The Resolvent Problem and H^∞-calculus of the Stokes Operator in Unbounded Cylinders with Several Exits to Infinity

Reinhard Farwig[1] and Ri Myong-Hwan[2,3]

Abstract

It is proved that the Stokes operator in L^q-space on an infinite cylindrical domain of $\mathbb{R}^n, n \geq 3$, with several exits to infinity generates a bounded and exponentially decaying analytic semigroup and admits a bounded H^∞-calculus. For the resolvent estimates, the Stokes resolvent system with a prescribed divergence in an infinite straight cylinder with bounded cross-section Σ is studied in $L^q(\mathbb{R}; L^r_\omega(\Sigma))$ where $1 < q, r < \infty$ and $\omega \in A_r(\mathbb{R}^{n-1})$ is an arbitrary Muckenhoupt weight. The proofs use cut-off techniques and the theory of Schauder decomposition of *UMD* spaces based on $\mathcal{R}$-boundedness of operator families and on square function estimates involving Muckenhoupt weights.

2000 Mathematical Subject Classification: 35Q30, 76D07
Keywords: H^∞-calculus; Stokes operator; Stokes resolvent estimate; Stokes semigroup; unbounded cylindrical domains; Muckenhoupt weights

1 Introduction

Let $\Omega = \bigcup_{i=0}^m \Omega_i$ be a cylindrical domain of $C^{1,1}$-class where Ω_0 is a bounded domain and $\Omega_i, i = 1, \dots, m$, are disjoint semi-infinite straight cylinders, that is, in possibly different coordinates,

$$\Omega_i = \{x^i = (x^i_1, \dots, x^i_n) \in \mathbb{R}^n : \, x^i_n > 0, \, (x^i_1, \dots, x^i_{n-1}) \in \Sigma^i\},$$

where $\Sigma^i \subset \mathbb{R}^{n-1}, i = 1, \dots, m$, is a bounded domain and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. Then we consider the Stokes operator $A_q = -P_q\Delta$ in $L^q_\sigma(\Omega)$ with domain

$$\mathcal{D}(A_q) = W^{2,q}(\Omega)^n \cap W^{1,q}_0(\Omega)^n \cap L^q_\sigma(\Omega),$$

[1]Reinhard Farwig: Department of Mathematics, Darmstadt University of Technology, 64289 Darmstadt, Germany,
email: farwig@mathematik.tu-darmstadt.de
[2]Ri Myong-Hwan: Institute of Mathematics, Academy of Sciences, DPR Korea,
email: ri@mathematik.tu-darmstadt.de
[3]Supported by the Gottlieb Daimler- und Karl Benz-Stiftung, grant no. S025/02-10/03.

where $L^q_\sigma(\Omega)$ is the completion of the set $C^\infty_{0,\sigma}(\Omega) = \{u \in C^\infty_0(\Omega)^n : \operatorname{div} u = 0\}$ in the norm of $L^q(\Omega)$ and where P_q is the Helmholtz projection of $L^q(\Omega)$ onto $L^q_\sigma(\Omega)$.

The Stokes operator is an important tool in the analysis of instationary Stokes and Navier-Stokes equations, and its properties have been studied for bounded domains and various kinds of unbounded domains. E.g., the Stokes resolvent system has been analyzed for half spaces, bounded and exterior domains, aperture domains and layer-like domains (see e.g. [1], [2], [6], [15], [20]–[24], [28]). For infinite cylindrical domains, one can find a result in the Bloch space of locally square integrable functions in [31]. Concerning the H^∞-calculus, see below, we mention that the Stokes operator admits a bounded H^∞-calculus for bounded and exterior domains [29], for half spaces [10], perturbed half spaces [29], aperture domains [5] and layer-like domains [3].

The goal of this paper is to show for the cylinder Ω that $-A_q$ generates a bounded and exponentially decaying analytic semigroup and that the Stokes operator A_q admits a bounded H^∞-calculus in $L^q_\sigma(\Omega)$. Actually, we show that $-\alpha+\Sigma_\varepsilon \subset \rho(-A_q)$ for some $\alpha > 0$ and arbitrary $\varepsilon \in (\pi/2, \pi)$, where

$$\Sigma_\varepsilon = \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg \lambda| < \varepsilon\}$$

and $\rho(-A_q)$ is the resolvent set of $-A_q$, and the resolvent estimate

$$\|(\lambda + A_q)^{-1}\|_{\mathcal{L}(L^q_\sigma(\Omega))} \leq \frac{C_\varepsilon}{|\lambda + \alpha|} \quad \forall \lambda \in -\alpha + \Sigma_\varepsilon. \tag{1.1}$$

Now we present the main results and ideas of this paper. Let $\widehat{W}^{1,q}(\Omega)$ be the homogeneous Sobolev space

$$\widehat{W}^{1,q}(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega)/\mathbb{R} : \nabla u \in L^q(\Omega)^n, \quad \|u\|_{\widehat{W}^{1,q}(\Omega)} := \|\nabla u\|_{L^q(\Omega)}\}.$$

We use the short notation $\|u, v\|_X$ for $\|u\|_X + \|v\|_X$, even if u and v are tensors of different order. Let $\bar{\alpha} = \min\{\alpha^{(i)} : i = 0, \dots, m\}$ where $\alpha^{(0)} > 0$ and $\alpha^{(i)} > 0$, $i = 1, \dots, m$, are the smallest eigenvalues of Dirichlet Laplacians in Ω_0 and in $\Sigma^i, i = 1, \dots, m$, respectively.

Theorem 1.1 *Let $1 < q < \infty$ and $\lambda \in -\alpha + \Sigma_\varepsilon$, where $\alpha \in (0, \bar{\alpha})$, and let $\varepsilon \in (\pi/2, \pi)$. If $f \in L^q(\Omega)^n$, then the resolvent problem*

$$\begin{aligned} \lambda u - \Delta u + \nabla p &= f && \text{in } \Omega \\ \operatorname{div} u &= 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

has a unique solution $(u, p) \in \big(W^{2,q}(\Omega)^n \cap W^{1,q}_0(\Omega)^n \cap L^q_\sigma(\Omega)\big) \times \widehat{W}^{1,q}(\Omega)$ satisfying the estimate

$$\|(\lambda + \alpha)u, \nabla^2 u, \nabla p\|_{L^q(\Omega)} \leq C\|f\|_{L^q(\Omega)} \tag{1.3}$$

with a constant $C = C(q, \alpha, \varepsilon, \Omega_0, \Sigma^1, \dots, \Sigma^m)$ independent of $\lambda \in -\alpha + \Sigma_\varepsilon$.

As a consequence, for every $\varepsilon \in (\pi/2, \pi)$ *and* $\alpha \in (0, \bar{\alpha})$ *the set* $-\alpha + \Sigma_\varepsilon$ *is contained in* $\rho(-A_q)$ *and the resolvent estimate* (1.1) *with* $C = C(q, \alpha, \varepsilon, \Omega_0, \Sigma^1, \dots, \Sigma^m)$ *holds. In particular,* $-A_q$ *generates a bounded analytic semigroup* e^{-tA_q} *in* $L^q_\sigma(\Omega)$ *satisfying*

$$\|e^{-tA_q}\|_{\mathcal{L}(L^q_\sigma(\Omega))} \leq Ce^{-\alpha t} \quad \text{for all} \;\; t \geq 0 \tag{1.4}$$

with a constant $C = C(q, \alpha, \varepsilon, \Omega_0, \Sigma^1, \dots, \Sigma^m)$ *independent of* $t \geq 0$.

The system (1.2) in an infinite straight cylinder $\Sigma \times \mathbb{R}$ was studied in vector-valued homogeneous Besov spaces $\dot{\mathcal{B}}^s_{pq}(\mathbb{R}; L^r(\Sigma))$, $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, ([17]) and in $L^q(\mathbb{R}; L^r_\omega(\Sigma))$ ([19]), where $1 < q, r < \infty$ and $\omega \in A_r(\mathbb{R}^{n-1})$ is an arbitrary Muckenhoupt weight. To be more concrete, using the partial Fourier transform $\mathcal{F} = \hat{}$ along the axis of the cylinder $\Sigma \times \mathbb{R}$, the authors obtained estimates for the parametrized Stokes resolvent system on Σ

$$(R_{\lambda,\xi}) \qquad \begin{aligned} (\lambda + \xi^2 - \Delta')U' + \nabla' P &= F' && \text{in } \Sigma \\ (\lambda + \xi^2 - \Delta')U_n + i\xi P &= F_n && \text{in } \Sigma \\ \operatorname{div}' U' + i\xi U_n &= G && \text{in } \Sigma \\ U' = 0, \quad U_n &= 0 && \text{on } \partial\Sigma \end{aligned}$$

in Fourier space. Then the solution u to (1.2) in $\Sigma \times \mathbb{R}$ is represented by

$$u = \mathcal{F}^{-1}(a_1(\xi)\hat{f}(\xi))$$

where $a_1(\xi)$ is the solution operator for $(R_{\lambda,\xi})$ with $G = 0$, i.e., $\hat{u} = U = (U', U_n) = a_1(\xi)F$, $F = \hat{f}$. Finally operator-valued Fourier multiplier theorems are applied to get the estimates of u.

The proof of Theorem 1.1 uses the technique of cut-off functions based on estimates for the resolvent system with prescribed divergence $\operatorname{div} u = g$ on an infinite cylinder, see (R_λ) in Section 2. With the solution operator $a_2(\xi)$ for $(R_{\lambda,\xi})$ with $F = 0$ the solution to (R_λ) with $f = 0$, $g \neq 0$ is represented by $u = \mathcal{F}^{-1}(a_2(\xi)\hat{g}(\xi))$. However, in this case, the application of Fourier multiplier theorems is not straightforward since the estimate for $(R_{\lambda,\xi})$ with $G \neq 0$ involves a complicated parameter-dependent norm, see (2.21)-(2.23). To get estimates for (R_λ) (Theorem 2.10) we use techniques of unconditional Schauder decompositions of *UMD* spaces combined with a property of Muckenhoupt weights (see Lemma 2.8).

The second main result of this paper concerns the H^∞-calculus of the Stokes operator in the cylinder Ω. For $\theta \in (0, \pi)$ let $\mathcal{H}^\infty(\Sigma_\theta)$ be the set of all holomorphic and bounded functions on the sector Σ_θ, and let ω_B be the *spectral angle* of B, i.e. $\omega_B = \inf\{\theta \in (0, \pi) : \sigma(B) \subset \Sigma_\theta\}$. Then for $\theta \in (\omega_B, \pi)$ a sectorial operator B on a Banach space X is said to admit a *bounded* $H^\infty(\Sigma_\theta)$*-calculus* (or, shortly, bounded H^∞-calculus) in X if there is a constant $C_\theta > 0$ such that for all

$$h \in \mathcal{H}^\infty_0(\Sigma_\theta) := \left\{ h \in \mathcal{H}^\infty(\Sigma_\theta) : \exists\, k, s > 0 : \; |h(z)| < k \frac{|z|^s}{1 + |z|^{2s}} \quad \forall z \in \Sigma_\theta \right\}$$

the bounded operator

$$h(B) = \frac{1}{2\pi i} \int_\Gamma h(\lambda)(\lambda - B)^{-1}\, d\lambda \in \mathcal{L}(X) \tag{1.5}$$

satisfies the estimate

$$\|h(B)\|_{\mathcal{L}(X)} \le C_\theta \|h\|_\infty; \tag{1.6}$$

here the integral curve Γ is the oriented boundary of $\Sigma_{\theta'}$ with $\theta' \in (\omega_B, \theta)$; note that $h(B)$ is independent of the choice of θ'. Furthermore, even for $h \in \mathcal{H}^\infty(\Sigma_\theta)$, we may define $h(B)$ with domain $D(B) \cap R(B)$ in X by

$$h(B) = \frac{1}{2\pi i} \int_\Gamma h(\lambda)\lambda(1+\lambda)^{-2}(\lambda - B)^{-1}\, d\lambda\, (1+B)^2 B^{-1}. \tag{1.7}$$

It is known that (1.7) is consistent with (1.5) and, if the operator B admits a bounded $H^\infty(\Sigma_\theta)$-calculus in X, then for $h \in \mathcal{H}^\infty(\Sigma_\theta)$ the operator $h(B)$ in (1.7) is bounded in X and (1.6) holds as well, cf. [12].

If a sectorial operator B admits a bounded $H^\infty(\Sigma_{\tilde\theta})$-calculus for some $\tilde\theta \in (\omega_B, \pi)$ in a Banach space X, then B has *bounded imaginary powers*, i.e., $B^{it} \in \mathcal{L}(X)$ and $\|B^{it}\|_{\mathcal{L}(X)} \le C$ with some $C > 0$ for all $|t| < 1$. Hence the domains of its fractional powers are represented by complex interpolation of the spaces $D(B)$ and X ([12] or [32], Theorem 1.15.3). Moreover, if $\tilde\theta < \pi/2$, then B has maximal regularity provided X is a *UMD* space ([12]), see also [9] and [14], Theorem 3.2. Note that the property of admitting a bounded H^∞-calculus is stable by small perturbation, see [11], Theorem 3.2.

A general theory for unbounded domains for which the shifted Stokes operator $c + A_q$ for some $c > 0$ admits a bounded H^∞-calculus was studied in [5], Theorem 1.3. We check that the unbounded cylinder Ω satisfies the assumptions on domains in that theory ([5], Assumption 1.1). Then, since the resolvent of A_q is bounded in a neighborhood of 0 by Theorem 1.1, we directly get the following theorem.

Theorem 1.2 *For $1 < q < \infty$ the Stokes operator A_q admits a bounded $H^\infty(\Sigma_\theta)$-calculus in $L^q_\sigma(\Omega)$ for any $\theta \in (0, \pi)$. In particular, the Stokes operator A_q has maximal regularity in $L^q_\sigma(\Omega)$.*

This paper is organized as follows. In Section 2 we deal with problems in an infinite cylinder. In §2.1 preliminaries for Muckenhoupt weights, $\mathcal{R}$-boundedness of operator families, Schauder decompositions and square function estimates are discussed. In §2.2 we get estimates for the Stokes resolvent system with a prescribed divergence on an infinite cylinder. Section 3 is devoted to the proof of the main result for cylindrical domains with several exits to infinity. In this paper, for notational convenience, constants appearing in the proofs may differ from line to line even though they may be denoted by the same letters.

2 Infinite Straight Cylinders: Stokes Resolvent System with a Prescribed Divergence in Weighted Spaces

In this section Ω is an infinite cylinder $\Sigma \times \mathbb{R} \subset \mathbb{R}^n$, $n \geq 3$, with a bounded cross-section $\Sigma \subset \mathbb{R}^{n-1}$. We consider the Stokes resolvent system (R_λ) on Ω with prescribed divergence:

$$(R_\lambda) \qquad \begin{aligned} \lambda u - \Delta u + \nabla p &= f && \text{in } \Omega \\ \operatorname{div} u &= g && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The system (R_λ) was studied in $L^q(\mathbb{R}; L^2(\Sigma)), 1 < q < \infty$, in [18]. Here we analyze (R_λ) in $L^q(\mathbb{R}; L^r_\omega(\Sigma))$ for $1 < q, r < \infty$ and arbitrary Muckenhoupt weight ω.

Let a generic point $x \in \Omega$ be written in the form $x = (x', x_n) \in \Omega$, where $x' \in \Sigma$ and $x_n \in \mathbb{R}$. Similarly, differential operators in $\mathbb{R}^n$ are split, in particular, $\nabla = (\nabla', \partial_n)$. The outward unit normal vector at $x' \in \Sigma$ is denoted by $N' \in \mathbb{R}^{n-1}$, whereas the exterior normal at $x \in \partial\Omega$ is denoted by N. First we recall some preliminaries on Muckenhoupt weights, $\mathcal{R}$-boundedness of operator families, Schauder decompositions and square function estimates for functions in weighted L^r-spaces.

2.1 Preliminaries

Let $1 < r < \infty$. A function $0 \leq \omega \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1})$ is called A_r-*weight* (*Muckenhoupt weight*) on $\mathbb{R}^{n-1}$ iff

$$\mathcal{A}_r(\omega) := \sup_Q \left(\frac{1}{|Q|} \int_Q \omega \, dx' \right) \cdot \left(\frac{1}{|Q|} \int_Q \omega^{-1/(r-1)} \, dx' \right)^{r-1} < \infty$$

where the supremum is taken over all cubes of $\mathbb{R}^{n-1}$ and $|Q|$ denotes the $(n-1)$-dimensional Lebesgue measure of Q. We call $\mathcal{A}_r(\omega)$ the A_r-*constant* of ω and we denote the set of all A_r-weights on $\mathbb{R}^{n-1}$ by $A_r = A_r(\mathbb{R}^{n-1})$. Note that

$$\omega \in A_r \quad \text{iff} \quad \omega' := \omega^{-1/(r-1)} \in A_{r'},\ r' = r/(r-1)$$

and $A_{r'}(\omega') = A_r(\omega)^{r'/r}$. A constant $C = C(\omega)$ is called A_r-*consistent* if for every $d > 0$

$$\sup\{C(\omega) : \ \omega \in A_r, \mathcal{A}_r(\omega) < d\} < \infty.$$

We write $\omega(Q)$ for $\int_Q \omega \, dx'$.

Given a Muckenhoupt weight ω, and an arbitrary domain Σ of $\mathbb{R}^{n-1}$ let

$$L^r_\omega(\Sigma) = \left\{ u \in L^1_{\mathrm{loc}}(\bar{\Sigma}) : \ \|u\|_{r,\omega} = \left(\int_\Sigma |u|^r \omega \, dx' \right)^{1/r} < \infty \right\}.$$

It is well-known that $L^r_\omega(\Sigma)$ is a separable reflexive Banach space with dense subspace $C_0^\infty(\Sigma)$; in particular, $L^r_\omega(\Sigma)^* = L^{r'}_{\omega'}(\Sigma)$ for $\omega \in A_r$. Moreover, we need the subspace

$$L^r_{m,\omega}(\Sigma) := \Big\{ u \in L^r_\omega(\Sigma) : \int_\Sigma u\,dx' = 0 \Big\}$$

of functions in $L^r_\omega(\Sigma)$ with vanishing mean. Let $\omega \in A_r$. As usual, $W^{k,r}_\omega(\Sigma)$, $k \in \mathbb{N}$, denotes the weighted Sobolev space with norm $\|u\|_{k,r,\omega} = \big(\sum_{|\alpha|\le k} \|D^\alpha u\|^r_{r,\omega}\big)^{1/r}$, where $|\alpha| = \alpha_1 + \cdots + \alpha_{n-1}$ is the length of the multi-index $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{N}_0^{n-1}$ and $D^\alpha = \partial_1^{\alpha_1} \cdot \ldots \cdot \partial_{n-1}^{\alpha_{n-1}}$. Moreover, let $W^{k,r}_{0,\omega}(\Sigma) := \overline{C_0^\infty(\Sigma)}^{\|\cdot\|_{k,r,\omega}}$ and $W^{-k,r}_{0,\omega}(\Sigma) := (W^{k,r'}_{0,\omega'}(\Sigma))^*$, where $r' = r/(r-1)$. We introduce the weighted homogeneous Sobolev space

$$\widehat{W}^{1,r}_\omega(\Sigma) = \big\{ u \in L^1_{\mathrm{loc}}(\bar\Sigma)/\mathbb{R} :\ \nabla' u \in L^r_\omega(\Sigma) \big\}$$

with norm $\|\nabla' u\|_{r,\omega}$ and its dual space $\widehat{W}^{-1,r'}_{\omega'} := (\widehat{W}^{1,r}_\omega)^*$ with norm $\|\cdot\|_{-1,r',\omega'} = \|\cdot\|_{-1,r',\omega';\Sigma}$.

Definition 2.1 *A Banach space X is called a* UMD *space if the Hilbert transform*

$$Hf(t) = -\frac{1}{\pi}\ \mathrm{PV} \int \frac{f(s)}{t-s}\,ds \qquad \textit{for } f \in \mathcal{S}(\mathbb{R}; X),$$

where $\mathcal{S}(\mathbb{R}; X)$ is the Schwartz space of all rapidly decreasing functions, extends to a bounded linear operator in $L^q(\mathbb{R}; X)$ for some $q \in (1, \infty)$.

It is well known that, if X is a *UMD* space, the Hilbert transform is bounded in $L^q(\mathbb{R}; X)$ for all $q \in (1, \infty)$ (see e.g. [30], Theorem 1.3). The dual space and closed subspaces of a *UMD* space are *UMD* spaces as well and for any open set Σ of $\mathbb{R}^{n-1}, 1 < r < \infty$, the weighted spaces $L^r_\omega(\Sigma), W^{1,r}_\omega(\Sigma)$ and $\widehat{W}^{1,r}_\omega(\Sigma)$ are *UMD* spaces.

Definition 2.2 *Let X be a Banach space and $(x_n)_{n=1}^\infty \subset X$. The series $\sum_{n=1}^\infty x_n$ is called unconditionally convergent if $\sum_{n=1}^\infty x_{\sigma(n)}$ is convergent in norm for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$.*

Definition 2.3 *A sequence of projections $(\Delta_j)_{j\in\mathbb{N}} \subset \mathcal{L}(\mathcal{X})$ on a Banach space $\mathcal{X}$ is called a Schauder decomposition of $\mathcal{X}$ if*

$$\Delta_i \Delta_j = 0 \qquad \textit{for all} \quad i \ne j$$

and

$$\sum_{j=1}^\infty \Delta_j x = x \qquad \textit{for each} \quad x \in \mathcal{X}.$$

A Schauder decomposition $(\Delta_j)_{j\in\mathbb{N}}$ of $\mathcal{X}$ is called unconditional if the series $\sum_{j=1}^\infty \Delta_j x$ converges unconditionally for each $x \in \mathcal{X}$.

If $(\Delta_j)_{j\in\mathbb{N}}$ is a Schauder decomposition of a Banach space $\mathcal{X}$, then the family $\{\sum_{j=l}^{k}\Delta_j\}_{l,k\in\mathbb{Z}}$ is uniform bounded in $\mathcal{X}$ due to the Banach-Steinhaus theorem. Moreover, if $(\Delta_j)_{j\in\mathbb{N}}$ is unconditional, then there is a constant $c>0$ such that

$$\Big\|\sum_{j=1}^{N}\varepsilon_j\Delta_j x\Big\|_{\mathcal{X}}\le c\Big\|\sum_{j=1}^{N}\Delta_j x\Big\|_{\mathcal{X}}\quad\text{for all } N\in\mathbb{N},\ x\in\mathcal{X},\ \varepsilon_j\in\{-1,1\},$$

see e.g. [12], Proposition 3.14. Moreover, there is a constant $c_\Delta>0$ such that for all u_j in the range $\mathcal{R}(\Delta_j)$ of Δ_j the inequalities

$$c_\Delta^{-1}\Big\|\sum_{j=l}^{k}u_j\Big\|_{\mathcal{X}}\quad\le\quad\Big\|\sum_{j=l}^{k}\varepsilon_j(s)u_j\Big\|_{L^p(0,1;\mathcal{X})}\le c_\Delta\Big\|\sum_{j=l}^{k}u_j\Big\|_{\mathcal{X}},\tag{2.1}$$

are valid for any sequence $(\varepsilon_j(s))$ of independent, symmetric $\{-1,1\}$-valued random variables defined on (0,1), for all $l\le k\in\mathbb{Z}$ and for each $p\in[1,\infty)$, see e.g. [12], (3.8). Given an interpolation couple $\mathcal{X}_1$, $\mathcal{X}_2$ of Banach spaces, it is easily seen that a Schauder decomposition of both $\mathcal{X}_1$ and $\mathcal{X}_2$ is a Schauder decomposition of $\mathcal{X}_1\cap\mathcal{X}_2$ and $\mathcal{X}_1+\mathcal{X}_2$ as well. We note that in the previous definitions and results the set of indices $\mathbb{N}$ may be replaced by $\mathbb{Z}$ without any further changes.

Let X be a *UMD* space and $\chi_{[a,b)}$ denote the characteristic function for the interval $[a,b)$. Let R be the Riesz projection, i.e.

$$R:=\mathcal{F}^{-1}\chi_{[0,\infty)}\mathcal{F},$$

and define

$$\Delta_j:=\mathcal{F}^{-1}\chi_{[2^j,2^{j+1})}\mathcal{F},\ j\in\mathbb{Z}.$$

It is well known that R and Δ_j, $j\in\mathbb{Z}$, are bounded in $L^q(\mathbb{R};X)$ for each $q\in(1,\infty)$ and that $\{\Delta_j : j\in\mathbb{Z}\}$ is an unconditional Schauder decomposition of $RL^q(\mathbb{R};X)$, the image of $L^q(\mathbb{R};X)$ by the Riesz projection R, see [12], proof of Theorem 3.19. Furthermore, $\{\Delta_j : j\in\mathbb{Z}\}$ is an unconditional Schauder decomposition of both $R\widehat{W}^{1,q}(\mathbb{R};X)$ and $R\widehat{W}^{-1,q}(\mathbb{R};X)$ for each $q\in(1,\infty)$ since for every permutation σ of $\mathbb{N}$, every $l<k\in\mathbb{Z}$ and any $u\in R\widehat{W}^{1,q}(\mathbb{R};X)$

$$\Big\|u-\sum_{j=l}^{k}\Delta_{\sigma(j)}u\Big\|_{\widehat{W}^{1,q}(\mathbb{R};X)}=\Big\|Du-\sum_{j=l}^{k}\Delta_{\sigma(j)}Du\Big\|_{L^q(\mathbb{R};X)},$$

as well as for any $v\in R\widehat{W}^{-1,q}(\mathbb{R};X)$

$$\Big\|v-\sum_{j=l}^{k}\Delta_{\sigma(j)}v\Big\|_{\widehat{W}^{-1,q}(\mathbb{R};X)}=\Big\|\mathcal{F}^{-1}(\xi^{-1}\hat{v})-\sum_{j=l}^{k}\Delta_{\sigma(j)}\mathcal{F}^{-1}(\xi^{-1}\hat{v})\Big\|_{L^q(\mathbb{R};X)}.$$

Definition 2.4 *Let X,Y be Banach spaces. An operator family $\mathcal{T}\subset\mathcal{L}(X;Y)$ is called $\mathcal{R}$-bounded if there is a constant $c>0$ such that for all $T_1,\cdots,T_N\in\mathcal{T}$, $x_1,\cdots,x_N\in X$ and $N\in\mathbb{N}$*

$$\Big\|\sum_{j=1}^{N}\varepsilon_j(\cdot)T_ju_j\Big\|_{L^q(0,1;Y)}\le c\Big\|\sum_{j=1}^{N}\varepsilon_j(\cdot)u_j\Big\|_{L^q(0,1;X)}\tag{2.2}$$

for some $q \in [1,\infty)$, *where* $(\varepsilon_j(\cdot))$ *is any sequence of independent, symmetric* $\{-1,1\}$*-valued random variables on* $[0,1]$. *The smallest constant* c *for which (2.2) holds is called* $\mathcal{R}$*-bound of* $\mathcal{T}$ *and denoted by* $\mathcal{R}_q(\mathcal{T})$.

Note that due to *Kahane's inequality*

$$\Big\| \sum_{j=1}^{N} \varepsilon_j(s) u_j \Big\|_{L^{q_1}(0,1;X)} \le c \Big\| \sum_{j=1}^{N} \varepsilon_j(s) u_j \Big\|_{L^{q_2}(0,1;X)}, \;\; 1 \le q_1, q_2 < \infty, \tag{2.3}$$

where $c = c(q_1, q_2, X) > 0$ ([13]), inequality (2.2) holds for all $q \in [1,\infty)$ if it holds for some $q \in [1,\infty)$.

Lemma 2.5 *Let* X *be a* UMD *space,* $1 < q < \infty$ *and* $R_{a,b} := \mathcal{F}^{-1}\chi_{[a,b)}\mathcal{F}$ *for* $-\infty < a < b < \infty$.

(1) If $g \in \widehat{W}^{-1,q}(\mathbb{R};X)$, *then* $R_{a,b}g \in L^q(\mathbb{R};X)$ *and there exists a constant* $c(q,X) > 0$ *such that*

$$\|R_{a,b}g\|_{L^q(\mathbb{R};X)} \le c(q,X) \max\{|a|,|b|\} \|R_{a,b}g\|_{\widehat{W}^{-1,q}(\mathbb{R};X)}.$$

In particular, if $a > 0$, *then*

$$\frac{1}{b\, c(q,X)} \|R_{a,b}g\|_{L^q(\mathbb{R};X)} \le \|R_{a,b}g\|_{\widehat{W}^{-1,q}(\mathbb{R};X)} \le \frac{c(q,X)}{a} \|R_{a,b}g\|_{L^q(\mathbb{R};X)}.$$

(2) There is a constant $c > 0$ *such that for all* $g \in L^q(\mathbb{R};X)$ *and for any* $l \le k \in \mathbb{Z}$ *the following two formulae hold:*

$$c^{-1} \Big\| \sum_{j=l}^{k} 2^j \Delta_j g \Big\|_{L^q(\mathbb{R};X)} \le \Big\| \sum_{j=l}^{k} \Delta_j g \Big\|_{\widehat{W}^{1,q}(\mathbb{R};X)} \le c \Big\| \sum_{j=l}^{k} 2^j \Delta_j g \Big\|_{L^q(\mathbb{R};X)} \tag{2.4}$$

$$c^{-1} \Big\| \sum_{j=l}^{k} 2^{-j} \Delta_j g \Big\|_{L^q(\mathbb{R};X)} \le \Big\| \sum_{j=l}^{k} \Delta_j g \Big\|_{\widehat{W}^{-1,q}(\mathbb{R};X)} \le c \Big\| \sum_{j=l}^{k} 2^{-j} \Delta_j g \Big\|_{L^q(\mathbb{R};X)}. \tag{2.5}$$

(3) The operator family $\{R_{a,b};\, -\infty < a < b < \infty\}$ *is* $\mathcal{R}$*-bounded in* $\mathcal{L}(L^q(\mathbb{R};X))$.

Proof: (1) and (2) were proved in [18], Lemma 2.7 and 2.8, respectively. Furthermore, (3) is well-known, see e.g. [12]. ∎

Lemma 2.6 *Let* $(H, (\cdot,\cdot), \|\cdot\|_H)$ *be a Hilbert space and let* $1 < q < \infty$. *Then there is a constant* $c > 0$ *such that for all* $u_j = \Delta_j u_j \in L^q(\mathbb{R};H)$ *the inequalities*

$$\frac{1}{c} \Big\| \Big(\sum_{j=l}^{k} \|u_j\|_H^2 \Big)^{1/2} \Big\|_{q,\mathbb{R}} \le \Big\| \sum_{j=l}^{k} u_j \Big\|_{L^q(\mathbb{R};H)} \le c \Big\| \Big(\sum_{j=l}^{k} \|u_j\|_H^2 \Big)^{1/2} \Big\|_{q,\mathbb{R}}$$

hold for all $l < k \in \mathbb{Z}$.

Proof: See [18], Lemma 2.6. ■

To generalize Lemma 2.6 to L^r-spaces, $r \neq 2$, we recall a crucial technical lemma from harmonic analysis ([27]).

Lemma 2.7 *Let* $1 < p < r < \infty, \dfrac{1}{s} = 1 - \dfrac{p}{r}$ *and* $\omega \in A_r$. *Then for every nonnegative function* $u \in L^s_\omega(\Sigma)$ *there is a nonnegative function* $v \in L^s_\omega(\mathbb{R}^{n-1})$ *such that*

(1) $u(x') \le v(x')$ *for a.a.* $x' \in \Sigma$.

(2) $\|v\|_{s,\omega;\mathbb{R}^{n-1}} \le 2\|u\|_{s,\omega;\Sigma}$.

(3) $\omega v \in A_p$ *and* $\mathcal{A}_p(\omega v) \le c$ *with* $c = c(\mathcal{A}_r(\omega)) > 0$ *depending only on the* A_r*-constant of* ω *and independent of* u, v.

If the function u *has a parameter* τ *in a Lebesgue measurable set* E *of* $\mathbb{R}^k, k \in \mathbb{N}$, *and is Lebesgue measurable w.r.t.* $(x', \tau) \in \Sigma \times E$, *then the function* v *is also Lebesgue measurable w.r.t.* $(x', \tau) \in \mathbb{R}^{n-1} \times E$.

Proof: We extend u onto $\mathbb{R}^{n-1}$ by 0 and again denote it by u. Then the assertion is a particular case of [27], Ch. IV, Lemma 5.18. Checking details of its proof, one can see that the constant in (2) may be taken as 2, cf. (2.6) below.

Let u have a parameter $\tau \in E$. By the proof of [27], Ch. IV, Lemma 5.18, the function v may be taken as

$$v(\cdot, \tau) = \sum_{j=0}^{\infty} (2\|S\|)^{-j} S^j u(\cdot, \tau), \tag{2.6}$$

where $Su = M(|u|\omega) \cdot \omega^{-1}$ with $M(|u|\omega)$ the Hardy-Littlewood maximal function of $|u|\omega$ on $\mathbb{R}^{n-1}$ and $\|S\|$ is the norm of the sublinear operator S in $L^s_\omega(\mathbb{R}^{n-1})$. Looking at the structure of the Hardy-Littlewood maximal function, $Su(\cdot, \tau)$ is seen to be Lebesgue measurable w.r.t. $(x', \tau) \in \mathbb{R}^{n-1} \times E$; hence each summand of the series in (2.6) is Lebesgue measurable w.r.t. (x', τ) as well. Then the function v as a limit of an increasing sequence of nonnegative measurable functions on $\mathbb{R}^{n-1} \times E$ is Lebesgue measurable on $\mathbb{R}^{n-1} \times E$. ■

Lemma 2.8 *Let* $1 < q < \infty$, $2 < r < \infty$, $\frac{1}{s} = 1 - \frac{2}{r}$ *and* $\omega \in A_r$. *Then there exist constants* $C_1 = C_1(\mathcal{A}_r(\omega)) > 0$ *and* $C_2 = C_2(q, r) > 0$ *independent of* ω *such that for* $l, k \in \mathbb{Z}$, $l \le k$, *and for each finite sequence* $u_j = \Delta_j u_j \in L^q(\mathbb{R}; L^r_\omega(\Sigma)), j = l, \ldots, k$, *there is some measurable function* v *on* $\mathbb{R}^n$ *satisfying* $v(\cdot, x_n) \in L^s_\omega(\mathbb{R}^{n-1})$ *for a.a.* $x_n \in \mathbb{R}$ *and*

$$\begin{aligned} &\|v(\cdot, x_n)\|_{s,\omega} \le 2, \;\; \omega v(\cdot, x_n) \in A_2(\mathbb{R}^{n-1}) \;\; \textit{and} \;\; \mathcal{A}_2(\omega v(\cdot, x_n)) \le C_1, \\ &\Big\| \sum_{j=l}^{k} u_j \Big\|_{L^q(\mathbb{R}; L^r_\omega(\Sigma))} \le C_2 c_\Delta \Big\| \Big(\sum_{j=l}^{k} \|u_j(\cdot, x_n)\|^2_{2,\omega v(x_n)} \Big)^{1/2} \Big\|_{q,\mathbb{R}}. \end{aligned} \tag{2.7}$$

Moreover, for all sequences $v_j = \Delta_j v_j \in L^q(\mathbb{R}; L^r_\omega(\Sigma)), j = l, \dots, k,$

$$\Big\| \Big(\sum_{j=l}^{k} \|v_j(\cdot, x_n)\|^2_{2,\omega v(x_n)} \Big)^{1/2} \Big\|_{q,\mathbb{R}} \le C_2 c_\Delta \Big\| \sum_{j=l}^{k} v_j \Big\|_{L^q(\mathbb{R}; L^r_\omega(\Sigma))}, \tag{2.8}$$

where c_Δ *is the constant in* (2.1). *In particular,* (2.8) *holds for* $(u_j)_{j=l}^k$ *as well.*

Proof: Choose a sequence $(\varepsilon_j(s))$ of $\{-1,1\}-$valued symmetric, independent random variables on $[0,1]$. By (2.1), Fubini's theorem and Kahane's inequality (2.3)

$$\begin{aligned} \Big\| \sum_{j=l}^{k} u_j \Big\|_{L^q(\mathbb{R}; L^r_\omega(\Sigma))} &\le c_\Delta \Big\| \sum_{j=l}^{k} \varepsilon_j u_j \Big\|_{L^q(0,1; L^q(\mathbb{R}; L^r_\omega(\Sigma)))} \\ = c_\Delta \Big\| \sum_{j=l}^{k} \varepsilon_j u_j \Big\|_{L^q(\mathbb{R}; L^q(0,1; L^r_\omega(\Sigma)))} &\le c \Big\| \sum_{j=l}^{k} \varepsilon_j u_j \Big\|_{L^q(\mathbb{R}; L^r(0,1; L^r_\omega(\Sigma)))}, \end{aligned} \tag{2.9}$$

where $c_\Delta = c_\Delta(q,r), c = c(q,r) > 0$; note that for $X = L^r_\omega(\Sigma)$ the constants c_Δ in (2.1) and c in (2.3) are independent of the weight ω, see [19], Remark 5.7, Remark 5.3, and even independent of Σ, which can easily be seen via the extension by 0 of functions on Σ onto $\mathbb{R}^{n-1}$. Let us recall *Khintchine's inequality* for complex numbers a_j, i.e.,

$$K^{-1} \Big\| \sum_{j=1}^{N} \varepsilon_j a_j \Big\|_{L^p(0,1)} \le \Big(\sum_{j=1}^{N} |a_j|^2 \Big)^{1/2} \le K \Big\| \sum_{j=1}^{N} \varepsilon_j a_j \Big\|_{L^p(0,1)}, \ p \in [1,\infty), \tag{2.10}$$

where the constant $K = K(p)$ does not depend on the choice of the sequence of independent, symmetric and $\{-1,1\}$-valued random variables $(\varepsilon_j(\cdot))$ on $[0,1]$ and on (a_j). By Fubini's theorem and (2.10) we get for a.a. $x_n \in \mathbb{R}$

$$\begin{aligned} \Big\| \sum_{j=l}^{k} \varepsilon_j u_j(\cdot, x_n) \Big\|_{L^r(0,1; L^r_\omega(\Sigma))} &= \Big(\int_\Sigma \int_0^1 \Big| \sum_{j=l}^{k} \varepsilon_j(s) u_j(x', x_n) \Big|^r ds\, \omega\, dx' \Big)^{1/r} \\ &\le K(r) \Big(\int_\Sigma \Big(\sum_{j=l}^{k} |u_j(\cdot, x_n)|^2 \Big)^{r/2} \omega(x')\, dx' \Big)^{1/r} \\ &= K(r) \Big\| \Big(\sum_{j=l}^{k} |u_j(\cdot, x_n)|^2 \Big)^{1/2} \Big\|_{r,\omega} = K(r) \Big\| \sum_{j=l}^{k} |u_j(\cdot, x_n)|^2 \omega^{1/s'} \Big\|_{s'}^{1/2}. \end{aligned} \tag{2.11}$$

For a.a. $x_n \in \mathbb{R}$ we have

$$\begin{aligned} \Big\| \sum_{j=l}^{k} |u_j(\cdot, x_n)|^2 \omega^{1/s'} \Big\|_{s'}^{1/2} &= \Big(\int_\Sigma \sum_{j=l}^{k} |u_j(\cdot, x_n)|^2 \omega^{1/s'} \tilde{u}(\cdot, x_n)\, dx' \Big)^{1/2} \\ &= \Big(\int_\Sigma \sum_{j=l}^{k} |u_j(\cdot, x_n)|^2 \omega u(\cdot, x_n)\, dx' \Big)^{1/2}, \end{aligned} \tag{2.12}$$

where $u(x_n) := \tilde{u}(\cdot, x_n)\omega^{-1/s}$ and, if $\sum_{j=l}^k |u_j(\cdot, x_n)|^2 \neq 0$,

$$\tilde{u}(\cdot, x_n) := \frac{\big(\sum_{j=l}^k |u_j(\cdot, x_n)|^2 \omega^{1/s'}\big)^{s'-1}}{\big\| \sum_{j=l}^k |u_j(\cdot, x_n)|^2 \omega^{1/s'} \big\|_{s'}^{s'-1}},$$

but if $\sum_{j=l}^k |u_j(\cdot, x_n)|^2 = 0$, then $\tilde{u}(\cdot, x_n) := |\Sigma|^{-1/s}$. Note that $\tilde{u}(x', x_n) \geq 0$ and $\tilde{u}(\cdot, x_n) \in L^s(\Sigma)$ with $\|\tilde{u}(\cdot, x_n)\|_{s;\Sigma} = 1$, and hence, for a.a. $x_n \in \mathbb{R}$ we get that $u(x_n) \in L^s_\omega(\Sigma), \|u(x_n)\|_{s,\omega} = 1$. Moreover, the function u is Lebesgue measurable w.r.t. $(x', x_n) \in \Sigma \times \mathbb{R}$. Therefore, by Lemma 2.7 there is a Lebesgue measurable function v on $\mathbb{R}^n$ such that $v(x_n) = v(\cdot, x_n) \in L^s_\omega(\mathbb{R}^{n-1})$ and

$$\begin{aligned} &u(x', x_n) \leq v(x', x_n) \quad \text{for a.a } x' \in \Sigma,\ \|v(x_n)\|_{s,\omega} \leq 2, \\ &\omega v(x_n) \in A_2(\mathbb{R}^{n-1}) \quad \text{and} \quad \mathcal{A}_2(\omega v(x_n)) \leq C, \end{aligned} \tag{2.13}$$

where the constant C in (2.13) depends only on the A_r-constant of ω and is independent of u, v; see Lemma 2.7. Now (2.9), (2.11) and (2.12) imply that (2.7) holds with the function v chosen above and some constant $C = C(q, r) > 0$.

Let $v_j = \Delta_j v_j \in L^q(\mathbb{R}; L^r_\omega(\Sigma)), j = l, \ldots, k$, be an arbitrary sequence. Then, by Hölder's inequality, (2.13), (2.10) and (2.3) we get for almost all $x_n \in \mathbb{R}$ that

$$\begin{aligned} \Big(\sum_{j=l}^k \|v_j(\cdot, x_n)\|^2_{2,\omega v(x_n)}\Big)^{1/2} &= \Big(\int_\Sigma \sum_{j=l}^k |v_j(x', x_n)|^2 \omega(x')^{1/s'} \cdot v(x', x_n)\omega(x')^{1/s}\, dx'\Big)^{1/2} \\ &\leq \Big\| \sum_{j=l}^k |v_j(\cdot, x_n)|^2 \Big\|^{1/2}_{s',\omega} \|v(x_n)\|^{1/2}_{s,\omega} \qquad \leq \sqrt{2} \Big\| \Big(\sum_{j=l}^k |v_j(\cdot, x_n)|^2\Big)^{1/2} \Big\|_{r,\omega} \\ &\leq K(r)\sqrt{2} \Big\| \sum_{j=l}^k \varepsilon_j v_j(\cdot, x_n) \Big\|_{L^r(0,1;L^r_\omega(\Sigma))} \leq c(q, r) \Big\| \sum_{j=l}^k \varepsilon_j v_j(\cdot, x_n) \Big\|_{L^q(0,1;L^r_\omega(\Sigma))}. \end{aligned}$$

Therefore, using a similar technique as in (2.9), by Fubini's theorem and (2.1) we get (2.8). ■

2.2 Generalized Resolvent Estimates on an Infinite Straight Cylinder

Let $1 < q, r < \infty$. On an infinite cylinder $\Omega = \Sigma \times \mathbb{R}$ we introduce the function space

$$L^q(L^r_\omega) := L^q(\mathbb{R}; L^r_\omega(\Sigma)), \quad \|u\|_{L^q(L^r_\omega)} = \bigg(\int_\mathbb{R} \Big(\int_\Sigma |u(x', x_n)|^r \omega(x')\, dx'\Big)^{q/r} dx_n\bigg)^{1/q}.$$

Furthermore, let $W^{k;q,r}_\omega(\Omega), k \in \mathbb{N}$, denote the Banach space of all functions on Ω whose partial derivatives of order up to k belong to $L^q(L^r_\omega)$ endowed with the norm $\|u\|_{W^{k;q,r}_\omega} = \big(\sum_{|\alpha| \leq k} \|D^\alpha u\|^2_{L^q(L^r_\omega)}\big)^{1/2}$, where $\alpha \in \mathbb{N}^n_0$ is a multi-index, and let $W^{1;q,r}_{0,\omega}(\Omega)$ be the completion of the set $C^\infty_0(\Omega)^n$ in $W^{1;q,r}_\omega(\Omega)$. The weighted homogeneous Sobolev space $\widehat{W}^{1;q,r}_\omega(\Omega)$ is defined by

$$\widehat{W}^{1;q,r}_\omega(\Omega) = \{u \in L^1_{\text{loc}}(\Omega)/\mathbb{R} : \ \nabla u \in L^q(L^r_\omega)\}$$

with norm $\|\nabla u\|_{L^q(L^r_\omega)}$; finally, $\widehat{W}^{-1;q,r}_\omega(\Omega) := \big(\widehat{W}^{1;q',r'}_{\omega'}(\Omega)\big)^*$. By the Hahn-Banach theorem it is easily seen that

$$\widehat{W}^{-1;q,r}_\omega(\Omega) = \widehat{W}^{-1,q}(\mathbb{R}; L^r_\omega(\Sigma)) + L^q(\mathbb{R}; \widehat{W}^{-1,q}_\omega(\Sigma)). \tag{2.14}$$

Lemma 2.9 *Let $1 < q, r < \infty$ and $\omega \in A_r(\mathbb{R}^{n-1})$.*

(1) For $d > 1$ let

$$\Omega_d = \{(x', x_n) \in \Omega : |x_n| < d\}.$$

Then Poincaré's inequality

$$\|\varphi\|_{L^q(-d,d;\, L^r_\omega(\Sigma))} \le C\, d \|\nabla\varphi\|_{L^q(-d,d;\, L^r_\omega(\Sigma))} \tag{2.15}$$

holds with an A_r-consistent constant $C = C(\mathcal{A}_r(\omega), \Sigma) > 0$ for all $\varphi \in C^\infty(\bar{\Omega}_d)$ with $\int_{\Omega_d} \varphi\, dx = 0$.

(2) The set $C^\infty_0(\bar{\Omega})$ is dense in $\widehat{W}^{1;q,r}_\omega(\Omega)$.

(3) The set $C^\infty_0(\mathbb{R}; W^{1,r}_\omega(\Sigma)) \cap \widehat{W}^{-1;q,r}_\omega(\Omega)$ is dense in the space $W^{1;q,r}_\omega(\Omega) \cap \widehat{W}^{-1;q,r}_\omega(\Omega)$.

Proof: (1) Let $\zeta(x_n) = \frac{1}{|\Sigma|}\int_\Sigma \varphi(x', x_n)\, dx'$, $x_n \in (-d, d)$, and define $\psi(x', x_n) = \varphi(x', x_n) - \zeta(x_n)$. Obviously, $\int_{-d}^d \zeta(x_n)\, dx_n = 0$ and $\int_\Sigma \psi(x', x_n)\, dx' = 0$ for all $x_n \in (-d, d)$. Therefore, by Poincare's inequalities on Σ and on $(-d, d)$ we get

$$\begin{aligned}
\|\varphi\|_{L^q(-d,d;\, L^r_\omega(\Sigma))} &\le \|\psi\|_{L^q(-d,d;\, L^r_\omega(\Sigma))} + \|\zeta\|_{L^q(-d,d;\, L^r_\omega(\Sigma))} \\
&\le \left(\int_{-d}^d \|\psi(\cdot, x_n)\|^q_{r,\omega}\, dx_n\right)^{1/q} + \omega(\Sigma)^{1/r}\|\zeta\|_{L^q(-d,d)} \\
&\le C(\mathcal{A}_r(\omega), \Sigma)\left(\int_{-d}^d \|\nabla'\psi(\cdot, x_n)\|^q_{r,\omega}\, dx_n\right)^{1/q} + dc_1\omega(\Sigma)^{1/r}\|\partial_n\zeta\|_{L^q(-d,d)}.
\end{aligned}$$

Note that $\nabla'\psi = \nabla'\varphi$ and, due to Hölder's inequality and $\omega(x)^{1/r}\omega'(x)^{1/r'} = 1$ for $x' \in \Sigma$,

$$\begin{aligned}
\omega(\Sigma)^{1/r}\|\partial_n\zeta\|_{L^q(-d,d)} &= \frac{\omega(\Sigma)^{1/r}}{|\Sigma|}\Big\|\int_\Sigma \partial_n\varphi(x', x_n)\, dx'\Big\|_{L^q(-d,d)} \\
&\le \frac{\omega(\Sigma)^{1/r}\omega'(\Sigma)^{1/r'}}{|\Sigma|}\|\partial_n\varphi\|_{L^q(-d,d;\, L^r_\omega(\Sigma))} \\
&\le c(\Sigma)\mathcal{A}_r(\omega)\|\partial_n\varphi\|_{L^q(-d,d;\, L^r_\omega(\Sigma))}.
\end{aligned}$$

Thus (2.15) is proved.

(2) The assertion is proved in the same way as [18], Lemma 2.1 (ii), where $\omega = 1$.

(3) Let $\{\rho_\varepsilon\}_{\varepsilon>0}$ be a *one-dimensional* mollifier defined by $\rho_\varepsilon(x_n) = \frac{1}{\varepsilon}\rho(\frac{x_n}{\varepsilon})$, $\varepsilon > 0$, with $\rho \in C^\infty_0(\mathbb{R})$ satisfying $\operatorname{supp}\rho \subset [-1, 1]$ and $\int_\mathbb{R} \rho(x_n)\, dx_n = 1$. In the subsequent proof, for a function f defined on Ω let $\rho_\varepsilon * f$ denote the convolution with respect to x_n, that is,

$$\rho_\varepsilon * f(x', x_n) := \int_\mathbb{R} f(x', x_n - y_n)\rho_\varepsilon(y_n)\, dy_n.$$

Further choose $\eta \in C_0^\infty(\mathbb{R})$ such that

$$\eta(x_n) := \begin{cases} 1 & \text{for } |x_n| < 1 \\ 0 & \text{for } |x_n| \geq 2, \end{cases}$$

and let $\eta_j(x_n) := \eta(\frac{x_n}{j})$ for $j \in \mathbb{N}$.

Now, for $g \in W_\omega^{1;q,r}(\Omega) \cap \widehat{W}_\omega^{-1;q,r}(\Omega)$, define the functions $g_j, \bar{g}_j$, $j \in \mathbb{N}$, by $g_j(x) := \eta_j(x_n)g(x)$, $x \in \Omega$, and

$$\bar{g}_j(x) := \begin{cases} g_j(x) - \dfrac{1}{|\Omega_{2j}|} \displaystyle\int_{\Omega_{2j}} g_j \, dx & \text{for } x \in \Omega_{2j} \\ 0 & \text{otherwise,} \end{cases}$$

respectively. Further let $g_{j\varepsilon} := \bar{g}_j * \rho_\varepsilon$ for $\varepsilon > 0$.

Evidently, $g_{j\varepsilon} \in C_0^\infty(\mathbb{R}; W_\omega^{1,r}(\Sigma)) \subset W_\omega^{1;q,r}(\Omega)$. To prove $g_{j\varepsilon} \in \widehat{W}_\omega^{-1;q,r}(\Omega)$ note that $\operatorname{supp} g_{j\varepsilon} \subset \Omega_{2j+\varepsilon}$ and that $\int_\Omega g_{j\varepsilon} \, dx = 0$ since $\int_\Omega \bar{g}_j \, dx = 0$. Therefore, by (2.15), for $\varphi \in C_0^\infty(\bar{\Omega})$

$$\begin{aligned} \int_\Omega g_{j\varepsilon}\varphi \, dx &= \int_{\Omega_{2j+\varepsilon}} g_{j\varepsilon}\varphi \, dx = \int_{\Omega_{2j+\varepsilon}} g_{j\varepsilon}\bar{\varphi} \, dx \\ &\leq \|g_{j\varepsilon}\|_{L^q(L^r_\omega)} \|\bar{\varphi}\|_{L^{q'}(-2j-\varepsilon, 2j+\varepsilon; L^{r'}_{\omega'}(\Sigma))} \\ &\leq c(2j+\varepsilon)\|g_{j\varepsilon}\|_{L^q(L^r_\omega)} \|\nabla\varphi\|_{L^{q'}(L^{r'}_{\omega'})}, \end{aligned}$$

where $\bar{\varphi} = \varphi - \frac{1}{|\Omega_{2j+\varepsilon}|} \int_{\Omega_{2j+\varepsilon}} \varphi \, dx$ and $c = c(\mathcal{A}_r(\omega), \Sigma) > 0$. Thus $g_{j\varepsilon} \in \widehat{W}_\omega^{-1;q,r}(\Omega)$.

Now we will show that the sequence $\{g_{j\varepsilon}\}$ with carefully chosen $\varepsilon = \varepsilon(j)$ converges to g in $W_\omega^{1;q,r}(\Omega) \cap \widehat{W}_\omega^{-1;q,r}(\Omega)$ as $j \to \infty$. First let us prove the convergence in $W_\omega^{1;q,r}(\Omega)$. Since $\operatorname{supp} g_j \subset \Omega_{2j}$, we obtain

$$g_{j\varepsilon} - g = (g * \rho_\varepsilon - g) + (g_j - g) * \rho_\varepsilon - \Big(\frac{1}{|\Omega_{2j}|} \int_{\Omega_{2j}} g_j \, dx\Big) \int_{-2j}^{2j} \rho_\varepsilon(x_n - y_n) \, dy_n \,. \quad (2.16)$$

Since $g \in \widehat{W}_\omega^{-1;q,r}(\Omega)$, by Hahn-Banach's theorem there is some $u \in L^q(L^r_\omega)$ such that

$$g = \operatorname{div} u, \quad u \cdot N|_{\partial\Omega} = 0 \quad \text{and} \quad \|u\|_{L^q(L^r_\omega)} = \|g\|_{\widehat{W}_\omega^{-1;q,r}(\Omega)}.$$

By elementary calculations we have

$$\begin{aligned} \Big| \int_{\Omega_{2j}} g_j \, dx \Big| &= \Big| \int_{\Omega_{2j}} \eta_j \operatorname{div} u \, dx \Big| = \Big| \int_{\Omega_{2j}} \nabla\eta_j \cdot u \, dx \Big| \\ &\leq \frac{1}{j} \Big\| (\partial_n \eta)(\frac{x_n}{j}) \Big\|_{L^{q'}(L^{r'}_{\omega'})} \|\chi_{j,2j} u\|_{L^q(L^r_\omega)} \\ &= c_1(q) j^{-1/q} \omega'(\Sigma)^{1/r'} \|\chi_{j,2j} u\|_{L^q(L^r_\omega)}, \end{aligned} \quad (2.17)$$

where $\chi_{j,2j}$ is the characteristic function of the set $[-2j, -j] \cup [j, 2j]$ and $c_1(q) = \big(\int_{-2}^2 |\partial_n \eta(y_n)|^{q'} \, dy_n\big)^{1/q'}$. Further we get

$$\Big\| \int_{-2j}^{2j} \rho_\varepsilon(x_n - y_n) \, dy_n \Big\|_{W_\omega^{1;q,r}(\Omega)} = \omega(\Sigma)^{1/r} \Big\| \int_{-2j}^{2j} \rho_\varepsilon(x_n - y_n) \, dy_n \Big\|_{W^{1,q}(\mathbb{R})}. \quad (2.18)$$

Note that, if $0 < \varepsilon < 2j$,

$$\Big\| \int_{-2j}^{2j} \rho_\varepsilon(x_n - y_n)\, dy_n \Big\|_{L^q(-2j-\varepsilon, 2j+\varepsilon)} \le (4j + 2\varepsilon)^{1/q} \le 8^{1/q} j^{1/q}$$

and that

$$\begin{aligned} &\Big\| \frac{\partial}{\partial x_n} \int_{-2j}^{2j} \rho_\varepsilon(x_n - y_n)\, dy_n \Big\|_{L^q(-2j-\varepsilon, 2j+\varepsilon)} \\ &= \|\rho_\varepsilon(x_n + 2j) - \rho_\varepsilon(x_n - 2j)\|_{L^q(-2j-\varepsilon, 2j+\varepsilon)} \\ &\le 2\|\rho_\varepsilon\|_{L^q(\mathbb{R})} = c_2(q)\varepsilon^{-1/q'}, \end{aligned}$$

where $c_2(q) = 2\|\rho\|_{L^q(-1,1)}$. Therefore, taking $\varepsilon = \varepsilon(j) := j^{-q'/q}$, it follows from (2.17),(2.18) that the $W^{1;q,r}_\omega(\Omega)$-norm of the third term of (2.16) is estimated by

$$\frac{c(q)\omega(\Sigma)^{1/r}\omega'(\Sigma)^{1/r'}}{|\Omega_{2j}|} \|\chi_{j,2j}\, u\|_{L^q(L^r_\omega)} \le \frac{c(q,\Sigma)\mathcal{A}_r(\omega)}{j} \|\chi_{j,2j}\, u\|_{L^q(L^r_\omega)}$$

which tends to 0 as $j \to \infty$.

Obviously $\|g * \rho_{\varepsilon(j)} - g\|_{W^{1;q,r}_\omega(\Omega)} \to 0$ and $\|(g_j - g) * \rho_{\varepsilon(j)}\|_{W^{1;q,r}_\omega(\Omega)} \to 0$ as $j \to \infty$. Summarizing the previous results we get that $\|g_{j\varepsilon(j)} - g\|_{W^{1;q,r}_\omega(\Omega)} \to 0$ as $j \to \infty$.

Next we will prove $\|g_{j\varepsilon(j)} - g\|_{\widehat{W}^{-1;q,r}_\omega(\Omega)} \to 0$ as $j \to \infty$. For $j \in \mathbb{N}$ define f_j on Ω by

$$f_j(x', x_n) = \begin{cases} u_n\, \partial_n \eta_j + \frac{1}{|\Omega_{2j}|} \int_{\Omega_{2j}} g_j\, dx, & |x_n| < 2j \\ 0, & |x_n| \ge 2j. \end{cases}$$

Then $\bar{g}_j = \operatorname{div}(\eta_j u) - f_j$ and, using (2.17), we have

$$\begin{aligned} \|f_j\|_{L^q(L^r_\omega)} &\le \|u_n\, \partial_n \eta_j\|_{L^q(-2j,2j;\, L^r_\omega)} + \Big\| \frac{1}{|\Omega_{2j}|} \int_{\Omega_{2j}} g_j\, dx \Big\|_{L^q(-2j,2j;\, L^r_\omega)} \\ &\le \|\partial_n \eta_j\|_\infty \|\chi_{j,2j}\, u\|_{L^q(L^r_\omega)} + \frac{(4j)^{1/q}\omega(\Sigma)^{1/r}}{|\Omega_{2j}|} \Big| \int_{\Omega_{2j}} g_j\, dx \Big| \\ &\le \Big(\frac{c}{j} + \frac{c(q)\omega(\Sigma)^{1/r}\omega'(\Sigma)^{1/r'}}{j|\Sigma|} \Big) \|\chi_{j,2j} u\|_{L^q(L^r_\omega)} \\ &\le \frac{c(q)}{j}(1 + \mathcal{A}_r(\omega)) \|\chi_{j,2j}\, u\|_{L^q(L^r_\omega)}. \end{aligned} \tag{2.19}$$

Note that $\int_{\Omega_{2j}} f_j\, dx = 0$. Therefore, defining $\langle f_j, \varphi \rangle := \int_\Omega f_j \varphi\, dx$ for $\varphi \in C^\infty(\bar{\Omega})$, we get by (2.15), (2.19) that

$$\begin{aligned} |\langle f_j, \varphi \rangle| &= \Big| \int_{\Omega_{2j}} f_j \varphi\, dx \Big| = \Big| \int_{\Omega_{2j}} f_j \bar{\varphi}\, dx \Big| \\ &\le \|f_j\|_{L^q(L^r_\omega)} \|\bar{\varphi}\|_{L^{q'}(-2j,2j;L^{r'}_{\omega'})} \\ &\le C(\mathcal{A}_r(\omega), \Sigma) \|\chi_{j,2j}\, u\|_{L^q(L^r_\omega)} \|\nabla \varphi\|_{L^{q'}(L^{r'}_{\omega'})}, \end{aligned}$$

where $\bar{\varphi} = \varphi - \frac{1}{|\Omega_{2j}|} \int_{\Omega_{2j}} \varphi\, dx$. Hence $\langle f_j, \cdot \rangle \in \widehat{W}^{-1;q,r}_\omega(\Omega)$ and

$$\|\langle f_j, \cdot \rangle\|_{\widehat{W}^{-1;q,r}_\omega(\Omega)} \le C(\mathcal{A}_r(\omega), \Sigma) \|\chi_{j,2j}\, u\|_{L^q(L^r_\omega)}.$$

By Hahn-Banach's theorem there exists some $w_j \in L^q(L^r_\omega)$ such that

$$\operatorname{div} w_j = f_j, \quad w_j \cdot N|_{\partial\Omega} = 0, \quad \text{and} \quad \|w_j\|_{L^q(L^r_\omega)} = \|\langle f_j, \cdot\rangle\|_{\widehat{W}_\omega^{-1;q,r}(\Omega)}.$$

Therefore, with $u_j := \eta_j u - w_j$, we get $\bar{g}_j = \operatorname{div} u_j$ and

$$\begin{aligned}\|g_{j\varepsilon(j)} - g\|_{\widehat{W}_\omega^{-1;q,r}(\Omega)} &= \|\operatorname{div}(u - u_j * \rho_{\varepsilon(j)})\|_{\widehat{W}_\omega^{-1;q,r}(\Omega)} \\ &\le \|u - u_j * \rho_{\varepsilon(j)}\|_{L^q(L^r_\omega)} \\ &\le \|u - u * \rho_{\varepsilon(j)}\|_{L^q(L^r_\omega)} + \|u - \eta_j u\|_{L^q(L^r_\omega)} + \|w_j\|_{L^q(L^r_\omega)} \to 0\end{aligned}$$

as $j \to \infty$.

The proof of this lemma is complete. ■

Now we are in a position to prove the main theorem of this section.

Theorem 2.10 *Let $\Sigma \subset \mathbb{R}^{n-1}$ be a bounded domain of $C^{1,1}$-class and let $\alpha_0 > 0$ be the smallest eigenvalue of the Dirichlet Laplacian in Σ. Moreover, let $1 < q < \infty$, $2 \le r < \infty$, $\omega \in A_r(\mathbb{R}^{n-1})$, $\alpha \in (0, \alpha_0)$ and let $\lambda \in -\alpha + \Sigma_\varepsilon$, $\varepsilon \in (\pi/2, \pi)$. Then, for every $f \in L^q(\mathbb{R}; L^r_\omega(\Sigma))^n$, $g \in W_\omega^{1;q,r}(\Omega) \cap \widehat{W}_\omega^{-1;q,r}(\Omega)$ there exists a unique solution $(u,p) \in \big(W_\omega^{2;q,r}(\Omega)^n \cap W_{0,\omega}^{1;q,r}(\Omega)^n\big) \times \widehat{W}_\omega^{1;q,r}(\Omega)$ to (R_λ) satisfying the estimate*

$$\begin{aligned}&\|(\lambda+\alpha)u, \nabla^2 u, \nabla p\|_{L^q(L^r_\omega)} \\ &\qquad \le C\big(\|f\|_{L^q(L^r_\omega)} + \|g\|_{W_\omega^{1;q,r}(\Omega)} + (|\lambda|+1)\|g\|_{\widehat{W}_\omega^{-1;q,r}(\Omega)}\big)\end{aligned} \tag{2.20}$$

with an A_r-consistent constant $C = C(q, r, \alpha, \varepsilon, \Sigma, \mathcal{A}_r(\omega))$ independent of λ.

Proof: For the special case $g = 0$ this theorem was treated in [19], Theorem 2.1. Therefore, we shall consider only the case $f = 0$ and assume, due to Lemma 2.9 (3), that $g \in C_0^\infty(\mathbb{R}; W_\omega^{1,r}(\Sigma)) \cap \widehat{W}_\omega^{-1;q,r}(\Omega)$.

By [19], Theorem 4.4, for every $\xi \in \mathbb{R}^*$ and $\lambda \in -\alpha + S_\varepsilon$ the parametrized Stokes resolvent system $(R_{\lambda,\xi})$, see the Introduction, with $F = \hat{f} = 0$ and $G = \hat{g} \in W^{1,r}(\Sigma)$, has a unique solution

$$(U_G, P_G) := (U_G(\xi), P_G(\xi)) \in (W_\omega^{2,r}(\Sigma) \cap W_{0,\omega}^{1,r}(\Sigma)) \times W_\omega^{1,r}(\Sigma)$$

such that

$$\begin{aligned}&\|(\lambda+\alpha)U_G, \xi^2 U_G, \xi\nabla' U_G, \nabla'^2 U_G, \xi P_G, \nabla' P_G\|_{r,\omega;\Sigma} \\ &\qquad \le c\big(\|\nabla' G, G, \xi G\|_{r,\omega;\Sigma} + (|\lambda|+1)\|G; L^r_{m,\omega} + L^r_{\omega,1/\xi}\|_0\big),\end{aligned} \tag{2.21}$$

and, by [19], Corollary 4.5,

$$\begin{aligned}&\big\|\xi \tfrac{d}{d\xi}\big((\lambda+\alpha)U_G, \xi^2 U_G, \xi\nabla' U_G, \nabla'^2 U_G, \xi P_G, \nabla' P_G\big)\big\|_{r,\omega;\Sigma} \\ &\qquad \le c\big(\|\nabla' G, G, \xi G\|_{r,\omega;\Sigma} + (|\lambda|+1)\|G; L^r_{m,\omega} + L^2_{\omega,1/\xi}\|_0\big);\end{aligned} \tag{2.22}$$

here the constant $c = c(r,\alpha,\varepsilon,\Sigma,\mathcal{A}_r(\omega)) > 0$ is independent of $\lambda \in -\alpha + S_\varepsilon, \xi \in \mathbb{R}^*$, and

$$\|G; L^r_{m,\omega} + L^r_{\omega,1/\xi}\|_0 := \inf\big\{\|G_0\|_{-1,r;\omega} + \|G_1/\xi\|_{r,\omega}; \\ G = G_0 + G_1,\ G_0 \in L^r_{m,\omega}(\Sigma), G_1 \in L^r_\omega(\Sigma)\big\}. \tag{2.23}$$

Moreover, the operator $M(\xi) : W^{1,r}_\omega(\Sigma) \to L^r_\omega(\Sigma)$, $\xi \in \mathbb{R}^*$, defined by

$$M(\xi)G := \big((\lambda+\alpha)U_G, \xi^2 U_G, \xi\nabla' U_G, \nabla'^2 U_G, \xi P_G, \nabla' P_G\big),$$

is Frechét differentiable, and (2.21), (2.22) yield the estimate

$$\|M(\xi)G, \xi M'(\xi)G\|_{r,\omega,\Sigma} \le c\big(\|\nabla' G, G, \xi G\|_{r,\omega;\Sigma} + (|\lambda|+1)\|G; L^r_{m,\omega} + L^r_{\omega,1/\xi}\|_0\big), \tag{2.24}$$

where $c = c(r,\alpha,\varepsilon,\Sigma,\mathcal{A}_r(\omega)) > 0$.

Obviously $(u,p) = (U_{\hat{g}(\xi)}(\xi)^\vee, P_{\hat{g}(\xi)}(\xi)^\vee)$ solves (R_λ) with right-hand side $(0,g)$ in the sense of distributions. Therefore, to prove (2.20) it is enough to show that

$$\|\big(M(\xi)\hat{g}(\xi)\big)^\vee\|_{L^q(\mathbb{R};L^r_\omega(\Sigma))} \le C\big(\|g\|_{W^{1;q,r}_\omega(\Omega)} + (|\lambda|+1)\|g\|_{\widehat{W}^{-1;q,r}_\omega(\Omega)}\big) \tag{2.25}$$

with an A_r-consistent constant $C = C(q,r,\alpha,\varepsilon,\Sigma,\mathcal{A}_r(\omega))$ independent of λ. We may assume without loss of generality that $\operatorname{supp}\hat{g} \subset [0,\infty)$ due to the relation

$$\begin{aligned} g(x',x_n) &= (\chi_{[0,\infty)}\hat{g}(\xi))^\vee(x',x_n) + (\chi_{(-\infty,0]}\hat{g}(\xi))^\vee(x',x_n) \\ &= (\chi_{[0,\infty)}\hat{g}(\xi))^\vee(x',x_n) + (\chi_{[0,\infty)}\hat{g}(-\xi))^\vee(x',-x_n) \end{aligned}$$

and due to the linearity of the problem (R_λ). For notational convenience, we introduce the space

$$\begin{aligned} \mathcal{X} &= W^{1;q,r}_\omega(\Omega) \cap \widehat{W}^{-1;q,r}_\omega(\Omega) \\ &= \big(W^{1,q}(\mathbb{R}; L^r_\omega(\Sigma)) \cap L^q(\mathbb{R}; W^{1,r}_\omega(\Sigma))\big) \cap \big(\widehat{W}^{-1,q}(\mathbb{R}; L^r_\omega(\Sigma)) + L^q(\mathbb{R}; \widehat{W}^{-1,r}_\omega(\Sigma))\big). \end{aligned}$$

As mentioned in §2.1 the operator family $\{\Delta_j = \mathcal{F}^{-1}\chi_{[2^j,2^{j+1})}(\xi)\mathcal{F} : j \in \mathbb{Z}\}$ is an unconditional Schauder decomposition of $R\mathcal{X}$, the image of $\mathcal{X}$ by the Riesz projection R; hence $g = \sum_{j\in\mathbb{Z}} \Delta_j g$ in $\mathcal{X}$. Moreover, for $s \in \mathbb{R}$ we define

$$R_s = \mathcal{F}^{-1}\chi_{[s,\infty)}\mathcal{F}.$$

Note that $M(\xi) = M(2^j) + \int_{2^j}^{\xi} M'(\tau)\,d\tau$ for $\xi \in [2^j, 2^{j+1}), j \in \mathbb{Z}$, and that obviously $(M(2^j)\widehat{\Delta_j g})^\vee = M(2^j)\Delta_j g$; furthermore,

$$\begin{aligned} \Big(\int_{2^j}^{\xi} M'(\tau)\,d\tau\,\widehat{\Delta_j g}(\xi)\Big)^\vee &= \Big(\int_{2^j}^{2^{j+1}} M'(\tau)\chi_{[2^j,\xi)}(\tau)\widehat{\Delta_j g}(\xi)\,d\tau\Big)^\vee \\ &= \Big(\int_0^1 2^j M'(2^j(1+t))\chi_{[2^j,\xi)}(2^j(1+t))\chi_{[2^j,2^{j+1})}(\xi)\hat{g}(\xi)\,dt\Big)^\vee \\ &= \int_0^1 2^j M'(2^j(1+t))(R_{2^j(1+t)} - R_{2^{j+1}})\Delta_j g\,dt. \end{aligned}$$

Thus we get

$$\begin{aligned}\big(M(\xi)\hat{g}(\xi)\big)^\vee &= \Big(\sum_{j\in\mathbb{Z}} \chi_{[2^j,2^{j+1})}(\xi) M(\xi)\widehat{\Delta_j g}\Big)^\vee \\ &= \sum_{j\in\mathbb{Z}} \big((M(2^j) + \int_{2^j}^{\xi} M'(\tau)\,d\tau\big)\,\widehat{\Delta_j g}\big)^\vee \\ &= \sum_{j\in\mathbb{Z}} M(2^j)\Delta_j g + \sum_{j\in\mathbb{Z}} \int_0^1 2^j M'(2^j(1+t))B_{j,t}\Delta_j g\,dt,\end{aligned} \tag{2.26}$$

where $B_{j,t} := R_{2^j(1+t)} - R_{2^{j+1}}$.

First let $2 < r < \infty$. To estimate the first term on the right-hand side of (2.26) in the norm of $L^q(\mathbb{R}; L^r_\omega(\Sigma))$, note that for each $j \in \mathbb{Z}$ the operator $M(2^j)$ commutes with Δ_j and that $\{\Delta_j : j \in \mathbb{Z}\}$ is a Schauder decomposition of $RL^q(\mathbb{R}; L^r_\omega(\Sigma))$. Then, by Lemma 2.8, for a.a. $x_n \in \mathbb{R}$ and for any $l, k \in \mathbb{Z}$ there is some $v(x_n) \in L^s_\omega(\mathbb{R}^{n-1})$ depending on $u_j = M(2^j)\Delta_j g$, $j = l,\dots,k$, such that (2.7), (2.8) are satisfied with $(u_j)_{j=l}^k$. Therefore, in view of (2.24), we get

$$\begin{aligned}\Big\|\sum_{j=l}^k M(2^j)\Delta_j g\Big\|_{L^q(\mathbb{R};L^r_\omega(\Sigma))} &\le c\Big\|\Big(\sum_{j=l}^k \|M(2^j)\Delta_j g\|^2_{2;\omega v(x_n)}\Big)^{1/2}\Big\|_{q,\mathbb{R}} \\ \le c\Big\{\Big\|\Big(\sum_{j=l}^k \|\Delta_j g\|^2_{W^{1,2}_{\omega v(x_n)}(\Sigma)}\Big)^{1/2}\Big\|_{q,\mathbb{R}} &+ \Big\|\Big(\sum_{j=l}^k 2^{2j}\|\Delta_j g\|^2_{2,\omega v(x_n)}\Big)^{1/2}\Big\|_{q,\mathbb{R}} \\ +(|\lambda|+1)\Big\|\Big(\sum_{j=l}^k \|\Delta_j g; L^2_{m,\omega v(x_n)} &+ L^2_{\omega v(x_n),1/2^j}\|_0^2\Big)^{1/2}\Big\|_{q,\mathbb{R}}\Big\}\end{aligned} \tag{2.27}$$

with $c = c(q, r, \alpha, \varepsilon, \Sigma, \mathcal{A}_r(\omega)) > 0$ independent of $l, k \in \mathbb{Z}$.

Now let us estimate each term on the right-hand side of (2.27). By (2.8) we get

$$\Big\|\Big(\sum_{j=l}^k \|\Delta_j g\|^2_{W^{1,2}_{\omega v(x_n)}(\Sigma)}\Big)^{1/2}\Big\|_{q,\mathbb{R}} \le c(q,r)\Big\|\sum_{j=l}^k \Delta_j g\Big\|_{L^q(\mathbb{R};W^{1,r}_\omega(\Sigma))}; \tag{2.28}$$

note that Δ_j is an operator with respect to the variable x_n. By analogy, exploiting Lemma 2.5 (2),

$$\begin{aligned}\big\|\big(\textstyle\sum_{j=l}^k 2^{2j}\|\Delta_j g\|^2_{2,\omega v(x_n)}\big)^{1/2}\big\|_{q,\mathbb{R}} &\le c(q,r)\big\|\textstyle\sum_{j=l}^k 2^j\Delta_j g\big\|_{L^q(\mathbb{R};L^r_\omega(\Sigma))} \\ &\le c(q,r)\big\|\textstyle\sum_{j=l}^k \Delta_j g\big\|_{\widehat{W}^{1,q}(\mathbb{R};L^r_\omega(\Sigma))}.\end{aligned} \tag{2.29}$$

In order to get an estimate of the last term on the right-hand side of (2.27), let

$$\sum_{j=l}^k \Delta_j g = g_0 + g_1, \quad g_0 \in L^q(\mathbb{R}; \widehat{W}^{-1,r}_\omega(\Sigma)),\ g_1 \in \widehat{W}^{-1,q}(\mathbb{R}; L^r_\omega(\Sigma)),$$

be any splitting of $\sum_{j=l}^k \Delta_j g$. Due to the properties of Δ_j we see that $\Delta_j g = \Delta_j g_0 + \Delta_j g_1$ for all $j = l,\dots,k$, and that, by Lemma 2.5 (1), $\Delta_j g_1 \in L^q(\mathbb{R}; L^r_\omega(\Sigma))$ and consequently even $\Delta_j g_0 \in L^q(\mathbb{R}; \widehat{W}^{-1,r}_\omega(\Sigma) \cap L^r_\omega(\Sigma)) = L^q(\mathbb{R}; L^r_{m,\omega}(\Sigma))$. Furthermore, by (2.7) and Hölder's inequality it is easily proved that for a.a $x_n \in \mathbb{R}$

$$L^r_\omega(\Sigma) \subset L^2_{\omega v(x_n)}(\Sigma), \quad \|\varphi\|_{2,\omega v(x_n)} \le \|\varphi\|_{r,\omega} \|v(x_n)\|^{1/2}_{s,\omega} \le \sqrt{2}\|\varphi\|_{r,\omega} \tag{2.30}$$

for all $\varphi \in L^r_\omega(\Sigma)$, and hence

$$\widehat{W}^{-1,r}_\omega(\Sigma) \subset \widehat{W}^{-1,2}_{\omega v(x_n)}(\Sigma), \quad \|h\|_{-1,2,\omega v(x_n)} \le \sqrt{2}\|h\|_{-1,r,\omega} \tag{2.31}$$

for all $h \in \widehat{W}^{-1,r}_\omega(\Sigma)$. By the triangle inequality,

$$\begin{aligned} &\Big\| \Big(\sum_{j=l}^k \|\Delta_j g; L^2_{m,\omega v(x_n)} + L^2_{\omega v(x_n), 1/2^j}\|_0^2 \Big)^{1/2} \Big\|_{q,\mathbb{R}} \\ &\qquad \le \Big\| \Big(\sum_{j=l}^k \|\Delta_j g_0\|^2_{-1,2,\omega v(x_n)} \Big)^{1/2} \Big\|_{q,\mathbb{R}} + \Big\| \Big(\sum_{j=l}^k 2^{-2j} \|\Delta_j g_1\|^2_{2,\omega v(x_n)} \Big)^{1/2} \Big\|_{q,\mathbb{R}} \end{aligned}$$

Then using the Hilbert space structure of $\widehat{W}^{-1,2}_{\omega v(x_n)}(\Sigma)$ and the properties of any independent symmetric $\{-1,1\}$-valued random variables $(\varepsilon_j(\cdot))$ on $(0,1)$ as well as (2.31), Kahane's inequality (2.3), Fubini's theorem and (2.1) we get that

$$\begin{aligned} \Big\| \Big(\sum_{j=l}^k \|\Delta_j g_0\|^2_{-1,2,\omega v(x_n)} \Big)^{1/2} \Big\|_{q,\mathbb{R}} &= \Big\| \Big\| \sum_{j=l}^k \varepsilon_j(s) \Delta_j g_0 \Big\|_{L^2(0,1; \widehat{W}^{-1,2}_{\omega v(x_n)}(\Sigma))} \Big\|_{q,\mathbb{R}} \\ &\le \sqrt{2} \Big\| \Big\| \sum_{j=l}^k \varepsilon_j(s) \Delta_j g_0 \Big\|_{L^2(0,1; \widehat{W}^{-1,r}_\omega(\Sigma))} \Big\|_{q,\mathbb{R}} \\ &\le c(q,r) \Big\| \Big\| \sum_{j=l}^k \varepsilon_j(s) \Delta_j g_0 \Big\|_{L^q(0,1; \widehat{W}^{-1,r}_\omega(\Sigma))} \Big\|_{q,\mathbb{R}} \\ &\le c(q,r) \Big\| \sum_{j=l}^k \Delta_j g_0 \Big\|_{L^q(\mathbb{R}; \widehat{W}^{-1,r}_\omega(\Sigma))}. \end{aligned}$$

Similarly, using (2.30) and (2.5), we get that

$$\begin{aligned} \big\| \big(\textstyle\sum_{j=l}^k 2^{-2j} \|\Delta_j g_1\|^2_{2,\omega v(x_n)} \big)^{1/2} \big\|_{q,\mathbb{R}} &\le c(q,r) \big\| \textstyle\sum_{j=l}^k 2^{-j} \Delta_j g_1 \big\|_{L^q(\mathbb{R}; L^r_\omega(\Sigma))} \\ &\le c(q,r) \big\| \textstyle\sum_{j=l}^k \Delta_j g_1 \big\|_{\widehat{W}^{-1,q}(\mathbb{R}; L^r_\omega(\Sigma))}. \end{aligned}$$

Then the uniform boundedness of $\{\sum_{j=l}^k \Delta_j\}_{l,k\in\mathbb{Z}}$ in $\mathcal{L}(L^q(\mathbb{R}; \widehat{W}^{-1,r}_\omega(\Sigma)))$ and in $\mathcal{L}(\widehat{W}^{-1,q}(\mathbb{R}; L^r_\omega(\Sigma)))$ implies the estimate

$$\begin{aligned} &\Big\| \Big(\sum_{j=l}^k \|\Delta_j g; L^2_{m,\omega v(x_n)} + L^2_{\omega v(x_n), 1/2^j}\|_0^2 \Big)^{1/2} \Big\|_{q,\mathbb{R}} \\ &\le c\Big(\Big\| \sum_{j=l}^k \Delta_j g_0 \Big\|_{L^q(\mathbb{R}; \widehat{W}^{-1,r}_\omega(\Sigma))} + \Big\| \sum_{j=l}^k \Delta_j g_1 \Big\|_{\widehat{W}^{-1,q}(\mathbb{R}; L^r_\omega(\Sigma))} \Big) \\ &\le c\big(\|g_0\|_{L^q(\mathbb{R}; \widehat{W}^{-1,r}_\omega(\Sigma))} + \|g_1\|_{\widehat{W}^{-1,q}(\mathbb{R}; L^r_\omega(\Sigma))} \big) \end{aligned}$$

with $c = c(q,r) > 0$ independent of $l, k \in \mathbb{Z}$. Now (2.14) implies the estimate

$$\Big\| \Big(\sum_{j=l}^{k} \|\Delta_j g; L^2_{m,\omega v(x_n)} + L^2_{\omega v(x_n),1/2^j}\|_0^2 \Big)^{1/2} \Big\|_{q,\mathbb{R}} \le c(q,r) \Big\| \sum_{j=l}^{k} \Delta_j g \Big\|_{\widehat{W}_\omega^{-1;q,r}(\Omega)}. \quad (2.32)$$

Summarizing (2.27)-(2.29) and (2.32) we get that

$$\begin{aligned} &\big\| \textstyle\sum_{j=l}^{k} M(2^j)\Delta_j g \big\|_{L^q(\mathbb{R};L^r_\omega(\Sigma))} \\ &\qquad \le c\big(\big\| \textstyle\sum_{j=l}^{k} \Delta_j g \big\|_{W^{1;q,r}_\omega(\Omega)} + (|\lambda|+1) \big\| \textstyle\sum_{j=l}^{k} \Delta_j g \big\|_{\widehat{W}_\omega^{-1;q,r}(\Omega)} \big) \end{aligned} \quad (2.33)$$

with $c = c(q,r,\alpha,\varepsilon,\Sigma,\mathcal{A}_r(\omega)) > 0$ for all $l,k \in \mathbb{Z}$ and for all $\lambda \in -\alpha + S_\varepsilon$. Since $(\Delta_j)_{j\in\mathbb{Z}}$ defines unconditional Schauder decompositions of $RW^{1;q,r}_\omega(\Omega)$ and of $R\widehat{W}_\omega^{-1;q,r}(\Omega)$, (2.33) implies that the series $\sum_{j\in\mathbb{Z}} M(2^j)\Delta_j g$ converges in $L^q(\mathbb{R};L^r_\omega(\Sigma))$ and

$$\Big\| \sum_{j\in\mathbb{Z}} M(2^j)\Delta_j g \Big\|_{L^q(\mathbb{R};L^r_\omega(\Sigma))} \le c\big(\|g\|_{W^{1;q,r}_\omega(\Omega)} + (|\lambda|+1)\|g\|_{\widehat{W}_\omega^{-1;q,r}(\Omega)} \big)$$

with $c = c(q,r,\alpha,\varepsilon,\Sigma,\mathcal{A}_r(\omega)) > 0$. This is the desired estimate of the first term on the right-hand side of (2.26).

Next let us estimate the second term on the right-hand side of (2.26). Note that the operator family

$$\{B_{j,t} : j \in \mathbb{N}, t \in (0,1)\} \subset \mathcal{L}(L^q(\mathbb{R};L^r_\omega(\Sigma)))$$

is $\mathcal{R}$-bounded, cf. Lemma 2.5 (3). Moreover, for $t \in (0,1)$, the operator $M'(2^j(1+t))$ commutes with the operator $B_{j,t}$ and the range of $B_{j,t}$ is contained in the range of Δ_j. Hence it follows from (2.1), (2.2) that for any independent symmetric $\{-1,1\}$-valued random variables $\{\varepsilon_j(\cdot)\}$ on $(0,1)$

$$\begin{aligned} &\Big\| \sum_{j=l}^{k} \int_0^1 2^j M'(2^j(1+t)) B_{j,t}\Delta_j g \, dt \Big\|_{L^q(\mathbb{R};L^r_\omega(\Sigma))} \\ &\qquad \le \int_0^1 \Big\| \sum_{j=l}^{k} 2^j(1+t) B_{j,t} M'(2^j(1+t))\Delta_j g \Big\|_{L^q(\mathbb{R};L^r_\omega(\Sigma))} dt \\ &\qquad \le c_\Delta \int_0^1 \Big\| \sum_{j=l}^{k} \varepsilon_j B_{j,t} 2^j(1+t) M'(2^j(1+t))\Delta_j g \Big\|_{L^q(0,1;L^q(\mathbb{R};L^r_\omega(\Sigma)))} dt \\ &\qquad \le c \int_0^1 \Big\| \sum_{j=l}^{k} \varepsilon_j 2^j(1+t) M'(2^j(1+t))\Delta_j g \Big\|_{L^q(0,1;L^q(\mathbb{R};L^r_\omega(\Sigma)))} dt \\ &\qquad \le c \int_0^1 \Big\| \sum_{j=l}^{k} 2^j(1+t) M'(2^j(1+t))\Delta_j g \Big\|_{L^q(\mathbb{R};L^r_\omega(\Sigma))} dt. \end{aligned} \quad (2.34)$$

By Lemma 2.8 (2.7) holds with $u_j = u_j(t) := 2^j(1+t)M'(2^j(1+t))\Delta_j g$ and with corresponding functions $v = v(\cdot, x_n, t) \in L^s_\omega(\mathbb{R}^{n-1})$ for $(x_n,t) \in \mathbb{R}\times(0,1)$, where v

is Lebesgue measurable w.r.t. $(x', x_n, t) \in \mathbb{R}^n \times (0,1)$ by Lemma 2.7, see the proof of Lemma 2.8. Therefore, using (2.24) we get that

$$
\begin{aligned}
&\text{the r.h.s. of (2.34)}\\
&\le c \int_0^1 \Big\| \Big(\sum_{j=l}^k \|2^j(1+t)M'(2^j(1+t))\Delta_j g(\cdot, x_n)\|^2_{2,\omega v(\cdot,x_n,t)} \Big)^{1/2} \Big\|_{q,\mathbb{R}}\, dt \\
&\le c \Big(\int_0^1 \Big\| \Big\{ \sum_{j=l}^k \big[\|\Delta_j g\|^2_{W^{1,2}_{\omega v(\cdot,x_n,t)}(\Sigma)} + 2^{2j}(1+t)^2 \|\Delta_j g\|^2_{2,\omega v(\cdot,x_n,t)} \\
&\qquad + |\lambda+1|^2 \, \|\Delta_j g; L^2_{m,\omega v(\cdot,x_n,t)} + L^2_{\omega v(\cdot,x_n,t),2^{-j}(1+t)^{-1}}\|_0^2 \big] \Big\}^{1/2} \Big\|_{q,\mathbb{R}} dt \Big) \\
&\le c \Big(\int_0^1 \Big\| \Big(\sum_{j=l}^k \|\Delta_j g\|^2_{W^{1,2}_{\omega v(\cdot,x_n,t)}(\Sigma)} \Big)^{1/2} \Big\|_{q,\mathbb{R}} + \Big\| \Big(\sum_{j=l}^k 2^{2j} \|\Delta_j g\|^2_{2,\omega v(\cdot,x_n,t)} \Big)^{1/2} \Big\|_{q,\mathbb{R}} \\
&\qquad + |\lambda+1| \, \Big\| \Big(\sum_{j=l}^k \|\Delta_j g; L^2_{m,\omega v(\cdot,x_n,t)} + L^2_{\omega v(\cdot,x_n,t),2^{-j}}\|_0^2 \Big)^{1/2} \Big\|_{q,\mathbb{R}} dt \Big),
\end{aligned}
$$

where $c = c(q, r, \alpha, \varepsilon, \Sigma, \mathcal{A}_r(\omega))$. Thus, by the same argument leading from (2.27) to (2.33) we get the estimate

$$
\begin{aligned}
&\Big\| \sum_{j=l}^k \int_0^1 2^j M'(2^j(1+t)) B_{j,t} \Delta_j g \, dt \Big\|_{L^q(\mathbb{R}; L^r_\omega(\Sigma))} \\
&\qquad \le c \Big(\Big\| \sum_{j=l}^k \Delta_j g \Big\|_{W^{1;q,r}_\omega(\Omega)} + (|\lambda|+1) \Big\| \sum_{j=l}^k \Delta_j g \Big\|_{\widehat{W}^{-1;q,r}_\omega(\Omega)} \Big)
\end{aligned}
$$

with $c = c(q, r, \alpha, \varepsilon, \Sigma, \mathcal{A}_r(\omega)) > 0$. Summarizing, we proved in the case $r > 2$ the existence of a solution to (R_λ) satisfying the estimate (2.20).

In the case $r = 2$ the same proof as before, but with $v \equiv 1$, may be used.

The uniqueness of solution is obvious from the uniqueness result for $f \neq 0, g = 0$, see [18]. Now the proof of the theorem is complete. ■

3 Cylindrical Domains with Several Exits to Infinity: Proof of the Main Results

In this section $\Omega \subset \mathbb{R}^n$ is the cylindrical domain $\Omega = \bigcup_{i=0}^m \Omega_i$ of $C^{1,1}$-class where Ω_0 is a bounded domain of class $C^{1,1}$ and Ω_i, $i = 1, \ldots, m$, are disjoint semi-infinite straight cylinders; that is, in possibly different coordinates,

$$
\Omega_i = \{x^i \in \mathbb{R}^n : \, x^i_n > 0, \, (x^i_1, \ldots, x^i_{n-1}) \in \Sigma^i\},
$$

where $\Sigma^i \subset \mathbb{R}^{n-1}$ is bounded, and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. Let $\bar\alpha = \min\{\alpha^{(i)} : \, i = 0, \ldots, m\}$ where $\alpha^{(0)} > 0$ and $\alpha^{(i)} > 0$, $i = 1, \ldots, m$, are the smallest eigenvalues of the Dirichlet Laplacians in Ω_0 and in Σ^i, $i = 1, \ldots, m$, respectively.

For fixed $\lambda \in \mathbb{C} \setminus (-\infty, -\bar{\alpha}]$ let us define the operator $S_{q,\lambda}$ by

$$\mathcal{D}(S_{q,\lambda}) = \big(W^{2,q}(\Omega)^n \cap W_0^{1,q}(\Omega)^n \cap L^q_\sigma(\Omega)\big) \times \widehat{W}^{1,q}(\Omega),$$
$$S_{q,\lambda}(u,p) = \lambda u - \Delta u + \nabla p.$$

Obviously the range $\mathcal{R}(S_{q,\lambda})$ of $S_{q,\lambda}$ is contained in $L^q(\Omega)^n$.

Lemma 3.1 *Let $2 \le q < \infty$, $\varepsilon \in (\pi/2, \pi)$ and $\lambda \in -\alpha + \Sigma_\varepsilon$, where $\alpha \in (0, \bar{\alpha})$.*

(i) If $(u,p) \in \big(W^{2,q}(\Omega)^n \cap W_0^{1,q}(\Omega)^n\big) \times \widehat{W}^{1,q}(\Omega)$ is a solution to the resolvent problem (1.2) with $f \in L^q(\Omega)^n$, then (u,p) satisfies the estimate

$$\begin{aligned} &\|(\lambda+\alpha)u, \nabla^2 u, \nabla p\|_{L^q(\Omega)} \\ &\quad \le C\big(\|f\|_{L^q(\Omega)} + \|\nabla u, u, p\|_{L^q(\Omega_0)} + (|\lambda|+1)\|u\|_{(W^{1,q'}(\Omega_0))^*}\big), \end{aligned} \tag{3.1}$$

with a constant $C = C(q, \alpha, \varepsilon, \Omega_0, \Sigma^1, \dots, \Sigma^m) > 0$ independent of $\lambda \in -\alpha + \Sigma_\varepsilon$; here $q' = q/(q-1)$.

(ii) The operator $S_{q,\lambda}$ is injective.

(iii) The range $\mathcal{R}(S_{q,\lambda})$ of $S_{q,\lambda}$ is dense in $L^q(\Omega)^n$.

Proof: The proof uses a cut-off technique and, in principle, follows the same argument as in the proof of Lemma 4.1 of [20]. Without loss of generality we may assume that there exist cut-off functions $\{\varphi_i\}_{i=0}^m$ such that

$$\begin{aligned} &\textstyle\sum_{i=0}^m \varphi_i(x) = 1, \quad 0 \le \varphi_i(x) \le 1 \quad \text{for } x \in \Omega, \\ &\varphi_i \in C^\infty(\bar{\Omega}_i), \quad \operatorname{dist}(\operatorname{supp}\varphi_i,\, \partial\Omega_i \cap \Omega) \ge \delta > 0,\ i = 0, \dots, m, \end{aligned} \tag{3.2}$$

where 'dist' means the distance. For $i = 1, \dots, m$ let $\widetilde{\Omega}_i$ be the infinite straight cylinder extending the semi-infinite cylinder Ω_i, and denote the zero extension of v to $\widetilde{\Omega}_i$ by $\tilde{v}$. Then $\{\varphi_0 u, \varphi_0 p\}$ on Ω_0 satisfies

$$(R_\lambda)_0 \qquad \begin{aligned} \lambda(\varphi_0 u) - \Delta(\varphi_0 u) + \nabla(\varphi_0 p) &= f^0 && \text{in } \Omega_0 \\ \operatorname{div}(\varphi_0 u) &= g^0 && \text{in } \Omega_0 \\ \varphi_0 u &= 0 && \text{on } \partial\Omega_0, \end{aligned}$$

and $\{\widetilde{\varphi_i u}, \widetilde{\varphi_i p}\}$ on $\widetilde{\Omega}_i$, $i = 1, \dots, m$, satisfy

$$(R_\lambda)_i \qquad \begin{aligned} \lambda(\widetilde{\varphi_i u}) - \Delta(\widetilde{\varphi_i u}) + \nabla(\widetilde{\varphi_i p}) &= \tilde{f}^i && \text{in } \widetilde{\Omega}_i \\ \operatorname{div}(\widetilde{\varphi_i u}) &= \tilde{g}^i && \text{in } \widetilde{\Omega}_i \\ \widetilde{\varphi_i u} &= 0 && \text{on } \partial\widetilde{\Omega}_i, \end{aligned}$$

where

$$f^i := \varphi_i f + (\nabla\varphi_i)p - (\Delta\varphi_i)u - 2\nabla\varphi_i \cdot \nabla u, \quad g^i := \nabla\varphi_i \cdot u, \quad i = 0, \dots, m.$$

Note that $\operatorname{supp} g^i \subset \Omega_0$ and $\int_{\Omega_0} g^i\,dx = 0$ for $i = 0, \dots, m$. Therefore,

$$\int_{\Omega_0} g^0 \psi\,dx = \int_{\Omega_0} u \cdot (\overline{\psi}\,\nabla\varphi_0)\,dx \qquad \text{for all } \psi \in C^\infty(\overline{\Omega}_0)$$

where $\overline{\psi} = \psi - \frac{1}{|\Omega_0|}\int_{\Omega_0}\psi\,dx$. Hence, using Poincaré's inequality, we get that $g^0 \in \widehat{W}^{-1,q}(\Omega_0)$ and

$$\|g^0\|_{\widehat{W}^{-1,q}(\Omega_0)} \le c(\Omega_0)\|\nabla^2\varphi_0, \nabla\varphi_0\|_{L^\infty(\Omega_0)}\|u\|_{(W^{1,q'}(\Omega_0))^*}.$$

In the same way it follows that $\tilde{g}^i \in \widehat{W}^{-1,q}(\widetilde{\Omega}_i)$ and

$$\|\tilde{g}^i\|_{\widehat{W}^{-1,q}(\widetilde{\Omega}_i)} \le c\|u\|_{(W^{1,q'}(\Omega_0))^*} \quad \text{for } i = 1, \dots, m.$$

Therefore, by [20], Theorem 1.2, for all $\lambda \in -\alpha + \Sigma_\varepsilon$

$$\begin{aligned}
&\|(\lambda+\alpha)(\varphi_0 u), \nabla^2(\varphi_0 u), \nabla(\varphi_0 p)\|_{L^q(\Omega_0)} \\
&\qquad \le c\big(\|f^0, \nabla g^0, g^0\|_{L^q(\Omega_0)} + |\lambda|\|g^0\|_{\widehat{W}^{-1,q}(\Omega_0)}\big) \\
&\qquad \le c\big(\|f\|_{L^q(\Omega)} + \|\nabla u, u, p\|_{L^q(\Omega_0)} + |\lambda|\|u\|_{(W^{1,q'}(\Omega_0))^*}\big)
\end{aligned} \tag{3.3}$$

with $c = c(q, \alpha, \varepsilon, \Omega_0) > 0$. Furthermore, by Theorem 2.10, for $i = 1, \dots, m$

$$\begin{aligned}
&\|(\lambda+\alpha)(\varphi_i u), \nabla^2(\varphi_i u), \nabla(\varphi_i p)\|_{L^q(\Omega_i)} \\
&\qquad = \|(\lambda+\alpha)(\widetilde{\varphi_i u}), \nabla^2(\widetilde{\varphi_i u}), \nabla(\widetilde{\varphi_i p})\|_{L^q(\tilde{\Omega}_i)} \\
&\qquad \le c\big(\|\tilde{f}^i, \nabla\tilde{g}^i, \tilde{g}^i\|_{L^q(\tilde{\Omega}_i)} + (|\lambda|+1)\|\tilde{g}^i\|_{\widehat{W}^{-1,q}(\tilde{\Omega}_i)}\big) \\
&\qquad \le c\big(\|f\|_{L^q(\Omega)} + \|\nabla u, u, p\|_{L^q(\Omega_0)} + (|\lambda|+1)\|u\|_{(W^{1,q'}(\Omega_0))^*}\big),
\end{aligned} \tag{3.4}$$

with $c = c(q, \alpha, \varepsilon, \Sigma^i) > 0$. Finally, summing (3.3) and (3.4) for $i = 1, \dots, m$, we get the estimate (3.1) for $u = \sum_{i=0}^m \varphi_i u$ and $p = \sum_{i=0}^m \varphi_i p$. Thus (i) is proved.

To prove the injectivity of $S_{q,\lambda}$ let $S_{q,\lambda}(u,p) = 0$ with $(u,p) \in \mathcal{D}(S_{q,\lambda})$. If $q = 2$, one directly gets $(u, \nabla p) = 0$ by testing with u.

Let $2 < q < \infty$. Looking at $(R_\lambda)_0$ and $(R_\lambda)_i$, $i = 1, \dots, m$, it is obvious that $f^0 \in L^2(\Omega_0)$, $g^0 \in W^{1,2}(\Omega_0) \cap \widehat{W}^{-1,2}(\Omega_0)$ and $\tilde{f}^i \in L^2(\widetilde{\Omega}_i)$, $\tilde{g}^i \in W^{1,2}(\widetilde{\Omega}_i) \cap \widehat{W}^{-1,2}(\widetilde{\Omega}_i)$; note that $f = 0$ and that f^i, g^i, $i = 0, \dots, m$ are compactly supported in Ω_0. Therefore, by [20], Theorem 1.2 and [19], Theorem 2.1, we get that

$$(\varphi_i u, \varphi_i p) \in \big(W^{2,2}(\Omega_i)^n \cap W_0^{1,2}(\Omega_i)^n\big) \times \widehat{W}^{1,2}(\Omega_i), \;\; i = 0, \dots, m.$$

Thus $(u,p) \in \mathcal{D}(S_{2,\lambda})$ yielding $(u,p) = 0$.

Next let us show that $\mathcal{R}(S_{q,\lambda})$ is dense in $L^q(\Omega)^n$. By the lemma of Lax-Milgram and regularity theory of the Stokes system we conclude that $\mathcal{R}(S_{2,\lambda}) = L^2(\Omega)^n$. For $q > 2$ and $f \in C_0^\infty(\Omega)^n$ which is dense in $L^q(\Omega)^n$, there is $(u,p) \in \mathcal{D}(S_{2,\lambda})$ such that $S_{2,\lambda}(u,p) = f$. Looking at $(R_\lambda)_0$ and $(R_\lambda)_i$ and using regularity results for Stokes resolvent systems on bounded domains and on infinite cylinders (Theorem 2.10), one can see that

$$(\varphi_i u, \varphi_i p) \in \big(W_\omega^{2;\tilde{q},r}(\tilde{\Omega}_i)^n \cap W_{0,\omega}^{1;\tilde{q},r}(\tilde{\Omega}_i)^n\big) \times \widehat{W}_\omega^{1;\tilde{q},r}(\Omega_i), \;\; i = 1, \dots, m,$$

with $\omega \equiv 1$ for all $\tilde{q} \in (1,\infty)$, $r \in [2,\infty)$, in particular,

$$(\varphi_i u, \varphi_i p) \in \big(W^{2,q}(\Omega_i)^n \cap W^{1,q}_0(\Omega_i)^n\big) \times \widehat{W}^{1,q}(\Omega_i), \quad i = 0,\dots,m,$$

yielding the denseness of $\mathcal{R}(S_{q,\lambda})$ in $L^q(\Omega)^n$

The proof of this lemma is complete. ■

Now we can prove the main theorem of this paper.

Proof of Theorem 1.1: First let $2 \le q < \infty$. Let us prove the *a priori* estimate (1.3) which will imply by Lemma 3.1 that the operator $S_{q,\lambda}$ is an isomorphism from $\mathcal{D}(S_{q,\lambda})$ to $L^q(\Omega)^n$. Instead of proving (1.3) we shall show a slightly stronger estimate

$$\|(\lambda+\beta)u, \nabla^2 u, \nabla p\|_{L^q(\Omega)} \le C\|f\|_{L^q(\Omega)} \quad \forall \lambda \in -\alpha + \Sigma_\varepsilon \tag{3.5}$$

with a constant $C = C(q,\alpha,\varepsilon,\Omega)$ independent of λ where $\beta = \frac{1}{2}(\alpha+\bar{\alpha})$; note that $|\lambda+\alpha| \le c(\varepsilon,\alpha)|\lambda+\beta|$ for all $\lambda \in -\alpha+\Sigma_\varepsilon$.

Assume that (3.5) does not hold. Then there are sequences $\{\lambda_j\} \subset -\alpha+\Sigma_\varepsilon$, $\{(u_j,p_j)\} \subset \mathcal{D}(S_{q,\lambda_j})$ such that

$$\|(\lambda_j+\beta)u_j, \nabla^2 u_j, \nabla p_j\|_{L^q(\Omega)} = 1, \quad \|f_j\|_{L^q(\Omega)} \to 0 \quad \text{as } j\to\infty, \tag{3.6}$$

where $f_j = S_{q,\lambda_j}(u_j,p_j)$. Without loss of generality we may assume that

$$(\lambda_j+\beta)u_j \rightharpoonup v, \quad \nabla^2 u_j \rightharpoonup \nabla^2 u, \quad \nabla p_j \rightharpoonup \nabla p \quad \text{as } j\to\infty \tag{3.7}$$

with some $v \in L^q(\Omega)$, $u \in \widehat{W}^{2,q}(\Omega)$ and $p \in \widehat{W}^{1,q}(\Omega)$. Moreover, we may assume $\int_{\Omega_0} p_j\,dx = 0$, $\int_{\Omega_0} p\,dx = 0$ and that $\lambda_j \to \lambda \in \{-\alpha+\bar{S}_\varepsilon\} \cup \{\infty\}$.

(i) Let $\lambda_j \to \lambda \in -\alpha+\bar{S}_\varepsilon$.

Note that $\lambda+\beta \ne 0$. Then by (3.7) $v = (\lambda+\beta)u$, $u_j \rightharpoonup u$ in $W^{2,q}(\Omega)$ and $u \in \mathcal{D}(S_{q,\lambda})$. It follows from (1.2), (3.6) that $S_{q,\lambda}(u,p) = 0$ yielding $(u,p) = 0$ by Lemma 3.1 (ii). On the other hand, we have the strong convergences

$$u_j \to 0 \text{ in } W^{1,q}(\Omega_0), \;\; p_j \to 0 \text{ in } L^q(\Omega_0), \;\; (|\lambda_j|+1)u_j \to 0 \text{ in } (W^{1,q'}(\Omega_0))^* \tag{3.8}$$

due to the compact embeddings $W^{2,q}(\Omega_0) \subset\subset W^{1,q}(\Omega_0) \subset\subset L^q(\Omega_0) \subset\subset (W^{1,q'}(\Omega_0))^*$, Poincaré's inequality on Ω_0 and (3.7). Thus Lemma 3.1 (i) together with (3.6) yields the contradiction $1 \le 0$.

(ii) Let $|\lambda_j| \to \infty$. Then, besides (3.7), we conclude that $\nabla^2 u = 0$, and consequently $v + \nabla p = 0$ where $v \in L^q_\sigma(\Omega)$. Note that this is the L^q-Helmholtz decomposition of the null vector field on Ω. Therefore, $v = 0$, $\nabla p = 0$. Again we get (3.8) and finally the contradiction $1 \le 0$.

Thus (3.5) holds true proving existence of a unique solution to (R_λ) in the case $2 \le q < \infty$.

The case $1 < q < 2$ can be proved by a duality argument. As is well known, (1.2) is equivalent to

$$(\lambda + A_q)u = P_q f$$

with the Stokes operator A_q and the Helmholtz decomposition P_q of $L^q(\Omega)$. Moreover, if $0 \in \rho(A_q)$, then the resolvent estimate of type (1.1) implies by the open mapping theorem the estimate (1.3) as well as the uniqueness and existence of a solution to (R_λ). If we show

$$A_q^* = A_{q'}, \tag{3.9}$$

where A_q^* is the dual of A_q in $L^q_\sigma(\Omega)$, then $-\alpha + \Sigma_\varepsilon \subset \rho(-A_{q'})$ and the estimate (1.1) for $2 < q' < \infty$ yield, by well-known theory on resolvents, that $-\alpha + \Sigma_\varepsilon \subset \rho(-A_q)$ and the estimate (1.1) for $1 < q < 2$.

Since $P_q^* = P_{q'}$, it is easily seen that $A_{q'} \subset A_q^*$. Let $v \in D(A_q^*)$ and let $w \in D(A_{q'})$ satisfy $A_{q'}w = A_q^* v$; note that $0 \in \rho(A_{q'})$ due to the result already proved for $q' > 2$. Then for all $u \in D(A_q)$

$$(A_q u, v)_{L^q, L^{q'}} = (u, A_q^* v)_{L^q, L^{q'}} = (u, A_{q'} w)_{L^q, L^{q'}} = (A_q u, w)_{L^q, L^{q'}}.$$

Since $R(A_q)$ is dense in $L^q_\sigma(\Omega)$ – for an argument see the last paragraph of the proof of Lemma 3.1 –, we conclude that $v = w \in D(A_{q'})$, and (3.9) is proved.

Finally, (1.4) follows from (1.1) by the well-known theory of analytic semigroups.

■

In [5] it is proved that the shifted Stokes operator $c + A_q$ with some $c > 0$ on $L^q_\sigma(G)$ admits a bounded H^∞-calculus provided the domain $G \subset R^n$, $n \geq 2$, satisfies the following assumptions (A1)-(A3):

(A1) There is a finite covering of $\bar{G}$ with relatively open sets $U_j, j = 1, \dots, l$, such that U_j coincides (after rotation) with a relatively open set of $\overline{\mathbb{R}^n_{\gamma_j}}$, where $\overline{\mathbb{R}^n_{\gamma_j}} := \{(x_1, \tilde{x}) \in \mathbb{R}^n : \ x_1 > \gamma_j(\tilde{x})\}$, $\gamma_j \in C^{1,1}$, $j = 1, \dots, l$. Moreover, suppose that there are cut-off functions $\varphi_j, \psi_j \in C_b^\infty(\bar{G})$, $j = 1, \dots, l$, such that $\{\varphi_j\}$ is a partition of unity subordinated to $\{U_j\}_{j=1}^l$, $\psi_j \equiv 1$ on $\operatorname{supp} \varphi_j$ and $\operatorname{supp} \psi_j \subset U_j$, $j = 1, \dots, l$; here $C_b^\infty(\bar{G})$ means the space of all infinitely differentiable and bounded functions on $\bar{G}$.

(A2) The Helmholtz decomposition is valid for $L^r(G)^n$ with $r = q$ and $r = q'$, i.e., for every $f \in L^r(G)^n$ there is a unique decomposition $f = f_0 + \nabla p$ with $f_0 \in L^r_\sigma(G)$ and $p \in \widehat{W}^{1,r}(G)$. Moreover,

$$L^q_\sigma(G) = \{f \in L^q(G)^n : \ \operatorname{div} f = 0, \ f \cdot N|_{\partial G} = 0\}. \tag{3.10}$$

(A3) For every $p \in \widehat{W}^{1,r}(G), r = q, q'$, there is a decomposition $p = p_1 + p_2$ such that $p_1 \in W^{1,r}(G), p_2 \in L^r_{\mathrm{loc}}(G)$ with $\nabla p_2 \in W^{1,r}(G)$ and $\|p_1, \nabla p_2\|_{W^{1,r}(G)} \leq C\|\nabla p\|_r$.

It is easily seen that the domain Ω satisfies the assumption (A1). Furthermore the Helmholtz decomposition of $L^q(\Omega)^n$ was proved in [7], Theorem 4(c). Through the following lemmata we shall see that the remaining assumptions are satisfied as well.

Lemma 3.2 *The set $C_0^\infty(\bar{\Omega})$ is dense in $\widehat{W}^{1,q}(\Omega)$ for $1 < q < \infty$.*

Proof: Fix $u \in \widehat{W}^{1,q}(\Omega)$. Using the same notation as in the proof of Lemma 3.1 and the cut-off functions φ_j, $j = 0, \dots, m$, see (3.2), we have $u = \sum_{j=0}^{m} \varphi_j u$. Without loss of generality assume that $\int_{\Omega_0} u\, dx = 0$. Thus, by Poincaré's inequality on the bounded domain Ω_0,

$$\varphi_0 u \in W^{1,q}(\Omega_0) \quad \text{and} \quad \varphi_j u \in \widehat{W}^{1,q}(\Omega_j), \quad \widetilde{\varphi_j u} \in \widehat{W}^{1,q}(\tilde{\Omega}_j),\ j = 1, \dots, m.$$

Then there are sequences $\{v_k^{(0)}\} \subset C_0^\infty(\bar{\Omega}_0)$, $\{v_k^{(j)}\} \subset C_0^\infty(\bar{\tilde{\Omega}}_j)$, $j = 1, \dots, m$, such that

$$\|v_k^{(0)} - \varphi_0 u\|_{W^{1,q}(\Omega_0)} \to 0, \quad \|v_k^{(j)} - \widetilde{\varphi_j u}\|_{\widehat{W}^{1,q}(\tilde{\Omega}_j)} \to 0 \tag{3.11}$$

as $k \to \infty$ due to the denseness of $C_0^\infty(\bar{\Omega}_0)$ in $W^{1,q}(\Omega_0)$ and Lemma 2.9 (2). Let

$$\Omega_j^\delta := \{x \in \Omega_j : \operatorname{dist}(x, \Omega \cap \partial\Omega_j) \ge \delta\} \quad \text{for } j = 0, \dots, m.$$

Note that

$$\operatorname{supp} \varphi_j u \subset \Omega_j^\delta, \quad j = 0, \dots, m, \tag{3.12}$$

due to the construction of $\{\varphi_j\}_{j=0}^m$. Without loss of generality we may assume that

$$\int_{\Omega_j \setminus \Omega_j^\delta} v_k^{(j)}\, dx = 0 \quad \text{for } j = 1, \dots, m. \tag{3.13}$$

Let us choose functions $\eta_0 \in C_0^\infty(\bar{\Omega})$ and $\eta_j \in C_0^\infty(\bar{\tilde{\Omega}}_j), j = 1, \dots, m$ such that

$$\begin{aligned} &\eta_0(x) \equiv 1 \text{ for } x \in \Omega_0^\delta \quad \text{and } \eta_0(x) \equiv 0 \text{ for } x \in \Omega \setminus \Omega_0^{\delta/2}, \\ &\eta_j(x) \equiv 1,\ x \in \Omega_j^\delta, \quad \text{and} \quad \eta_j(x) \equiv 0,\ x \in \tilde{\Omega}_j \setminus \Omega_j,\ j = 1, \dots, m. \end{aligned} \tag{3.14}$$

For $k \in \mathbb{N}$ let $w_k^{(0)} = \eta_0 v_k^{(0)}$ and let $w_k^{(j)}$ be the zero extension of $\eta_j v_k^{(j)}$ onto Ω.

Now let $w_k := \sum_{j=0}^m w_k^{(j)}$. Obviously $w_k \in C_0^\infty(\bar{\Omega})$, $k \in \mathbb{N}$, and

$$\|\nabla(u - w_k)\|_{L^q(\Omega)} \le \sum_{j=0}^m \|\nabla(\varphi_j u - w_k^{(j)})\|_{L^q(\Omega)}. \tag{3.15}$$

Due to (3.12) and (3.14) we get for each $j = 0, \dots, m$ that

$$\begin{aligned} \|\nabla(\varphi_j u - w_k^{(j)})\|_{L^q(\Omega)} &\le \|\nabla(\varphi_j u - v_k^{(j)})\|_{L^q(\Omega_j^\delta)} + \|\nabla(\eta_j v_k^{(j)})\|_{L^q(\Omega_j \setminus \Omega_j^\delta)} \\ &\le \|\nabla(\varphi_j u - v_k^{(j)})\|_{L^q(\Omega_j^\delta)} + c_j \|v_k^{(j)}, \nabla v_k^{(j)}\|_{L^q(\Omega_j \setminus \Omega_j^\delta)}. \end{aligned} \tag{3.16}$$

Note that for $j = 1, \dots, m$, using (3.13) and Poincaré's inequality, $\|v_k^{(j)}\|_{L^q(\Omega_j \setminus \Omega_j^\delta)} \le c(q, \Omega_0)\|\nabla v_k^{(j)}\|_{L^q(\Omega_j \setminus \Omega_j^\delta)}$. Therefore, by (3.11), (3.12) the right-hand side of (3.16) for $j = 0, \dots, m$ tends to 0 as $k \to \infty$, and so does the right-hand side of (3.15).

The proof of the lemma is complete. ■

Corollary 3.3 *For the domain Ω the assertion* (3.10) *holds.*

Proof: Obviously,

$$L^q_\sigma(\Omega) \subset \{f \in L^q(\Omega)^n : \ \mathrm{div}\, f = 0, \ f \cdot N|_{\partial\Omega} = 0\}.$$

Since the right-hand side of (3.10) is 'orthogonal' to $\{\nabla h : h \in C_0^\infty(\bar\Omega)\}$, the same result holds for $\{\nabla h : h \in \widehat{W}^{1,q'}(\Omega)\}$ by Lemma 3.2. Therefore, [25], Ch. III, Lemma 2.1, accomplishes the proof. ∎

Lemma 3.4 *The assumption (A3) is satisfied for the domain Ω.*

Proof: First consider the case of Ω being an infinite straight cylinder $\Sigma \times \mathbb{R}$ with $\Sigma \subset \mathbb{R}^{n-1}$, a bounded domain of $C^{1,1}$-class. For $p \in \widehat{W}^{1,q}(\Omega)$ let $p_0(x', x_n) \equiv p_0(x_n) := \frac{1}{|\Sigma|}\int_\Sigma p(x', x_n)\, dx'$ and $\tilde p := p - p_0$. Then it follows that

$$\begin{aligned} &p_0 \in \widehat{W}^{1,q}(\Sigma \times \mathbb{R}), \quad \|p_0\|_{\widehat{W}^{1,q}(\Sigma\times\mathbb{R})} \le c(\Sigma, q)\|p\|_{\widehat{W}^{1,q}(\Sigma\times\mathbb{R})},\\ &\tilde p \in W^{1,q}(\Sigma \times \mathbb{R}), \quad \|\tilde p\|_{W^{1,q}(\Sigma\times\mathbb{R})} \le c(\Sigma, q)\|p\|_{\widehat{W}^{1,q}(\Sigma\times\mathbb{R})}; \end{aligned} \tag{3.17}$$

here we used Poincaré's inequality for $\tilde p(\cdot, x_n)$ on Σ. On the other hand the whole space $\mathbb{R}^k$, $k \in \mathbb{N}$, was proved to satisfy assumption (A3), see [4], Remark 2.7. Therefore, as a function on $\mathbb{R}$, p_0 is decomposed by

$$p_0 = p_{01} + p_{02}, \quad \|p_{01}, \partial_1 p_{02}\|_{W^{1,q}(\mathbb{R})} \le c\|p_0\|_{\widehat{W}^{1,q}(\mathbb{R})}.$$

Then $p_1 := \tilde p + p_{01}, p_2 := p_{02}$ satisfy assumption (A3) due to (3.17).

Next let Ω be the general unbounded cylinder introduced in the beginning of this section. We use the same notation for $\{\varphi_j\}_{j=0}^m, \Omega_j, \tilde\Omega_j$ and Ω_j^δ as in the proof of Lemma 3.2. Fix $p \in \widehat{W}^{1,q}(\Omega)$ and write it in the form $p = \sum_{j=0}^m \varphi_j p$. Without loss of generality we assume that $\int_{\Omega_0} p\, dx = 0$; therefore, by Poincaré's inequality

$$\|p\|_{W^{1,q}(\Omega_0)} \le c\|p\|_{\widehat{W}^{1,q}(\Omega)}. \tag{3.18}$$

By the fact already proved for infinite straight cylinders, we have for $j = 1, \dots, m$, a decomposition $\widetilde{\varphi_j p} = p_{j1} + p_{j2}$ such that $p_{j1}, \nabla p_{j2} \in W^{1,q}(\tilde\Omega_j)$ and

$$\|p_{j1}, \nabla p_{j2}\|_{W^{1,q}(\Omega_j)} \le \|p_{j1}, \nabla p_{j2}\|_{W^{1,q}(\tilde\Omega_j)} \le c\|\widetilde{\varphi_j p}\|_{\widehat{W}^{1,q}(\tilde\Omega_j)} \le c\|p\|_{\widehat{W}^{1,q}(\Omega)}; \tag{3.19}$$

here we used $\int_{\Omega_0} p\, dx = 0$. Now define the functions $\eta \in C^\infty(\Omega)$ by

$$\eta(x) = \begin{cases} 1, & x \in \Omega_j^{2\delta},\ j = 1, \dots, m\\ 0, & x \in \Omega \setminus \bigcup_{j=1}^m \Omega_j^\delta, \end{cases}$$

with $\delta > 0$ as in (3.2), and w_i, $i = 1, 2$, on Ω by

$$w_i(x) = \begin{cases} p_{ji}(x), & x \in \Omega_j,\ j = 1, \dots, m\\ 0, & \text{otherwise.} \end{cases}$$

Then we get the decomposition

$$p = p_1 + p_2 \quad \text{with} \;\; p_1 = \psi p + \eta w_1, \; p_2 = \eta w_2, \tag{3.20}$$

where $\psi = (1-\eta)\sum_{j=1}^m \varphi_j + \varphi_0$; note that $\psi \in C^\infty(\Omega)$ and $\operatorname{supp}\psi \in \bar{\Omega}_0$. Hence, in view of (3.18), $\psi p \in W^{1,q}(\Omega)$ and $\|\psi p\|_{W^{1,q}(\Omega)} \le c\|p\|_{\widehat{W}^{1,q}(\Omega)}$. Moreover, $\eta w_1 \in W^{1,q}(\Omega)$ and, due to (3.19), $\|\eta w_1\|_{W^{1,q}(\Omega)} \le c\|p\|_{\widehat{W}^{1,q}(\Omega)}$. Thus we conclude that

$$p_1 \in W^{1,q}(\Omega), \quad \|p_1\|_{W^{1,q}(\Omega)} \le c\|p\|_{\widehat{W}^{1,q}(\Omega)}. \tag{3.21}$$

On the other hand, we have $\nabla p_2 = \nabla(\eta w_2) = \eta\nabla w_2 + w_2\nabla\eta$ and, due to (3.19),

$$\|\eta\nabla w_2\|_{W^{1,q}(\Omega)} = \|\eta\nabla w_2\|_{W^{1,q}(\cup_{j=1}^m \Omega_j^\delta)} \le c\sum_{j=1}^m \|\nabla p_{j2}\|_{W^{1,q}(\Omega_j^\delta)} \le c\|p\|_{\widehat{W}^{1,q}(\Omega)};$$

moreover, $\operatorname{supp}\nabla\eta \subset \bigcup_{j=1}^m \left(\Omega_j^\delta \setminus \Omega_j^{2\delta}\right) \subset \Omega_0$ and obviously $w_2 = p - \varphi_0 p - w_1 \in W^{1,q}\big(\bigcup_{j=1}^m \left(\Omega_j^\delta \setminus \Omega_j^{2\delta}\right)\big)$ implying that

$$\|w_2\nabla\eta\|_{W^{1,q}(\Omega)} \le c\sum_{j=1}^m \|p, p_{j1}\|_{W^{1,q}(\Omega_j^\delta\setminus\Omega_j^{2\delta})} \le c\|p\|_{\widehat{W}^{1,q}(\Omega)},$$

due to (3.19). Therefore we get that

$$\nabla p_2 \in W^{1,q}(\Omega), \quad \|\nabla p_2\|_{W^{1,q}(\Omega)} \le c\|p\|_{\widehat{W}^{1,q}(\Omega)},$$

which together with (3.20), (3.21) completes the proof of this lemma. ■

Now we can prove Theorem 1.2.

Proof of Theorem 1.2: By Theorem 1.1 the spectral angle ω_{A_q} of A_q is 0. Fix $\theta \in (0,\pi)$ arbitrarily. We must show that there is a constant $C > 0$ depending on θ such that for all $h \in \mathcal{H}^\infty(\Sigma_\theta)$ the operator

$$h(A_q) = \int_\Gamma h(\lambda)(\lambda - A_q)^{-1}\,d\lambda \in \mathcal{L}(L^q_\sigma(\Omega))$$

satisfies the estimate

$$\|h(A_q)\|_{\mathcal{L}(L^q_\sigma(\Omega))} \le C_\theta\|h\|_\infty, \tag{3.22}$$

where Γ is the oriented boundary of the sector $\Sigma_{\theta'}$ for any fixed $\theta' \in (0,\theta)$.

Since the domain Ω has been shown to satisfy the assumptions (A1)-(A3), by [5], Theorem 1.3, there are constant $R = R(q,\theta) > 0$ and $C = C(q,\theta) > 0$ such that

$$\left\|\int_{\Gamma_{R,\infty}} h(\lambda)(\lambda - A_q)^{-1}\,d\lambda\right\|_{\mathcal{L}(L^q_\sigma(\Omega))} \le C\|h\|_\infty,$$

where $\Gamma_{R,\infty} = \{\lambda \in \Gamma : \; |\lambda| > R\}$. On the other hand, due to Theorem 1.1, we get

$$\left\|\int_{\Gamma\setminus\Gamma_{R,\infty}} h(\lambda)(\lambda - A_q)^{-1}\,d\lambda\right\|_{\mathcal{L}(L^q_\sigma(\Omega))} \le C_{q,\theta}\|h\|_\infty.$$

Thus we proved (3.22).

Maximal regularity of A_q in $L^q_\sigma(\Omega)$ follows directly, since A_q admits a bounded $H^\infty(\Sigma_\theta)$-calculus for $\theta \in (0, \pi/2)$ and $L^q_\sigma(\Omega)$ is a *UMD* space, see Introduction.

Now the proof is complete. ∎

References

[1] T. Abe and Y. Shibata, On a resolvent estimate of the Stokes equation on an infinite layer, II: $\lambda = 0$ case, J. Math. Fluid Mech. 5 (2003), 245-274

[2] H. Abels, Reduced and generalized Stokes resolvent estimates in asymptotically flat layers, Part I: unique solvability. J. Math. Fluid Mech. 1 (2005), 201-222

[3] H. Abels, Reduced and generalized Stokes resolvent estimates in asymptotically flat layers, Part II: H^∞-calculus. J. Math. Fluid Mech. 7 (2005), 223-260

[4] H. Abels, Stokes Equations in Asymptotically Flat Domains and the Motion of a Free Surface, Dissertation, Shaker, 164 p., 2003

[5] H. Abels, Bounded imaginary powers and H_∞-calculus of the Stokes operator in unbounded domains, Preprint No. 2362, FB Mathematik, TU Darmstadt, 2004

[6] H. Abels and M. Wiegner, Resolvent estimates for the Stokes operator on an infinite layer. Preprint no. 2363, FB Mathematik, TU Darmstadt, 2004

[7] M. E. Bogovskii, Decomposition of $L_p(\Omega; \mathbb{R}^n)$ into the direct sum of subspaces of solenoidal and potential vector fields, Sov. Math. Dokl. 33 (1986), 161-165; translation from Dokl. Akad. Nauk SSSR 286 (1986), 781-786

[8] W. Borchers and H. Sohr, On the semigroup of the Stokes operator for exterior domains in L^q-spaces, Math. Z. 196 (1987), 415-425

[9] M. Cowling, I. Doust, A. McIntosh, and A. Yagi, Banach space operators with a bounded H^∞ functional calculus, J. Aust. Math. Soc., Ser. A 60 (1996), 51-89

[10] W. Desch, M. Hieber and J. Prüss, L^p-theory of the Stokes equation in a half-space, J. Evol. Equ., 1 (2001), 115-142

[11] R. Denk, G. Dore, M. Hieber, J. Prüss and A. Venni, New thoughts on old results of R. T. Seeley, Math. Ann. 328 (2004), 545-583

[12] R. Denk, M. Hieber and J. Prüss, $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type, Mem. Amer. Math. Soc. 788, 114 p., 2003

[13] J. Diestel, H. Jarchow and A. Tonge, Absolutely summing operators, Cambridge University Press, 1995

[14] G. Dore and A. Venni, On the closedness of the sum of two closed operators, Math. Z. 196 (1987), 189-201

[15] R. Farwig, Note on the flux condition and pressure drop in the resolvent Problem of the Stokes System, Manuscripta Math. 89 (1996), 139–158

[16] R. Farwig, Weighted L^q-Helmholtz decompositions in infinite cylinders and in infinite layers, Adv. Differ. Equ. 8 (2003), 357-384

[17] R. Farwig and Ri Myong-Hwan, Stokes resolvent estimates in an infinite cylinder, Preprint no. 2410, FB Mathematik, TU Darmstadt, 2005; to appear in: Math. Nachr.

[18] R. Farwig and Ri Myong-Hwan, An $L^q(L^2)$-theory of generalized Stokes resolvent estimates in an infinite cylinder, Preprint no. 2415, FB Mathematik, TU Darmstadt, 2005

[19] R. Farwig and Ri Myong-Hwan, Resolvent estimates and maximal regularity in weighted L^q-spaces of the Stokes operator in an infinite cylinder, Preprint no. 2431, FB Mathematik, TU Darmstadt, 2005

[20] R. Farwig and H. Sohr, Generalized resolvent estimates for the Stokes system in bounded and unbounded domains, J. Math. Soc. Japan 46 (1994), 607-643

[21] R. Farwig and H. Sohr, Helmholtz decomposition and Stokes resolvent system for aperture domains in L^q–spaces, Analysis 16 (1996), 1–26

[22] R. Farwig and H. Sohr, Weighted L^q-theory for the Stokes resolvent in exterior domains, J. Math. Soc. Japan 49 (1997), 251-288

[23] A. Fröhlich, The Stokes operator in weighted L^q-spaces I: Weighted estimates for the Stokes resolvent problem in a half space, J. Math. Fluid Mech. 5 (2003), 166-199

[24] A. Fröhlich, The Stokes operator in weighted L^q-spaces II: Weighted resolvent estimates and maximal regularity, Preprint no. 2173, FB Mathematik, TU Darmstadt, 2001

[25] G. P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations, Vol. 1: Linearized steady problems, Springer Tracts in Natural Philosophy, 38, Springer, 1994

[26] G. P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations, Vol. II: Nonlinear steady problems, Springer Tracts in Natural Philosophy, 39, Springer, 1994

[27] J. Garcia-Cuerva, J. L. Rubio de Francia, Weighted norm inequalities and related topics, North Holland, Amsterdam, 1985

[28] Y. Giga, Analyticity of the semigroup generated by the Stokes operator in L_r spaces, Math. Z. 178 (1981), 297-329

[29] A. Noll and J. Saal, H^∞-calculus for the Stokes operator on L_q-spaces, Math. Z. 244 (2003), 651-688

[30] J. L. Rubio de Francia, F. J. Ruiz and J. L. Torrea, Calderón-Zygmund theory for operator-valued kernels, Advances in Math. 62 (1986), 7-48

[31] G. Schneider, Nonlinear stability of Taylor vortices in infinite cylinders, Arch. Rational Mech. Anal. 144, (1998), 121-200

[32] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North Holland, 1978