathbb{K}$-vector spaces, $U \subseteq E$ be a subset with dense interior and $f\colon U \to F$ be C^1. Then $(x + ty_2, y_1 - y_2, t) \in U^{[1]}$ for all $x \in U$, $y_1, y_2 \in E$ and $t \in \mathbb{K}$ such that $(x, y_1, t), (x, y_2, t) \in U^{[1]}$, and*

$$f^{[1]}(x, y_1, t) - f^{[1]}(x, y_2, t) \;=\; f^{[1]}(x + ty_2, y_1 - y_2, t)\,. \tag{2}$$

Proof. The first assertion is clear since $x + ty_2 \in U$ (because $(x, y_2, t) \in U^{[1]}$) and $x + ty_2 + t(y_1 - y_2) = x + ty_1 \in U$ (because $(x, y_1, t) \in U^{[1]}$). If $t \neq 0$, then

$$\begin{aligned}
f^{[1]}(x, y_1, t) - f^{[1]}(x, y_2, t) &= \frac{f(x + ty_1) - f(x)}{t} - \frac{f(x + ty_2) - f(x)}{t} \\
&= \frac{f(x + ty_1) - f(x + ty_2)}{t} \\
&= \frac{f(x + ty_2 + t(y_1 - y_2)) - f(x + ty_2)}{t} \\
&= f^{[1]}(x + ty_2, y_1 - y_2, t)\,,
\end{aligned}$$

as desired. If $t = 0$, then (2) turns into the identity $df(x, y_1) - df(x, y_2) = df(x, y_1 - y_2)$, which is valid by linearity of $df(x, \bullet)$. □

Also the following slightly more complicated symmetry properties will be used. They shall enable us to shrink certain entries of $f^{[1]}$ while inflating others in a controlled way.

Lemma 1.13 *Let E and F be topological vector spaces over a topological field $\mathbb{K}$, and $f\colon U \to F$ be a C^1-map, defined on a subset $U \subseteq E$ with dense interior. If f is C^1, $t \in \mathbb{K}^\times$, and $(x, y, s) \in E \times E \times \mathbb{K}$ such that $(x, y, ts) \in U^{[1]}$, then also $(x, ty, s) \in U^{[1]}$, and*

$$t\, f^{[1]}(x, y, ts) = f^{[1]}(x, ty, s)\,. \tag{3}$$

Proof. Since $x + (ts)y = x + s(ty)$, it is obvious that $(x, ty, s) \in U^{[1]}$ if and only if $(x, y, ts) \in U^{[1]}$. In this case, we have

$$tf^{[1]}(x, y, ts) = \tfrac{1}{s}(f(x + tsy) - f(x)) = f^{[1]}(x, ty, s)$$

provided $s \neq 0$; if $s = 0$, then $f^{[1]}(x, ty, s) = f^{[1]}(x, ty, 0) = df(x, ty) = tdf(x, y) = tf^{[1]}(x, y, 0) = tf^{[1]}(x, y, ts)$. Thus (3) holds. □

Vector spaces over valued fields, seminorms and gauges

We now fix terminology and notation concerning topological vector spaces over valued fields, and compile some simple observations which are analogous to well-known facts concerning locally convex spaces.

Definition 1.14 A *valued field* is a field $\mathbb{K}$, together with an absolute value $|.|\colon \mathbb{K} \to [0, \infty[$ (see [36]); we require furthermore that the absolute value be non-trivial (meaning that it gives rise to a non-discrete topology on $\mathbb{K}$). An *ultrametric field* is a valued field $(\mathbb{K}, |.|)$ whose absolute value satisfies the ultrametric inequality

$$|x + y| \leq \max\{|x|, |y|\} \quad \text{for all } x, y \in \mathbb{K}.$$

Locally compact, totally disconnected, non-discrete topological fields will be referred to as *local fields*. It is well known that every local field $\mathbb{K}$ admits an ultrametric absolute value defining its topology [34]. Fixing such an absolute value on $\mathbb{K}$, we can consider $\mathbb{K}$ as an ultrametric field.

Remark 1.15 Note that we do not require that valued fields (nor ultrametric fields) be complete (with respect to the metric induced by the absolute value). Of course, we are interested exclusively in complete valued fields – but none of our results will depend on completeness of $\mathbb{K}$.

Given a valued field $(\mathbb{K}, |.|)$, we can speak of *seminorms* on $\mathbb{K}$-vector spaces (as in [4, Ch. II, § 1, no. 1, Defn. 1]). If $(\mathbb{K}, |.|)$ is an ultrametric field, we can also speak of *ultrametric seminorms* (Bourbaki's "ultra-semi-norms"), satisfying the ultrametric inequality.

1.16 Recall that a topological vector space E over an ultrametric field $\mathbb{K}$ is called *locally convex* if every 0-neighborhood of E contains an open $\mathbb{O}$-submodule of E, where $\mathbb{O} := \{t \in \mathbb{K}\colon |t| \leq 1\}$ is the valuation ring of $\mathbb{K}$. Equivalently, E is locally convex if and only if its vector topology is defined by a family of ultrametric continuous seminorms $\gamma\colon E \to [0,\infty[$ on E (cf. [29] for more information, or also the discussions of Minkowski functionals given below). Let $\mathbb{K}$ be a valued field. We call a topological $\mathbb{K}$-vector space *polynormed* if its vector topology is defined by a family of continuous seminorms (which need not be ultrametric seminorms if $\mathbb{K}$ is an ultrametric field).

1.17 A *Banach space* over a valued field $\mathbb{K}$ is a normed $\mathbb{K}$-vector space $(E, \|.\|)$ (see [4, Ch. I, §1, no. 2]) which is complete in the metric associated with $\|.\|$.

1.18 We shall not presume that normed spaces (nor Banach spaces) over ultrametric fields be ultrametric, unless saying so explicitly. For example, $\ell^1(\mathbb{Q}_p)$ is a non-ultrametric (and non-locally convex) Banach space over $\mathbb{Q}_p$.

We are mainly interested in mappings between polynormed (or even locally convex) spaces, but all of our results can be proved just as well for mappings on arbitrary topological vector spaces over valued fields, without (or with very little) additional effort. To make the general proofs look like those for polynormed vector spaces, we now introduce a convenient replacement for continuous seminorms, namely the more general concept of a *gauge*.

Definition 1.19 Let E be a topological vector space over a valued field $(\mathbb{K}, |.|)$. A *gauge on* E is an upper semicontinuous map $\gamma\colon E \to [0,\infty[$ (also written $\|.\|_\gamma := \gamma$) which satisfies $\gamma(tx) = |t|\gamma(x)$ for all $t \in \mathbb{K}$ and $x \in E$.

Remark 1.20 (a) The upper semicontinuity of a gauge $\gamma\colon E \to [0,\infty[$ means that $\gamma^{-1}([0,r[)$ is open in E, for each $r > 0$. This is equivalent to the following condition: For each $x \in E$ and net (x_α) in E that converges to x, we have

$$\limsup_\alpha \|x_\alpha\|_\gamma \;\leq\; \|x\|_\gamma\,. \tag{4}$$

(b) Every gauge is continuous at 0. Indeed: Since $\gamma(0) = 0$ and $\gamma \geq 0$ pointwise, this follows from the upper semicontinuity.

(c) Sums of gauges and non-negative multiples $r\gamma$ of gauges are gauges.

Remark 1.21 Typical examples of gauges are Minkowski functionals of balanced, open 0-neighborhoods. We recall: If E is a topological vector space over a valued field $\mathbb{K}$, then a subset $U \subseteq E$ is called *balanced* if $tU \subseteq U$ for all $t \in \mathbb{K}$ such that $|t| \leq 1$. It is easy to see that the filter of 0-neighborhoods of E has a basis of balanced, open 0-neighborhoods. For any balanced, open 0-neighborhood U, we define its *Minkowski functional* $\mu_U\colon E \to [0,\infty[$ via

$$\mu_U(x) \;:=\; \inf\{|t|\colon t \in \mathbb{K}^\times \text{ such that } x \in tU\}\,.$$

Clearly $\mu_U(sx) = |s|\mu_U(x)$ for all $x \in E$ and $s \in \mathbb{K}$. To see that μ_U is a gauge, it only remains to check its upper semicontinuity. To this end, let $x \in E$ and

$\varepsilon > 0$. There is $t \in \mathbb{K}^\times$ such that $x \in tU$ and $|t| \leq \mu_U(x) + \varepsilon$. Then tU is a neighborhood of x such that $\mu_U(y) \leq |t| \leq \mu_U(x) + \varepsilon$ for all $y \in tU$.

For later use, we observe that

$$\{x \in E \colon \mu_U(x) < 1\} \subseteq U \subseteq \{x \in E \colon \mu_U(x) \leq 1\} \subseteq tU \tag{5}$$

for each $t \in \mathbb{K}$ such that $|t| > 1$. Note that, in contrast to the familiar real case, tU on the right hand side of (5) cannot be replaced by the closure $\overline{U}$ in general, as the example $E := \mathbb{K} := \mathbb{C}_p$, $U := \{x \in \mathbb{C}_p \colon |x| < 1\}$ shows. In this case, U is closed, and it is a proper subset of $\{x \in \mathbb{C}_p \colon |x| \leq 1\}$.

Definition 1.22 Given a topological $\mathbb{K}$-vector space E, $x \in E$, $r > 0$ and a gauge γ on E, we set $B_r^\gamma(x) := \{y \in E \colon \|y - x\|_\gamma < r\}$ and $\overline{B}_r^\gamma(x) := \{y \in E \colon \|y - x\|_\gamma \leq r\}$. If $(E, \|.\|)$ is a normed space over $\mathbb{K}$ and the norm $\gamma := \|.\|$ is understood, we simply write $B_r^E(x) := B_r^\gamma(x)$ and $\overline{B}_r^E(x) := \overline{B}_r^\gamma(x)$. We abbreviate $B_r(x) := B_r^E(x)$ when no confusion is possible, and $B_r(x) := B_r^E(x)$.

Given a balanced, open 0-neighborhood $V \subseteq E$ and $t \in \mathbb{K}$ such that $|t| > 1$, the set $U := t^{-1}V$ is a balanced, open 0-neighborhood. The right hand side of (5) being V, we deduce that the balls $\overline{B}_1^\gamma(0)$ form a basis of 0-neighborhoods for E, if γ ranges through the set of all gauges on E.

Definition 1.23 Let E be a topological vector space over a valued field $\mathbb{K}$. A set Γ of gauges on E is called a *fundamental system of gauges* if

$$\{B_r^\gamma(0) \colon \gamma \in \Gamma, r \in \,]0, \infty[\}$$

is a basis for the filter of 0-neighborhoods in E.

It is useful to single out a simple argument which will be used repeatedly.

Lemma 1.24 *Let E be a vector space over a valued field $\mathbb{K}$ and $\gamma, \eta \colon E \to [0, \infty[$ be mappings such that $\gamma(tx) = |t|\gamma(x)$ and $\eta(tx) = |t|\eta(x)$ for all $x \in E$, $t \in \mathbb{K}$. We assume that there are $r, s > 0$ such that $B_s^\eta(0) \subseteq B_r^\gamma(0)$, using notation as in* Definition 1.22. *Then*

$$\gamma \leq rs^{-1}|a|^{-1}\eta$$

for each $a \in \mathbb{K}^\times$ such that $|a| < 1$.

Proof. Let $x \in E$. If $\eta(x) \neq 0$, pick $k \in \mathbb{Z}$ such that $|a|^{k+1} \leq s^{-1}\eta(x) < |a|^k$. If $\eta(x) = 0$, pick any $k \in \mathbb{Z}$. Then $\eta(a^{-k}x) < s$ and thus $|a|^{-k}\gamma(x) = \gamma(a^{-k}x) < r$, whence

$$\gamma(x) \leq r|a|^k. \tag{6}$$

If $\eta(x) > 0$, then the right hand side of (6) is $\leq rs^{-1}|a|^{-1}\eta(x)$, as required. If $\eta(x) = 0$, we can choose k arbitrarily large, and thus (6) entails that $\gamma(x) = 0 \leq rs^{-1}|a|^{-1}\eta(x)$ also in this case. □

Remark 1.25 If γ is a gauge on E, $a \in \mathbb{K}^\times$ such that $|a| < 1$, and $U := B^\gamma_{|a|}(0)$, then $B^{\mu_U}_1(0) \subseteq U$ by (5), whence $\gamma \leq \mu_U$ by Lemma 1.24. Therefore Minkowski functionals form a fundamental system of gauges in particular.

Remark 1.26 If Γ is a fundamental system of gauges for E, and γ is a gauge on E, then there exists a gauge $\eta \in \Gamma$ and $c > 0$ such that $\gamma \leq c\eta$. In fact, there exists $\eta \in \Gamma$ and $r > 0$ such that $B^\eta_r(0) \subseteq B^\gamma_1(0)$. Then $\gamma \leq r^{-1}|a|^{-1}\eta$, by Lemma 1.24.

Lemma 1.27 *Let E be a topological vector space over a valued field $\mathbb{K}$, and Γ be a fundamental system of gauges on E. Then the following holds:*

(a) *The set of balls $\{B^\gamma_r(x)\colon \gamma \in \Gamma,\, r \in \,]0,\infty[\}$ is a basis for the filter of neighborhoods of x in E, and so is $\{\overline{B}^\gamma_r(x)\colon \gamma \in \Gamma,\, r \in \,]0,\infty[\}$. If $]0,\infty[\,\Gamma \subseteq \Gamma$, then also $\{B^\gamma_1(x)\colon \gamma \in \Gamma\}$ and $\{\overline{B}^\gamma_1(x)\colon \gamma \in \Gamma\}$ are bases.*

(b) *A map $f\colon X \to E$ from a topological space to E is continuous at $x \in X$ if and only if, for each gauge $\gamma \in \Gamma$ and $\varepsilon > 0$, there exists a neighborhood U of x in X such that $\|f(y) - f(x)\|_\gamma < \varepsilon$ for all $y \in U$. Or equivalently: $\|f(x_\alpha) - f(x)\|_\gamma \to 0$, for each $\gamma \in \Gamma$ and each net (x_α) in X that converges to x.*

(c) *Suppose that also E_1 is a topological $\mathbb{K}$-vector space, Γ_1 a fundamental set of gauges on E_1, and $\alpha\colon E \to E_1$ a linear map. Then α is continuous if and only if, for each $\gamma \in \Gamma_1$, there exists $\eta \in \Gamma$ and a constant $c \in [0,\infty[$ such that*

$$\|\alpha(x)\|_\gamma \;\leq\; c\|x\|_\eta \qquad \text{for all } x \in E.$$

If $[0,\infty[\,\Gamma \subseteq \Gamma$ and α is continuous, then η and c can always be chosen such that $c = 1$.

Proof. (a) Given $x \in E$, the map $E \to E$, $y \mapsto x + y$ is a homeomorphism. Hence, if Γ is a fundamental system of gauges on E, then $\{B^\gamma_r(x) = x + B^\gamma_r(0)\colon \gamma \in \Gamma,\, r \in \,]0,\infty[\}$ is a basis for the filter of neighborhoods of x in E. Similarly, so is $\{\overline{B}^\gamma_r(x) = x + B^\gamma_r(0)\colon \gamma \in \Gamma,\, r \in \,]0,\infty[\}$. Since $B^{\gamma/r}_1(x) = B^\gamma_r(x)$ and $\overline{B}^{\gamma/r}_1(x) = \overline{B}_r(x)$ for each $\gamma \in \Gamma$ and $r > 0$, the final assertions follow.

(b) As a consequence of (a), the map f is continuous at x if and only if $f^{-1}(B^\gamma_r(f(x))) = \{y \in X\colon \|f(y) - f(x)\|_\gamma < r\}$ is a neighborhood of x, for each $\gamma \in \Gamma$ and $r > 0$. Hence the first assertion holds. It also follows from (a) that $f(x_\alpha) \to f(x)$ if and only if $f(x_\alpha) \in B^\gamma_r(f(x))$ eventually for all $\gamma \in \Gamma$ and $r > 0$. Since $f(x_\alpha) \in B^\gamma_r(f(x))$ if and only if $\|f(x_\alpha) - f(x)\|_\gamma < r$, the second assertion follows.

(c) Suppose that, for each γ, a gauge η can be chosen as described. Then $\alpha(B^\eta_{c^{-1}r}(0)) \subseteq B^\gamma_r(0)$ for each $r > 0$, entailing that α is continuous at 0 and hence continuous. If α is continuous and γ is a gauge on E_1, then $U := \alpha^{-1}(B^\gamma_1(0)) =$

$B_1^{\gamma\circ\alpha}(0)$ is a balanced open 0-neighborhood in E. There exists $\eta \in \Gamma$ and $r > 0$ such that $B_r^\eta(0) \subseteq U$. Applying Lemma 1.24 to $\gamma \circ \alpha$ and η, we obtain $\gamma \circ \alpha \leq r^{-1}|a|^{-1}\eta = c\eta$ with $c := r^{-1}|a|^{-1}$. □

Definition 1.28 In the situation of Lemma 1.27 (c), we set

$$\|\alpha\|_{\gamma,\eta} := \min\{c \geq 0\colon \|\alpha(x)\|_\gamma \leq c\|x\|_\eta \text{ for all } x \in E\}\,.$$

Note that the triangle inequality need not hold for gauges. The following lemma provides a certain substitute.

Lemma 1.29 *If E is a topological vector space over a valued field $\mathbb{K}$ and $U, V \subseteq E$ are balanced open 0-neighborhoods such that $V + V \subseteq U$, then*

$$\mu_U(x+y) \;\leq\; \max\{\mu_V(x), \mu_V(y)\} \quad \textit{for all } x, y \in E. \tag{7}$$

As a consequence, for each gauge $\|.\|_\gamma$ on E, there is a gauge $\|.\|_\eta$ on E such that

$$\|x+y\|_\gamma \;\leq\; \max\{\|x\|_\eta, \|y\|_\eta\} \quad \textit{for all } x, y \in E. \tag{8}$$

Proof. Let $x, y \in E$. Given $\varepsilon > 0$, there exists $t \in \mathbb{K}^\times$ such that $|t| \leq \mu_V(x) + \varepsilon$ and $x \in tV$, and $s \in \mathbb{K}^\times$ such that $|s| \leq \mu_V(y) + \varepsilon$ and $y \in sV$. Assume that $|s| \leq |t|$ (the case $|s| > |t|$ is similar). Then $x, y \in tV$ and thus $x + y \in tV + tV = t(V + V) \subseteq tU$, entailing that $\mu_U(x + y) \leq |t| \leq \max\{\mu_V(x), \mu_V(y)\} + \varepsilon$. As $\varepsilon > 0$ was arbitrary, (7) follows.

Given γ, by Remark 1.25 there exists U such that $\gamma \leq \mu_U$. Choosing V as before we then have (8) with $\eta := \mu_V$. □

Recall that a subset $B \subseteq E$ of a topological vector space E over a valued field $\mathbb{K}$ is called *bounded* if, for each 0-neighborhood $U \subseteq E$, there exists $t \in \mathbb{K}^\times$ such that $B \subseteq tU$.

Definition 1.30 If E and F are topological vector spaces over a valued field $\mathbb{K}$, we equip the space $\mathcal{L}(E, F)$ of continuous $\mathbb{K}$-linear maps $E \to F$ with the topology of uniform convergence on bounded subsets of E. Thus, we equip $\mathcal{L}(E, F)$ with the unique vector topology which has the sets

$$[B, U] \;:=\; \{\alpha \in \mathcal{L}(E, F)\colon \alpha(B) \subseteq U\}$$

as a filter basis of 0-neighborhoods, where B ranges through the bounded subsets of E and U through the 0-neighborhoods of F (it easily follows from [4, Ch. I, § 1, no. 5, Prop. 4] that such a vector topology exists). As usual, we abbreviate $\mathcal{L}(E) := \mathcal{L}(E, E)$ and set

$$\mathrm{GL}(E) \;:=\; \mathcal{L}(E)^\times \;:=\; \{\alpha \in \mathcal{L}(E)\colon \exists \beta \in \mathcal{L}(E) \text{ s.t. } \alpha \circ \beta = \beta \circ \alpha = \mathrm{id}_E\}\,.$$

Remark 1.31 Note that the gauges

$$\|.\|_{\gamma,B}\colon \mathcal{L}(E,F)\to [0,\infty[\,, \quad \|\alpha\|_{\gamma,B}:=\sup\{\|\alpha(x)\|_\gamma\colon x\in B\}$$

define the vector topology on $\mathcal{L}(E,F)$, for B and γ ranging through the bounded subsets of E and the gauges on F, respectively. If F is polynormed (resp., locally convex), then also $\mathcal{L}(E,F)$ is polynormed (resp., locally convex), because $\|.\|_{\gamma,B}$ is a seminorm (resp., an ultrametric seminorm) if so is γ. If $(E,\|.\|_E)$ is normed and F is polynormed, then the vector topology on $\mathcal{L}(E,F)$ arises from the family of continuous seminorms $\|.\|_\gamma\colon \mathcal{L}(E,F)\to[0,\infty[$ defined for $\alpha\in\mathcal{L}(E,F)$ via

$$\|\alpha\|_\gamma := \sup\{\|\alpha(v)\|_\gamma\cdot\|v\|_E^{-1}\colon\ 0\neq v\in E\}\in[0,\infty[\tag{9}$$

where γ ranges through the continuous seminorms on F (see Lemma 1.32 below; cf. also [30, note on p. 59]). By definition, we have

$$\|\alpha(x)\|_\gamma \;\leq\; \|\alpha\|_\gamma\|x\|_E \qquad \text{for all } x\in E. \tag{10}$$

If both $(E,\|.\|_E)$ and $(F,\|.\|_F)$ are normed, then $\mathcal{L}(E,F)$ is normable; its vector topology arises from the *operator norm* $\|.\|:=\|.\|_\gamma$ defined in (9), with $\gamma:=\|.\|_F$. It is easy to see that the operator norm $\|.\|$ is ultrametric if the norm $\|.\|_F$ is ultrametric. If F is a Banach space here, then also $\mathcal{L}(E,F)$ is complete and hence a Banach space (this can be shown as in the real case). It easily follows from the definition of the operator norm on $\mathcal{L}(E)$ that $\|\alpha\circ\beta\|\leq\|\alpha\|\cdot\|\beta\|$ for all $\alpha,\beta\in\mathcal{L}(E)$.

The following observation is occasionally useful.

Lemma 1.32 *Let $\alpha\colon E\to F$ be a linear map, γ be a gauge on F and η be a gauge on E such that*

$$\|\alpha\|_{\gamma,B} \;:=\; \sup\{\|\alpha.v\|_\gamma\colon v\in B\}\;<\infty\,,$$

where $B:=B_1^\eta(0)$. Then

$$\|\alpha\|_{\gamma,B} \;\leq\; \|\alpha\|_{\gamma,\eta} \;\leq\; |a|^{-1}\|\alpha\|_{\gamma,B} \tag{11}$$

for each $a\in\mathbb{K}^\times$ such that $|a|<1$. If $|\mathbb{K}^\times|$ is dense in $[0,\infty[$, then $\|\alpha\|_{\gamma,B}=\|\alpha\|_{\gamma,\eta}$.

Proof. For each $x\in B$, we have $\|\alpha(x)\|_\gamma\leq\|\alpha\|_{\gamma,\eta}\|x\|_\eta\leq\|\alpha\|_{\gamma,\eta}$ and thus $\|\alpha\|_{\gamma,B}:=\sup\{\alpha(x)\|_\gamma\colon x\in B\}\leq\|\alpha\|_{\gamma,\eta}$. Hence the first half of (11) holds. To prove the second, let $r>\|\alpha\|_{\gamma,B}$. Then $\alpha(B)\subseteq B_r^\gamma(0)$ and thus $B\subseteq B_r^{\gamma\circ\alpha}(0)$. Then $\gamma\circ\alpha\leq r^{-1}|a|^{-1}\eta$, by Lemma 1.24, entailing that $\|\alpha\|_{\gamma,\eta}\leq r|a|^{-1}$. Letting $r\to\|\alpha\|_{\gamma,B}$, the second half of (11) follows. The other assertions are now immediate. □

Note that the notation $\|.\|_{\gamma,B}$ is abused in Lemma 1.32 for a set B which need not be bounded (unless E is normed).

If E is a Banach space, then $\mathrm{GL}(E)$ is open in $\mathcal{L}(E)$ and is a topological group.

Proposition 1.33 *If $(E, \|.\|)$ is a Banach space, then* $\mathrm{GL}(E)$ *is open in* $\mathcal{L}(E)$ *and the inversion map* $\iota\colon \mathrm{GL}(E) \to \mathrm{GL}(E)$, $\iota(\alpha) := \alpha^{-1}$ *is continuous. For each* $\alpha \in \mathcal{L}(E)$ *with operator norm* $\|\alpha\| < 1$, *we have* $\mathrm{id}_E - \alpha \in \mathrm{GL}(E)$,

$$(\mathrm{id}_E - \alpha)^{-1} \;=\; \sum_{k=0}^{\infty} \alpha^k\,, \tag{12}$$

and

$$\|(\mathrm{id}_E - \alpha)^{-1}\| \;\leq\; \frac{1}{1 - \|\alpha\|}\,. \tag{13}$$

If $\mathbb{K}$ *is an ultrametric here and the norm* $\|.\|$ *on* E *is ultrametric, then the set*

$$\Omega \;:=\; \{\mathrm{id}_E - \alpha \colon \alpha \in \mathcal{L}(E) \text{ such that } \|\alpha\| < 1\}$$

is an open subgroup of $\mathrm{GL}(E)$ *and each* $\alpha \in \Omega$ *is an isometry. As a consequence, in the ultrametric case also the set*

$$\mathrm{Iso}(E) \;:=\; \{\alpha \in \mathrm{GL}(E) \colon (\forall u \in E)\; \|\alpha(u)\| = \|u\|\}$$

of surjective linear isometries is an open subgroup of $\mathrm{GL}(E)$.

Proof. Since $\|\alpha^k\| \leq \|\alpha\|^k$ and $\mathcal{L}(E)$ is complete, the Neumann series $\sum_{k=0}^{\infty} \alpha^k$ converges for all $\alpha \in \mathcal{L}(E)$ such that $\|\alpha\| < 1$. Then $(\mathrm{id}_E - \alpha)\sum_{k=0}^{\infty} \alpha^k = \mathrm{id}_E = \big(\sum_{k=0}^{\infty} \alpha^k\big)(\mathrm{id}_E - \alpha)$, showing that (12) holds. The convergence of the Neumann series being uniform on the set $\{\alpha \in \mathcal{L}(E) \colon \|\alpha\| < \frac{1}{2}\}$, we see that ι is continuous on an identity neighborhood and hence continuous (since $\mathrm{GL}(E)$ is a topological monoid). The calculation

$$\|(\mathrm{id}_E - \alpha)^{-1}\| \;=\; \Big\| \sum_{k=0}^{\infty} \alpha^k \Big\| \;\leq\; \sum_{k=0}^{\infty} \|\alpha\|^k \;=\; \frac{1}{1 - \|\alpha\|}$$

establishes (11). In the ultrametric case, given $\alpha \in \mathcal{L}(E)$ with $\|\alpha\| < 1$ we have $\|\alpha(x)\| < \|x\|$ and hence $\|(\mathrm{id}_E - \alpha).x\| = \|x\|$ for all $0 \neq x \in E$, whence $\mathrm{id}_E - \alpha$ is an isometry. Furthermore, $\mathrm{id}_E - \alpha$ is invertible by the preceding, with inverse $(\mathrm{id}_E - \alpha)^{-1} = \mathrm{id}_E - (-\sum_{k=1}^{\infty} \alpha^k) \in \Omega$ as $\|-\sum_{k=1}^{\infty} \alpha^k\| \leq \max\{\|\alpha^k\| \colon k \in \mathbb{N}\} < 1$. Given $\mathrm{id}_E - \alpha, \mathrm{id}_E - \beta \in \Omega$, we have $(\mathrm{id}_E - \alpha) \circ (\mathrm{id}_E - \beta) = \mathrm{id}_E - (\alpha + \beta - \alpha \circ \beta) \in \Omega$. Thus Ω is an open subgroup of $\mathrm{GL}(E)$. Since $\Omega \subseteq \mathrm{Iso}(E)$, also $\mathrm{Iso}(E)$ is an open subgroup. □

Actually, $\mathrm{GL}(E)$ is a Lie group, by [13, Proposition 2.2].

Lemma 1.34 *Let* E_1, E_2 *and* F *be topological vector spaces over a valued field and* $\lambda\colon E_1 \to E_2$ *be a continuous linear map. Then the mappings*

$$\mathcal{L}(\lambda, F)\colon \mathcal{L}(E_2, F) \to \mathcal{L}(E_1, F)\,, \quad \alpha \mapsto \alpha \circ \lambda$$

and

$$\mathcal{L}(F, \lambda)\colon \mathcal{L}(F, E_1) \to \mathcal{L}(F, E_2)\,, \quad \alpha \mapsto \lambda \circ \alpha$$

are continuous and linear.

Proof. Clearly both maps are linear. If $B \subseteq E_1$ is a bounded set and $U \subseteq F$ a 0-neighborhood, then $\lambda(B) \subseteq E_2$ is bounded and $\mathcal{L}(\lambda, F)(\lfloor \lambda(B), U \rfloor) \subseteq \lfloor B, U \rfloor$. Hence $\mathcal{L}(\lambda, F)$ is continuous at 0 and hence continuous.

If $B \subseteq F$ is bounded and $U \subseteq E_2$ a 0-neighborhood, then $\lambda^{-1}(U)$ is a 0-neighborhood in E_1 and $\mathcal{L}(F, \lambda)(\lfloor B, \lambda^{-1}(U) \rfloor) \subseteq \lfloor B, U \rfloor$. Hence $\mathcal{L}(F, \lambda)$ is continuous at 0 and hence continuous. □

2 Lipschitz continuous and Lipschitz differentiable mappings

In this section, we set up our terminology and prove basic facts concerning Lipschitz conditions, Lipschitz continuity and Lipschitz differentiability.

Lipschitz conditions and Lipschitz continuity

We first consider functions between normed spaces that satisfy a global Lipschitz condition.

Definition 2.1 Let E and F be normed spaces over a valued field $\mathbb{K}$ and $U \subseteq E$ be a subset. We say that a map $f: U \to F$ is *Lipschitz* if there exists $L \in [0, \infty[$ such that

$$\|f(x) - f(y)\| \leq L \|x - y\| \qquad \text{for all } x, y \in U. \tag{14}$$

In this case, we define

$$\operatorname{Lip}(f) := \sup \left\{ \frac{\|f(x) - f(y)\|}{\|x - y\|} : x, y \in U,\, x \neq y \right\} \in [0, \infty[\,. \tag{15}$$

Thus $\operatorname{Lip}(f)$ is the smallest possible choice for the Lipschitz constant L.

Lemma 2.2 *Let E and F be normed spaces over a valued field $\mathbb{K}$, $U \subseteq E$ be a subset with dense interior, and $f: U \to E$ be a mapping which is C^1 and Lipschitz. Then $\|f'(x)\| \leq \operatorname{Lip}(f)$ for each $x \in U$.*

Proof. Given $x \in U^0$ and $y \in E$, we have

$$\begin{aligned} \|f'(x).y\| &= \lim_{t \to 0} \|t^{-1}(f(x + ty) - f(x))\| = \lim_{t \to 0} |t|^{-1} \|f(x + ty) - f(x)\| \\ &\leq \operatorname{Lip}(f)\|y\| \end{aligned}$$

because $\|f(x + ty) - f(x)\| \leq \operatorname{Lip}(f)\|ty\| = |t| \operatorname{Lip}(f)\|y\|$. Hence $\|f'(x).y\| = \|df(x, y)\| \leq \operatorname{Lip}(f)\|y\|$ for all $x \in U$, because df is continuous and U^0 is dense in U. As a consequence, $\|f'(x)\| = \sup\{\|f'(x).y\|/\|y\| : 0 \neq y \in E\} \leq \operatorname{Lip}(f)$. □

While the Lipschitz maps just introduced satisfy a quite restrictive condition, we shall use the term "Lipschitz continuity" for a much weaker property, which amounts to a local Lipschitz condition in the case of mappings between normed spaces.

Definition 2.3 Let E and F be topological vector spaces over a valued field $\mathbb{K}$, and $U \subseteq E$ be a subset. A map $f\colon U \to F$ is called *Lipschitz continuous* if, for every $x_0 \in U$ and gauge γ on F, there exists a gauge η on E and $\delta > 0$ such that

$$\|f(y) - f(x)\|_\gamma \;\leq\; \|y - x\|_\eta \quad \text{for all } x, y \in B^\eta_\delta(x_0) \cap U.$$

After replacing η with a suitable multiple, we may always assume that $\delta = 1$. If U is open, we may also assume that $B^\eta_\delta(x_0) \subseteq U$, whenever this is convenient.

Remark 2.4 For example, every continuous linear map is Lipschitz continuous, by Lemma 1.27 (c).

Lemma 2.5 *Let E, F and H be topological vector spaces over a valued field $\mathbb{K}$, $U \subseteq E$ and $V \subseteq F$ be subsets, and $f\colon U \to V \subseteq F$ and $g\colon V \to H$ be mappings. Then the following holds:*

(a) *If f is Lipschitz continuous then f is continuous.*

(b) *If f and g are Lipschitz continuous, then $g \circ f$ is Lipschitz continuous.*

(c) *If U has dense interior and f is C^1, then f is Lipschitz continuous.*

(d) *If f is Lipschitz continuous and both E and F are Banach spaces, then every point $x \in U$ has a neighborhood W in U such that $f|_W$ is Lipschitz.*

Proof. (a) Given $x_0 \in U$ and a gauge γ on F, choose a gauge η on E and $\delta \in \,]0,1]$ as in Definition 2.3. Then $f(U \cap B^\eta_\delta(x_0)) \subseteq \overline{B}^\gamma_\delta(f(x_0)) \subseteq \overline{B}^\gamma_1(f(x_0))$, whence f is continuous at x_0 by Lemma 1.27 (a).

(b) Let $x \in U$. Given a gauge ζ on H, there exists a gauge γ on F and $\theta > 0$ such that $\|g(z) - g(y)\|_\zeta \leq \|z - y\|_\gamma$ for all $y, z \in V \cap B^\gamma_\theta(f(x))$. There exists a gauge η on E and $\delta \in \,]0,\theta]$ such that $\|f(z) - f(y)\|_\gamma \leq \|z - y\|_\eta$ for all $z, y \in U \cap B^\eta_\delta(x)$. Then $f(y) \in V \cap B^\gamma_\theta(f(x))$ for all $y \in B^\eta_\delta(x)$ and thus $\|g(f(z)) - g(f(y))\|_\zeta \leq \|f(z) - f(y)\|_\gamma \leq \|z - y\|_\eta$ for all $z, y \in B^\eta_\delta(x)$.

(c) We use the first order Taylor expansion

$$f(x + ty) - f(x) \;=\; t\,df(x, y) + tR_1(x, y, t)$$

of the C^1-map $f\colon E \supseteq U \to F$ (cf. [2, Theorem 5.1]). Here $R_1\colon U^{[1]} \to F$ is a continuous map and

$$R_1(x, y, 1) = tR_1(x, t^{-1}y, t) \text{ for } t \in \mathbb{K}^\times \text{ and } (x, y) \in U \times E \text{ such that } x + y \in U.$$

Fix $x_0 \in U$. Let γ be a gauge on F. Pick $a \in \mathbb{K}^\times$ such that $|a| < 1$, and a gauge η on F such that $\|u + v\|_\gamma \leq \max\{\|u\|_\eta, \|v\|_\eta\}$ for all $u, v \in F$. Since $df(x_0, 0) = 0$, using the continuity of df we find a gauge ζ on E such that $\|df(x, y)\|_\eta \leq |a|$ for all $x \in B^\zeta_1(x_0) \cap U$ and $y \in B^\zeta_1(0)$, whence $\|df(x, y)\|_\eta \leq \|y\|_\zeta$ for all $x \in B^\zeta_1(x_0) \cap U$ and $y \in E$ (cf. Lemma 1.32). Since $R_1(x_0, 0, 0) =$

0, we find a gauge σ on E and $r \in]0,1]$ such that $\|R_1(x,y,t)\|_\eta \leq 1$ for all $x \in B^\sigma_r(x_0)$, $y \in B^\sigma_r(0)$ and $t \in B_r(0) \subseteq \mathbb{K}$ such that $(x,y,t) \in U^{[1]}$; we may assume that $\sigma \geq \zeta$. Let τ be a gauge on E such that $\tau \geq |a|^{-1}r^{-1}\sigma$ and $\|u+v\|_\sigma \leq \max\{\|u\|_\tau, \|v\|_\tau\}$ for all $u,v \in E$. Define $\delta := \frac{1}{2}r^2|a|$. Given $x,y \in B^\tau_\delta(x_0) \cap U$, set $z := y - x$. If $\|z\|_\sigma > 0$, there is $k \in \mathbb{Z}$ such that $|a|^{k+1} \leq r^{-1}\|z\|_\sigma < |a|^k$. Then $\|a^{-k}z\|_\sigma < r$ and $|a^k| \leq |a|^{-1}r^{-1}\|z\|_\sigma \leq |a|^{-1}r^{-1}\delta < r$, whence $\|R_1(x,z,1)\|_\eta = |a^k|\,\|R_1(x,a^{-k}z,a^k)\|_\eta \leq |a^k| \leq |a|^{-1}r^{-1}\|z\|_\sigma \leq \|z\|_\tau$ and thus $\|f(y)-f(x)\|_\gamma = \|f(x+z)-f(x)\|_\gamma = \|df(x,z)+R_1(x,z,1)\|_\gamma \leq \max\big\{\|df(x,z)\|_\eta, \|R_1(x,z,1)\|_\eta\big\} \leq \|z\|_\tau$. Hence

$$\|f(y)-f(x)\|_\gamma \;\leq\; \|y-x\|_\tau\,. \tag{16}$$

If $\|z\|_\sigma = 0$, given $\varepsilon > 0$ pick $t \in \mathbb{K}^\times$ such that $|t| < \min\{r,\varepsilon\}$. Then $\|df(x,z)\|_\eta = 0$ and $\|R_1(x,z,1)\|_\eta = |t|\,\|R_1(x,t^{-1}z,t)\|_\eta \leq |t| \leq \varepsilon$, whence $\|R_1(x,z,1)\|_\eta = 0$ (as ε was arbitrary). Thus (16) also holds if $\|z\|_\sigma = 0$.

(d) Let $\|.\|_E$ and $\|.\|_F$ be the norms on the normed spaces E and F, respectively. Given $x_0 \in U$, by Lipschitz continuity there exists a neighborhood $W \subseteq U$ of x_0 and a gauge γ on E such that $\|f(z)-f(y)\|_F \leq \|z-y\|_\gamma$ for all $z,y \in W$. Since $\{\|.\|_E\}$ is a fundamental system of gauges for E, Remark 1.26 provides $L > 0$ such that $\gamma \leq L\|.\|_E$. Thus $\|f(z)-f(y)\|_F \leq L\|z-y\|_E$ for all $z,y \in W$, showing that $f|_W$ is Lipschitz. □

Lipschitz differentiable maps

We now strengthen the C^k-property by imposing Lipschitz continuity of the extended difference quotient maps. The LC^k-maps so obtained are valuable, for example, in the context of p-adic differential equations. They are also used in [14] to study the parameter dependence of stable manifolds.

Definition 2.6 Let $\mathbb{K}$ be a valued field, E and F be topological $\mathbb{K}$-vector spaces, $U \subseteq E$ be a subset with dense interior, and $f\colon U \to F$ be a mapping. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. We say that f is *k times Lipschitz differentiable* (or an *LC^k-map*) if f is C^k and $f^{[j]}\colon U^{[j]} \to F$ is Lipschitz continuous for all $j \in \mathbb{N}_0$ such that $j \leq k$ (where $f^{[0]} := f$).

Note that the Lipschitz continuity of $f^{[j]}$ is automatic for $j < k$, by Lemma 2.5 (c). In particular, f is LC^∞ if and only if f is C^∞. Also note that LC^0-maps are precisely the Lipschitz continuous maps (on subsets with dense interior).

Proposition 2.7 *Let $\mathbb{K}$ be a valued field, E, F and H be topological $\mathbb{K}$-vector spaces, $U \subseteq E$ and $V \subseteq F$ be subsets with dense interior, and $f\colon U \to V$ as well as $g\colon V \to H$ be LC^k-maps. Then also $g \circ f\colon U \to H$ is LC^k.*

Proof. In [2], so-called "$\mathcal{C}^0$-concepts" were introduced, dealing (in particular) with mappings between open subsets of Hausdorff topological vector spaces. As mentioned before, this idea can directly be adapted to mappings between subsets with dense interior (cf. also Appendix A). It is clear that Lipschitz continuous maps between subsets with dense interior of Hausdorff topological $\mathbb{K}$-vector spaces define a $\mathcal{C}^0$-concept, and that LC^k-maps are precisely the $\mathcal{C}^k$-maps with respect to this $\mathcal{C}^0$-concept. We now use that compositions of $\mathcal{C}^k$-maps are always $\mathcal{C}^k$ (cf. [2, Proposition 4.5]). □

Lemma 2.8 *Let E and F be topological vector spaces over a complete valued field $\mathbb{K}$ and $f\colon U \to F$ be an LC^1-map on a subset $U \subseteq E$ with dense interior. Then the following holds:*

(a) *The map $f'\colon U \to \mathcal{L}(E,F)$, $x \mapsto f'(x) = df(x,\bullet)$ is Lipschitz continuous and hence continuous.*

(b) *For each gauge γ on F and $x \in U$, there exists a gauge ξ on E such that*

$$\|f'(z) - f'(y)\|_{\gamma,\xi} \;\leq\; \|z-y\|_\xi \quad \text{for all } z,y \in B_1^\xi(x) \cap U. \tag{17}$$

Proof. (a) Given $x \in U$, let γ be a gauge on F and $B \subseteq E$ be a bounded set. Since $f^{[1]}$ is Lipschitz continuous, also its partial map $df\colon U \times E \to F$ is Lipschitz continuous. As a consequence, there exists a gauge η on E such that

$$\|df(z,v) - df(y,u)\|_\gamma \leq \max\{\|z-y\|_\eta, \|v-u\|_\eta\} \tag{18}$$

for all (y,u), $(z,v) \in (U \cap B_1^\eta(x)) \times B_1^\eta(0)$. By boundedness of B, there exists $t \in \mathbb{K}^\times$ such that $tB \subseteq B_1^\eta(0)$. We may assume that $|t| \leq 1$. Then also $\zeta := |t|^{-1}\eta$ is a gauge on E. Given $z, y \in U \cap B_1^\zeta(x) \subseteq B_1^\eta(x)$ and $u \in B$, we then have $\|f'(z).u - f'(y).u\|_\gamma = |t^{-1}|\,\|f'(z).tu - f'(y).tu\|_\gamma \leq |t^{-1}|\,\|z-y\|_\eta = \|z-y\|_\zeta$. Hence $\|f'(z) - f'(y)\|_{\gamma,B} \leq \|z-y\|_\zeta$ for all $z,y \in U \cap B_1^\zeta(x)$. As the gauges $\|.\|_{\gamma,B}$ form a fundamental system, we deduce that f' is Lipschitz continuous.

(b) Taking $u = v$ in (18), we see that

$$\|df(z,u) - df(y,u)\|_\gamma \;\leq\; \|z-y\|_\eta \tag{19}$$

for all $z,y \in U \cap B_1^\eta(x)$ and all $u \in B_1^\eta(0)$. Pick $a \in \mathbb{K}^\times$ such that $|a| < 1$. Then (17) is satisfied with $\xi := |a|^{-1}\eta$. This follows from (19) and Lemma 1.32. □

The following technical lemma will be needed when we consider families of contractions of a Banach space and study differentiable dependence of fixed points on parameters.

Lemma 2.9 *Let $\mathbb{K}$ be a valued field, E and F be topological vector spaces over $\mathbb{K}$ and $f\colon U \to F$ be an LC^1-map on a subset $U \subseteq E$ with dense interior. Let $x_0 \in U$, $y_0 \in E$ and γ be a gauge on F. Then there exists a gauge η on E with the following property: For each $\varepsilon > 0$, there is $\delta > 0$ such that*

$$\|f^{[1]}(x,y_1,t) - f^{[1]}(x,y_2,t) - f'(x_0).(y_1-y_2)\|_\gamma \;\leq\; \varepsilon\,\|y_1-y_2\|_\eta \tag{20}$$

for all elements $x \in B^\eta_\delta(x_0) \cap U$, $y_1, y_2 \in B^\eta_\delta(y_0)$ *and* $t \in B_\delta(0) \subseteq \mathbb{K}$ *satisfying* $(x, y_1, t), (x, y_2, t) \in U^{[1]}$.

Proof. We shall prove the following stronger assertion, from which the Lemma readily follows by taking $\delta := \min\{\rho, \frac{\varepsilon}{C}\}$:

2.10 *There exists a gauge* η *on* E *and* $\rho, C > 0$ *such that*

$$\begin{aligned} &\|f^{[1]}(x, y_1, t) - f^{[1]}(x, y_2, t) - f'(x_0).(y_1 - y_2)\|_\gamma \\ &\quad\leq\quad C \cdot \max\big\{\|y_1 - y_0\|_\eta, \|y_2 - y_0\|_\eta, \|x - x_0\|_\eta, |t|\big\} \cdot \|y_1 - y_2\|_\eta \end{aligned} \tag{21}$$

for all elements $x \in B^\eta_\rho(x_0) \cap U$, $y_1, y_2 \in B^\eta_\rho(y_0)$ *and* $t \in B_\rho(0) \subseteq \mathbb{K}$ *satisfying* $(x, y_1, t), (x, y_2, t) \in U^{[1]}$.

Let ζ be a gauge on F such that $\|u+v\|_\gamma \leq \max\{\|u\|_\zeta, \|v\|_\zeta\}$. Given any elements $x \in U$, $y_1, y_2 \in E$ and $t \in \mathbb{K}$ such that $(x, y_1, t), (x, y_2, t) \in U^{[1]}$, we have

$$\begin{aligned} &\|f^{[1]}(x, y_1, t) - f^{[1]}(x, y_2, t) - f'(x_0).(y_1 - y_2)\|_\gamma \\ &\quad=\quad \|f^{[1]}(x + ty_2, y_1 - y_2, t) - f'(x_0).(y_1 - y_2)\|_\gamma \\ &\quad=\quad \|f^{[1]}(x + ty_2, y_1 - y_2, t) - f'(x + ty_2).(y_1 - y_2) \\ &\qquad\qquad +(f'(x + ty_2) - f'(x_0)).(y_1 - y_2)\|_\gamma \\ &\quad\leq\quad \max\big\{\|f^{[1]}(x + ty_2, y_1 - y_2, t) - f^{[1]}(x + ty_2, y_1 - y_2, 0)\|_\zeta, \\ &\qquad\qquad \|(f'(x + ty_2) - f'(x_0)).(y_1 - y_2)\|_\zeta\big\}, \end{aligned} \tag{22}$$

using Lemma 1.12 to obtain the first equality.

By Lemma 2.8 (b), there exists a gauge ξ on E such that

$$\|f'(x_1) - f'(x_2)\|_{\zeta,\xi} \leq \|x_1 - x_2\|_\xi \quad \text{for all } x_1, x_2 \in B^\xi_1(x_0) \cap U. \tag{23}$$

There exists $r \in \,]0, 1]$ and a gauge κ on E such that $\|u+v\|_\xi \leq \max\{\|u\|_\kappa, \|v\|_\kappa\}$ for all $u, v \in E$, and such that $a + sb \in B^\xi_1(x_0)$ for all $a \in B^\kappa_r(x_0)$, $b \in B^\kappa_r(y_0)$ and $s \in B^{\mathbb{K}}_r(0)$. As κ is upper semicontinuous, after shrinking r we may assume that $\kappa(b) < \kappa(y_0) + 1 =: C_1$ for all $b \in B^\kappa_r(y_0)$. Then

$$\begin{aligned} &\|(f'(x + ty_2) - f'(x_0)).(y_1 - y_2)\|_\zeta \\ &\quad\leq\quad \|f'(x + ty_2) - f'(x_0)\|_{\zeta,\xi}\|y_1 - y_2\|_\xi \\ &\quad\leq\quad \|x - x_0 + ty_2\|_\xi\|y_1 - y_2\|_\xi \\ &\quad\leq\quad \max\{\|x - x_0\|_\kappa, \|y_2\|_\kappa|t|\} \cdot \|y_1 - y_2\|_\kappa \\ &\quad\leq\quad C_1 \max\{\|x - x_0\|_\kappa, |t|\} \cdot \|y_1 - y_2\|_\kappa \end{aligned} \tag{24}$$

whenever $x \in B^\kappa_r(x_0)$, $y_2 \in B^\kappa_r(y_0)$ and $t \in B^{\mathbb{K}}_r(0)$ in the above situation. Hence, we have established estimates of the desired form for the second term in (22).

By Lipschitz continuity of $f^{[1]}$, there exists a gauge $\theta \geq \kappa$ on E, $L > 0$ and $\sigma \in]0, r]$ such that

$$\|f^{[1]}(x_1, u_1, t_1) - f^{[1]}(x_2, u_2, t_2)\|_\zeta \leq L \max\{\|x_1 - x_2\|_\theta, \|u_1 - u_2\|_\theta, |t_1 - t_2|\}$$

for all $x_1, x_2 \in B^\theta_\sigma(x_0) \cap U$, $u_1, u_2 \in B^\theta_\sigma(0)$, and $t_1, t_2 \in B^{\mathbb{K}}_\sigma(0)$. We choose $a \in \mathbb{K}^\times$ such that $|a| < 1$. There exists a gauge η on E and $\rho \in]0, \sigma]$ such that $\|u + v\|_\theta \leq \max\{\|u\|_\eta, \|v\|_\eta\}$ for all $u, v \in E$ and

$$B^\eta_\rho(x_0) + B^{\mathbb{K}}_\rho(0) B^\eta_\rho(y_0) \subseteq B^\theta_\sigma(x_0)\,. \tag{25}$$

To see that η and ρ have the desired properties, let x, y_1, y_2 and t be as described in **2.10**. If $\|y_1 - y_2\|_\theta \neq 0$, we let $k \in \mathbb{N}_0$ be the unique element such that $|a|^{k+1} \leq \rho^{-1}\|y_1 - y_2\|_\theta < |a|^k$. If $\|y_1 - y_2\|_\theta = 0$, we let $k \in \mathbb{N}_0$ be arbitrary. In either case, we abbreviate $s := a^k$. Then $|s| \leq 1$. Using the difference quotient identity (3) from Lemma 1.13, we obtain the following estimates for the first term in (22):

$$\begin{aligned}
&\|f^{[1]}(x + ty_2, y_1 - y_2, t) - f^{[1]}(x + ty_2, y_1 - y_2, 0)\|_\zeta \\
&\quad = \ |s| \cdot \|f^{[1]}(x + ty_2, s^{-1}(y_1 - y_2), st) - f^{[1]}(x + ty_2, s^{-1}(y_1 - y_2), 0)\|_\zeta \\
&\quad \leq \ |s| L |st| \ = \ |s|^2 L |t| \ \leq \ |s|^2 L \ = \ |a|^{2k} L
\end{aligned} \tag{26}$$

because $\|s^{-1}(y_1 - y_2)\|_\theta < \rho$ by definition of s, and $x + ty_2 \in B^\theta_\sigma(x_0)$ by (25). If $\|y_1 - y_2\|_\theta \neq 0$, then

$$\begin{aligned}
|a|^{2k} L &\leq L |a|^{-2} \rho^{-2} \|y_1 - y_2\|^2_\theta \\
&\leq L |a|^{-2} \rho^{-2} \max\{\|y_1 - y_0\|_\eta, \|y_2 - y_0\|_\eta\} \cdot \|y_1 - y_2\|_\theta
\end{aligned}$$

and thus

$$\begin{aligned}
&\|f^{[1]}(x + ty_2, y_1 - y_2, t) - f^{[1]}(x + ty_2, y_1 - y_2, 0)\|_\zeta \\
&\quad \leq \ C_2 \max\{\|y_1 - y_0\|_\eta, \|y_2 - y_0\|_\eta\} \cdot \|y_1 - y_2\|_\eta
\end{aligned} \tag{27}$$

with $C_2 := L|a|^{-2}\rho^{-2}$. If $\|y_1 - y_2\|_\theta = 0$, then k can be chosen arbitrarily large in (26). Hence $\|f^{[1]}(x + ty_2, y_1 - y_2, t) - f^{[1]}(x + ty_2, y_1 - y_2, 0)\|_\zeta = 0$, and thus (27) also holds in this case. Since $\rho \leq r$ and $\eta \geq \kappa$, we deduce from (24) and (27) that (22) holds, with $C := \max\{C_1, C_2\}$. □

3 Strictly differentiable mappings

In this section, we discuss a second class of maps which are k-times differentiable in a stronger sense then mere C^k-maps, namely k times strictly differentiable maps (SC^k-maps). These SC^k-maps resemble to some extent the familiar continuously Fréchet differentiable mappings between real Banach spaces (for example, it is known that a map between real Banach spaces is once strictly differentiable if and only if it is continuously Fréchet differentiable, see [5]). The SC^k-property is weaker than the LC^k-property, but yet sufficiently strong for many purposes. For example, because mere C^1-maps need not be approximated well enough by their linearization around a given point, we shall not be able to prove inverse function theorems for C^1-maps in general (only for C^k-maps with $k \geq 2$, or in the presence of locally compactness). Strict differentiability, by contrast, provides exactly the quality of approximation needed to make the construction of inverse functions work.

Definition of strictly differentiable maps

Before we define strictly differentiable maps in general, let us consider the simpler special case of mappings between normed spaces.

Definition 3.1 Let $\mathbb{K}$ be a valued field, E and F be normed $\mathbb{K}$-vector spaces, $U \subseteq E$ be a subset with dense interior, and $f\colon U \to F$ be a map. Given $x \in U$, we say that f is *strictly differentiable at* x if there exists a continuous linear map $f'(x) \in \mathcal{L}(E,F)$ such that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f(z) - f(y) - f'(x).(z-y)\| \ < \ \varepsilon\,\|z-y\| \tag{28}$$

for all $y, z \in U$ such that $\|z - x\| < \delta$ and $\|y - x\| < \delta$. The map f is called *strictly differentiable* if it is strictly differentiable at each $x \in U$.

Remark 3.2 Clearly strict differentiability at x implies total differentiability at x in the conventional sense (fixing $z = x$). But it is a stronger condition, as we are even allowed to let two elements z and y pass to x simultaneously.

Remark 3.3 It is illuminating to interpret strict differentiability in terms of Lipschitz conditions. Writing $\tilde{f}\colon U \to F$, $\tilde{f}(y) := f(y) - f(x) - f'(x).(y-x)$, we have $f(y) = f(x) + f'(x).(y-x) + \tilde{f}(y)$, i.e., $\tilde{f}$ is the remainder term of the affine linear approximation (first order Taylor expansion) of f at x. Strict differentiability at x means that, for each $\varepsilon > 0$, we can find $\delta > 0$ such that $\tilde{f}|_{B_\delta(x)\cap U}$ is a Lipschitz map with $\mathrm{Lip}(\tilde{f}|_{B_\delta(x)\cap U}) \leq \varepsilon$. To see this, note that the left hand side of (28) can be written as $\|\tilde{f}(z) - \tilde{f}(y)\|$.

We now state the appropriate generalization of Definition 3.1 for mappings between arbitrary topological vector spaces over valued fields.

Definition 3.4 Let $\mathbb{K}$ be a valued field, E and F be topological $\mathbb{K}$-vector spaces, $U \subseteq E$ be a subset with dense interior, and $f\colon U \to F$ be a map. Given $x \in U$, we say that f is *strictly differentiable at* x if there exists a continuous linear map $f'(x) \in \mathcal{L}(E,F)$ such that, for each gauge $\|.\|_\gamma$ on F there exists a gauge $\|.\|_\eta$ on E with the following property: For each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f(z) - f(y) - f'(x).(z-y)\|_\gamma < \varepsilon\,\|z-y\|_\eta \tag{29}$$

for all $y, z \in U$ such that $\|z - x\|_\eta < \delta$ and $\|y - x\|_\eta < \delta$. The map f is called *strictly differentiable* if it is strictly differentiable at each $x \in U$.

Lemma 3.5 *$f'(x)$ is uniquely determined in the preceding situation.*

Proof. Suppose that also $\alpha \in \mathcal{L}(E, F)$ satisfies the property of $f'(x)$. If $\alpha \neq f'(x)$, we find $u \in E$ such that $f'(x).u \neq \alpha(u)$, whence $\|\alpha(u) - f'(x).u\|_\zeta \neq 0$ for some gauge $\|.\|_\zeta$ on F. By Lemma 1.29, there exists a gauge $\|.\|_\gamma$ on F such that $\|v + w\|_\zeta \leq \max\{\|v\|_\gamma, \|w\|_\gamma\}$ for all $v, w \in F$. Choose a gauge $\|.\|_\eta$ on E as in Definition 3.4, which works for both $f'(x)$ and α. Pick $\varepsilon > 0$ such that $\varepsilon\|u\|_\eta < \|\alpha(u) - f'(x).u\|_\zeta$ and let $\delta > 0$ be such that (29) and its analog with α in place of $f'(x)$ hold. Since U^0 is dense in E, we find $y \in U^0 \cap B^\eta_\delta(0)$. By openness of $U^0 \cap B^\eta_\delta(0)$, there exists $t \in \mathbb{K}^\times$ such that $y + tu \in U^0 \cap B^\eta_\delta(0)$. Then

$$\begin{aligned} |t| \cdot \|\alpha(u) - f'(x).u\|_\zeta &= \|\alpha(tu) - f'(x).tu\|_\zeta \\ &\leq \max\{\|f(y+tu) - f(y) - f'(x).tu)\|_\gamma, \\ &\qquad \|f(y+tu) - f(y) - \alpha(tu)\|_\gamma\} \\ &\leq \varepsilon\,\|tu\|_\eta = |t|\,\varepsilon\,\|u\|_\eta \end{aligned}$$

and thus $\|\alpha(u) - f'(x).u\|_\zeta \leq \varepsilon\|u\|_\eta$, contradicting our choice of ε. □

Remark 3.6 Of course, equivalently we can use gauges in any given fundamental systems Γ_E and Γ_F of gauges for E and F in Definition 3.4. In particular, if E (resp., F) is polynormed, we may replace $\|.\|_\eta$ (resp., $\|.\|_\gamma$) in the definition by a continuous seminorm. If $(E, \|.\|_E)$ is a normed space, we can always take $\|.\|_\eta = \|.\|_E$, and if $(F, \|.\|_F)$ is normed, we only need to test the condition for $\|.\|_\gamma = \|.\|_F$.

Strictly differentiable maps are C^1

We now verify that strict differentiability is a stronger differentiability property than being C^1.

A simple lemma by Bourbaki and Dieudonné [7] (see also [8, Exercise 3.2 A (b)]) will be useful:

Lemma 3.7 *Let X be a topological space, $X_0 \subseteq X$ be a dense subset and $f: X_0 \to Y$ be a continuous map to a regular topological space Y. Then f has a continuous extension to X if and only if f has a continuous extension to $X_0 \cup \{x\}$ for each $x \in X$.* □

Lemma 3.8 *Let $\mathbb{K}$ be a valued field, E and F be topological $\mathbb{K}$-vector spaces, $U \subseteq E$ be a subset with dense interior, and $f: U \to F$ be a strictly differentiable map. Then f is C^1, its strict differential is given by $f'(x) = df(x, \bullet)$ for all $x \in U$, and the map $f': U \to \mathcal{L}(E, F)$, $x \mapsto f'(x)$ is continuous.*

Proof. *f' is continuous at each $x \in U$.* To see this, given x let $B \subseteq E$ be bounded and $\|.\|_\zeta$ be a gauge on F. Choose a gauge $\|.\|_\gamma$ on F such that $\|u+v\|_\zeta \leq \max\{\|u\|_\gamma, \|v\|_\gamma\}$ for all $u, v \in F$. Pick $\|.\|_\eta$ as in Definition 3.4. The set B being bounded, $M_\eta := \sup \eta(B)$ is finite. Given $\varepsilon' > 0$, set $\varepsilon := \varepsilon'/(1+M_\eta)$. Choose $\delta > 0$ such that (29) holds. Let $y \in B^\eta_\delta(x) \cap U$ be arbitrary; we show that

$$\|f'(y) - f'(x)\|_{\zeta,B} \;\leq\; \varepsilon' \tag{30}$$

(with notation as in Remark 1.31). By strict differentiability of f at y, there exists a gauge $\xi \geq \eta$ on E such that, for each $\varepsilon'' > 0$, there exists $\rho > 0$ such that

$$\|f(w) - f(v) - f'(y).(w-v)\|_\gamma \;\leq\; \varepsilon''\|w-v\|_\xi$$

for all $v, w \in B^\xi_\rho(y) \cap U$. We set $M_\xi := \sup \xi(B) < \infty$ and choose ρ as before for $\varepsilon'' := \varepsilon'/(M_\xi + 1)$. There exists $z \in B^\eta_\delta(x) \cap B^\xi_\rho(y) \cap U^0$, and $t \in \mathbb{K}^\times$ such that $z + tB \subseteq B^\eta_\delta(x) \cap B^\xi_\rho(y) \cap U^0$. For $u \in B$, we then have

$$\begin{aligned}
\|f'(y).u - f'(x).u\|_\zeta &= t^{-1}\|f'(y).tu - f'(x).tu\|_\zeta \\
&\leq t^{-1}\max\big\{\|f(z+tu) - f(z) - f'(y).tu\|_\gamma, \\
&\qquad\qquad \|f(z+tu) - f(z) - f'(x).tu\|_\gamma\big\} \\
&\leq t^{-1}\max\big\{\varepsilon''\|tu\|_\xi, \varepsilon\|tu\|_\eta\big\} \\
&\leq \max\big\{\varepsilon'' M_\xi, \varepsilon M_\eta\big\} \;\leq\; \varepsilon'\,.
\end{aligned}$$

Hence (30) holds. The continuity of f' at x follows.

f is $C^1_\mathbb{K}$. Note first that f is continuous. In fact, given $x \in U$ and a gauge ζ on F, we let γ be a gauge on F such that $\|y+z\|_\zeta \leq \max\{\|y\|_\gamma, \|z\|_\gamma\}$ for all $y, z \in F$. Choose a gauge $\|.\|_\eta$ as in Definition 3.4. After replacing $\|.\|_\eta$ by a larger gauge, we may assume that $\|f'(x).w\|_\gamma \leq \|w\|_\eta$ for all $w \in E$. Given $\varepsilon > 0$, there is $\delta \in \,]0, \min\{\varepsilon, 1\}]$ such that (29) holds. Then

$$\begin{aligned}
\|f(y) - f(x)\|_\zeta &\leq \max\{\|f(y) - f(x) - f'(x).(y-x)\|_\gamma, \|f'(x).(y-x)\|_\gamma\} \\
&\leq \max\{\varepsilon\|y-x\|_\eta, \|y-x\|_\eta\} \;\leq\; \varepsilon
\end{aligned}$$

for all $y \in B^\eta_\delta(x) \cap U$. We deduce that f is continuous.

Next, let $W := \{(x,y,t) \in U^{[1]} \colon t \neq 0\}$. Define $g \colon U^{[1]} \to F$ via $g(x,y,t) := \frac{1}{t}(f(x+ty) - f(x))$ for $(x,y,t) \in W$, while we set $g(x,y,0) := f'(x).y$ for $(x,y) \in U \times E$. Then $g|_W$ is continuous since f is continuous. Hence, by Lemma 3.7, g will be continuous if we can show that $g(x_\alpha, y_\alpha, t_\alpha) \to g(x,y,0)$, for each net $((x_\alpha, y_\alpha, t_\alpha))_{\alpha \in I}$ in W which converges to some $(x,y,0) \in U^{[1]}$. To this end, given a gauge $\|.\|_\zeta$ on F let $\|.\|_\gamma$ be a gauge on F such that $\|y+z\|_\zeta \leq \max\{\|y\|_\gamma, \|z\|_\gamma\}$. Let $\|.\|_\eta$ be a gauge on E as in Definition 3.4. After replacing $\|.\|_\eta$ by a larger gauge if necessary, we may assume that $\|f'(x).w\|_\gamma \leq \|w\|_\eta$ for all $w \in E$. Given $\varepsilon > 0$, let $\delta \in \,]0, \varepsilon]$ be such that (29) holds. Since $x_\alpha \to x$

and $x_\alpha + t_\alpha y_\alpha \to x$, we have $x_\alpha \in U \cap B^\eta_\delta(x)$ and $x_\alpha + t_\alpha y_\alpha \in U \cap B^\eta_\delta(x)$ eventually. Furthermore, $\|y_\alpha\|_\eta \leq \|y\|_\eta + 1$ and $\|y_\alpha - y\|_\eta \leq \varepsilon$ eventually, as $y_\alpha \to y$. For any such α, we obtain

$$\begin{aligned}
&\|g(x_\alpha, y_\alpha, t_\alpha) - g(x, y, 0)\|_\zeta \\
&= \left\| \frac{f(x_\alpha + t_\alpha y_\alpha) - f(x_\alpha)}{t_\alpha} - f'(x).y \right\|_\zeta \\
&\leq \max\left\{ \frac{1}{|t_\alpha|} \|f(x_\alpha + t_\alpha y_\alpha) - f(x_\alpha) - f'(x).t_\alpha y_\alpha\|_\gamma,\, \|f'(x).(y_\alpha - y)\|_\gamma \right\} \\
&\leq \max\{\varepsilon \|y_\alpha\|_\eta,\, \|y_\alpha - y\|_\eta\} \;\leq\; \varepsilon(\|y\|_\eta + 1)\,,
\end{aligned}$$

which can be made arbitrarily small. Thus $g(x_\alpha, y_\alpha, t_\alpha) \to g(x, y, 0)$ in F, which completes the proof. □

LC^1-maps are strictly differentiable

We now show that every Lipschitz differentiable map is strictly differentiable. As a consequence, every C^2-map is strictly differentiable.

Proposition 3.9 *Let E and F be topological vector spaces over a valued field $\mathbb{K}$ and $f\colon U \to F$ be an LC^1-map on a subset $U \subseteq E$ with dense interior. Then f is SC^1, with strict differential $f'(x) = df(x, \bullet)$ at $x \in U$.*

Proof. Let $x_0 \in U$ and $\|.\|_\gamma$ be a gauge on F. Abbreviate $f'(x_0) := df(x_0, \bullet)$. Choose a gauge $\|.\|_\zeta$ on F such that $\|v+w\|_\gamma \leq \max\{\|v\|_\zeta, \|w\|_\zeta\}$ for all $v, w \in F$. Then

$$\begin{aligned}
\|f(z) - f(y) - f'(x_0).(z - y)\|_\gamma \;\leq\; & \max\big\{\|f(z) - f(y) - f'(y).(z - y)\|_\zeta, \\
& \|(f'(y) - f'(x_0)).(z - y)\|_\zeta\big\}. \qquad (31)
\end{aligned}$$

The first order Taylor remainder $R_1\colon U^{[1]} \to F$ (as in the proof of Lemma 2.5 (c)) being Lipschitz continuous (cf. proof of Proposition 2.7 and [2, Theorem 5.1]), there exists an open neighborhood $V \subseteq U^{[1]}$ of $(x_0, 0, 0)$, a gauge $\|.\|_\xi$ on E and $L > 0$ such that

$$\begin{aligned}
&\|R_1(x_1, y_1, t_1) - R_1(x_2, y_2, t_2)\|_\zeta \\
&\quad\leq\; L \max\{\|x_1 - x_2\|_\xi, \|y_1 - y_2\|_\xi, |t_1 - t_2|\} \qquad (32)
\end{aligned}$$

for all $(x_1, y_1, t_1), (x_2, y_2, t_2) \in V$. After replacing ξ by a larger gauge if necessary, there exists $r > 0$ such that $U^{[1]} \cap (B^\xi_r(x_0) \times B^\xi_r(0) \times B^{\mathbb{K}}_r(0)) \subseteq V$. After replacing ξ with a larger gauge, by Lemma 2.8 (b) we may assume that

$$\|f'(z) - f'(y)\|_{\zeta,\xi} \;\leq\; \|z - y\|_\xi \qquad \text{for all } z, y \in B^\xi_r(x_0).$$

There exists a gauge $\|.\|_\eta$ on E such that $\|u+v\|_\xi \leq \max\{\|u\|_\eta, \|v\|_\eta\}$ for all $u, v \in E$. Pick $a \in \mathbb{K}^\times$ such that $|a| < 1$. Given $\varepsilon > 0$, let $\rho := \min\{1, r, \varepsilon\}$. Choose $\delta \in \left]0, \min\{\rho^2|a|, \frac{|a|^2\varepsilon\rho^2}{L}\}\right[$. For all $z, y \in B^\eta_\delta(x_0) \cap U$, we then have

$$\begin{aligned}
\|(f'(y) - f'(x_0)).(z-y)\|_\zeta &\leq \|f'(y) - f'(x_0)\|_{\zeta,\xi}\|z-y\|_\xi \\
&\leq \|y - x_0\|_\xi \|z-y\|_\xi \;\leq\; \varepsilon\|z-y\|_\xi \,,
\end{aligned}$$

whence the second term of on the right hand side of (31) is no larger than ε. We have $\|z-y\|_\xi \leq \max\{\|z-x_0\|_\eta, \|y-x_0\|_\eta\} < \delta \leq \rho$. If $\|z-y\|_\xi \neq 0$, let $k \in \mathbb{N}_0$ be the unique element such that $|a|^{k+1} \leq \frac{\|z-y\|_\xi}{\rho} < |a|^k$. If $\|z-y\|_\xi = 0$, let $k \in \mathbb{N}_0$ be arbitrary. Set $t := a^k$. Then $|t| \leq |a|^{-1}\|z-y\|_\xi\,\rho^{-1} < |a|^{-1}\delta\rho^{-1} < \rho$. Hence

$$\begin{aligned}
\|f(z) - f(y) - f'(y).(z-y)\|_\zeta &= \|R_1(y, z-y, 1)\|_\zeta \;=\; |t| \cdot \left\|R_1\big(y, \tfrac{z-y}{t}, t\big)\right\|_\zeta \\
&= |t| \cdot \left\|R_1\big(y, \tfrac{z-y}{t}, t\big) - R_1\big(y, \tfrac{z-y}{t}, 0\big)\right\|_\zeta \\
&\leq L|t|^2 \,,
\end{aligned}$$

by (32). If $\|z-y\|_\xi \neq 0$, then

$$\begin{aligned}
L|t|^2 &\leq L|a|^{-2}\rho^{-2}\|z-y\|_\xi^2 \;\leq\; L|a|^{-2}\rho^{-2}\|z-y\|_\xi \max\{\|z-x_0\|_\eta, \|y-x_0\|_\eta\} \\
&\leq L|a|^{-2}\rho^{-2}\delta\|z-y\|_\xi \;\leq\; \varepsilon\|z-y\|_\eta \,.
\end{aligned}$$

If $\|z-y\|_\xi = 0$, then $L|t|^2 = L|a|^{2k}$, where k can be chosen arbitrarily large, and thus $\|f(z) - f(y) - f'(y).(z-y)\|_\zeta = 0 \leq \varepsilon\|z-y\|_\eta$ also in this case. Hence both terms on the right hand side of (31) are $\leq \varepsilon\|z-y\|_\eta$, and thus

$$\|f(z) - f(y) - f'(x_0).(z-y)\|_\gamma \;\leq\; \varepsilon\|z-y\|_\eta$$

for all $y, z \in B^\eta_\delta(x_0) \cap U$. Thus f is strictly differentiable at x_0, with $f'(x_0)$ as before. □

Corollary 3.10 *Let E and F be topological vector spaces over a valued field and $f\colon U \to F$ be a C^2-map on a subset $U \subseteq E$ with dense interior. Then f is LC^1 and hence also SC^1.*

Proof. Applying Lemma 2.5 (c) to f and $f^{[1]}$, we find that f is LC^1 and hence SC^1, by Proposition 3.9. □

Strictly differentiable maps on locally compact domains

For mappings on open subsets of finite-dimensional topological vector spaces over locally compact topological fields, the preceding result can be strengthened: such a map is C^1 if and only if it is strictly differentiable. More generally, this conclusion remains valid for mappings on locally compact domains.

Lemma 3.11 *Let $\mathbb{K}$ be a locally compact field, E be a finite-dimensional $\mathbb{K}$-vector space, F be a topological $\mathbb{K}$-vector space, $U \subseteq E$ be a locally compact subset with dense interior, and $f: U \to F$ be a map. Then f is C^1 if and only if f is strictly differentiable.*

Proof. We already know that every strictly differentiable map is C^1. Conversely, assume that f is C^1. Let $|.|$ be an absolute value on $\mathbb{K}$ defining its topology, $\|.\|$ be a norm on E, and $0 \neq a \in \mathbb{K}$ such that $|a| < 1$. Given $x_0 \in U$, let $V \subseteq U$ be an open neighborhood of x_0 with compact closure $\overline{V} \subseteq U$. Define a map $f': U \to \mathcal{L}(E,F)$ via $f'(x) := df(x, \bullet) = f^{[1]}(x, \bullet, 0)$. Given a gauge γ on F, choose gauges η and ζ on F such that $\|u+v\|_\gamma \leq \|u\|_\eta + \|v\|_\eta$ and $\|u_1 + \cdots + u_n\|_\eta \leq \sum_{i=1}^n \|u_i\|_\zeta$ for all $u, v, u_1, \ldots, u_n \in F$, where $n := \dim_{\mathbb{K}}(E)$. Given $\varepsilon > 0$, consider the continuous function

$$g: U^{[1]} \to F, \quad g(x,y,t) := f^{[1]}(x,y,t) - f^{[1]}(x,y,0)\,.$$

Then $\overline{V} \times \overline{B^E_{\frac{1}{|a|}}(0)} \times \{0\} \subseteq U \times E \times \{0\} \subseteq U^{[1]}$ is a compact subset on which g vanishes identically. Using a compactness argument, we find $\sigma > 0$ such that $\|g(x,y,t)\|_\eta < \frac{\varepsilon}{2}$ for all $(x,y,t) \in U^{[1]} \cap \big(\overline{V} \times \overline{B^E_{\frac{1}{|a|}}(0)} \times B^{\mathbb{K}}_\sigma(0)\big)$. Let $e_1, \ldots, e_n$ be a basis of E, and $e_1^*, \ldots, e_n^* \in E'$ be its dual basis.

Given $\alpha \in \mathcal{L}(E,F)$, for each $v \in E$ we have $\|\alpha(v)\|_\eta = \|\sum_{i=1}^n e_i^*(v)\alpha(e_i)\|_\eta \leq \sum_{i=1}^n |e_i^*(v)| \cdot \|\alpha(e_i)\|_\zeta \leq \sum_{i=1}^n \|e_i^*\| \cdot \|\alpha(e_i)\|_\zeta \|v\|$. Thus

$$\|\alpha\|_\eta := \sup\{\|\alpha(v)\|_\eta / \|v\| : 0 \neq v \in E\} \leq \sum_{i=1}^n \|e_i^*\| \cdot \|\alpha(e_i)\|_\zeta \tag{33}$$

for all $\alpha \in \mathcal{L}(E,F)$. Let $i \in \{1, \ldots, n\}$. The map $\overline{V} \to F$, $x \mapsto df(x, e_i)$ being uniformly continuous, we find $\delta_i > 0$ such that $\|df(y,e_i) - df(x,e_i)\|_\zeta < \frac{\varepsilon}{2n\|e_i^*\|}$ for all $x, y \in \overline{V}$ such that $\|x - y\| < \delta_i$. Define $\delta := \frac{1}{2}\min\{\sigma, \delta_1, \ldots, \delta_n\}$. By (33) and the choice of δ_i, we have $\|df(y, \bullet) - df(x, \bullet)\|_\eta < \frac{\varepsilon}{2}$ for all $x, y \in \overline{V}$ such that $\|x - y\| < 2\delta$.

Let $x, y, z \in V$ be given such that $y \neq z$, $\|y - x\| < \delta$, and $\|z - x\| < \delta$. There exists $k \in \mathbb{Z}$ such that $|a|^{k+1} \leq \|z - y\| < |a|^k$. We set $s := a^{k+1}$. Then $\|\frac{1}{s}(z-y)\| < \frac{1}{|a|}$, $|s| = |a|^{k+1} \leq \|z - y\| < 2\delta \leq \sigma$, and $\|z - y\| < 2\delta$. Thus

$$\begin{aligned}
&\frac{\|f(z) - f(y) - f'(x).(z-y)\|_\gamma}{\|z-y\|} \\
&\leq \frac{\|f(z) - f(y) - f'(y).(z-y)\|_\eta}{\|z-y\|} + \frac{\|(f'(y) - f'(x)).(z-y)\|_\eta}{\|z-y\|} \\
&< \tfrac{|s|}{\|z-y\|} \cdot \left\| \tfrac{1}{s}(f(z) - f(y)) - f'(y).\tfrac{1}{s}(z-y) \right\|_\eta + \tfrac{\varepsilon}{2} \\
&\leq \left\| f^{[1]}\left(y, \tfrac{1}{s}(z-y), s\right) - f^{[1]}\left(y, \tfrac{1}{s}(z-y), 0\right) \right\|_\eta + \tfrac{\varepsilon}{2} \\
&= \left\| g\left(y, \tfrac{1}{s}(z-y), s\right) \right\|_\eta + \tfrac{\varepsilon}{2} \leq \varepsilon\,.
\end{aligned}$$

Hence $f|_V$ is strictly differentiable at each $x \in V$. As the proof shows, given γ and ε we can even choose δ independently of $x \in V$. $\square$

Also a variant of Lemma 3.11 involving parameters will be needed later.

Lemma 3.12 *Let $\mathbb{K}$ be a locally compact topological field and $|.|$ be an absolute value on $\mathbb{K}$ defining its topology. Let E be a finite-dimensional normed $\mathbb{K}$-vector space, $U \subseteq E$ be a locally compact subset with dense interior, F be a topological $\mathbb{K}$-vector space, and P be a topological space. Let $f\colon P \times U \to F$ be a continuous map such that $f_p := f(p, \bullet)\colon U \to F$ is C^1 for all $p \in P$, and such that the map*

$$P \times U^{[1]} \to F, \quad (p, y) \mapsto (f_p)^{[1]}(y)$$

is continuous. Let $p \in P$ and $u \in U$ be given. Then, for every $\varepsilon > 0$ and gauge γ on F, there is a neighborhood Q of p in P and $\delta > 0$ such that

$$\|f_q(z) - f_q(y) - f_q'(u).(z-y)\|_\gamma < \varepsilon\|z-y\|$$

for all $q \in Q$ and $y, z \in B_\delta(u) \cap U$, where $f_q'(u) := d(f_q)(u, \bullet)$.

Proof. Given $\varepsilon > 0$ and γ, let η and ζ be as in the preceding proof. Pick $0 \neq a \in \mathbb{K}$ such that $|a| < 1$. Let $V \subseteq U$ be an open neighborhood of u with compact closure $\overline{V} \subseteq U$. Consider the continuous mapping

$$g\colon P \times U^{[1]} \to F, \quad g(q, x, y, t) := f_q^{[1]}(x, y, t) - f_q^{[1]}(x, y, 0)\,.$$

Then $\{p\} \times \overline{V} \times \overline{B^E_{\frac{1}{|a|}}(0)} \times \{0\} \subseteq P \times U \times E \times \{0\} \subseteq P \times U^{[1]}$ is a compact subset on which g vanishes identically. Using a compactness argument, we find $\sigma > 0$ and a neighborhood P_0 of p in P such that

$$\|g(q, x, y, t)\|_\eta < \tfrac{\varepsilon}{2} \quad \text{for all } (q, x, y, t) \in P_0 \times \big(U^{[1]} \cap (\overline{V} \times \overline{B^E_{\frac{1}{|a|}}(0)} \times B^{\mathbb{K}}_\sigma(0))\big).$$

Let $e_1, \ldots, e_n$ be a basis of E, and $e_1^*, \ldots, e_n^*$ be its dual basis. Using the compactness of $\overline{V}$, we find a neighborhood $Q \subseteq P_0$ of p and $\kappa > 0$ such that $\|df_q(z, e_i) - df_q(y, e_i)\|_\eta < \frac{\varepsilon}{2n\|e_i^*\|}$ for all $q \in Q$, $i \in \{1, \ldots, n\}$, and all $y, z \in \overline{V}$ such that $\|z - y\| < \kappa$. Let $\delta := \min\{\frac{\sigma}{2}, \frac{\kappa}{2}\}$. Re-using the estimates from the proof of Lemma 3.11, we see that the current assertion holds for Q and δ. $\square$

Recall that on a finite-dimensional vector space F over a topological field $\mathbb{K}$, of dimension n, there is a unique Hausdorff vector topology making F isomorphic to the direct product $\mathbb{K}^n$ as a topological vector space. It is called the *canonical vector topology on F*.

In the context of our current discussions, the following observation is useful.

Lemma 3.13 *Let E and H be topological vector spaces over a valued field $\mathbb{K}$, and F be a finite-dimensional $\mathbb{K}$-vector space, equipped with its canonical vector* *topology. Let $U \subseteq E$ and $V \subseteq F$ be subsets with dense interior and $f\colon U \times V \to H$ be a C^1-map. Then the map*

$$U \times V \to \mathcal{L}(F, H)\,, \quad (x, y) \mapsto f'_x(y) := df((x, y), (0, \bullet))$$

is continuous.

Proof. Let $e_1, \ldots, e_n$ be a basis of F and $e_1^*, \ldots, e_n^* \in F'$ be its dual basis, determined by $e_i^*(e_j) = \delta_{ij}$. Then

$$f'_x(y).w \;=\; \sum_{j=1}^{n} e_j^*(w)\, df(x, y, 0, e_j) \tag{34}$$

for all $x \in U$, $y \in V$ and $w \in F$. The assertion can now easily be derived. In fact, if $\|.\|_\gamma$ is a gauge on H and $B \subseteq F$ a bounded subset, choose a gauge $\|.\|_\zeta$ on H such that $\|u_1 + \cdots + u_n\|_\gamma \leq \max\{\|u_j\|_\zeta \colon j = 1, \ldots, n\}$. Pick $C \in \,]0, \infty[$ such that

$$|e_j^*(w)| \;\leq\; C \qquad \text{for all } w \in B \text{ and } j = 1, \ldots, n. \tag{35}$$

By continuity, for $(x_0, y_0) \in U \times V$ there exist neighborhoods $U_0 \subseteq U$ of x_0 and $V_0 \subseteq V$ of y_0 such that

$$\|df(x, y, 0, e_j) - df(x_0, y_0, 0, e_j)\|_\zeta \;\leq\; \frac{1}{C} \quad \text{for all } (x, y) \in U_0 \times V_0. \tag{36}$$

Combining (34), (35) and (36), we see that $\|(f'_x(y) - f'_{x_0}(y_0)).w\|_\zeta \leq 1$ for all $(x, y) \in U_0 \times V_0$ and $w \in B$, and thus $\|f'_x(y) - f'_{x_0}(y_0)\|_{\zeta, B} \leq 1$ (using the notation from Remark 1.31). As a consequence, the map under consideration is continuous at (x_0, y_0). □

Remark 3.14 In the situation of Lemma 3.12, we can achieve that furthermore

$$\|f_q(z) - f_q(y) - f'_p(u).(z - y)\|_\gamma < \varepsilon \|z - y\|$$

for all $q \in Q$ and $y, z \in B_\delta(u) \cap U$.

Indeed, given $\varepsilon > 0$ and a gauge $\|.\|_\gamma$ on F, let $\|.\|_\xi$ be a gauge on F such that $\|u + v\|_\gamma \leq \max\{\|u\|_\xi, \|v\|_\xi\}$ for all $u, v \in F$. By Lemma 3.12, there is a neighborhood Q of p in P and $\delta > 0$ such that

$$\|f_q(z) - f_q(y) - f'_q(u).(z - y)\|_\xi < \varepsilon \|z - y\|$$

for all $q \in Q$ and $y, z \in B_\delta(u) \cap U$. By Lemma 3.13, after shrinking Q we may assume that $\|f'_q(u) - f'_p(u)\|_{\xi, \nu} \leq \varepsilon$ for all $q \in Q$, where $\nu := \|.\|$ is the norm on F. Hence

$$\begin{aligned} &\|f_q(z) - f_q(y) - f'_p(u).(z - y)\|_\gamma \\ &\quad\leq\; \max\big\{\|f_q(z) - f_q(y) - f'_q(u).(z - y)\|_\xi, \|(f'_q(u) - f'_p(u)).(z - y)\|_\xi\big\} \\ &\quad\leq\; \max\big\{\varepsilon\|z - y\|, \|f'_q(u) - f'_p(u)\|_{\xi,\nu}\|z - y\|\big\} \;\leq\; \varepsilon\|z - y\|\,. \end{aligned}$$

Strict differentiability of higher order

We now define and discuss k times strictly differentiable mappings between subsets of topological vector spaces over valued fields.

Definition 3.15 Let $\mathbb{K}$ be a valued field, E and F be topological $\mathbb{K}$-vector spaces, and $U \subseteq E$ be a subset with dense interior. A map $f\colon U \to F$ is called an *SC^0-map* if it is continuous; it is called an *SC^1-map* is it is strictly differentiable (and hence C^1 in particular). Inductively, having defined SC^k-maps for some $k \in \mathbb{N}$ (which are C^k in particular), we call f an *SC^{k+1}-map* if it is an SC^k-map and the map $f^{[k]}\colon U^{[k]} \to F$ is SC^1. The map f is SC^∞ if it is an SC^k-map for all $k \in \mathbb{N}_0$.

Remark 3.16 In other words, f is SC^k if and only if f is C^k and $f^{[j]}\colon U^{[j]} \to F$ is strictly differentiable for all $j \in \mathbb{N}_0$ such that $j < k$. It follows from this and Remark 1.7 that f is SC^k if and only if f is SC^1 and $f^{[1]}$ is SC^{k-1}.

Remark 3.17 If $f\colon E \supseteq U \to F$ is C^{k+1} in the preceding situation, then f is an SC^k-map. In fact, for every $j \in \mathbb{N}_0$ such that $j < k$, the map $f^{[j]}$ is C^{k+1-j}, where $k + 1 - j \geq 2$. Thus $f^{[j]}$ is strictly differentiable, by Corollary 3.10. It is also clear from the definitions that every LC^k map is SC^k, since every LC^1-map is SC^1. Hence, the relations between the various differentiability properties can be summarized as follows:

$$C^{k+1} \implies LC^k \implies SC^k \implies C^k\,.$$

Remark 3.18 If $\mathbb{K}$ is a locally compact topological field, then a mapping from an open subset of a finite-dimensional $\mathbb{K}$-vector space to a topological $\mathbb{K}$-vector space is C^k if and only if it is an SC^k-map, by a simple induction based on Lemma 3.11 and Remark 3.16. The same conclusion holds for mappings on locally compact subsets with dense interior.

Compositions of composable SC^k-maps are SC^k.

Proposition 3.19 *Let $\mathbb{K}$ be a valued field, E, F and H be topological $\mathbb{K}$-vector spaces, and $U \subseteq E$, $V \subseteq F$ be subsets with dense interior. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and suppose that $f\colon U \to V \subseteq F$ and $g\colon V \to H$ are SC^k. Then also $g \circ f\colon U \to H$ is SC^k.*

Proof. The case $k = 0$ is trivial. The case $k = 1$ can be shown as follows: Given $x \in U$, let γ be a gauge on H. There exists a gauge ξ on H such that $\|u + v\|_\gamma \leq \|u\|_\xi + \|v\|_\xi$ for all $u, v \in H$. By strict differentiability of g at $f(x)$, there exists a gauge ζ on F such that, for each $\varepsilon > 0$, there exists $\theta > 0$ such that

$$\|g(z) - g(y) - g'(f(x)).(z - y)\|_\xi \leq \varepsilon\, \|z - y\|_\zeta \quad \text{for all } y, z \in B^\zeta_\theta(f(x)) \cap V. \tag{37}$$

Let κ be a gauge on F such that $\|u+v\|_\zeta \leq \max\{\|u\|_\kappa, \|v\|_\kappa\}$ for all $u, v \in F$. After increasing κ if necessary, we may assume that $\|g'(f(x))\|_{\xi,\kappa} \leq 1$, using the notation from Definition 1.28. By strict differentiability of f at x and continuity of f at x, there exists a gauge η on E such that, for each $\theta > 0$, there exists $\delta > 0$ such that

$$\|f(z) - f(y) - f'(x).(z-y)\|_\kappa \leq \theta\, \|z-y\|_\eta \quad \text{for all } z, y \in B^\eta_\delta(x) \cap U. \tag{38}$$

After increasing η, we may assume that $\|f'(x)\|_{\kappa,\eta} \leq 1$. Given $\varepsilon > 0$, choose $\theta \in]0, \min\{1, \varepsilon\}]$ and $\delta \in]0, \theta[$ such that (37) and (38) hold. Given $y, z \in B^\eta_\delta(x) \cap U$, we have

$$\begin{aligned} \|f(z) - f(x)\|_\zeta &\leq \max\{\|f(z) - f(x) - f'(x).(z-x)\|_\kappa, \|f'(x).(z-x)\|_\kappa\} \\ &\leq \max\{\theta\|z-x\|_\eta, \|z-x\|_\eta\} \;<\; \theta \end{aligned}$$

and likewise $\|f(y) - f(x)\|_\zeta < \theta$. Hence

$$\begin{aligned} &\|g(f(z)) - g(f(y)) - g'(f(x)).f'(x).(z-y)\|_\gamma \\ &\quad\leq \|g(f(z)) - g(f(y)) - g'(f(x)).(f(z) - f(y))\|_\xi \\ &\qquad\quad + \|g'(f(x)).(f(z) - f(y) - f'(x).(z-y))\|_\xi \\ &\quad\leq \varepsilon\|f(z) - f(y)\|_\zeta + \|g'(f(x))\|_{\xi,\kappa} \cdot \|f(z) - f(y) - f'(x).(z-y)\|_\kappa \\ &\quad\leq \varepsilon\|f(z) - f(y) - f'(x).(z-y)\|_\kappa + \varepsilon\|f'(x).(z-y)\|_\kappa + \theta\|z-y\|_\eta \\ &\quad\leq \varepsilon\theta\|z-y\|_\eta + \varepsilon\|z-y\|_\eta + \theta\|z-y\|_\eta \;\leq\; 3\varepsilon\|z-y\|_\eta \end{aligned}$$

Hence indeed $g \circ f$ is strictly differentiable at x, with differential $(g \circ f)'(x) = g'(f(x)) \circ f'(x)$.

The case $k \geq 2$. Let us call a map between subsets with dense interior of Hausdorff topological $\mathbb{K}$-vector spaces $\mathcal{C}^0$ if it is SC^1. It is clear from the case $k = 1$ that we obtain a $\mathcal{C}^0$-concept in the sense of [2] in this way (suitable adapted to non-open sets, e.g. as in Appendix A). Furthermore, SC^{k+1}-maps are precisely the $\mathcal{C}^k$-maps for this $\mathcal{C}^0$-concept, for each $k \in \mathbb{N}_0 \cup \{\infty\}$. The assertion therefore reduces to the Chain Rule for $\mathcal{C}^k$-maps (cf. [2, Proposition 4.5]). □

4 Dependence of fixed points on parameters

We now study the dependence of fixed points of contractions on parameters. In particular, we shall establish C^k-, SC^k- and LC^k-dependence under natural hypotheses. These results will be used in Section 5 to prove generalizations of the inverse- and implicit function theorems.

We recall the notion of a contraction.

Definition 4.1 A mapping $f\colon X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is called a *contraction* if there exists $\theta \in [0, 1[$ (a "contraction constant") such that

$$d_Y(f(x), f(y)) \;\leq\; \theta\, d_X(x, y) \qquad \text{for all } x, y \in X.$$

Banach's Contraction Theorem (see, e.g., [31, Appendix A]) is a paradigmatic fixed point theorem for contractions. We recall it as a model for the slight generalizations which we actually need for our purposes:

Lemma 4.2 *Let (X, d) be a (non-empty) complete metric space and $f\colon X \to X$ be a contraction, with contraction constant $\theta \in [0, 1[$. Thus*

$$d(f(x), f(y)) \;<\; \theta\, d(x, y) \quad \textit{for all } x, y \in X.$$

Then $f(p) = p$ for a unique point $p \in X$. Given any $x_0 \in X$, we have $\lim_{n\to\infty} f^n(x_0) = p$. Furthermore, the a priori estimate

$$d(f^n(x_0), p) \leq \frac{\theta^n}{1-\theta}\, d(f(x_0), x_0)$$

holds, for each $n \in \mathbb{N}_0$. □

Unfortunately, we are not always in the situation of this theorem. But the simple variants compiled in the next proposition are flexible enough for our purposes.

Proposition 4.3 *Let (X, d) be a metric space, $U \subseteq X$ be a subset and $f\colon U \to X$ be a contraction, with contraction constant θ. Then the following holds:*

(a) *f has at most one fixed point.*

(b) *If $x_0 \in U$ is a point and $n \in \mathbb{N}_0$ such that $f^{n+1}(x_0)$ is defined, then*

$$d(f^{k+1}(x_0), f^k(x_0)) \;\leq\; \theta^k\, d(f(x_0), x_0) \tag{39}$$

for all $k \in \{0, \ldots, n\}$, and $d(f^{n+1}(x_0), x_0) \leq \frac{1-\theta^{n+1}}{1-\theta}\, d(f(x_0), x_0)$.

(c) *If $x_0 \in U$ is a point such that $f^n(x_0)$ is defined for all $n \in \mathbb{N}$, then $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence in U, and*

$$d(f^{n+k}(x_0), f^n(x_0)) \;\leq\; \frac{\theta^n(1-\theta^k)}{1-\theta}\, d(f(x_0), x_0) \quad \textit{for all } n, k \in \mathbb{N}_0. \tag{40}$$

If $(f^n(x_0))_{n\in\mathbb{N}}$ converges to some $x \in U$, then x is a fixed point of f, and

$$d(x, f^n(x_0)) \;\leq\; \frac{\theta^n}{1-\theta}\, d(f(x_0), x) \quad \textit{for all } n \in \mathbb{N}_0. \tag{41}$$

If $f^n(x_0)$ is defined for all $n \in \mathbb{N}$ and f has a fixed point x, then $f^n(x) \to x$ as $n \to \infty$.

(d) *Assume that* $U = \overline{B}_r(x_0)$ *is a closed ball of radius* r *around a point* $x_0 \in X$*, and* $d(f(x_0), x_0) \leq (1-\theta)r$*. Then* $f^n(x_0)$ *is defined for all* $n \in \mathbb{N}_0$*. Hence* f *has a fixed point inside* $\overline{B}_r(x_0)$*, provided* X *is complete. Likewise,* f *has a fixed point in the open ball* $B_r(x_0)$ *if* X *is complete,* $U = B_r(x_0)$*, and* $d(f(x_0), x_0) < (1-\theta)r$*.*

Proof. (a) If $x, y \in U$ are fixed points of f, then $d(x,y) = d(f(x), f(y)) \leq \theta d(x,y)$, entailing that $d(x,y) = 0$ and thus $x = y$.

(b) For $k = 0$, the formula (39) is trivial. If $k < n$ and $d(f^{k+1}(x_0), f^k(x_0)) \leq \theta^k\, d(f(x_0), x_0)$, then $d(f^{k+2}(x_0), f^{k+1}(x_0)) = d(f(f^{k+1}(x_0)), f(f^k(x_0))) \leq \theta\, d(f^{k+1}(x_0), f^k(x_0)) \leq \theta^{k+1}\, d(f(x_0), x_0)$. Thus (39) holds in general.

Using the triangle inequality and the summation formula for the geometric series, we obtain the estimates $d(f^{n+1}(x_0), x_0) \leq \sum_{k=0}^n d(f^{k+1}(x_0), f^k(x_0)) \leq \sum_{k=0}^n \theta^k\, d(f(x_0), x_0) = \frac{1-\theta^{n+1}}{1-\theta}\, d(f(x_0), x_0)$, as asserted.

(c) Using both of the estimates from (b), obtain

$$d(f^{n+k}(x_0), f^n(x_0)) \;\leq\; \frac{1-\theta^k}{1-\theta}\, d(f^{n+1}(x_0), f^n(x_0)) \;\leq\; \frac{1-\theta^k}{1-\theta}\, \theta^n d(f(x_0), x_0)\,.$$

Thus (40) holds, and thus $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence. If $f^n(x_0) \to x$ for some $x \in U$, then $x = \lim_{n\to\infty} f^{n+1}(x_0) = f(\lim_{n\to\infty} f^n(x_0)) = f(x)$ by continuity of f, whence indeed x is a fixed point of f. Letting now $k \to \infty$ in (40), we obtain (41).

To prove the final assertion, assume that f has a fixed point x and that $f^n(x_0)$ is defined for all n. We choose a completion $\overline{X}$ of X (with $X \subseteq \overline{X}$) and let $\overline{U}$ be the closure of U in $\overline{X}$. Then f extends to a contraction $\overline{U} \to \overline{X}$, which we also denote by f. Since $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\overline{U}$ and $\overline{U}$ is complete, we deduce that $f^n(x_0) \to y$ for some $y \in \overline{U}$. Then both y and x are fixed points of f and hence $x = y$.

(d) We show by induction that $f^n(x_0)$ is defined for all $n \in \mathbb{N}$. For $n = 1$, this is trivial. If $f^n(x_0)$ is defined, then

$$d(f^n(x_0), x_0) \;\leq\; \frac{1-\theta^n}{1-\theta}\, d(f(x_0), x_0) \;\leq\; \frac{1-\theta^n}{1-\theta}\,(1-\theta)r \;\leq\; r$$

and thus $f^n(x_0) \in \overline{B}_r(x_0)$, whence also $f^{n+1}(x_0) = f(f^n(x_0))$ is defined. Then $f^n(x_0) \in \overline{B}_r(x_0)$ for each $n \in \mathbb{N}$. By (c), $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence. If X is complete, then so is $\overline{B}_r(x_0)$ and thus $(f^n(x_0))_{n\in\mathbb{N}}$ converges to some point $x \in \overline{B}_r(x_0)$, which is a fixed point of f by (c). Finally, if $U = B_r(x_0)$ and $d(f(x_0), x_0) < (1-\theta)r$, there exists $s \in \,]0, r[$ such that $d(f(x_0), x_0) \leq (1-\theta)s$. By the preceding, $f^n(x_0) \in \overline{B}_s(x_0) \subseteq B_r(x_0)$ for all $n \in \mathbb{N}$, and f has a fixed point in $\overline{B}_s(x_0) \subseteq B_r(x_0)$. □

We are interested in uniform families of contractions.

Definition 4.4 Let F be a Banach space over a valued field $\mathbb{K}$, and $U \subseteq F$ be a subset. A family $(f_p)_{p\in P}$ of mappings $f_p\colon U \to F$ is called a *uniform family* *of contractions* if there exists $\theta \in [0,1[$ (a "uniform contraction constant") such that

$$\|f_p(x) - f_p(y)\| \le \theta\|x - y\| \quad \text{for all } x, y \in U \text{ and } p \in P.$$

If U is closed and each f_p is a self-map of U here, then Banach's Contraction Theorem ensures that, for each $p \in P$, the map f_p has a unique fixed point x_p. Our goal is to understand the dependence of x_p on the parameter p. In particular, for P a subset of a topological $\mathbb{K}$-vector space, we want to find conditions ensuring that the map $P \to F$, $p \mapsto x_p$ is continuously differentiable. We discuss dependence of fixed points on parameters in two steps.

Proposition 4.5 *Let P be a topological space and F be a Banach space over a valued field $\mathbb{K}$. Let $U \subseteq F$ be a subset with dense interior and $f\colon P \times U \to F$ be a map such that $(f_p)_{p\in P}$ is a uniform family of contractions, where $f_p := f(p, \bullet)\colon U \to F$. We assume that f_p has a fixed point x_p, for each $p \in P$. Furthermore, we assume that U is open or $f(P \times U) \subseteq U$ (whence every f_p is a self-map of U). Then the following holds:*

(a) *If f is continuous, then also the map $\phi\colon P \to F$, $\phi(p) := x_p$ is continuous.*

(b) *If P is a subset of a topological $\mathbb{K}$-vector space E and f is Lipschitz continuous, then also ϕ is Lipschitz continuous.*

(c) *If P is a subset with dense interior of a topological $\mathbb{K}$-vector space E and f is SC^1, then also ϕ is SC^1.*

(d) *If P is a subset with dense interior of a topological $\mathbb{K}$-vector space E and f is C^1, then also ϕ is C^1.*

Remark 4.6 We shall see that the differential of ϕ at $p \in P$ is given by

$$\phi'(p) = (\mathrm{id}_F - \beta_2)^{-1} \circ \beta_1 \tag{42}$$

in the situation of Proposition 4.5 (c) and (d), where $\beta_1 := d_1 f(p, x_p, \bullet) := df(p, x_p, \bullet, 0) \in \mathcal{L}(E, F)$ and $\beta_2 := d_2 f(p, x_p, \bullet) := df(p, x_p, 0, \bullet) \in \mathcal{L}(F)$.

Proof. Let $\theta \in [0,1[$ be a uniform contraction constant for $(f_p)_{p\in P}$.

(a) If f is continuous, $p \in P$ and $\varepsilon > 0$, we find a neighborhood $Q \subseteq P$ of p such that $\|x_p - f_q(x_p)\| \le (1-\theta)\varepsilon$ for all $q \in Q$. If $f(P \times U) \subseteq U$, then $\|x_p - x_q\| \le \frac{1}{1-\theta}\|x_p - f_q(x_p)\| \le \varepsilon$, by (41) in Proposition 4.3 (c). If U is open, we may assume that $\overline{B}_\varepsilon(x_p) \subseteq U$ after shrinking ε. Then Proposition 4.3 (d) applies to f_q as a map $\overline{B}_\varepsilon(x_p) \to E$ for each $q \in Q$, showing that $f_q^n(x_p)$ is defined for each $n \in \mathbb{N}$ and $x_q = \lim_{n\to\infty} f_q^n(x_p) \in \overline{B}_\varepsilon(x_p)$, that is, $\|x_p - x_q\| \le \varepsilon$.

(b) Given $s \in P$, there exists a gauge γ on E and $R, L > 0$ such that

$$\|f_p(x) - f_q(y)\| \le L \max\{\|p - q\|_\gamma, \|x - y\|\} \tag{43}$$

for all $p, q \in P \cap B^\gamma_R(s)$ and $x, y \in U \cap B^F_{2R}(x_s)$. Let η be a gauge on E such that $\|u+v\|\gamma \leq \max\{\|u\|_\eta, \|v\|_\eta\}$ for all $u, v \in E$. If U is open, we assume that $\overline{B}_{2R}(x_s) \subseteq U$. Set $r := \min\{(1-\theta)R, (1-\theta)R/L\}$. Given $p \in P \cap B^\eta_r(s)$, we have $\|f_p(x_s) - x_s\| = \|f_p(x_s) - f_s(x_s)\| \leq L\|p-s\|_\gamma \leq (1-\theta)R$ by (43) and hence $x_p \in \overline{B}_R(x_s)$, by the proof of (a). Given $p, q \in P \cap B^\eta_r(s)$, we either have $\|p-q\|_\gamma = 0$; then $\|f_q(x_p) - x_p\| = \|f_q(x_p) - f_p(x_p)\| = 0$ by (43), whence $f_q(x_p) = x_p$. Hence $x_q = x_p$, whence $\|x_q - x_p\| = 0$ and thus

$$\|x_p - x_q\| \ \leq \ \frac{L}{1-\theta}\, \|p-q\|_\gamma \tag{44}$$

in particular. Otherwise, $0 < \varepsilon := \|p-q\|_\gamma \leq \max\{\|p-s\|_\eta, \|q-s\|_\eta\} < r$ and $\|f_q(x_p) - x_p\| = \|f_q(x_p) - f_p(x_p)\| \leq L\|p-q\|_\gamma = \varepsilon L$, by (43). If U is open, then $\varepsilon L < \varepsilon L/(1-\theta) \leq rL/(1-\theta) \leq R$ and thus $\overline{B}_{\varepsilon L/(1-\theta)}(x_p) \subseteq U$. Hence $\|x_p - x_q\| \leq \varepsilon L/(1-\theta) = \|p-q\|_\gamma L/(1-\theta)$, by the proof of (a). Thus (44) holds for all $p, q \in P \cap B^\eta_r(x_s)$. We deduce that ϕ is Lipschitz continuous.

(c) Assume that f is strictly differentiable. Given $s \in P$, by strict differentiability of f at (s, x_s), there exists a gauge γ on E such that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f(p,x) - f(q,y) - f'(s,x_s).(p-q, x-y)\| \ \leq \ \varepsilon \, \max\{\|p-q\|_\gamma, \|x-y\|\} \tag{45}$$

for all $p, q \in P \cap B^\gamma_\delta(s)$ and $x, y \in U \cap B^F_\delta(x_s)$. Since f is strictly differentiable, it is Lipschitz continuous, whence also ϕ is Lipschitz continuous, by (b). Hence, after replacing γ by a larger gauge, we may assume that

$$\|x_p - x_q\| \ \leq \ \|p-q\|_\gamma \quad \text{for all } p, q \in P \cap B^\gamma_1(x_s). \tag{46}$$

Given $\varepsilon > 0$, choose $\delta \in \,]0,1]$ such that (45) holds. Taking $q = s$ in (46), it follows that $x_p \in U \cap B^F_\delta(x_s)$ for each $p \in B^\gamma_\delta(s)$.

It is useful to write $f'(s, x_s)(u, v) = \beta_1(u) + \beta_2(v)$ in terms of the partial differentials $\beta_1 := d_1 f(x, x_s, \bullet)\colon E \to F$ and $\beta_2 := d_2 f(x, x_s, \bullet)\colon F \to F$. Then $\|\beta_2\| \leq \theta < 1$ by Lemma 2.2. Abbreviate

$$R(p,q) := f(p, x_p) - f(q, x_q) - f'(s, x_s).(p-q, x_p - x_q)$$

for $p, q \in P \cap B^\gamma_\delta(s)$. Combining (45) and (46), we find that

$$\|R(p,q)\| \ \leq \ \varepsilon \max\{\|p-q\|_\gamma, \|x_p - x_q\|\} \ \leq \ \varepsilon\|p-q\|_\gamma \tag{47}$$

for all $p, q \in P \cap B^\gamma_\delta(s)$. Now

$$\begin{aligned} x_p - x_q \ &= \ f(p, x_p) - f(q, x_q) \ = \ f'(s, x_s).(p-q, x_p - x_q) + R(p,q) \\ &= \ \beta_1(p-q) + \beta_2(x_p - x_q) + R(p,q) \end{aligned}$$

and therefore $x_p - x_q - (\mathrm{id}_F - \beta_2)^{-1}.\beta_1(p-q) = (\mathrm{id}_F - \beta_2)^{-1} R(p,q)$, where $\|(\mathrm{id}_F - \beta_2)^{-1} R(p,q)\| \leq \|(\mathrm{id}_F - \beta_2)^{-1}\| \, \|R(p,q)\| \leq \varepsilon(1-\theta)^{-1}\|p-q\|_\gamma$, by (13) in Proposition 1.33 and (47). We have shown that ϕ is strictly differentiable at s, with the desired differential.

(d) Being C^1, f is Lipschitz continuous, whence ϕ is Lipschitz continuous, by (b). Thus $\phi^{]1[}$ is continuous. To see that ϕ is C^1, it only remains to show that, for all $p_0 \in P$ and $q_0 \in E$, there exists an open neighborhood $W \subseteq P^{[1]}$ of $(p_0, q_0, 0)$ and a continuous map $g\colon W \to F$ which extends the difference quotient map $\phi^{]1[}|_{W \cap P^{]1[}}\colon W \cap P^{]1[} \to F$. Then $\phi^{]1[}$ has a continuous extension $\phi^{[1]}$ to all of $P^{[1]}$, by Lemma 3.7, and thus ϕ will be C^1. Our strategy is the following: We write

$$(f_{p+tq}^{n+1}(x_p) - x_p)/t \;=\; \sum_{k=0}^{n} (f_{p+tq}^{k+1}(x_p) - f_{p+tq}^{k}(x_p))/t \tag{48}$$

for (p, q, t) in a suitable neighborhood W of $(p_0, q_0, 0)$. For W sufficiently small, the left hand side converges to $\frac{x_{p+tq} - x_p}{t} = \frac{\phi(p+tq) - \phi(p)}{t}$ as $n \to \infty$. Furthermore, we can achieve that each term on the right hand side extends continuously to all of W, and that the series converges uniformly to a continuous function on W. This will be our desired continuous extension g.

Let us carry this out in detail. Case 1. If $f(P \times U) \subseteq U$, we set $W_0 := P^{[1]}$. Case 2. Otherwise, U is open, whence there exists $\varepsilon > 0$ such that $\overline{B}_{2\varepsilon}^F(x_{p_0}) \subseteq U$. Since f and ϕ are continuous and $f_{p_0}(x_{p_0}) = x_{p_0}$, we find an open neighborhood $Q \subseteq P$ of p_0 such that $\|x_p - f_q(x_p)\| \leq (1-\theta)\varepsilon$ for all $p, q \in Q$. Then $f_q^k(x_p)$ is defined for all $k \in \mathbb{N}_0$, $f_q^k(x_p) \in B_\varepsilon^F(x_p)$, and $x_q \in \overline{B}_\varepsilon^F(x_p)$ (cf. proof of (a)). In particular, $x_p \in \overline{B}_\varepsilon^F(x_{p_0})$. We now set $W_0 := Q^{[1]}$ and note that, if $(p, q, t) \in Q^{[1]}$, then $p, p+tq \in Q$, whence $f_{p+tq}^k(x_p)$ is defined for all $k \in \mathbb{N}_0$ and $\lim_{k\to\infty} f_{p+tq}^k(x_p) = x_{p+tq}$ (by the preceding considerations).

In either case, we define

$$h_0\colon W_0 \to F\,, \qquad h_0(p, q, t) \;=\; f^{[1]}(p, x_p, q, 0, t)$$

and note that h_0 is a continuous map such that

$$h_0(p, q, t) = (f_{p+tq}(x_p) - f_p(x_p))/t = (f_{p+tq}(x_p) - x_p)/t \quad \text{if } t \neq 0. \tag{49}$$

For all $k \in \mathbb{N}$ and $(p, q, t) \in W_0$, we have

$$\begin{aligned} &\frac{f_{p+tq}^{k+1}(x_p) - f_{p+tq}^{k}(x_p)}{t} \\ &\quad= \frac{f\big(p+tq, f_{p+tq}^{k-1}(x_p) + t\,\frac{f_{p+tq}^{k}(x_p) - f_{p+tq}^{k-1}(x_p)}{t}\big) - f(p+tq, f_{p+tq}^{k-1}(x_p))}{t} \\ &\quad= f^{[1]}\Big(p+tq, f_{p+tq}^{k-1}(x_p), 0, \frac{f_{p+tq}^{k}(x_p) - f_{p+tq}^{k-1}(x_p)}{t}, t\Big)\,. \end{aligned} \tag{50}$$

Recursively, we define

$$h_k\colon W_0 \to F\,, \quad h_k(p, q, t) \;:=\; f^{[1]}\big(p+tq, f_{p+tq}^{k-1}(x_p), 0, h_{k-1}(p, q, t), t\big)$$

for $k \in \mathbb{N}$. A simple induction based on (49) and (50) shows that the definition of h_k makes sense for each $k \in \mathbb{N}_0$, and that

$$h_k(p,q,t) \;=\; \frac{f_{p+tq}^{k+1}(x_p) - f_{p+tq}^k(x_p)}{t} \quad \text{for all } (p,q,t) \in W_0 \text{ with } t \neq 0. \tag{51}$$

The function $h_0 \colon W_0 \to F$, $(p,q,t) \mapsto f^{[1]}(p,x_p,q,0,t)$ being continuous, we find an open neighborhood $W \subseteq W_0$ of $(p_0,q_0,0)$ and $C \in [0,\infty[$ such that

$$\|f^{[1]}(p,x_p,q,0,t)\| \;\leq\; C \qquad \text{for all } (p,q,t) \in W.$$

For all $(p,q,t) \in W$ such that $t \neq 0$, we have

$$\begin{aligned}
\|f_{p+tq}^{k+1}(x_p) - f_{p+tq}^k(x_p)\| &= \|f_{p+tq}^k(f_{p+tq}(x_p)) - f_{p+tq}^k(x_p)\| \\
&\leq \theta^k \|f_{p+tq}(x_p) - x_p\| \\
&= |t|\theta^k \,\|(f_{p+tq}(x_p) - f_p(x_p))/t\| \\
&= |t|\theta^k \|f^{[1]}(p,x_p,q,0,t)\| \;\leq\; |t|\theta^k C\,.
\end{aligned}$$

Combining this with (51), for each $k \in \mathbb{N}_0$ we see that $\|h_k(p,q,t)\| \leq \theta^k C$ for all $(p,q,t) \in W$ such that $t \neq 0$, and thus

$$\|h_k(p,q,t)\| \;\leq\; \theta^k C \quad \text{for all } (p,q,t) \in W, \tag{52}$$

because h_k is continuous and $W \cap P^{]1[}$ is dense in W. As a consequence, $\sum_{k=0}^\infty \|h_k|_W\|_\infty \leq \sum_{k=0}^\infty \theta^k C = \frac{1}{1-\theta} C < \infty$, whence the series $\sum_{k=0}^\infty h_k|_W$ of bounded continuous functions converges uniformly and absolutely. Thus

$$g(p,q,t) \;:=\; \sum_{k=0}^\infty h_k(p,q,t)$$

exists for all $(p,q,t) \in W$, and $g \colon W \to F$ is continuous. It only remains to observe that

$$\frac{f_{p+tq}^{n+1}(x_p) - x_p}{t} \;=\; \sum_{k=0}^n \frac{f_{p+tq}^{k+1}(x_p) - f_{p+tq}^k(x_p)}{t} \;=\; \sum_{k=0}^n h_k(p,q,t)$$

for all $(p,q,t) \in W$ such that $t \neq 0$. Since the left hand side converges to $\frac{x_{p+tq} - x_p}{t}$ and the right hand side converges to $g(p,q,t)$, we obtain

$$\frac{\phi(p+tq) - \phi(p)}{t} \;=\; \sum_{k=0}^\infty h_k(p,q,t) \;=\; g(p,q,t)\,.$$

Thus $g \colon W \to F$ is a continuous map which extends $\phi^{]1[}|_{W \cap P^{]1[}}$, as desired. This completes the proof. □

We now state and prove the main result of this section.

Theorem 4.7 (Dependence of Fixed Points of Parameters) *Let $\mathbb{K}$ be a valued field, E be a topological $\mathbb{K}$-vector space, and F be a Banach space over $\mathbb{K}$. Let $P \subseteq E$ be a subset with dense interior, $U \subseteq F$ be open, and $f\colon P \times U \to F$ be a continuous map such that $f_p := f(p, \bullet)\colon U \to F$ defines a uniform family $(f_p)_{p \in P}$ of contractions. Then the following holds:*

(a) *The set Q of all $p \in P$ such that f_p has a fixed point x_p is open in P.*

(b) *If f is C^k for some $k \in \mathbb{N}_0 \cup \{\infty\}$ (resp., SC^k, resp., LC^k), then also $\phi\colon Q \to U$, $\phi(p) := x_p$ is C^k, SC^k, resp., LC^k.*

Proof. Let $\theta \in [0, 1[$ be a uniform contraction constant for $(f_p)_{p \in P}$.

(a) If $p \in Q$, there is $r > 0$ such that $\overline{B}_r(x_p) \subseteq U$. There is a neighborhood $S \subseteq Q$ of p such that $\|f_q(x_p) - x_p\| = \|f_q(x_p) - f_p(x_p)\| \leq (1 - \theta)r$ for all $q \in S$. Now Proposition 4.3 (d) shows that f_q has a fixed point x_q in $\overline{B}_r(x_p)$, for each $q \in S$. Thus $S \subseteq Q$ and we deduce that Q is open.

(b) We may assume that $k < \infty$. The proof is by induction on $k \in \mathbb{N}_0$. The case $k = 0$ is covered by Proposition 4.5 (a) and (b). Now assume that our assertion holds for some $k \in \mathbb{N}_0$ and assume that f is C^{k+1} (resp., SC^{k+1}, resp., LC^{k+1}). Then ϕ is C^k (resp., SC^k, resp., LC^k) by the induction hypothesis. Furthermore, ϕ is C^1 by Proposition 4.5 (d), and if f is SC^1, then ϕ is SC^1, by Proposition 4.5 (c). If $k = 0$, this already completes the induction step for continuously differentiable and for strictly differentiable maps. If f is C^{k+1} or SC^{k+1} with $k \geq 1$ (whence f is LC^1 in particular) or if $k \in \mathbb{N}_0$ and f is LC^{k+1}, it remains to show that $\phi^{[1]}$ is C^k, SC^k and LC^k, respectively. It suffices to prove that $\phi^{[1]}|_Z$ is C^k (resp., SC^k, resp., LC^k) for Z ranging through an open cover of $Q^{[1]}$ (cf. [2, Lemma 4.9]). Because

$$\phi^{[1]}(p, q, t) = \frac{\phi(p + tq) - \phi(p)}{t} \quad \text{for } (p, q, t) \in Q^{]1[}, \tag{53}$$

we observe first that $\phi^{[1]}$ is C^k (resp., SC^k, resp., LC^k) on the open subset $Q^{]1[}$ of $Q^{[1]}$. Given $(p_0, q_0) \in Q \times E$, let us find a description of $\phi^{[1]}$ on a neighborhood of $(p_0, q_0, 0)$ in $Q^{[1]}$. Given $(p, q, t) \in Q^{]1[}$, we have

$$\begin{aligned} \phi^{[1]}(p, q, t) &= \frac{\phi(p + tq) - \phi(p)}{t} = \frac{f(p + tq, \phi(p + tq)) - f(p, \phi(p))}{t} \\ &= \frac{f(p + tq, \phi(p) + t\frac{\phi(p+tq) - \phi(p)}{t}) - f(p, \phi(p))}{t} \end{aligned}$$

and thus

$$\phi^{[1]}(p, q, t) = f^{[1]}(p, \phi(p), q, \phi^{[1]}(p, q, t), t) \tag{54}$$

for all $(p, q, t) \in Q^{]1[}$. As both the left and right hand side of (54) make sense for all $(p, q, t) \in Q^{[1]}$ and are continuous there, they coincide on all of $Q^{[1]}$.

Observe that (54) means that, for each parameter $(p, q, t) \in Q^{[1]}$, the element

$\phi^{[1]}(p,q,t) \in F$ is a fixed point of the function $f^{[1]}(p,\phi(p),q,\bullet,t)$. We make this more precise now and show that we are dealing with a C^k- (resp., SC^k-, resp., LC^k-) family of uniform contractions, which will enable us to deduce from the induction hypothesis that $\phi^{[1]}$ is C^k (resp., SC^k, resp., LC^k).

Abbreviate $v_0 := \phi(p_0)$ and $w_0 := \phi^{[1]}(p_0,q_0,0)$. Because $U^{[1]}$ is open in $F\times F\times\mathbb{K}$ and $(v_0,w_0,0) \in U^{[1]}$, there exist open neighborhoods $V \subseteq U$ of v_0, $W \subseteq F$ of w_0 and a 0-neighborhood $S \subseteq \mathbb{K}$ such that $V\times W\times S \subseteq U^{[1]}$ and thus $V+SW \subseteq U$. Then $P_1 := (Q^{[1]}\cap(Q\times E\times S))\times V$ is a subset of $E\times E\times\mathbb{K}\times F$ with dense interior, W is an open subset of the Banach space F, and $(p,v,q,w,t) \in (Q\times U)^{[1]}$ for all $(p,q,t,v,w) \in P_1\times W$. We can therefore define a C^k-map (resp., SC^k-map, resp., LC^k-map)

$$g\colon P_1\times W \to F\,,\quad g(p,q,t,v,w) := f^{[1]}(p,v,q,w,t)\,. \tag{55}$$

Choose $\varepsilon \in \,]0,1-\theta[$. Because f is LC^1, Lemma 2.9 entails that there exists $s>0$ such that $B^F_s(w_0) \subseteq W$ and an open neighborhood $P_2 \subseteq P_1$ of $(p_0,q_0,0,v_0)$ such that

$$\|f^{[1]}(p,v,q,w_1,t) - f^{[1]}(p,v,q,w_2,t) - f'_{p_0}(v_0).(w_1-w_2)\| \,\leq\, \varepsilon\|w_1-w_2\| \tag{56}$$

whenever $(p,q,t,v,w_1),(p,q,t,v,w_2) \in P_2\times B^F_s(w_0)$. Since $\|f'_{p_0}(v_0)\| \leq \theta$ by Lemma 2.2, (56) entails that

$$\begin{aligned}\|f^{[1]}(p,v,q,w_1,t) - f^{[1]}(p,v,q,w_2,t)\| &\leq (\|f'_{p_0}(v_0)\|+\varepsilon)\cdot\|w_1-w_2\| \\ &\leq (\theta+\varepsilon)\cdot\|w_1-w_2\| \end{aligned} \tag{57}$$

whenever $(p,q,t,v,w_1),(p,q,t,v,w_2) \in P_2\times B^F_s(w_0)$. Since $\theta+\varepsilon<1$, we see that the restriction of g to a map $g\colon P_2\times B^F_s(w_0) \to F$ is a uniform family of contractions. The map g being C^k (resp., SC^k, resp., LC^k), we know by induction that the set $Q_2 \subseteq P_2$ of all $(p,q,t,v) \in P_2$ such that $g(p,q,t,v,\bullet)\colon B^F_s(w_0) \to F$ has a fixed point $\psi(p,q,t,v)$ is open in P_2, and that the map $\psi\colon Q_2 \to F$ so obtained is C^k (resp., SC^k, resp., LC^k). By continuity of ϕ and $\phi^{[1]}$, there exists an open neighborhood Z of $(p_0,q_0,0)$ in $P^{[1]}$ such that $(p,,q,t,\phi(t)) \in Q_2$ for all $(p,q,t) \in Z$, and $\phi^{[1]}(p,q,t) \in B^F_s(w_0)$. Then (54) entails that

$$\phi^{[1]}(p,q,t) = \psi(p,q,t,\phi(p)) \qquad \text{for all } (p,q,t) \in Z,$$

whence $\phi^{[1]}|_Z$ is C^k (resp., SC^k, resp., LC^k). This completes the proof. □

5 Inverse and implicit function theorems

In this section, we prove inverse- and implicit function theorems for mappings into Banach spaces over valued fields. In particular, we obtain the following analog of the classical Inverse Function Theorem:

Theorem 5.1 (Inverse Function Theorem) *Let E be a Banach space over a valued field $\mathbb{K}$ and $f\colon U \to E$ be a mapping on an open subset $U \subseteq E$ which is C^k, SC^k and LC^k, respectively, where $k \in \mathbb{N} \cup \{\infty\}$. In the C^k-case, we require that $k \geq 2$ or that $\mathbb{K}$ is locally compact and E has finite dimension. If $f'(x) \in \mathrm{GL}(E)$ for some $x \in U$, then there exists an open neighborhood $V \subseteq U$ of x such that $f(V)$ is open in E and $f|_V\colon V \to f(V)$ is a C^k-diffeomorphism (resp., an SC^k-diffeomorphism, resp., an LC^k-diffeomorphism).*

While the classical implicit function theorem deals with functions between real Banach spaces, we can discuss implicit functions from arbitrary topological vector spaces to Banach spaces:

Theorem 5.2 (Generalized Implicit Function Theorem) *Let $\mathbb{K}$ be a valued field, E be a topological $\mathbb{K}$-vector space, F be a Banach space over $\mathbb{K}$, and $f\colon U \times V \to F$ be a mapping, where $U \subseteq E$ is a subset with dense interior and $V \subseteq F$ is open. We assume that f is C^k, SC^k, respectively, LC^k for some $k \in \mathbb{N} \cup \{\infty\}$. In the C^k-case, we require that $k \geq 2$ or that $\mathbb{K}$ is locally compact and F has finite dimension. Given $x \in U$, abbreviate $f_x := f(x, \bullet)\colon V \to F$. If $f(x_0, y_0) = 0$ for some $(x_0, y_0) \in U \times V$ and $f'_{x_0}(y_0) \in \mathrm{GL}(F)$, then there exist open neighborhoods $U_0 \subseteq U$ of x_0 and $V_0 \subseteq V$ of y_0 such that*

$$\{(x, y) \in U_0 \times V_0\colon f(x, y) = 0\} \;=\; \operatorname{graph} \lambda$$

for a map $\lambda\colon U_0 \to V_0$ which is C^k, SC^k, resp., LC^k.

We shall deduce the Generalized Implicit Function Theorem from an "Inverse Function Theorem with Parameters." It is useful to get an idea of the main steps of the proof (for SC^k-maps, say) before we carry them out in detail.

Strategy of the proof. Since $f'_{x_0}(y_0) \in \mathrm{GL}(F)$, where $\mathrm{GL}(F)$ is open in $\mathcal{L}(F)$ and the map $U \to \mathcal{L}(F)$, $x \mapsto f'_x(y_0)$ is continuous, we see that $f'_x(y_0) \in \mathrm{GL}(F)$ for x close to x_0. Then f_x^{-1} exists locally around $f_x(y_0)$, by the Inverse Function Theorem (Theorem 5.1). Now the essential point is that the map

$$(x, z) \;\mapsto\; (f_x)^{-1}(z) \tag{58}$$

actually makes sense on a whole neighborhood $U_0 \times W$ of $(x_0, 0)$ in $U \times F$, and is SC^k there. This assertion is the main content of the "Inverse Function Theorem with Parameters" just announced. Once we have this, the rest is easy: The map

$$\lambda\colon U_0 \to F\,, \quad \lambda(x) \;:=\; (f_x)^{-1}(0)$$

is SC^k, and $f(x, \lambda(x)) = 0$. □

Motivated by these considerations, we now proceed as follows: First, we formulate and prove versions of the inverse function theorem which provide quantitative information on the "size" of the domain of the local inverse f^{-1}, and the size of the images of balls (under f). These studies can be carried out under quite weak hypotheses (for Lipschitz maps). They will enable us to see that the domain of definition in (58) is a neighborhood. The second step, then, is the exact formulation and proof of the Inverse Function Theorem with Parameters.

Lipschitz Inverse Function Theorems

We now prove a Lipschitz Inverse Function Theorem for self-maps of general Banach spaces, and a variant for ultrametric Banach spaces. As a rule, inverse function theorems are available in much stronger form in the ultrametric case. This is a general phenomenon which we shall encounter repeatedly.

Theorem 5.3 (Lipschitz Inverse Function Theorem) *Let $(E, \|.\|)$ be a Banach space over a valued field $(\mathbb{K}, |.|)$. Let $r > 0$, $x \in E$, and $f\colon B_r^E(x) \to E$ be a mapping. We suppose that there exists $A \in \mathrm{GL}(E) := \mathcal{L}(E)^\times$ such that*

$$\sigma := \sup\left\{ \frac{\|f(z) - f(y) - A.(z-y)\|}{\|z-y\|} \colon\ y, z \in B_r(x),\ y \neq z \right\} < \frac{1}{\|A^{-1}\|}\,. \tag{59}$$

Then the following holds:

(a) *f has open image and is a homeomorphism onto its image.*

(b) *The inverse map $f^{-1}\colon f(B_r(x)) \to B_r(x)$ is Lipschitz, with*

$$\mathrm{Lip}(f^{-1}) \ \leq\ \frac{1}{\|A^{-1}\|^{-1} - \sigma}\,. \tag{60}$$

(c) *Abbreviating $a := \|A^{-1}\|^{-1} - \sigma > 0$ and $b := \|A\| + \sigma$, we have*

$$a\|z-y\| \ \leq\ \|f(z) - f(y)\| \ \leq\ b\|z-y\| \quad \text{for all } y, z \in B_r(x). \tag{61}$$

(d) *The following estimates for the size of images of balls are available: For every $y \in B_r(x)$ and $s \in\,]0, r - \|y - x\|]$,*

$$B_{as}(f(y)) \ \subseteq\ f(B_s(y)) \ \subseteq\ B_{bs}(f(y)) \tag{62}$$

holds. In particular, $B_{ar}(f(x)) \subseteq f(B_r(x)) \subseteq B_{br}(f(x))$.

Remark 5.4 Note that the condition (59) means that the remainder term

$$\tilde{f}\colon B_r(x) \to E, \quad \tilde{f}(y) := f(y) - f(x) - A.(y-x)$$

in the affine-linear approximation $f(y) = f(x) + A.(y-x) + \tilde{f}(y)$ is a Lipschitz map, with $\mathrm{Lip}(\tilde{f}) = \sigma < \|A^{-1}\|^{-1}$.

Remark 5.5 To understand the constants in Theorem 5.3 better, we recall that $\|A^{-1}\|^{-1}$ can be interpreted as a minimal distortion factor, in the following sense: For each $u \in E$, we have $\|u\| = \|A^{-1}.(A.u)\| \leq \|A^{-1}\| \cdot \|A.u\|$ and thus

$$\|A.u\| \;\geq\; \|A^{-1}\|^{-1}\|u\| \quad \text{for all } u \in E. \tag{63}$$

Thus A increases the norm of each element by a factor of at least $\|A^{-1}\|^{-1}$. Furthermore, $\|A^{-1}\|^{-1}$ is maximal among such factors, as one verifies by going backwards through the preceding lines. Similarly, since $A^{-1}B_s(0) \subseteq B_{\|A^{-1}\|s}(0)$ and thus $B_s(0) \subseteq A.B_{\|A^{-1}\|s}(0)$ for each $s > 0$, we find that

$$A.B_s(0) \;\supseteq\; B_{\|A^{-1}\|^{-1}s}(0) \quad \text{for all } s > 0. \tag{64}$$

Remark 5.6 The proof of Theorem 5.3 also provides the following information. Set $\alpha := a\|A^{-1}\| = 1 - \sigma\|A^{-1}\| \in \,]0,1]$ and $\beta := 1 + \sigma\|A^{-1}\| \in [1,2[$. Then $\beta \leq b\|A^{-1}\|$ and

$$\alpha\|z-y\| \leq \|A^{-1}.f(z) - A^{-1}.f(y)\| \leq \beta\|z-y\| \quad \text{for all } y,z \in B_r(x). \tag{65}$$

For every $y \in B_r(x)$ and $s \in \,]0, r - \|y-x\|]$, we have

$$f(y) + A.B_{\alpha s}(0) \;\subseteq\; f(B_s(y)) \;\subseteq\; f(y) + A.B_{\beta s}(0)\,. \tag{66}$$

Here $\alpha, \beta \to 1$ as $\sigma \to 0$.

Proof of Theorem 5.3. (c) Given $y, z \in B_r(x)$, we have

$$\begin{aligned}
\|f(z) - f(y)\| &= \|f(z) - f(y) - A.(z-y) + A.(z-y)\| \\
&\leq \|f(z) - f(y) - A.(z-y)\| + \|A.(z-y)\| \\
&\leq (\sigma + \|A\|)\|z-y\| \;=\; b\|z-y\|
\end{aligned}$$

and

$$\begin{aligned}
\|z-y\| &= \|A^{-1}.(f(z) - f(y) - A.(z-y)) - (A^{-1}.f(z) - A^{-1}.f(y))\| \\
&\leq \|A^{-1}\| \cdot \|f(z) - f(y) - A.(z-y)\| + \|A^{-1}.f(z) - A^{-1}.f(y)\| \\
&\leq \sigma\|A^{-1}\| \cdot \|z-y\| + \|A^{-1}\| \cdot \|f(z) - f(y)\|\,,
\end{aligned}$$

whence (61) holds. Likewise, (65) from Remark 5.6 follows from

$$\begin{aligned}
\|A^{-1}.f(z) - A^{-1}.f(y)\| &= \|A^{-1}.(f(z) - f(y) - A.(z-y)) + z - y\| \\
&\leq \|A^{-1}\| \cdot \|f(z) - f(y) - A.(z-y)\| + \|z-y\| \\
&\leq (\sigma\|A^{-1}\| + 1)\|z-y\| = \beta\|z-y\|
\end{aligned}$$

and

$$\begin{aligned}
\|z-y\| &\leq \|A^{-1}\| \cdot \|f(z) - f(y) - A.(z-y)\| + \|A^{-1}.f(z) - A^{-1}.f(y)\| \\
&\leq \sigma\|A^{-1}\| \cdot \|z-y\| + \|A^{-1}.f(z) - A^{-1}.f(y)\|\,.
\end{aligned}$$

(b) As a consequence of (61), f is injective, a homeomorphism onto its image, and $\mathrm{Lip}(f^{-1}) \leq a^{-1} = (\|A^{-1}\|^{-1} - \sigma)^{-1}$.

(d) Suppose that $y \in B_r(x)$ and $s \in \,]0, r - \|y - x\|]$. By (61), we have $f(B_s(y)) \subseteq B_{bs}(f(y))$, proving the second half of (62). The second half of (66) can be shown similarly: By (65), we have $A^{-1}.f(B_s(y)) \subseteq A^{-1}.f(y) + B_{\beta s}(0)$ and thus $f(B_s(y)) \subseteq f(y) + A.B_{\beta s}(0)$. We now show the first half of (66), namely

$$f(y) + A.B_{\alpha s}(0) \;\subseteq\; f(B_s(y))\,. \tag{67}$$

Then also the first half of (62) will hold, as $A.B_{\alpha s}(0) \supseteq B_{\|A^{-1}\|^{-1}\alpha s}(0) = B_{as}(0)$ by (64). To prove (67), let $c \in f(y) + A.B_{\alpha s}(0)$. There exists $t \in \,]0,1[$ such that $c \in f(y) + A.\overline{B}_{t\alpha s}(0)$. For $v \in \overline{B}_{st}(y)$, we define

$$g(v) \;:=\; v - A^{-1}.(f(v) - c)\,.$$

Then $g(v) \in \overline{B}_{st}(y)$, because

$$\begin{aligned}\|g(v) - y\| \;&\leq\; \underbrace{\|v - y - A^{-1}.f(v) + A^{-1}.f(y)\|}_{\leq \|A^{-1}\|\sigma\|v-y\| \leq \|A^{-1}\|\sigma st} + \underbrace{\|A^{-1}.c - A^{-1}.f(y)\|}_{\leq t\alpha s}\\ &\leq\; (\|A^{-1}\|\sigma + \alpha)st = st\,.\end{aligned}$$

Thus $g(\overline{B}_{st}(y)) \subseteq \overline{B}_{st}(y)$. The map $g\colon \overline{B}_{st}(y) \to \overline{B}_{st}(y)$ is a contraction, since

$$\begin{aligned}\|g(v) - g(w)\| \;&=\; \|v - w - A^{-1}.(f(v) - f(w))\|\\ &\leq\; \|A^{-1}\| \cdot \|f(v) - f(w) - A.(v - w)\|\\ &\leq\; \sigma \cdot \|A^{-1}\| \cdot \|v - w\| \end{aligned} \tag{68}$$

for all $v, w \in \overline{B}_{st}(y)$, where $\sigma\|A^{-1}\| < 1$. By Banach's Contraction Theorem (Lemma 4.2), there exists a unique element $v_0 \in \overline{B}_{st}(y)$ such that $g(v_0) = v_0$ and hence $f(v_0) = c$.

(a) We have already seen that f is a homeomorphism onto its image. As a consequence of (d), the image of f is open. □

Remark 5.7 Let E be a Banach space, $U \subseteq E$ be an open subset, $x \in U$ and $f\colon U \to E$ be a map which is strictly differentiable at x (for example, an SC^1-map, LC^1-map or C^2-map). If $f'(x) \in \mathrm{GL}(E)$, then the hypothesis of Theorem 5.3 is satisfied on $B_r(x)$ for some $r > 0$, with $A := f'(x)$.

Stronger results are available for ultrametric Banach spaces. In this case, f behaves like an affine-linear map, as far as the distortion of balls is concerned:

Theorem 5.8 (Ultrametric Lipschitz Inverse Function Theorem) *Let* $(E, \|.\|)$ *be an ultrametric Banach space over an ultrametric field* $(\mathbb{K}, |.|)$. *Let*

$r > 0$, $x \in E$, and $f\colon B_r(x) \to E$ be a mapping. We suppose that there exists $A \in \mathrm{GL}(E)$ such that

$$\sigma := \sup \left\{ \frac{\|f(z) - f(y) - A.(z-y)\|}{\|z-y\|} : \; y, z \in B_r(x),\; y \neq z \right\} < \frac{1}{\|A^{-1}\|}\,. \tag{69}$$

Then the following holds:

(a) *$A^{-1} \circ f\colon B_r(x) \to E$ is an isometry onto an open subset of E.*

(b) *f is Lipschitz, with $\mathrm{Lip}(f) \leq \|A\|$, and $f^{-1}\colon f(B_r(x)) \to B_r(x)$ is Lipschitz, with $\mathrm{Lip}(f^{-1}) \leq \|A^{-1}\|$.*

(c) *For all $y, z \in B_r(x)$, we have*

$$\|A^{-1}\|^{-1} \cdot \|z-y\| \;\leq\; \|f(z) - f(y)\| \;\leq\; \|A\| \cdot \|z-y\|\,. \tag{70}$$

(d) *For each $y \in B_r(x)$ and $s \in \,]0, r]$, we have $B_s(y) \subseteq B_r(x)$ and*

$$f(B_s(y)) \;=\; f(y) + A.B_s(0). \tag{71}$$

Proof. (a) For all $y, z \in B_r(x)$ with $y \neq z$, we have

$$\begin{aligned} \|A^{-1}.f(z) - A^{-1}.f(y) - (z-y)\| &\leq \|A^{-1}\| \cdot \|f(z) - f(y) - A.(z-y)\| \\ &< \|z-y\|\,, \end{aligned}$$

using (69) to obtain the final inequality. Hence, the norm $\|.\|$ being ultrametric, we must have $\|A^{-1}.f(z) - A^{-1}.f(y)\| = \|z-y\|$. Thus $A^{-1} \circ f$ is in fact isometric. As a consequence of Theorem 5.3, $A^{-1} \circ f$ has open image.

(b) Since $f = A \circ (A^{-1} \circ f)$ where $A^{-1} \circ f$ is an isometry, f is Lipschitz with $\mathrm{Lip}(f) \leq \mathrm{Lip}(A) = \|A\|$. Likewise, $f^{-1} = (A^{-1} \circ f)^{-1} \circ A^{-1}|_{f(B_r(x))}$ is Lipschitz, with $\mathrm{Lip}(f^{-1}) \leq \mathrm{Lip}(A^{-1}) = \|A^{-1}\|$.

(c) is a mere reformulation of (b).

(d) If $y \in B_r(x)$ and $s \in \,]0, r]$, then $B_s(y) \subseteq B_r(y) = B_r(x)$, as $\|.\|$ is ultrametric. The map $A^{-1} \circ f$ being isometric, we have $f(B_s(y)) = A.(A^{-1} \circ f)(B_s(y)) \subseteq A.B_s(A^{-1}.f(y)) = f(y) + A.B_s(0)$. If $c \in f(y) + A.B_s(0)$ is given, define

$$g(z) \;:=\; z - A^{-1}.(f(z) - c) \;\; \text{for } z \in B_s(y).$$

Then

$$\begin{aligned} \|g(z) - y\| &= \|(z-y) - (A^{-1}.f(z) - A^{-1}.f(y)) + A^{-1}.(c - f(y))\| \\ &\leq \max\{\|z-y\|, \|A^{-1}.f(z) - A^{-1}.f(y)\|, \|A^{-1}.(c - f(y))\|\} < s \end{aligned}$$

for $z \in B_s(y)$, whence $g(z) \in B_s(y)$. The map $g\colon B_s(y) \to B_s(y)$ is a contraction, by the calculation from (68). Recall that, the norm on E being ultrametric, the open ball $B_s(y)$ is also closed and therefore complete in the induced metric. By Banach's Contraction Theorem (Lemma 4.2), there is a unique element $z_0 \in B_s(y)$ such that $g(z_0) = z_0$ and thus $f(z_0) = c$. □

The following consequence of Theorem 5.8 (a) and (d) is particularly useful.

Corollary 5.9 *If A is an isometry in the situation of Theorem* 5.8 *(for example, if $A = \mathrm{id}_E$), then $f(B_r(x)) = B_r(f(x))$ and $f\colon B_r(x) \to B_r(f(x))$ is an isometry.* □

We are now in the position to formulate the first version of an inverse function theorem with parameters. The result, and its proof, can be re-used later to prove the corresponding results for C^k-maps, SC^k-maps, and LC^k-maps.

Theorem 5.10 (Lipschitz Inverse Function Theorem with Parameters) *Let $(F, \|.\|)$ be a Banach space over a valued field $\mathbb{K}$, and P be a topological space. Let $r > 0$, $x \in F$, and $f\colon P \times B \to F$ be a continuous mapping, where $B := B_r^F(x)$. Given $p \in P$, we abbreviate $f_p := f(p, \bullet)\colon B \to F$. We suppose that there exists $A \in \mathrm{GL}(F)$ such that*

$$\sigma := \sup\left\{\frac{\|f_p(z) - f_p(y) - A.(z-y)\|}{\|z-y\|}\colon\ p \in P,\ y, z \in B,\ y \neq z\right\} < \frac{1}{\|A^{-1}\|}\,. \tag{72}$$

Then the following holds:

(a) *$f_p(B)$ is open in F and $f_p|_B$ is a homeomorphism onto its image, for each $p \in P$.*

(b) *The set $W := \bigcup_{p\in P} \{p\} \times f_p(B)$ is open in $P \times F$, and the map $\psi\colon W \to F$, $\psi(p,z) := (f_p|_B^{f_p(B)})^{-1}(z)$ is continuous.*

(c) *The map $\xi\colon P \times B \to W$, $\xi(p,y) := (p, f(p,y))$ is a homeomorphism, with inverse given by $\xi^{-1}(p,z) = (p, \psi(p,z))$.*

If P is a subset of a topological $\mathbb{K}$-vector space E here and f is Lipschitz continuous, then also ψ, ξ, ξ^{-1} and each of the maps $(f_p|_B)^{-1}$ are Lipschitz continuous.

Proof. (a)–(c): By Theorem 5.3, applied to f_p, the set $f_p(B)$ is open in F and $f_p|_B$ a homeomorphism onto its image. Define $\alpha := 1 - \sigma\|A^{-1}\|$. Let us show openness of W and continuity of h. If $(p,z) \in W$, there exists $y \in B$ such that $f_p(y) = z$. Let $\varepsilon \in \,]0, r - \|y - x\|]$ be given. There is an open neighborhood Q of p in P such that $f(q,y) \in f(p,y) + A.B_{\frac{\alpha\varepsilon}{2}}(0)$ for all $q \in Q$, by continuity of f. Then, as a consequence of Remark 5.6, Equation (66),

$$f_q(B_\varepsilon(y)) \supseteq f(q,y) + A.B_{\alpha\varepsilon}(0) \supseteq f(p,y) + A.\overline{B}_{\frac{\alpha\varepsilon}{2}}(0) = z + A.\overline{B}_{\frac{\alpha\varepsilon}{2}}(0)\,.$$

By the preceding, $Q \times (z + A.\overline{B}_{\frac{\alpha\varepsilon}{2}}(0)) \subseteq W$, whence W is a neighborhood of (p,z). Furthermore, $\psi(q,z') = (f_q)^{-1}(z') \in B_\varepsilon(y) = B_\varepsilon((f_p)^{-1}(z)) = B_\varepsilon(\psi(p,z))$ for all (q,z') in the neighborhood $Q \times (z + A.\overline{B}_{\frac{\alpha\varepsilon}{2}}(0))$ of (p,z). Thus W is open and ψ is continuous. The assertions concerning ξ follow immediately.

Final assertions. Assume now that P is a subset of a topological $\mathbb{K}$-vector space E and f is Lipschitz continuous. We only need to show that ψ is Lipschitz continuous (the remaining assertions are then immediate). To this end, let

$(p,z) \in W$. Let y, ε and Q be as before. By continuity of f, after shrinking Q and ε, we may assume that

$$\|A^{-1}\| \cdot \|f_q(v) - f_p(v)\| \leq \frac{1-\theta}{8}\frac{\varepsilon}{2} \quad \text{for all } q \in Q \text{ and } v \in \overline{B}_{\frac{\varepsilon}{2}}(y). \tag{73}$$

Abbreviate $\theta := \sigma\|A^{-1}\| < 1$. Our goal is to see that ψ is Lipschitz continuous on the neighborhood $Q \times (z + A\overline{B}_{(1-\theta)\varepsilon/4}(0))$ of (p,z), where $1-\theta = \alpha$ from above. To achieve this, for each (q,c) in this neighborhood we interpret $\psi(q,c)$ as a fixed point of a suitable contraction $g_{(q,c)}$ and then apply Proposition 4.5 on the Lipschitz continuous dependence of fixed points on parameters. The Lipschitz continuous map

$$g\colon Q \times \big(z + A\overline{B}_{(1-\theta)\varepsilon/4}(0)\big) \times \overline{B}_{\frac{\varepsilon}{2}}(y) \to F\,, \quad g(q,c,v) := v - A^{-1}.(f_q(v) - c)$$

will serve our purpose. Note first that for all $(q,c) \in Q \times (z + A\overline{B}_{(1-\theta)\varepsilon/4}(0))$, the map $g_{(q,c)} := g(q,c,\bullet)\colon \overline{B}_{\frac{\varepsilon}{2}}(y) \to F$ satisfies

$$\begin{aligned}
\|g_{(q,c)}(v) - g_{(q,c)}(w)\| &= \|g(q,c,v) - g(q,c,w)\| \\
&= \|A^{-1}.f_q(v) - A^{-1}.f_q(w) - (v-w)\| \\
&\leq \|A^{-1}\|\,\|f_q(v) - f_q(w) - A.(v-w)\| \\
&\leq \|A^{-1}\|\sigma\|v-w\| \;=\; \theta\|v-w\|\,;
\end{aligned} \tag{74}$$

we are thus dealing with a uniform family of contractions. Each $g_{(q,c)}$ is a self-map of $\overline{B}_{\frac{\varepsilon}{2}}(y)$, because, for each $v \in \overline{B}_{\frac{\varepsilon}{2}}(y)$,

$$\begin{aligned}
&\|g(q,c,v) - y\| \\
&\quad= \|g(q,c,v) - g(p,z,y)\| \\
&\quad\leq \|g(q,c,v) - g(p,c,v)\| + \|g(p,c,v) - g(p,z,v)\| + \|g(p,z,v) - g(p,z,y)\| \\
&\quad\leq \|A^{-1}.(f_q(v) - f_p(v))\| + \|A^{-1}.(c-z)\| + \theta\|v-y\| \\
&\quad\leq \tfrac{1-\theta}{8}\tfrac{\varepsilon}{2} + \tfrac{(1-\theta)\varepsilon}{4} + \theta\tfrac{\varepsilon}{2} \;\leq\; \tfrac{\varepsilon}{2}\,.
\end{aligned} \tag{75}$$

Here, we used (74) to pass to the penultimate line and then (73). By Banach's Contraction Theorem (Lemma 4.2) and Proposition 4.5, $g_{(q,c)}$ has a unique fixed point $\phi(q,c)$ for each (q,c), and the map $\phi\colon Q \times (z + A\overline{B}_{(1-\theta)\varepsilon/4}(0)) \to F$ is Lipschitz continuous. But $g_{(q,c)}(v) = v$ if and only if $f_q(v) = c$, i.e., if and only if $v = \psi(q,c)$. Thus $\phi(q,c) = \psi(q,c)$ and thus ψ is Lipschitz continuous on $Q \times (z + A\overline{B}_{(1-\theta)\varepsilon/4}(0))$. □

Remark 5.11 Note that if $\varepsilon' \in [\frac{\varepsilon}{2}, \varepsilon]$, $q \in Q$ and $c \in z + A\overline{B}_{(1-\theta)\varepsilon'/4}(0)$, then the calculation (75) shows that $\|g(q,c,v) - y\| \leq \frac{\varepsilon'}{2}$. This will be useful later.

As an immediate consequence, we obtain an implicit function theorem.

Corollary 5.12 (Lipschitz Implicit Function Theorem) *In the situation of Theorem* 5.10*, let* $(p_0, y_0) \in P \times B$*. Then there exists an open neighborhood* $Q \subseteq P$ *of* p_0 *such that* $z_0 := f(p_0, y_0) \in f_p(B)$ *for all* $p \in Q$*. The mapping* $\lambda\colon Q \to B$*,* $\lambda(p) := \psi(p, z_0)$ *is continuous (resp., Lipschitz continuous if so is* f*), satisfies* $\lambda(p_0) = y_0$*, and*

$$\{(p, y) \in Q \times B\colon f(p, y) = z_0\} \;=\; \operatorname{graph}(\lambda)\,.$$

Proof. Because W is an open neighborhood of (p_0, z_0) in $P \times F$, there exists an open neighborhood Q of p_0 in P such that $Q \times \{z_0\} \subseteq W$. Then $\lambda(p) := \psi(p, z_0)$ makes sense for all $p \in Q$. The rest is now obvious from Theorem 5.10. □

Inverse Function Theorem with Parameters

We are now in the position to formulate and prove our main result, an Inverse Function Theorem with Parameters for various types of differentiable mappings.

Theorem 5.13 (Inverse Function Theorem with Parameters) *Let* $\mathbb{K}$ *be a valued field,* $k \in \mathbb{N} \cup \{\infty\}$*,* E *be a topological* $\mathbb{K}$*-vector space, and* F *be a Banach space over* $\mathbb{K}$*. Let* $P_0 \subseteq E$ *be a subset with dense interior,* $U \subseteq F$ *be open, and* $f\colon P_0 \times U \to F$ *be a map. Assume that*

(i) f *is* LC^k*, respectively,* SC^k*; or:*

(ii) f *is* C^k *and* $k \geq 2$*, or* f *is* C^k*,* $\mathbb{K}$ *is locally compact and* F *has finite dimension.*

Abbreviate $f_p := f(p, \bullet)\colon U \to F$ *for* $p \in P_0$*. Suppose that* $(p_0, x_0) \in P_0 \times U$ *is given such that* $f'_{p_0}(x_0) \in \mathrm{GL}(F)$*. Then there exists an open neighborhood* $P \subseteq P_0$ *of* p_0 *and* $r > 0$ *such that* $B := B_r(x_0) \subseteq U$ *and the following holds:*

(a) $f_p(B)$ *is open in* F*, for each* $p \in P$*, and* $\phi_p\colon B \to f_p(B)$*,* $\phi_p(x) := f_p(x) = f(p, x)$ *is an* LC^k*-diffeomorphism (resp., an* SC^k*-diffeomorphism; resp., a* C^k*-diffeomorphism).*

(b) $W := \bigcup_{p \in P}(\{p\} \times f_p(B))$ *is open in* $P_0 \times F$*, and the map*

$$\psi\colon W \to B, \quad \psi(p, z) \;:=\; \phi_p^{-1}(z)$$

is LC^k *(resp.,* SC^k*, resp.,* C^k*). Furthermore, the map*

$$\xi\colon P \times B \to W, \quad \xi(p, x) := (p, f(p, x))$$

is an LC^k*-diffeomorphism (resp., an* SC^k*-diffeomorphism, resp., a* C^k*-diffeomorphism), with inverse* $\xi^{-1}(p, z) = (p, \psi(p, z))$*.*

(c) $P \times B_\delta(f_{p_0}(x_0)) \subseteq W$ *for some* $\delta > 0$*.*

In particular, for each $p \in P$ there is a unique element $\lambda(p) \in B$ such that $f(p,\lambda(p)) = f(p_0, x_0)$, and the map $\lambda\colon P \to B$ so obtained is LC^k (resp., SC^k, resp., C^k).

Remark 5.14 Given $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < 1 < \beta$ in the situation of Theorem 5.13, one can furthermore achieve that

$$f_p(x) + A.B_{\alpha s}(0) \;\subseteq\; f_p(B_s(x)) \;\subseteq\; f_p(x) + A.B_{\beta s}(0) \tag{76}$$

for all $p \in Q$, $x \in B$ and $s \in \,]0, r - \|x - x_0\|]$.

Proof. Given $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < 1 < \beta$, define

$$\tau \;:=\; \min\left\{\tfrac{\beta-1}{\|A^{-1}\|}, \tfrac{1-\alpha}{\|A^{-1}\|}\right\} \;<\; \tfrac{1}{\|A^{-1}\|}\,,$$

where $A := f'_{p_0}(x_0)$. Then $1 - \tau\|A^{-1}\| \geq \alpha$ and $1 + \tau\|A^{-1}\| \leq \beta$. By strict differentiability of f at (p_0, x_0) (resp., by Remark 3.14), there exists an open neighborhood $P \subseteq P_0$ of p_0 and $r > 0$ such that $B := B_r(x) \subseteq U$ and

$$\|f_p(z) - f_p(y) - f'_{p_0}(x).(z-y)\| \;\leq\; \tau\,\|z - y\| \tag{77}$$

for all $p \in P$ and $y \neq z \in B$. Hence

$$\begin{aligned} \sigma \;&:=\; \sup\left\{\frac{\|f_p(z) - f_p(y) - f'_{p_0}(x_0).(z-y)\|}{\|z-y\|} : \; p \in P,\, z \neq y \in B\right\} \\ &\leq\; \tau \;<\; \frac{1}{\|A^{-1}\|}\,. \end{aligned} \tag{78}$$

Thus Theorem 5.10 applies to $f|_{P\times B}$ with $A := f'_{p_0}(x_0)$, whence $f_p(B)$ is open in F and $\phi_p := f_p|_B^{f_q(B)}$ a homeomorphism onto its image, for each $p \in P$; the set $W := \bigcup_{p\in P}\{p\} \times f_p(B)$ is open in $P_0 \times F$; the map $\psi\colon W \to B$, $\psi(p,z) := \phi_p^{-1}(z)$ is continuous; and the mapping $\xi\colon P \times B \to W$, $\xi(p,y) := (p, f(p,y))$ is a homeomorphism, with inverse given by $\xi^{-1}(p,z) = (p, \psi(p,z))$. In view of (78), Lemma 5.3 applies to $f_p|_B$, for all $p \in P$, whence (76) in Remark 5.14 holds.

Also (c) is easily established: we set $\delta := \|A^{-1}\|^{-1}\frac{\alpha r}{2}$. After shrinking P, we may assume that $\|f(p,x_0) - f(p_0,x_0)\| < \delta$ for all $p \in P$. Then, using (76) with $x := x_0$ and $s := r$, we get $f_p(B) \supseteq f_p(x_0) + A.B_{\alpha r}(0) \supseteq B_{2\delta}(f_p(x_0)) \supseteq B_\delta(f_{p_0}(x_0))$, for all $p \in P$. Thus (c) holds.

(a) and (b): If can show that ψ is LC^k (resp., SC^k, resp., C^k), then clearly all of the maps ψ, ξ, λ and ϕ_q will have the desired properties. Given $(p,z) \in W$, we let $A := f'_{p_0}(x_0)$ as before and define y, θ, Q, ε and the map

$$g\colon Q \times \big(z + AB_{(1-\theta)\varepsilon/4}(0)\big) \times B_{\frac{\varepsilon}{2}}(y) \to F\,, \quad g(q,c,v) \;:=\; v - A^{-1}.(f_q(v) - c)$$

as in the proof of Theorem 5.10 (using now open balls instead of closed balls). Because the arguments from the proof of Theorem 5.10 apply to the restriction of g to $Q \times \big(z + A\overline{B}_{(1-\theta)\varepsilon'/4}(0)\big) \times \overline{B}_{\frac{\varepsilon'}{2}}(y)$ for each $\varepsilon' \in [\frac{\varepsilon}{2}, \varepsilon[$ (see Remark 5.11), we deduce that $g(q, c, \bullet)$ has a fixed point in $B_{\frac{\varepsilon}{2}}(y)$, for each $q \in Q$ and each $c \in z + AB_{(1-\theta)\varepsilon/4}(0)$. Repeating the arguments used in the proof of Theorem 5.10, we see that $(g(q, c, \bullet))_{q,c}$ is a uniform family of contractions for $q \in Q$, $c \in z + AB_{(1-\theta)\varepsilon/4}(0)$, and that $\psi(q, c)$ is the unique fixed point of the contraction $g(q, c, \bullet)\colon B_{\frac{\varepsilon}{2}}(y) \to B_{\frac{\varepsilon}{2}}(y)$. Since g is LC^k (resp., SC^k, resp., C^k), Proposition 4.5 shows that ψ is LC^k (resp., SC^k, resp., C^k) on the open neighborhood $Q \times \big(z + AB_{(1-\theta)\varepsilon/4}(0)\big)$ of (p, z) in W. This completes the proof. □

Remark 5.15 If f is C^1 in the situation of Theorem 5.13 but $\mathbb{K}$ is not locally compact or F is infinite-dimensional, then the conclusions of the theorem still remain intact if we assume that (72) is satisfied by f with $A := f'_{p_0}(x_0)$. This ensures that we are in the situation of Theorem 5.10, and we can now complete the proof (with $\alpha := 1 - \sigma\|A^{-1}\|$ and $\beta = 1 + \sigma\|A^{-1}\|$) as before, noting that the C^1-dependence of fixed points on parameters established in Theorem 4.7 requires neither local compactness of $\mathbb{K}$ nor finite-dimensionality of F.

The same reasoning shows that if f is C^k, SC^k or LC^k for some $k \in \mathbb{N} \cup \{\infty\}$ in the situation of the Lipschitz Inverse Function Theorem (Theorem 5.3), then $f^{-1}\colon f(B_r(x)) \to B_r(x)$ is C^k (resp., SC^k, resp., LC^k).

Remark 5.16 Note that Theorem 5.13 subsumes as its final assertion the Generalized Implicit Function Theorem announced above (Theorem 5.2). Using a singleton set of parameters, we also obtain the ordinary Inverse Function Theorem (Theorem 5.1) as a special case.

For ultrametric Banach spaces, the Inverse Function Theorem with Parameters attains a simpler form:

Theorem 5.17 *Let $(\mathbb{K}, |.|)$ be an ultrametric field, $k \in \mathbb{N} \cup \{\infty\}$, E be a topological $\mathbb{K}$-vector space, and F be an ultrametric Banach space over $\mathbb{K}$. Let $P_0 \subseteq E$ be a subset with dense interior, $U \subseteq F$ be open, and $f\colon P_0 \times U \to F$ be a map. Assume that*

(i) *f is LC^k, respectively, SC^k; or:*

(ii) *f is C^k and $k \geq 2$, or f is C^k, $\mathbb{K}$ is locally compact and F has finite dimension.*

Abbreviate $f_p := f(p, \bullet)\colon U \to F$ for $p \in P_0$. Suppose that $(p_0, x_0) \in P_0 \times U$ is given such that $A := f'_{p_0}(x_0) \in \mathrm{GL}(F)$. Then there exists an open neighborhood $P \subseteq P_0$ of p_0 and $r > 0$ such that $B := B_r(x_0) \subseteq U$ and the following holds:

(a) *$f_p(B) = f(p_0, x_0) + A.B_r(0) =: V$, for each $p \in P$, and $\phi_p\colon B \to V$, $\phi_p(y) := f(p, y)$ is an LC^k-diffeomorphism (resp., an SC^k-diffeomorphism, resp., a C^k-diffeomorphism).*

(b) $f_p(B_s(y)) = f_p(y) + A.B_s(0)$ *for all* $p \in P$, $y \in B$ *and* $s \in]0, r]$.

(c) *The map* $\psi\colon P \times V \to B$, $\psi(p, v) := \phi_p^{-1}(v)$ *is* LC^k, SC^k, *resp.,* C^k.

(d) $\xi\colon P \times B \to P \times V$, $\xi(p, y) := (p, f(p, y))$ *is an* LC^k*-diffeomorphism (resp.,* SC^k*-diffeomorphism, resp.,* C^k*-diffeomorphism), with inverse given by* $\xi^{-1}(p, v) = (p, \psi(p, v))$.

Proof. We let P and r be as in the proof of Theorem 5.13, choosing P so small that

$$\|A^{-1}.f(p, x_0) - A^{-1}.f(p_0, x_0)\| \; < \; r \tag{79}$$

for all $p \in P$. Since (b) holds by Theorem 5.8 (d), taking $s := r$ we deduce that

$$\begin{aligned} f_p(B_r(x_0)) &= f_p(x_0) + A.B_r(0) \; = \; f_{p_0}(x_0) + (f_p(x_0) - f_{p_0}(x_0)) + A.B_r(0) \\ &= f_{p_0}(x_0) + A.B_r(0) \; =: \; V\,. \end{aligned}$$

The final equality holds because $f_p(x_0) - f_{p_0}(x_0) \in A.B_r(0)$ by (79) and $B_r(0)$ is an additive subgroup of F. Thus (a) holds by Theorem 5.13 (a). Furthermore, $W := \bigcup_{p \in P} \{p\} \times f_p(B) = P \times V$ by (a), whence (c) and (d) hold by Theorem 5.13 (b). □

A Appendix: $\mathcal{C}^0$-concepts

In this appendix, we describe a version of the notion of "$\mathcal{C}^0$-concept" introduced in [2]. While only mappings between open sets were considered in [2], we now define $\mathcal{C}^0$-concepts for mappings between subsets of topological vector spaces with dense interior. In other respects, our $\mathcal{C}^0$-concepts are more restrictive than those from [2]. In particular, we are working only with Hausdorff topological vector spaces over Hausdorff topological fields, whereas suitable topologized modules over suitable topologized rings provided the general framework in [2].

Definition A.1 Let $\mathbb{K}$ be a (non-discrete, Hausdorff) topological field and $\mathcal{E}$ be a class of (Hausdorff) topological $\mathbb{K}$-vector spaces satisfying the following axioms:

(E1) $\mathbb{K} \in \mathcal{E}$ and $\{0\} \in \mathcal{E}$ hold:

(E2) If $E \in \mathcal{E}$ and F is a topological $\mathbb{K}$-vector space isomorphic to E, then $F \in \mathcal{E}$;

(E3) If $E_1, E_2 \in \mathcal{E}$, then also $E_1 \times E_2 \in \mathcal{E}$ (when equipped with the product topology).

A $\mathcal{C}^0$*-concept* over $\mathbb{K}$ (with underlying class of topological vector spaces $\mathcal{E}$) assigns a set $\mathcal{C}^0(U, V) \subseteq C(U, V)$ of continuous maps to all $E, F \in \mathcal{E}$ and subsets $U \subseteq E$ and $V \subseteq F$ with dense interior, such that the following axioms are satisfied:

(C01) If $E, F, H \in \mathcal{E}$ and $U \subseteq E$, $V \subseteq F$, $W \subseteq H$ are subsets with dense interior, then $g \circ f \in \mathcal{C}^0(U, W)$ for all $f \in \mathcal{C}^0(U, V)$ and $g \in \mathcal{C}^0(V, W)$. Furthermore, $\mathrm{id}_U \in \mathcal{C}^0(U, U)$.

(C02) For each $E \in \mathcal{E}$ and subset $U \subseteq E$ with dense interior, the inclusion map $i_U \colon U \to E$ is $\mathcal{C}^0$.

(C03) For all $E, F \in \mathcal{E}$ and subsets $U \subseteq E$, $V \subseteq F$ with dense interior, a map $f \colon U \to V$ is $\mathcal{C}^0$ if and only if it is $\mathcal{C}^0$ as a map into F, i.e., if and only if $i_V \circ f \in \mathcal{C}^0(U, F)$.

(C04) If $n \in \mathbb{N}$, $E_1, \ldots, E_n, F \in \mathcal{E}$ and $\beta \colon E_1 \times \cdots \times E_n \to F$ is a continuous n-linear map, then $\beta \in \mathcal{C}^0(E_1 \times \cdots \times E_n, F)$.

(C05) Given $F, E_1, E_2 \in \mathcal{E}$ and a subset $U \subseteq F$ with dense interior, a mapping $f = (f_1, f_2) \colon U \to E_1 \times E_2$ is $\mathcal{C}^0$ if and only if both components f_1 and f_2 are $\mathcal{C}^0$.

(C06) (Locality). If $E, F \in \mathcal{E}$, $U \subseteq E$ is a subset with dense interior and $f \colon U \to F$ a mapping such that $f|_{U_i} \in \mathcal{C}^0(U_i, F)$ for an open cover $(U_i)_{i \in I}$ of U, then $f \in \mathcal{C}^0(U, F)$.

Remark A.2 Whenever a $\mathcal{C}^0$-concept is used in the present article, $\mathcal{E}$ simply is the class of all Hausdorff topological $\mathbb{K}$-vector spaces. But, of course, there are other interesting classes of topological $\mathbb{K}$-vector spaces, for example the classes of locally convex (polynormed, normable, resp., complete normable) spaces over valued fields, or classes of spaces with certain completeness properties, etc.

Remark A.3 Here are some consequences of the axioms.

(a) Property (C01) means that the pairs (E, U), where $E \in \mathcal{E}$ and $U \subseteq E$ is a subset with dense interior, form a category with $\mathrm{Hom}((U, E), (V, F)) = \mathcal{C}^0(U, V)$ as respective set of morphisms.[4]

(b) Properties (C01) and (C02) guarantee in particular that $f|_U = f \circ i_U \in \mathcal{C}^0(U, F)$ for each $\mathcal{C}^0$-map $f \colon E \to F$ and subset $U \subseteq E$ with dense interior.

(c) (C04) ensures that continuous linear maps and continuous bilinear maps are $\mathcal{C}^0$. Hence the addition map $E \times E \to E$ and the scalar multiplication map $\mathbb{K} \times E \to E$ are $\mathcal{C}^0$, for each $E \in \mathcal{E}$.

(d) Note that if $U \subseteq \mathbb{K}$ has dense interior and $t \in \mathbb{K}$, then $U \setminus \{t\}$ is dense in U, whence a $\mathcal{C}^0$-map $f \colon U \to F$ is uniquely determined by its restriction to $U \setminus \{t\}$. This property ("determination axiom") plays an essential role in the discussions of [2] (in the case of open subsets).

[4]We prefer to suppress E and F in the notation $\mathcal{C}^0(U, V)$, with little risk of misunderstanding.

(e) Given a $\mathcal{C}^0$-concept in the present sense, restricting attention to mappings between open subsets we obtain a $\mathcal{C}^0$-concept in the sense of [2].

(f) In [2], Property (C04) is not required in full, but it is satisfied by all interesting examples of $\mathcal{C}^0$-concepts based on topological vector spaces, and makes it unnecessary to distinguish between continuous linear maps and linear maps which are $\mathcal{C}^0$ (and similar nuisances).

(g) Usually, one only specifies the $\mathcal{C}^0$-maps $U \to F$ on subsets $U \subseteq E$ with dense interior, for all $E, F \in \mathcal{E}$. One then tacitly uses (C03) as the *definition* of $\mathcal{C}^0$-maps to a subset $V \subseteq F$ with dense interior.

Definition A.4 Given a $\mathcal{C}^0$-concept over a topological field $\mathbb{K}$ based on a class $\mathcal{E}$ of topological $\mathbb{K}$-vector spaces, we define $\mathcal{C}^1$-maps as follows: Let $E, F \in \mathcal{E}$ and $f\colon U \to F$ be a $\mathcal{C}^0$-map on a subset $U \subseteq E$ with dense interior. We say that f is $\mathcal{C}^1$ if there exists a $\mathcal{C}^0$-map $f^{[1]}\colon U^{[1]} \to F$ which extends $f^{]1[}\colon U^{]1[} \to F$ (where $U^{]1[}$ and $f^{]1[}$ are as in Section 1). Recursively, we say that f is $\mathcal{C}^k$ if f is $\mathcal{C}^1$ and $f^{[1]}$ is $\mathcal{C}^{k-1}$.

Then all relevant results from [2] (and their proofs) remain valid. In particular, the Chain Rule holds for $\mathcal{C}^k$-maps; being $\mathcal{C}^k$ is a local property; finite-order Taylor expansions are available, etc.

B Appendix: A variant of Lemma 2.9

In this appendix, we prove the following variant of Lemma 2.9 for C^1-maps (which is not needed in the main text).

Lemma B.1 *Let $\mathbb{K}$ be a locally compact field, E and H be topological $\mathbb{K}$-vector spaces and F be a finite-dimensional normed $\mathbb{K}$-vector space. Let $U \subseteq E$ and $V \subseteq F$ be subsets with dense interior and $f\colon U \times V \to H$ be a C^1-map. Let $u_0 \in U$, $v_0 \in V$, $x_0 \in E$, $y_0 \in F$, γ be a gauge on H, and $\varepsilon > 0$. Then there exist neighborhoods $U_0 \subseteq U$ of u_0, $V_0 \subseteq V$ of v_0, $X_0 \subseteq E$ of x_0, $Y_0 \subseteq F$ of y_0 and a 0-neighborhood $S_0 \subseteq \mathbb{K}$ such that*

$$\|f^{[1]}(u, v, w, y_1, t) - f^{[1]}(u, v, w, y_2, t) - f'_{u_0}(v_0).(y_1 - y_2)\|_\gamma \;\leq\; \varepsilon\,\|y_1 - y_2\| \quad (80)$$

for all elements $u \in U_0$, $v \in V_0$, $x \in X_0$, $y_1, y_2 \in Y_0$ and $t \in S_0$ such that $(u, v, x, y_1, t), (u, v, x, y_2, t) \in (U \times V)^{[1]}$.

The proof of Lemma B.1 uses a variant of Lemma 1.13 for $f^{[2]}$:

Lemma B.2 *Let E and F be topological vector spaces over a topological field $\mathbb{K}$, and $f\colon U \to F$ be a C^2-map, defined on a subset of E with dense interior. If $t \in \mathbb{K}^\times$, $x, x_1, y, y_1 \in E$ and $s, s_1, s_2 \in \mathbb{K}$ such that*

$$((x, y, ts),\, (x_1, y_1, ts_1),\, ts_2) \in U^{[2]}\,,$$

then also $((x,t^2y,\frac{s}{t}),\,(tx_1,t^3y_1,s_1),\,s_2)\in U^{[2]}$*, and*

$$t^3\,f^{[2]}((x,y,ts),\,(x_1,y_1,ts_1),\,ts_2)=f^{[2]}((x,t^2y,\tfrac{s}{t}),\,(tx_1,t^3y_1,s_1),\,s_2)\,. \tag{81}$$

Proof. See [12, Lemma 3.3 (b)] for the case of open domains. The proof carries over verbatim. □

Proof of Lemma B.1. We may assume that $\|F\|\subseteq|\mathbb{K}|$ (because all norms on F are equivalent, and we can take a maximum norm with respect to some basis). Let Ω be the set of all $(u,v,x,t,y_1,y_2)\in U\times V\times E\times\mathbb{K}\times F\times F$ such that $(u,v,x,y_1,t),(u,v,x,y_2,t)\in(U\times V)^{[1]}$. We choose a gauge $\|.\|_\zeta$ on H such that $\|a+b\|_\gamma\leq\max\{\|a\|_\zeta,\|b\|_\zeta\}$ for all $a,b\in H$. Given $(u,v,x,t,y_1,y_2)\in\Omega$, we have

$$\begin{aligned}
&\|f^{[1]}(u,v,x,y_1,t)-f^{[1]}(u,v,x,y_2,t)-f'_{u_0}(v_0).(y_1-y_2)\|_\gamma\\
&\quad=\;\|f^{[1]}(u+tx,v+ty_2,0,y_1-y_2,t)-f'_{u_0}(v_0).(y_1-y_2)\|_\gamma\\
&\quad=\;\|f^{[1]}(u+tx,v+ty_2,0,y_1-y_2,t)-f'_{u+tx}(v).(y_1-y_2)\\
&\qquad\qquad+(f'_{u+tx}(v)-f'_{u_0}(v_0)).(y_1-y_2)\|_\gamma\\
&\quad\leq\;\max\big\{\|f^{[1]}(u+tx,v+ty_2,0,y_1-y_2,t)-f'_{u+tx}(v).(y_1-y_2)\|_\zeta,\\
&\qquad\;\|(f'_{u+tx}(v)-f'_{u_0}(v_0)).(y_1-y_2)\|_\zeta\big\}\,,
\end{aligned}\tag{82}$$

using Lemma 1.12 to obtain the first equality. Abbreviate

$$\Omega_1:=\{(u,v,x,t,y_1,y_2,w,r)\in\Omega\times F\times\mathbb{K}\colon(u+tx,v+ty_2,0,w,r)\in(U\times V)^{[1]}\}.$$

The function

$$g\colon\Omega_1\to H\,,\quad g(u,v,x,t,y_1,y_2,w,r):=f^{[1]}(u+tx,v+ty_2,0,w,r)-f'_{u+tx}(v).w$$

is continuous and vanishes on the compact set $\{(u_0,v_0,x_0,0,y_0,y_0)\}\times K\times\{0\}$, where $K:=\overline{B}^F_1(0)$. Hence, there exist neighborhoods $U_0\subseteq U$ of u_0, $V_0\subseteq V$ of v_0, $X_0\subseteq E$ of x_0, $Y_0\subseteq F$ of y_0 and a balanced 0-neighborhood $S_0\subseteq\mathbb{K}$ such that

$$\|g(u,v,x,t,y_1,y_2,w,r)\|_\zeta\;\leq\;\varepsilon$$

for all $(u,v,x,t,y_1,y_2,w,r)\in\Omega_1\cap(U_0\times V_0\times X_0\times S_0\times Y_0\times Y_0\times K\times S_0)=:\Omega_2$. Given $(u,v,x,t,y_1,y_2,w,r)\in\Omega_2$ and $s\in\mathbb{K}^\times$ such that $\|w\|\leq|s|\leq 1$, we have $(u,v,x,t,y_1,y_2,s^{-1}w,sr)\in\Omega_2$ and

$$\begin{aligned}
&\|g(u,v,x,t,y_1,y_2,w,r)\|_\zeta\\
&\quad=\;\|f^{[1]}(u+tx,v+ty_2,0,w,r)-f'_{u+tx}(v).w\|_\zeta\\
&\quad=\;|s|\cdot\|f^{[1]}(u+tx,v+ty_2,0,s^{-1}w,sr)-f'_{u+tx}(v).s^{-1}w\|_\zeta\\
&\quad=\;|s|\cdot\|g(u,v,x,t,y_1,y_2,s^{-1}w,sr)\|_\zeta\leq|s|\cdot\varepsilon\,.
\end{aligned}\tag{83}$$

If $\|w\| > 0$, we can choose s such that $|s| = \|w\|$; if $\|w\| = 0$, we can let s pass to 0. In either case, (83) entails that $\|g(u,v,x,t,y_1,y_2,w,r)\|_\zeta \leq \varepsilon\|w\|$. After shrinking Y_0, we may assume that $Y_0 - Y_0 \subseteq K$. For all $(u,v,x,t,y_1,y_2) \in \Omega \cap (U_0 \times V_0 \times X_0 \times S_0 \times Y_0 \times Y_0)$, we then have $(u,v,x,t,y_1,y_2,y_1-y_2,t) \in \Omega_2$ and

$$\begin{aligned} &\|f^{[1]}(u+tx, v+ty_2, 0, y_1-y_2, t) - f'_{u+tx}(v).(y_1-y_2)\|_\zeta \\ &\quad = \ \|g(u,v,x,t,y_1,y_2,y_1-y_2,t)\|_\zeta \ \leq \ \varepsilon\|y_1-y_2\| \,. \end{aligned} \tag{84}$$

By Lemma 3.13, after shrinking U_0, V_0, X_0, Y_0 and S_0, we can achieve that also

$$\|(f'_{u+tx}(v) - f'_{u_0}(v_0)).(y_1-y_2)\|_\zeta \ \leq \ \varepsilon\, \|y_1-y_2\| \tag{85}$$

for all $(u,v,x,t,y_1,y_2) \in \Omega \cap (U_0 \times V_0 \times X_0 \times S_0 \times Y_0 \times Y_0)$. Combining (82), (84) and (85), we now see that (80) holds. □

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Helge Glöckner, TU Darmstadt, FB Mathematik AG 5, Schlossgartenstr. 7,
64289 Darmstadt, Germany. E-Mail: gloeckner@mathematik.tu-darmstadt.de