Fundamental Problems in the Theory of Infinite-Dimensional Lie Groups

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Abstract

In a preprint from 1982, John Milnor formulated various fundamental questions concerning infinite-dimensional Lie groups. In this note, we describe some of the answers (and partial answers) obtained in the preceding years.

Introduction

Specific classes of infinite-dimensional Lie groups (like groups of operators, gauge groups, and diffeomorphism groups) have been studied extensively and are well understood. But much less is known about general infinite-dimensional Lie groups, and many fundamental problems are still unsolved. Typical problems were recorded in John Milnor's preprint [19], which preceded his well-known survey article [20]. In this note, we recall Milnor's questions and their background and describe some of the answers (or partial answers) obtained so far.

1 Basic definitions

To define infinite-dimensional Lie groups, John Milnor uses the following notion of smooth maps between locally convex spaces (which are known as "Keller C^∞_c-maps" [15] in the literature):

Definition. Let E and F be real locally convex spaces, $U \subseteq E$ be open, and $f\colon U \to F$ be a map. For $x \in U$ and $y \in E$, let $(D_y f)(x) := \frac{d}{dt}\big|_{t=0} f(x+ty)$ be the directional derivative (if it exists). Given $k \in \mathbb{N} \cup \{\infty\}$, the map f is called C^k if it is continuous, the iterated directional derivatives

$$d^j f(x, y_1, \ldots, y_j) := (D_{y_j} \cdots D_{y_1} f)(x)$$

exist for all $j \in \mathbb{N}$ such that $j \leq k$, $x \in U$ and $y_1, \ldots, y_j \in E$, and all of the maps $d^j f\colon U \times E^j \to F$ are continuous. As usual, C^∞-maps are also called *smooth*.

A *smooth manifold* modeled on a locally convex topological vector space E is a Hausdorff topological space M, together with a set $\mathcal{A}$ of homeomorphisms from open subsets of M onto open subsets of E, such that the domains cover M and the transition maps are smooth. Smoothness of maps between manifolds is defined as in the finite-dimensional case (it can be tested in local charts). Also products of manifolds are defined as usual. A *Lie group* is a group, equipped with a smooth manifold structure modelled on a locally convex space E such that the group operations are smooth maps. Lie groups modelled on Banach spaces are called *Banach-Lie groups*. As in finite dimensions, the tangent space $L(G) := T_1(G) \cong E$ at the identity element of a Lie group G can be made a topological Lie algebra via the identification with the Lie algebra of left invariant vector fields on G.

Milnor requires in [19] that the modelling space is complete, and relaxes the condition to sequential completeness (convergence of Cauchy sequences) in [20]. We follow his custom here (unless we explicitly state the contrary).

Occasionally, we shall also encounter analytic mappings and the corresponding Lie groups. Given complex locally convex spaces E and F, a map $f\colon U \to F$ on an open subset $U \subseteq E$ is called *complex analytic* if it is continuous and for each $x \in U$ there exist a 0-neighbourhood $Y \subseteq E$ and continuous homogeneous polynomials $\beta_n\colon E \to F$ of degree n such that $x + Y \subseteq U$ and

$$f(x+y) \;=\; \sum_{n=0}^{\infty} \beta_n(y) \quad \text{for all } y \in Y$$

as a pointwise limit (see [1]). Given real locally convex spaces, following [20] a map $f\colon U \to F$ on an open subset $U \subseteq E$ is called *real analytic* if it extends to a complex analytic map between open subsets of the complexifications $E_{\mathbb{C}}$ and $F_{\mathbb{C}}$. We remark that the above definition of C^k-maps also makes sense over the complex field of scalars; it is known that mappings to sequentially complete complex locally convex spaces are complex analytic if and only if they are C^1 in the complex sense. Further information can be found in [11]. In particular, complex analytic maps are real analytic, and real analytic maps are smooth.

2 Existence of an exponential map

Let G be a Lie group. Given $X \in L(G)$, there is at most one smooth homomorphism $\gamma_X \colon \mathbb{R} \to G$ with $\gamma'_X(0) = X$. If γ_X always exists, G is said to *have an exponential map*, and we define it via $\exp_G \colon L(G) \to G$, $\exp_G(X) := \gamma_X(1)$.

Milnor asked [19, p. 1] whether every Lie group has a smooth exponential map. This question is still wide open: neither is it known whether an exponential map always exists, nor whether smoothness is automatic.

In the absence of completeness properties of the modelling space of a Lie group (which Milnor requires), an exponential mapping need not exist (see [4, §6]):

Example. Consider the algebra $\mathbb{R}[X]$ of polynomial functions $[0,1] \to \mathbb{R}$ and the algebra of fractions $A := S^{-1}\mathbb{R}[X] \subseteq C[0,1]$, where S is the set of all polynomial functions without zeros in $[0,1]$. Then A is a non-complete topological algebra in the topology induced by the Banach algebra $C[0,1]$. Since $A^\times = A \cap C[0,1]^\times$, the unit group $A^\times$ is open in A. Hence $A^\times$ is a Lie group. It does not have an exponential map since γ_f only exists for $f \in \mathbb{R}\mathbf{1} \subseteq A = L(A^\times)$.

Of course, a smooth exponential map does exist for all typical classes of Lie groups. But the construction of $\exp_G$ (and its particular properties) strongly depend on the type of Lie group:

Banach-Lie groups. As a consequence of the local existence and uniqueness of solutions to ordinary differential equations in Banach spaces, every Banach-Lie group G has a smooth exponential map (cf. also Section 4). Since $T_0(\exp_G) = \mathrm{id}_{L(G)}$, the inverse function theorem for smooth maps between Banach spaces implies that $\exp_G$ is a local diffeomorphism at 0.

Linear Lie groups. Let A be a *continuous inverse algebra*, viz. a locally convex topological algebra whose unit group $A^\times$ is open and such that the inversion map $\iota \colon A^\times \to A$, $x \mapsto x^{-1}$ is continuous. Then ι is analytic and thus $A^\times$ is an analytic Lie group. If A is sequentially complete, then the exponential series converges and defines an analytic map $\exp \colon A \to A^\times$, $\exp(x) := \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ which is the exponential map of $A^\times$ (see [4, Theorem 5.6]). After replacing A with $A_{\mathbb{C}}$ if necessary, this follows from the fact that

$$\exp(x) \;=\; \frac{1}{2\pi i}\int_{|\zeta|=r} e^{\zeta}\cdot(\zeta-x)^{-1}\,d\zeta$$

for $x \in A$ in terms of holomorphic functional calculus, where r is chosen so large that the circle $|\zeta| = r$ surrounds the spectrum of x. Here exp is a local diffeomorphism, with $\exp^{-1}(x) = \log(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n$ for x near 1.

Mapping groups. Let M be a compact manifold, G a Lie group with a smooth exponential map $\exp_G$ (e.g., a finite-dimensional Lie group). Then $C^\infty(M, G)$ is a group with pointwise group operations, and can be made a Lie group modelled on $C^\infty(M, L(G))$. The map $C^\infty(M, L(G)) \to C^\infty(M, G)$, $\gamma \mapsto \exp_G \circ \gamma$ is smooth and is easily seen to be the exponential map of $C^\infty(M, G)$ (cf. [20], [3]). If G has a locally diffeomorphic exponential map, then also $C^\infty(M, G)$.

Diffeomorphism groups. For each compact smooth manifold M, the group $G := \mathrm{Diff}(M)$ of C^∞-diffeomorphisms of M can be made a Lie group (with composition as the group multiplication), modelled on the space $\mathcal{V}(M)$ of smooth vector fields. It has a smooth exponential map given by

$$\exp_G\colon \mathcal{V}(M) \to G\,, \quad X \mapsto \Phi_X(1, \bullet)\,,$$

where $\Phi_X\colon \mathbb{R} \times M \to M$ is the flow of vector field X (see [17] and [20]). Already for $M := \mathbb{S}^1$, $\exp_G$ is not a local diffeomorphism at 0 (see [20, p. 1017]).

Direct limit groups. Given an ascending sequence $G_1 \subseteq G_2 \subseteq \cdots$ of finite-dimensional Lie groups such that the inclusion maps are smooth homomorphisms, we consider $L(G_n)$ as a Lie subalgebra of $L(G_{n+1})$. Then $G := \bigcup_{n\in\mathbb{N}} G_n = \varinjlim G_n$ is a group in a natural way, which can be given a smooth manifold structure modelled on the locally convex direct limit $\mathfrak{g} := \varinjlim L(G_n)$ making it Lie group (see [10]; cf. [22] for an earlier, more restricted method). The map $\mathfrak{g} \to G$, $x \mapsto \exp_{G_n}(x)$ if $x \in L(G_n)$ is the exponential map of G; it is smooth.

3 Analyticity of multiplication in exponential coordinates

As illustrated by the examples above, many (but not all) infinite-dimensional Lie groups G are *locally exponential* in the sense that $\exp_G$ exists and is a local C^∞-diffeomorphism at 0. A locally exponential Lie group G is called a *BCH-Lie group* if the group multiplication is analytic in exponential coordinates, i.e., if $(x, y) \mapsto x * y := \exp_G^{-1}(\exp_G(x) \exp_G(y))$ is analytic on some open 0-neighbourhood in $L(G) \times L(G)$. Then $x * y$ is given by the Baker-Campbell-Hausdorff (BCH-) series [12]. In our terminology, Milnor asked (cf. [19, p. 31]):

If (a) *G is locally exponential, or* (b) *G is real or complex analytic, does it follow that G is BCH?*

The answers to both questions are negative.

(a) A counterexample for (a) is mentioned in [29, p. 823]. Slightly simpler is

$$G := \mathbb{R}^{\mathbb{N}} \rtimes_\alpha \mathbb{R} \quad \text{with} \quad \alpha(t).(x_n)_{n\in\mathbb{N}} := (e^{nt}x_n)_{n\in\mathbb{N}}.$$

Using the identity map onto the Fréchet space $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}$ as a global chart, G becomes a real analytic Lie group. Its exponential map $\exp_G \colon \mathbb{R}^{\mathbb{N}} \rtimes \mathbb{R} \to G$, $\exp_G((x_n)_n, t) = \big(\big(\frac{e^{nt}-1}{nt}x_n\big)_n, t\big)$ is a C^∞-diffeomorphism and real analytic (whence G is locally exponential), but $\exp_G^{-1}$ is not real analytic. It can be shown that the map $(x, y) \mapsto x*y$ is not real analytic on any 0-neighbourhood in $L(G) \times L(G)$, and thus G is not a BCH-Lie group (cf. [12] for details).

(b) The group $G := \mathbb{C}^{(\mathbb{N})} \rtimes_\alpha \mathbb{R}$ with $\alpha(t).(x_n)_{n\in\mathbb{N}} := (e^{int}x_n)_{n\in\mathbb{N}}$ is a real analytic Lie group with a global chart, the identity map onto $\mathbb{C}^{(\mathbb{N})} \times \mathbb{R}$ (equipped with the finest locally convex vector topology). As shown in [7, Example 5.5], the exponential map $\exp_G((x_n)_n, t) = \big(\big(\frac{e^{int}-1}{int}x_n\big)_n, t\big)$ is not injective on any 0-neighbourhood, and the exponential image is not an identity neighbourhood. Hence G is not BCH (not even locally exponential). The corresponding semidirect product $\mathbb{C}^{(\mathbb{N})} \rtimes \mathbb{C}$ provides a complex analytic counterexample.

The counterexample for (a) was stimulated by Neeb's discussions of projective limits of finite-dimensional Lie groups [12]. Our counterexamples show that neither projective nor direct limits of finite-dimensional (and hence BCH-) Lie groups need to be locally exponential. For further information on BCH-Lie groups and locally exponential Lie groups, see [3], [4], [12], [19], [29] and [30].

4 Regularity questions

Roughly speaking, a Lie group G is called regular if all ODEs of interest for Lie theory can be solved in G, and the solutions depend smoothly on parameters. Formally, a Lie group is *regular* if the following holds (see [20, Definition 7.6]):

(a) Every smooth curve $\gamma\colon [0,1] \to L(G)$ arises as the left logarithmic derivative of a (necessarily unique) smooth curve $\eta\colon [0,1] \to G$, that is, $\gamma(t) = \eta(t)^{-1} \cdot \eta'(t)$ for all $t \in [0,1]$ (taking the product in the Lie group TG);

(b) The mapping $C^\infty([0,1], L(G)) \to G$ taking γ to $\eta(1)$ is smooth, where $C^\infty([0,1], L(G))$ is equipped with its usual locally convex topology.

Regularity is a useful property. For example, every regular Lie group has a smooth exponential map. Also, every continuous homomorphism

$$\phi\colon L(G) \to L(H)\,,$$

where G is a simply connected Lie group and H regular, gives rise to a unique smooth homomorphism $\psi\colon G \to H$ with $T_1\psi = \phi$ (see [20, Theorem 8.1]; cf. [19, Theorem 5.4] for a precursor by Thurston for so-called "receptive" Lie groups, which coincide with regular Lie groups by [20, Lemma 8.8]).

It is unknown whether every Lie group is regular, although all typical examples are regular: Regularity of Banach-Lie groups follows from the smooth dependence of solutions to ODEs in Banach spaces on parameters; regularity of $\mathrm{Diff}(M)$ for compact M was proved in [20] (see also [28], where an earlier, stronger notion of regularity was used); and regularity of $C^\infty(M,G) = \bigcap_{k\in\mathbb{N}_0} C^k(M,G)$ with finite-dimensional G can be reduced to the Banach case.

Also the group $\mathrm{Diff}_c(M)$ of compactly supported smooth diffeomorphisms of a σ-compact finite-dimensional smooth manifold M can be made a Lie group, and in fact in two ways: It can be modelled either on the LF-space $\mathcal{V}_c(M) = \lim_{\to K} \mathcal{V}_K(M)$ of compactly supported smooth vector fields, equipped with the locally convex direct limit topology; or on the same vector space, but equipped with the coarser topology making it the projective limit

$$\mathcal{V}_c(M) \;=\; \bigcap_{k\in\mathbb{N}_0} \mathcal{V}_c^k(M) \;=\; \lim_{\leftarrow\, k\in\mathbb{N}_0} \mathcal{V}_c^k(M)$$

of the LB-spaces of compactly supported C^k-vector fields. The first discussion of $\mathrm{Diff}_c(M)$ was given in [17] (even for paracompact manifolds). A different, more elementary construction was described later in [5]. The regularity of both Lie group structures on $\mathrm{Diff}_c(M)$ was asserted in [19] (using other terminology) and fully proved in [5].

Also every direct limit group (as described in Section 2) is regular, by [10, Theorem 8.1]. The unit groups of sequentially complete continuous inverse algebras are regular as a consequence of results by Robart [30], who addressed the question whether every BCH-Lie group is regular and achieved essential progress in this direction.

See [16] for a counterpart of regularity in the convenient setting of analysis. As in the case of convenient regularity [18], an abelian Lie group G modelled on a Mackey complete locally convex space E is regular if and only if $G \cong E/\Gamma$ for a discrete subgroup $\Gamma \subseteq E$ (see [26, Proposition V.1.9] or [12]). Neeb also showed that every solvable Lie group with smooth exponential map is regular [12].

Criteria for convenient regularity were given in [31] and applied to the "strong ILB-Lie groups" of Omori and collaborators (as in [27]).

Related to regularity is another question by Milnor [19, p. 1]: *If two simply connected Lie groups G and H have isomorphic Lie algebras, does it follow that $G \cong H$?* The theorem by Thurston and Milnor just described implies that the answer is positive if both G and H are regular [20, Corollary 8.2]. The general case remains open.

5 Properties of a Lie group compared to those of its Lie algebra

Lie theory derives its strength from the interplay between properties of a Lie group and properties of its Lie algebra. In the infinite-dimensional case, the study of links between G and $L(G)$ has just begun. It was shown that a connected Lie group G is abelian if and only if $L(G)$ is abelian (see [6, Proposition 22.15], [26, Proposition IV.1.10] or [12]). Milnor knew this for regular G. But for general G with $L(G)$ abelian, Milnor stated he could not prove that G is commutative [19, p. 36]. Neeb achieved essential further progress: A connected Lie group G is solvable (resp., nilpotent) if and only if $L(G)$ is solvable (resp., nilpotent) [12].

6 Smoothness of continuous homomorphisms

Milnor asked [19, p. 1]: *Is a continuous homomorphism between Lie groups necessarily smooth?* For special types of Lie groups, this is known:

- Banach-Lie groups (classical);
- Locally exponential Lie groups [19, Lemma 4.3];
- Countable direct limits of finite-dimensional Lie groups [10, Prop. 4.6 (c)].

Also continuous homomorphisms from finite-dimensional Lie groups to diffeomorphism groups are smooth (handwritten notes in [19], also [5]; cf. [21, p. 212]). Although Milnor's question remains open, a positive answer is available under stronger hypotheses: If a homomorphism $\phi\colon G \to H$ is Hölder continuous at 1, then ϕ is smooth [8, Theorem 3.2]. See [8, Definition 1.7] for the appropriate concept of Hölder continuity. A similar result holds for the Lie groups of convenient differential calculus [9, Theorem 9.1].

7 Kernels, Lie subgroups, quotients and homogeneous spaces

Milnor asked whether the kernel of a homomorphism necessarily is a Lie subgroup [19, p. 1], and proved this for homomorphisms between locally exponential Lie groups (they are "embedded" Lie subgroups in the terminology described below). It is also known that kernels of smooth homomorphisms from direct limit groups to Lie groups are Lie subgroups, like all closed subgroups of such groups [10, Proposition 7.5]. But the general answer to Milnor's question remains open.

We remark that a wide range of possible concepts of Lie subgroups is available in the theory of infinite-dimensional Lie groups, each of which can be preferable in certain situations. To describe the most basic concept, let M be a smooth manifold modelled on a locally convex space E. A subset $N \subseteq M$ is called a *submanifold* if there is a sequentially closed vector subspace $F \subseteq E$ such that each $x \in N$ is contained in the domain of some chart $\phi\colon U \to V \subseteq E$ of M which takes $U \cap N$ onto $V \cap F$. Then the restrictions $\phi|_{N \cap U}\colon U \cap N \to V \cap F$ define a smooth atlas for N. Note that we do not require that F is complemented in E as a topological vector space (beyond Banach manifolds, this property loses much of its usefulness). Given a Lie group G, a *Lie subgroup* is a subgroup $H \leq G$ which also is a submanifold.

Also weaker concepts are needed, analogous to the "analytic subgroups" in finite-dimensional Lie theory. In the terminology of [12], an *initial Lie subgroup* is a subgroup $H \leq G$ which can be given a Lie group structure which makes the inclusion map $i\colon H \to G$ a smooth homomorphism with injective differential $T_1(i)$, and such that mappings to H are smooth if and only if they are smooth as mappings to G. It may happen that a subgroup of a (non-separable) Banach-Lie group can be made an analytic subgroup in two different ways; then one of the Lie group structures is not the initial one (cf. [14, p. 157]). Furthermore, it is not clear whether all subgroups of interest are initial Lie subgroups. As a substitute, one still has the concept of an *integral subgroup*, referring to an injective smooth homomorphism $i\colon H \to G$ from a (connected) Lie group to G such that $T_1(i)$ is injective. Milnor used the notion of an *immersed Lie subgroup*: this is an injective smooth homomorphism of Lie groups $i\colon H \to G$ taking some open identity neighbourhood in H onto a submanifold of G (cf. [19, p. 22]).

Also stronger notions of Lie subgroups are needed. A Lie subgroup $H \leq G$ is called a *split* if G/H can be given a smooth manifold structure making the canonical map $q\colon G \to G/H$ a smooth H-principal bundle (i.e., q is smooth and admits smooth local sections). For G locally exponential, the concept of an *embedded Lie subgroup* H is particularly useful. This is a sequentially closed subgroup such that $H \cap \exp_G(U) = \exp_G(U \cap \mathfrak{h})$ for a 0-neighbourhood $U \subseteq L(G)$ on which $\exp_G$ is injective, where

$$\mathfrak{h} := \{X \in L(G)\colon \exp_G(\mathbb{R}X) \subseteq H\}.$$

For example, it can be shown that the topological quotient group G/N of a BCH-Lie group G modulo a closed normal subgroup N of G is a BCH-Lie group if and only if N is an embedded Lie subgroup of G ([3, Corollary 2.21]; cf. [11] for Banach-Lie groups). This result was extended to locally exponential Lie groups G by Neeb [12]. In this case, G/N is a locally exponential Lie group if and only if N is an embedded Lie subgroup whose Lie algebra $L(N)$ is "locally exponential." Neeb also showed that every locally compact subgroup of a locally exponential Lie group is an embedded Lie subgroup [12], as in the case of Banach-Lie groups (first discussed by Birkhoff).

It would be very useful to find tangible criteria ensuring that a homogenous space G/H can be made a smooth manifold with reasonable properties,[1] for a closed subgroup H of a locally exponential Lie group (which does not happen to be normal or a split Lie subgroup). Such criteria are not even known in the case of Banach-Lie groups, and any progress in this direction would be most valuable.

Also, it would be desirable to clarify the precise relations between the various concepts of Lie subgroups, and to find examples which clearly distinguish the concepts. For instance, it is well known that embedded Lie subgroups and ordinary Lie subgroups of Banach-Lie groups coincide. But it is unclear whether these concepts still agree in the case of BCH- (or locally exponential) Lie groups.

8 Integrability questions

Milnor [19, p. 1] asked whether every closed subalgebra of $L(G)$ corresponds to some immersed Lie subgroup of G. In [20, Warning 8.5], he described a counterexample (due to Omori). The integrability question of Lie subalgebras was analyzed further in [29] and [12], notably for locally exponential Lie groups.

It is a classical result by van Est and Korthagen that a Banach-Lie algebra $\mathfrak{g}$ need not be "integrable" (or "enlargible") – there need not be a (Banach-) Lie group G with $L(G) \cong \mathfrak{g}$. As they showed, $\mathfrak{g}$ is integrable if and only if a certain subgroup $\Pi(\mathfrak{g}) \leq \mathfrak{z}(\mathfrak{g})$ of the center of $\mathfrak{g}$ (the period group) is discrete [2]. Related to this work is a question by Milnor [19, pp. 31-32],

[1] A minimal requirement is that the smooth manifold structure on G/H is final with respect to the quotient map $q\colon G \to G/H$. In addition to this, one would certainly like to require that $T_1(q)$ is a quotient homomorphism with kernel $T_1(H)$.

which can be re-phrased as follows: If G is a BCH-Lie group, does it follow that $L(G)_{\mathbb{C}}$ is integrable to a Lie group? The answer is negative, even for Banach-Lie groups (see [11, Example VI.4]).

Milnor remarks that it would be interesting to know which topological Lie algebras $\mathfrak{g}$ correspond to BCH-Lie groups. A necessary condition is that the BCH-series converges on a 0-neighbourhood of $\mathfrak{g} \times \mathfrak{g}$ to an analytic function (see [30] for characterizations of this property). A full solution to Milnor's question was given by Neeb, even for the wider class of "locally exponential" Lie algebras. Any such Lie algebra $\mathfrak{g}$ can be associated a certain period group $\Pi(\mathfrak{g}) \leq \mathfrak{z}(\mathfrak{g})$; it is integrable to a locally exponential Lie group if and only if $\Pi(\mathfrak{g})$ is discrete [13].

Integrability questions have also been studied for other types of Lie algebras. It was shown that every locally finite Lie algebra of countable dimension is integrable [10, Theorem 5.1]. In [23], Neeb described the obstructions to integrate a central extension of topological Lie algebras to a Lie group extension. Later, he extended his methods to abelian [24] and non-abelian extensions [25].

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