The Samelson Product and Rational Homotopy for Gauge Groups

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Abstract

This paper is on the connecting homomorphism in the long exact homotopy sequence of the evaluation fibration $\mathrm{ev}_{p_0} : C(P,K)^K \to K$, where $C(P,K)^K \cong \mathrm{Gau}(\mathcal{P})$ is the gauge group of a continuous principal K-bundle P over a closed orientable surface or a sphere. We show that in this cases the connecting homomorphism in the corresponding long exact homotopy sequence is given in terms of the Samelson product. As applications, we exploit this correspondence to get an explicit formula for $\pi_2(\mathrm{Gau}(\mathcal{P}_k))$, where $\mathcal{P}_k$ denotes the principal $\mathbb{S}^3$-bundle over $\mathbb{S}^4$ of Chern number k and derive explicit formulae for the rational homotopy groups $\pi_n(\mathrm{Gau}(\mathcal{P})) \otimes \mathbb{Q}$.

Keywords: bundles over spheres, bundles over surfaces, gauge groups, pointed gauge groups, homotopy groups of gauge groups, rational homotopy groups of gauge groups, evaluation fibration, connecting homomorphism, Samelson product, Whitehead product

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Introduction

The topological properties of gauge groups play an important role in the analysis of the configuration space in quantum field theory. There one analyses the moduli space $\mathrm{Conn}(\mathcal{P})/\mathrm{Gau}(\mathcal{P})$ of connections on a principal K-bundle $\mathcal{P}$ modulo $\mathrm{Gau}(\mathcal{P})$ the group of gauge transformations, shortly called gauge group (cf. [Sin78]). Since $\mathrm{Conn}(\mathcal{P})$ is an affine space, the exact homotopy sequence gives detailed information on the homotopy groups of the configuration space in terms of the homotopy groups of the gauge group. On the other hand, $\pi_1(\mathrm{Gau}(\mathcal{P}))$ and $\pi_2(\mathrm{Gau}(\mathcal{P}))$ carry crucial information on central extensions of $\mathrm{Gau}(\mathcal{P})$ (cf. [Nee02]), which are important for an understanding of the relation between the projective and unitary representations of $\mathrm{Gau}(\mathcal{P})$. Furthermore, if $\mathcal{P}$ is a bundle over $\mathbb{S}^1$, then $\mathrm{Gau}(\mathcal{P})$ is isomorphic to a twisted loop goup, and thus gauge groups are closely related to Kac-Moody groups (cf. [Mic87]).

We now describe our results in some detail. In the first section, we recall some basic facts from elementary topology and from the classification of principal K-bundles over spheres and surfaces. The latter are the types of bundles this text deals with since they have explicit descriptions in terms of $\pi_m(K)$. In the case of a principal K-bundle $\mathcal{P} = (K, \eta : P \to \mathbb{S}^m)$ over $\mathbb{S}^m$, this leads to an explicit description of the gauge group $\mathrm{Gau}(\mathcal{P})$ as a subgroup of $C(\mathbb{B}^m, K)$ and of $\mathrm{Gau}_*(\mathcal{P})$ as $C_*(\mathbb{S}^m, K)$, where $\mathrm{Gau}_*(\mathcal{P})$ denotes the group of gauge transformations fixing $\eta^{-1}(x_0)$ pointwise and $\mathbb{B}^m := \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$. This description of $\mathrm{Gau}(\mathcal{P})$ leads directly to the main result of this paper.

Theorem. *If $\mathcal{P} = (K, \eta : P \to \mathbb{S}^m)$ is a continuous principal K-bundle over $\mathbb{S}^m$, K is locally contractible and $b \in \pi_{m-1}(K)$ is characteristic for $\mathcal{P}$, then the connecting homomorphism $\delta_n : \pi_n(K) \to \pi_{n+m-1}(K)$ in the exact sequence*

$$\ldots \to \pi_{n+1}(K) \xrightarrow{\delta_{n+1}} \pi_{n+m}(K) \to \pi_n(\mathrm{Gau}(\mathcal{P})) \to \pi_n(K) \xrightarrow{\delta_n} \pi_{n+m-1}(K) \to \ldots$$

is given by $\delta_n(a) = -\langle a, b\rangle$, where $\langle \cdot, \cdot \rangle$ denotes the Samelson product.

The connection between the Samelson Product and the evaluation fibration is not new (cf. [Whi46, Th. 3.2] and [BJS60, Sect. 1] and Remark II.11). The remarkable thing in this paper is that the above theorem can be proven by using only very elementary facts on fibrations. However, we give an alternative proof of the theorem in terms of more involved facts from homotopy theory.

As an application of the above theorem we obtain a new proof of [Kon91] providing an explicit formula for $\pi_2(\mathrm{Gau}(\mathcal{P}_k))$, where $\mathcal{P}_k$ denotes the principal $\mathrm{SU}_2(\mathbb{C})$-bundle over $\mathbb{S}^4$ of Chern number k. Furthermore, we show that the connecting homomorphism of the evaluation fibration for bundles over closed compact orientable surfaces is also given in terms of the Samelson product, since the situation there reduces to the situation of bundles over $\mathbb{S}^2$.

Since the rational Samelson produce $\langle \cdot, \cdot \rangle \otimes \mathrm{id}_{\mathbb{Q}}$ between the rational homotopy groups $\pi_n(K) \otimes \mathbb{Q}$ and $\pi_m(K) \otimes \mathbb{Q}$ vanishes for a connected Lie group K, this leads to the following explicit description of the rational homotopy groups of $\mathrm{Gau}(\mathcal{P})$ for a large class of bundles.

Theorem. *Let K be a connected Lie group and $\mathcal{P} = (K, \eta : P \to M)$ be a continuous principal K-bundle over $\mathbb{S}^m$ or a compact orientable surface Σ.*

i) If $M = \mathbb{S}^m$, then $\pi_n^{\mathbb{Q}}(\mathrm{Gau}(\mathcal{P})) \cong \pi_{n+m}^{\mathbb{Q}}(K) \oplus \pi_n^{\mathbb{Q}}(K)$.

ii) If $M = \Sigma$, then $\pi_n^{\mathbb{Q}}(\mathrm{Gau}(\mathcal{P})) \cong \pi_{n+2}^{\mathbb{Q}}(K) \oplus \pi_{n+1}^{\mathbb{Q}}(K)^{2g} \oplus \pi_n^{\mathbb{Q}}(K)$.

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I General Remarks and Notation

Remark I.1. Throughout this paper, we denote by $\mathbb{B}^n := \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$ the closed unit ball of radius 1, where $\|\cdot\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$ denotes the infinity-norm (we use this somewhat uncommon setting since then the proof of Theorem II.10 becomes less cryptic). Furthermore we set $I = [-1, 1] = \mathbb{B}^1$ and thus have $\mathbb{B}^n = \mathbb{B}^{n-1} \times I$. By $\mathbb{S}^n$, we denote the n-sphere and identify it interchangeably with $\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ (where $\|\cdot\|$ denots the euclidean norm), with $\{x \in \mathbb{R}^{n+1} : \|x\|_\infty = 1\}$ or with $\mathbb{B}^n/\partial\mathbb{B}^n$, depending on what is convenient in the considered situation. When dealing with pointed spaces, we take $(1, 0, \ldots, 0)$ as the base-point in $\mathbb{B}^n$, $\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ or $\{x \in \mathbb{R}^{n+1} : \|x\|_\infty = 1\}$ and $\partial\mathbb{B}^n$ as base-point in $\mathbb{B}^n/\partial\mathbb{B}^n$.

If $\sim$ is an equivalence relation on the topological space X and $X/\sim$ is the quotient X by this relation, then the continuous functions on $X/\sim$ are in on-to-one correspondence with the continuous functions on X which are constant on the equivalence classes of $\sim$ [Bou89, §I.3.4].

If $f : X \times Y \to Z$ is a function, then we denote for each $x \in X$ by f_x the function $f_x : Y \to Z$, $y \mapsto f(x, y)$, and for each $y \in Y$ by f_y the function $f_y : X \to Z$, $y \mapsto f(x, y)$.

If X,Y are spaces with base-points x_0, y_0, then $C_*(X, Y) := \{f \in C(X, Y) : f(x_0) = y_0\}$. If $X = I$ we set $PY := C_*(I, Y)$ and if $X = \mathbb{S}^1$ we set $\Omega Y := C_*(\mathbb{S}^1, Y)$

Remark I.2. If X and Y are topological spaces, then we equip $C(X,Y)$ with the compact-open topology. If $Y = K$ is a topological group, then the compact-open topology on $C(X,K)$ coincides with the topology of compact convergence (cf. [Bou89, Th. X.3.4.2]) and this turns $C(X,K)$ into a topological group.

The elementary facts on the compact open topology on $C(X,K)$ we use throughout this paper are the following (cf. [Bou89]):

- If $x \in X$, then the *exaluation map* $\mathrm{ev}_x : C(X,Y) \to Y$ is continuous.
- If Z is a topological space and $f : X \to Z$ is continuous, then the *pull-back* $f^* : C(Z,Y) \to C(X,Y)$, $\gamma \mapsto \gamma \circ f$ is continuous.
- We have a continuous map $^\wedge : C(X \times Y, Z) \to C(X, C(Y,Z))$, $f^\wedge(x)(y) = f(x,y)$ and if Y is locally compact, then this map is a homeomorphism. This property is called the *exponential law* or *Cartesian closedness principle.*

Remark I.3. If $m \in \mathbb{N}^+$, then the equivalence classes of continuous principal K-bundles over $\mathbb{S}^m$ are in one-to-one correspondence with the orbits of the $\pi_0(K)$-action on $\pi_{m-1}(K)$, where $\pi_0(K)$ acts on $\pi_{m-1}(K)$ by $([\gamma],k) \mapsto [k\gamma k^{-1}]$ (cf. [Ste51, §18.5]).

A characteristic map for a fixed bundle $\mathcal{P} = (K, \eta : P \to \mathbb{S}^m)$ can be obtained as follows. Take $\mathbb{S}^n := \{x \in \mathbb{R}^n : \|x\| = 1\}$ and $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^{n-1} : \|x\| = 1\} = \mathbb{S}^n \cap \{x \in \mathbb{R}^n : x_n = 0\}$ and let $V_{N/S} \subseteq \mathbb{S}^n$ denote open n-cells with $\mathbb{S}^{n-1} \subseteq V_{N/S}$ and $(0,\ldots,0,1) \in V_N$ and $(0,\ldots,0,-1) \in V_S$. Then there exist sections $\sigma_{N/S} : V_{N/S} \to P$ and $\sigma_S(x) = \sigma_N(x)\gamma(x)$ defines a continuous map $\gamma : \mathbb{S}^{n-1} \to K$. If we substitute σ_N by $\sigma_N \cdot \gamma(x_0)$ we may assume that $\gamma(x_0) = e$. Then $[\gamma] \in \pi_{n-1}(K)$ represents the equivalence class of $\mathcal{P}$ (cf. [Ste51, §18.1]).

Remark I.4. Let K be a connected topological group and Σ be a closed compact orientable surface. For the set of equivalence classes of continuous principal K-bundles over Σ we have that it is equal to $[\Sigma, BK]$, where BK is the classifying space of K (cf. [Hus66, Th. 4.13.1]). Furthermore we have

$$[\Sigma, BK] \cong H^2(\Sigma, \pi_2(BK)) \cong \mathrm{Hom}(H_2(\Sigma), \pi_1(K)) \cong \pi_1(K).$$

The first isomorphism is a consequence of [Bre93, Cor. VII.13.16] and [Bre93, Th. VII6.7], the second is [Bre93, Th. V.7.2] which applies since $H_1(\Sigma) \cong \mathbb{Z}^{2g}$ is free, and the last isomorphism follows from $H_2(\Sigma) \cong \mathbb{Z}$.

Remark I.5. We recall the construction of the connecting homomorphism for a fibration $p : Y \to B$ with fibre $F = p^{-1}(\{x_0\})$. This fibration yields a long exact homotopy sequence

$$\ldots \to \pi_{n+1}(B) \xrightarrow{\delta_{n+1}} \pi_n(F) \xrightarrow{\pi_n(i)} \pi_n(Y) \xrightarrow{\pi_n(q)} \pi_n(B) \xrightarrow{\delta_n} \pi_{n-1}(F) \to \ldots$$

and the construction of the connecting homomorphism δ_n is as follows (cf. [Bre93, Th. VII.6.7]): Let $f \in C_*(\mathbb{B}^n, B)$ represent an element of $\pi_n(B)$, i.e. $f|_{\partial \mathbb{B}^n} \equiv x_0$. Then f can be lifted to a map $F : \mathbb{B}^n \to Y$ with $q \circ F = f$ since q is a fibration. Then F takes $\partial \mathbb{B}^n \cong \mathbb{S}^{n-1}$ into $q^{-1}(x_0) = F$, and $F|_{\partial \mathbb{B}^n}$ represents $\delta([f])$.

II The Connecting Homomorphism

Definition II.1 (Bundle Map, Automorphism Group, Gauge Group)**.** If $\mathcal{P} = (K, \eta : P \to M)$ and $\mathcal{P}' = (K, \eta' : P' \to B')$ are principal K-bundles, then

$$\mathrm{Bun}(\mathcal{P}, \mathcal{P}') := \{f \in C(P,P') : (\forall p \in P)(\forall k \in K)\ f(p \cdot k) = f(p) \cdot k\}$$

are called *bundle maps* from $\mathcal{P}$ to $\mathcal{P}'$. Furthermore, $\mathrm{Aut}(\mathcal{P}) := \mathrm{Bun}(\mathcal{P},\mathcal{P}) \cap \mathrm{Homeo}(P)$ is called the group of bundle automorphism or *automorphism group* of $\mathcal{P}$ and $\mathrm{Gau}(\mathcal{P}) := \{f \in \mathrm{Aut}(P) : \eta \circ f = \eta\}$ is called the group of bundle equivalences or *gauge group* of $\mathcal{P}$.

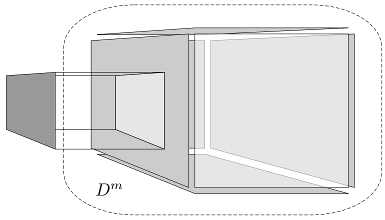

Figure 1: Illustration of D^m

Remark II.2. The gauge group of $\mathcal{P}$ is isomorphic to the space of continuous K-equivariant mappings

$$C(P,K)^K := \{f \in C(P,K) : (\forall p \in P)(\forall k \in K)\ f(p\cdot k) = k^{-1}\cdot f(p)\cdot k\}$$

under the isomorphism $C(P,K)^K \ni f \mapsto \big(p \mapsto p\cdot f(p)\big) \in \mathrm{Gau}(\mathcal{P})$, and we endow $C(P,K)^K$ with the subspace topology induced from the compact-open topology on $C(P,K)$. This turns $C(P,K)^K$ and thus $\mathrm{Gau}(\mathcal{P})$ into topological groups.

Remark II.3. We recall the description of principal K-bundles over $\mathbb{S}^m$ by its characteristic maps (also called clutching functions). Given a principal K-bundle $\mathcal{P} = (K, \eta : P \to \mathbb{S}^m)$ over $\mathbb{S}^m$ and denoting by $q : \mathbb{B}^m \to \mathbb{S}^m$ the quotient map identifying $\partial\mathbb{B}^m$ with the base-point in $\mathbb{S}^m$, [Bre93, Cor. VII.6.12] provides a map $\sigma : \mathbb{B}^m \to P$ satisfying $\eta\circ\sigma = q$. Thus $\sigma(y)\cdot\gamma(y) = \sigma(y_0)$ for $y \in \partial\mathbb{B}^m$ and we thus obtain a continuous map $\gamma : \partial\mathbb{B}^m \cong \mathbb{S}^m \to K$ satisfying $\gamma(y_0) = e$ which is called the *clutching function* or *characteristic map* describing $\mathcal{P}$. Furthermore γ is a representative of $[\mathcal{P}]$ since we may identify $\mathrm{int}(\mathbb{B}^m)$ with V_N, $(\mathbb{B}^m/\partial\mathbb{B}^m)\backslash\{0\}$ with V_S and then

$$\sigma_N : V_N \to P,\quad x \mapsto \sigma(x)$$
$$\sigma_S : V_S \to P,\quad x \mapsto \sigma(x)\cdot\gamma\left(\frac{x}{\|x\|_\infty}\right)$$

denote corresponding sections (cf. Remark I.3). Set $P/\gamma := \mathbb{B}^m \times K/\sim$ with $(x,k)\sim(y,k') :\Leftrightarrow x,y \in \partial B^m$ and $\gamma(x)\cdot k = \gamma(y)\cdot k'$ and endow it with the quotient topology. Then K acts continuously on P/γ by $([(x,k)],k)' \mapsto [(x,kk')]$ and

$$P/\gamma \to P,\quad [(x,k)] \mapsto \begin{cases} \sigma_N(x)\cdot k & \text{if } x \in V_N \\ \sigma_S(x)\cdot\gamma(\frac{x}{\|x\|_\infty})\cdot k & \text{if } x \in V_S \end{cases}$$

is an isomorphism between the K-spaces P and P/γ, whence a bundle isomorphism.

Lemma II.4. *Let $\mathcal{P} = (K, \eta : P \to \mathbb{S}^m)$ be a continuous principal K-bundle with characteristic map $\gamma : \partial\mathbb{B}^m \to K$ and set*

$$D^m = (I\times\partial\mathbb{B}^{m-1}) \cup (\{1\}\times\mathbb{B}^{m-1}) \cup \{(t,x) \in I\times\mathbb{R}^{m-1} : t = -1 \text{ and } \frac{1}{2} \leq \|x\|_\infty \leq 1\} \subseteq \partial\mathbb{B}^m$$

if $m \geq 2$ (cf. Figure 1) and $D^1 = \{1\}$. Then

$$C(P,K)^K \cong G(\mathcal{P}) := \{f \in C(\mathbb{B}^m, K) : (\exists k \in K)\ f|_{D^m} \equiv k,$$
$$(\forall x \in \partial\mathbb{B}^m\backslash D^m)\ \gamma(x)^{-1}\cdot f(x)\cdot\gamma(x) = f(x_0)\}.$$

and thus $G_*(\mathcal{P}) := \{f \in G(\mathcal{P}) : f(x_0) = e\} \cong C_*(\mathbb{S}^m, K)$.

Proof. Let γ be determined by $\sigma : \mathbb{B}^m \to P$ with $\gamma(x_0) = e$ as in the preceding remark. Since $\partial\mathbb{B}^m = (I\times\partial\mathbb{B}^{m-1}) \cup (\{-1,1\}\times\mathbb{B}^{m-1})$ and since D^m is contractible in $\partial\mathbb{B}^m$, [Hat02, Prop. 0.17] implies that γ is homotopic to a map which is the identity on D^m. Since homotopic maps yield equivalent bundles (cf. [Ste51, Th. 18.3]) we may assume that $\gamma|_{D^m} \equiv e$. Furthermore, $f \mapsto f \circ \sigma$ provides a map $\sigma^* : C(P,K)^K \to G(\mathcal{P})$ since $\gamma|_{D^m} \equiv e$. We claim that σ^* is an isomorphism and that an inverse map can be constructed with σ_N and σ_S in terms of pull-backs and multiplication in spaces of continuous mappings. In fact, for $f \in G(\mathcal{P})$ we set $f_N := f|_{V_N}$ and $f_S : V_S \to K,\ x \mapsto \gamma(\frac{x}{\|x\|_\infty})^{-1} f(x) \gamma(\frac{x}{\|x\|_\infty})$. Furthermore, $p = \sigma_{N/S}(\eta(p)) \cdot k_{N/S}(p)$ determines continuous maps $k_{N/S} : \eta^{-1}(V_{S/N}) \to K$ satisfying $k_N(p) = \gamma\left(\frac{\eta(p)}{\|\eta(p)\|_\infty}\right) k_S(p)$ for $p \in \eta^{-1}(V_S \cap V_N)$. Then

$$f' : P \to K,\ \ p \mapsto k_{N/S}(p)^{-1} f_{N/S}(\eta(p)) k_{N/S}(p) \ \text{ if } \ \eta(p) \in V_{N/S}$$

determines an element of $C(P,K)^K$ and the assignment $f \mapsto f'$ defines a continuous inverse of σ^*. □

Remark II.5. (cf. [PS86, 3.7]) Note that the preceding lemma implies that if $\mathcal{P} = (K, \eta : P \to \mathbb{S}^1)$ is a principal K-bundle over the circle given by $[k] \in \pi_0(K)$, then the gauge group is isomorphic to the twisted loop group

$$C_k(\mathbb{S}, K) := \{f \in C(\mathbb{R}, K) : f(x+n) = k^{-n} f(x) k^n\}.$$

In fact, since a characteristic map for a bundle over $\mathbb{S}^1$ is represented by an element $k \in K$ we have $G(\mathcal{P}) = \{f \in C(I,K) : k^{-1} \cdot f(-1) \cdot k = f(1)\}$ and the isomorphism

$$G(\mathcal{P}) \ni f \mapsto \big(x \mapsto k^{-1} \cdot f(2(x-n)-1) \cdot k^n\big) \in C_k(\mathbb{S}, K),$$

where $n \in \mathbb{Z}$ such that $x - n \in [0,1]$.

Definition II.6 (Evaluation Map)**.** If $\mathcal{P} = (K, \eta : P \to \mathbb{S}^m)$ is a continuous principal K-bundle, then $\mathrm{ev}_{x_0} : G(\mathcal{P}) \to K,\ f \mapsto f(x_0)$ is called the *the evaluation map.*

Lemma II.7. *If $\mathcal{P} = (K, \eta : P \to \mathbb{S}^m)$ is a continuous principal K-bundle and K is locally contractible, then the evaluation map is a fibration with kernel $G_*(\mathcal{P}) \cong C_*(\mathbb{S}^m, K)$. Furthermore, $K_{\mathcal{P}} := \mathrm{im}(\mathrm{ev}_{x_0})$ is open and thus contains the identity component K_0.*

Proof. Since K is locally contractible, there exist open unit neighbourhoods $V \subseteq U$ and a continuous map $F : [0,1] \times V \to U$ such that $F(0,k) = e$, $F(1,k) = k$ for all $k \in V$ and $F(t,e) = e$ for all $t \in [0,1]$. For $k \in V$ we set $\tau_k := F(\cdot, k)$, which is a continuous path and satisfies $\tau_k(0) = e$ and $\tau_k(1) = k$. Furthermore, the map $V \ni k \mapsto \tau_k \in C(I,K)$ is continuous as an easy calculation in the topology of compact convergence shows.

Now $V \ni k \mapsto f_{\tau_k} \in G(\mathcal{P})$ defines a continuous section of the evaluation map and since ev_{x_0} is surjective this shows that $(G_*(\mathcal{P}), \mathrm{ev}_{x_0} : G(\mathcal{P}) \to K)$ is a continuous principal $G_*(\mathcal{P})$-bundle and thus a fibration (cf. [Bre93, Cor. VII.6.12]). Since the bundle projection of a locally trivial bundle is open it follows in particular that ev_{x_0} is open and thus that its image is open. □

Lemma II.8. *If $\mathcal{P} = (K, \eta : P \to \mathbb{S}^m)$ is a continuous principal K-bundle over $\mathbb{S}^m$ and K is locally contractible, then the evaluation map ev_{x_0} induces a long exact homotopy sequence*

$$(1) \qquad \ldots \to \pi_{n+1}(K) \xrightarrow{\delta_{n+1}} \pi_{n+m}(K) \to \pi_n(\mathrm{Gau}(\mathcal{P})) \to \pi_n(K) \xrightarrow{\delta_n} \pi_{n+m-1}(K) \to \ldots$$

Proof. Since $K_{\mathcal{P}}$ contains the identity component K_0 we have $\pi_n(K_0) = \pi_n(K_{\mathcal{P}}) = \pi_n(K)$, and since $\pi_{n+m}(K) = \pi_0(C_*(\mathbb{S}^{n+m}, K)) \cong \pi_0(C_*(\mathbb{S}^n, C_*(\mathbb{S}^m, K))) = \pi_n(C_*(\mathbb{S}^m, K))$ this a direct consequence of the long exact homotopy sequence (cf. [Bre93, Th. VII.6.7]) for $\mathrm{ev}_{x_0} : G(\mathcal{P}) \cong C(P,K)^K \cong \mathrm{Gau}(\mathcal{P}) \to K_{\mathcal{P}}$ and the preceding lemma. □

Definition II.9 (Samelson Product)**.** If K is a topological group, $a \in \pi_n(K)$ is represented by $\alpha \in C_*(\mathbb{S}^n, K)$ and $b \in \pi_m(K)$ is represented by $\beta \in C_*(\mathbb{S}^m, K)$, then the commutator map

$$\alpha \# \gamma : \mathbb{S}^n \times \mathbb{S}^m \to K,\ \ (x,y) \mapsto \alpha(x)\beta(y)\alpha(x)^{-1}\beta(y)^{-1}$$

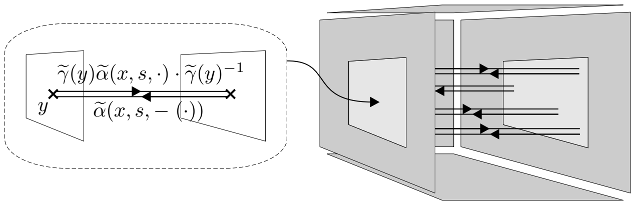

Figure 2: Construction of A'

maps $\mathbb{S}^n \vee \mathbb{S}^m$ to e. Hence it may be viewed as an element of $C_*(\mathbb{S}^n \wedge \mathbb{S}^m, K)$ and thus determines an element $\langle a, b\rangle := [\alpha\#\beta] \in \pi_0(C_*(\mathbb{S}^{n+m}, K)) \cong \pi_{n+m}(K)$. The map

$$\pi_n(K) \times \pi_m(K) \to \pi_{n+m}(K), \quad (a,b) \mapsto \langle a,b\rangle$$

is biadditive [Whi78, Th. X.5.1] and is called the *Samelson Product* (cf. [Whi78, Sect. X.5]).

Theorem II.10. *If $\mathcal{P} = (K, \eta : P \to \mathbb{S}^m)$ is a continuous principal K-bundle over $\mathbb{S}^m$, K is locally contractible and $b \in \pi_{m-1}(K)$ is characteristic for $\mathcal{P}$ (cf. Remark I.3), then the connecting homomorphism $\delta_n : \pi_n(K) \to \pi_{n+m-1}(K)$ in* (1) *is given by $\delta_n(a) = -\langle a, b\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the Samelson product.*

Proof. Let b be represented by $\gamma \in C_*(\partial\mathbb{B}^m, K)$ with $\gamma|_{D^m} \equiv e$, $a \in \pi_n(K)$ be represented by $\alpha \in C(\mathbb{B}^n, K)$ with $\alpha|_{\partial\mathbb{B}^n} \equiv e$. Due to the construction of the connecting homomorphism (cf. Remark I.5), we have to construct a lift $A : \mathbb{B}^n \to G(\mathcal{P})$ of α.

We set $\widetilde{\alpha}(x,s,t) := \alpha(x, \frac{t+1}{2}s - (1 - \frac{t+1}{2}))$ and note that $\widetilde{\alpha}(x,s,1) = \alpha(x,s)$ and $\widetilde{\alpha}(x,s,-1) = \alpha(x,-1) = e$. If $m = 1$, then $[\gamma] = [k] \in \pi_0(K)$ for some $k \in K$, and we set

$$A : \mathbb{B}^n \times I \times I \to K, \quad (x,s,t) \mapsto \widetilde{\alpha}(x,s,-t)\cdot k \cdot \widetilde{\alpha}(x,s,t)\cdot k^{-1}.$$

If $m \geq 2$ the construction of A is as follows. First we set

$$A' : \mathbb{B}^{n-1} \times \mathbb{B}^2 \times \frac{1}{2}\mathbb{B}^{m-1} \to K, \quad (x,s,t,y) \mapsto \widetilde{\alpha}(x,s,-t)\widetilde{\gamma}(y)\widetilde{\alpha}(x,s,t)\widetilde{\gamma}(y)^{-1},$$

where $\widetilde{\gamma}(y) := \gamma(-1,y)$ (cf. Figure 2). Note that due to $\gamma|_{I\times\partial\mathbb{B}^{m-1}\cup\{1\}\times\mathbb{B}^{m-1}} \equiv e$, we have that $\widetilde{\gamma} : \mathbb{B}^{m-1} \to K$ represents the same element of $\pi_{m-1}(K)$ as γ does if we identify $\mathbb{S}^{m-1}$ with $\mathbb{B}^{m-1}/\partial\mathbb{B}^{m-1}$ instead of $\partial\mathbb{B}^m$.

Then $t \mapsto A'_{x,s,y}(t)$ satisfies $A'_{x,s,y}(t) = \widetilde{\alpha}(x,s,-t)\widetilde{\alpha}(x,s,t)$ if $\|y\|_\infty = \frac{1}{2}$ since then $\widetilde{\gamma}(y) = e$ and this map is homotopic to the map which is constantly $\alpha(x,s)$. We take a standard homotopy $F_{x,s}$ between $t \mapsto \widetilde{\alpha}(x,s,-t)\cdot\widetilde{\alpha}(x,s,t)$ and the constant map $\alpha(x,s)$.

Then $(x,s,r,t) \mapsto F_{x,s}(r,t)$ is continuous and thus

$$A : \mathbb{B}^{n-1} \times \mathbb{B}^2 \times \mathbb{B}^{m-1} \to K, \quad (x,s,t,y) \mapsto \begin{cases} A'(x,s,t,y) & \text{if } \|y\|_\infty \leq \frac{1}{2} \\ F_{x,s}(3 - 4\|y\|_\infty, t) & \text{if } \|y\|_\infty \geq \frac{1}{2} \end{cases}$$

defines a continuous map (cf. Figure 3) such that $A_{x,s}$ is an element of $G(\mathcal{P})$ (note that $F_{x,s}$ satisfies $F_{x,s}|_{I\times\{-1,1\}} \equiv \alpha(x,s)$). Furthermore $(x,s) \mapsto A_{x,s}$ is a lift of α since it is continuous by the exponential law and satisfies $A_{x,s}(1,0,\ldots,0) = \alpha(x,s)$.

We now restrict the lift to $\partial\mathbb{B}^m = \partial\mathbb{B}^{m-1} \times I \cup \mathbb{B}^m \times \{-1,1\}$. For $x \in \partial\mathbb{B}^{m-1}$ or $s = -1$ we see that $F_{x,s} \equiv e$ since then $\widetilde{\alpha}(x,s,t) = e$ and thus that in this case $A_{x,s} \equiv e$. Identifying $\mathbb{S}^{n-1}$ with $\{x \in \partial\mathbb{B}^n : x_n = 1\}$ modulo boundary it thus suffices to evaluate the lift for $s = 1$. Note that we have $\widetilde{\alpha}(x,1,t) = \alpha(x,t)$. If $m = 1$ we take a homotopy $G : I \times I \to K$ between $t \mapsto \alpha(x,-t)$ and $t \mapsto \alpha(x,t)^{-1}$. Then $(r',x,t,y) \mapsto G(r',t)A(x,1,t,y)$ defines a homotopy in $G_*(\mathcal{P})$ between $A|_{s=1}$ and $\alpha^{-1}\#\widetilde{\gamma}$.

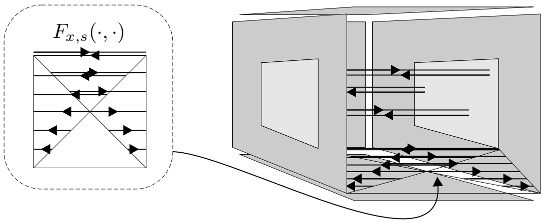

Figure 3: Construction of A

If $m \geq 2$, we define $\widetilde{F}_x : I^3 \to K$, with $\widetilde{F}_x(1,r,t) = \widetilde{F}_x(r,1,t) = F_{x,1}(r,t)$, constant on straight lines joining $(1,r,t)$ with $(r,1,t)$ and $\widetilde{F}_x(r',r,t) = e$ if $r'+r \leq 0$. Then $\widetilde{F}_x(1,1,t) = F_{x,1}(1,t) = \alpha(x,-t)\alpha(x,t)$, $F_x|_{\{-1\}\times I\times I} \equiv e$ and F_x depends continuously on x. Thus

$$G : I \times \mathbb{B}^{n-1} \times I \times \mathbb{B}^{m-1},$$

$$(r',x,t,y) \mapsto \begin{cases} \widetilde{F}_x(r',1,t)\alpha(x,t)^{-1}\widetilde{\gamma}(y)\alpha(x,t)\widetilde{\gamma}(y)^{-1} & \text{if } \|y\|_\infty \leq \frac{1}{2} \\ \widetilde{F}_x(r',3-4\|y\|_\infty,t) & \text{if } \|y\|_\infty \geq \frac{1}{2} \end{cases}$$

defines a homotopy in $G_*(\mathcal{P})$ between $A|_{s=1} = G_1$ and $\alpha^{-1}\#\widetilde{\gamma} = G_{-1}$. Thus we have $[\alpha^{-1}\#\widetilde{\gamma}] = [\alpha^{-1}\#\gamma] = -\langle a,b\rangle$ in $\pi_{n+m-1}(K)$. □

Remark II.11. The above sequence can also be obtained as follows. Let $\mathcal{P}_K = (K, \eta_K : EK \to BK)$ be a universal bundle for K, i.e. a continuous principal K-bundle such that $\pi_n(EK)$ vanishes for $n \in \mathbb{N}^+$. Furthermore let $\gamma : \mathbb{S}^m \to BK$ be a classifying map for $\mathcal{P}$ and denote by $\Gamma : P \to EK$ the corresponding bundle map.

Now each $f \in \operatorname{Bun}(\mathcal{P},\mathcal{P}_K)$ induces a map $\bar{f} : \mathbb{S}^m \to BK$ and the map

$$\operatorname{Bun}(\mathcal{P},\mathcal{P}_K,\Gamma) \ni f \mapsto \bar{f} \in C(B,BK,\gamma)$$

is a fibration [Got72, Prop. 3.1], where $\operatorname{Bun}(\mathcal{P},\mathcal{P}_K,\Gamma)$ (respectively $C(B,BK,\gamma)$) denotes the connected component of Γ (respectively γ). Then the fibre $F = \{\operatorname{Bun}(\mathcal{P},\mathcal{P}_K) : \bar{f} = \gamma\}$ of this map is homeomorphic to $\operatorname{Gau}(\mathcal{P})$ [Got72, Prop. 4.3]. Since $\operatorname{Bun}(\mathcal{P},\mathcal{P}_K)$ is essentially contractible [Got72, Th. 5.2] $\pi_n(\operatorname{Bun}(\mathcal{P},\mathcal{P}_K))$ vanishes, and thus the long exact homotopy sequence of the above fibration leads to $\pi_{n-1}(\operatorname{Gau}(\mathcal{P})) \cong \pi_n(C(B,BK,\gamma))$ (cf. [Tsu85, Th. 1.5]).

We now consider the evaluation map $\operatorname{ev}_{x_0} : C(\mathbb{S}^m, BK) \to BK$ in the base-point x_0 of $\mathbb{S}^m$. This map is a fibration ([Bre93, Th. VII.6.13]) and we thus get a long exact homotopy sequence

$$(2) \quad \ldots \to \pi_{n+1}(BK) \xrightarrow{\delta_{n+1}} \pi_n(C_*(\mathbb{S}^m,BK,\gamma)) \to \pi_n(C(\mathbb{S}^m,BK,\gamma)) \to \pi_n(BK) \xrightarrow{\delta_n} \pi_{n-1}(C_*(\mathbb{S}^m,BK,\gamma)) \to \ldots$$

If we identify $\pi_n(C_*(\mathbb{S}^m,BK,\gamma))$ with $\pi_{n+m}(BK)$ (cf.[Whi46, 2.10]), then the connecting homomorphism in this sequence is given by $\delta_{n+1}(a) = -[a,b]$, where $b = [\gamma] \in \pi_m(BK)$ and $[\cdot,\cdot]$ denotes the Whitehead product (cf. [Whi46, Th. 3.2] and [Whi53, (3.1)]).

Since $\pi_n(EK)$ vanishes, the connecting homomorphism $\Delta : \pi_{n+1}(BK) \to \pi_n(K)$ from the long exact homotopy sequence for $\mathcal{P}_K$ is an isomorphism. Since we have

$$\Delta([a,b]) = (-1)^n \langle \Delta(a), \Delta(b)\rangle$$

for $a \in \pi_{n+1}(BK)$ by [BJS60, Sect. 1], (2) yields a long exact sequence

$$\ldots \pi_n(K) \xrightarrow{\delta'_n} \pi_{n+m-1}(K) \to \pi_{n-1}(\operatorname{Gau}(\mathcal{P})) \to \pi_{n-1}(K) \xrightarrow{\delta'_{n-1}} \pi_{n+m-2}(K) \to \ldots$$

with connecting homomotphism $\delta'_n(a) = (-1)^n\langle a,b\rangle$ if we identify $\pi_{n-1}(\operatorname{Gau}(\mathcal{P}))$ with $\pi_n(C_*(\mathbb{S}^m,BK,\gamma))$ as described above and $\pi_{n+1}(BK)$ with $\pi_n(K)$ and $\pi_{n+m}(BK)$ with $\pi_{n+m-1}(K)$ by Δ.

III Applications

Proposition III.1. *(cf. [Kon91, Lem. 1.3]) If $\mathcal{P}_k$ is a principal $\mathrm{SU}_2(\mathbb{C})$-bundle over $\mathbb{S}^4$ of Chern number $k \in \mathbb{Z}$, then $\pi_2(\mathrm{Gau}(\mathcal{P}_k)) \cong \mathbb{Z}_{\gcd(k,12)}$. In particular, if $\mathcal{P}_1 = \mathcal{H}$ is the quaterionic Hopf fibration, then $\pi_2(\mathrm{Gau}(\mathcal{H}))$ vanishes.*

Proof. Since by [Nab00, Th. 6.4.2] $\mathcal{P}_k$ is classified by its Chern number $k \in \mathbb{Z} \cong \pi_3(\mathrm{SU}_2(\mathbb{C}))$, Theorem II.10 provides an exact sequence

$$\ldots \to \pi_3(\mathrm{SU}_2(\mathbb{C})) \xrightarrow{\delta_2^k} \pi_6(\mathrm{SU}_2(\mathbb{C})) \xrightarrow{\pi_2(i)} \pi_2(\mathrm{Gau}(\mathcal{P}_k)) \to \pi_2(\mathrm{SU}_2(\mathbb{C})) \to \ldots \quad ,$$

where $\delta_2^k : \pi_3(\mathrm{SU}_2(\mathbb{C})) \to \pi_6(\mathrm{SU}_2(\mathbb{C}))$ is given by $a \mapsto -\langle a, k\rangle$. Since $\pi_3(\mathrm{SU}_2(\mathbb{C})) \cong \mathbb{Z}$, $\pi_6(\mathrm{SU}_2(\mathbb{C})) \cong \mathbb{Z}_{12}$ and $\langle 1,1\rangle$ generates $\mathbb{Z}_{12}$, we may assume that $\delta_2^k : \mathbb{Z} \to \mathbb{Z}_{12}$ is the map $\mathbb{Z} \ni z \mapsto -[kz] \in \mathbb{Z}_{12}$ due to the biadditivity of $\langle \cdot, \cdot \rangle$. Since $\pi_2(\mathrm{SU}_2(\mathbb{C}))$ is trivial we have that $\pi_2(i)$ is surjective and

$$\mathrm{im}(\pi_2(i)) \cong \mathbb{Z}_{12}/\ker(\pi_2(i)) = \mathbb{Z}_{12}/\mathrm{im}(\delta_2^k) = \mathbb{Z}_{12}/k\mathbb{Z}_{12} \cong \mathbb{Z}_{\gcd(k,12)}. \qquad \square$$

Corollary III.2. *If $\mathcal{P}_k$ is a smooth principal $\mathrm{SU}_2(\mathbb{C})$-bundle over $\mathbb{S}^4$ with Chern number k, then $\pi_2(\mathrm{Gau}^\infty(\mathcal{P})) \cong \mathbb{Z}_{\gcd(12,k)}$, where $\mathrm{Gau}^\infty(\mathcal{P})$ denotes the group of smooth gauge transformations on $\mathcal{P}$.*

Proof. This is the preceding proposition and [Woc05, Th. III.11] $\square$

Remark III.3. We recall that a closed compact orientable surface Σ of genus g with $\partial\Sigma = \emptyset$ can be described as a CW-complex by starting with a bouquet

$$B_g = \underbrace{\mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1}_{2g}$$

of $2g$ circles. We write $a_1, b_1, \ldots, a_g, b_g$ for the corresponding generators of the fundamental group of B_g, which is a free group of $2g$ generators [Bre93, Th. III.V.14]. Then we consider a continuous map $f : \mathbb{S}^1 \to B_g$ representing

$$a_1 \cdot b_1 \cdot a_1^{-1} \cdot b_1^{-1} \cdots a_g \cdot b_g \cdot a_g^{-1} \cdot b_g^{-1} \in \pi_1(B_g).$$

Now Σ is homeomorphic to the space obtained by identifying the points on $\partial\mathbb{B}^2 \cong \mathbb{S}^1$ with their images in B_g under f, i.e.

$$\Sigma \cong B_g \cup_f \partial\mathbb{B}^2 \tag{3}$$

and we denote by q_Σ the corresponding quotient map $q_\Sigma : \mathbb{B}^2 \to \Sigma$.

Remark III.4. Let $(K, \eta : P \to \Sigma)$ be a continuous principal K-bundle over a closed, compact and orientable surface with, K be connected, and denote by $q_\Sigma : \mathbb{B}^2 \to \Sigma$ the quotient map from Remark III.3. Then [Bre93, Cor. VII.6.12] provides a map $\sigma : \Sigma \to P$ satisfying $\eta \circ \sigma = q_\Sigma$ and since $\mathcal{P}|_{\eta^{-1}(B_g)}$ is trivial, we have a continuous map $\gamma : \partial\mathbb{B}^2 \to K$ satisfying $\sigma(x) \cdot \gamma(x) = \sigma(y) \cdot \gamma(y)$ if $x, y \in \partial\mathbb{B}^2$ and $f(x) = f(y)$. We may also require w.l.o.g. that $\gamma(x_0) = e$ and then $[\gamma]$ may be viewed as a representative of $\mathcal{P}$.

Denote by $\sigma' : B_g \to P$ a continuous section. Then $p \sim p'$ wherever $p = \sigma'(x) \cdot k$ and $p' = \sigma'(y) \cdot k$ for some $x, y \in B_g$ and $k \in K$ defines an equivalence relation on P. Then $P/\sim$ is isomorphic to P/γ from Remark I.3 (by a similar construction) and we thus set $P/\gamma := P/\sim$.

Proposition III.5. *Let $\mathcal{P} = (K, \eta : P \to M)$ be a continuous principal K-bundle over a closed, compact orientable surface, let K be locally contractible and connected and let $b \in \pi_1(K)$ be characteristic for $\mathcal{P}$ (cf. Remark I.4). If $\mathrm{ev}_{p_0} : C(P,K)^K \to K$ is the evaluation fibration at the base-point of P, then we have a long exact sequence*

$$\begin{aligned} \ldots \to \pi_{n+1}(K) \xrightarrow{\delta_{n+1}} \pi_{n+1}(K)^{2g} \oplus \pi_{n+2}(K) \to \pi_n(C(P,K)^K) \\ \to \pi_n(K) \xrightarrow{\delta_n} \pi_n(K)^{2g} \oplus \pi_{n+1}(K) \to \ldots \end{aligned} \tag{4}$$

with connecting homomorphisms $\delta_n : \pi_n(K) \to \pi_n(K)^{2g} \oplus \pi_{n+1}(K)$ *given by* $a \mapsto (0, -\langle a, b\rangle)$, *where* $\langle \cdot, \cdot \rangle$ *denotes the Samelson product.*

Proof. Recall the notation for surfaces from Remark III.3 and consider the restriction map $r : C(P,K)^K \to C(\eta^{-1}(B_g),K)^K$. Furthermore, [Woc05, Lem. IV.4] provides a continuous map $S : C_*(\eta^{-1}(B_g),K)^K \cong C_*(B_g,K) \cong C_*(\mathbb{S}^1,K)^{2g} \to C_*(P,K)^K$ satisfying $r \circ S = \mathrm{id}_{C_*(\eta^{-1}(B_g),K)^K}$. Furthermore, $\mathcal{P}|_{\eta^{-1}(B_g)}$ is trivial and we may assume S to be defined on $C(\eta^{-1}(B_g,K))^K$ such that $r \circ S = \mathrm{id}_{C_*(\eta^{-1}(B_g),K)^K}$ still holds.

Now take the long exact homotopy sequence for the fibration $\mathrm{ev}_{p_0} : C(P,K)^K \to K$ [Woc05, Prop. IV.8] and recall the construction of the connecting homomorphism from Remark I.5. If $\alpha : \mathbb{B}^n \to K$ represents $a \in \pi_n(K)$ and $A : \mathbb{B}^n \to C(P,K)^K$ is a lift of α, then $A' := A \cdot (S \circ r \circ A)$ is also a lift of α and $A'(x)(p) = e$ holds for $x \in \partial\mathbb{B}^n$ and $p \in \eta^{-1}(B_g)$. Hence $A'|_{\partial\mathbb{B}^n}$ factors through a map on P/γ (where $\gamma : \mathbb{S}^1 \to K$ is supposed to represent the equivalence class of $\mathcal{P}$) and thus represents $(0, -\langle a,b\rangle) \in \pi_n(K) \oplus \pi_{n+1}(K)$ due to Theorem II.10. □

Remark III.6. In infinite-dimensional Lie theory one often considers (period-) homomorphisms $\varphi : \pi_n(G) \to V$ for an infinite-dimensional Lie Group G and an $\mathbb{R}$-vector space V, which we consider here as a $\mathbb{Q}$-vector space. If $n \geq 1$, then $\pi_n(G)$ is abelian and this homomorphism factors through the canonical map $\psi : \pi_n(G) \to \pi_n(G) \otimes \mathbb{Q}$, $a \mapsto a \otimes 1$ and

$$\widetilde{\varphi} : \pi_n(G) \otimes \mathbb{Q} \to V, \quad a \otimes x \mapsto x\, \varphi(a).$$

It thus suffices for many interesting questions arising from infinite-dimensional Lie theory to consider the *rational homotopy groups* $\pi_n^{\mathbb{Q}}(G) := \pi_n(G) \otimes \mathbb{Q}$ for $n \geq 1$.

Furthermore, the functor $\otimes\mathbb{Q}$ in the category of abelian groups, sending A to $A^{\mathbb{Q}} := A \otimes \mathbb{Q}$ and $\varphi : A \to B$ to $\varphi^{\mathbb{Q}} := \varphi \otimes \mathrm{id}_{\mathbb{Q}} : A \otimes \mathbb{Q} \to B \otimes \mathbb{Q}$, preserves exact sequences since $\mathbb{Q}$ is torsion free and hence flat.

Lemma III.7. *If K is a (possibly infinite-dimensioanl) connected Lie group, then the rational Samelson product*

$$\langle \cdot, \cdot \rangle^{\mathbb{Q}} : \pi_n^{\mathbb{Q}}(G) \times \pi_m^{\mathbb{Q}}(G) \to \pi_{n+m}^{\mathbb{Q}}(G), \quad a \otimes x, b \otimes y \mapsto \langle a, b\rangle \otimes xy$$

vanishes.

Proof. Since each connected Lie group is homeomorphic to a compact group and a vector space, is has finite-dimensional rational homology and thus the rational Whitedead product in BK vanishes (cf [FHT01, Prop. 15.15 f.]). Since the Whitehead product in BK and the Samelson product in K correspond to each other via the connecting homomorphism from the classifying bundle $EK \to BK$ (cf. Remark II.11 and [BJS60, Sect. 1]), it follows that the rational Samelson product vanishes either. □

Theorem III.8. *Let K be a connected Lie group and $\mathcal{P} = (K, \eta : P \to M)$ be a continuous principal K-bundle over $\mathbb{S}^m$ or a compact orientable surface Σ.*

i) If $M = \mathbb{S}^m$, then $\pi_n^{\mathbb{Q}}(\mathrm{Gau}(\mathcal{P})) \cong \pi_{n+m}^{\mathbb{Q}}(K) \oplus \pi_n^{\mathbb{Q}}(K)$.

ii) If $M = \Sigma$, then $\pi_n^{\mathbb{Q}}(\mathrm{Gau}(\mathcal{P})) \cong \pi_{n+2}^{\mathbb{Q}}(K) \oplus \pi_{n+1}^{\mathbb{Q}}(K)^{2g} \oplus \pi_n^{\mathbb{Q}}(K)$.

Proof. With Remark III.6 we obtain exact rational homotopy sequences from the exact sequences (1) and (4). Then the preceding Lemma implies that the connecting homomorphisms in these sequences vanish and the long exact sequences decay into short ones. Furthermore, the short exact sequences split linearly since each of them involves just vector spaces. □

Remark III.9. Since the rational homotopy groups of finite-dimensioanl Lie groups are those of odd-dimensional spheres [FHT01, Sect. 15.f], which are well known [FHT01, Ex. 15.d.1] the preceding theorem gives an explicit formula for $\pi_n^{\mathbb{Q}}(\mathrm{Gau}(\mathcal{P}))$ in the case of finite-dimensional structure groups. E.g., if $M = \mathbb{S}^m$ and m is even, then $\pi_n^{\mathbb{Q}}(\mathrm{Gau}(\mathcal{P}))$ vanishes for even n.

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