Monastir Summer School: Infinite-Dimensional Lie Groups

Karl-Hermann Neeb

Abstract. These are lecture notes of a course given at a summer school in Monastir in July 2005. The main purpose of this course is to present some of the main ideas of infinite-dimensional Lie theory and to explain how it differs from the finite-dimensional theory. In the introductory section, we present some of the main types of infinite-dimensional Lie groups: linear Lie groups, groups of smooth maps and groups of diffeomorphisms. We then turn in some more detail to manifolds modeled on locally convex spaces and the corresponding calculus (Section II). In Section III, we present some basic Lie theory for locally convex Lie groups. The Fundamental Theorem for Lie group-valued-functions on manifolds and some of its immediate applications are discussed in Section IV. For many infinite-dimensional groups, the exponential function behaves worse than for finite-dimensional ones or Banach–Lie groups. Section V is devoted to the class of locally exponential Lie groups, i.e., those for which the exponential function is a local diffeomorphism in 0. We conclude these notes with a brief discussion of the integrability problem for locally convex Lie algebras: When is a locally convex Lie algebra the Lie algebra of a global Lie group?
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I. Introduction

Symmetries play a decisive role in the natural sciences and throughout mathematics. Infinite-dimensional Lie theory deals with symmetries depending on infinitely many parameters. Such symmetries may be studied on an infinitesimal, local or global level, which amounts to studying Lie algebras, local Lie groups and global Lie groups, respectively.

Finite-dimensional Lie theory was created in the late 19th century by Marius Sophus Lie, who showed that in finite dimensions the local and the infinitesimal theory are essentially equivalent. The differential-geometric approach to finite-dimensional global Lie groups (as smooth or analytic manifolds) is naturally complemented by the theory of algebraic groups with which it interacts most fruitfully. A crucial point of the finite-dimensional theory is that finiteness conditions permit to develop a full-fledged structure theory of finite-dimensional Lie groups in terms of the Levi splitting and the fine structure of semisimple groups.

In infinite dimensions, the passage from the infinitesimal to the local level and from there to the global level is not possible in general, whence the theory splits into three properly distinct levels. A substantial part of the literature on infinite-dimensional Lie theory exclusively deals with the level of Lie algebras, their structure, and their representations. However, only special classes of groups, such as Kac–Moody groups or certain direct limit groups, can be approached by purely algebraic methods. In particular, this is relevant for many applications in mathematical physics, where the infinitesimal approach is convenient for calculations, but a global perspective would be most desirable to understand global phenomena. We think that a similar statement applies to non-commutative geometry, where derivations and covariant derivatives are ubiquitous, but global symmetry groups have been neglected.

In these lectures, we concentrate on the local and global level of infinite-dimensional Lie theory, as well as the mechanisms allowing or preventing to pass from one level to another. Our studies are based on a notion of Lie group which is both simple and very general: A Lie group simply is a manifold, endowed with a group structure such that multiplication and inversion are smooth maps. The main difference compared to the finite-dimensional theory concerns the notion of a manifold: The manifolds we consider shall not be finite-dimensional, but modeled on an arbitrary locally convex space. It is quite useful to approach Lie groups from such a general perspective, because this enables a unified discussion of all basic aspects of the theory. To obtain more specific results, it is essential to focus on individual classes of Lie groups. In this introduction, we discuss several classes of infinite-dimensional Lie groups without going into details. The main purpose is to give an impression of the enormous variety of infinite-dimensional Lie groups.

Some history

The concept of a Banach–Lie group, i.e., a Lie group modeled on a Banach space, has been introduced by G. Birkhoff in [Bi38]. The step to more general classes of infinite-dimensional Lie groups modeled on complete locally convex spaces occurs first in an article of Marsden and Abraham [MA70] in the context of hydrodynamics. This Lie group concept has been worked out by J. Milnor in his Les Houches lecture notes [Mil83] which provide many basic results of the general theory. The observation that the completeness condition on the underlying locally convex space can be omitted for the basic theory is due to H. Glöckner ([Gl02a]). This is important for quotient constructions because quotients of complete locally convex spaces need not be complete.

There are other, weaker, concepts of Lie groups, resp., infinite-dimensional manifolds. One is based on the "convenient setting" for global analysis developed by Fröhlicher, Kriegl and Michor ([FK88] and [KM97]). In the context of Fréchet manifolds, this setting does not differ from the one mentioned above, but for more general model spaces it provides a concept of a smooth map which does not necessarily imply continuity, hence leads to Lie groups which are not topological groups. Another approach is based on the concept of a diffeological space due to J.-M. Souriau ([So85]) which can be used to study spaces like quotients of $\mathbb{R}$ by non-discrete subgroups in a differential geometric context. It has the important advantage that the category of diffeological spaces is cartesian closed and that any quotient of a diffeological space carries a natural diffeology. On the other hand, this incredible freedom creates some quite ugly creatures.

Throughout these notes, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and all vector spaces are real or complex. For two topological vector spaces V, W, we write $\mathcal{L}(V, W)$ for the space of continuous linear operators $V \to W$ and put $\mathcal{L}(V) := \mathcal{L}(V, V)$.

I.1. Linear Lie groups

In finite-dimensional Lie theory, a natural approach to Lie groups is via matrix groups, i.e., subgroups of the group $\mathrm{GL}_n(\mathbb{R})$ of invertible real $n \times n$-matrices. Since every finite-dimensional algebra can be embedded into a matrix algebra, this is equivalent to considering subgroups of the unit groups $A^\times := \{a \in A \colon (\exists b \in A) ab = ba = \mathbf{1}\}$ of finite-dimensional unital associative algebras A. The advantage of this approach is that, under mild completeness assumptions, one can define the exponential function quite directly via the exponential series and thus take a shortcut to several deeper results on Lie groups. This approach also works quite well in the context of Banach-Lie groups. Here the linear Lie groups are subgroups of unit groups of Banach algebras, but this setting is too restrictive for many applications of infinite-dimensional Lie theory.

Let V be a locally convex space and $A := \mathcal{L}(V)$ the unital associative algebra of all continuous linear endomorphisms of V. Its unit group is the general linear group $\mathrm{GL}(V)$ of V, but unfortunately there is no natural manifold structure on $\mathrm{GL}(V)$ if V is not a Banach space. In particular, it is far from being open, as follows from the fact that if the spectrum of the operator A is unbounded, then $\mathbf{1} + tA$ is not invertible for all sufficiently small values of t. Therefore it is much more natural to consider a class of well-behaved associative algebras instead of the algebras of the form $\mathcal{L}(V)$ for general locally convex spaces.

We shall see that the most natural class of algebras for infinite-dimensional Lie theory are the so-called *continuous inverse algebras* (CIAs). These are unital locally convex algebras A with continuous multiplication such that the unit group $A^\times$ is open and the inversion is a continuous map $A^\times \to A$.

Remark I.1.1. (a) Each unital Banach algebra A is a continuous inverse algebra. In fact, if $\|\cdot\|$ is a sub-multiplicative norm on A with $\|\mathbf{1}\| = 1$, then for each $x \in A$ with $\|x\| < 1$ we have $\mathbf{1} - x \in A^\times$ with

$$(\mathbf{1} - x)^{-1} = \sum_{k=0}^{\infty} x^k,$$

and the geometric series, also called the *Neumann series*, converges uniformly on each ball $B_r(0)$ with $r < 1$. We conclude that $A^\times$ contains $B_1(\mathbf{1})$ and that inversion is continuous on this ball. Now elementary arguments imply that $A^\times$ is open and that inversion is continuous (Exercise I.1).

(b) For each Banach space V, the algebra $\mathcal{L}(V)$ of continuous linear operators on V is a unital Banach algebra with respect to the operator norm

$$\|\varphi\| := \sup\{\|\varphi(v)\| \colon \|v\| \leq 1\},$$

hence in particular a CIA.

(c) For each CIA A and $n \in \mathbb{N}$, the matrix algebra $M_n(A)$ also is a CIA when endowed with the product topology obtained by identifying it with A^{n^2} (cf. [Bos90], Exercise I.3).

(d) If M is a compact manifold, then the algebra $C^\infty(M, \mathbb{C})$ is a continuous inverse algebra (cf. Section II for the topology on this algebra).

(e) Let B be a Banach algebra and $\alpha \colon G \times B \to B$ a strongly continuous action of the finite-dimensional Lie group G on B by isometric automorphisms. Then the space $A := B^\infty$ of smooth vectors for this action is a dense subalgebra and a Fréchet CIA (cf. [Bos90, Prop. A.2.9]). ■

We shall see below that the unit group of a CIA is a Lie group, when endowed with its natural manifold structures as an open subset. This property clearly shows that in the context of infinite-dimensional Lie theory over locally convex spaces, CIAs form the natural class of algebras to be considered.

In view of Remark I.1.1(c), $\mathrm{GL}_n(A)$ is a Lie group for each CIA A. We think of "Lie subgroups" of these groups as *linear Lie groups*, but we shall only see later in Section III how and in how many ways the notion of a Lie subgroup can be made more precise. Note that most classical Lie groups are defined as centralizers of certain matrices or as the set of fixed points for a group of automorphisms. All these constructions have natural generalizations to matrices with entries in CIAs.

I.2. Groups of continuous and smooth maps

In the context of Banach–Lie groups, one constructs Lie groups of mappings as follows. For a compact space X and a Banach–Lie group K, the group $C(X, K)$ of continuous maps is a Banach–Lie group with Lie algebra $C(X, \mathfrak{k})$, where $\mathfrak{k} := \mathbf{L}(K)$ is the Lie algebra of K.

In the larger context of locally convex Lie groups, one also obtains for each Lie group K and a compact smooth manifold M a Lie group structure on the group $C^\infty(M, K)$ of smooth maps from M to K. This is a Fréchet–Lie group if K is a Fréchet–Lie group and its Lie algebra is $C^\infty(M, \mathfrak{k})$.

The passage from continuous maps to smooth maps is motivated by the behavior of central extensions of these groups. The groups $C^\infty(M, K)$ have much more central extensions than the groups $C(M, K)$, hence exhibit a richer geometric structure. Closely related is the fact that algebras of smooth functions have much more derivations than algebras of continuous functions (cf. also the discussion in Section I.3).

A larger class of groups of smooth maps is obtained as *gauge groups* of principal bundles. If $q\colon P \to B$ is a smooth principal bundle with structure group K and $\sigma\colon P \times K \to P, (p,k) \mapsto \sigma_k(p) = p.k$ denotes the right action of K on P, then

$$\mathrm{Gau}(P) := \{\varphi \in \mathrm{Diff}(P)\colon q \circ \varphi = q,\ (\forall k \in k)\ \varphi \circ \sigma_k = \sigma_k \circ \varphi\}$$

is called the *gauge group of the bundle* and its elements are called *gauge transformations*. In view of $q \circ \varphi = q$, each gauge transformation φ can be written as $\varphi(p) = p.f(p)$ for some smooth function $f\colon P \to K$, and from $\varphi \circ \sigma_k = \sigma_k \circ \varphi$ we derive that $kf(p.k) = f(p)k$, i.e.,

$$f(p.k) = k^{-1}f(p)k, \quad p \in P, k \in K. \tag{1.1.1}$$

Conversely, every smooth function $f\colon P \to K$ satisfying (1.1.1) defines a gauge transformation by $\varphi_f(p) := p.f(p)$. Moreover,

$$\varphi_{f_1}(\varphi_{f_2}(p)) = \varphi_{f_2}(p).f_1\big(\varphi_{f_2}(p)\big) = p.(f_2(p)f_1(p.f_2(p))) = p.(f_1(p)f_2(p)) = \varphi_{f_1f_2}(p)$$

implies that we obtain an isomorphism of groups

$$C^\infty(P,K)^K := \{f \in C^\infty(P,K)\colon (\forall p \in P)(\forall k \in K)\ f(p.k) = k^{-1}f(p)k\} \to \mathrm{Gau}(P), \quad f \mapsto \varphi_f.$$

We may therefore view $\mathrm{Gau}(P)$ as a subgroup of the group $C^\infty(P,K)$, endowed with the pointwise product, and we shall see below under which requirements on the bundle and the structure group K one can show that $\mathrm{Gau}(P)$ is a Lie group.

If the bundle P is trivial, then there exists a smooth global section $\sigma\colon B \to P$, and the map

$$C^\infty(P,K)^K \to C^\infty(M,K), \quad f \mapsto f \circ \sigma$$

is an isomorphism of groups.

I.3. Groups of homeomorphisms and diffeomorphisms

Interesting groups arise naturally from geometric or other structures on spaces as their automorphism groups. In the spirit of Felix Klein's Erlangen Program, geometric structures are even defined in terms of their automorphism groups. In this section, we take a closer look at the homeomorphism group $\mathrm{Homeo}(X)$ of a topological space X, the diffeomorphism group $\mathrm{Diff}(M)$ of a smooth manifold M and relate them to the automorphism groups of the corresponding algebras of continuous and smooth functions.

I.3.1. If X is a topological space, then the group $\mathrm{Homeo}(X)$ acts naturally by automorphisms on the algebra $C(X,\mathbb{R})$ of continuous real-valued functions on X by algebra automorphisms via

$$(\varphi.f)(x) := f(\varphi^{-1}(x)).$$

If, in addition, X is compact, then $C(X,\mathbb{R})$ has a natural Banach algebra structure given by the sup-norm, and with Gelfand duality the space X can be recovered from this algebra as

$$X \cong \mathrm{Hom}_{\mathrm{alg}}(C(X,\mathbb{R}),\mathbb{R}) \setminus \{0\}$$

in the sense that every non-zero algebra homomorphism $C(X,\mathbb{R}) \to \mathbb{R}$ (which is automatically continuous) is given by a point evaluation $\delta_p(f) = f(p)$. The topology on X can be recovered from $C(X,\mathbb{R})$ by endowing $\mathrm{Hom}_{\mathrm{alg}}(C(X,\mathbb{R}),\mathbb{R})$ with the topology of pointwise convergence on $C(X,\mathbb{R})$.

For any Banach algebra A, the group $\mathrm{Aut}(A)$ carries a natural Lie group structure (as a Lie subgroup of $\mathrm{GL}(A)$), so that $\mathrm{Homeo}(X) \cong \mathrm{Aut}(C(X,\mathbb{R}))$ inherits a natural Lie group structure when endowed with the topology inherited from the Banach algebra $\mathcal{L}(C(X,\mathbb{R}))$. We claim that this topology turns $\mathrm{Homeo}(X)$ into a discrete group. In fact, if φ is a non-trivial homeomorphism of X and $p \in X$ is moved by φ, then there exists a continuous function $f \in C(X,\mathbb{R})$ with $\|f\| = 1$, $f(p) = 0$ and $f(\varphi^{-1}(p)) = 1$. Then $\|\varphi.f - f\| \geq 1$ implies that $\|\varphi - \mathbf{1}\| \geq 1$. Therefore the group $\mathrm{Homeo}(X)$ is discrete. Since exponentials of continuous derivations yield one-parameter groups of automorphisms, it follows that $\mathrm{der}(C(X,\mathbb{R})) = \{0\}$.

Nevertheless, one considers continuous actions of connected Lie groups G on X, where the continuity of the action means that the action map $\alpha: G \times X \to X$ is continuous. But this does not mean that the corresponding homomorphism $G \to \mathrm{Homeo}(X)$ is continuous. We will see that this phenomenon, i.e., that certain automorphism groups are endowed with Lie group structures which are too fine for many purposes, occurs at several levels of the theory (cf. also Exercise I.5).

I.3.2. Now let M be a compact smooth manifold and consider the Fréchet algebra $A := C^\infty(M,\mathbb{R})$ of smooth functions on M (cf. Example II.1.4). In this context, we also have

$$M \cong \mathrm{Hom}(C^\infty(M,\mathbb{R}),\mathbb{R}) \setminus \{0\}$$

in the sense that every non-zero algebra homomorphism $C^\infty(M,\mathbb{R}) \to \mathbb{R}$ is given by a point evaluation $\delta_p(f) := f(p)$ for some $p \in M$ (see Theorem A.1). The smooth structure on M is completely determined by the requirement that the maps $M \to \mathbb{R}, p \mapsto \delta_p(f)$ are smooth. This implies that the group $\mathrm{Aut}(C^\infty(M,\mathbb{R}))$ of automorphisms of $C^\infty(M,\mathbb{R})$ can be identified with the group $\mathrm{Diff}(M)$ of all diffeomorphisms of M.

In sharp contrast to the topological context, the group $\mathrm{Diff}(M)$ has a non-trivial structure as a Lie group modeled on the space $\mathcal{V}(M)$ of (smooth) vector fields on M, which then is the Lie algebra of (the opposite of) this group. Moreover, for a finite-dimensional Lie group G, smooth left actions $\alpha: G \times M \to M$ correspond to Lie group homomorphisms $G \to \mathrm{Diff}(M)$. For $G = \mathbb{R}$, we obtain in particular the correspondence between smooth flows on M, smooth vector fields on M, and one-parameter subgroups of $\mathrm{Diff}(M)$. If $X \in \mathcal{V}(M)$ is a vector field and $\mathrm{Fl}_X: \mathbb{R} \to \mathrm{Diff}(M)$ the corresponding flow, then

$$\exp: \mathcal{V}(M) \to \mathrm{Diff}(M), \quad X \mapsto \mathrm{Fl}_X(1)$$

is the exponential function of the Fréchet–Lie group $\mathrm{Diff}(M)$.

Other important groups of diffeomorphisms arise as subgroups of $\mathrm{Diff}(M)$. Of particular importance is the stabilizer subgroup $\mathrm{Diff}(M,\mu)$ of a volume form μ on M (if M is orientable), and the stabilizer $\mathrm{Sp}(M,\omega)$ of a symplectic form ω if (M,ω) is symplectic (cf. [KM97]).

I.3.3. If M is a paracompact finite-dimensional smooth manifold, then we still have

$$M \cong \mathrm{Hom}(C^\infty(M,\mathbb{R}),\mathbb{R}) \setminus \{0\} \quad \text{and} \quad \mathrm{Diff}(M) \cong \mathrm{Aut}(C^\infty(M,\mathbb{R}))$$

(Theorem A.1), but then there is no natural Lie group structure on $\mathrm{Diff}(M)$ such that smooth actions of Lie groups G on M correspond to Lie group homomorphisms $G \to \mathrm{Diff}(M)$.

It is possible to turn $\mathrm{Diff}(M)$ into a Lie group with Lie algebra $\mathcal{V}_c(M)$, the Lie algebra of all smooth vector fields with compact support. If M is compact, this yields the aforementioned Lie group structure on $\mathrm{Diff}(M)$, but if M is not compact, then the corresponding topology on $\mathrm{Diff}(M)$ is so fine that the global flow generated by a vector field whose support is not compact does not lead to a continuous homomorphism $\mathbb{R} \to \mathrm{Diff}(M)$. For this Lie group structure, the normal subgroup $\mathrm{Diff}_c(M)$ of all diffeomorphisms which coincide with id_M outside a compact set is an open subgroup.

I.3.4. The situation for non-compact manifolds is similar to the situation we encounter in the theory of unitary group representations. Let H be a Hilbert space and $U(H)$ its unitary group. This group has two natural topologies. The uniform topology on $U(H)$ inherited from the Banach algebra $\mathcal{L}(H)$ turns it into a Banach–Lie group, but this topology is rather fine. The strong operator topology (the topology of pointwise convergence) turns $U(H)$ into a topological group such that continuous unitary representations of a topological group G correspond to continuous group homomorphisms $G \to U(H)$. If G is a finite-dimensional Lie group, then a continuous unitary representation is continuous with respect to the uniform topology on $U(H)$ if and only if all operators of the derived representation are bounded, but this implies already that the representation factors through a Lie group with compact Lie algebra (cf. [Si52], [Gu80], Exercise I.6). In some sense, the condition that the operators of the derived representation are bounded is analogous to the requirement that the vector fields corresponding to a smooth action on a manifold have compact support. In this sense, the uniform topology on $U(H)$ shows similarities to the Lie group structure from (I.3.3) on $\mathrm{Diff}(M)$ if M is non-compact. The case of a compact manifold M corresponds to the case of a finite-dimensional Hilbert space H, for which the two topologies on $U(H)$ coincide.

I.3.5. Clearly, the situation becomes worse if M is an infinite-dimensional manifold. Then $\mathrm{Diff}(M)$ has no natural group topology, but we can still make sense of smooth maps $f\colon N \to \mathrm{Diff}(M)$, where N is a smooth manifold, by requiring that the corresponding map

$$N \times M \to M^2, \quad (n,m) \mapsto (f(n)(m), f(n)^{-1}(m))$$

is smooth. In this sense, a smooth action of a Lie group G on M is a *smooth homomorphism* $G \to \mathrm{Diff}(M)$.

Similar statements hold for the group $\mathrm{GL}(V)$, where V is a general locally convex space.

Exercises for Section I

Exercise I.1. For an associative algebra A, we write A_+ for the algebra $A \times \mathbb{K}$ with the multiplication

$$(a,s)(b,t) := (ab + sb + ta, st).$$

(1) Verify that A_+ is a unital algebra with unit $\mathbf{1} = (0,1)$.
(2) Show that $\mathrm{GL}_1(A) := A_+^\times \cap (A \times \{1\})$ is a group.
(3) If $e \in A$ is an identity element, then A_+ is isomorphic to the direct product algebra $A \times \mathbb{K}$ with the product $(a,s)(b,t) = (ab, st)$. ∎

Exercise I.2. A *topological ring* is a ring R endowed with a topology for which addition and multiplication are continuous. Let R be a unital topological ring. Show that:
(1) For $x \in R^\times$, the left and right multiplications $\lambda_x(y) := xy$ and $\rho_x(y) := yx$ are homeomorphisms of R.
(2) The unit group $R^\times$ is open if and only if it is a neighborhood of $\mathbf{1}$.
(3) The inversion $R^\times \to R$ is continuous, i.e., $(R^\times, \cdot)$ is a topological group, if it is continuous in $\mathbf{1}$. ∎

Exercise I.3. Let R be a unital ring, $n \in \mathbb{N}$ and $M_n(R)$ the ring of all $(n \times n)$-matrices with entries in R. In the following, we write elements $x \in M_n(R)$ as

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_n(R) = \begin{pmatrix} M_{n-1}(R) & M_{n-1,1}(R) \\ M_{1,n-1}(R) & R \end{pmatrix} = \begin{pmatrix} M_{n-1}(R) & R^{n-1} \\ (R^{n-1})^\top & R \end{pmatrix}.$$

(1) Show that a matrix x is of the form

$$\begin{pmatrix} \mathbf{1} & \beta \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ \gamma & \mathbf{1} \end{pmatrix} \quad \text{with} \quad \alpha \in \mathrm{GL}_{n-1}(R), \beta, \gamma^\top \in R^{n-1}, \delta \in R^\times$$

if and only if $d \in R^\times, a - bd^{-1}c \in \mathrm{GL}_{n-1}(R)$, and that in this case

$$\delta = d, \quad \beta = bd^{-1}, \quad \gamma = d^{-1}c, \quad \alpha = a - bd^{-1}c.$$

(2) Assume, in addition, that R is a topological ring with open unit group and continuous inversion. Show by induction on n that
  (a) $\mathrm{GL}_n(R)$ is open in $M_n(R)$.
  (b) Inversion in $\mathrm{GL}_n(R)$ is continuous, i.e., $\mathrm{GL}_n(R)$ is a topological group. ∎

Exercise I.4. Let R be a unital ring and consider the right R-module R^n, where the module structure is given by $(x_1, \ldots, x_n).r := (x_1 r, \ldots, x_n r)$. Let M be a right R-module, $\sigma: r \mapsto r^\sigma$ an involution on R, i.e., an involutive anti-automorphism and $\varepsilon \in \{\pm 1\}$. A biadditive map $\beta: M \times M \to R$ is called *σ-sesquilinear* if

$$\beta(x.r, y.s) = r^\sigma \beta(x,y) s \quad \text{for} \quad x, y \in M, r, s \in R.$$

It is called *σ-ε-hermitian* if, in addition,

$$\sigma(x,y)^\sigma = \varepsilon \sigma(y,x) \quad \text{for} \quad x, y \in M.$$

For $\varepsilon = 1$, we call the form *σ-hermitian* and *σ-antihermitian* for $\varepsilon = -1$. For a σ-ε-hermitian form β on M,

$$U(M, \beta) := \{\varphi \in \mathrm{Aut}_R(M) : (\forall x, y \in M)\ \beta(\varphi(x), \varphi(y)) = \beta(x,y)\}$$

is called the corresponding *unitary group*. Show that:

(1) $\mathrm{End}_R(R^n) \cong M_n(R)$, where $M_n(R)$ operates by left multiplication on column vectors on R^n.

(2) $\mathrm{Aut}_R(R^n) \cong \mathrm{GL}_n(R)$.

(3) $\beta(x,y) := \sum_{i=1}^n x_i^\sigma y_i$ is a σ-hermitian form on R^n. Describe the corresponding unitary group in terms of matrices.

(4) $\beta(x,y) := \sum_{i=1}^n x_i^\sigma y_{n+i} - x_{n+i}^\sigma y_i$ is a σ-antihermitian form on R^{2n}. Describe the corresponding unitary group in terms of matrices. ■

Exercise I.5. Let X be a topological space and endow the set $C(X,X)$ of continuous self-maps of X with the compact open topology, i.e., the topology generated by the sets $W(K,O) := \{f \in C(X,X) : f(K) \subseteq O\}$, where $K \subseteq X$ is compact and $O \subseteq X$ is open (cf. Appendix B). We endow the group $\mathrm{Homeo}(X)$ with the initial topology with respect to the map

$$\mathrm{Homeo}(X) \to C(X,X)^2, \varphi \mapsto (\varphi, \varphi^{-1}).$$

Show that if X is locally compact, then this topology turns $\mathrm{Homeo}(X)$ into a topological group. Hint: If $f \circ g \in W(K,O)$ choose a compact subset K' and an open subset O' with $g(K) \subseteq O' \subseteq K' \subseteq f^{-1}(O)$. ■

Exercise I.6. Let G be a finite-dimensional connected Lie group and $\pi: G \to \mathrm{GL}(X)$ be a faithful representation which is continuous when $\mathrm{GL}(X)$ carries the uniform topology inherited from the Banach algebra $\mathcal{L}(X)$ and for which $\pi(G)$ is bounded. Show that $\mathfrak{g} := \mathbf{L}(G)$ is a compact Lie algebra by using the following steps:

(1) π is a smooth homomorphism of Lie groups. In particular, we have a representation of the Lie algebra $\mathbf{L}(\pi): \mathfrak{g} \to \mathcal{L}(X)$.

(2) $\|x\| := \|\mathbf{L}(\pi)(x)\|$ defines a norm on $\mathfrak{g}$, and $\mathrm{Ad}(G)$ is bounded with respect to this norm.

(3) $\mathrm{Ad}(G)$ has compact closure, so that $\mathfrak{g}$ is a compact Lie algebra.

If, in addition, X is a Hilbert space, then one can even show that there exists a scalar product compatible with the topology which is invariant under G, so that π becomes a unitary representation with respect to this scalar product. This can be achieved by showing that the set of all compatible scalar products is a Bruhat–Tits space and then applying the Bruhat–Tits Fixed Point Theorem. ■

II. Infinite-dimensional manifolds

In this section, we turn to some more details on infinite-dimensional manifolds. First we briefly discuss the concept of a locally convex space, then the basics and the peculiarities of calculus on these spaces, and finally manifolds modeled on locally convex spaces.

In this section, V always denotes a $\mathbb{K}$-vector space and $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

II.1. Locally convex spaces

Definition II.1.1. (a) If p is a seminorm on a $\mathbb{K}$-vector space V, then $N_p := p^{-1}(0)$ is a subspace of V, and $V_p := V/N_p$ is a normed space with $\|v + N_p\| := p(v)$. Let $\alpha_p: V \to V_p$ denote the corresponding quotient map.

(b) We call a set $\mathcal{P}$ of seminorms on V *separating* if $p(v) = 0$ for all $p \in \mathcal{P}$ implies $v = 0$. This is equivalent to the linear map

$$\alpha: V \to \prod_{p\in\mathcal{P}} V_p, \quad v \mapsto (\alpha_p(v))_{p\in\mathcal{P}}$$

being injective.

(c) If X is a set and $f_j: X \to X_j$, $j \in J$, are mappings into topological spaces, then the coarsest topology on X for which all these maps are continuous is called the *initial topology on X with respect to the family* $(f_j)_{j\in J}$. This topology is generated by the inverse images of open subsets of the spaces X_j under the maps f_j. Combining the functions f_j to a single function

$$f: X \to \prod_{j\in J} X_j, \quad x \mapsto (f_j(x))_{j\in J},$$

the initial topology on X is nothing but the inverse image of the product topology under f.

(d) To each separating family $\mathcal{P}$ of seminorms on V we associate the initial topology $\tau_{\mathcal{P}}$ on V defined by the maps $\alpha_p: V \to V_p$ to the normed spaces V_p. We call it the *locally convex topology* on V defined by $\mathcal{P}$.

Since the family $\mathcal{P}$ is separating, V is a Hausdorff space. Further it is easy to show that V is a topological vector space in the sense that addition and scalar multiplication on V are continuous maps.

A *locally convex space* is a vector space endowed with a topology defined by a separating family of seminorms. The preceding argument shows that each locally convex space is in particular a topological vector space which can be embedded into a product $\prod_{p\in\mathcal{P}} V_p$ of normed spaces.

(e) A locally convex space V is called a *Fréchet space* if its topology can be defined by a countable family $\mathcal{P} = \{p_n: n \in \mathbb{N}\}$ of seminorms and if V is complete with respect to the compatible metric

$$d(x,y) := \sum_{n\in\mathbb{N}} 2^{-n} \frac{p_n(x-y)}{1 + p_n(x-y)}. \qquad \blacksquare$$

Remark II.1.2. (a) A sequence $(x_n)_{n\in\mathbb{N}}$ in a locally convex space V is said to be a *Cauchy sequence* if each sequence $\alpha_p(x_n)$, $p \in \mathcal{P}$, is a Cauchy sequence in V_p. We say that V is *sequentially complete* if every Cauchy sequence in V converges.

(b) One has a natural notion of completeness for locally convex spaces (every Cauchy filter converges). Complete locally convex spaces can be characterized as those isomorphic to closed subspaces of products of Banach spaces. In fact, let $\overline{V}_p$ denote the completion of the normed space V_p. We then have an embedding

$$\alpha: V \to \prod_{p\in\mathcal{P}} \overline{V}_p, \quad v \mapsto (\alpha_p(v))_{p\in\mathcal{P}},$$

and the completeness of V is equivalent to the closedness of $\alpha(V)$ in the product of the Banach spaces $\overline{V}_p$, which is a complete space (Exercise II.8). ∎

Examples II.1.3. (a) Let X be a topological space. For each compact subset $K \subseteq X$, we obtain a seminorm p_K on $C(X,\mathbb{R})$ by

$$p_K(f) := \sup\{|f(x)| : x \in K\}.$$

The family $\mathcal{P}$ of these seminorms defines on $C(X,\mathbb{R})$ the locally convex *topology of uniform convergence on compact subsets of* X.

If X is compact, then we may take $K = X$ and obtain a norm on $C(X,\mathbb{R})$ which defines the topology; all other seminorms p_K are redundant (cf. Exercise II.1). In this case, $C(X,\mathbb{R})$ is a Banach space.

(b) The preceding example can be generalized to the space $C(X,V)$, where X is a topological space and V is a locally convex space. Then we define for each compact subset $K \subseteq X$ and each continuous seminorm q on V a seminorm

$$p_{K,q}(f) := \sup\{q(f(x)) : x \in K\}.$$

The family of these seminorms defines a locally convex topology on $C(X,V)$, the *topology of uniform convergence on compact subsets of* X (cf. Appendix B).

(c) If X is locally compact and countable at infinity, then there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X with $\bigcup_n K_n$ and $K_n \subseteq K_{n+1}^0$. We call such a sequence $(K_n)_{n\in\mathbb{N}}$ an *exhaustion of* X. Then each compact subset $K \subseteq X$ lies in some K_n, so that each seminorm p_K is dominated by some p_{K_n}. This implies that $C(X,\mathbb{R})$ is metrizable, and since it is also complete, it is a Fréchet space. It even is a Fréchet algebra in the sense that the algebra multiplication is continuous (cf. Exercise II.4).

(d) For any set X, the space $\mathbb{R}^X$ of all real-valued function $X \to \mathbb{R}$ is a locally convex space with respect to the product topology. The topology is defined by the seminorms p_x defined by $p_x(f) := |f(x)|$, $x \in X$. This space is complete, and it is metrizable if and only if X is countable. ∎

Example II.1.4. (a) Let $U \subseteq \mathbb{R}^n$ be an open subset and consider the algebra $C^\infty(U,\mathbb{R})$. For each multiindex $m = (m_1,\ldots,m_n) \in \mathbb{N}_0$ with $|m| := m_1 + \ldots + m_n$, we consider the differential operator

$$D^m := D_1^{m_1}\cdots D_n^{m_n} := \frac{\partial^{|m|}}{\partial_1^{m_1}\cdots\partial_n^{m_n}}.$$

We now obtain for each m and each compact subset $K \subseteq U$ a seminorm on $C^\infty(U,\mathbb{R})$ by

$$p_{K,m}(f) := \sup\{|D^m f(x)| : x \in K\}.$$

The family of all these seminorms defines a locally convex topology on $C^\infty(U,\mathbb{R})$.

To obtain an exhaustion of U, we choose a norm $\|\cdot\|$ on $\mathbb{R}^n$ and consider the compact subsets

$$K_n := \{x \in U : \|x\| \le n, \operatorname{dist}(x,U^c) \ge \tfrac{1}{n}\},$$

where $U^c := \mathbb{R}^n \setminus U$ denotes the complement of U and $\operatorname{dist}(x,U^c) := \inf\{\|x-y\| : y \in U^c\}$ is a continuous function (Exercise II.5). It is easy to see that $(K_n)_{n\in\mathbb{N}}$ is an exhaustion of U, so that the topology on $C^\infty(U,\mathbb{R})$ can be defined by a countable set of seminorms. Moreover, $C^\infty(U,\mathbb{R})$

is complete with respect to the corresponding metric, and the multiplication on this algebra is continuous, so that it is a Fréchet algebra (Exercise II.6).

(b) Let M be a smooth n-dimensional manifold and consider the algebra $C^\infty(M,\mathbb{R})$. If (φ,U) is a chart of M, then $\varphi(U)$ is an open subset of some $\mathbb{R}^n$, so that, in view of (a), we have already a Fréchet algebra structure on $C^\infty(\varphi(U),\mathbb{R})$. We now consider the map

$$\Phi\colon C^\infty(M,\mathbb{R}) \hookrightarrow \prod_{(\varphi,U)} C^\infty(\varphi(U),\mathbb{R}), \quad f \mapsto (f|_U \circ \varphi^{-1})_{(\varphi,U)}$$

and endow the right hand side with the product topology, turning it into a locally convex algebra (Exercise II.8). Therefore the inverse image of this topology turns $C^\infty(M,\mathbb{R})$ into a locally convex algebra.

This description is convenient, but not very explicit. To see how it can be defined by seminorms, note that for each compact subset $K \subseteq M$ for which there exists a chart $\varphi\colon U \to \mathbb{R}^n$ with $K \subseteq U$ and for each multiindex $m \in \mathbb{N}_0^n$ we have a seminorm

$$p_{K,m}(f) := \sup\{|D^m(f \circ \varphi^{-1})(x)|\colon x \in \varphi(K)\}.$$

It is easy to see that these seminorms define the topology on $C^\infty(M,\mathbb{R})$ and that we thus obtain the structure of a Fréchet algebra on $C^\infty(M,\mathbb{R})$. The topology is called the *topology of local uniform convergence of all partial derivatives.*

(c) If M is a finite-dimensional paracompact complex manifold, then we consider the algebra $\mathrm{Hol}(M,\mathbb{C})$ of holomorphic functions on M as a subalgebra of $C(M,\mathbb{C})$, endowed with the topology of uniform convergence on compact subsets of M (Example II.1.3). This topology turns $\mathrm{Hol}(M,\mathbb{C})$ into a Fréchet algebra. Moreover, one can show that the injective map $\mathrm{Hol}(M,\mathbb{C}) \hookrightarrow C^\infty(M,\mathbb{C})$is also a topological embedding (Exercise II.9). ■

Definition II.1.5. Let V be a vector space and $\alpha_j\colon V_j \to V$ linear maps, defined on locally convex spaces V_j. We consider the system $\mathcal{P}$ of all those seminorms p on V for which all compositions $p \circ \alpha_j$ are continuous seminorms on the spaces V_j. By means of $\mathcal{P}$, we obtain on V a locally convex topology called the *final locally convex topology* defined by the mappings $(\alpha_j)_{j\in J}$.

This locally convex topology has the universal property that a linear map $\varphi\colon V \to W$ into a locally convex space W is continuous if and only if all the maps $\varphi \circ \alpha_j$, $j \in J$, are continuous (Exercise). ■

Example II.1.6. (a) Let X be a locally compact space and $C_c(X,\mathbb{R})$ the space of compactly supported continuous functions. For each compact subset $K \subseteq X$, we then have a natural inclusion

$$\alpha_K\colon C_K(X,\mathbb{R}) := \{f \in C_c(X,\mathbb{R})\colon \mathrm{supp}(f) \subseteq K\} \hookrightarrow C_c(X,\mathbb{R}).$$

Each space $C_K(X,\mathbb{R})$ is a Banach space with respect to the norm

$$\|f\|_\infty := \sup\{|f(x)|\colon x \in X\} = \sup\{|f(x)|\colon x \in K\}.$$

We endow $C_c(X,\mathbb{R})$ with the final locally convex topology defined by the maps α_K (Definition II.1.5).

(b) Let M be a smooth manifold and consider the space $C_c^\infty(M,\mathbb{R})$ of smooth functions with compact support. For each compact subset $K \subseteq M$, we then have a natural inclusion

$$\alpha_K\colon C_K^\infty(M,\mathbb{R}) := \{f \in C_c^\infty(M,\mathbb{R})\colon \mathrm{supp}(f) \subseteq K\} \hookrightarrow C_c^\infty(M,\mathbb{R}).$$

We endow each space $C_K^\infty(M,\mathbb{R})$ with the subspace topology inherited from $C^\infty(M,\mathbb{R})$, which turns it into a Fréchet space. We endow $C_c^\infty(M,\mathbb{R})$ with the final locally convex topology defined by the maps α_K. ■

II.2. Calculus on locally convex spaces

In this section, we briefly explain the cornerstones of calculus in locally convex spaces. The main point is that one uses an appropriate notion of differentiability, resp., smoothness which for the special case of Banach spaces differs from Fréchet differentiability but which is more convenient in the setup of locally convex spaces. Our basic references are [Ha82] and [Gl02a], and in particular the forthcoming book [GN05], where one finds detailed proofs. One readily observes that once one has the Fundamental Theorem of Calculus, then the proofs of the finite-dimensional case carry over.

A different approach to differentiability in infinite-dimensional spaces is provided by the so-called convenient setting, which can be found in [FK88] and [KM97]. A central feature of this approach is that smooth maps are no longer required to be continuous, but for calculus over Fréchet spaces one finds the same class of smooth maps. The concept of a diffeological space due to J.-M. Souriau ([So85]) goes much further. It is primarily designed to study spaces with pathologies like quotients of $\mathbb{R}$ by non-discrete subgroups in a differential geometric context.

Definition II.2.1. Let X and Y be topological vector spaces, $U \subseteq X$ open and $f\colon U \to Y$ a map. Then the *derivative of* f *at* x *in the direction of* h is defined as

$$df(x)(h) := \lim_{t\to 0} \frac{1}{t}\big(f(x+th) - f(x)\big)$$

whenever the limit exists. The function f is called *differentiable at* x if $df(x)(h)$ exists for all $h \in X$. It is called *continuously differentiable* if it is differentiable at all points of U and

$$df\colon U \times X \to Y, \quad (x,h) \mapsto df(x)(h)$$

is a continuous map. It is called a C^1*-map* if it is continuous and continuously differentiable; for $n \geq 2$, a C^n*-map* if df is a C^{n-1}-map, and C^∞ (or *smooth*) if it is C^n for each $n \in \mathbb{N}$. This is the notion of differentiability used in [Mil83], [Ha82], [Gl02a] and [Ne01].

(b) If X and Y are complex vector spaces, then the map f is called *holomorphic* if it is C^1 and for all $x \in U$ the map $df(x)\colon X \to Y$ is complex linear (cf. [Mil83, p. 1027]). We will see below that the maps $df(x)$ are always real linear (Lemma II.2.3).

(c) Higher derivatives are defined for C^n-maps by

$$d^n f(x)(h_1, \ldots, h_n) := \lim_{t\to 0} \frac{1}{t}\big(d^{n-1}f(x+th_n)(h_1,\ldots,h_{n-1}) - d^{n-1}f(x)(h_1,\ldots,h_{n-1})\big). \qquad \blacksquare$$

Remark II.2.2. (a) If X and Y are Banach spaces, then the notion of continuous differentiability is weaker than the usual notion of continuous Fréchet-differentiability in Banach spaces, which requires that the map $x \mapsto df(x)$ is continuous with respect to the operator norm. Nevertheless, one can show that a C^2-map in the sense defined above is C^1 in the sense of Fréchet differentiability, so that the two concepts lead to the same class of C^∞-functions (cf. [Ne01, I.6 and I.7]).

(b) We also note that the existence of linear maps which are not continuous shows that the continuity of f does not follow from the differentiability of f because each linear map $f\colon X \to Y$ is differentiable at each $x \in X$ in the sense of Definition II.2.1(a). $\blacksquare$

Now we recall the precise statements of the most fundamental facts on calculus on locally convex spaces needed in the following.

Lemma II.2.3. *Let X and Y be locally convex spaces, $U \subseteq X$ an open subset, and $f\colon U \to Y$ a continuously differentiable function.*

(i) *For any $x \in U$, the map $df(x)\colon X \to Y$ is real linear and continuous.*

(ii) (Fundamental Theorem of Calculus) *If* $x+[0,1]h \subseteq U$, *then*

$$f(x+h) = f(x) + \int_0^1 df(x+th)(h)\,dt.$$

In particular, f *is locally constant if and only if* $df = 0$.

(iii) f *is continuous.*

(iv) *If* f *is* C^n, $n \geq 2$, *then the functions* $(h_1,\ldots,h_n) \mapsto d^n f(x)(h_1,\ldots,h_n)$, $x \in U$, *are symmetric* n*-linear maps.*

(v) *If* $x+[0,1]h \subseteq U$, *then we have the Taylor Formula*

$$f(x+h) = f(x) + df(x)(h) + \ldots + \frac{1}{(n-1)!} d^{n-1} f(x)(h,\ldots,h) + \frac{1}{(n-1)!}\int_0^1 (1-t)^{n-1} d^n f(x+th)(h,\ldots,h)\,dt.$$

Proof. (i) For each linear functional $\lambda \in Y'$ and $h_1, h_2 \in X$, the map

$$F(t_1,t_2) := \lambda(f(x+t_1h_1+t_2h_2))$$

is defined on an open 0-neighborhood in $\mathbb{R}^2$ and has continuous partial derivatives

$$\frac{\partial F}{\partial t_1}(t_1,t_2) = df(x+t_1h_1+t_2h_2)(h_1), \quad \frac{\partial F}{\partial t_2}(t_1,t_2) = df(x+t_1h_1+t_2h_2)(h_2).$$

From finite-dimensional calculus we know that F is a C^1-map and $dF(0,0)\colon \mathbb{R}^2 \to \mathbb{R}$ is linear. This implies that $\lambda \circ df(x)$ is linear on $\operatorname{span}\{h_1,h_2\}$. Since E' separates the points of Y and h_1, h_2 are arbitrary, the map $df(x)$ is real linear. Its continuity follows from the continuity of df.

(ii) We consider for $\lambda \in Y'$ the C^1-map

$$F\colon I \to \mathbb{R}, \quad F(t) := \lambda(f(x+th))$$

and obtain from the Fundamental Theorem in one variable calculus

$$\lambda(f(x+h) - f(x)) = F(1) - F(0) = \int_0^1 F'(t)\,dt = \int_0^1 \lambda(df(x+th)(h))\,dt.$$

Since Y' separates the points of Y, this implies that the weak integral $\int_0^1 df(x+th)(h)\,dt$, which a priori exists only in the completion of Y, actually defines an element of Y which coincides with $f(x+h)-f(x)$.

(iii) Let p be a continuous seminorm on Y and $\varepsilon > 0$. Then there exists a balanced 0-neighborhood $U_1 \subseteq X$ with $x + U_1 \subseteq U$ and $p\big(df(x+th)(h)\big) < \varepsilon$ for $t \in [0,1]$ and $h \in U_1$. Hence

$$p\big(f(x+h)-f(x)\big) \leq \int_0^1 p\big(df(x+th)(h)\big)\,dt \leq \varepsilon$$

(Exercise II.12), and thus f is continuous.

(iv) Arguing as in (i), we may w.l.o.g. assume that $Y = \mathbb{R}$. That the maps $d^n f(x)$ are symmetric and n-linear follows by considering maps of the form

$$(t_1,\ldots,t_n) \to f(x+t_1h_1+\ldots+t_nh_n)$$

on open 0-neighborhood in $\mathbb{R}^n$ and then applying the corresponding finite-dimensional result.

(v) We consider the C^n-map

$$F\colon I = [0,1] \to \mathbb{R}, \quad F(t) := f(x+th) \quad \text{with} \quad F^{(n)}(t) = d^n f(x+th)(h,\ldots,h)$$

and apply the Taylor Formula for C^n-functions $I \to \mathbb{R}$. ∎

The following characterization of C^1-functions is particularly convenient for the proof of the Chain Rule.

Proposition II.2.4. *Let X and Y be locally convex spaces, $U \subseteq X$ an open subset and $f: U \to Y$ a map. Then*

$$U^{[1]} := \{(x,h,t) \in U \times X \times \mathbb{K}: x + th \in U\}$$

is an open subset of $U \times X \times \mathbb{K}$ and f is C^1 if and only if there exists a continuous function $f^{[1]}: U^{[1]} \to Y$ with

$$f^{[1]}(x,h,t) := \frac{1}{t}(f(x+th) - f(x)) \quad \text{for} \quad t \neq 0.$$

If this is the case, then

$$df(x)(h) = f^{[1]}(x,h,0).$$

Proof. The openness of $U^{[1]}$ follows from the continuity of the map $U \times X \times \mathbb{K} \to X, (x,h,t) \mapsto x + th$, because $U^{[1]}$ is the inverse image of U under this map.

If a continuous function $f^{[1]}$ exists with the required properties, then clearly $df(x)(h) = f^{[1]}(x,h,0)$, which implies that f is a C^1-function.

Suppose, conversely, that f is C^1. Since U is open, there exists for each $x \in U$ a convex balanced 0-neighborhood $V \subseteq X$ with $x + V \subseteq U$. For $y, th \in \frac{1}{2}V$, we then have $y + [0,1]th \subseteq U$, so that Lemma II.2.3(ii) implies that

$$\frac{1}{t}(f(y+th) - f(y)) = \int_0^1 df(y+sth)(h)\, ds.$$

Since the right hand side defines a continuous function on the neighborhood

$$\{(y,h,t) \in U^{[1]}: y + [0,1]th \subseteq U\}$$

of $U \times X \times \{0\}$, we see that

$$f^{[1]}(x,h,t) := \begin{cases} \int_0^1 df(y+sth)(h)\, ds & \text{if } x + [0,1]th \subseteq U \\ \frac{1}{t}(f(x+th) - f(x)) & \text{otherwise} \end{cases}$$

is a continuous function on $U^{[1]}$ satisfying all requirements. ∎

Proposition II.2.5. (Chain Rule) *If X, Y and Z are locally convex spaces, $U \subseteq X$ and $V \subseteq Y$ are open, and $f_1: U \to V$, $f_2: V \to Z$ are C^1, then $f_2 \circ f_1: U \to Z$ is C^1 with*

$$d(f_2 \circ f_1)(x) = df_2\big(f_1(x)\big) \circ df_1(x) \quad \text{for} \quad x \in U.$$

Proof. We use the characterization of C^1-function from Proposition II.2.4. For $(x,h,t) \in U^{[1]}$, we have

$$\begin{aligned} \frac{1}{t}\big((f_2 \circ f_1)(x+th) - (f_2 \circ f_1)(x)\big) &= \frac{1}{t}\big(f_2(f_1(x) + tf_1^{[1]}(x,h,t)) - f_2(f_1(x))\big) \\ &= f_2^{[1]}(f_1(x), f_1^{[1]}(x,h,t), t). \end{aligned}$$

Since this is a continuous function on $U^{[1]}$, Proposition II.2.4 implies that $f_2 \circ f_1$ is C^1. For $t = 0$, we obtain in particular

$$d(f_2 \circ f_1)(x)(h) = f_2^{[1]}(f_1(x), f_1^{[1]}(x,h,0), 0) = df_2(f_1(x))(df_1(x)(h)).$$ ∎

Proposition II.2.6. *If X_1, X_2 and Y are locally convex spaces, $X = X_1 \times X_2$, $U \subseteq X$ is open, and $f: U \to Y$ is continuous, then the partial derivatives*

$$d_1 f(x_1, x_2)(h) := \lim_{t \to 0} \frac{1}{t}\big(f(x_1 + th, x_2) - f(x_1, x_2)\big)$$

and

$$d_2 f(x_1,x_2)(h) := \lim_{t\to 0} \frac{1}{t}\big(f(x_1,x_2+th) - f(x_1,x_2)\big)$$

exist and are continuous if and only if f *is* C^1. *In this case, we have*

$$df(x_1,x_2)(h_1,h_2) = d_1 f(x_1,x_2)(h_1) + d_2 f(x_1,x_2)(h_2).$$

Proof. If f is C^1, then the existence and continuity of the partial derivatives $d_1 f$ and $d_2 f$ follows by restricting df.

Suppose, conversely, that the partial derivatives df_1 and df_2 exist and that they are continuous, so that they are also linear in the last argument (Lemma III.2.3). For

$$(x_1,x_2) + ([0,1]h_1,[0,1]h_2) \subseteq U,$$

we then have

$$\begin{aligned} & f(x_1+th_1,x_2+th_2) - f(x_1,x_2) \\ &= f(x_1+th_1,x_2+th_2) - f(x_1+th_1,x_2) + f(x_1+th_1,x_2) - f(x_1,x_2) \\ &= \int_0^1 df_2(x_1+th_1,x_2+sth_2)(th_2)\,ds + \int_0^1 d_1 f(x_1+sth_1,x_2)(th_1)\,ds \\ &= t\Big(\int_0^1 df_2(x_1+th_1,x_2+sth_2)(h_2)\,ds + \int_0^1 d_1 f(x_1+sth_1,x_2)(h_1)\,ds\Big). \end{aligned}$$

Using the continuous dependence of integrals on parameters (Exercise II.12(c)), we conclude that all directional derivatives of f exist and equal

$$\begin{aligned} df(x_1,x_2)(h_1,h_2) &= \int_0^1 df_2(x_1,x_2)(h_2)\,ds + \int_0^1 d_1 f(x_1,x_2)(h_1)\,ds \\ &= d_2 f(x_1,x_2)(h_2) + d_1 f(x_1,x_2)(h_1). \end{aligned}$$

■

Remark II.2.7. (a) If $f\colon X \to Y$ is a continuous linear map, then f is smooth with

$$df(x)(h) = f(h)$$

for all $x,h \in X$, and $d^n f = 0$ for $n \geq 2$.

(b) From (a) and Proposition II.2.6 it follows that a continuous k-linear map

$$m\colon X_1 \times \ldots \times X_k \to Y$$

is continuously differentiable with

$$dm(x)(h_1,\ldots,h_k) = m(h_1,x_2,\ldots,x_k) + \cdots + m(x_1,\ldots,x_{k-1},h_k).$$

Inductively one obtains that m is smooth with $d^{k+1}m = 0$ (cf. Exercise II.21).

(c) The addition map $a\colon X \times X \to X$ of a topological vector space is smooth. In fact, we have

$$da(x,y)(v,w) = v + w = a(v,w),$$

so that a is a C^1-map. Inductively it follows that a is smooth.

(d) If $f\colon U \to Y$ is C^{n+1}, then Lemma II.2.3(iv) and Proposition II.2.6 imply that

$$\begin{aligned} d(d^n f)(x,h_1,\ldots,h_n)(y,k_1,\ldots,k_n) &= d^{n+1} f(x)(h_1,\ldots,h_n,y) \\ &\quad + d^n f(x)(k_1,h_2,\ldots,h_n) + \ldots + d^n f(x)(h_1,\ldots,h_{n-1},k_n). \end{aligned}$$

It follows in particular that, whenever f is C^n, then f is C^{n+1} if and only if $d^n f$ is C^1.

(e) If $f\colon U \to Y$ is holomorphic, then the finite-dimensional theory shows that for each $h \in X$, the function $U \to Y, x \mapsto df(x)(h)$ is holomorphic. Hence $d^2 f(x)$ is complex bilinear and therefore $d(df)$ is complex linear. Thus $df\colon U \times X \to Y$ is also holomorphic. ■

Example II.2.8. In the definition of C^1-maps, we have not required the underlying topological vector spaces to be locally convex and one may wonder whether this assumption is made for convenience or if there are some serious underlying reasons. The following example shows that local convexity is crucial for the validity of the Fundamental Theorem.

Let V denote the space of measurable functions $f: [0,1] \to \mathbb{R}$ for which

$$|f| := \int_0^1 |f(x)|^{\frac{1}{2}}\, dx$$

is finite and identify functions which coincide on a set whose complement has measure zero. Then $d(f,g) := |f-g|$ defines a metric on this space (Exercise II.3). We thus obtain a metric topological vector space (V,d).

For a subset $E \subseteq [0,1]$, let χ_E denote its characteristic function. Consider the curve

$$\gamma: [0,1] \to V, \quad \gamma(t) := \chi_{[0,t]}.$$

Then

$$|h^{-1}\big(\gamma(t+h) - \gamma(t)\big)| = |h|^{-\frac{1}{2}}|h| \to 0$$

for each $t \in [0,1]$ as $h \to 0$. Hence γ is C^1 with $d\gamma = 0$. Since γ is not constant, the Fundamental Theorem of Calculus does not hold in V.

The defect in this example is caused by the non-local convexity of V. In fact, one can even show that all continuous linear functionals on V vanish. ∎

Remark II.2.9. (a) In the context of Banach spaces, one has an Inverse Function Theorem and also an Implicit Function Theorem ([La99]). Such results cannot be expected in general for Fréchet spaces (cf. the exponential function of $\mathrm{Diff}(\mathbb{S}^1)$). Nevertheless, Glöckner's recent paper [Gl03] contains implicit function theorems for maps of the type $f: E \to F$, where F is a Banach space and E is locally convex.

(b) Another remarkable pathology occurring already for Banach spaces is that a closed subspace F of a Banach space E need not have a closed complement. A simple example is the subspace $F := c_0(\mathbb{N}, \mathbb{R})$ in $E := \ell^\infty(\mathbb{N}, \mathbb{R})$ ([Wer95, Satz IV.6.5]) (cf. Exercise II.13).

This has the consequence that the quotient map $q: E \to E/F$ has no smooth sections because the existence of a smooth local section $\sigma: U \to E$ around $0 \in E/F$ implies the existence of a closed complement $\mathrm{im}(d\sigma(0)) \cong E/F$ to F in E. Nevertheless, the map $q: E \to E/F$ defines the structure of a topological F-principal bundle over E/F which has a continuous global section by Michael's Selection Theorem ([Mi59]). ∎

Remark II.2.10. (Pathologies of linear ODEs in Fréchet spaces) (a) First we give an example of a linear ODE for which solutions to initial value problems exist, but are not unique. We consider the Fréchet space $V := C^\infty([0,1], \mathbb{R})$ and the continuous linear operator $Lf := f'$ on this space. We are asking for solutions of the initial value problem

$$\gamma'(t) = L\gamma(t), \quad \gamma(0) = \gamma_0. \tag{2.2.1}$$

As a consequence of E. Borel's Theorem that each power series is the Taylor series of a smooth function, each γ_0 has a smooth extension to a function on $\mathbb{R}$. Let h be such a function and consider

$$\gamma: \mathbb{R} \to V, \quad \gamma(t)(x) := h(t+x).$$

Then $\gamma(0) = h|_{[0,1]} = \gamma_0$ and $\gamma'(t)(x) = h'(t+x) = (L\gamma(t))(x)$. It is clear that these solutions of (2.2.1) depend on the choice of the extension h of γ_0. Different choices lead to different extensions.

(b) Now we consider the space $V := C^\infty(\mathbb{S}^1, \mathbb{C})$ which we identify with the space of 2π-periodic smooth functions on the real line. We consider the linear operator $Lf := -f''$ and the equation (2.2.1), which in this case is the heat equation with reversed time. It is easy to analyze this equation in terms of the Fourier expansion of γ. So let

$$\gamma(t)(x) = \sum_{n\in\mathbb{Z}} a_n(t)e^{inx}$$

be the Fourier expansion of $\gamma(t)$. Then (2.2.1) implies $a_n'(t) = n^2 a_n(t)$ for each $n \in \mathbb{Z}$, so that $a_n(t) = a_n(0)e^{tn^2}$ holds for any solution γ of (2.2.1). If the Fourier coefficients $a_n(0)$ of γ_0 do not satisfy

$$\sum_n |a_n(0)|e^{\varepsilon n^2} < \infty$$

for some $\varepsilon > 0$ (which need not be the case for a smooth function γ_0), then (2.2.1) does not have a solution on $[0,\varepsilon]$.

As a consequence, the operator $\exp(tL)$ is never defined for $t \neq 0$. Nevertheless, we may use the Fourier series expansion to see that $\beta(t) := (1+it^2)\mathbf{1} + tL$ defines a curve $\beta\colon \mathbb{R} \to \mathrm{GL}(V)$ which is smooth in the sense that

$$\mathbb{R} \times V \to V \times V, \quad (t,v) \mapsto (\beta(t)(v), \beta(t)^{-1}(v))$$

is smooth. We further have $\beta'(0) = L$, so that L arises as the tangent vector of a smooth curve in $\mathrm{GL}(V)$, but not for a one-parameter group. ■

Definition II.2.11. A locally convex space E is said to be *Mackey complete* if for each smooth curve $\xi\colon [0,1] \to E$ there exists a smooth curve $\eta\colon [0,1] \to E$ with $\eta' = \xi$. ■

For a more detailed discussion of Mackey completeness and equivalent conditions we refer to [KM97, Th. 2.14].

Remark II.2.12. If E is a sequentially complete locally convex space, then it is Mackey complete because the sequential completeness implies the existence of Riemann integrals of continuous E-valued functions on compact intervals, hence that for each continuous curve $\xi\colon [0,1] \to E$ there exists a smooth curve $\eta\colon [0,1] \to E$ with $\eta' = \xi$. ■

Remark II.2.13. (a) We briefly recall the basic definitions underlying the convenient calculus in [KM97]. Let E be a locally convex space. The c^∞*-topology* on E is the final topology with respect to the set $C^\infty(\mathbb{R}, E)$. Let $U \subseteq E$ be an open subset and $f\colon U \to F$ a function, where F is a locally convex space. Then we call f *conveniently smooth* if

$$f \circ C^\infty(\mathbb{R}, U) \subseteq C^\infty(\mathbb{R}, F).$$

This implies nice cartesian closedness properties of the class of smooth maps (cf. [KM97, p.30]).

(b) If E is a Fréchet space, then the c^∞-topology coincides with the original topology ([KM97, Th. 4.11]), so that each conveniently smooth map is continuous.

We claim that for an open subset U of a Fréchet space, a map $f\colon U \to F$ is conveniently smooth if and only if it is smooth in the sense of Definition II.2.1. This can be shown as follows. Since $C^\infty(\mathbb{R}, E)$ is the same space for both notions of differentiability, the Chain Rule shows that smoothness in the sense of Definition II.2.1 implies smoothness in the sense of convenient calculus. Now we assume that $f\colon U \to F$ is conveniently smooth. Then the derivative $df\colon U \times E \to F$ exists and defines a conveniently smooth map $df\colon U \to \mathcal{L}(E,F) \subseteq C^\infty(E,F)$ ([KM97, Th. 3.18]). Hence $df\colon U \times E \to F$ is also conveniently smooth, and thus continuous with respect to the c^∞-topology. As $E \times E$ is a Fréchet space, it follows that df is continuous. Therefore f is C^1 in the sense of Definition II.2.1, and now one can iterate the argument. ■

II.3. Differentiable manifolds

Since we have a chain rule for C^1-maps between locally convex spaces, hence also for smooth maps, we can define smooth manifolds as in the finite-dimensional case (cf. [Ha82], [Mil83], [Gl02a], [GN05]):

Definition II.3.1. Let M be a Hausdorff topological space and E a locally convex space. An E-*chart* of an open subset $U \subseteq M$ is a homeomorphism $\varphi: U \to \varphi(U) \subseteq E$ onto an open subset $\varphi(U)$ of E. We denote such a chart as a pair (φ, U). Two charts (φ, U) and (ψ, V) are said to be *smoothly compatible* if the map

$$\psi \circ \varphi^{-1}|_{\varphi(U\cap V)}: \varphi(U \cap V) \to \psi(U \cap V)$$

is smooth. From the chain rule it follows that compatibility of charts is an equivalence relation on the set of all E-charts of M. An E-*atlas of* M is a family $\mathcal{A} := (\varphi_i, U_i)_{i\in I}$ of pairwise compatible E-charts of M for which $\bigcup_i U_i = M$. A *smooth* E-*structure on* M is a maximal E-atlas and a smooth E-manifold is a pair $(M, \mathcal{A})$, where $\mathcal{A}$ is a maximal E-atlas on M.

We call a manifold modeled on a locally convex, resp., Fréchet space, resp., Banach space a *locally convex*, resp., *Fréchet*, resp., *Banach manifold.* ∎

Remark II.3.2. (a) Locally convex spaces are *regular* in the sense that each point has a neighborhood base consisting of closed sets, and this property is inherited by manifolds modeled on these spaces (cf. [Mil83]).

(b) If $M_1, \ldots, M_n$ are smooth manifolds modeled on the spaces E_i, $i = 1, \ldots, n$, then the product set $M := M_1 \times \ldots \times M_n$ carries a natural manifold structure with model space $E = \prod_{i=1}^n E_i$. ∎

Definition II.3.3. (a) One defines the tangent bundle $\pi_{TM}: TM \to M$ as follows. Let $\mathcal{A} := (\varphi_i, U_i)_{i\in I}$ be an E-atlas of M. On the disjoint union of the set $\varphi(U_i) \times E$, we define an equivalence relation by

$$(x, v) \sim \big((\varphi_j \circ \varphi_i^{-1})(x), d(\varphi_j \circ \varphi_i^{-1})(x)(v)\big)$$

for $x \in \varphi_i(U_i \cap U_j)$ and $v \in E$ and write $[x, v]$ for the equivalence class of (x, v). Let $p \in U_i$. Then the equivalence classes of the form $[\varphi_i(p), v]$ are called *tangent vectors in* p. Since all the differentials $d(\varphi_j \circ \varphi_i^{-1})(x)$ are invertible linear maps, it easily follows that the set $T_p(M)$ of all tangent vectors in p forms a vector space isomorphic to E under the map $E \to T_p(M), v \mapsto [x, v]$. Now we turn the *tangent bundle*

$$TM := \bigcup_{p\in M} T_p(M)$$

into a manifold by the charts

$$\psi_i: TU_i := \bigcup_{p\in U_i} T_p(M) \to \varphi(U_i) \times E, \quad [\varphi_i(x), v] \mapsto (\varphi_i(x), v).$$

It is easy to see that for each open subset U of a locally convex space E, we have $TU \cong U \times E$ (as smooth manifolds) and in particular $TU_j \cong U_j \times E$ in the setting from above.

(b) Let M and N be smooth manifolds modeled on locally convex spaces and $f: M \to N$ a smooth map. We write $Tf: TM \to TN$ for the corresponding map induced on the level of tangent vectors. Locally this map is given by

$$Tf(x, h) = \big(f(x), df(x)(h)\big),$$

where $df(p) := T_p(f): T_p(M) \to T_{f(p)}(N)$ denotes the differential of f at p. In view of Remark II.2.7(d), the tangent map Tf is smooth if f is smooth. In the following, we will always identify M with the zero section in TM. In this sense, we have $Tf\,|_M^N = f$. If V is a locally convex space, then $TV \cong V \times V$ and the map Tf can accordingly be written as $Tf = (f, df)$, where we think of df as a map $TM \to V$.

From the relations

$$T(\mathrm{id}_M) = \mathrm{id}_{TM} \quad \text{and} \quad T(f_1 \circ f_2) = Tf_1 \circ Tf_2$$

for smooth maps $f_2: M_1 \to M_2$ and $f_2: M_2 \to M_3$ it follows that T is an endofunctor on the category of smooth manifolds. Moreover, it preserves finite products in the sense that for smooth manifolds $M_1, \ldots, M_n$, there is a natural isomorphism

$$T(M_1 \times \cdots \times M_n) \cong TM_1 \times \cdots \times TM_n.$$

(c) A *(smooth) vector field* X on M is a smooth section of the tangent bundle $q_{TM}: TM \to M$, i.e. a smooth map $X: M \to TM$ with $\pi_{TM} \circ X = \mathrm{id}_M$. We write $\mathcal{V}(M)$ for the space of all vector fields on M. If $f \in C^\infty(M, V)$ is a smooth function on M with values in some locally convex space V and $X \in \mathcal{V}(M)$, then we obtain a smooth function on M via

$$X.f := df \circ X: M \to TM \to V. \qquad \blacksquare$$

Remark II.3.4. If $M = U$ is an open subset of the locally convex space E, then $TU = U \times E$ with the bundle projection $\pi_{TU}: U \times E \to U, (x, v) \mapsto x$. Then each smooth vector field is of the form $X(x) = (x, \widetilde{X}(x))$ for some smooth function $\widetilde{X}: U \to E$, and we may thus identify $\mathcal{V}(U)$ with the space $C^\infty(U, E)$. $\blacksquare$

Remark II.3.5. (a) One can also define for each E-manifold M a cotangent bundle $T^*(M) = \bigcup_{m \in M} T_m(M)'$ and endow it with a vector bundle structure over M, but to endow it with a smooth manifold structure we need a locally convex topology on the dual space E' such that for each local diffeomorphism $f: U \to E$, U open in E, the map $U \times E' \to E', (x, \lambda) \mapsto \lambda \circ df(x)$ is smooth. If E is a Banach space, then the norm topology on E' has this property, and the author of these notes is not aware of any other example where this is the case.

In Section II.4, we shall introduce differential forms directly, without reference to any cotangent bundle.

(b) The following modification might be useful to construct a replacement for a cotangent bundle. Instead of the, mostly badly behaved, duality $E \times E' \to \mathbb{K}$, one may also start with another locally convex space F for which we have a non-degenerate continuous pairing $E \times F \to \mathbb{K}, (e, f) \mapsto \langle e, f \rangle$, so that we may think of F as a subspace of E'. Then we may consider E-manifolds with an atlas for which all coordinate changes

$$f := \psi \circ \varphi^{-1}: \varphi(U \cap V) \to \psi(U \cap V) \subseteq E$$

have the property that for each x, the continuous linear map $df(x): E \to E$ has an adjoint map $df(x)^\top$ on F, satisfying

$$\langle df(x)v, w \rangle = \langle v, df(x)^\top w \rangle \quad \text{for} \quad v \in E, w \in F,$$

and for which the map

$$\varphi(U \cap V) \times F \to \psi(U \cap V) \times F, \quad (x, w) \mapsto (f(x), (df(x)^\top)^{-1} w)$$

is smooth. Then one can use these maps as glueing maps to obtain an F-vector bundle over M which is a subbundle of $T^*(M)$ with a natural differentiable structure. $\blacksquare$

Lemma II.3.6. *If $X, Y \in \mathcal{V}(M)$, then there exists a vector field $[X,Y] \in \mathcal{V}(M)$ which is uniquely determined by the property that on each open subset $U \subseteq M$ we have*

$$[X,Y].f = X.(Y.f) - Y.(X.f) \tag{2.3.1}$$

for all $f \in C^\infty(U, \mathbb{R})$.

Proof. Locally the vector fields X and Y are given as $X(p) = \big(p, \widetilde{X}(p)\big)$ and $Y(p) = \big(p, \widetilde{Y}(p)\big)$. We define a vector field by

$$[X,Y]\widetilde{\ }(p) := d\widetilde{Y}(p)\big(\widetilde{X}(p)\big) - d\widetilde{X}(p)\big(\widetilde{Y}(p)\big). \tag{2.3.2}$$

Then the smoothness of the right hand side follows from the chain rule. The requirement that (2.3.1) holds on continuous linear functionals f determines $[X,Y]\widetilde{\ }$ uniquely. Clearly, (2.3.2) defines a smooth vector field on M. Now the assertion follows because locally (2.3.1) is a consequence of the Chain Rule (Proposition II.2.5). ∎

Proposition II.3.7. $(\mathcal{V}(M), [\cdot,\cdot])$ *is a Lie algebra.*

Proof. The crucial part is to check the Jacobi identity. This follows from the observation that if U is an open subset of a locally convex space, then the mapping

$$\Phi\colon \mathcal{V}(U) \to \mathrm{der}\big(C^\infty(U,\mathbb{R})\big), \quad \Phi(X)(f) = X.f$$

is injective and satisfies $\Phi([X,Y]) = [\Phi(X), \Phi(Y)]$ (Exercise II.17). Therefore the Jacobi identity in $\mathcal{V}(U)$ follows from the Jacobi identity in the associative algebra $\mathrm{End}\big(C^\infty(U,\mathbb{R})\big)$. ∎

For the applications to Lie groups we will need the following lemma.

Lemma II.3.8. *Let M and N be smooth manifolds and $\varphi\colon M \to N$ a smooth map. Suppose that $X_N, Y_N \in \mathcal{V}(N)$ and $X_M, Y_M \in \mathcal{V}(M)$ are φ-related in the sense that $X_N \circ \varphi = T\varphi \circ X_M$ and $Y_N \circ \varphi = T\varphi \circ Y_M$. Then $[X_N, Y_N] \circ \varphi = T\varphi \circ [X_M, Y_M]$.*

Proof. It suffices to perform a local calculation. Therefore we may w.l.o.g. assume that $M \subseteq F$ is open, where F is a locally convex space and that N is a locally convex space. Then

$$[X_N, Y_N]\widetilde{\ }\big(\varphi(p)\big) = d\widetilde{Y}_N\big(\varphi(p)\big).\widetilde{X}_N\big(\varphi(p)\big) - d\widetilde{X}_N\big(\varphi(p)\big).\widetilde{Y}_N\big(\varphi(p)\big).$$

Next we note that our assumption implies that $\widetilde{Y}_N \circ \varphi = d\varphi \circ (\mathrm{id}_F \times \widetilde{Y}_M)$. Using the Chain Rule we obtain

$$d\widetilde{Y}_N\big(\varphi(p)\big)d\varphi(p) = d(d\varphi)\big(p, \widetilde{Y}_M(p)\big) \circ \big(\mathrm{id}_F, d\widetilde{Y}_M(p)\big)$$

which, in view of Remark II.2.7(d), leads to

$$\begin{aligned} d\widetilde{Y}_N\big(\varphi(p)\big).\widetilde{X}_N\big(\varphi(p)\big) &= d\widetilde{Y}_N\big(\varphi(p)\big)d\varphi(p).\widetilde{X}_M(p) \\ &= d(d\varphi)\big(p, \widetilde{Y}_M(p)\big) \circ \big(\mathrm{id}_F, d\widetilde{Y}_M(p)\big).\widetilde{X}_M(p) \\ &= d^2\varphi(p)\big(\widetilde{Y}_M(p), \widetilde{X}_M(p)\big) + d\varphi(p)\big(d\widetilde{Y}_M(p).\widetilde{X}_M(p)\big). \end{aligned}$$

Now the symmetry of the second derivative (Lemma II.2.3(iv)) implies that

$$[X_N, Y_N]\widetilde{\ }\big(\varphi(p)\big) = d\varphi(p)\big(d\widetilde{Y}_M(p).\widetilde{X}_M(p) - d\widetilde{X}_M(p).\widetilde{Y}_M(p)\big) = d\varphi(p)\big([X_M, Y_M]\widetilde{\ }(p)\big).$$ ∎

II.4. Differential forms

Differential forms play a significant role throughout infinite-dimensional Lie theory; either as differential forms on Lie groups or as differential forms on manifolds on which certain Lie groups act. In the present section, we describe a natural approach to differential forms on manifolds modeled on locally convex spaces. The main difference to the finite-dimensional case is that in local charts there is no natural coordinate description of differential forms and that for general locally convex manifolds (not even for all Banach manifolds), smooth partitions of unity are available, so that one has to be careful with localization arguments.

We have already seen that for each smooth manifold M, the space $\mathcal{V}(M)$ of smooth vector fields on M carries a natural Lie algebra structure. We shall see below that each smooth p-form $\omega \in \Omega^p(M,V)$ with values in a locally convex space V defines an alternating p-linear map

$$\mathcal{V}(M)^p \to C^\infty(M,V), \quad (X_1,\ldots,X_p) \mapsto \omega(X_1,\ldots,X_p).$$

If M has the property that each tangent vector extends to a smooth vector field, which is always the case locally, then this leads to an inclusion of $\Omega^p(M,V)$ into the space of Lie algebra cochains for $\mathcal{V}(M)$ with values in the $\mathcal{V}(M)$-module $C^\infty(M,V)$. We shall define the exterior derivative on differential forms in such a way that with respect to this identification, it corresponds to the Lie algebra differential (Appendix C). This point of view will prove very useful, and in this section we use it to derive geometric structures such as the Lie derivative and the exterior differential from the abstract setting of Lie algebra cochains.

Definition II.4.1. (a) If M is a differentiable manifold and V a locally convex space, then a *V-valued p-form* ω on M is a function ω which associates to each $x \in M$ a k-linear alternating map $\omega_x = \omega(x)\colon T_x(M)^p \to V$ such that in local coordinates the map $(x,v_1,\ldots,v_p) \mapsto \omega_x(v_1,\ldots,v_p)$ is smooth. We write $\Omega^p(M,V)$ for the space of smooth V-valued p-forms on M with values in V and identify $\Omega^0(M,V)$ with the space $C^\infty(M,V)$ of smooth V-valued functions on M.

(b) Let V_1,V_2,V_3 be locally convex spaces and $\beta\colon V_1 \times V_2 \to V_3$ be a continuous bilinear map. Then the wedge product

$$\Omega^p(M,V_1) \times \Omega^q(M,V_2) \to \Omega^{p+q}(M,V_3), \quad (\omega,\eta) \mapsto \omega \wedge \eta$$

is defined by $(\omega \wedge \eta)_x := \omega_x \wedge \eta_x$, where

$$(\omega_x \wedge \eta_x)(v_1,\ldots,v_{p+q}) := \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \mathrm{sgn}(\sigma)\beta\big(\omega_x(v_{\sigma(1)},\ldots,v_{\sigma(p)}),\eta_x(v_{\sigma(p+1)},\ldots,v_{\sigma(p+q)})\big).$$

For $p = q = 1$, we have in particular

$$(\omega \wedge \eta)_x(v_1,v_2) = \beta(\omega_x(v_1),\eta_x(v_2)) - \beta(\omega_x(v_2),\eta_x(v_1)).$$

Important special cases where such wedge products are used are:

(1) $\beta\colon \mathbb{R} \times V \to V$ is the scalar multiplication of V.

(2) $\beta\colon A \times A \to A$ is the multiplication of an associative algebra.

(3) $\beta\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the Lie bracket of a Lie algebra. In this case, we also write $[\omega,\eta] := \omega \wedge \eta$.

(c) The pull-back $\varphi^*\omega$ of $\omega \in \Omega^p(M,V)$ with respect to a smooth map $\varphi\colon N \to M$ is the smooth p-form in $\Omega^p(N,V)$ defined by

$$(\varphi^*\omega)_x(v_1,\ldots,v_p) := \omega_{\varphi(x)}(d\varphi(x)v_1,\ldots,d\varphi(x)v_p) = \omega_{\varphi(x)}(T_x(\varphi)v_1,\ldots,T_x(\varphi)v_p).$$

Note that the chain rule implies that

$$\mathrm{id}_M^*\,\omega = \omega \quad \text{and} \quad \varphi_1^*(\varphi_2^*\omega) = (\varphi_2 \circ \varphi_1)^*\omega \tag{2.4.1}$$

holds for compositions of smooth maps. Moreover,

$$\varphi^*(\omega \wedge \eta) = \varphi^*\omega \wedge \varphi^*\eta \tag{2.4.2}$$

follows directly from the definitions. For $f = \omega \in \Omega^0(M,V)$, we simply have $\varphi^* f = f \circ \varphi$. ■

The definition of the exterior differential

$$d\colon \Omega^p(M,V) \to \Omega^{p+1}(M,V)$$

is a bit more subtle than in finite dimensions where one usually uses local coordinates to define it in charts.

Proposition II.4.2. *For $\omega \in \Omega^p(M,V)$, $x \in M$ and $v_0,\ldots,v_p \in T_x(M)$, we choose smooth vector fields X_i defined on a neighborhood of x satisfying $X_i(x) = v_i$. Then*

$$(d\omega)_x(v_0,\ldots,v_p) := \sum_{i=0}^{p}(-1)^i\big(X_i.\omega(X_0,\ldots,\widehat{X}_i,\ldots,X_p)\big)(x) + \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_0,\ldots,\widehat{X}_i,\ldots,\widehat{X}_j,\ldots,X_p)(x) \tag{2.4.3}$$

does not depend on the choice of the vector fields X_i and defines a smooth $(p+1)$-form $d\omega \in \Omega^{p+1}(M,V)$.

The definition of the differential is designed in such a way that for $X_0,\ldots,X_p \in \mathcal{V}(M)$ we have in $C^\infty(M,V)$ the identity

$$(d\omega)(X_0,\ldots,X_p) := \sum_{i=0}^{p}(-1)^i X_i.\omega(X_0,\ldots,\widehat{X}_i,\ldots,X_p) + \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_0,\ldots,\widehat{X}_i,\ldots,\widehat{X}_j,\ldots,X_p). \tag{2.4.4}$$

Proof. We have to verify that the right hand side of (2.4.3) does not depend on the choice of the vector fields X_k and that it is alternating in the v_k. First we show that $d\omega$ does not depend on the choice of the vector fields X_k, which amounts to showing that if one vector field X_k vanishes in x, then the right hand side of (2.4.3) vanishes.

Suppose that $X_k(x) = 0$. Then the only terms not obviously vanishing in x are

$$\sum_{i\neq k}^{p}(-1)^i\big(X_i.\omega(X_0,\ldots,\widehat{X}_i,\ldots,X_p)\big)(x), \tag{2.4.5}$$

$$\sum_{i<k}(-1)^{i+k}\omega([X_i,X_k],X_0,\ldots,\widehat{X}_i,\ldots,\widehat{X}_k,\ldots,X_p)(x), \tag{2.4.6}$$

and

$$\sum_{k<i}(-1)^{i+k}\omega([X_k,X_i],X_0,\ldots,\widehat{X}_k,\ldots,\widehat{X}_i,\ldots,X_p)(x). \tag{2.4.7}$$

In local coordinates, we have

$$\begin{aligned}
&\big(X_i.\omega(X_0,\ldots,\widehat{X}_i,\ldots,X_p)\big)(x)\\
&= (d_1\omega)(x,X_i(x))(X_1(x),\ldots,\widehat{X}_i(x),\ldots,X_p(x))\\
&\qquad + \sum_{j<i}\omega_x(X_0(x),\ldots,dX_j(x)X_i(x),\ldots,\widehat{X}_i(x),\ldots,X_p(x))\\
&\qquad + \sum_{j>i}\omega_x(X_0(x),\ldots,\widehat{X}_i(x),\ldots,dX_j(x)X_i(x),\ldots,X_p(x)).
\end{aligned}$$

For a fixed $i > k$, the assumption $X_k(x) = 0$ implies that only the term

$$\omega_x(X_0(x), \ldots, dX_k(x)X_i(x), \ldots, \widehat{X}_i(x), \ldots, X_p(x))$$

contributes. In view of $X_k(x) = 0$, we have

$$dX_k(x)X_i(x) = dX_k(x)X_i(x) - dX_i(x)X_k(x) = [X_i, X_k](x).$$

This leads to

$$\begin{aligned}&(-1)^k\omega([X_k, X_i], X_0, \ldots, \widehat{X}_k, \ldots, \widehat{X}_i, \ldots, X_p)(x)\\ &= -\omega_x(X_0(x), \ldots, dX_k(x)X_i(x), \ldots, \widehat{X}_i(x), \ldots, X_p(x)),\end{aligned}$$

so that corresponding terms in (2.4.5) and (2.4.7) cancel, and the same happens for $i < k$ for terms in (2.4.5) and (2.4.6). This proves that $d\omega$ is independent of the choice of the vector fields X_i.

To see that we obtain a smooth $(p+1)$-form, we use local coordinates and choose the vector fields X_i as constant vector fields. Then

$$(d\omega)_x(v_0, \ldots, v_p) = \sum_{i=0}^{p}(-1)^i(d_1\omega)(x, v_i)(v_0, \ldots, \widehat{v}_i, \ldots, v_p) \tag{2.4.8}$$

is a smooth function of $(x, v_0, \ldots, v_p)$.

It remains to show that $d\omega$ is alternating. If $v_i = v_j$ for some $i < j$, then the argument above shows that we may assume that $X_i = X_j$. Since ω is alternating, it suffices to observe that

$$\begin{aligned}&(d\omega)_x(v_0, v_1, \ldots, v_p)\\ &= (-1)^i(d_1\omega)(x, v_i)(v_0, \ldots, \widehat{v}_i, \ldots, v_p) + (-1)^j(d_1\omega)(x, v_j)(v_0, \ldots, \widehat{v}_j, \ldots, v_p)\\ &= (-1)^i(d_1\omega)(x, v_i)(v_0, \ldots, \widehat{v}_i, \ldots, v_p) + (-1)^{i+1}(d_1\omega)(x, v_i)(v_0, \ldots, \widehat{v}_i, \ldots, v_p) = 0.\end{aligned}$$

■

Proposition II.4.3. *For each $\omega \in \Omega^p(M, V)$, we have $d^2\omega = 0$.*

Proof. It clearly suffices to verify this for the case where M is an open subset of a locally convex space E.

Each p-form $\omega \in \Omega^p(M, V)$ defines a p-linear map $\omega_{\mathfrak{g}} \colon \mathcal{V}(M)^p \to C^\infty(M, V)$. In this sense, we may consider $\omega_{\mathfrak{g}}$ as a p-cochain for the Lie algebra $\mathfrak{g} := \mathcal{V}(M)$ with values in the $\mathcal{V}(M)$-module $C^\infty(M, V)$, where the module structure is the natural one given by $(X.f)(x) := df(x)X(x)$. The map $\omega \mapsto \omega_{\mathfrak{g}}$ is injective, as we see by evaluating p-forms on constant vector fields. Moreover, the definition of d implies that $d_{\mathfrak{g}}\omega_{\mathfrak{g}} = (d\omega)_{\mathfrak{g}}$. Now $(d^2\omega)_{\mathfrak{g}} = d_{\mathfrak{g}}^2\omega_{\mathfrak{g}} = 0$ implies that $d^2\omega = 0$ (Appendix C). ■

Remark II.4.4. Another way to verify that $d^2\omega = 0$ is to calculate directly in local coordinates using formula (2.4.8). Then $d^2\omega = 0$ easily follows from the symmetry of second derivatives of ω (Lemma II.2.3(iv)) (Exercise II.10). ■

Definition II.4.5. Extending d to a linear map on the space $\Omega(M, V) := \bigoplus_{p\in\mathbb{N}_0}\Omega^p(M, V)$ of all V-valued differential forms on M, the relation $d^2 = 0$ implies that the space

$$Z^p_{\mathrm{dR}}(M, V) := \ker(d|_{\Omega^p(M,V)})$$

of *closed forms* contains the space $B^p_{\mathrm{dR}}(M, V) := d(\Omega^{p-1}(M, V))$ of exact forms, so that the *V-valued de Rham cohomology space*

$$H^p_{\mathrm{dR}}(M, V) := Z^p_{\mathrm{dR}}(M, V)/B^p_{\mathrm{dR}}(M, V)$$

is well-defined. ■

Remark II.4.6. We consider smooth functions $f\colon M \to V$ as differential forms of degree 0. Then df is the 1-form with $df(x)(v) = T_x(f)v$, where df is the differential of f, as defined above. Since M is locally convex, the vanishing of df means that the function f is locally constant (Lemma II.2.3(ii)). Thus $H^0_{\rm dR}(M,V) = Z^0_{\rm dR}(M,V)$ is the space of locally constant functions on M. If M has d connected components, then $H^0_{\rm dR}(M,V) \cong V^d$. ■

Lemma II.4.7. *If $\varphi\colon N \to M$ is a smooth map and $\omega \in \Omega^p(M,V)$, then $d(\varphi^*\omega) = \varphi^* d\omega$.*

Proof. First we assume that φ is a diffeomorphism. Let $X_0,\ldots,X_p \in \mathcal{V}(N)$ and define $Y_0,\ldots,Y_p \in \mathcal{V}(M)$ by $Y_i(\varphi(x)) := d\varphi(x)(X_i(x))$, so that $Y_i \circ \varphi = T\varphi \circ X_i$. In view of Lemma II.3.6, this implies that $[Y_i,Y_j]\circ\varphi = T\varphi\circ[X_i,X_j]$ for $i,j = 0,\ldots,p$. Moreover, we have

$$\varphi^*(\omega(Y_0,\ldots,\widehat{Y}_i,\ldots,Y_p)) = (\varphi^*\omega)(X_0,\ldots,\widehat{X}_i,\ldots,X_p).$$

We further have for each smooth function f on M the relation

$$\varphi^*(Y_i.f)(x) = df(\varphi(x))Y_i(\varphi(x)) = df(\varphi(x))d\varphi(x)X_i(x) = (X_i.(\varphi^*f))(x),$$

so that we obtain with (2.4.3)

$$\varphi^*(d\omega)(X_0,\ldots,X_p) = d(\varphi^*\omega)(X_0,\ldots,X_p).$$

Since this relation also holds on each open subset of M, resp., N, we conclude that $d(\varphi^*\omega) = \varphi^*(d\omega)$. The preceding argument applies in particular to local diffeomorphisms defined by charts.

To complete the proof of the general case, we may now assume w.l.o.g. that M and N are open subsets of locally convex spaces. Using constant vector fields, we then have

$$(d\omega)_x(v_0,\ldots,v_p) = \sum_{i=0}^p (-1)^i (d_1\omega(x,v_i))(v_0,\ldots,\widehat{v}_i,\ldots,v_p)$$

and therefore

$$(\varphi^*(d\omega))_x(v_0,\ldots,v_p) = \sum_{i=0}^p (-1)^i (d_1\omega(\varphi(x), d\varphi(x)v_i))(d\varphi(x)v_0,\ldots,d\varphi(x)\widehat{v}_i,\ldots,d\varphi(x)v_p).$$

On the other hand, the Chain Rule leads to

$$\begin{aligned}
&d(\varphi^*\omega)_x(v_0,\ldots,v_p)\\
&= \sum_{i=0}^p (-1)^i (d_1\omega(\varphi(x), d\varphi(x)v_i))(d\varphi(x)v_0,\ldots,d\varphi(x)\widehat{v}_i,\ldots,d\varphi(x)v_p)\\
&\quad + \sum_{i=0}^p (-1)^i \sum_{j<i} \omega_{\varphi(x)}(d\varphi(x)v_0,\ldots,d^2\varphi(x)(v_i,v_j),\ldots,d\varphi(x)\widehat{v}_i,\ldots,d\varphi(x)v_p)\\
&\quad + \sum_{i=0}^p (-1)^i \sum_{j>i} \omega_{\varphi(x)}(d\varphi(x)v_0,\ldots,d\varphi(x)\widehat{v}_i,\ldots,d^2\varphi(x)(v_i,v_j),\ldots,d\varphi(x)v_p)\\
&= \sum_{i=0}^p (-1)^i (d_1\omega(\varphi(x), d\varphi(x)v_i))(d\varphi(x)v_0,\ldots,d\varphi(x)\widehat{v}_i,\ldots,d\varphi(x)v_p),
\end{aligned}$$

where the terms in the last two lines cancel because of the symmetry of the bilinear maps $d^2\varphi(x)$ (Lemma II.2.3(iv)). This proves the assertion. ■

For finite-dimensional manifolds, one usually defines the Lie derivative of a differential form in the direction of a vector field X by using its local flow $t \mapsto \mathrm{Fl}^t_X$:

$$\mathcal{L}_X\omega := \frac{d}{dt}\Big|_{t=0} (\mathrm{Fl}^{-t}_X)^*\omega.$$

Since vector fields on infinite-dimensional manifold need not have a local flow (cf. Remark II.2.10), we introduce the Lie derivative more directly, resembling its algebraic counterpart (cf. Appendix C).

Definition II.4.8. (a) For any smooth manifold M and each locally convex space, we have a natural representation of the Lie algebra $\mathcal{V}(M)$ on the space $\Omega^p(M,V)$ of V-valued p-forms on M, given by the *Lie derivative.* For $Y \in \mathcal{V}(M)$, the Lie derivative $\mathcal{L}_Y\omega$ is defined on $v_1,\ldots,v_p \in T_x(M)$ by

$$\begin{aligned}
&(\mathcal{L}_Y.\omega)_x(v_1,\ldots,v_p)\\
&= (Y.\omega(X_1,\ldots,X_p))(x) - \sum_{j=1}^p \omega(X_1,\ldots,[Y,X_j],\ldots,X_p)(x)\\
&= (Y.\omega(X_1,\ldots,X_p))(x) + \sum_{j=1}^p (-1)^j \omega([Y,X_j],X_1,\ldots,\widehat{X}_j,\ldots,X_p)(x),
\end{aligned}$$

where $X_1,\ldots,X_p$ are vector fields on a neighborhood of x satisfying $X_i(x) = v_i$. To see that the right hand side does not depend on the choice of the vector fields X_i, suppose that $X_i(x) = 0$ for some i. Then evaluation of the right hand side in x yields in local coordinates

$$\begin{aligned}
&(Y.\omega(X_1,\ldots,X_p))(x) - \omega(X_1,\ldots,[Y,X_i],\ldots,X_p)(x)\\
&= \omega_x(X_1(x),\ldots,dX_i(x)Y(x),\ldots,X_p(x))\\
&\quad - \omega_x(X_1(x),\ldots,dX_i(x)Y(x) - dY(x)X_i(x),\ldots,X_p(x)) = 0.
\end{aligned}$$

Therefore $\mathcal{L}_Y\omega$ is well-defined. In local coordinates, we have

$$\begin{aligned}
(\mathcal{L}_Y\omega)_x(v_1,\ldots,v_p) &= (Y.\omega(v_1,\ldots,v_p))(x) + \sum_{j=1}^p \omega(v_1,\ldots,dY(x)v_i,\ldots,v_p)\\
&= (d_1\omega)(x,Y(x))(v_1,\ldots,v_p) + \sum_{j=1}^p \omega(v_1,\ldots,dY(x)v_i,\ldots,v_p),
\end{aligned}$$

which immediately implies that $\mathcal{L}_Y\omega$ defines a smooth V-valued p-form on M.

(b) We further obtain for each $X \in \mathcal{V}(M)$ and $p \geq 1$ a linear map

$$i_X\colon \Omega^p(M,V) \to \Omega^{p-1}(M,V) \quad \text{with} \quad (i_X\omega)_x = i_{X(x)}\omega_x,$$

where

$$(i_v\omega_x)(v_1,\ldots,v_{p-1}) := \omega_x(v,v_1,\ldots,v_{p-1}).$$

For $\omega \in \Omega^0(M,V) = C^\infty(M,V)$, we put $i_X\omega := 0$. ■

Proposition II.4.9. *For $X,Y \in \mathcal{V}(M)$, we have on $\Omega(M,V)$:*

(1) $[\mathcal{L}_X,\mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$, *i.e., the Lie derivative defines a representation of the Lie algebra $\mathcal{V}(M)$ on $\Omega^p(M,V)$.*
(2) $[\mathcal{L}_X, i_Y] = i_{[X,Y]}$.
(3) $\mathcal{L}_X = d \circ i_X + i_X \circ d$ (Cartan formula).
(4) $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$.
(5) $\mathcal{L}_X(Z^p_{\mathrm{dR}}(M,V)) \subseteq B^p_{\mathrm{dR}}(M,V)$.

Proof. (1)-(3) It suffices to verify these formulas locally in charts, so that we may assume that M is an open subset of a locally convex space. Then (1)-(3) follow from the corresponding formulas in Appendix C, applied to the Lie algebra $\mathfrak{g} = \mathcal{V}(M)$ and the module $C^\infty(M,V)$.

(4) follows from (3) and $d^2 = 0$.

(5) follows from (3). ■

Remark II.4.10. Clearly integration of differential forms $\omega \in \Omega^p(M,V)$ only makes sense if M is a finite-dimensional oriented manifold (possibly with boundary) of dimension p and V is Mackey complete. We need the Mackey completeness to insure that each smooth function $f\colon Q \to V$ on a cube $Q := \prod_{i=1}^p [a_i,b_i] \subseteq \mathbb{R}^p$ has an iterated integral

$$\int_Q f dx := \int_{a_1}^{b_1} \cdots \int_{a_p}^{b_p} f(x_1,\ldots,x_p)\, dx_1 \cdots dx_p.$$

If $\varphi\colon U \to \mathbb{R}^p$ is a chart of M compatible with the orientation and $\mathrm{supp}(\omega)$ is a compact subset of U, then we define

$$\int_M \omega := \int_{\varphi(U)} (\varphi^{-1})^*\omega = \int_{\varphi(U)} f\ dx,$$

where $f \in C^\infty(\varphi(U),V)$ is the compactly supported function determined by

$$((\varphi^{-1})^*\omega)(x) = f(x)\ dx_1 \wedge \ldots \wedge dx_p.$$

If, more generally, ω has compact support and $(\chi_i)_{i\in I}$ is a smooth partition of unity with the property that $\mathrm{supp}(\chi_i)$ is contained in a chart domain, then we define

$$\int_M \omega := \sum_{i\in I} \int_M \chi_i\omega$$

and observe that the right hand side is a finite sum, where each summand is defined since $\mathrm{supp}(\chi_i\omega)$ is contained in a chart domain. Using the transformation formula for p-dimensional integrals, it is easy to see that the definition of the integral $\int_M \omega$ does not depend on the choice of the charts and the partitions of unity.

We also note that Stokes' Theorem

$$\int_M d\eta = \int_{\partial M} \eta$$

holds for V-valued $(p-1)$-forms, where it is understood that the boundary ∂M carries the induced orientation. ■

The assumption that V is Mackey complete is crucial in the following lemma to ensure the existence of the Riemann integral defining φ. For a conceptual proof we refer to [GN05, Ch. III].

Lemma II.4.11. (Poincaré Lemma) *Let E be locally convex, V a Mackey complete locally convex space and $U \subseteq E$ an open subset which is star-shaped with respect to 0. Let $\omega \in \Omega^{k+1}(U,V)$ be a V-valued closed $(k+1)$-form. Then $\omega = d\varphi$ for some $\varphi \in \Omega^k(U,V)$ satisfying $\varphi(0) = 0$ which is given by*

$$\varphi(x)(v_1,\ldots,v_k) = \int_0^1 t^k\omega(tx)(x,v_1,\ldots,v_k)\, dt. \qquad ■$$

Remark II.4.12. (a) The Poincaré Lemma is the first step to de Rham's Theorem. To obtain de Rham's Theorem for finite-dimensional manifolds, one makes heavy use of smooth partitions of unity which do not always exist for infinite-dimensional manifolds, not even for all Banach manifolds.

(b) We call a smooth manifold M *smoothly paracompact* if every open cover has a subordinated smooth partition of unity. De Rham's Theorem holds for every smoothly paracompact Fréchet manifold ([KM97,Thm. 34.7]). Smoothly Hausdorff second countable manifolds modeled on a smoothly regular space are smoothly paracompact ([KM97, 27.4]). Typical examples of smoothly regular spaces are nuclear Fréchet spaces ([KM97, Th. 16.10]).

(c) Examples of Banach spaces which are not smoothly paracompact are $C([0,1],\mathbb{R})$ and $\ell^1(\mathbb{N},\mathbb{R})$. On these spaces, there exists no non-zero smooth function supported in the unit ball ([KM97, 14.11]). ■

Exercises for Section II

Exercise II.1. Let $(V, \tau_{\mathcal{P}})$ be a locally convex space.

(1) Show that a seminorm q on V is continuous if and only if there exists a $\lambda > 0$ and $p_1, \ldots, p_n \in \mathcal{P}$ such that $q \leq \lambda \max(p_1, \ldots, p_n)$. Hint: A seminorm is continuous if and only if it is bounded on some 0-neighborhood.

(2) Two sets $\mathcal{P}_1$ and $\mathcal{P}_2$ of seminorms on V define the same locally convex topology if and only if all seminorms in $\mathcal{P}_2$ are continuous w.r.t. $\tau_{\mathcal{P}_1}$ and vice versa. ■

Exercise II.2. Show that the set of all seminorms on a vector space V is separating. The corresponding locally convex topology is called the *finest locally convex topology*. Hint: Every vector space has a basis (provided one believes in the Axiom of Choice, resp., Zorn's Lemma). ■

Exercise II.3. Fix $p \in]0,1[$ and let V denote the space of measurable functions $f\colon [0,1] \to \mathbb{R}$ (we identify functions which coincide on a set whose complement has measure zero), for which

$$|f| := \int_0^1 |f(x)|^p \, dx$$

is finite. Show that $d(f,g) := |f-g|$ defines a metric on this space. Hint: The function $[0,\infty[\to \mathbb{R}, x \mapsto x^p$ is sub-additive. This is turn follows from its concavity. ■

Exercise II.4. Let X be a locally compact space which is countable at infinity, i.e., there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X with $X = \bigcup_n K_n$ and $K_n \subseteq K_{n+1}^0$. We call such a sequence $(K_n)_{n\in\mathbb{N}}$ an *exhaustion of* X. Show that:

(1) Each compact subset $K \subseteq X$ lies in some K_n.

(2) The topology of uniform convergence on compact subsets of X on the space $C(X,\mathbb{R})$ is given by the sequence of seminorms $(p_{K_n})_{n\in\mathbb{N}}$ (Hint: Exercise II.1).

(3) $C(X,\mathbb{R})$ is metrizable.

(4) $C(X,\mathbb{R})$ is complete.

(5) The multiplication on $C(X,\mathbb{R})$ is continuous.

(6) $C(X,\mathbb{R})$ is a Fréchet algebra. ■

Exercise II.5. Let (M,d) be a metric space and $\emptyset \neq S \subseteq M$ a subset. Show that the function

$$f\colon M \to \mathbb{R}, \quad x \mapsto \mathrm{dist}(x,S) := \inf\{d(x,s)\colon s \in S\}$$

is a contraction, hence in particular continuous. ■

Exercise II.6. Let $U \subseteq \mathbb{R}^n$ be an open subset and $K_n := \{x \in U\colon \|x\| \leq n, \mathrm{dist}(x,U^c) \geq \frac{1}{n}\}$.

(1) Each compact subset $K \subseteq U$ lies in some K_n.

(2) The topology on the space $C^\infty(U,\mathbb{R})$ is given by the countable family of seminorms $(p_{K_n,m})_{n,m\in\mathbb{N}}$ (cf. Example II.1.4).

(3) $C^\infty(U,\mathbb{R})$ is metrizable.

(4) $C^\infty(U,\mathbb{R})$ is complete.

(5) The multiplication on $C^\infty(U,\mathbb{R})$ is continuous. Hint: Leibniz Rule.

(6) $C^\infty(U,\mathbb{R})$ is a Fréchet algebra. ■

Exercise II.7. Let X be a locally compact space. The unit group $C(X,\mathbb{R})^\times = C(X,\mathbb{R}^\times)$ is open in $C(X,\mathbb{R})$ if and only if X is compact. Hint: If X is not compact, then there exists for each compact subset $K \subseteq X$ a continuous function $f_K \in C(X,\mathbb{R})$ with $f_K|_K = 1$. Show that the net (f_K) converges to 1. ■

Exercise II.8. Let $(X_i)_{i\in I}$ be a family of locally convex spaces. Show that:
(1) The product topology on $X := \prod_{i\in I} X_i$ defines on X the structure of a locally convex space.
(2) This space is complete if and only if all the spaces X_i are complete.
(3) If, in addition, each X_i is a locally convex unital algebra, then X is a locally convex unital algebra. ■

Exercise II.9. Let M be a paracompact finite-dimensional complex manifold and endow the space $\mathrm{Hol}(M,\mathbb{C})$ with the topology of uniform convergence on compact subsets. Show that:
(1) $\mathrm{Hol}(M,\mathbb{C})$ is a Fréchet algebra.
(2) The mapping $\mathrm{Hol}(M,\mathbb{C}) \to C^\infty(M,\mathbb{C})$ is a topological embedding. Hint: Cauchy estimates in several variables. ■

Exercise II.10. Verify that $d^2\omega = 0$ for the exterior differential on $\Omega^p(M,V)$ (M a smooth manifold modeled on X, V a locally convex space) directly in local coordinates, using formula (2.4.8). Hint: For each $x \in M$, the map

$$X^2 \to \mathrm{Alt}^p(X,V), \quad (v,w) \mapsto d_1^2\omega(x)(v,w)$$

(second derivative with respect to the first argument of ω) is symmetric (Lemma II.1.3). ■

Exercise II.11. Let X be a locally convex space and p a continuous seminorm on X. Show that

$$p = \sup\{\lambda \in X' : \lambda \le p\}.$$

Hint: Consider the closed convex subset $B := \{x \in X : p(x) \le 1\}$. Then $\lambda|_B \le 1$ is equivalent to $\lambda \le p$ and if $p(x) > 1$, then there exists a continuous linear functional $\lambda \in Y'$ with $\lambda|_B \le 1$ and $\lambda(x) > 1$ (Hahn–Banach Separation Theorem). ■

Exercise II.12. Let Y be a locally convex space and $\gamma: [a,b] \to Y$ a continuous curve. Assume that the integral $I(\gamma) := \int_a^b \gamma(t)\,dt$ exists in the sense that there exists an element $I \in Y$ such that $\lambda(I(\gamma)) = \int_a^b \lambda(\gamma(t))\,dt$ holds for each continuous linear functional $\lambda \in Y'$. Show that:
(a) For each continuous seminorm p on Y, we have

$$p\Big(\int_a^b \gamma(t)\,dt\Big) \le \int_a^b p(\gamma(t))\,dt.$$

Hint: Use Exercise II.11.
(b) The map $I: C([a,b],Y) \to Y$ is continuous, when $C([a,b],Y)$ is endowed with the topology of uniform convergence (which coincides with the compact open topology; cf. Appendix B).
(c) If X is a topological space and $\gamma: X \times [a,b] \to Y$ a continuous map, then the map

$$X \to Y, \quad x \mapsto \int_a^b \gamma(x,t)\,dt$$

is continuous. ■

Exercise II.13. Let X be a complete metric topological vector space (f.i. a Fréchet space) and $Y \subseteq X$ a closed subspace. Show that the following are equivalent:
(1) There exists a closed subspace $Z \subseteq X$ for which the map $S: Y \oplus Z \to X, (y,z) \mapsto y + z$ is bijective.
(2) There exists a closed subspace $Z \subseteq X$ for which the map $S: Y \oplus Z \to X$ is a topological isomorphism.
(3) There exists a continuous projection $p: X \to X$ with $p(X) = Y$.
Hint: Open Mapping Theorem. ■

Exercise II.14. Let V be a $\mathbb{K}$-vector space and $\mathfrak{g}$ a $\mathbb{K}$-Lie algebra, where $\mathbb{K}$ is a field of characteristic zero. We write $\mathrm{Alt}^p(V,\mathfrak{g})$ for the linear space of p-linear alternating maps $V^p \to \mathfrak{g}$ and put $\mathrm{Alt}^0(V,\mathfrak{g}) := \mathfrak{g}$ and $\mathrm{Alt}^1(V,\mathfrak{g}) := \mathrm{Lin}(V,\mathfrak{g})$. On the space $\mathrm{Alt}(V,\mathfrak{g}) := \bigoplus_{p\in\mathbb{N}_0} \mathrm{Alt}^p(V,\mathfrak{g})$, we then define a bilinear product by

$$[\alpha,\beta](v_1,\ldots,v_{p+q}) := \frac{1}{p!q!}\sum_{\sigma\in S_{p+q}} \mathrm{sgn}(\sigma)[\alpha(v_{\sigma(1)},\ldots,v_{\sigma(p)}),\beta(v_{\sigma(p+1)},\ldots,v_{\sigma(p+q)})]$$

for $\alpha \in \mathrm{Alt}^p(V,\mathfrak{g})$ and $\beta \in \mathrm{Alt}^q(V,\mathfrak{g})$. Show that this multiplication has the following properties for $\alpha \in \mathrm{Alt}^p(V,\mathfrak{g})$, $\beta \in \mathrm{Alt}^q(V,\mathfrak{g})$ and $\gamma \in \mathrm{Alt}^r(V,\mathfrak{g})$:

(1) $[\alpha,\beta] = (-1)^{pq+1}[\beta,\alpha]$.

(2) $(-1)^{pr}[[\alpha,\beta],\gamma] + (-1)^{qp}[[\beta,\gamma],\alpha] + (-1)^{qr}[[\gamma,\alpha],\beta] = 0$ (graded Jacobi identity).

(3) $\mathrm{Alt}(V,\mathfrak{g})$ is a Lie superalgebra with respect to the 2-grading defined by

$$\mathrm{Alt}(V,\mathfrak{g}) := \mathrm{Alt}^{\mathrm{even}}(V,\mathfrak{g}) \oplus \mathrm{Alt}^{\mathrm{odd}}(V,\mathfrak{g}). \qquad \blacksquare$$

Exercise II.15. Let M be a smooth manifold and $\mathfrak{g}$ a locally convex Lie algebra. Then the product on the space $\Omega(M,\mathfrak{g}) := \bigoplus_{p\in\mathbb{N}_0} \Omega^p(M,\mathfrak{g})$, defined in Proposition II.4.1(3) satisfies for $\alpha \in \Omega^p(M,\mathfrak{g}), \beta \in \Omega^q(M,\mathfrak{g})$ and $\gamma \in \Omega^r(M,\mathfrak{g})$:

(1) $[\alpha,\beta] = (-1)^{pq+1}[\beta,\alpha]$.

(2) $(-1)^{pr}[[\alpha,\beta],\gamma] + (-1)^{qp}[[\beta,\gamma],\alpha] + (-1)^{qr}[[\gamma,\alpha],\beta] = 0$ (super Jacobi identity).

(3) $\Omega(M,\mathfrak{g})$ is a Lie super-algebra with respect to the 2-grading defined by

$$\Omega(M,\mathfrak{g}) := \Omega^{\mathrm{even}}(M,\mathfrak{g}) \oplus \Omega^{\mathrm{odd}}(M,\mathfrak{g}).$$

Hint: If M is an open subset of a locally convex space, then we have the canonical embedding $\Omega^p(M,\mathfrak{g}) \hookrightarrow \mathrm{Alt}^p(\mathcal{V}(M), C^\infty(M,\mathfrak{g}))$ which is compatible with the product, and Exercise II.14 applies. $\blacksquare$

Exercise II.16. Let $f\colon M \to N$ be a smooth map between manifolds, $\pi_{TM}\colon TM \to M$ the tangent bundle projection and $\sigma_M\colon M \to TM$ the zero section. Show that

$$\pi_{TN} \circ Tf = f \circ \pi_{TM} \quad \text{and} \quad \sigma_N \circ f = Tf \circ \sigma_M. \qquad \blacksquare$$

Exercise II.17. Let M be a smooth manifold. Show that:

(a) For each vector field, the map $C^\infty(M,\mathbb{K}) \to C^\infty(M,\mathbb{K}), f \mapsto \mathcal{L}_X f := X.f$ is a derivation.

(b) The map $\mathcal{V}(M) \to \mathrm{der}(C^\infty(M,\mathbb{K})), X \mapsto \mathcal{L}_X$ from (a) is a homomorphism of Lie algebras.

(c) If M is an open subset of some locally convex space, then the map under (b) is injective. $\blacksquare$

Exercise II.18. Let M and N be smooth manifolds. The C^k*-topology* on the set $C^k(M,N)$ of smooth maps $M \to N$ is the topology obtained from the embedding

$$C^k(M,N) \hookrightarrow C(T^kM,T^kN), \quad f \mapsto T^kf,$$

where the space $C(T^kM,T^kN)$ is endowed with the compact open topology. Show that:

(1) If $M = U$ is open in a locally convex space E and $N = F$ is a locally convex space, then the C^k-topology on the space $C^k(U,F)$ coincides with the topology defined by the embedding

$$C^k(U,F) \hookrightarrow \prod_{j=0}^k C(U\times E^j,F), \quad f \mapsto (f,df,\ldots,d^kf),$$

where each factor on the right hand side carries the compact open topology.

(2) If $M = U$ is open in $E := \mathbb{K}^n$ and $N = F$ is a locally convex space, then the C^k-topology on the space $C^k(U,F)$ coincides with the topology defined by the seminorms

$$q_{K,j}(f) := \sup\{(q\circ D^jf)(x)\colon x\in K\},$$

for $j \leq m$, $K \subseteq U$ compact and q a continuous seminorm on F (cf. Example II.1.4). Hint: Use that $T^jU \cong U \times E^{2^j-1}$ and $T^jF \cong F^{2^j}$ and describe the 2^j-components of the map T^jf in terms of higher derivatives of f. $\blacksquare$

Exercise II.19. If E and F are Banach spaces and $\mathcal{L}(E,F)$ is endowed with the operator norm, then the subset $\mathrm{Iso}(E,F) \subseteq \mathcal{L}(E,F)$ of all topological isomorphisms $E \to F$ is an open subset. ■

Exercise II.20. Let M be a smooth compact manifold. We endow the set $C^1(M,M)$ with the C^1-topology (cf. Exercise II.18). Show that:

(1) The set $\mathrm{Diff}^1_{\mathrm{loc}}(M)$ of all maps $f \in C^1(M,M)$ for which each map $df(x)\colon T_x(M) \to T_{f(x)}(N)$ is a linear isomorphism (the set of local diffeomorphisms) is open. Hint: $\mathrm{GL}_n(\mathbb{K})$ is open in $M_n(\mathbb{K})$.

(2) If $f\colon M \to M$ is a local diffeomorphism, then it is a covering map. It is a diffeomorphism if and only if it is one-to-one.

(3)* For a local diffeomorphism f, the number $n(f) := |f^{-1}(x)|$ does not depend on x and it defines a continuous function $\mathrm{Diff}^1_{\mathrm{loc}}(M) \to \mathbb{N}$. Hint: Let $q\colon \widehat{M} \to M$ denote the orientation cover of M. Then f lifts to a map $\widehat{f}\colon \widehat{M} \to \widehat{M}$ and $n(f) = |\deg(\widehat{f})|$ holds for the mapping degree $\deg(\widehat{f})$ of $\widehat{f}$ which can be defined by $\widehat{f}^*\mu = \deg(\widehat{f})\mu$ for a volume form μ on $\widehat{M}$.

(4)* Show that the set of all local diffeomorphisms f with $n(f) \geq 2$ is closed in the C^1-topology. Hint: Use a Riemannian metric on M to see that for each $c \in]0,1[$, the set of all f with $\|df(x)v\| \geq c\|v\|$ for all $x \in M$, $v \in T_x(M)$, is closed and a neighborhood of each g with $\|dg(x)v\| > \frac{c}{2}\|v\|$ for all $x \in M$, $0 \neq v \in T_x(M)$. For any sequence $f_n \to f$ with $f_n(x_n) = f_n(y_n)$ and $f_n \to f$, we may assume that $x_n \to x$, $y_n \to y$. Show that if $x_n \neq y_n$ for all n, then $x \neq y$ and $f(x) = f(y)$.

(5) Show that the group $\mathrm{Diff}^1(M)$ of C^1-diffeomorphisms is an open subset of $C^1(M,M)$. Hint: Use (3) or (4). ■

Exercise II.21. Let $X_1,\ldots,X_k$ and Y be locally convex spaces. Show that for a k-linear map $m\colon X_1 \times \ldots \times X_k \to Y$, the following are equivalent:

(1) m is continuous.

(2) m is continuous in $(0,0,\ldots,0)$.

(3) m is continuous in some k-tuple $(x_1,\ldots,x_k)$. ■

III. Infinite-dimensional Lie groups

In this section, we give the definition of an infinite-dimensional (locally convex) Lie group and explain how its Lie algebra can be defined in such a way that it defines a functor from the category of Lie groups to the category of locally convex Lie algebras.

In our treatment of Lie groups, we basically follow [Mil83], but we do not assume that the model space of a Lie group is complete (cf. also [GN05]).

Notation: Let G be a group and $g \in G$. We write
$\lambda_g\colon G \to G, x \mapsto gx$ for the *left multiplication* by g,
$\rho_g\colon G \to G, x \mapsto xg$ for the *right multiplication* by g,
$m_G\colon G \times G \to G, (x,y) \mapsto xy$ for the *multiplication map*, and
$\eta_G\colon G \to G, x \mapsto x^{-1}$ for the *inversion*.

In the following, $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

III.1. Infinite-dimensional Lie groups and their Lie algebras

Definition III.1.1. A *locally convex Lie group* G is a locally convex manifold endowed with a group structure such that the multiplication map and the inversion map are smooth. We shall often write $\mathfrak{g} := T_{\mathbf{1}}(G)$ for the tangent space in $\mathbf{1}$.

A morphism of Lie groups is a smooth group homomorphism. In the following, we shall call locally convex Lie groups simply *Lie groups.* We write *LieGrp* for the so obtained category of Lie groups. ∎

Example III.1.2. (Vector groups) Each locally convex space V is an abelian Lie group with respect to addition. In fact, we endow V with the obvious manifold structure and observe that addition and inversion are smooth maps. ∎

Example III.1.3. (Unit groups of CIAs) Let A be a continuous inverse algebra over $\mathbb{K}$ and $A^\times$ its unit group. As an open subset of A, the group $A^\times$ carries a natural manifold structure. The multiplication on A is bilinear and continuous, hence a smooth map (Remark II.2.7(b)). Therefore the multiplication of $A^\times$ is smooth.

It remains to see that the inversion $\eta\colon A^\times \to A^\times$ is smooth. Its continuity follows from the assumption that A is a CIA. For $a, b \in A^\times$, we have $b^{-1} - a^{-1} = a^{-1}(a-b)b^{-1}$, which implies that for $t \in \mathbb{K}$ sufficiently close to 0, we get

$$\eta(a+th) - \eta(a) = (a+th)^{-1} - a^{-1} = a^{-1}(-th)(a+th)^{-1} = -ta^{-1}h(a+th)^{-1}.$$

Therefore the continuity of η implies that η is everywhere differentiable with

$$d\eta(a)(h) = \lim_{t\to 0} \frac{1}{t}(\eta(a+th) - \eta(a)) = \lim_{t\to 0} -a^{-1}h(a+th)^{-1} = -a^{-1}ha^{-1}.$$

Now the continuity of η implies that $d\eta\colon A^\times \times A \to A$ is continuous, hence that η is a C^1-map. With the Chain Rule and the smoothness of the multiplication, this in turn implies that $d\eta$ is a C^1-map, hence that η is C^2. Iterating this argument, we conclude that η is smooth. ∎

Lemma III.1.4. *Let G be a Lie group.*

(a) *The tangent map*

$$Tm_G\colon T(G\times G) \cong TG \times TG \to TG, \quad (v,w) \mapsto v\cdot w := Tm_G(v,w)$$

defines a Lie group structure on TG with identity element $0 \in T_{\mathbf{1}}(G) = \mathfrak{g}$ and inversion $T\eta_G$. The canonical projection $\pi_{TG}\colon TG \to G$ is a morphism of Lie groups with kernel $(\mathfrak{g}, +)$ and the zero section $\sigma\colon G \to TG, g \mapsto 0_g \in T_g(G)$ is a homomorphism of Lie groups with $\pi_{TG}\circ\sigma = \mathrm{id}_G$.

(b) *Identifying $g \in G$ with $\sigma(g) \in TG$, we write*

$$g.v := 0_g \cdot v, \quad v.g := v \cdot 0_g \quad \text{for} \quad g \in G, v \in TG. \tag{3.1.1}$$

Then the map

$$\Phi\colon G \times \mathfrak{g} \to TG, \quad (g,x) \mapsto g.x$$

is a diffeomorphism.

Proof. (a) Since the multiplication map $m_G\colon G\times G \to G$ is smooth, the same holds for its tangent map

$$Tm_G\colon T(G\times G) \cong TG\times TG \to TG.$$

Let $\{\mathbf{1}\}$ denote the trivial group, $\varepsilon_G\colon G \to \{\mathbf{1}\}$ the constant homomorphism and $u_G\colon \{\mathbf{1}\} \to G$ the group homomorphism representing the identity element. Then the group axioms for G are encoded in the relations

(1) $m_G \circ (m_G \times \mathrm{id}) = m_G \circ (\mathrm{id} \times m_G)$ (associativity),

(2) $m_G \circ (\eta_G \times \mathrm{id}) = m_G \circ (\mathrm{id} \times \eta_G) = \varepsilon_G$ (inversion), and

(3) $m_G \circ (u_G \times \mathrm{id}) = m_G \circ (\mathrm{id} \times u_G) = \mathrm{id}$ (unit element).

Using the functoriality of T, we see that these properties carry over to the corresponding maps on TG and show that TG is a Lie group with multiplication Tm_G, inversion $T\eta_G$, and unit element $0 = Tu_G(0) \in T_{\mathbf{1}}(G) = \mathfrak{g}$.

For the zero section $\sigma: G \to TG$, we have $Tm_G \circ (\sigma \times \sigma) = \sigma \circ m_G$, which means that it is a morphism of Lie groups. That π_{TG} is a morphism of Lie groups follows likewise from

$$\pi_{TG} \circ Tm_G = m_G \circ (\pi_{TG} \times \pi_{TG})$$

(cf. Exercise II.16).

We have for $v, v' \in \mathfrak{g}$:

$$Tm_G(g.v, g'.v') = Tm_G(g.v, g'.0) + Tm_G(g.0, g'.v') = (g.v).g' + gg'.v'$$

and in particular $Tm_G(v, v') = v + v'$, showing that $\ker \pi_{TG} \cong (\mathfrak{g}, +)$.

That the smooth map Φ is a diffeomorphism follows from $\Phi^{-1}(v) = (\pi_{TG}(v), \pi_{TG}(v)^{-1}.v)$. ■

Definition III.1.5. A vector field $X \in \mathcal{V}(G)$ is called *left invariant* if

$$X \circ \lambda_g = T(\lambda_g) \circ X$$

holds for each $g \in G$ if we consider X as a section $X : G \to TG$ of the tangent bundle TG. We write $\mathcal{V}(G)^l$ for the set of left invariant vector fields in $\mathcal{V}(G)$. The left invariance of a vector field X implies in particular that for each $g \in G$, we have $X(g) = g.X(\mathbf{1})$ in the sense of (3.1.1) in Lemma III.1.4. For each $x \in \mathfrak{g}$, we have a unique left invariant vector field $x_l \in \mathcal{V}(G)^l$ defined by $x_l(g) := g.x$, and the map

$$\mathcal{V}(G)^l \to T_{\mathbf{1}}(G) = \mathfrak{g}, \quad X \mapsto X(\mathbf{1})$$

is a linear bijection. If X, Y are left invariant, then they are λ_g-related to themselves, and Lemma II.3.8 implies that their Lie bracket $[X, Y]$ inherits this property, hence that $[X, Y] \in \mathcal{V}(G)^l$. We thus obtain a unique Lie bracket $[\cdot,\cdot]$ on $\mathfrak{g}$ satisfying

$$[x, y]_l = [x_l, y_l] \quad \text{for all} \quad x, y \in \mathfrak{g}. \tag{3.1.2}$$

■

Lemma III.1.6. *For each $\mathfrak{g}$-chart (φ, U) of G with $\mathbf{1} \in U$ and $\varphi(\mathbf{1}) = 0$, the second order Taylor polynomial in $(0,0)$ of the multiplication $x * y := \varphi(\varphi^{-1}(x)\varphi^{-1}(y))$ is of the form*

$$x + y + b(x, y),$$

where $b: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a continuous bilinear map satisfying

$$[x, y] = b(x, y) - b(y, x). \tag{3.1.3}$$

In particular, the Lie bracket on $\mathfrak{g} = T_{\mathbf{1}}(G)$ is continuous.

Proof. We consider a chart $\varphi: V \to \mathfrak{g}$ of G, where $V \subseteq G$ is an open $\mathbf{1}$-neighborhood and $\varphi(\mathbf{1}) = 0$. Let $W \subseteq V$ be an open symmetric $\mathbf{1}$-neighborhood with $WW \subseteq V$. Then we have on the open set $\varphi(W) \subseteq \mathfrak{g}$ the smooth multiplication

$$x * y := \varphi(\varphi^{-1}(x)\varphi^{-1}(y)), \quad x, y \in \varphi(W).$$

From $Tm(v, w) = v + w$ for $v, w \in T_{\mathbf{1}}(G)$ we immediately see that the second order Taylor polynomial of $*$ has the form $x + y + b(x, y)$, where $b: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is quadratic map, hence can be written as

$$b(x, y) = \beta((x, y), (x, y))$$

for some continuous symmetric bilinear map $\beta: (\mathfrak{g} \times \mathfrak{g})^2 \to \mathfrak{g}$ (Lemma II.2.3(iv)). Comparing Taylor expansions of $x * 0 = 0 * x = x$ up to second order implies that $b(x, 0) = b(0, x) = 0$, so that

$$b(x, y) = \beta((x, 0), (0, y)) + \beta((0, y), (x, 0)).$$

It follows in particular that b is bilinear.

For $x \in W$, let $\lambda_x: \varphi(W) \to \mathfrak{g}, y \mapsto x * y$. Then the left invariant vector field v_l corresponding to $v \in \mathfrak{g}$ is given on $\varphi(W)$ by $v_l(x) = d\lambda_x(0).v$, and in 0 its first order Taylor polynomial in 0 is $v + b(x, v)$. Therefore, the Lie bracket on $\mathfrak{g}$ satisfies

$$[v, w] = [v_l, w_l](0) = dw_l(0).v_l(0) - dv_l(0).w_l(0) = dw_l(0).v - dv_l(0).w = b(v, w) - b(w, v). \quad ■$$

Definition III.1.7. The locally convex Lie algebra $\mathbf{L}(G) := (\mathfrak{g}, [\cdot,\cdot])$ is called *the Lie algebra of G*. ■

Proposition III.1.8. (Functoriality of the Lie algebra) *If $\varphi: G \to H$ is a homomorphism of Lie groups, then the tangent map*

$$\mathbf{L}(\varphi) := T_{\mathbf{1}}(\varphi): \mathbf{L}(G) \to \mathbf{L}(H)$$

is a homomorphism of Lie algebras.

Proof. Let $x, y \in \mathfrak{g}$ and x_l, y_l be the corresponding left invariant vector fields. Then $\varphi \circ \lambda_g = \lambda_{\varphi(g)} \circ \varphi$ for each $g \in G$ implies that

$$T\varphi \circ x_l = \mathbf{L}(\varphi)(x)_l \circ \varphi \quad \text{and} \quad T\varphi \circ y_l = \mathbf{L}(\varphi)(y)_l \circ \varphi,$$

and therefore

$$T\varphi \circ [x_l, y_l] = [\mathbf{L}(\varphi)(x)_l, \mathbf{L}(\varphi)(y)_l] \circ \varphi$$

(Lemma II.3.8). Evaluating at $\mathbf{1}$, we obtain $\mathbf{L}(\varphi).[x, y] = [\mathbf{L}(\varphi)(x), \mathbf{L}(\varphi)(y)]$. ■

Remark III.1.9. We obviously have $\mathbf{L}(\mathrm{id}_G) = \mathrm{id}_{\mathbf{L}(G)}$, and for two morphisms $\varphi_1: G_1 \to G_2$ and $\varphi_2: G_2 \to G_3$ of Lie groups, we have

$$\mathbf{L}(\varphi_2 \circ \varphi_1) = \mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1),$$

as a consequence of the Chain Rule.

The preceding lemma implies that the assignments $G \mapsto \mathbf{L}(G)$ and $\varphi \mapsto \mathbf{L}(\varphi)$ define a functor

$$\mathbf{L}: \underline{\mathit{LieGrp}} \to \underline{\mathit{lcLieAlg}}$$

from the category $\underline{\mathit{LieGrp}}$ of (locally convex) Lie groups to the category $\underline{\mathit{lcLieAlg}}$ of locally convex Lie algebras.

Since each functor maps isomorphisms to isomorphisms, for each isomorphism of Lie groups $\varphi: G \to H$, the map $\mathbf{L}(\varphi)$ is an isomorphism of locally convex Lie algebras. ■

Definition III.1.10. A locally convex Lie algebra $\mathfrak{g}$ is said to be *integrable* if there exists a Lie group G with $\mathbf{L}(G) \cong \mathfrak{g}$. ■

Although every finite-dimensional Lie algebra is integrable, integrability of infinite-dimensional Lie algebras turns out to be a very subtle property. We shall discuss some interesting examples in Section VI below.

We now have a look at the Lie algebras of the Lie groups from Examples II.1.2/3.

Examples III.1.11. (a) If G is an abelian Lie group, then the map $b: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ in Lemma III.1.6 is symmetric, which implies that $\mathbf{L}(G)$ is abelian. This applies in particular to the additive Lie group $(V, +)$ of a locally convex space.

(b) Let A be a CIA. Then the map

$$\varphi: A^\times \to A, \quad x \mapsto x - \mathbf{1}$$

is a chart of $A^\times$ satisfying $\varphi(\mathbf{1}) = 0$. In this chart, the group multiplication is given by

$$x * y := \varphi(\varphi^{-1}(x)\varphi^{-1}(y)) = (x + \mathbf{1})(y + \mathbf{1}) - \mathbf{1} = x + y + xy.$$

In the terminology of Lemma III.1.6, we then have $b(x, y) = xy$ and therefore

$$[x, y] = xy - yx$$

is the commutator bracket in the associative algebra A. ■

Using the Lie group structures on tangent bundles, we can now also deal with groups of smooth maps and diffeomorphism groups.

Example III.1.12. (Groups of smooth maps) Let M be a manifold (possibly infinite-dimensional) and K a Lie group with Lie algebra $\mathfrak{k}$. Then we obtain a natural topology on the group $G := C^\infty(M,K)$ as follows.

The tangent bundle TK of K is a Lie group (Lemma III.1.4). Iterating this procedure, we obtain a Lie group structure on all higher tangent bundles T^nK.

For each $n \in \mathbb{N}_0$, we thus obtain topological groups $C(T^nM, T^nK)$ by using the topology of uniform convergence on compact subsets of T^nM (Lemma B.3), which coincides with the compact open topology (Proposition B.4). We also observe that for two smooth maps $f_1, f_2 \colon M \to K$, the functoriality of T yields

$$T(f_1 \cdot f_2) = T(m_G \circ (f_1 \times f_2)) = T(m_G) \circ (Tf_1 \times Tf_2) = Tf_1 \cdot Tf_2.$$

Therefore the canonical inclusion map

$$C^\infty(M,K) \hookrightarrow \prod_{n \in \mathbb{N}_0} C(T^nM, T^nK), \quad f \mapsto (T^nf)_{n \in \mathbb{N}_0}$$

is a group homomorphism, so that the inverse image of the product topology on the right hand side is a group topology on $C^\infty(M,K)$. Therefore $C^\infty(M,K)$ always carries a natural structure of a topological group, even if M and K are infinite-dimensional.

Now we assume that M is compact. Then these topological groups can even be turned into Lie groups modeled on the space $\mathfrak{g} := C^\infty(M, \mathfrak{k})$. The charts of G are obtained from those of K as follows. If $\varphi_K \colon U_K \to \mathfrak{k}$ is a chart of K, i.e., a diffeomorphism of an open subset $U_K \subseteq K$ onto an open subset $\varphi(U_K)$ of $\mathfrak{k}$, then the set $U_G := \{f \in G \colon f(M) \subseteq U_K\}$ is an open subset of G (cf. Appendix B). Assume, in addition, that $\mathbf{1} \in U_K$ and $\varphi_K(\mathbf{1}) = 0$. Then we use the map

$$\varphi_G \colon U_G \to \mathfrak{g}, \quad f \mapsto \varphi_K \circ f$$

as a chart of a $\mathbf{1}$-neighborhood of G, and by combining it with left translates, we obtain an atlas of G defining a Lie group structure (cf. Theorem II.2.1 below). For details we refer to [Gl01b], resp., [GN05].

To calculate the Lie algebra of this group, we observe that for $m \in M$, we have for the multiplication in local coordinates

$$\begin{aligned}(f *_G g)(m) &= \varphi_G\Big(\varphi_G^{-1}(f)\varphi_G^{-1}(g)\Big)(m) = \varphi_K\big(\varphi_K^{-1}(f(m))\varphi_K^{-1}(g(m))\big) \\ &= f(m) *_K g(m) = f(m) + g(m) + b_{\mathfrak{k}}(f(m), g(m)) + \cdots.\end{aligned}$$

In view of Lemma III.1.5, this implies that $\big(b_{\mathfrak{g}}(f,g)\big)(m) = b_{\mathfrak{k}}(f(m), g(m))$, and hence that

$$[f,g](m) = b_{\mathfrak{g}}(f,g)(m) - b_{\mathfrak{g}}(g,f)(m) = b_{\mathfrak{k}}(f(m),g(m)) - b_{\mathfrak{k}}(g(m),f(m)) = [f(m),g(m)].$$

Therefore $\mathbf{L}(C^\infty(M,K)) = C^\infty(M,\mathfrak{k})$, endowed with the pointwise defined Lie bracket. ■

Remark III.1.13. If M is a non-compact finite-dimensional manifold, then one cannot expect the topological groups $C^\infty(M,K)$ to be Lie groups. A typical example arises for $M = \mathbb{N}$ (a 0-dimensional manifold) and $K = \mathbb{T} := \mathbb{R}/\mathbb{Z}$. Then $C^\infty(M,K) \cong \mathbb{T}^{\mathbb{N}}$ is a topological group for which no $\mathbf{1}$-neighborhood is contractible, so that it carries no smooth manifold structure. ■

Remark III.1.14. (The Lie algebra of a local Lie group) There is also a natural notion of a local Lie group. The corresponding algebraic concept is that of a local group: Let G be a set and $D \subseteq G \times G$ a subset on which we are given a map

$$m_G \colon D \to G, \quad (x,y) \mapsto xy.$$

We say that the product xy of two elements $x, y \in G$ is *defined* if $(x,y) \in D$. The quadruple $(G, D, m_G, \mathbf{1})$, where $\mathbf{1}$ is a distinguished element of G, is called a *local group* if the following conditions are satisfied:

(1) Suppose that xy and yz are defined. If $(xy)z$ or $x(yz)$ is defined, then the other product is also defined and both are equal.
(2) For each $x \in G$, the products $x\mathbf{1}$ and $\mathbf{1}x$ are defined and equal to x.
(3) For each $x \in G$, there exists a unique element $x^{-1} \in G$ such that xx^{-1} and $x^{-1}x$ are defined and $xx^{-1} = x^{-1}x = \mathbf{1}$.
(4) If xy is defined, then $y^{-1}x^{-1}$ is defined.

If $(G, D, m_G, \mathbf{1})$ is a local group and, in addition, G has a smooth manifold structure, D is open, and the maps

$$m_G: D \to G, \quad \eta_G: G \to G, x \mapsto x^{-1}$$

are smooth, then G, resp., $(G, D, m_G, \mathbf{1})$ is called a *local Lie group.*

Let G be a local Lie group and $\mathfrak{g} := T_{\mathbf{1}}(G)$. For each $x \in \mathfrak{g} = T_{\mathbf{1}}(G)$, we then obtain a *left invariant vector field* $x_l(g) := g.x := 0_g \cdot x$. One can show that the Lie bracket of two left invariant vector fields is again left invariant and that we thus obtain a Lie algebra structure on $\mathfrak{g}$ (Exercise III.1). Describing the multiplication in a local chart $\varphi: V \to \mathfrak{g}$ with $\varphi(\mathbf{1}) = 0$, we can argue as in the proof of Lemma III.1.6 that its second order Taylor polynomial is of the form $x + y + b(x, y)$ with a continuous bilinear map $b: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

$$[x, y] = b(x, y) - b(y, x).$$

We conclude that $\mathbf{L}(G) := \mathbf{L}(G, D, m_G, \mathbf{1}) := (\mathfrak{g}, [\cdot, \cdot])$ is a locally convex Lie algebra. For more details on local Lie groups we refer to [GN05]. ■

The adjoint representation

The adjoint action is a crucial structure element of a Lie group G. It is the representation of G on $\mathbf{L}(G)$ obtained by taking derivatives in $\mathbf{1}$ for the conjugation action of G on itself. In this sense, it is a linearized picture of the non-commutativity of G.

Definition III.1.15. Let G be a Lie group. Then for each $g \in G$ the map

$$c_g: G \to G, \quad x \mapsto gxg^{-1},$$

is a smooth automorphism, hence induces a continuous linear automorphism

$$\mathrm{Ad}(g) := \mathbf{L}(c_g): \mathfrak{g} \to \mathfrak{g}.$$

We thus obtain an action $G \times \mathfrak{g} \to \mathfrak{g}, (g, x) \mapsto \mathrm{Ad}(g).x$ called the *adjoint action* of G on $\mathfrak{g}$.

If $\mathfrak{g}' := \mathcal{L}(\mathfrak{g}, \mathbb{K})$ denotes the *topological dual of* $\mathfrak{g}$, then we also obtain a representation on $\mathfrak{g}'$ by $\mathrm{Ad}^*(g).f := f \circ \mathrm{Ad}(g)^{-1}$, called the *coadjoint action.* Since we do not endow $\mathfrak{g}'$ with a topology, we will not specify any smoothness or continuity properties of this action. ■

Proposition III.1.16. *The adjoint action* $\mathrm{Ad}: G \times \mathfrak{g} \to \mathfrak{g}, (g, x) \mapsto \mathrm{Ad}(g).x$ *is smooth. The operators*

$$\mathrm{ad}\, x: \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{ad}\, x(y) := T\,\mathrm{Ad}(x, 0_y) \quad \textit{satisfy} \quad \mathrm{ad}\, x(y) = [x, y].$$

Proof. The smoothness of the adjoint action of G on $\mathfrak{g}$ follows directly from the smoothness of the multiplication of the Lie group TG because $\mathrm{Ad}(g).x = (g.x).g^{-1}$ (Lemma III.1.4).

To calculate the linear maps $\mathrm{ad}\, x: \mathfrak{g} \to \mathfrak{g}$, we consider a local chart $\varphi: V \to \mathfrak{g}$ of G, where $V \subseteq G$ is an open $\mathbf{1}$-neighborhood and $\varphi(\mathbf{1}) = 0$.

For $x \in \varphi(W)$, we write $\alpha_1(x) + \alpha_2(x)$ for the second order Taylor polynomial of the inversion map $x \mapsto x^{-1}$, where α_1 is linear and α_2 is quadratic. Comparing Taylor expansions in 0 of

$$0 = x * x^{-1} = x + \alpha_1(x) + \alpha_2(x) + b(x, \alpha_1(x)) + \dots$$

(Lemma III.1.6), we get $\alpha_1(x) = -x$ and $\alpha_2(x) = -b(x, -x) = b(x, x)$. Therefore

$$\begin{aligned}(x * y) * x^{-1} &= \big(x + y + b(x, y)\big) + \big(-x + b(x, x)\big) + b(x + y, -x) + \cdots \\ &= y + b(x, y) - b(y, x) + \cdots\end{aligned}$$

by the Chain Rule for Taylor polynomials, and by taking the derivative w.r.t. x in 0 in the direction z, we eventually get $\mathrm{ad}\, z(y) = b(z, y) - b(y, z) = [z, y]$. ■

The diffeomorphism group

Proposition III.1.17. *Let G be a Lie group and $\sigma: M \times G \to M, (m,g) \mapsto m.g$ a smooth right action of G on the smooth manifold M. Then the map $T\sigma: TM \times TG \to TM$ is a smooth right action of TG on TM. The assignment*

$$\dot{\sigma}: \mathfrak{g} \to \mathcal{V}(M), \quad \text{with} \quad \dot{\sigma}(x)(m) := d\sigma(m, \mathbf{1})(0, x) = T\sigma(0_m, x)$$

is a homomorphism of Lie algebras.

Proof. That $T\sigma$ defines an action of TG on TM follows in the same way as in Lemma III.1.4 above by applying T to the commutative diagrams defining a right action of a group.

To see that $\dot{\sigma}$ is a homomorphism of Lie algebras, we pick $m \in M$ and write $\varphi_m: G \to M, g \mapsto m.g$ for the smooth orbit map of m. Then the equivariance of φ_m means that $\varphi_m \circ \lambda_g = \varphi_{m.g}$. From this we derive

$$d\varphi_m(g)x_l(g) = d\varphi_m(g)d\lambda_g(\mathbf{1})x = d\varphi_{m.g}(\mathbf{1})x = \dot{\sigma}(x)(m.g),$$

i.e., the left invariant vector field x_l is φ_m-related to $\dot{\sigma}(x)$. Therefore Lemma II.3.8 implies that

$$\dot{\sigma}([x,y])(m) = d\varphi_m(\mathbf{1})[x,y]_l(\mathbf{1}) = d\varphi_m(\mathbf{1})[x_l, y_l](\mathbf{1}) = [\dot{\sigma}(x), \dot{\sigma}(x)](m). \qquad \blacksquare$$

Corollary III.1.18. *If $\sigma: G \times M \to M$ is a smooth left aftion of G on M, then*

$$\dot{\sigma}: \mathfrak{g} \to \mathcal{V}(M), \quad \text{with} \quad \dot{\sigma}(x)(m) := -T\sigma(x, 0_m)$$

is a homomorphism of Lie algebras.

Proof. If σ is a smooth left action, then $\widetilde{\sigma}(m,g) := \sigma(m, g^{-1})$ is a smooth right action and $T\widetilde{\sigma}(0_m, x) = -T\sigma(x, 0_m)$ follows from the Chain Rule and $d\eta_G(\mathbf{1})x = -x$. $\blacksquare$

Example III.1.19. Let M be a compact manifold and $\mathfrak{g} = \mathcal{V}(M)$, the Lie algebra of smooth vector fields on M. We now explain how the group $\mathrm{Diff}(M)$ can be turned into a Lie group, modeled on $\mathfrak{g}$.

We shall see in Section IV below that, although $\mathrm{Diff}(M)$ has a smooth exponential function, it is not a local diffeomorphism of a 0-neighborhood in $\mathfrak{g}$ onto an identity neighborhood in G. Therefore we cannot use it to define charts for G. But there is an easy way around this problem.

Let g be a Riemannian metric on M and $\mathrm{Exp}: TM \to M$ be its exponential function, which assigns to $v \in T_m(M)$ the point $\gamma(1)$, where $\gamma: [0,1] \to M$ is the geodesic segment with $\gamma(0) = m$ and $\gamma'(0) = v$. We then obtain a smooth map

$$\Phi: TM \to M \times M, \quad v \mapsto (m, \mathrm{Exp}\, v), \quad v \in T_m(M).$$

There exists an open neighborhood $U \subseteq TM$ of the zero section such that Φ maps U diffeomorphically onto an open neighborhood of the diagonal in $M \times M$. Now

$$U_\mathfrak{g} := \{X \in \mathcal{V}(M): X(M) \subseteq U\}$$

is an open subset of the Fréchet space $\mathcal{V}(M)$, and we define a map

$$\varphi: U_\mathfrak{g} \to C^\infty(M, M), \quad \varphi(X)(m) := \mathrm{Exp}(X(m)).$$

It is clear that $\varphi(0) = \mathrm{id}_M$. One can show that after shrinking $U_\mathfrak{g}$ to a sufficiently small 0-neighborhood in the C^1-topology on $\mathcal{V}(M)$, we may achieve that $\varphi(U_\mathfrak{g}) \subseteq \mathrm{Diff}(M)$. To see that $\mathrm{Diff}(M)$ carries a Lie group structure for which φ is a chart, one has to verify that the group operations are smooth in a 0-neighborhood when transfered to $U_{\mathfrak g}$ via φ, so that Theorem III.2.1 below applies. We thus obtain a Lie group structure on $\mathrm{Diff}(M)$ (cf. [GN05]).

From the smoothness of the map $U_{\mathfrak g} \times M \to M, (X,m) \mapsto \varphi(X)(m) = \mathrm{Exp}(X(m))$ it follows that the canonical left action $\sigma\colon \mathrm{Diff}(M) \times M \to M, (\varphi,m) \mapsto \varphi(m)$ is smooth in an identity neighborhood of $\mathrm{Diff}(M)$, and hence smooth, because it is an action by smooth maps. The corresponding homomorphism of Lie algebras $\dot\sigma\colon \mathbf{L}(\mathrm{Diff}(M)) \to \mathcal{V}(M)$ is given by

$$\dot\sigma(X)(m) = -T\sigma(X, 0_m) = -(d\,\mathrm{Exp})_{0_m}(X(m)) = -X(m),$$

i.e., $\dot\sigma = -\mathrm{id}_{\mathcal{V}(M)}$. This leads to

$$\mathbf{L}(\mathrm{Diff}(M)) = (\mathcal{V}(M), [\cdot,\cdot])^{\mathrm{op}}.$$

This “wrong” sign is caused by the fact that we consider $\mathrm{Diff}(M)$ as a group acting on M from the left. If we consider it as a group acting on the right, we obtain the opposite multiplication

$$\varphi * \psi := \psi \circ \varphi,$$

and

$$\mathbf{L}(\mathrm{Diff}(M)^{\mathrm{op}}) \cong (\mathcal{V}(M), [\cdot,\cdot])$$

follows from Proposition III.1.17.

The tangent bundle of $\mathrm{Diff}(M)$ can be identified with the set

$$T(\mathrm{Diff}(M)) := \{X \in C^\infty(M,TM)\colon \pi_{TM} \circ X \in \mathrm{Diff}(M)\},$$

where the map

$$\pi\colon T(\mathrm{Diff}(M)) \to \mathrm{Diff}(M), \quad X \mapsto \pi_{TM} \circ X$$

is the bundle projection. Then

$$T_\varphi(\mathrm{Diff}(M)) := \pi^{-1}(\varphi) = \{X \in C^\infty(M,TM)\colon \pi_{TM} \circ X = \varphi\}$$

is the tangent space in the diffeomorphism φ. The multiplication in the group $T(\mathrm{Diff}(M))$ is given by the formula

$$X \cdot Y := \pi_{T^2M} \circ TX \circ Y,$$

where $\pi_{T^2M}\colon T^2M \to TM$ is the natural projection. Note that

$$\pi_{TM} \circ (X \cdot Y) = \pi_{TM} \circ \pi_{T^2M} \circ TX \circ Y = \pi_{TM} \circ X \circ \pi_{TM} \circ Y$$

shows that π is a group homomorphism. Identifying $\varphi \in \mathrm{Diff}(M)$ with the origin in $T_\varphi(\mathrm{Diff}(M))$, we get

$$X \cdot \varphi = \pi_{T^2M} \circ TX \circ \varphi = X \circ \varphi \quad \text{and} \quad \varphi \cdot X = \pi_{T^2M} \circ T\varphi \circ X = T\varphi \circ X.$$

In particular, this leads to the formula

$$\mathrm{Ad}(\varphi).X = T\varphi \circ X \circ \varphi^{-1}$$

for the adjoint action of $\mathrm{Diff}(M)$ on $T_0(\mathrm{Diff}(M)) = \mathcal{V}(M)$. ■

III.2. From local data to global Lie groups

The following theorem is helpful to obtain Lie group structures on groups.

Theorem III.2.1. *Let G be a group and $U = U^{-1}$ a symmetric subset. We further assume that U is a smooth manifold such that*

(L1) *there exists an open $\mathbf{1}$-neighborhood $V \subseteq U$ with $V^2 = V \cdot V \subseteq U$ such that the group multiplication $m_V : V \times V \to U$ is smooth,*

(L2) *the inversion map $\eta_U : U \to U, u \mapsto u^{-1}$ is smooth, and*

(L3) *for each $g \in G$ there exists an open $\mathbf{1}$-neighborhood $U_g \subseteq U$ with $c_g(U_g) \subseteq U$ and such that the conjugation map $c_g : U_g \to U, x \mapsto gxg^{-1}$ is smooth.*

Then there exists a unique Lie group structure on G for which there exists an open $\mathbf{1}$-neighborhood $U_0 \subseteq U$ such that the inclusion map $U_0 \to G$ induces a diffeomorphism onto an open subset of G.

Proof. (cf. [Ch46, §14, Prop. 2] or [Ti83, p.14] for the finite-dimensional case) First we consider the filter basis $\mathcal{F}$ consisting of all $\mathbf{1}$-neighborhoods in U. In the terminology of Lemma B.2, (L1) implies (U1), (L2) implies (U2), and (L3) implies (U3). Moreover, the assumption that U is Hausdorff implies that $\bigcap \mathcal{F} = \{\mathbf{1}\}$. Therefore Lemma B.2 implies that G carries a unique structure of a (Hausdorff) topological group for which $\mathcal{F}$ is a basis of $\mathbf{1}$-neighborhoods.

After shrinking V and U, we may assume that there exists a diffeomorphism $\varphi : U \to \varphi(U) \subseteq E$, where E is a topological $\mathbb{K}$-vector space, $\varphi(U)$ an open subset, that V satisfies $V = V^{-1}$, $V^4 \subseteq U$, and that $m_V : V^2 \times V^2 \to U$ is smooth. For $g \in G$, we consider the maps

$$\varphi_g : gV \to E, \quad \varphi_g(x) = \varphi(g^{-1}x)$$

which are homeomorphisms of gV onto $\varphi(V)$. We claim that $(\varphi_g, gV)_{g\in G}$ is an E-atlas of G.

Let $g_1, g_2 \in G$ and put $W := g_1V \cap g_2V$. If $W \neq \emptyset$, then $g_2^{-1}g_1 \in VV^{-1} = V^2$. The smoothness of the map

$$\psi := \varphi_{g_2} \circ \varphi_{g_1}^{-1}|_{\varphi_{g_1}(W)} : \varphi_{g_1}(W) \to \varphi_{g_2}(W)$$

given by

$$\psi(x) = \varphi_{g_2}(\varphi_{g_1}^{-1}(x)) = \varphi_{g_2}(g_1\varphi^{-1}(x)) = \varphi(g_2^{-1}g_1\varphi^{-1}(x))$$

follows from the smoothness of the multiplication $V^2 \times V^2 \to U$. This proves that $(\varphi_g, gU)_{g\in G}$ is an atlas of G. Moreover, the construction implies that all left translations of G are smooth maps.

The construction also shows that for each $g \in G$, the conjugation $c_g : G \to G$ is smooth in a neighborhood of $\mathbf{1}$. Since all left translations are smooth, and $c_g \circ \lambda_x = \lambda_{c_g(x)} \circ c_g$, the smoothness of c_g in a neighborhood of $x \in G$ follows. Therefore all conjugations and hence also all right multiplications are smooth. The smoothness of the inversion follows from its smoothness on V and the fact that left and right multiplications are smooth. Finally the smoothness of the multiplication follows from the smoothness in $\mathbf{1} \times \mathbf{1}$ because

$$m_G(g_1x, g_2y) = g_1xg_2y = g_1g_2c_{g_2^{-1}}(x)y = g_1g_2m_G(c_{g_2^{-1}}(x), y).$$

The uniqueness of the Lie group structure is clear, because each locally diffeomorphic bijective homomorphism between Lie groups is a diffeomorphism. ■

Remark III.2.2. Suppose that the group G in Theorem III.2.1 is generated by each $\mathbf{1}$-neighborhood V in U. Then condition (L3) can be omitted. Indeed, the construction of the Lie group structure shows that for each $g \in V$, the conjugation $c_g : G \to G$ is smooth in a neighborhood of $\mathbf{1}$. Since the set of all these g is a submonoid of G containing V, it contains V^n for each $n \in \mathbb{N}$, hence all of G because G is generated by V. Therefore all conjugations are smooth, and one can proceed as in the proof of Theorem III.2.1. ■

Corollary III.2.3. *Let G be a group and $N \trianglelefteq G$ a normal subgroup that carries a Lie group structure. Then there exists a Lie group structure on G for which N is an open subgroup if and only if for each $g \in G$ the restriction $c_g|_N$ is a smooth automorphism of N.*

Proof. If N is an open normal subgroup of the Lie group G, then clearly all inner automorphisms of G restrict to smooth automorphisms of N.

Suppose, conversely, that N is a normal subgroup of the group G which is a Lie group and that all inner automorphisms of G restrict to smooth automorphisms of N. Then we can apply Theorem III.2.1 with $U = N$ and obtain a Lie group structure on G for which the inclusion $N \to G$ is a local diffeomorphism, hence a diffeomorphism onto an open subgroup of G. ∎

For the following corollary we recall that a surjective morphism $\varphi: G \to H$ of topological groups is called a *covering* if it is an open map with discrete kernel.

Corollary III.2.4. *Let $\varphi: G \to H$ be a covering of topological groups. If G or H is a Lie group, then the other group has a unique Lie group structure for which φ is a morphism of Lie groups which is a local diffeomorphism.*

Proof. Since φ is a covering, it is a local homeomorphism, so that there exists an open symmetric $\mathbf{1}$-neighborhood $W \subseteq G$ such that $\varphi_W := \varphi|_W: W \to \varphi(W)$ is a homeomorphism. We only have to choose W so small that we have $WW^{-1} \cap \ker\varphi = \{\mathbf{1}\}$ to ensure that φ_W is injective.

Suppose first that G is a Lie group. Then we apply Theorem III.2.1 with $U := \varphi(W)$. To verify (L1), we choose $W_1 \subseteq W$ open with $W_1 W_1 \subseteq W$ and put $V := \varphi(W_1)$, and for (L3) we note that the surjectivity of φ implies that for each $h \in H$, there is an element $g \in G$ with $\varphi(g) = h$. Now we choose an open $\mathbf{1}$-neighborhood $W_{\mathfrak{g}} \subseteq W$ with $c_g(W_g) \subseteq W$ and put $U_h := \varphi(W_g)$.

If, conversely, H is a Lie group, then we put $U := W$, as V we choose any open $\mathbf{1}$-neighborhood with $VV \subseteq U$, and as U_g we may also choose any open $\mathbf{1}$-neighborhood with $c_g(U_g) \subseteq U$. ∎

Corollary III.2.5. *Let G be a Lie group.*

(1) *If $N \trianglelefteq G$ is a discrete subgroup, then the quotient G/N carries a unique Lie group structure for which the quotient map $q: G \to G/N$ is a local diffeomorphism.*

(2) *If G is connected and $q_G: \widetilde{G} \to G$ the universal covering group, then $\widetilde{G}$ carries a unique Lie group structure for which q_G is a local diffeomorphism.*

Proof. (1) follows directly from Corollary III.2.4, because the quotient map $G \to G/N$ is a covering.

(2) We first have to construct a topological group structure on the universal covering space $\widetilde{G}$. Pick an element $\widetilde{\mathbf{1}} \in q_G^{-1}(\mathbf{1})$. Then the multiplication map $m_G: G \times G \to G$ lifts uniquely to a continuous map $\widetilde{m}_G: \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ with $\widetilde{m}_G(\widetilde{\mathbf{1}}, \widetilde{\mathbf{1}}) = \widetilde{\mathbf{1}}$. To see that the multiplication map $\widetilde{m}_G$ is associative, we observe that

$$\begin{aligned} q_G \circ \widetilde{m}_G \circ (\mathrm{id}_{\widetilde{G}} \times \widetilde{m}_G) &= m_G \circ (q_G \times q_G) \circ (\mathrm{id}_{\widetilde{G}} \times \widetilde{m}_G) = m_G \circ (\mathrm{id}_G \times m_G) \circ (q_G \times q_G \times q_G) \\ &= m_G \circ (m_G \times \mathrm{id}_G) \circ (q_G \times q_G \times q_G) = q_G \circ \widetilde{m}_G \circ (\widetilde{m}_G \times \mathrm{id}_{\widetilde{G}}), \end{aligned}$$

so that the two continuous maps

$$\widetilde{m}_G \circ (\mathrm{id}_{\widetilde{G}} \times \widetilde{m}_G), \quad \widetilde{m}_G \circ (\widetilde{m}_G \times \mathrm{id}_{\widetilde{G}}): \widetilde{G}^3 \to G,$$

are lifts of the same map $G^3 \to G$ and both map $(\widetilde{\mathbf{1}}, \widetilde{\mathbf{1}}, \widetilde{\mathbf{1}})$ to $\widetilde{\mathbf{1}}$. Hence the uniqueness of lifts implies that $\widetilde{m}_G$ is associative. We likewise obtain that the unique lift $\widetilde{\eta}_G: \widetilde{G} \to \widetilde{G}$ of the inversion map $\eta_G: G \to G$ with $\widetilde{\eta}_G(\widetilde{\mathbf{1}}) = \widetilde{\mathbf{1}}$ satisfies

$$\widetilde{m}_G \circ (\eta_G \times \mathrm{id}_{\widetilde{G}}) = \widetilde{\mathbf{1}} = \widetilde{m}_G \circ (\mathrm{id}_{\widetilde{G}} \times \eta_G).$$

Therefore $\widetilde{m}_G$ defines on $\widetilde{G}$ a topological group structure such that $q_G: \widetilde{G} \to G$ is a covering morphism of topological groups. Now Corollary III.2.4 applies. ∎

Remark III.2.6. If $q_G\colon \widetilde{G} \to G$ is the universal covering morphism of a connected Lie group G, then $\ker q_G$ is a discrete normal subgroup of the connected group $\widetilde{G}$, hence central (Exercise III.3). Left multiplications by elements of this group lead to deck transformations of the covering $\widetilde{G} \to G$, and this shows that $\pi_1(G) \cong \ker q_G$ as groups.

Clearly, $G \cong \widetilde{G}/\ker q_G$. If, conversely, $\Gamma \subseteq \widetilde{G}$ is a discrete central subgroup, then $\widetilde{G}/\Gamma$ is a Lie group with the same universal covering group as G. Two such groups $\widetilde{G}/\Gamma_1$ and $\widetilde{G}/\Gamma_2$ are isomorphic if and only if there exists a Lie group automorphism $\varphi \in \mathrm{Aut}(\widetilde{G})$ with $\varphi(\Gamma_1) = \Gamma_2$. Therefore the isomorphism classes of Lie groups with the same universal covering group G are parametrized by the orbits of the group $\mathrm{Aut}(\widetilde{G})$ in the set S of discrete central subgroups of $\widetilde{G}$. Since the normal subgroup $\mathrm{Inn}(\widetilde{G}) := \{c_g\colon g \in \widetilde{G}\}$ of inner automorphisms acts trivially on this set, the action of $\mathrm{Aut}(\widetilde{G})$ on S factors through an action of the group $\mathrm{Out}(\widetilde{G}) := \mathrm{Aut}(\widetilde{G})/\mathrm{Inn}(\widetilde{G})$.

Since each automorphism $\varphi \in \mathrm{Aut}(G)$ lifts to a unique automorphism $\widetilde{\varphi} \in \mathrm{Aut}(\widetilde{G})$ (Exercise!), we have a natural embedding $\mathrm{Aut}(G) \hookrightarrow \mathrm{Aut}(\widetilde{G})$, and the image of this homomorphism consists of the stabilizer of the subgroup $\ker q_G \subseteq Z(\widetilde{G})$. ■

Exercises for Section III

Exercise III.1. Let $(G, D, m_G, \mathbf{1})$ be a local Lie group. Show that:

(1) For $g, h, u \in G$ with $(g,h), (h,u), (gh, u) \in D$, we have
$$d\lambda_g(h) \circ d\lambda_h(u) = d\lambda_{gh}(u).$$
Hint: Show that $\lambda_g \circ \lambda_h = \lambda_{gh}$ on a neighborhood of u.

(2) For the open set $D_g := \{h \in G\colon (g,h) \in D\}$ and the smooth map
$$\lambda_g\colon D_g \to G, \quad h \mapsto gh$$
the vector field defined by $x_l(u) := d\lambda_u(\mathbf{1}).x$ satisfies the left invariance condition
$$x_l \circ \lambda_g = T(\lambda_g) \circ x_l|_{D_g}.$$

(3) Show that the set $\mathcal{V}(G)^l$ of left invariant vector fields on G is a Lie subalgebra of the Lie algebra $\mathcal{V}(G)$ and show that this leads to a Lie bracket on $\mathfrak{g} = T_{\mathbf{1}}(G)$.

(4) The tangent bundle TG of G carries a local Lie group structure $(TG, TD, Tm_G, 0_{\mathbf{1}})$.

(5) If $\varphi\colon G \to H$ is a morphism of local Lie groups, then $\mathbf{L}(\varphi) := d\varphi(\mathbf{1})$ is a homomorphism of Lie algebras.

(6) For $x \in G$ and $(x,y), (y, x^{-1}), (xy, x^{-1}) \in D$, we put $c_x(y) := (xy)x^{-1}$ and note that this map is defined on some neighborhood of $\mathbf{1}$. If $(x,y) \in D$, then $c_x \circ c_y = c_{xy}$ holds on a neighborhood of $\mathbf{1}$.

(7) $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g}), g \mapsto \mathbf{L}(c_g)$ is a homomorphism of the local group G to the group $\mathrm{Aut}(\mathfrak{g})$. ■

Exercise III.2. Let G be an abelian group and $N \leq G$ a subgroup carrying a Lie group structure. Then there exists a unique Lie group structure on G for which N is an open subgroup. Hint: Corollary III.2.3. ■

Exercise III.3. Let G be a connected topological group and $\Gamma \trianglelefteq G$ a discrete normal subgroup. Then Γ is central. ■

Exercise III.4. Let A be a CIA and M a compact smooth manifold. Show that $C^\infty(M, A)$ is a CIA with respect to the natural topology on this algebra which is obtained from the embedding
$$C^\infty(M, A) \hookrightarrow \prod_{p \in \mathbb{N}_0} C(T^p M, T^p A),$$
where the right hand side carries the product topology and on each factor the topology of compact convergence (which, in view of Appendix B, coincides with the compact open topology). ■

Exercise III.5. Let G be a Lie group and T^nG, $n \in \mathbb{N}$, its iterated tangent bundles. Show that:

(1) $TG \cong (\mathfrak{g}, +) \rtimes_{\mathrm{Ad}} G$.

(2) The adjoint action of G on $\mathfrak{g}$ induces an action $T\,\mathrm{Ad}$ of $TG \cong \mathfrak{g} \rtimes G$ on $T\mathfrak{g} \cong \mathfrak{g} \times \mathfrak{g}$, given by

$$(T\,\mathrm{Ad})(x,g)(v,w) = (\mathrm{Ad}(g).v + [x, \mathrm{Ad}(g).w], \mathrm{Ad}(g).w).$$

(3) $T^2G \cong (\mathfrak{g} \times \mathfrak{g}) \rtimes_{T\,\mathrm{Ad}} (\mathfrak{g} \rtimes G)$. The multiplication in this group satisfies

$$(x_2, x_1, x_0, \mathbf{1})(x_2', x_1', x_0', \mathbf{1}) = (x_2 + x_2' + [x_0, x_1'], x_1 + x_1', x_0 + x_0').$$

(4) Generalize (3) to T^3G.

(5) $T^nG \cong N \rtimes G$, where N is a nilpotent Lie group diffeomorphic to $\mathfrak{g}^{2^n-1}$. ∎

Exercise III.6. (a) Let $m: G \times G \to G$ be a smooth associative multiplication on the manifold G with identity element $\mathbf{1}$. Show that the differential in $(\mathbf{1}, \mathbf{1})$ is given by

$$dm(\mathbf{1}, \mathbf{1}): T_{\mathbf{1}}(G) \times T_{\mathbf{1}}(G) \to T_{\mathbf{1}}(G), \quad (v,w) \mapsto v + w.$$

(b) Show that the smoothness of the inversion in the definition of a Banach–Lie group is redundant because the Inverse Function Theorem can be applied to the map

$$G \times G \to G \times G, \quad (x,y) \mapsto (x, xy)$$

whose differential in $(\mathbf{1}, \mathbf{1})$ is given by the map $(v,w) \mapsto (v, v+w)$. ∎

Exercise III.7. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\varphi: U_G \to \mathfrak{g}$ a local chart with $\varphi(\mathbf{1}) = 0$. Show that:

(1) For the local multiplication $x * y := \varphi(\varphi^{-1}(x)\varphi^{-1}(y))$, the second order Taylor polynomial of $x * y * x^{-1} * y^{-1}$ in $(0,0)$ is the Lie bracket $[x,y]$.

(2) Use (1) to show that for each morphism of Lie groups $\varphi: G \to H$, the map $d\varphi(\mathbf{1})$ is a homomorphism of Lie algebras. Hint: Compare the second order Taylor polynomials of $\varphi(x) * \varphi(y) * \varphi(x)^{-1} * \varphi(y)^{-1}$ and $\varphi(x * y * x^{-1} * y^{-1})$ by using the Chain Rule for Taylor polynomials. ∎

Exercise III.8. Let G be a Lie group, V a locally convex space and $\sigma: G \times V \to V$ a smooth linear action of G on V. Then all vector fields $\dot\sigma(x)$, $x \in \mathfrak{g}$, are linear, and we thus obtain a representation of Lie algebras $\mathbf{L}(\sigma): \mathfrak{g} \to \mathfrak{gl}(V)$ with $\mathbf{L}(\sigma)(x)v = -\dot\sigma(x)(v)$. ∎

Exercise III.9. Let G and N be Lie groups and $\varphi: G \to \mathrm{Aut}(N)$ be a homomorphism such that the map $G \times N \to N, (g,n) \mapsto \varphi(g)(n)$ is smooth. Then the semi-direct product group $N \rtimes G$ with the multiplication

$$(n,g)(n',g') := (n\varphi(g)(n'), gg')$$

is a Lie group with Lie algebra $\mathfrak{n} \rtimes_{\mathbf{L}(\varphi)} \mathfrak{g}$, where $\mathbf{L}(\varphi): \mathfrak{g} \to \mathrm{der}(\mathfrak{n})$ is the derived representations (cf. Exercise III.8). ∎

IV. The Fundamental Theorem for Lie group-valued functions

In this section, we undertake a systematic study of Lie group-valued functions. In the same way as a smooth function $f: M \to V$ on a connected manifold M with values in a locally convex space V is determined by a value in one-point and the differential form $df \in \Omega^1(M,V)$, we can associate to a smooth function $f: M \to G$ with values in a Lie group a smooth 1-form

$\delta(f) \in \Omega^1(M, \mathfrak{g})$. We shall see that if M is connected, then $\delta(f)$ determines f up to left multiplication by a constant. Conversely, we can ask which $\mathfrak{g}$-valued 1-forms α are integrable in the sense that $\alpha = \delta(f)$ for some smooth function $f\colon M \to G$. For the special case $M = [0,1]$, this leads to the concept of a regular Lie group and finally the Fundamental Theorem for Lie group-valued functions gives necessary and sufficient conditions for $\alpha \in \Omega^1(M, \mathfrak{g})$ to be integrable in the sense that it is of the form $\delta(f)$.

The main point of this setup is that $\mathfrak{g}$-valued 1-forms are much simpler objects than Lie group-valued functions. In particular, each Lie algebra homomorphism $\varphi\colon \mathbf{L}(G) \to \mathbf{L}(H)$ defines an $\mathbf{L}(H)$-valued 1-form on G which is integrable if and only if there exists a Lie group homomorphism $\psi\colon G \to H$ with $\mathbf{L}(\psi) = \varphi$. If G is 1-connected and H is regular, such a homomorphism always exists.

IV.1. Logarithmic derivatives and their applications

Equivariant differential forms and Lie algebra cohomology

Definition IV.1.1. Let G be a Lie group and V a *smooth locally convex G-module*, i.e., V is a locally convex space and the action map $\rho_V\colon G \times V \to V, (g,v) \mapsto g.v$ is smooth. We write $\rho_V(g)(v) := g.v$ for the corresponding continuous linear automorphisms of V.

We call a p-form $\alpha \in \Omega^p(G,V)$ *equivariant* if we have for each $g \in G$ the relation

$$\lambda_g^*\alpha = \rho_V(g) \circ \alpha.$$

We write $\Omega^p(G,V)^G$ for the subspace of equivariant p-forms in $\Omega^p(G,V)$ and note that this is the space of G-fixed elements with respect to the action given by $g.\alpha := \rho_V(g) \circ (\lambda_{g^{-1}})^*\alpha$. *

If V is a trivial module, then an equivariant form is a left invariant V-valued form on G. An equivariant p-form α is uniquely determined by the corresponding element $\alpha_{\mathbf{1}} \in C^p_c(\mathfrak{g}, V) = \mathrm{Alt}^p(\mathfrak{g}, V)$ (cf. Appendix C):

$$\alpha_g(g.x_1, \ldots, g.x_p) = \rho_V(g) \circ \alpha_{\mathbf{1}}(x_1, \ldots, x_p) \tag{4.1.1}$$

for $g \in G, x_i \in \mathfrak{g}$.

Conversely, (4.1.1) provides for each $\omega \in C^p_c(\mathfrak{g}, V)$ a unique equivariant p-form ω^{eq} on G with $\omega^{\mathrm{eq}}_{\mathbf{1}} = \omega$. ■

The following proposition shows that the complex of equivariant differential forms is the same as the Lie algebra complex associated to the $\mathfrak{g}$-module V.

Proposition IV.1.2. *For each $\omega \in C^p_c(\mathfrak{g}, V)$, we have $d(\omega^{\mathrm{eq}}) = (d_{\mathfrak{g}}\omega)^{\mathrm{eq}}$. In particular, the evaluation map*

$$\mathrm{ev}_{\mathbf{1}}\colon \Omega^p(G,V)^G \to C^p_c(\mathfrak{g}, V), \quad \omega \mapsto \omega_{\mathbf{1}}$$

defines an isomorphism from the chain complex $(\Omega^\bullet(G,V)^G, d)$ of equivariant V-valued differential forms on G to the continuous V-valued Lie algebra complex $(C^\bullet_c(\mathfrak{g}, V), d_{\mathfrak{g}})$.

Proof. (cf. [ChE48, Th. 10.1]) For $g \in G$, we have

$$\lambda_g^* d\omega^{\mathrm{eq}} = d\lambda_g^*\omega^{\mathrm{eq}} = d(\rho_V(g) \circ \omega^{\mathrm{eq}}) = \rho_V(g) \circ (d\omega^{\mathrm{eq}}),$$

showing that $d\omega^{\mathrm{eq}}$ is equivariant.

---

* The complex $(\Omega^\bullet(G,V)^G, d)$ of equivariant differential forms has been introduced in the finite-dimensional setting by Chevalley and Eilenberg in [ChE48].

For $x \in \mathfrak{g}$, we write $x_l(g) := g.x$ for the corresponding left invariant vector field on G. In view of (4.1.1), it suffices to calculate the value of $d\omega^{\mathrm{eq}}$ on $(p+1)$-tuples of left invariant vector fields in the identity element. From

$$\omega^{\mathrm{eq}}(x_{1,l}, \ldots, x_{p,l})(g) = \rho_V(g).\omega(x_1, \ldots, x_p),$$

we obtain

$$\big(x_{0,l}.\omega^{\mathrm{eq}}(x_{1,l}, \ldots, x_{p,l})\big)(\mathbf{1}) = x_0.\omega(x_1, \ldots, x_p),$$

and therefore

$$\begin{aligned}
&\big(d\omega^{\mathrm{eq}}(x_{0,l}, \ldots, x_{p,l})\big)(\mathbf{1}) \\
&= \sum_{i=0}^{p} (-1)^i x_{i,l}.\omega^{\mathrm{eq}}(x_{0,l}, \ldots, \widehat{x_{i,l}}, \ldots, x_{p,l})(\mathbf{1}) \\
&\quad + \sum_{i<j} (-1)^{i+j} \omega^{\mathrm{eq}}([x_{i,l}, x_{j,l}], x_{0,l}, \ldots, \widehat{x_{i,l}}, \ldots, \widehat{x_{j,l}}, \ldots, x_{p,l})(\mathbf{1}) \\
&= \sum_{i=0}^{p} (-1)^i x_i.\omega(x_0, \ldots, \widehat{x}_i, \ldots, x_p) + \sum_{i<j} (-1)^{i+j} \omega([x_i, x_j], x_0, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_p) \\
&= (d_{\mathfrak{g}}\omega)(x_0, \ldots, x_p).
\end{aligned}$$

This proves our assertion. ■

Maurer–Cartan forms and logarithmic derivatives

For the following definition, we recall from Lemma III.1.4 that for each Lie group G, the tangent bundle TG has a natural Lie group structure containing G as the zero section. Restricting the multiplication of TG to $G \times TG$, we obtain in particular a smooth left action of G on TG which we simply write $(g, v) \mapsto g.v$.

Definition IV.1.3. (a) For $v \in T_g(G)$, we define $\kappa_G(v) := g^{-1}.v \in \mathfrak{g} = T_{\mathbf{1}}(G)$ and note that this defines a smooth 1-form $\kappa_G \in \Omega^1(G, \mathfrak{g})$ because the multiplication in the Lie group TG is smooth. This form is called the *(left) Maurer–Cartan form of G*. It is a left invariant $\mathfrak{g}$-valued 1-form on G.

(b) Let M be a smooth manifold and G a Lie group with Lie algebra $\mathbf{L}(G) = \mathfrak{g}$. For an element $f \in C^\infty(M, G)$, we define the *(left) logarithmic derivative* as the $\mathfrak{g}$-valued 1-form

$$\delta(f) := f^*\kappa_G \in \Omega^1(M, \mathfrak{g}).$$

For $v \in T_m(M)$, this means that $\delta(f)_m(v) = f(m)^{-1}.(df)_m(v) = f(m)^{-1}Tf(v)$.

We call $\alpha \in \Omega^1(M, \mathfrak{g})$ *G-integrable* if there exists a smooth function $f\colon M \to G$ with $\delta(f) = \alpha$.

(c) If $M = I$ is an interval, then we identify $\Omega^1(I, \mathfrak{g})$ with $C^\infty(I, \mathfrak{g})$ by identifying the smooth function $\xi\colon I \to \mathfrak{g}$ with the 1-form $\xi \cdot dt$. In this sense, we can interprete for a smooth curve $\gamma\colon I \to G$ the logarithmic derivative $\delta(\gamma) = \gamma^*\kappa_G$ as a smooth curve in $\mathfrak{g}$. Explicitly, we have

$$\delta(\gamma)(t) = \gamma(t)^{-1}.\gamma'(t).$$ ■

We recall from Definition II.4.1 that on the space $\Omega^\bullet(M, \mathfrak{g})$ of $\mathfrak{g}$-valued differential forms on M we have a natural bracket

$$\Omega^p(M, \mathfrak{g}) \times \Omega^q(M, \mathfrak{g}) \to \Omega^{p+q}(M, \mathfrak{g}), \quad (\alpha, \beta) \mapsto [\alpha, \beta]$$

which for $\alpha, \beta \in \Omega^1(M, \mathfrak{g})$ satisfies for $v, w \in T_m(M)$

$$[\alpha, \beta]_m(v, w) = [\alpha_m(v), \beta_m(w)] - [\alpha_m(w), \beta_m(v)] = 2[\alpha_m(v), \beta_m(w)]$$

(Exercise II.15).

Lemma IV.1.4. (Product and Quotient Rule) *For smooth functions* $f, g\colon M \to G$, *we have*

$$\delta(fg) = \delta(g) + \mathrm{Ad}(g)^{-1}.\delta(f), \tag{4.1.2}$$

where $(\mathrm{Ad}(g)^{-1}.\delta(f))_m := \mathrm{Ad}(g(m))^{-1} \circ \delta(f)_m$. *In particular, we have*

$$\delta(f^{-1}) = -\,\mathrm{Ad}(f).\delta(f). \tag{4.1.3}$$

Proof. Clearly the pointwise product is a smooth function $fg\colon M \to G$. With the Chain Rule we obtain

$$d(fg)_m = f(m).(dg)_m + (df)_m.g(m),$$

and this leads to

$$\begin{aligned}\delta(fg)_m &= (fg)(m)^{-1}.d(fg)_m = g(m)^{-1}.(dg)_m + g(m)^{-1}f(m)^{-1}.(df)_m.g(m)\\ &= \delta(g)_m + \mathrm{Ad}(g(m))^{-1} \circ \delta(f)_m,\end{aligned}$$

which is (4.1.2). Putting $g = f^{-1}$, we obtain (4.1.3). ■

The following lemma provides a uniqueness result for the equation $\delta(f) = \alpha$.

Lemma IV.1.5. (Uniqueness Lemma) *If two smooth functions* $f_1, f_2\colon M \to G$ *have the same left logarithmic derivative and* M *is connected, then there exists* $g \in G$ *with* $f_1 = \lambda_g \circ f_2$.

Proof. We have to show that the function $x \mapsto f_1(x)f_2(x)^{-1}$ is locally constant, hence constant, because M is connected. First we obtain with Lemma IV.1.4

$$\delta(f_1f_2^{-1}) = \delta(f_2^{-1}) + \mathrm{Ad}(f_2)\delta(f_1) = \delta(f_2^{-1}) + \mathrm{Ad}(f_2)\delta(f_2) = \delta(f_2f_2^{-1}) = 0.$$

This implies that $d(f_1f_2^{-1})$ vanishes, and hence that $f_1f_2^{-1}$ is locally constant. ■

For the existence of a solution of the equation $\delta(f) = \alpha$, the following lemma provides a necessary condition.

Lemma IV.1.6. *If* $\alpha = \delta(f)$ *for some* $f \in C^\infty(M,G)$, *then* α *satisfies the Maurer–Cartan equation*

$$d\alpha + \frac{1}{2}[\alpha,\alpha] = 0. \tag{MC}$$

Proof. We first show that κ_G satisfies the MC equation. For that, we observe that the isomorphism of chain complexes

$$\mathrm{ev}_{\mathbf{1}}\colon \Omega^p(G,\mathfrak{g})^G \to C^p_c(\mathfrak{g},\mathfrak{g}), \quad \omega \mapsto \omega_{\mathbf{1}},$$

corresponding to the trivial action of G on $\mathfrak{g}$ is compatible with the bracket defined on both sides (cf. Exercise II.15). Since $\kappa_G = (\mathrm{id}_{\mathfrak{g}})^{\mathrm{eq}}$ and

$$(d_{\mathfrak{g}}\,\mathrm{id}_{\mathfrak{g}})(x,y) = -\,\mathrm{id}_{\mathfrak{g}}([x,y]) = -[x,y] = -\frac{1}{2}[\mathrm{id}_{\mathfrak{g}},\mathrm{id}_{\mathfrak{g}}](x,y),$$

we derive

$$d_{\mathfrak{g}}\,\mathrm{id}_{\mathfrak{g}} + \frac{1}{2}[\mathrm{id}_{\mathfrak{g}},\mathrm{id}_{\mathfrak{g}}]$$

in $C^2_c(\mathfrak{g},\mathfrak{g})$, and with Proposition IV.1.2 this leads to

$$d\kappa_G + \frac{1}{2}[\kappa_G,\kappa_G] = 0.$$

Therefore $\alpha = f^*\kappa_G$ satisfies

$$d\alpha = f^*d\kappa_G = -\frac{1}{2}f^*([\kappa_G,\kappa_G]) = -\frac{1}{2}[f^*\kappa_G, f^*\kappa_G] = -\frac{1}{2}[\alpha,\alpha],$$

which is the Maurer–Cartan equation for α. ■

Remark IV.1.7. If M is one-dimensional, then each $\mathfrak{g}$-valued 2-form on M vanishes, so that $[\alpha,\beta] = 0 = d\alpha$ for $\alpha,\beta \in \Omega^1(M,\mathfrak{g})$. Therefore all 1-forms trivially satisfy the Maurer–Cartan equation. ■

Proposition IV.1.8. *Let G and H be Lie groups.*

(1) *If $\varphi: G \to H$ is a morphism of Lie groups, then $\delta(\varphi) = \mathbf{L}(\varphi) \circ \kappa_G$. For any smooth function $f: M \to G$, we have $\delta(\varphi \circ f) = \mathbf{L}(\varphi) \circ \delta(f)$.*

(2) *If G is connected and $\varphi_1, \varphi_2 : G \to H$ are morphisms of Lie groups with $\mathbf{L}(\varphi_1) = \mathbf{L}(\varphi_2)$, then $\varphi_1 = \varphi_2$.*

(3) *Suppose that we are given a smooth action of the connected Lie group G on H by automorphisms, so that we also obtain a smooth action of G on $\mathfrak{h} = \mathbf{L}(H)$. Then for a smooth function $f: G \to H$ with $f(\mathbf{1}) = \mathbf{1}$ the following are equivalent:*
   (a) *$\delta(f)$ is an equivariant $\mathfrak{h}$-valued 1-form on G.*
   (b) *$f(gx) = f(g) \cdot g.f(x)$ for $g, x \in G$, i.e., f is a crossed homomorphism.*

Proof. (1) For $g \in G$, we have $\varphi \circ \lambda_g = \lambda_{\varphi(g)} \circ \varphi$, so that

$$\delta(\varphi)_g = d(\lambda_{\varphi(g)}^{-1} \circ \varphi)_g = d(\varphi \circ \lambda_g^{-1})_g = (d\varphi)(\mathbf{1}) \circ d\lambda_g^{-1}(g) = \mathbf{L}(\varphi) \circ (\kappa_G)_g.$$

For any smooth function $f: M \to G$, we now get

$$\delta(\varphi \circ f) = f^*\varphi^*\kappa_H = f^*(\mathbf{L}(\varphi) \circ \kappa_G) = \mathbf{L}(\varphi) \circ f^*\kappa_G = \mathbf{L}(\varphi) \circ \delta(f).$$

(2) In view of (1), $\delta(\varphi_1) = \delta(\varphi_2)$, so that the assertion follows from $\varphi_1(\mathbf{1}) = \varphi_2(\mathbf{1})$ and Lemma IV.1.3.

(3) We write $g.x = \rho_{\mathfrak{h}}(g).x$ for the action of G on $\mathfrak{h}$ and $g.h = \rho_H(g).h$ for the action of G on H and note that $\mathbf{L}(\rho_H(g)) = \rho_{\mathfrak{h}}(g)$ holds for each $g \in G$.

Let $g \in G$. Then the logarithmic derivative of $\lambda_{f(g)}^{-1} \circ f \circ \lambda_g$ is $\lambda_g^*\delta(f)$, and, in view of (1), the logarithmic derivative of $\rho_H(g) \circ f$ is $\rho_{\mathfrak{h}}(g) \circ \delta(f)$. Since both functions map $\mathbf{1}$ to $\mathbf{1}$, they coincide if and only if their logarithmic derivatives coincide (Lemma IV.1.5). This implies (3). ■

Corollary IV.1.9. *If G is a connected Lie group, then* $\ker \mathrm{Ad} = Z(G)$.

Proof. Let $c_g(x) = gxg^{-1}$. In view of Proposition IV.1.8(2), for $g \in G$ the conditions $c_g = \mathrm{id}_G$ and $\mathbf{L}(c_g) = \mathrm{Ad}(g) = \mathrm{id}_{\mathfrak{g}}$ are equivalent. This implies the assertion. ■

Proposition IV.1.10. *A connected Lie group G is abelian if and only if its Lie algebra is abelian.*

Proof. That the Lie algebra of an abelian Lie group is abelian is a direct consequence of Lemma III.1.6, which implies that in any chart the second order Taylor polynomial of the multiplication has the form $x + y + b(x,y)$ with $[x,y] = b(x,y) - b(y,x)$. If G is abelian, then b is symmetric, and therefore $\mathbf{L}(G)$ is abelian.

In view of the preceding corollary, we have to show that for each $g \in G$ we have $\mathrm{Ad}(g) = \mathbf{1}$. Let $x \in \mathfrak{g}$ and consider a smooth curve $\gamma: [0,1] \to G$ with $\gamma(0) = \mathbf{1}$ and $\gamma(1) = g$. For $\eta(t) := \mathrm{Ad}(\gamma(t)).x$, we then have by Proposition III.1.16

$$\eta'(t) = T\,\mathrm{Ad}(\gamma(t).\delta(\gamma)(t), 0) = \mathrm{Ad}(\gamma(t)).[\delta(\gamma)(t), x] = 0$$

for each t, so that η is constant. This implies that $\mathrm{Ad}(g).x = \eta(1) = \eta(0) = x$. ■

Problem IV.1. Show that a connected Lie group G is nilpotent/solvable if and only if its Lie algebra $\mathfrak{g}$ is nilpotent/solvable. A promising strategy should be to show that certain commutators vanishing for nilpotent/solvable Lie algebras can be expressed as derivatives of certain commutator expressions in the group. Such an argument would imply thay nilpotence/solvability of G entails the corresponding property of $\mathfrak{g}$.

If, conversely, $\mathfrak{g}$ is nilpotent/solvable, then the adjoint representation has certain properties which have to be "integrated" to the group. ■

IV.2. Regular Lie groups and the Fundamental Theorem

If $M = I = [0,1]$, then the Maurer–Cartan equation is satisfied by each $\xi \in \Omega^1(I,\mathfrak{g}) \cong C^\infty(I,\mathfrak{g})$, because each 2-form on I vanishes. The requirement that for each smooth curve $\xi \in C^\infty(I,\mathfrak{g})$, the ordinary differential equation

$$\gamma'(t) = \gamma(t).\xi(t) \quad \text{for} \quad t \in I,$$

has a solution depending smoothly on ξ leads to the concept of a regular Lie group.

Definition IV.2.1. A Lie group G is called *regular* if for each $\xi \in C^\infty(I,\mathfrak{g})$ the initial value problem (IVP)

$$\gamma(0) = \mathbf{1}, \quad \delta(\gamma) = \xi, \tag{4.2.1}$$

has a solution $\gamma_\xi \in C^\infty(I,G)$ and the evolution map

$$\mathrm{evol}_G: C^\infty(I,\mathfrak{g}) \to G, \quad \xi \mapsto \gamma_\xi(1)$$

is smooth.

For a regular Lie group G, we define the *exponential function*

$$\exp: \mathbf{L}(G) = \mathfrak{g} \to G \quad \text{by} \quad \exp(x) := \gamma_x(1) = \mathrm{evol}_G(x),$$

where $x \in \mathfrak{g}$ is considered as a constant function $I \to \mathfrak{g}$. As a restriction of the smooth function evol_G, the exponential function is smooth. ■

For a general Lie group G, we call a smooth function $\exp_G: \mathfrak{g} \to G$ an *exponential function for* G if for each $x \in \mathfrak{g}$ the curve $\gamma_x(t) := \exp(tx)$ is a solution of the IVP (4.2.1). According to Lemma IV.1.5, such a solution is unique whenever it exists. Therefore a Lie group has at most one exponential function.

Remark IV.2.2. (a) As a direct consequence of the existence of solutions to ordinary differential equations on open domains of Banach spaces and their smooth dependence on parameters, every Banach–Lie group is regular.

(b) Let A be a unital Banach algebra and $A^\times$ its unit group. Since A is a CIA, $A^\times$ is a Lie group. For $x \in A$, the corresponding left invariant vector field is given on $A^\times$ by $x_l(a) = ax$, and the unique solutions of the IVP (4.2.1) are given by $\gamma(t) = \exp(tx)$, where

$$\exp_A: A \to A^\times, \quad x \mapsto \sum_{k=0}^\infty \frac{1}{k!}x^k$$

is the exponential function of A. This implies that $\exp_A$ is a smooth exponential function of the Lie group A.

This remains true for each Mackey complete CIA A: For each $x \in A$, the exponential series converges and $\exp_A$ defines a smooth exponential function of $A^\times$ (cf. [Gl02b]).

(c) Although it might be hard to verify it in concrete situations, all "known" Lie groups modeled on Mackey complete spaces are regular. For example we do not know if all unit groups of Mackey complete CIAs are regular, but we have just seen in (b) that they always have a smooth exponential function.

If the model space is no longer assumed to be Mackey complete, one obtains non-regular Lie groups as follows (cf. [Gl02b, Sect. 7]): Let $A \subseteq C([0,1],\mathbb{R})$ denote the subalgebra of all rational functions, i.e., of all quotients $p(x)/q(x)$, where $q(x)$ is a polynomial without zero in $[0,1]$. We endow A with the induced norm $\|f\| := \sup_{0\leq t\leq 1}|f(t)|$. If an element $f \in A$ is invertible in $C([0,1],\mathbb{R})$, then it has no zero in $[0,1]$, which implies that it is also invertible in A, i.e.,

$$A^\times = C([0,1],\mathbb{R})^\times \cap A.$$

This shows that $A^\times$ is open in A, and since the Banach algebra $C([0,1],\mathbb{R})$ is a CIA, the smoothness of the inversion is inherited by A, so that A is a CIA. Hence $A^\times$ is a Lie group (Example III.1.3).

If $A^\times$ is regular, then it also has a smooth exponential function, and from Lemma IV.1.3 we derive that it is the restriction of the exponential function of $C([0,1],\mathbb{R})^\times$ to A, which leads to

$$\exp_A(f) = e^f, \quad t \mapsto e^{f(t)}.$$

This contradicts the observation that for the function $f(t) = t$, the function e^f is not rational. Therefore the Lie group $A^\times$ does not have an exponential function, hence is not regular.

(d) If V is a locally convex space, then $(V,+)$ is a regular Lie group if and only if it is Mackey complete because this means that for each smooth curve $\xi\colon I \to V$, there is a smooth curve $\gamma_\xi\colon I \to V$ with $\gamma_\xi' = \xi$. Regularity is inherited by all abelian Lie groups of the form $Z = V/\Gamma$, where Γ is a discrete subgroup of V (Exercise III.4) (cf. Corollary I.1.17 for the Lie group structure on V/Γ).

(e) If K is a Lie group with a smooth exponential function $\exp_K\colon \mathfrak{k} \to K$ and M is a compact smooth manifold, then we obtain an exponential function of the group $C^\infty(M,K)$ by

$$\exp_G\colon \mathfrak{g} = C^\infty(M,\mathfrak{k}) \to G = C^\infty(M,K), \quad \xi \mapsto \exp_K \circ \xi. \qquad \blacksquare$$

The following theorem is an important tool to verify that given Lie groups are regular.

Theorem IV.2.3. *Let $\widehat{G}$ be a Lie group extension of the Lie groups G and N, i.e., there exists a surjective morphism $q\colon \widehat{G} \to G$ with $\ker q \cong N$, where $\widehat{G}$ carries the structure of an N-principal bundle. Then the group $\widehat{G}$ is regular if and only if the groups G and N are regular.* $\blacksquare$

The Fundamental Theorem

Lemma IV.2.4. (Omori) *If G is a regular Lie group, $x \in \mathfrak{g}$ and $\xi \in C^\infty(I,\mathfrak{g})$, then the initial value problem*

$$\eta'(t) = [\eta(t),\xi(t)], \quad \eta(0) = x \tag{E1}$$

has a unique solution given by

$$\eta(t) = \mathrm{Ad}(\gamma_\xi(t))^{-1}.x. \tag{E2}$$

Proof. For $\gamma(t) := \gamma_\xi(t)$, we get with Lemma IV.1.4

$$\delta(\gamma^{-1}) = -\mathrm{Ad}(\gamma).\delta(\gamma) = -\mathrm{Ad}(\gamma).\xi.$$

We define η by (E2). Then η is a smooth curve with

$$\eta'(t) = \mathrm{Ad}(\gamma(t))^{-1}[-\mathrm{Ad}(\gamma(t)).\xi(t), x] = [\mathrm{Ad}(\gamma(t))^{-1}.x, \xi(t)] = [\eta(t),\xi(t)]$$

(Proposition III.1.16).

Now let β be another solution of (E1) and consider the curve

$$\widetilde{\beta}(t) := \mathrm{Ad}(\gamma(t)).\beta(t).$$

Then $\widetilde{\beta}(0) = \beta(0) = x$, and Proposition III.1.16 leads to

$$\widetilde{\beta}'(t) = \mathrm{Ad}(\gamma(t)).[\delta(\gamma)(t),\beta(t)] + \mathrm{Ad}(\gamma(t)).\beta'(t) = \mathrm{Ad}(\gamma(t)).\big([\xi(t),\beta(t)] + \beta'(t)\big) = 0.$$

Therefore $\widetilde{\beta}$ is constant equal to x, and we obtain $\beta(t) = \mathrm{Ad}(\gamma(t))^{-1}.\widetilde{\beta}(t) = \mathrm{Ad}(\gamma(t))^{-1}.x = \eta(t)$. $\blacksquare$

Remark IV.2.5. Let G be regular. Then the map

$$S: I \times C^\infty(I, \mathfrak{g}) \to C^\infty(I, \mathfrak{g}), \quad S(s, \xi)(t) = s\xi(st)$$

is smooth. For $\xi \in C^\infty(I, \mathfrak{g})$ and $\gamma_{\xi,s}(t) := \gamma_\xi(st)$, $0 \le s \le 1$, we have $\delta(\gamma_{\xi,s})(t) = s\xi(st) = S(s, \xi)(t)$. Therefore

$$\gamma_\xi(s) = \mathrm{evol}_G\big(S(s, \xi)\big),$$

so that the map

$$\mathrm{evol}_G \circ S: I \times C^\infty(I, \mathfrak{g}) \to G, \quad (s, \xi) \mapsto \gamma_\xi(s)$$

is smooth. ∎

Remark IV.2.6. Now we consider smooth functions $I^2 \to G$, where $I = [0,1]$ is the unit interval and G is a regular Lie group. A smooth $\mathfrak{g}$-valued 1-form $\alpha \in \Omega^1(I^2, \mathfrak{g})$ can be written as

$$\alpha = v \cdot dx + w \cdot dy \quad \text{with} \quad v, w \in C^\infty(I^2, \mathfrak{g}).$$

To evaluate the Maurer–Cartan equation for α, we first observe that

$$\frac{1}{2}[\alpha, \alpha]\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big) = \Big[\alpha\Big(\frac{\partial}{\partial x}\Big), \alpha\Big(\frac{\partial}{\partial y}\Big)\Big] = [v, w] \in C^\infty(I^2, \mathfrak{g}),$$

and obtain

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = \frac{\partial v}{\partial y} dy \wedge dx + \frac{\partial w}{\partial x} dx \wedge dy + [v, w] dx \wedge dy = \Big(\frac{\partial w}{\partial x} - \frac{\partial v}{\partial y} + [v, w]\Big) dx \wedge dy.$$

Therefore the MC equation for α is equivalent to the partial differential equation

$$\frac{\partial v}{\partial y} - \frac{\partial w}{\partial x} = [v, w]. \tag{4.1.5}$$

Suppose that the two smooth functions $v, w: I^2 \to \mathfrak{g}$ satisfy (4.1.5). Then we define a smooth function $f: I^2 \to G$ by

$$f(x, 0) := \gamma_{v(\cdot, 0)}(x) \quad \text{and} \quad f(x, y) := f(x, 0) \cdot \gamma_{w(x, \cdot)}(y).$$

Since the map $I \to C^\infty(I, \mathfrak{g}), x \mapsto w(x, \cdot)$ is smooth (Exercise!), f is a smooth function. We have

$$\delta(f) = \widehat{v} \cdot dx + w \cdot dy \quad \text{with} \quad \widehat{v}(x, 0) = v(x, 0), x \in I.$$

The Maurer–Cartan equation for $\delta(f)$ reads $\frac{\partial \widehat{v}}{\partial y} - \frac{\partial w}{\partial x} = [\widehat{v}, w]$, so that subtraction of this equation from (4.1.5) leads to

$$\frac{\partial(v - \widehat{v})}{\partial y} = [v - \widehat{v}, w].$$

As $(v - \widehat{v})(x, 0) = 0$, the uniqueness assertion of Lemma IV.2.3, applied with $\xi(t) := w(x, t)$, implies that $(v - \widehat{v})(x, y) = 0$ for all $x, y \in I$. We conclude that $v = \widehat{v}$, which means that $\delta(f) = v \cdot dx + w \cdot dy$. ∎

Lemma IV.2.7. *Let U be an open convex subset of the locally convex space V, G a regular Lie group and and $\alpha \in \Omega^1(U, \mathfrak{g})$ satisfy the Maurer–Cartan equation. Then there exists a smooth function $f: U \to G$ satisfying $\delta(f) = \alpha$.*

Proof. We may w.l.o.g. assume that $x_0 = 0 \in U$. For $x \in U$, we then consider the smooth curve

$$\xi_x: I \to \mathfrak{g}, \quad t \mapsto \alpha(tx)(x).$$

Then the map $U \to C^\infty(I, \mathfrak{g})$ is smooth (Exercise), so that the function

$$f : U \to G, \quad x \mapsto \mathrm{evol}_G(\xi_x)$$

is smooth.

First we show that $f(sx) = \gamma_{\xi_x}(s)$ holds for each $s \in I$. From Remark IV.2.4 we derive that

$$S(s, \xi_x)(t) = s\xi_x(st) = \alpha(stx)(sx) = \xi_{sx}(t),$$

hence $S(s, \xi_x) = \xi_{sx}$, which leads to $f(sx) = \gamma_{\xi_x}(s)$.

For $x, x + h \in U$, we consider the smooth map

$$\varphi : I \times I \to U, \quad (s,t) \mapsto t(x + sh)$$

and the smooth function $F := f \circ \varphi$. Then the preceding considerations imply $F(s, 0) = f(0) = \mathbf{1}$,

$$\begin{aligned} \frac{\partial F}{\partial t}(s,t) &= \frac{d}{dt} f(t(x+sh)) = \frac{d}{dt}\gamma_{\xi_{x+sh}}(t) = F(s,t).\xi_{x+sh}(t) \\ &= F(s,t).\alpha(t(x+sh))(x+sh) = F(s,t).(\varphi^*\alpha)_{(s,t)}\big(\frac{\partial}{\partial t}\big). \end{aligned}$$

As we have seen in Remark IV.2.5, these relations imply already that $\delta(F) = \varphi^*\alpha$ holds on the square I^2. We therefore obtain

$$\frac{\partial}{\partial s} f(x+sh) = \frac{\partial}{\partial s} F(s,1) = F(s,1).\alpha_{x+sh}(h) = f(x+sh).\alpha_{x+sh}(h),$$

and for $s = 0$, this leads to $(df)_x(h) = f(x).\alpha_x(h)$, which means that $\delta(f) = \alpha$. ■

The following theorem is a version of the Fundamental Theorem of calculus for functions with values in regular Lie groups.

Theorem IV.2.8. (Fundamental Theorem for Lie group valued functions) *Let M be a simply connected manifold and G a regular Lie group. Then $\alpha \in \Omega^1(M, \mathfrak{g})$ is integrable if and only if*

$$(MC) \qquad d\alpha + \frac{1}{2}[\alpha, \alpha] = 0.$$

Proof. We have already seen in Lemma IV.1.6 that the MC equation is necessary for the existence of a smooth function $f : M \to G$ with $\delta(f) = \alpha$.

We consider the product set $P := M \times G$ with the two projection maps $F : P \to G$ and $q : P \to M$. We define a topology on P as follows. For each pair (U, f), consisting of an open subset $U \subseteq M$ and a smooth function $f : U \to G$ with $\delta(f) = \alpha\,|_U$, the graph $\Gamma(f, U) := \{(x, f(x)) : x \in U\}$ is a subset of P. These sets form a basis for a topology τ on P.

With respect to this topology, the mapping $q : P \to M$ is a covering map. To see this, let $x \in M$. Since M is a manifold, there exists a neighborhood U of x which is diffeomorphic to a convex subset of a locally convex space. In view of Lemmas IV.1.10 and IV.1.5, for each $g \in G$ and each $x \in U$, the equation $\delta(f) = \alpha\,|_U$ has a unique solution f_g with $f_g(x) = g$. Now $q^{-1}(U) = U \times G = \bigcup_{g \in G} \Gamma(f_g, U)$ is a disjoint union of open subsets of P (here we use the connectedness of U), and therefore q is a covering. We conclude that P carries a natural manifold structure, for which q is a local diffeomorphism. For this manifold structure, the function $F : P \to G$ is smooth with $\delta(F) = q^*\alpha$.

Now we fix a point $x_0 \in M$ and an element $g \in G$. Then the connected component $\widehat{M}$ of (x, g) in P is a connected covering manifold of M, hence diffeomorphic to M, so that we may put $f := F \circ (q|_{\widehat{M}})^{-1}$. ■

Remark IV.2.9. (a) If M is a complex manifold, G is a complex regular Lie group and $\alpha \in \Omega^1(M, \mathfrak{g})$ is a holomorphic 1-form, then for any smooth function $f: M \to G$ with $\delta(f) = \alpha$, the differential of f is complex linear in each point, so that f is holomorphic. Conversely, the left logarithmic derivative of any holomorphic function f is a holomorphic 1-form.

If, in addition, M is a complex curve, i.e., a one-dimensional complex manifold, then for each holomorphic 1-form $\alpha \in \Omega^1(M, \mathfrak{g})$ the 2-forms $d\alpha$ and $[\alpha, \alpha]$ are holomorphic, which implies that they vanish, because M is a one-dimensional. Therefore the Maurer–Cartan equation is automatically satisfied by all holomorphic 1-forms. ∎

One of the main points of the notion of regularity is provided by the following theorem.

Theorem IV.2.10. *If H is a regular Lie group, G is a simply connected Lie group, and $\varphi: \mathbf{L}(G) \to \mathbf{L}(H)$ is a continuous homomorphism of Lie algebras, then there exists a unique Lie group homomorphism $f: G \to H$ with $\mathbf{L}(f) = \varphi$.*

Proof. This is Theorem 8.1 in [Mil83] (see also [KM97, Th. 40.3]). The uniqueness assertion follows from Proposition IV.1.5 and does not require the regularity of H.

On G, we consider the smooth $\mathfrak{h}$-valued 1-form given by $\alpha := \varphi \circ \kappa_G$. That it satisfies the Maurer–Cartan equation follows from

$$d\alpha = \varphi \circ d\kappa_G = -\frac{1}{2}\varphi \circ [\kappa_G, \kappa_G] = -\frac{1}{2}[\varphi \circ \kappa_G, \varphi \circ \kappa_G] = -\frac{1}{2}[\alpha, \alpha].$$

Therefore the Fundamental Theorem implies the existence of a unique smooth function $f: G \to H$ with $\delta(f) = \alpha$ and $f(\mathbf{1}_G) = \mathbf{1}_H$. In view of Proposition IV.1.8(3), the function f is a morphism of Lie groups, and we clearly have $\mathbf{L}(f) = \alpha_{\mathbf{1}} = \varphi$. ∎

Corollary IV.2.11. *If G_1 and G_2 are regular simply connected Lie groups with isomorphic Lie algebras, then G_1 and G_2 are isomorphic.* ∎

The non-simply connected case

For a locally convex Lie algebra $\mathfrak{g}$, we write

$$Z^1_{\mathrm{dR}}(M, \mathfrak{g}) := \{\alpha \in \Omega^1(M, \mathfrak{g}): d\alpha + \frac{1}{2}[\alpha, \alpha] = 0\}$$

for the set of solutions of the MC equation. Note that if $\mathfrak{g}$ is abelian, then $Z^1_{\mathrm{dR}}(M, \mathfrak{g})$ is the space of closed $\mathfrak{g}$-valued 1-forms, but that for non-abelian Lie algebras $\mathfrak{g}$, the set $Z^1_{\mathrm{dR}}(M, \mathfrak{g})$ does not have any natural vector space structure.

We are now looking for a sufficient condition on $\alpha \in Z^1_{\mathrm{dR}}(M, \mathfrak{g})$ to be G-integrable. In the remainder of this section, we shall assume that G is regular and that M is connected, but not that M is simply connected. We fix a base point $m_0 \in M$.

Let $\alpha \in Z^1_{\mathrm{dR}}(M, \mathfrak{g})$. If $\gamma: I = [0, 1] \to M$ is a piecewise smooth loop, then $\gamma^*\alpha \in \Omega^1(I, \mathfrak{g}) \cong C^\infty(I, \mathfrak{g})$, so that $\mathrm{evol}_G(\gamma^*\alpha) \in G$ is defined, because G is regular.

Lemma IV.2.12. *If α satisfies the MC equation, then $\mathrm{evol}_G(\gamma^*\alpha)$ does not change under homotopies with fixed endpoints and*

$$\mathrm{per}^{m_0}_\alpha: \pi_1(M, m_0) \to G, \quad [\gamma] \mapsto \mathrm{evol}_G(\gamma^*\alpha)$$

is a group homomorphism.

Proof. Let $q_M: \widetilde{M} \to M$ denote a universal covering manifold of M and choose a base point $\widetilde{m}_0 \in \widetilde{M}$ with $q_M(\widetilde{m}_0) = m_0$. Then the $\mathfrak{g}$-valued 1-form $q_M^*\alpha$ on $\widetilde{M}$ also satisfies the Maurer–Cartan equation, so that the Fundamental Theorem for simply connected manifolds

(Theorem IV.2.7) implies the existence of a unique smooth function $\widetilde{f}\colon \widetilde{M} \to G$ with $\delta(\widetilde{f}) = q_M^*\alpha$ and $\widetilde{f}(\widetilde{m}_0) = \mathbf{1}$.

We write

$$\sigma\colon \pi_1(M, m_0) \times \widetilde{M} \to \widetilde{M}, \quad (d, m) \mapsto d.m = \sigma_d(m)$$

for the left action of the fundamental group $\pi_1(M, m_0)$ on $\widetilde{M}$. Then $\sigma_d^* q_M^*\alpha = q_M^*\alpha$ for each $d \in \pi_1(M, m_0)$ implies the existence of a function

$$\varphi\colon \pi_1(M, m_0) \to G \quad \text{with} \quad \widetilde{f} \circ \sigma_d = \varphi(d) \cdot \widetilde{f}, \quad d \in \pi_1(M, m_0),$$

because

$$\delta(\widetilde{f} \circ \sigma_d) = \sigma_d^* q_M^*\alpha = q_M^*\alpha = \delta(\widetilde{f}).$$

For $d_1, d_2 \in \pi_1(M, m_0)$, we then have

$$\widetilde{f} \circ \sigma_{d_1 d_2} = \widetilde{f} \circ \sigma_{d_1} \circ \sigma_{d_2} = (\varphi(d_1) \cdot \widetilde{f}) \circ \sigma_{d_2} = \varphi(d_1) \cdot (\widetilde{f} \circ \sigma_{d_2}) = \varphi(d_1)\varphi(d_2) \cdot \widetilde{f},$$

so that φ is a group homomorphism.

We now pick a continuous lift $\widetilde{\gamma}\colon I \to \widetilde{M}$ with $q_M \circ \widetilde{\gamma} = \gamma$ and observe that

$$\delta(\widetilde{f} \circ \widetilde{\gamma}) = \widetilde{\gamma}^* q_M^*\alpha = \gamma^*\alpha,$$

which entails that

$$\varphi([\gamma]) = \widetilde{f}([\gamma].\widetilde{m}_0) = \widetilde{f}(\widetilde{\gamma}(1)) = \mathrm{evol}(\gamma^*\alpha).$$

This completes the proof. ∎

Definition IV.2.13. For $\alpha \in Z^1_{\mathrm{dR}}(M, \mathfrak{g})$, the homomorphism

$$\mathrm{per}_\alpha^{m_0}\colon \pi_1(M, m_0) \to G \quad \text{with} \quad \mathrm{per}_\alpha^{m_0}([\gamma]) = \mathrm{evol}(\gamma^*\alpha)$$

for each piecewise smooth loop $\gamma\colon I \to M$ in m_0 is called the *period homomorphism of α with respect to m_0*. ∎

Clearly, the function $\widetilde{f}$ in the proof of Lemma IV.2.12 factors through a smooth function on M if and only if the period homomorphism is trivial. This leads to the following version of the fundamental theorem for manifolds which are not simply connected.

Theorem IV.2.14. (Fundamental Theorem; non-simply connected case) *Let M be a connected manifold, $m_0 \in M$, G a regular Lie group and $\alpha \in \Omega^1(M, \mathfrak{g})$. There exists a smooth function $f\colon M \to G$ with $\alpha = \delta(f)$ if and only if α satisfies*

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0 \quad \textit{and} \quad \mathrm{per}_\alpha^{m_0} = \mathbf{1}.$$ ∎

Exercises for Section IV

Exercise IV.1. Let V be a Mackey complete space and $\Gamma \subseteq V$ a discrete subgroup. Show that the quotient Lie group V/Γ is regular. ∎

Exercise IV.2. Let M be a smooth manifold, H a regular Lie group and $\alpha \in Z^1_{\mathrm{dR}}(M, \mathfrak{h})$. Show that:

(1) For any diffeomorphism $\varphi \in \mathrm{Diff}(M)$, we have

$$\mathrm{per}_\alpha^{m_0}(\varphi^*\alpha) = \mathrm{per}_\alpha^{\varphi(m_0)}(\alpha) \circ \pi_1(\varphi, m_0)\colon \pi_1(M, m_0) \to H.$$

(2) Let G be a Lie group, acting smoothly on M from the left by $g.m = \sigma_g(m)$ and also on H, resp., $\mathfrak{h}$, by automorphisms $\rho_H(g)$, resp., $\rho_{\mathfrak{h}}(g)$. We call α an equivariant form if

$$\sigma_g^*\alpha = \rho_{\mathfrak{h}}(g) \circ \alpha$$

holds for each $g \in G$. Show that if α is equivariant, then

$$\rho_H(g) \circ \mathrm{per}_\alpha^{m_0}(\alpha) = \mathrm{per}_\alpha^{g.m_0}(\alpha) \circ \pi_1(\sigma_g, m_0)\colon \pi_1(M, m_0) \to G.$$

If, in addition, m_0 is fixed by G and G is connected, then

$$\mathrm{im}(\mathrm{per}_\alpha^{m_0}) \subseteq H^G.$$

∎

V. Locally exponential Lie groups and Lie subgroups

In this section, we turn to Lie groups with an exponential function $\exp: \mathbf{L}(G) \to G$ which is well-behaved in the sense that it maps a 0-neighborhood in $\mathbf{L}(G)$ diffeomorphically onto a $\mathbf{1}$-neighborhood in G. We call such Lie groups *locally exponential.*

The assumption of local exponentiality has important structural consequences, the most important ones of which are that it permits us to develop a good theory of Lie subgroups and that there even is a characterization of those subgroup for which we may form Lie group quotients.

Unfortunately, not all regular Lie groups are locally exponential. As an important example we discuss the group $\mathrm{Diff}(\mathbb{S}^1)$ in some detail.

V.1. Locally exponential Lie groups

Definition V.1.1. We call a Lie group G *locally exponential* if it has a smooth exponential function $\exp: \mathfrak{g} = \mathbf{L}(G) \to G$ and there exists an open 0-neighborhood $U \subseteq \mathfrak{g}$ such that $\exp|_U: U \to \exp(U)$ is a diffeomorphism onto an open $\mathbf{1}$-neighborhood of G. A Lie group is called *exponential* if it has an exponential function which is a diffeomorphism $\mathfrak{g} \to G$. ■

Lemma V.1.2. *If G is a Lie group with exponential function* $\exp: \mathfrak{g} \to G$, *then*

$$d\exp(0) = \mathrm{id}_{\mathfrak{g}}.$$

Proof. For $x \in \mathfrak{g}$, we have $\exp(x) = \gamma_x(1)$, where γ_x is a solution of the IVP

$$\gamma(0) = \mathbf{1}, \quad \delta(\gamma) = x.$$

This implies in particular that $\exp(tx) = \gamma_{tx}(1) = \gamma_x(t)$ (Remark IV.2.4), and hence

$$(d\exp)(0)(x) = \gamma_x'(0) = x. \qquad ■$$

The preceding lemma is not as useful in the infinite-dimensional context as it is in the finite-dimensional or Banach context. For Banach–Lie groups, it follows from the Inverse Function Theorem that exp restricts to a diffeomorphism of some open 0-neighborhood in $\mathfrak{g}$ to an open $\mathbf{1}$-neighborhood in G, so that we can use the exponential function to obtain charts around $\mathbf{1}$. We will see below that this conclusion does not work for Fréchet–Lie groups, because in this context there is no general Inverse Function Theorem. This observation also implies that to integrate Lie algebra homomorphisms to group homomorphisms, it is in general not enough to start with the prescription $\alpha(\exp_G x) := \exp_H \varphi(x)$ to prove Theorem IV.2.10, because the image of $\exp_G$ need not contain an identity neighborhood in G (cf. Theorem V.1.6 below).

Remark V.1.3. (a) In view of Lemma V.1.2, the Inverse Function Theorem implies that each Banach–Lie group is locally exponential. This also covers all finite-dimensional Lie groups.

(b) Unit groups of Mackey complete CIAs are locally exponential (cf. [Gl02b]). In fact, if A is a Mackey complete complex CIA, then the fact that $A^\times$ is open implies that for each $a \in A$, the spectrum $\mathrm{Spec}(a)$ is a compact subset (which also is non-empty), and it is shown in [Gl02b] that the holomorphic calculus works as for Banach algebras. We only have to use partially smooth countours around spectra. We thus obtain an exponential function

$$\exp_A: A \to A^\times, \quad x \mapsto \frac{1}{2\pi i}\oint_\Gamma e^\zeta(\zeta\mathbf{1} - x)^{-1}\, d\zeta,$$

where Γ is a piecewise smooth contour around $\mathrm{Spec}(x)$. Then exp is a holomorphic function $A \to A^\times$.

Let $\rho(a) := \sup\{|\lambda|: \lambda \in \mathrm{Spec}(a)\}$ denote the spectral radius of $a \in A$. Then

$$\Omega := \{a \in A: \rho(a - \mathbf{1}) < 1\}$$

is an open $\mathbf{1}$-neighborhood in $A^\times$, and with the complex logarithm function

$$\log: \{z \in \mathbb{C}: |1 - z| < 1\} \to \mathbb{C}$$

satisfying $\log(1) = 0$, we get the holomorphic function

$$\log_A: \Omega \to A, \quad x \mapsto \frac{1}{2\pi i} \oint_\Gamma \log(\zeta)(\zeta\mathbf{1} - x)^{-1}\, d\zeta,$$

where Γ is a contour around $\mathrm{Spec}(x)$, lying in the open disc of radius 1 around 1. Now functional calculus implies that $(\log_A \circ \exp_A)(x) = x$ for $\rho(x)$ sufficiently small, and $(\exp_A \circ \log_A)(x) = x$ for each $x \in \Omega$. We conclude that the unit group $A^\times$ is locally exponential.

If A is a real CIA, then one uses the fact that its complexification $A_\mathbb{C}$ is a CIA to see that $\log_{A_\mathbb{C}}(\Omega \cap A^\times) \subseteq A$, and that $\log_A := \log_{A_\mathbb{C}}|_\Omega$ is a smooth local inverse to $\exp_A := \exp_{A_\mathbb{C}}|_A$.

(c) If K is a locally exponential Lie group and M is a compact manifold, then the Lie group $G := C^\infty(M, K)$ (Example III.1.12) is locally exponential.

In fact, if $\exp_K: \mathfrak{k} \to K$ is an exponential function of K, then

$$\exp_G: \mathfrak{g} = C^\infty(M, \mathfrak{k}) \to G = C^\infty(M, K), \quad \xi \mapsto \exp_K \circ \xi$$

is a smooth exponential function of G. Since we may use the exponential function $\exp_K: \mathfrak{k} \to K$ to get a local chart of K, the construction of the local charts of G implies that G is locally exponential.

(d) Recent results of Ch. Wockel ([Wo03]) imply that the preceding theorem generalizes even to gauge groups: If K is locally exponential and $q: P \to M$ is a smooth principal K-bundle over the compact manifold M, then $\mathrm{Gau}(P)$ carries a natural Lie group structure, turning it into a locally exponential Lie group. In fact, one shows that

$$\mathfrak{gau}(P) := C^\infty(P, \mathfrak{k})^K \to C^\infty(P, K)^K \cong \mathrm{Gau}(P), \quad \xi \mapsto \exp_K \circ \xi$$

is a local homeomorphism, and that it can be used to define a Lie group structure on $\mathrm{Gau}(P)$.

(e) If $\mathfrak{g}$ is a nilpotent locally convex Lie algebra, then we can use the BCH series $x * y := x + y + \frac{1}{2}[x, y] + \cdots$ to define a polynomial Lie group structure $(\mathfrak{g}, *)$ with $\mathbf{L}(\mathfrak{g}, *) = \mathfrak{g}$.

More generally, if $\mathfrak{g} = \varprojlim \mathfrak{g}_j$ is a projective limit of a family of nilpotent Lie algebras $(\mathfrak{g}_j)_{j \in J}$ (a so-called *pro-nilpotent Lie algebra*), then the corresponding morphisms of Lie algebras are also morphisms for the corresponding group structures, so that $(\mathfrak{g}, *) := \varprojlim (\mathfrak{g}_j, *)$ defines on the space $\mathfrak{g}$ a Lie group structure with $\mathbf{L}(\mathfrak{g}, *) = \mathfrak{g}$. We thus obtain an exponential Lie group $G = (\mathfrak{g}, *)$ with $\exp_G = \mathrm{id}_\mathfrak{g}$.

This construction can be used in many situations to see that certain groups can be turned into Lie groups. An important class of examples arises as follows. Let V be a finite-dimensional $\mathbb{K}$-vector space, let $P_d(V, V)$ denote the space of all polynomials functions $V \to V$ of degree d. Then for each $n \geq 2$, the space $\mathfrak{g}_n := \bigoplus_{k=2}^n P_k(V, V)$ carries a natural Lie algebra structure given for $f \in P_i(V, V)$ and $g \in P_j(V, V)$ by

$$[f, g](x) := \begin{cases} dg(x)f(x) - df(x)g(x) & \text{for } i + j - 1 \leq n \\ 0 & \text{for } i + j - 1 > n. \end{cases}$$

This is a modification of the natural Lie bracket on the space $C^\infty(V, V) \cong \mathcal{V}(V)$, obtained by cutting of all terms of degree $> n$. From

$$[P_i(V, V), P_j(V, V)] \subseteq P_{i+j-1}(V, V)$$

it immediately follows that each $\mathfrak{g}_n$ is a nilpotent Lie algebra. For $n < m$, we have natural projections

$$\varphi_{nm}\colon \mathfrak{g}_m \to \mathfrak{g}_n,$$

which are actually homomorphisms of Lie algebras. The projective limit Lie algebra $\mathfrak{g} := \varprojlim \mathfrak{g}_n$ can be identified with the space of V-valued formal power series starting in degree 2.

A natural Lie group corresponding to $\mathfrak{g}_n$ is the set of all polynomial maps $f\colon V \to V$ with $f - \mathrm{id}_V \in \mathfrak{g}_n$. The group structure is given by composition and then omitting all terms of order $> n$:

$$f * g = (f \circ g)_{\leq n}.$$

This turns G_n into a nilpotent Lie group with Lie algebra $\mathfrak{g}_n$. The corresponding exponential function

$$\exp_{G_n}\colon \mathfrak{g}_n \to G_n$$

is given by "integrating" a vector field $X \in \mathfrak{g}_n$ modulo terms of order $> n$. Since G_n is diffeomorphic to a vector space, its exponential function is a diffeomorphism $\mathfrak{g}_n \to G_n$.

We can now form the projective limit group $G := \varprojlim G_n$ whose manifold structure is obtained from the fact that it is an affine space with translation group $\mathfrak{g}$. Since the exponential functions are compatible with the limiting process, we see that G is an exponential Lie group with a pro-nilpotent Lie algebra. The group G can be defined with the set of all formal diffeomorphisms of V fixing 0 and with first order term given by id_V. Likewise, $\mathfrak{g}$ can be identified with a Lie algebra of formal vector fields.

(f) We describe a Fréchet–Lie group G which is analytic, for which $\exp\colon \mathfrak{g} \to G$ is a diffeomorphism and analytic, but $\exp^{-1}$ is not an analytic map, and the corresponding multiplication on $\mathfrak{g}$ is not analytic.

Let $\mathrm{Aff}(\mathbb{R})$ denote the affine group of $\mathbb{R}$, which is isomorphic to $\mathbb{R}^2$, endowed with the multiplication

$$(x,y)(x',y') = (x + e^y x', y + y')$$

and the exponential map

$$\exp\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad \exp(x,y) = \Big(\frac{e^y - 1}{y} x, y\Big),$$

whose inverse is given by

$$\log\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad \log(x,y) = \Big(\frac{y}{e^y - 1} x, y\Big).$$

On the Lie algebra level, we have

$$[(x,y),(x',y')] = (yx' - y'x, 0).$$

This means that

$$\mathrm{ad}(0,y)^n.(x',y') = (y^n x', 0),$$

so that $\sum_{n=1}^\infty \mathrm{ad}(0,y)^n$ converges if and only if $|y| < 1$.

We put

$$G := \mathrm{Aff}(\mathbb{R})^{\mathbb{N}} \cong (\mathbb{R}^2)^{\mathbb{N}}$$

with the multiplication

$$(x_n, y_n)_{n\in\mathbb{N}}(x_n', y_n')_{n\in\mathbb{N}} := (x_n + e^{y_n} x_n', y_n + y_n')_{n\in\mathbb{N}}.$$

We endow G with the manifold structure we obtain by identifying it with the product space $(\mathbb{R}^2)^{\mathbb{N}}$ which is a Fréchet space (cf. Exercise II.8). This turns G into an analytic manifold. As the power series defining the multiplication converges globally, the multiplication of G is analytic, and the same holds for the inversion map, because in $\mathrm{Aff}(\mathbb{R})$ we have

$$(x,y)^{-1} = (-e^{-y}x, -y).$$

Therefore G is an analytic Lie group.

The exponential map of G is given by

$$\exp((x_n,y_n))_n = \Big(\frac{e^{y_n}-1}{y_n}x_n, y_n\Big),$$

and again we see that $\exp$ is analytic because the corresponding power series converges globally. For the inverse function, we obtain

$$\exp^{-1}((x_n,y_n))_n = \Big(\frac{y_n}{e^{y_n}-1}x_n, y_n\Big),$$

but this map is not analytic, because the power series of the real analytic function $y \mapsto \frac{y}{e^y-1}$ converges only on the interval from -2π to 2π, and the product of infinitely many such intervals is not an open subset in $\mathfrak{g} \cong (\mathbb{R}^2)^{\mathbb{N}}$.

For the multiplication on the Lie algebra $\mathfrak{aff}(\mathbb{R})$ obtained from the exponential chart, we have

$$\begin{aligned}(x,y) * (x',y') &= \log(\exp(x,y)\exp(x',y')) = \log\Big(\frac{e^y-1}{y}x + e^y\frac{e^{y'}-1}{y'}x', y+y'\Big)\\ &= \Big(\frac{y+y'}{e^{y+y'}-1}\Big(\frac{e^y-1}{y}x + e^y\frac{e^{y'}-1}{y'}x'\Big), y+y'\Big)\end{aligned}$$

and in particular

$$(0,y)*(1,0) = \log(e^y, y) = \Big(\frac{ye^y}{e^y-1}, y\Big) = \Big(\frac{y}{1-e^{-y}}, y\Big).$$

Therefore the argument form above also shows that the multiplication on the product Lie algebra $\mathfrak{g}$ is not analytic. ∎

For the following results we refer to [GN05].

Theorem V.1.4. *Each continuous homomorphism $\varphi: G \to H$ between locally exponential groups is smooth.*

Proof. (Idea) Using exponential charts, we obtain open 0-neighborhoods $U_{\mathfrak{g}} \subseteq \mathfrak{g} = \mathbf{L}(G)$ and $U_{\mathfrak{h}} \subseteq \mathfrak{h} = \mathbf{L}(H)$ together with a continuous map $\psi: U_{\mathfrak{g}} \to U_{\mathfrak{h}}$ satisfying

$$\psi(x*y) = \psi(x)*\psi(y), \quad x,y \in U_{\mathfrak{g}}.$$

Then one shows that

$$f(x) := \lim_{n\to\infty} n\psi(\tfrac{1}{n}x)$$

converges for each $x \in \mathfrak{g}$, that f coincides on a 0-neighborhood with ψ, and that f is linear. As f is continuous in a 0-neighborhood, it is smooth, and from $\exp_H \circ f = \varphi \circ \exp_G$ on a 0-neighborhood in $U_{\mathfrak{g}}$, we derive that φ is smooth. ∎

Theorem V.1.5. *Let G and H be locally exponential groups, $\psi\colon \mathbf{L}(G) \to \mathbf{L}(H)$ a continuous homomorphism of Lie algebras, and assume that G is connected and simply connected. Then there exists a unique morphism of Lie groups $\varphi\colon G \to H$ with $\mathbf{L}(\varphi) = \psi$.*

Proof. (Idea) Let $U_{\mathfrak g} \subseteq \mathfrak g = \mathbf{L}(G)$ be a convex balanced 0-neighborhood mapped diffeomorphically by the exponential function to an open subset of G.

First one shows that the local Maurer–Cartan form on $U_{\mathfrak g}$ is given by

$$(\kappa_{\mathfrak g})_x := (\exp^* \kappa_G)_x = \int_0^1 e^{-t\,\mathrm{ad}\,x}\,dt.$$

This implies that $\psi^*\kappa_{\mathfrak h} = \psi \circ \kappa_{\mathfrak g}$ on some 0-neighborhood in $\mathfrak g$. For the map $f\colon U_G \to H, x \mapsto \exp_H(\psi(x))$, this leads to

$$f^*\kappa_H = \psi \circ \kappa_G,$$

showing that the $\mathfrak h$-valued 1-form $\psi \circ \kappa_G$ is locally integrable. Since this form on G is left invariant and G is simply connected, it is globally integrable (for that one can argue as in the proof of the Fundamental Theorem IV.2.7), so that we find a smooth function $\varphi\colon G \to H$ with $\varphi(\mathbf{1}) = \mathbf{1}$ and $\delta(\varphi) = \psi\circ\kappa_G$. Now Proposition IV.1.8(3) implies that φ is a group homomorphism with $\mathbf{L}(\varphi) = \alpha_{\mathbf 1} = \psi$. ■

Corollary V.1.6. *If G_1 and G_2 are locally exponential simply connected Lie groups with isomorphic Lie algebras, then G_1 and G_2 are isomorphic.* ■

It is instructive to compare the preceding corollary with Corollary IV.2.10, which makes a similar statement for regular Lie groups. Although all known Lie groups are regular, there is no theorem saying that all locally exponential groups are regular. That the converse is false is clear from the example $G = \mathrm{Diff}(\mathbb S^1)$, which is regular but not locally exponential.

$\mathrm{Diff}(\mathbb S^1)$ is not locally exponential

Below we show that the exponential function

$$\exp\colon \mathcal V(\mathbb S^1) \to \mathrm{Diff}(\mathbb S^1)$$

is not a local diffeomorphism by proving that every identity neighborhood of $\mathrm{Diff}(\mathbb S^1)$ contains elements which do not lie on a one-parameter group, hence are not contained in the image of $\exp$.

Let $G := \mathrm{Diff}_+(\mathbb S^1)$ denote the group of orientation preserving diffeomorphisms of $\mathbb S^1$, i.e., the identity component of $\mathrm{Diff}(\mathbb S^1)$. To get a better picture of this group, we first construct its universal covering group $\widetilde G$. Let

$$\widetilde G := \{\varphi \in \mathrm{Diff}(\mathbb R)\colon (\forall x \in \mathbb R)\ \varphi(x + 2\pi) = \varphi(x), \varphi' > 0\}.$$

We consider the map

$$q\colon \mathbb R \to \mathbb S^1 := \mathbb R/2\pi\mathbb Z, \quad x \mapsto x + 2\pi\mathbb Z$$

as the universal covering map of $\mathbb S^1$. Then every orientation preserving diffeomorphism $\psi \in \mathrm{Diff}_+(\mathbb S^1)$ lifts to a diffeomorphism $\widetilde\psi$ of $\mathbb R$, commuting with the translation action of the group $2\pi\mathbb Z \cong \pi_1(\mathbb S^1)$, which means that $\widetilde\psi(x + 2\pi) = \widetilde\psi(x) + 2\pi$ for each $x \in \mathbb R$. The diffeomorphism $\widetilde\psi$ is uniquely determined by the choice of an element in $q^{-1}(\psi(q(0)))$. That ψ is orientation preserving means that $\widetilde\psi' > 0$. Hence we have a surjective homomorphism

$$q_G\colon \widetilde G \to G, \quad q_G(\varphi)(q(x)) := q(\varphi(x))$$

with kernel isomorphic to $\mathbb Z$.

The Lie group structure of $\widetilde{G}$ is rather simple. It can be defined by a global chart. Let $C^\infty_{2\pi}(\mathbb{R},\mathbb{R})$ denote the Fréchet space of 2π-periodic smooth functions on $\mathbb{R}$, which is considered as a closed subspace of the Fréchet space $C^\infty(\mathbb{R},\mathbb{R})$. In this space,

$$U := \{\varphi \in C^\infty_{2\pi}(\mathbb{R},\mathbb{R})\colon \varphi' > -1\}$$

is an open convex subset, and the map

$$\Phi\colon U \to \widetilde{G}, \quad \Phi(f)(x) := x + f(x)$$

is a bijection.

In fact, let $f \in U$. Then $\Phi(f)(x+2\pi) = \Phi(f)(x)+2\pi$ follows directly from the requirement that f is 2π-periodic, and $\Phi(f)' > 0$ follows from $f' > -1$. Therefore $\Phi(f)$ is strictly increasing, hence a diffeomorphism of $\mathbb{R}$ onto the interval $\Phi(f)(\mathbb{R})$. As the latter interval is invariant under translation by 2π, we see that $\Phi(f)$ is surjective and therefore $\Phi(f) \in \widetilde{G}$. Conversely, it is easy to see that $\Phi^{-1}(\psi)(x) = \psi(x) - x$ yields an inverse of Φ. We define the manifold structure on $\widetilde{G}$ by declaring Φ to be a global chart. With respect to this chart, the group operations in $\widetilde{G}$ are given by

$$m(f,g)(x) := f(g(x)+x) - x \quad \text{and} \quad \eta(f)(x) = (f + \mathrm{id}_{\mathbb{R}})^{-1}(x) - x,$$

which can be shown directly to be smooth maps. We thus obtain on $\widetilde{G}$ the structure of a Lie group such that $\Phi\colon U \to \widetilde{G}$ is a diffeomorphism. In particular, $\widetilde{G}$ is contractible and therefore simply connected, so that the map $q_G\colon \widetilde{G} \to G$ turns out to be the universal covering map of G.

Theorem V.1.7. *Every identity neighborhood in* $\mathrm{Diff}(\mathbb{S}^1)$ *contains elements not contained in the image of the exponential function.*

Proof. First we construct certain elements in $\widetilde{G}$ which are close to the identity. For $0 < \varepsilon < \frac{1}{n}$, we consider the function

$$f\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto x + \frac{\pi}{n} + \varepsilon \sin^2(nx)$$

and observe that $f \in \widetilde{G}$ follows from $f'(x) = 1 + 2\varepsilon n \sin(nx)\cos(nx) = 1 + \varepsilon n \sin(2nx) > 0$.

Step 1. For n large fixed and $\varepsilon \to 0$, we get elements in $\widetilde{G}$ which are arbitrarily close to $\mathrm{id}_{\mathbb{R}}$.

Step 2. $q_G(f)$ has a unique periodic orbit of order $2n$ on $\mathbb{S}^1$: Under $q_G(f)$, the point $q(0) \in \mathbb{S}^1$ is mapped to $\frac{\pi}{n}$ etc., so that we obtain the orbit

$$q(0) \to q(\tfrac{\pi}{n}) \to q(\tfrac{2\pi}{n}) \to \ldots \to q(\tfrac{(2n-1)\pi}{n}) \to q(0).$$

For $0 < x_0 < \frac{\pi}{n}$, we have for $x_1 := f(x_0)$:

$$x_0 + \frac{\pi}{n} < x_1 < \frac{2\pi}{n},$$

and for $x_n := f(x_{n-1})$, the relations

$$0 < x_0 < x_1 - \frac{\pi}{n} < x_2 - \frac{2\pi}{n} < \cdots < \frac{\pi}{n}.$$

Therefore $x_k - x_0 \notin 2\pi\mathbb{Z}$ for each $k \in \mathbb{N}$, and hence the orbit of $q(x_0)$ under $q_G(f)$ is not finite. This proves that $q_G(f)$ has a unique periodic orbit and that the order of this orbit is $2n$.

Step 3. $q_G(f) \neq g^2$ for all $g \in \mathrm{Diff}(\mathbb{S}^1)$: We analyze the periodic orbits. Every periodic point of g is a periodic point of g^2 and vice versa. If the period of x under g is odd, then the period of x under g and g^2 is the same. If the period of x is $2m$, then its orbit under g breaks up into two orbits under g^2, each of order m. Therefore g^2 can never have a single periodic orbit of even order, and this proves that $q_G(f)$ has no square root in $\mathrm{Diff}(\mathbb{S}^1)$. It follows in particular that $q_G(f)$ does not lie on any one-parameter subgroup, i.e., $q_G(f) \neq \exp X$ for each $X \in \mathcal{V}(M)$. ■

Remark V.1.8. (a) If M is a compact manifold, then one can show that the identity component $\mathrm{Diff}(M)_0$ of $\mathrm{Diff}(M)$ is a simple group (Epstein, Hermann and Thurston; see [Ep70]). Being normal in $\mathrm{Diff}(M)_0$, the subgroup $\langle \exp \mathcal{V}(M)\rangle$ coincides with $\mathrm{Diff}(M)_0$. Hence every diffeomorphism homotopic to the identity is a finite product of exponentials.

(b) Although $\mathrm{Diff}(M)_0$ is a simple Lie group, its Lie algebra $\mathcal{V}(M)$ is far from being simple. For each subset $K \subseteq M$, the set $\mathcal{V}_K(M)$ of all vector fields supported in the set K is a Lie algebra ideal which is proper if K is not dense. ■

The structure of abelian Lie groups

Proposition V.1.9. (Michor–Teichmann, 1999) *Let A be a connected abelian Lie group modeled on a Mackey complete space $\mathfrak{a}$. Then A has a smooth exponential function if and only if $A \cong \mathfrak{a}/\Gamma_A$ holds for a discrete subgroup Γ_A of $\mathfrak{a}$.*

Proof. For each abelian Lie group of the form $A = \mathfrak{a}/\Gamma_A$, the Lie algebra is $\mathbf{L}(A) = \mathfrak{a}$ and the quotient map $\mathfrak{a} \to A$ is a smooth exponential function.

Therefore it remains to see that the existence of a smooth exponential function implies that A is of the form $\mathfrak{a}/\Gamma_A$. First we claim that $\exp_A$ is surjective. Since the adjoint action of A is trivial (Corollary IV.1.9), Lemma IV.1.4 implies that $\exp\colon (\mathfrak{a}, +) \to A$ is a group homomorphism, hence a morphism of Lie groups. Let $a \in A$ and consider a smooth path $\gamma\colon [0,1] \to A$ with $\gamma(0) = \mathbf{1}$ and $\gamma(1) = a$. Then the logarithmic derivative $\xi := \delta(\gamma)$ is a smooth map $[0,1] \to \mathfrak{a}$, and we consider the smooth path

$$\eta(t) := \exp_A \Big(\int_0^t \xi(s)\, ds \Big)$$

that also satisfies $\delta(\eta) = \xi$ (Proposition IV.1.8(1)). Here we have used the Mackey completeness of $\mathfrak{a}$ to ensure the existence of the Riemann integral of the smooth curve ξ. Now $\eta(0) = \gamma(0) = \mathbf{1}$ implies that

$$a = \gamma(1) = \eta(1) = \exp \Big(\int_0^1 \xi(s)\, ds \Big) \in \mathrm{im}(\exp)$$

(Lemma IV.1.5).

Let $q_A\colon \widetilde{A} \to A$ denote a universal covering homomorphism with $\mathbf{L}(q_A) = \mathrm{id}_{\mathfrak{a}}$. Then the exponential function of A lifts to a smooth exponential function $\exp_{\widetilde{A}}\colon \mathfrak{a} \to \widetilde{A}$ with $\exp_A = q_A \circ \exp_{\widetilde{A}}$. Since $\widetilde{A}$ is simply connected, the Lie algebra homomorphism $\mathrm{id}_{\mathfrak{a}}\colon \mathfrak{a} \to \mathfrak{a}$ integrates to a Lie group homomorphism $L\colon \widetilde{A} \to \mathfrak{a}$ with $\mathbf{L}(L) = \mathrm{id}_{\mathfrak{a}}$ (Theorem IV.2.10). We now have

$$L \circ \exp_{\widetilde{A}} = \exp_{\mathfrak{a}} \circ \mathbf{L}(L) = \mathrm{id}_{\mathfrak{a}} \circ \mathrm{id}_{\mathfrak{a}} = \mathrm{id}_{\mathfrak{a}},$$

and hence $\exp_{\widetilde{A}} \circ L$ restricts to the identity on $\mathrm{im}(\exp_{\widetilde{A}}) = \mathfrak{a}$ (apply the reasing above to $\widetilde{A}$), which also leads to

$$\exp_{\widetilde{A}} \circ L = \mathrm{id}_{\widetilde{A}}.$$

Hence $\widetilde{A} \cong \mathfrak{a}$ as Lie groups, which implies that $\exp_A$ is a covering morphism and therefore that $\Gamma_A := \ker(\exp_A) \subseteq \mathfrak{a}$ is discrete with $A \cong \mathfrak{a}/\Gamma_A$. ■

V.2. Lie subgroups

It is a well known result in finite-dimensional Lie theory that for each subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of a finite-dimensional Lie group G, there exists a Lie group H with Lie algebra $\mathfrak{h}$ together with an injective morphism of Lie groups $\iota\colon H \to G$ for which $\mathbf{L}(\iota)\colon \mathfrak{h} \to \mathfrak{g}$ is the inclusion map. As a group H coincides with $\langle \exp \mathfrak{h}\rangle$, the analytic subgroup corresponding to $\mathfrak{h}$, and $\mathfrak{h}$ can be recovered from this subgroup as the set

$$\{x \in \mathfrak{g} \colon \exp(\mathbb{R}x) \subseteq H\}.$$

This nice and simple theory of analytic subgroups is no longer valid in full generality for infinite-dimensional Lie groups, not even for locally exponential ones. As we shall see below, it has to be refined in several respects.

Proposition V.2.1. *Let G be a locally exponential Lie group. For $x, y \in \mathbf{L}(G)$, we have the Trotter Product Formula*

$$\exp(x+y) = \lim_{n\to\infty} \Big(\exp\big(\frac{x}{n}\big)\exp\big(\frac{y}{n}\big)\Big)^n$$

and the Commutator Formula

$$\exp([x,y]) = \lim_{n\to\infty} \Big(\exp\big(\frac{x}{n}\big)\exp\big(\frac{y}{n}\big)\exp\big(-\frac{x}{n}\big)\exp\big(-\frac{y}{n}\big)\Big)^{n^2}. \qquad \blacksquare$$

As an immediate consequence, we can assign to each closed subgroup $H \leq G$ a Lie subalgebra of $\mathbf{L}(G)$:

Corollary V.2.2. *For every closed subgroup H of the locally exponential Lie group G the subset*

$$\mathbf{L}(H) := \{X \in \mathbf{L}(G) \colon \exp(\mathbb{R}X) \subseteq H\}$$

is a closed Lie subalgebra of $\mathbf{L}(G)$. ■

Since the range of a morphism of Lie algebras need not be closed, it is quite restrictive to consider only closed subgroups, resp., closed Lie subalgebras.

Definition V.2.3. A closed subgroup H of a locally exponential Lie group G is called a *Lie subgroup* if there exists an open 0-neighborhood $V \subseteq \mathbf{L}(G)$ such that $\exp|_V$ is a diffeomorphism onto an open subset $\exp(V)$ of G and

$$\exp(V \cap \mathbf{L}(H)) = (\exp V) \cap H. \qquad \blacksquare$$

Remark V.2.4. (a) In [La99], S. Lang calls a subgroup H of a Banach–Lie group G a *Lie subgroup* if H carries a Lie group structure for which there exists an immersion $\eta \colon H \to G$. In view of the definition of an immersion, this concept requires the Lie algebra $\mathfrak{h} = \mathbf{L}(H)$ of $\mathfrak{g} = \mathbf{L}(G)$ to be a closed subalgebra of $\mathfrak{g}$ which is complemented in the sense that there exists a closed vector space complement. Conversely, it is shown in [La99] that for every complemented closed subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, there exists a Lie subgroup in this sense ([La99, Th. VI.5.4]). For a finite-dimensional Lie group G, this concept describes the analytic subgroups of G, because every subalgebra of a finite-dimensional Lie algebra is closed and complemented. As the dense wind in the two-dimensional torus $G = \mathbb{T}^2$ shows, subgroups of this type need not be closed. We also note that the closed subspace

$$c_0(\mathbb{N}, \mathbb{R}) \subseteq \ell^\infty(\mathbb{N}, \mathbb{R})$$

of sequences converging to 0 is not complemented (see [Wer95, Satz IV.6.5] for an elementary proof), hence not a Lie subgroup in the sense of Lang.

(b) The most restrictive concept of a Lie subgroup is the one used in [Bou89, Ch. 3]. Here a Lie subgroup H is required to be a submanifold, which implies in particular that it is locally closed and therefore closed. On the other hand, this implies that the quotient space G/H has a natural manifold structure for which the quotient map $q \colon G \to G/H$ is a submersion ([Bou89, Ch. 3, §1.6, Prop. 11]).

(c) For finite-dimensional Lie groups, closed subgroups are Lie subgroups, but for Banach–Lie groups this is no longer true. What remains true is that locally compact subgroups (which are in particular closed) are Lie subgroups (cf. [HoMo98, Th. 5.41(vi)]). How bad closed subgroups may behave is illustrated by the following example due to K. H. Hofmann: We consider the real Hilbert space $G := L^2([0,1],\mathbb{R})$ as a Banach–Lie group. Then the subgroup $H := L^2([0,1],\mathbb{Z})$ of all those functions which almost everywhere take values in $\mathbb{Z}$ is a closed subgroup. Since the one-parameter subgroups of G are of the form $\mathbb{R}f$, $f \in G$, we have $\mathbf{L}(H) = \{0\}$. On the other hand, the group H is arcwise connected and even contractible,because the map $F\colon [0,1]\times H \to H$ given by

$$F(t,f)(x) := \begin{cases} f(x) & 0 \leq x \leq t \\ 0 & t < x \leq 1 \end{cases}$$

is continuous with $F(1,f) = f$ and $F(0,f) = 0$. ■

The following proposition shows that Lie subgroups carry natural Lie group structures.

Proposition V.2.5. *Let G be a locally exponential Lie group and $H \subseteq G$ a Lie subgroup. Then H carries a natural locally exponential Lie group structure such that $\mathbf{L}(H)$ is the Lie algebra of H,and the exponential map of H is given by the restriction*

$$\exp_H = \exp_G|_{\mathbf{L}(H)}\colon \mathbf{L}(H) \to H.$$

Moreover, the inclusion map $\iota\colon H \to G$ is a morphism of Lie groups which is a homeomorphism onto its image, and $\mathbf{L}(\iota)\colon \mathbf{L}(H) \to \mathbf{L}(G)$ is the inclusion map.

Proof. (Idea) The idea is to apply Theorem III.2.1 to the subgroup H where $U = \exp V$ holds for some suitable open symmetric subset $V \subseteq \mathbf{L}(H)$. ■

Proposition V.2.6. *If $\varphi\colon G' \to G$ is a morphism of locally exponential Lie groups and $H \subseteq G$ is a Lie subgroup, then $H' := \varphi^{-1}(H)$ is a Lie subgroup. In particular, $\ker\varphi$ is a Lie subgroup of G'.* ■

Corollary V.2.7. *If $N \trianglelefteq G$ is a normal subgroup of the locally exponential Lie group G such that the quotient group G/N carries a locally exponential Lie group structure for which the quotient map $q\colon G \to G/N$ is a morphism of Lie groups, then N is a Lie subgroup.* ■

Theorem V.2.8. (Quotient Theorem for locally exponential groups) *Let $N \trianglelefteq G$ be a normal Lie subgroup and $\mathfrak{n} \subseteq \mathfrak{g} = \mathbf{L}(G)$ its Lie algebra. Then the quotient group G/N is a locally exponential Lie group if and only if there exists a 0-neighborhood $U \subseteq \mathfrak{g}$ such that the operator*

$$\kappa_{\mathfrak{g}}(x) := \int_0^1 e^{-t\,\mathrm{ad}\,x}\,dt$$

on $\mathfrak{g}$ satisfies

$$\kappa_{\mathfrak{g}}(x)(\mathfrak{n}) = \mathfrak{n} \quad \text{for all} \quad x \in U.$$ ■

Corollary V.2.9. (Quotient Theorem for Banach–Lie groups) *Let $N \trianglelefteq G$ be a closed subgroup of the Banach–Lie group G. Then the quotient group G/N is a Banach–Lie group if and only N is a normal Lie subgroup.*

Proof. Since $\mathfrak{g} = \mathbf{L}(G)$ is a Banach–Lie algebra, the ideal $\mathfrak{n} = \mathbf{L}(N)$ is invariant under all operators

$$\kappa_{\mathfrak{g}}(x) = \int_0^1 e^{-t\,\mathrm{ad}\,x}\,dt = \frac{\mathbf{1} - e^{-\mathrm{ad}\,x}}{\mathrm{ad}\,x} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(-1)^n(\mathrm{ad}\,x)^n.$$

For $\mathrm{Spec}(\mathrm{ad}\,x) \subseteq B_{2\pi}(0)$ (which is the case on some 0-neighborhood of $\mathfrak{g}$), this operator is invertible, and its inverse can be expressed by a power series in $\mathrm{ad}\,x$. Therefore we also get $\kappa_{\mathfrak{g}}(x)^{-1}(\mathfrak{n}) \subseteq \mathfrak{n}$, which implies $\kappa_{\mathfrak{g}}(x)(\mathfrak{n}) = \mathfrak{n}$. ■

Algebraic subgroups

We will now discuss a very convenient criterion which in many concrete cases can be used to verify that a closed subgroup H of a Banach–Lie group is a Lie subgroup. To this end, we will need the concept of a polynomial function and of an algebraic subgroup.

Definition V.2.10. Let A be a Banach algebra. A subgroup $G \subseteq A^\times$ is called *algebraic* if there exists a $d \in \mathbb{N}_0$ and a set $\mathcal{F}$ of Banach space valued polynomial functions on $A \times A$ of degree $\leq d$ such that

$$G = \{g \in A^\times : (\forall f \in \mathcal{F})\ f(g, g^{-1}) = 0\}. \qquad \blacksquare$$

Theorem V.2.11. (Harris/Kaup) [Ne04c, Prop. IV.14] *Every algebraic subgroup $G \subseteq A^\times$ of the unit group $A^\times$ of a Banach algebra A is a Lie subgroup.* $\blacksquare$

Proposition V.2.12. *Let E be a Banach space and $F \subseteq E$ a closed subspace. Then*

$$H := \{g \in \mathrm{GL}(E) : g.F \subseteq F\}$$

is a Lie subgroup of $\mathrm{GL}(E)$.

Proof. Let $V \subseteq \mathfrak{g}$ be an open 0-neighborhood such that $\exp|_V : V \to \exp V$ is a diffeomorphism and $\|\exp x - \mathbf{1}\| < 1$ for all $x \in V$. Then the inverse function

$$\log := (\exp|_V)^{-1} : \exp V \to \mathfrak{g}$$

is given by the convergent power series

$$\log(g) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}(g - \mathbf{1})^n$$

(this requires a proof!). For $g = \exp x \in (\exp V) \cap H$, we then obtain $x.F \subseteq F$ directly from the power series. $\blacksquare$

Analytic subgroups

Definition V.2.13. Let G be a Lie group with an exponential function, so that we obtain for each $x \in \mathfrak{g} := \mathbf{L}(G)$ an automorphism $e^{\mathrm{ad}\, x} := \mathrm{Ad}(\exp x) \in \mathrm{Aut}(\mathfrak{g})$. A subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is called *stable* if

$$e^{\mathrm{ad}\, x}.\mathfrak{h} = \mathrm{Ad}(\exp x).\mathfrak{h} = \mathfrak{h} \quad \text{for all } x \in \mathfrak{h}.$$

An ideal $\mathfrak{n} \trianglelefteq \mathfrak{g}$ is called *a stable ideal* if

$$e^{\mathrm{ad}\, x}.\mathfrak{n} = \mathfrak{n} \quad \text{for all } x \in \mathfrak{g}. \qquad \blacksquare$$

The following lemma shows that stability of kernel and range is a necessary reqirement for the integrability of a homomorphism of Lie algebras.

Lemma V.2.14. *If $\varphi : G \to H$ is a morphism of Lie groups with an exponential function, then* $\mathrm{im}(\mathbf{L}(\varphi))$ *is a stable subalgebra of* $\mathbf{L}(H)$, *and* $\ker(\mathbf{L}(\varphi))$ *is a stable ideal of* $\mathbf{L}(G)$.

Proof. For $\alpha := \mathbf{L}(\varphi)$, we have $\varphi \circ \exp_G = \exp_H \circ \alpha$, which leads to

$$\alpha \circ e^{\mathrm{ad}\, x} = \mathbf{L}(\varphi) \circ \mathrm{Ad}(\exp x) = \mathbf{L}(\varphi \circ c_{\exp x}) = \mathbf{L}(c_{\varphi(\exp x)} \circ \varphi) = \mathrm{Ad}(\exp \mathbf{L}(\varphi).x) \circ \mathbf{L}(\varphi) = e^{\mathrm{ad}\, \alpha(x)} \circ \alpha.$$

We conclude in particular that $\mathrm{im}(\alpha)$ is a stable subalgebra and that $\ker \alpha$ is a stable ideal. $\blacksquare$

Example V.2.15. Let $V := C^\infty(\mathbb{R}, \mathbb{R})$ and consider the one-parameter group $\alpha\colon \mathbb{R} \to \mathrm{GL}(V)$, given by $\alpha(t)(f)(x) = f(x+t)$. Then $\mathbb{R}$ acts smoothly on V, so that we may form the corresponding semidirect product group

$$G := V \rtimes_\alpha \mathbb{R}.$$

This is a Lie group with a smooth exponential function given by

$$\exp(v,t) = \Big(\int_0^1 \alpha(st).v\,ds, t\Big),$$

where

$$\Big(\int_0^1 \alpha(st).v\,ds\Big)(x) = \int_0^1 v(x+st)\,ds.$$

The Lie algebra $\mathfrak{g}$ has the corresponding semidirect product structure $\mathfrak{g} = V \rtimes_D \mathbb{R}$ with $Dv = v'$, i.e.,

$$[(f,t),(g,s)] = (tg' - sf', 0).$$

In $\mathfrak{g} \cong V \rtimes \mathbb{R}$, we now consider the subalgebra $\mathfrak{h} := V_{[0,1]} \rtimes \mathbb{R}$, where

$$V_{[0,1]} := \{f \in V\colon \mathrm{supp}(f) \subseteq [0,1]\}.$$

Then $\mathfrak{h}$ clearly is a closed subalgebra of $\mathfrak{g}$. It is not stable because $\alpha(-t)V_{[0,1]} = V_{[t,t+1]}$. The subgroup of G generated by $\exp\mathfrak{h}$ contains $\{0\} \rtimes \mathbb{R}$, $V_{[0,1]}$, and hence all intervales $V_{[t,t+1]}$, which implies that $\langle \exp\mathfrak{h}\rangle = C^\infty_c(\mathbb{R}) \rtimes \mathbb{R}$.

The preceding lemma implies that the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ does not integrate to a homomorphism $\varphi\colon H \to G$ of Lie group with an exponential function, for which $\mathbf{L}(\varphi)$ is the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$. ■

Definition V.2.16. Let G be a locally exponential Lie group. An *analytic subgroup* is an injective morphism $\iota\colon H \to G$ of locally exponential Lie groups for which H is connected and the differential $\mathbf{L}(\iota)$ of ι is injective. ■

Remark V.2.17. If $\iota\colon H \to G$ is an analytic subgroup, then the relation

$$\exp_G \circ \mathbf{L}(\iota) = \iota \circ \exp_H \tag{5.2.1}$$

implies that

$$\ker(\mathbf{L}(\iota)) = \mathbf{L}(\ker\iota) = \{0\},$$

so that $\mathbf{L}(\iota)\colon \mathbf{L}(H) \to \mathbf{L}(G)$ is an injective morphism of locally exponential Lie algebras, which implies in particular that $\mathfrak{h} := \mathrm{im}(\mathbf{L}(\iota))$ is a stable subalgebra of $\mathbf{L}(G)$ (Lemma V.2.14). Moreover, (5.2.1) shows that the subgroup $\iota(H)$ of G coincides, as a set, with the subgroup $\langle \exp_G \mathfrak{h}\rangle$ of G generated by $\exp_G \mathfrak{h}$. Therefore an analytic subgroup can be viewed as a locally exponential Lie group structure on the subgroup of G generated by $\exp_G \mathfrak{h}$. ■

Definition V.2.18. A locally convex Lie algebra $\mathfrak{g}$ is called *locally exponential* if there exists a symmetric convex open 0-neighborhood $U \subseteq \mathfrak{g}$ and an open subset $D \subseteq U \times U$ on which we have a smooth map

$$m_U\colon D \to U, \quad (x,y) \mapsto x * y$$

such that $(U, D, m_U, 0)$ is a local Lie group with the additional property that
(E1) For $x \in U$ and $|t|, |s|, |t+s| \leq 1$, we have $(tx, sx) \in D$ with

$$tx * sx = (t+s)x.$$

(E2) The second order term in the Taylor expansion of m_U is $b(x,y) = \frac{1}{2}[x,y]$. ■

Since any local Lie group $(U, D, m_U, 0)$ on an open subset of a locally convex space V leads to a Lie algebra structure on V (Remark III.1.12), condition (E2) only insures that $\mathfrak{g}$ is the Lie algebra of the local group.

Lemma V.2.19. *The Lie algebra of a locally exponential Lie group is locally exponential.* ■

Theorem V.2.20. (Analytic Subgroup Theorem) *Let G be a locally exponential Lie group and $\mathfrak{g}$ its Lie algebra. Then an injective morphism $\alpha\colon \mathfrak{h} \to \mathfrak{g}$ of locally convex Lie algebras integrates to an analytic subgroup if and only if $\mathfrak{h}$ is a locally exponential Lie algebra.* ■

The condition that a closed subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is locally exponential is quite subtle. It means that for x, y sufficiently close to 0 in $\mathfrak{h}$, we have $x * y \in \mathfrak{h}$. To verify this condition, one would like to show that the integral curve $\gamma(t) := x * ty$ of the left invariant vector field y_l through x does not leave the closed subspace $\mathfrak{h}$ of $\mathfrak{g}$. This leads to the additional condition that $d\lambda_x(0)(\mathfrak{h}) \subseteq \mathfrak{h}$, which, under the assumption that $\mathfrak{h}$ is stable, means that the operator $\kappa_{\mathfrak{g}}(x) = \int_0^1 e^{-t \operatorname{ad} x}\, dt$ satisfies $\kappa_{\mathfrak{g}}(x)(\mathfrak{h}) = \mathfrak{h}$ for $x \in \mathfrak{h}$ sufficiently close to 0 (Theorem V.2.8). For a closed ideal $\mathfrak{n} \trianglelefteq \mathfrak{g}$ of the locally exponential Lie algebra $\mathfrak{g}$, the corresponding condition $\kappa_{\mathfrak{g}}(x)(\mathfrak{n}) = n$ for all x sufficiently close 0 is sufficient for $\mathfrak{n}$ to be locally exponential. This result is used in the proof of the Quotient Theorem V.2.8.

Corollary V.2.21. (Analytic Subgroup Theorem for Banach–Lie groups) *Let G be a locally exponential Lie group and $\mathfrak{g}$ its Lie algebra. Then an injective morphism $\alpha\colon \mathfrak{h} \to \mathfrak{g}$ of Banach algebras always integrates to an analytic subgroup.*

Proof. Using the BCH multiplication on a 0-neighborhood of $\mathfrak{h}$, it follows that $\mathfrak{h}$ is locally exponential. ■

Remark V.2.22. If G is a Banach–Lie group and $\mathfrak{h} \subseteq \mathfrak{g} := \mathbf{L}(G)$ a closed separable subalgebra, then the analytic subgroup $H := \langle \exp \mathfrak{h} \rangle \subseteq G$ satisfies

$$\mathbf{L}(H) = \{x \in \mathfrak{g} \colon \exp(\mathbb{R}x) \subseteq H\} = \mathfrak{h},$$

i.e., $\exp \mathbb{R}x \subseteq H$ implies $x \in \mathfrak{h}$ (Theorem 5.52 in [HoMo98]).

For non-separable subalgebras $\mathfrak{h}$, this is no longer true in general, as the following counterexample shows ([HoMo98, p.157]): We consider the abelian Lie group $\mathfrak{g} := \ell^1(\mathbb{R}, \mathbb{R}) \times \mathbb{R}$, where the group structure is given by the addition. We write $(e_r)_{r \in \mathbb{R}}$ for the canonical topological basis elements of $\ell^1(\mathbb{R}, \mathbb{R})$. Then the subgroup D generated by the pairs $(e_r, -r)$, $r \in \mathbb{R}$, is closed and discrete, so that $G := \mathfrak{g}/D$ is an abelian Lie group. Now we consider the closed subalgebra $\mathfrak{h} := \ell^1(\mathbb{R}, \mathbb{R})$ of $\mathfrak{g}$. As $\mathfrak{h} + D = \mathfrak{g}$, we have $H := \exp \mathfrak{h} = G$, and therefore $(0,1) \in \mathbf{L}(H) \setminus \mathfrak{h}$. ■

Exercises for Section V

Exercise V.1. Let V be a locally convex space. Show that every continuous group homomorphism $\gamma\colon (\mathbb{R}, +) \to (V, +)$ can be written as $\gamma(t) = tv$ for some $v \in E$. ■

Exercise V.2. Let E be a Banach space.

(1) If F is a closed subspace of E and $H := \{g \in \mathrm{GL}(E) \colon g(F) \subseteq F\}$ (cf. Proposition V.2.12), then

$$\mathbf{L}(H) = \{Y \in \mathcal{L}(E) \colon Y(F) \subseteq F\}.$$

(2) For each $v \in E$ and $H := \{g \in \mathrm{GL}(E) \colon g(v) = v\}$, we have

$$\mathbf{L}(H) = \{Y \in \mathcal{L}(E) \colon Y.v = 0\}.$$ ■

Exercise V.3. Let A be a Banach space and $m\colon A \times A \to A$ a continuous bilinear map. Then the group

$$\mathrm{Aut}(A, m) := \{g \in \mathrm{GL}(A) \colon (\forall a, b \in A)\ m(g.a, g.b) = g.m(a, b)\}$$

of automorphisms of the (not necessarily associative) algebra (A, m) is a Lie group whose Lie algebra is the space

$$\mathrm{der}(A, m) := \{X \in \mathcal{L}(A) \colon (\forall a, b \in A)\ X.m(a, b) = m(X.a, b) + m(a, X.b)\}$$

of derivations of (A, m). Hint: Theorem V.2.11. ■

Exercise V.4. Let J be a set. For a tuple $x = (x_j)_{j\in J} \in (\mathbb{R}^+)^J$, we define

$$\sum_{j\in J} x_j := \sup\Big\{\sum_{j\in F} x_j : F \subseteq J \text{ finite}\Big\}.$$

Show that

$$\ell^1(J,\mathbb{R}) := \Big\{x = (x_j)_{j\in J} : \sum_{j\in J} |x_j| < \infty\Big\}$$

is a Banach space with respect to the norm $\|x\|_1 := \sum_{j\in J}|x_j|$. Define $e_j \in \ell^1(J,\mathbb{R})$ by $(e_j)_i = \delta_{ij}$. Show that the subgroup Γ generated by $\{e_j : j \in J\}$ is discrete. ■

VI. More on integrability of Lie algebras

We recall that a locally convex Lie algebra $\mathfrak{g}$ is said to integrable if there exists some Lie group G with $\mathbf{L}(G) = \mathfrak{g}$ (Definition III.1.9).

Examples VI.1. If $\mathfrak{g}$ is a finite-dimensional Lie algebra, endowed with its unique locally convex topology, then $\mathfrak{g}$ is integrable. This is Lie's Third Theorem. One possibility to prove this is first to use Ado's Theorem to find an embedding $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n(\mathbb{R})$ and then to endow the analytic subgroup $G := \langle \exp \mathfrak{g}\rangle \subseteq \mathrm{GL}_n(\mathbb{R})$ with a Lie group structure such that $\mathbf{L}(G) = \mathfrak{g}$ (cf. Corollary V.2.20). ■

Proposition VI.2. *Let G be a connected complex Lie group. Then each closed ideal of $\mathbf{L}(G)$ is invariant under $\mathrm{Ad}(G)$.*

Proof. Let $\mathfrak{a} \trianglelefteq \mathfrak{g} = \mathbf{L}(G)$ be a closed ideal. Since G is assumed to be connected, it suffices to show that there exists a $\mathbf{1}$-neighborhood $U \subseteq G$ with $\mathrm{Ad}(U).\mathfrak{a} \subseteq \mathfrak{a}$. We may w.l.o.g. assume that U is diffeomorphic to an open convex 0-neighborhood in $\mathfrak{g}$. Then we find for every $g \in U$ a connected open subset $V \subseteq \mathbb{C}$ with $0, 1 \in V$ and a holomorphic map $p: V \to G$ with $p(0) = \mathbf{1}$ and $p(1) = g$.

Let $w_0 \in \mathfrak{a}$ and $w(t) := \mathrm{Ad}(p(t)).w_0$ for $t \in V$. We have to show that $w(1) = \mathrm{Ad}(g).w_0 \in \mathfrak{a}$. For the right logarithmic derivative $v := \mathrm{Ad}(p).\delta(p): V \to \mathfrak{g}$, we obtain the differential equation

$$w'(t) = \mathrm{Ad}(p(t)).[p^{-1}(t).p'(t), w_0] = \mathrm{Ad}(p(t)).[\delta(p)(t), w_0] = [v(t), w(t)]. \tag{6.1}$$

Since the maps v and w are holomorphic, their Taylor expansions converge for t close to 0:

$$v(t) = \sum_{n=0}^\infty v_n t^n \quad \text{and} \quad w(t) = \sum_{n=0}^\infty w_n t^n$$

in $\mathfrak{g}$. Then the differential equation (6.1) for w can be written as

$$\sum_{n=0}^\infty (n+1) w_{n+1} t^n = w'(t) = [v(t), w(t)] = \sum_{n=0}^\infty t^n \sum_{k=0}^n [v_k, w_{n-k}].$$

Comparing coefficients now leads to

$$w_{n+1} = \frac{1}{n+1}\sum_{k=0}^n [v_k, w_{n-k}],$$

so that we obtain inductively $w_n \in \mathfrak{a}$ for each $n \in \mathbb{N}$. Since $\mathfrak{a}$ is closed, we get $w(t) \in \mathfrak{a}$ for t close to 0. Applying the same argument in other points $t_0 \in V$, we see that the set $w^{-1}(\mathfrak{a})$ is an open closed subset of V, and therefore that $a(1) \in \mathfrak{a}$ because $a(0) \in \mathfrak{a}$ and V is connected. ■

Corollary VI.3. *If $\mathfrak{g}$ is a complex Fréchet–Lie algebra containing a closed ideal which is not stable, then $\mathfrak{g}$ is not integrable to a complex Lie group with an exponential function.* ■

Remark VI.4. The preceding proposition can be generalized to the larger class of real analytic Lie groups, where it can be used to conclude that the Lie group $\mathrm{Diff}(M)$ does not possess an analytic Lie group structure. Indeed, for each non-dense open subset $K \subseteq M$, the subspace

$$\mathcal{V}(M)_K := \{X \in \mathcal{V}(M): X|_K = 0\}$$

is a closed ideal of $\mathcal{V}(M)$ not invariant under $\mathrm{Diff}(M)$ because $\mathrm{Ad}(\varphi).\mathcal{V}(M)_K = \mathcal{V}(M)_{\varphi(K)}$ for $\varphi \in \mathrm{Diff}(M)$. ■

Theorem VI.5. (Lempert) *If M is a compact manifold, then the Fréchet–Lie algebra $\mathcal{V}(M)_{\mathbb{C}}$ is not integrable to a regular complex Lie group.*

Proof. (Sketch; see [Mil83]) Let $\mathfrak{g} := \mathcal{V}(M)_{\mathbb{C}}$ and $K \subseteq M$ be an open non-empty subset of M which is not dense. Then

$$\mathfrak{i}_K := \{x \in \mathfrak{g}: x|_K = 0\}$$

is a closed ideal of $\mathfrak{g}$.

Let G be a regular complex Lie group with Lie algebra $\mathfrak{g}$ and let $q: \widetilde{\mathrm{Diff}}(M) \to \mathrm{Diff}(M)_0$ denote the universal covering homomorphism of $\mathrm{Diff}(M)_0$. Then the inclusion homomorphism $\mathcal{V}(M) \hookrightarrow \mathfrak{g}$ can be integrated to a Lie group homomorphism $\varphi: \widetilde{\mathrm{Diff}}(M) \to G$. For $g \in \widetilde{\mathrm{Diff}}(M)$, we then have

$$\mathrm{Ad}(\varphi(g)).\mathfrak{i}_K = \mathfrak{i}_{\varphi(g)(K)},$$

contradicting the invariance of $\mathfrak{i}_K$ under $\mathrm{Ad}(G)$ (Proposition VI.2). ■

Remark VI.6. (a) In [Omo81], Omori shows that for any non-compact smooth manifold M, the Lie algebra $\mathcal{V}(M)$ is not integrable.

(b) Theorem VI.5 holds without the regularity assumption, resulting in the fact that $\mathcal{V}(M)_{\mathbb{C}}$ is not integrable to any group G with an exponential function. The main point is that for any such group G and $X \in \mathcal{V}(M) \subseteq \mathfrak{g}$, the one-parameter group $\exp(\mathbb{R}X)$ acts on $\mathfrak{g}$ precisely as the corresponding one-parameter group of $\mathrm{Diff}(M)$. This argument requires a uniqeness lemma for "smooth" maps with values in $\mathrm{Aut}(\mathfrak{g})$, which is far from being a Lie group (cf. [GN05]). ■

Example VI.7. To construct an example of a non-integrable Banach–Lie algebra, we proceed as follows.

Let H be an infinite-dimensional complex Hilbert space and $U(H)$ its unitary group. This is a Banach–Lie group with Lie algebra

$$\mathbf{L}(U(H)) = \mathfrak{u}(H) := \{X \in \mathcal{L}(H): X^* = -X\}.$$

The center of this Lie algebra is given by $\mathfrak{z}(\mathfrak{u}(H)) = \mathbb{R}i\mathbf{1}$. We consider the Banach–Lie algebra

$$\mathfrak{g} := \big(\mathfrak{u}(H) \oplus \mathfrak{u}(H)\big)/\mathbb{R}i(\mathbf{1}, \sqrt{2}\mathbf{1}).$$

We claim that $\mathfrak{g}$ is not integrable. Let us assume to the contrary that G is a connected Lie group with Lie algebra $\mathfrak{g}$. Let

$$q: \mathfrak{u}(H) \oplus \mathfrak{u}(H) \to \mathfrak{g}$$

denote the quotient homomorphism. According to Kuiper's Theorem, the group $U(H)$ and hence the group $G_1 := U(H) \times U(H)$ is contractible ([Ku65]) and therefore in particular simply connected. Hence there exists a unique Lie group homomorphism

$$f: G_1 \to G \quad \text{with} \quad \mathbf{L}(f) = q.$$

We then have $\exp_G \circ q = f \circ \exp_{G_1}$, and in particular $\exp(\ker q) \subseteq \ker f$. As $Z(G_1) \cong \mathbb{T}^2$ is a two-dimensional torus and $\exp(\ker q)$ is a dense one-parameter subgroup of $Z(G_1)$, the continuity of f implies that $Z(G_1) \subseteq \ker f$ and hence that $\mathfrak{z}(\mathfrak{g}_1) \subseteq \ker \mathbf{L}(f) = \ker q$, which is a contradiction. ■

The following theorem is an immediate consequence of Corollary V.2.20.

Theorem VI.8. (van Est–Korthagen, 1964) *Let $\mathfrak{h}$ and $\mathfrak{g}$ be Banach–Lie algebras. If $\mathfrak{g}$ is integrable and $\varphi: \mathfrak{h} \hookrightarrow \mathfrak{g}$ is injective, then $\mathfrak{h}$ is integrable.* ∎

Corollary VI.9. *If $\mathfrak{g}$ is a Banach–Lie algebra, then $\mathfrak{g}/\operatorname{ad}\mathfrak{z}(\mathfrak{g}) \cong \operatorname{ad}\mathfrak{g}$ is integrable.*

Proof. The adjoint representation $\operatorname{ad}: \mathfrak{g} \to \operatorname{der}\mathfrak{g}$ factors through an injective homomorphism $\mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \hookrightarrow \operatorname{der}\mathfrak{g}$, and

$$\operatorname{der}\mathfrak{g} := \{D \in \mathcal{L}(\mathfrak{g}): (\forall x, y \in \mathfrak{g})\ D([x,y]) = [D(x), y] + [x, D(y)]\}$$

is the Lie algebra of the Banach–Lie group $\operatorname{Aut}(\mathfrak{g})$ (cf. Exercise V.3). ∎

The following theorem generalizes Corollary VI.9. It requires more refined machinery because for a locally convex Lie algebra $\mathfrak{g}$ the group $\operatorname{Aut}(\mathfrak{g})$ carries no natural Lie group structure. Nevertheless, the technique of the proof is to endow the subgroup generated by $e^{\operatorname{ad}\mathfrak{g}}$, which makes sense for locally exponential Lie algebras, with a Lie group structure.

Theorem VI.10. *For any locally exponential Lie algebra $\mathfrak{g}$, the quotient $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is integrable to a locally exponential Lie group.* ∎

The preceding corollary reduces the integrability problem for Banach–Lie algebras, and even for locally exponential Lie algebras, to the question when a central extension of an integrable Lie algebra is again integrable. In this context, a central extension is a quotient morphism $q: \widehat{\mathfrak{g}} \to \mathfrak{g}$ of Lie algebras for which $\mathfrak{z} := \ker q$ is central in $\widehat{\mathfrak{g}}$. Now the question is the following: given a connected Lie group G with Lie algebra $\mathfrak{g}$, when is there a central group extension $Z \hookrightarrow \widehat{G} \to G$ "integrating" the corresponding Lie algebra extension? Without going too much into details, we cite the following theorem which points into a direction which can be followed with success for general Lie groups (see [Ne02a]). Earlier versions of the following theorem for Banach–Lie algebras have been obtained by van Est and Korthagen in their systematic discussion of the non-integrability problem for Banach–Lie algebras in [EK64].

Theorem VI.11. *Let G be a simply connected locally exponential Lie group with Lie algebra $\mathfrak{g}$. Then one can associate to each central Lie algebra extension $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} \to \mathfrak{g}$ a singular cohomology class $c \in H^2(G, \mathfrak{z}) \cong \operatorname{Hom}(\pi_2(G), \mathfrak{z})$ which we interpret as a period homomorphism*

$$\operatorname{per}_c: \pi_2(G) \to \mathfrak{z}.$$

Then a corresponding central extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ exists for a Lie group Z with Lie algebra $\mathfrak{z}$ if and only if $\operatorname{im}(\operatorname{per}_c) \subseteq \mathfrak{z}$ is discrete. ∎

Remark VI.12. (a) Let $\mathfrak{g}$ be a locally exponential Lie algebra and G_{ad} a simply connected Lie group with Lie algebra $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ (Theorem VI.9). Then the preceding theorem implies in particular that $\mathfrak{g}$ is integrable if and only if the period homomorphism $\operatorname{per}_{\mathfrak{g}}: \pi_2(G_{\mathrm{ad}}) \to \mathfrak{z}(\mathfrak{g})$ associated to the central extension $\operatorname{ad}: \mathfrak{g} \to \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ has discrete image.

The problem with this characterization is that in general it might be quite hard to determine the image of the period homomorphism.

(b) For any quotient morphism $G \to G/N$ of Banach–Lie groups, Michael's Selection Theorem ([Mi59]) implies that G is a locally trivial topological N-principal bundle over G/N, which implies the existence of a corresponding long exact homotopy sequence.

If $\mathfrak{g}$ is an integrable Banach–Lie algebra and G is a simply connected Banach–Lie group with Lie algebra $\mathfrak{g}$, then the long exact homotopy sequence associated to the homomorphism $q: G \to G_{\mathrm{ad}}$ with kernel $Z(G)_0$ induces a surjective connecting homomorphism

$$\pi_2(G_{\mathrm{ad}}) \to \pi_1(Z(G)),$$

and by identifying the universal covering group of $Z(G)_0$ with $(\mathfrak{z}(\mathfrak{g}), +)$, one can show that this connecting homomorphism coincides with the period map. Its image is the group $\pi_1(Z(G))$, considered as a subgroup of $\mathfrak{z}$. With this picture in mind, one may think that the non-integrability on a Banach–Lie algebra $\mathfrak{g}$ is caused by the non-existence of a Lie group Z with Lie algebra $\mathfrak{z}(\mathfrak{g})$ and fundamental group $\operatorname{im}(\operatorname{per}_{\mathfrak{g}})$.

(c) If $\mathfrak{g}$ is finite-dimensional, then G_{ad} is also finite-dimensional, and therefore $\pi_2(G_{\mathrm{ad}})$ vanishes by a theorem of E. Cartan ([Mi95, Th. 3.7]). Hence the period homomorphism $\operatorname{per}_{\mathfrak{g}}$ is trivial for every finite-dimensional Lie algebra $\mathfrak{g}$. ∎

Example VI.13. We consider the Lie algebra

$$\mathfrak{g} := \big(\mathfrak{u}(H) \oplus \mathfrak{u}(H)\big)/\mathbb{R}i(\mathbf{1}, \sqrt{2}\mathbf{1})$$

from Example VI.7. Then $\mathfrak{z}(\mathfrak{g}) \cong i\mathbb{R}$, and one can show that the image of the period map is given by

$$2\pi i(\mathbb{Z} + \sqrt{2}\mathbb{Z}) \subseteq i\mathbb{R},$$

which is not discrete. ■

Appendix A. Characters of the algebra of smooth functions

Theorem A.1. *Let M be a finite-dimensional smooth paracompact manifold and $A := C^\infty(M, \mathbb{R})$ the unital Fréchet algebra of smooth functions on M.*

(1) *If M is compact, then each maximal ideal of A is closed.*

(2) *Each closed maximal ideal of A is the kernel of an evaluation homomorphism $\delta_p: A \to \mathbb{R}, f \mapsto f(p)$.*

(3) *Each character $\chi: A \to \mathbb{R}$ is an evaluation in some point $p \in M$.*

Proof. (1) If M is compact, then the unit group $A^\times = C^\infty(M, \mathbb{R}^\times)$ is an open subset of A. If $I \subseteq A$ is a maximal ideal, then I intersects $A^\times$ trivially, and since $A^\times$ is open, the same holds for the closure $\overline{I}$. Hence $\overline{I}$ also is a proper ideal, so that the maximality of I implies that I is closed.

(2) Let $I \subseteq A$ be a closed maximal ideal. If all functions in I vanish in the point $p \in M$, then the maximality of I implies that $I = \ker \delta_p$. So we have to show that such a point exists. Let us assume that this is not the case. From that we shall derive the contradiction $I = A$.

Let $K \subseteq M$ be a compact set. Then for each $p \in K$, there exists a function $f_p \in I$ with $f_p(p) \neq 0$. The family $(f_p^{-1}(\mathbb{R}^\times))_{p \in K}$ is an open cover of K, so that there exist $p_1, \ldots, p_n \in K$ with $f_K := \sum_j f_{p_j}^2 > 0$ on K.

If M is compact, then we thus obtain a function $f_M \in I$ with no zeros, which leads to the contradiction $f_M \in A^\times \cap I$. Suppose that M is non-compact. Then there exists a sequence $(M_n)_{n \in \mathbb{N}}$ of compact subsets with $M = \bigcup_n M_n$ and $M_n \subseteq M_{n+1}^0$. Let $f_n \in I$ be a non-negative function supported by $M_{n+1} \setminus M_{n-1}^0$ with $f_n > 0$ on the compact set $M_n \setminus M_{n-1}^0$. Here the requirement on the support can be achieved by multiplying with a smooth function supported by $M_{n+1} \setminus M_{n-1}^0$ which equals 1 on $M_n \setminus M_{n-1}^0$. Then the series $\sum_n f_n$ converges, because on each set M_n it is eventually constant and each compact subset of M is contained in some M_n. Now $f := \sum_n f_n$ is a smooth function in $\overline{I} = I$ with $f > 0$. Hence f is invertible, which is a contradiction.

(3) Let $\chi: A \to \mathbb{R}$ be a character. If $f \in A$ is non-negative, then for each $c > 0$ we have $f + c = h^2$ for some $h \in A^\times$, and this implies that $\chi(f) + c = \chi(f + c) = \chi(h)^2 \geq 0$, which leads to $\chi(f) \geq -c$, and consequently $\chi(f) \geq 0$.

Now let $F: M \to \mathbb{R}$ be a smooth function for which the sets $F^{-1}(]-\infty, c])$, $c \in \mathbb{R}$, are compact. Such a function can easily be constructed from a sequence $(M_n)_{n \in \mathbb{N}}$ as above using a smooth version of Urysohn's Lemma (Exercise).

We consider the ideal $I := \ker \chi$. If I has a zero, then $I = \ker \delta_p$ for some $p \in M$ and this implies that $\chi = \delta_p$. Hence we may assume that I has no zeros. Then the argument under (2) provides for each compact subset $K \subseteq M$ a compactly supported function $f_K \in I$ with $f_K > 0$ on K. If $h \in A$ is supported by K, we therefore find a $\lambda > 0$ with $\lambda f_K - h \geq 0$, which leads to

$$0 \leq \chi(\lambda f_K - h) = \chi(-h),$$

and hence to $\chi(h) \geq 0$. Replacing h by $-h$, we also get $\chi(h) \leq 0$ and hence $\chi(h) = 0$. Therefore χ vanishes on all compactly supported functions.

For $c > 0$, we now pick $f_c \in I$ with $f_c > 0$ on the compact subset $F^{-1}(]-\infty, c])$ and $f_c \geq 0$. Then there exists a $\mu > 0$ with $\mu f_c + F \geq c$ on $F^{-1}(]-\infty, c])$. Now $\mu f_c + F \geq c$ holds on all of M, and therefore

$$\chi(F) = \chi(F + \mu f_c) \geq c.$$

Since $c > 0$ was arbitrary, we arrive at a contradiction. ∎

Appendix B. The compact open topology

In this appendix, we discuss some properties of the compact open topology on the space $C(X,Y)$ of continuous maps between two topological spaces X and Y.

Definition B.1. If X and Y are topological spaces, then the topology on $C(X,Y)$ generated by the sets

$$W(K,O) := \{f \in C(X,Y) : f(K) \subseteq O\},$$

$K \subseteq X$ compact and $O \subseteq Y$ open, is called the *compact open topology.* ∎

The following lemma is extremely useful to construct group topologies from a filter basis of identity neighborhoods. Here we shall use it to see that for a topological group G, the compact open topology turns $C(X,G)$ into a topological group.

Lemma B.2. *Let G be a group and $\mathcal{F}$ a filter basis of subsets of G satisfying*
(U0) $\bigcap \mathcal{F} = \{\mathbf{1}\}$.
(U1) $(\forall U \in \mathcal{F})(\exists V \in \mathcal{F})\ VV \subseteq U$.
(U2) $(\forall U \in \mathcal{F})(\exists V \in \mathcal{F})\ V^{-1} \subseteq U$.
(U3) $(\forall U \in \mathcal{F})(\forall g \in G)(\exists V \in \mathcal{F})\ gVg^{-1} \subseteq U$.
Then there exists a unique group topology on G such that $\mathcal{F}$ is a basis of $\mathbf{1}$-neighborhoods in G. This topology is given by $\{U \subseteq G : (\forall g \in U)(\exists V \in \mathcal{F})\ gV \subseteq U\}$.

Proof. ([Bou88, Ch. III, §1.2, Prop. 1]) Let

$$\tau := \{U \subseteq G : (\forall g \in U)(\exists V \in \mathcal{F})\ gV \subseteq U\}.$$

First we show that τ is a topology. Clearly $\emptyset, G \in \tau$. Let $(U_j)_{j \in J}$ be a family of elements of τ and $U := \bigcup_{j \in J} U_j$. For each $g \in U$, there exists a $j_0 \in J$ with $g \in U_j$ and a $V \in \mathcal{F}$ with $gV \subseteq U_{j_0} \subseteq U$. Thus $U \in \tau$, and we see that τ is stable under arbitrary unions.

If $U_1, U_2 \in \tau$ and $g \in U_1 \cap U_2$, then there exist $V_1, V_2 \in \mathcal{F}$ with $gV_i \subseteq U_i$. Since $\mathcal{F}$ is a filter basis, there exists $V_3 \in \mathcal{F}$ with $V_3 \subseteq V_1 \cap V_2$, and then $gV_3 \subseteq U_1 \cap U_2$. We conclude that $U_1 \cap U_2 \in \tau$, and hence that τ is a topology on G.

We claim that the interior of a subset $U \subseteq G$ is given by

$$U^0 = U_1 := \{u \in U : (\exists V \in \mathcal{F})\ uV \subseteq U\}.$$

In fact, if there exists a $V \in \mathcal{F}$ with $uV \subseteq U$, then we pick a $W \in \mathcal{F}$ with $WW \subseteq V$ and obtain $uWW \subseteq U$, so that $uW \subseteq U_1$. Hence U_1 is open, and it clearly is the largest open subset contained in U, i.e., $U_1 = U^0$. It follows in particular that U is a neighborhood of g if and only if $g \in U^0$, and we see in particular that $\mathcal{F}$ is a basis of the neighborhood filter of $\mathbf{1}$. The property $\bigcap \mathcal{F} = \{\mathbf{1}\}$ implies that for $x \neq y$, there exists $U \in \mathcal{F}$ with $y^{-1}x \notin U$. For $V \in \mathcal{F}$ with $VV \subseteq U$ and $W \in \mathcal{F}$ with $W^{-1} \subseteq V$, we then obtain $y^{-1}x \notin WW^{-1}$, i.e., $xW \cap yW = \emptyset$. Thus (G, τ) is a Hausdorff space.

To see that G is a topological group, we have to verify that the map

$$f : G \times G \to G, \quad (x,y) \mapsto xy^{-1}$$

is continuous. So let $x, y \in G$, $U \in \mathcal{F}$ and pick $V \in \mathcal{F}$ with $yVy^{-1} \subseteq U$ and $W \in \mathcal{F}$ with $WW^{-1} \subseteq V$. Then

$$f(xW, yW) = xWW^{-1}y^{-1} = xy^{-1}y(WW^{-1})y^{-1} \subseteq xy^{-1}yVy^{-1} \subseteq xy^{-1}U,$$

implies that f is continuous in (x,y). ∎

Lemma B.3. *Let X be a non-empty topological space and G a topological group. Then the set $C(X,G)$ of all continuous maps $X \to G$ is a group with respect to pointwise multiplication. The unit element of this group is the constant function $\mathbf{1}$. The system $\mathcal{F}$ of all sets $W(K,U) \subseteq C(X,G)$, where $K \subseteq X$ is compact and $U \subseteq G$ is an open $\mathbf{1}$-neighborhood, is a filter basis, and there exists a unique group topology on $C(X,G)$ for which $\mathcal{F}$ is a basis of $\mathbf{1}$-neighborhoods.*

This topology is called the *topology of compact convergence* or the *topology of uniform convergence on compact sets.*

Proof. First we show that $\mathcal{F}$ is a filter basis:

For each $x \in X$, the set $W(\{x\},G)$ is contained in $\mathcal{F}$, so that $\mathcal{F}$ is not empty. Since each set $W(K,U)$ contains the constant map $\mathbf{1}$, it is non-empty. We further have $W(K_1,U_1) \cap W(K_2,U_2) \supseteq W(K_1 \cup K_2, U_1 \cap U_2)$. This proves that $\mathcal{F}$ is a filter basis of subsets of G. We now verify the conditions in Lemma B.2:

(U0): If $f \in C(X,G)$ is contained in $W(\{x\},U)$ for all $\mathbf{1}$-neighborhoods U in G, it follows from the fact that G is Hausdorff that $f(x) = \mathbf{1}$, so that $\bigcap \mathcal{F}$ consists only of the constant function $\mathbf{1}$.

(U1): For each $W(K,U) \in \mathcal{F}$, we find a $\mathbf{1}$-neighborhood $V \subseteq G$ with $VV \subseteq U$. Then $W(K,V)W(K,V) \subseteq W(K,U)$.

(U2): $W(K,U)^{-1} = W(K,U^{-1})$.

(U3): For $f \in C(X,G)$ and $W(K,U) \in \mathcal{F}$, we consider the open set

$$E := \{(x,g) \in X \times G \colon f(x)gf(x)^{-1} \in U\}.$$

Then $K \times \{\mathbf{1}\} \subseteq E$ and the compactness of K imply the existence of a $\mathbf{1}$-neighborhood V in G with $K \times V \subseteq E$. Then $fW(K,V)f^{-1} \subseteq W(K,U)$.

Now Lemma B.2 shows that there exists a unique group topology on $C(X,G)$ for which $\mathcal{F}$ is a basis of $\mathbf{1}$-neighborhoods. ■

Proposition B.4. *For a topological space X and a topological group G, the compact open topology coincides on $C(X,G)$ with the topology of compact convergence for which the sets $W(K,O)$, $K \subseteq X$ compact and O an open $\mathbf{1}$-neighborhood in G, form a basis of identity neighborhoods.*

Proof. **Step 1:** The topology of compact convergence is finer than the compact open topology because each set $W(K,O)$ is open in the topology of compact convergence. In fact, for $f \in W(K,O)$ the set $f(K) \subseteq O \subseteq G$ is compact, so that there exists a $\mathbf{1}$-neighborhood $U \subseteq G$ with $f(K)U \subseteq O$. This implies that $f \cdot W(K,U) \subseteq W(K,O)$, and hence that $W(K,O)$ is open in the topology of uniform convergence on compact subsets of X.

Step 2: Let $f_0 \in C(X,G)$. We claim that each set of the form $f_0W(K,V)$ contains a neighborhood of f_0 in the compact open topology.

Let $W = W^{-1} \subseteq G$ be an open $\mathbf{1}$-neighborhood. Since f_0 is continuous, each $k \in K$ has a compact neighborhood U_k in K with $f_0(U_k) \subseteq f_0(k)W$. The compactness of K implies that it is covered by finitely many of the sets U_k, so that there exist $k_1, \ldots, k_n \in K$ with

$$K \subseteq U_{k_1} \cup \ldots \cup U_{k_n}.$$

Then the sets $Q_j := f_0(U_{k_j})W$ are open in G with $f_0 \in W(U_{k_j}, Q_j)$. Therefore $P := \bigcap_{j=1}^n W(U_{k_j}, Q_j)$ is a neighborhood of f_0 with respect to the compact open topology. For $f \in P$ and $x \in U_{k_j}$, we have $f_0(x) \in Q_j$ and $f(x) \in Q_j$, which implies that

$$f_0(x)^{-1}f(x) \in Q_j^{-1}Q_j \subseteq W^{-1}f_0(U_{k_j})^{-1}f_0(U_{k_j})W \subseteq W^{-1}W^{-1}f_0(k_j)^{-1}f_0(k_j)WW \subseteq W^4 \subseteq V.$$

We conclude that $f \in f_0W(K,V)$ and therefore $P \subseteq f_0W(K,V)$. This completes the proof. ■

Remark B.5. (a) If G is a fixed topological group, then $C(\cdot, G)$ is a contravariant functor from the category of Hausdorff topological spaces and continuous maps to the category of topological groups.

In fact, for each continuous map $f\colon X \to Y$, we have a group homomorphism

$$f^* = C(f,G)\colon C(Y,G) \to C(X,G), \quad \xi \mapsto \xi \circ f.$$

For each compact subset $K \subseteq X$ and each open subset $O \subseteq G$, we have

$$(f^*)^{-1}(W(K,O)) \supseteq W(f(K),O),$$

which implies the continuity of $C(f,G)$.

(b) If X is a fixed Hausdorff space and $\varphi\colon G \to H$ a morphism of topological groups, then the map

$$\varphi_* = C(X,\varphi)\colon C(X,G) \to C(X,H), \quad \xi \mapsto \varphi \circ \xi$$

is a group homomorphism. For each compact subset $K \subseteq X$ and each open subset $O \subseteq H$, we have

$$(\varphi_*)^{-1}(W(K,O)) \supseteq W(K,\varphi^{-1}(O)),$$

which implies the continuity of $C(X,\varphi)$. ∎

Proposition B.6. *Let X and Y be topological spaces. On $C(X,Y)$, the compact open topology coincides with the graph topology, i.e., the topology generated by the sets of the form*

$$C(X,Y)_{U,K} := \{f \in C(X,Y)\colon \Gamma(f|_K) \subseteq U\},$$

where $U \subseteq X \times Y$ is open, $K \subseteq X$ is compact, and $\Gamma(f) \subseteq X \times Y$ is the graph of f.

If, in addition, X is compact, then a basis for the graph topology is given by the sets

$$C(X,Y)_U := \{f \in C(X,Y)\colon \Gamma(f) \subseteq U\},$$

where $U \subseteq X \times Y$ is open.

Proof. Let $f \in C(X,Y)$, $K \subseteq X$ compact and $U \supseteq \Gamma(f|_K)$ be an open subset of $X \times Y$. Then there exists for each $x \in X$ a compact neighborhood K_x of x in K and an open neighborhood $U_{f(x)}$ of $f(x)$ in Y with $K_x \times U_{f(x)} \subseteq U$ and $f(K_x) \subseteq U_{f(x)}$. Covering K with finitely many sets K_{x_i}, $i = 1, \ldots, n$, we see that

$$\bigcap_{i=1}^{n} W(K_{x_i}, U_{f(x_i)}) \subseteq C(X,Y)_{U,K}.$$

This implies that each set $C(X,Y)_{U,K}$ is open in the compact open topology.

Conversely, let $K \subseteq X$ be compact and $O \subseteq Y$ open. Then

$$W(K,O) = \{f \in C(X,Y)\colon \Gamma(f|_K) \subseteq X \times O\} = C(X,Y)_{X \times O,K}$$

is open in the graph topology. We conclude that the graph topology coincides with the compact open topology.

Assume, in addition, that X is compact. The system of the sets $C(X,Y)_U$ is stable under intersections, hence a basis for the topology it generates. Each set $C(X,Y)_U = C(X,Y)_{U,X}$ is open in the graph topology. If, conversely, $K \subseteq X$ is compact and $U \subseteq X \times Y$ is open with $f \in C(X,Y)_{U,K}$, then $V := \big((X \setminus K) \times Y\big) \cup U$ is an open subset of $X \times Y$ with $f \in C(X,Y)_V \subseteq C(X,Y)_{U,K}$. This completes the proof. ∎

Appendix C. Lie algebra cohomology

The cohomology of Lie algebras is the natural tool to understand how we can build new Lie algebras $\widehat{\mathfrak{g}}$ from given Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$ in such a way that $\mathfrak{a} \trianglelefteq \widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{g}}/\mathfrak{a} \cong \mathfrak{g}$. An important special case of this situation arises if $\mathfrak{a}$ is assumed to be abelian. We will see in particular how the abelian extensions of Lie algebras can be parametrized by a certain cohomology space.

Cohomology with values in topological modules

Let $\mathbb{K}$ be a topological field of characteristic zero (all field operations are assumed to be continuous). A *topological Lie algebra* $\mathfrak{g}$ is a $\mathbb{K}$-Lie algebra which is a topological vector space for which the Lie bracket is a continuous bilinear map. A *topological* $\mathfrak{g}$*-module* is a $\mathfrak{g}$-module V which is a topological vector space for which the module structure, viewed as a map $\mathfrak{g} \times V \to V$, $(x,v) \mapsto x.v$ is continuous. Note that every module V of a Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ becomes a topological module if we endow $\mathbb{K}$, $\mathfrak{g}$ and V with the discrete topology. In this sense, all the following applies in particular to general modules of Lie algebra over fields of characteristic zero.

Definition C.1. Let V be a *topological module of the topological Lie algebra* $\mathfrak{g}$. For $p \in \mathbb{N}_0$, let $C^p_c(\mathfrak{g}, V)$ denote the space of continuous alternating maps $\mathfrak{g}^p \to V$, i.e., the *Lie algebra* p*-cochains with values in the module* V. We write $C^\bullet(\mathfrak{g}, V) := \bigoplus_{p\in\mathbb{N}_0} C^p_c(\mathfrak{g}, V)$. Note that $C^1_c(\mathfrak{g}, V) = \mathcal{L}(\mathfrak{g}, V)$ is the space of continuous linear maps $\mathfrak{g} \to V$. We use the convention $C^0_c(\mathfrak{g}, V) = V$. We then obtain a chain complex with the differential

$$d_{\mathfrak{g}}\colon C^p_c(\mathfrak{g}, V) \to C^{p+1}_c(\mathfrak{g}, V)$$

given on $f \in C^p_c(\mathfrak{g}, V)$ by

$$\begin{aligned}(d_{\mathfrak{g}}f)(x_0, \ldots, x_p) := &\sum_{j=0}^{p} (-1)^j x_j.f(x_0, \ldots, \widehat{x}_j, \ldots, x_p)\\ &+ \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_0, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_p),\end{aligned}$$

where $\widehat{x}_j$ indicates omission of x_j. Note that the continuity of the bracket on $\mathfrak{g}$ and the action on V imply that $d_{\mathfrak{g}}f$ is continuous and an element of $C^{p+1}_c(\mathfrak{g}, V)$.

For elements of low degree, we have in particular:

$$\begin{aligned}
p = 0:&\quad d_{\mathfrak{g}}f(x) = x.f\\
p = 1:&\quad d_{\mathfrak{g}}f(x,y) = x.f(y) - y.f(x) - f([x,y])\\
p = 2:&\quad d_{\mathfrak{g}}f(x,y,z) = x.f(y,z) - y.f(x,z) + z.f(x,y) - f([x,y],z) + f([x,z],y) - f([y,z],x)\\
&\quad = \sum_{\text{cyc.}} x.f(y,z) - f([x,y],z),
\end{aligned}$$

where we have used the notation

$$\sum_{\text{cyc.}} \gamma(x,y,z) := \gamma(x,y,z) + \gamma(y,z,x) + \gamma(z,x,y).$$

In this sense, the Jacobi identity reads $\sum_{\text{cyc.}} [[x,y],z] = 0$.

Below we shall show that $d_{\mathfrak{g}}^2 = 0$, so that the space $Z^p_c(\mathfrak{g}, V) := \ker(d_{\mathfrak{g}} \mid_{C^p_c(\mathfrak{g},V)})$ of p-*cocycles* contains the space $B^p_c(\mathfrak{g}, V) := d_{\mathfrak{g}}(C^{p-1}_c(\mathfrak{g}, V))$ of p-*coboundaries*. The quotient

$$H^p_c(\mathfrak{g}, V) := Z^p_c(\mathfrak{g}, V)/B^p_c(\mathfrak{g}, V)$$

is the p-*th continuous cohomology space of* $\mathfrak{g}$ *with values in the* $\mathfrak{g}$*-module* V. We write $[f] := f + B^p_c(\mathfrak{g}, V)$ for the cohomology class of the cocycle f. ■

On $C_c^\bullet(\mathfrak{g}, V)$, we have a natural representation of $\mathfrak{g}$, given for $x \in \mathfrak{g}$ and $f \in C_c^p(\mathfrak{g}, V)$ by the *Lie derivative*

$$\begin{aligned}(\mathcal{L}_x f)(x_1, \ldots, x_p) &= x.f(x_1, \ldots, x_p) - \sum_{j=1}^p f(x_1, \ldots, [x, x_j], \ldots, x_p) \\ &= x.f(x_1, \ldots, x_p) + \sum_{j=1}^p (-1)^j f([x, x_j], x_1, \ldots, \widehat{x}_j, \ldots, x_p).\end{aligned}$$

We further have for each $x \in \mathfrak{g}$ an insertion map

$$i_x \colon C_c^p(\mathfrak{g}, V) \to C_c^{p-1}(\mathfrak{g}, V), \quad \big(i_x.f\big)(x_1, \ldots, x_{p-1}) = f(x, x_1, \ldots, x_{p-1}),$$

where we define i_x to be 0 on $C_c^0(\mathfrak{g}, V) \cong V$.

Lemma C.2. *For $x, y \in \mathfrak{g}$, we have the following identities:*
(1) $\mathcal{L}_x = d_\mathfrak{g} \circ i_x + i_x \circ d_\mathfrak{g}$ (Cartan formula).
(2) $[\mathcal{L}_x, i_y] = i_{[x,y]}$.
(3) $[\mathcal{L}_x, d_\mathfrak{g}] = 0$.
(4) $d_\mathfrak{g}^2 = 0$.
(5) $\mathcal{L}_x(Z_c^p(\mathfrak{g}, V)) \subseteq B_c^p(\mathfrak{g}, V)$. *In particular, the natural $\mathfrak{g}$-action on $H_c^p(\mathfrak{g}, V)$ is trivial.*

Proof. (1) Using the insertion map i_{x_0}, we can rewrite the formula for the coboundary operator as

$$\begin{aligned}\big(i_{x_0}.d_\mathfrak{g} f\big)(x_1, \ldots, x_p) = x_0.f(x_1, \ldots, x_p) &- \sum_{j=1}^p (-1)^{j-1} x_j.f(x_0, \ldots, \widehat{x}_j, \ldots, x_p) \\ &+ \sum_{j=1}^p (-1)^j f([x_0, x_j], x_1, \ldots, \widehat{x}_j, \ldots, x_p) \\ &+ \sum_{1 \le i < j} (-1)^{i+j} f([x_i, x_j], x_0, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_p) \\ =& x_0.f(x_1, \ldots, x_p) - \sum_{j=1}^p (-1)^{j-1} x_j.f(x_0, \ldots, \widehat{x}_j, \ldots, x_p) \\ &- \sum_{j=1}^p f(x_1, \ldots, x_{j-1}, [x_0, x_j], x_{j+1}, \ldots, x_p) \\ &- \sum_{1 \le i < j} (-1)^{i+j} f(x_0, [x_i, x_j], \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_p) \\ =& (\mathcal{L}_{x_0} f)(x_1, \ldots, x_p) - d_\mathfrak{g}\big(i_{x_0} f\big)(x_1, \ldots, x_p).\end{aligned}$$

This proves our assertion.

(2) The explicit formula for $\mathcal{L}_x$ implies that for $y = x_1$, we have $i_y \mathcal{L}_x = \mathcal{L}_x i_y - i_{[x,y]}$.

(3),(4) Let $\varphi \colon C_c^\bullet(\mathfrak{g}, V) \to C_c^\bullet(\mathfrak{g}, V)$ be a linear map for which there exists an $\varepsilon \in \{\pm 1\}$ with $\varphi \circ i_x = \varepsilon i_x \circ \varphi$ for all $x \in \mathfrak{g}$ and a $k \in \mathbb{N}$ with $\varphi(C_c^p(\mathfrak{g}, V)) \subseteq C_c^{p+k}(\mathfrak{g}, V)$ for each $p \in \mathbb{N}_0$. We claim that $\varphi = 0$. Since the operators $i_x \colon C_c^p(\mathfrak{g}, V) \to C_c^{p-1}(\mathfrak{g}, V)$, $x \in \mathfrak{g}$, separate the points, it suffices to show that $i_x \circ \varphi = \varepsilon \varphi \circ i_x$ vanishes for each $x \in \mathfrak{g}$. On $C_c^0(\mathfrak{g}, V)$, this follows from the definition of i_x, and on $C_c^p(\mathfrak{g}, V)$, $p \in \mathbb{N}$, we obtain it by induction.

Now we prove (3). From (2) we get

$$\begin{aligned}\mathcal{L}_{[x,y]} = [\mathcal{L}_x, \mathcal{L}_y] &= [d_\mathfrak{g} \circ i_x, \mathcal{L}_y] + [i_x \circ d_\mathfrak{g}, \mathcal{L}_y] \\ &= [d_\mathfrak{g}, \mathcal{L}_y] \circ i_x + d_\mathfrak{g} \circ i_{[x,y]} + i_{[x,y]} \circ d_\mathfrak{g} + i_x \circ [d_\mathfrak{g}, \mathcal{L}_y] \\ &= [d_\mathfrak{g}, \mathcal{L}_y] \circ i_x + \mathcal{L}_{[x,y]} + i_x \circ [d_\mathfrak{g}, \mathcal{L}_y],\end{aligned}$$

so that $\varphi := [d_{\mathfrak g}, \mathcal{L}_y]$ anticommutes with the operators i_x ($\varepsilon = -1$ and $k = 1$). Therefore the argument in the preceding paragraph shows that φ vanishes, which is (3).

To obtain (4), we consider the operator $\varphi = d_{\mathfrak g}^2$. Combining (3) with the Cartan Formula, we get

$$0 = [d_{\mathfrak g}, \mathcal{L}_x] = d_{\mathfrak g}^2 \circ i_x - i_x \circ d_{\mathfrak g}^2, \tag{C.2}$$

so that the argument above applies with $\varepsilon = 1$ and $k = 2$. This proves that $d_{\mathfrak g}^2 = 0$.

(5) follows immediately from the Cartan formula (1). ■

Definition C.3. A linear subspace W of a topological vector space V is called *(topologically) split* if it is closed and there is a continuous linear map $\sigma: V/W \to V$ for which the map

$$W \times V/W \to V, \quad (w, x) \mapsto w + \sigma(x)$$

is an isomorphism of topological vector spaces. Note that the closedness of W guarantees that the quotient topology turns V/W into a Hausdorff space which is a topological vector space with respect to the induced vector space structure. A continuous linear map $f: V \to W$ between topological vector spaces is said to be *(topologically) split* if the subspaces $\ker(f) \subseteq V$ and $\mathrm{im}(f) \subseteq W$ are topologically split. ■

Remark C.4. Let $\mathfrak g$ be a Lie algebra and

$$0 \to V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3 \to 0$$

be a topologically split short exact sequence of $\mathfrak g$-modules. Identifying V_1 with $\alpha(V_1) \subseteq V_2$, we then obtain injective maps $\alpha_p: C_c^p(\mathfrak g, V_1) \to C_c^p(\mathfrak g, V_2)$ and surjective maps $\beta_p: C_c^p(\mathfrak g, V_2) \to C_c^p(\mathfrak g, V_3)$ which lead a short exact sequence

$$0 \to C_c^\bullet(\mathfrak g, V_1) \xrightarrow{\alpha_*} C_c^\bullet(\mathfrak g, V_2) \xrightarrow{\beta_*} C_c^\bullet(\mathfrak g, V_3) \to 0$$

of cochain complexes. These maps can be combined to a long exact sequence

$$0 \to H_c^0(\mathfrak g, V_1) \to H_c^0(\mathfrak g, V_2) \to H_c^0(\mathfrak g, V_3) \to H_c^1(\mathfrak g, V_1) \to H_c^1(\mathfrak g, V_2) \to H_c^1(\mathfrak g, V_3) \to \ldots,$$

where, for $p \in \mathbb{N}_0$, the connecting map

$$\delta: H_c^p(\mathfrak g, V_3) \to H^{p+1}(\mathfrak g, V_1)$$

is defined by $\delta([f]) = [d_{\mathfrak g}\widetilde f]$, where $\widetilde f \in C^p(\mathfrak g, V_2)$ satisfies $\beta \circ \widetilde f = f$, which implies that $\mathrm{im}(d_{\mathfrak g}\widetilde f) \subseteq V_1$ if f is a cocycle. ■

Affine actions of Lie algebras and 1-cocycles

Definition C.5. Let $\mathfrak g$ be a (topological) Lie algebra and $\mathfrak n$ another (topological) Lie algebra, which is a (topological) $\mathfrak g$-module on which $\mathfrak g$ acts by derivations. A linear map $f: \mathfrak g \to \mathfrak n$ is called a *crossed homomorphism* if

$$f([x, y]) = x.f(y) - y.f(x) + [f(x), f(y)]$$

holds for $x, y \in \mathfrak g$. With respect to the bracket on $C^\bullet(\mathfrak g, \mathfrak n)$, this is the Maurer Cartan equation

$$d_{\mathfrak g} f + \frac{1}{2}[f, f] = 0$$

(cf. Exercise II.14).

If $V := \mathfrak n$ is abelian, hence simply a $\mathfrak g$-module, then the crossed homomorphisms are the 1-cocycles. The elements of the subspace $B^1(\mathfrak g, V)$ (the 1-coboundaries) are called *principal crossed homomorphisms.* ■

In the following, we write $\mathfrak{aff}(V) = V \rtimes \mathfrak{gl}(V)$ for the *affine Lie algebra of* V, where $\mathfrak{gl}(V) := \mathcal{L}(V)$, endowed with the commutator bracket. A *continuous affine action* of a Lie algebra $\mathfrak g$ on V is a homomorphism $\pi: \mathfrak g \to \mathfrak{aff}(V)$ satisfying the following continuity condition: We associate to each pair $(v, A) \in \mathfrak{aff}(V)$ the affine map $x \mapsto A.x + v$ and we require the map

$$\mathfrak g \times V \to V, \quad (x, v) \mapsto \pi(x).v$$

to be continuous.

Proposition C.6. *Let (ρ, V) be a topological $\mathfrak{g}$-module. An element $f \in C^1_c(\mathfrak{g}, V)$ is in $Z^1_c(\mathfrak{g}, V)$ if and only if the map*

$$\rho_f = (f, \rho)\colon \mathfrak{g} \to \mathfrak{aff}(V) \cong V \rtimes \mathfrak{gl}(V), \quad x \mapsto \big(f(x), \rho(x)\big)$$

is a homomorphism of Lie algebras. The space $H^1_c(\mathfrak{g}, V)$ parametrizes the $e^{\mathrm{ad}\, V}$-conjugacy classes of continuous affine actions of $\mathfrak{g}$ on V whose corresponding linear action coincides with ρ.

The coboundaries correspond to those affine actions which are conjugate to a linear action, i.e., which have a fixed point. The relation $f = -d_{\mathfrak{g}} v$ is equivalent to $\rho_f(x).v = 0$ for all $x \in \mathfrak{g}$.

Proof. The first assertion is easily checked. For $v \in V$, we consider the automorphism of $\mathfrak{aff}(V)$ given by $\eta_v = e^{\mathrm{ad}\, v} = \mathbf{1} + \mathrm{ad}\, v$. Then $\eta_v(w, x) = (w - x.v, x)$, showing that $\eta_v \circ \rho_f = \rho_{f - d_{\mathfrak{g}} v}$, where $d_{\mathfrak{g}} v(x) = x.v$. Thus two affine actions ρ_f and $\rho_{f'}$ are conjugate under some η_v if and only if the cohomology classes of f and f' coincide. In this sense, $H^1_c(\mathfrak{g}, V)$ parametrizes the $e^{\mathrm{ad}\, V}$-conjugacy classes of affine actions of $\mathfrak{g}$ on V whose corresponding linear action coincides with ρ, and the coboundaries correspond to those affine actions which are conjugate to a linear action. Moreover, it is clear that an affine action ρ_f is linearizable, i.e., conjugate to a linear action, if and only if there exists a *fixed point* $v \in V$, i.e., $\rho_f(x).v = 0$ holds for all $x \in \mathfrak{g}$. This condition means that $f = -d_{\mathfrak{g}} v$. ∎

Abelian extensions and 2-cocycles

Definition C.7. Let $\mathfrak{g}$ and $\mathfrak{n}$ be topological Lie algebras. A topologically split short exact sequence

$$\mathfrak{n} \hookrightarrow \widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$$

is called a *(topologically split) extension of* $\mathfrak{g}$ *by* $\mathfrak{n}$. We identify $\mathfrak{n}$ with its image in $\widehat{\mathfrak{g}}$, and write $\widehat{\mathfrak{g}}$ as a direct sum $\widehat{\mathfrak{g}} = \mathfrak{n} \oplus \mathfrak{g}$ of topological vector spaces. Then $\mathfrak{n}$ is a topologically split ideal and the quotient map $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ corresponds to $(n, x) \mapsto x$. If $\mathfrak{n}$ is abelian, then the extension is called *abelian*.

Two extensions $\mathfrak{n} \hookrightarrow \widehat{\mathfrak{g}}_1 \twoheadrightarrow \mathfrak{g}$ and $\mathfrak{n} \hookrightarrow \widehat{\mathfrak{g}}_2 \twoheadrightarrow \mathfrak{g}$ are called *equivalent* if there exists a morphism $\varphi\colon \widehat{\mathfrak{g}}_1 \to \widehat{\mathfrak{g}}_2$ of topological Lie algebras such that the diagram

$$\begin{array}{ccccc} \mathfrak{n} & \hookrightarrow & \widehat{\mathfrak{g}}_1 & \twoheadrightarrow & \mathfrak{g} \\ \big\downarrow{\scriptstyle \mathrm{id}_{\mathfrak{n}}} & & \big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle \mathrm{id}_{\mathfrak{g}}} \\ \mathfrak{n} & \hookrightarrow & \widehat{\mathfrak{g}}_2 & \twoheadrightarrow & \mathfrak{g} \end{array}$$

commutes. It is easy to see that this implies that φ is an isomorphism of topological Lie algebras, hence defines an equivalence relation. We write $\mathrm{Ext}(\mathfrak{g}, \mathfrak{n})$ for the set of equivalence classes of extensions of $\mathfrak{g}$ by $\mathfrak{n}$.

We call an extension $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ with $\ker q = \mathfrak{n}$ *trivial*, or say that the extension *splits*, if there exists a continuous Lie algebra homomorphism $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ with $q \circ \sigma = \mathrm{id}_{\mathfrak{g}}$. In this case, the map

$$\mathfrak{n} \rtimes_S \mathfrak{g} \to \widehat{\mathfrak{g}}, \quad (n, x) \mapsto n + \sigma(x)$$

is an isomorphism, where the semi-direct sum is defined by the homomorphism

$$S\colon \mathfrak{g} \to \mathrm{der}(\mathfrak{n}), \quad S(x)(n) := [\sigma(x), n]. \qquad ∎$$

Definition C.8. Let $\mathfrak{a}$ be a topological $\mathfrak{g}$-module. To each continuous 2-cocycle $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{a})$, we associate a topological Lie algebra $\mathfrak{a} \oplus_\omega \mathfrak{g}$ as the topological product vector space $\mathfrak{a} \times \mathfrak{g}$ endowed with the Lie bracket

$$[(a, x), (a', x')] := (x.a' - x'.a + \omega(x, x'), [x, x']).$$

The quotient map $q\colon \mathfrak{a} \oplus_\omega \mathfrak{g} \to \mathfrak{g}, (a, x) \mapsto x$ is a continuous homomorphism of Lie algebras with kernel $\mathfrak{a}$, hence defines an $\mathfrak{a}$-extension of $\mathfrak{g}$. The map $\sigma\colon \mathfrak{g} \to \mathfrak{a} \oplus_\omega \mathfrak{g}, x \mapsto (0, x)$ is a continuous linear section of q. ∎

Proposition C.9. *Let* $(\mathfrak{a}, \rho_{\mathfrak{a}})$ *be a topological* $\mathfrak{g}$*-module and write* $\mathrm{Ext}_{\rho_{\mathfrak{a}}}(\mathfrak{g}, \mathfrak{a})$ *for the set of all equivalence classes of* $\mathfrak{a}$*-extensions* $\widehat{\mathfrak{g}}$ *of* $\mathfrak{g}$ *for which the adjoint action of* $\widehat{\mathfrak{g}}$ *on* $\mathfrak{a}$ *induces the given* $\mathfrak{g}$*-module structure on* $\mathfrak{a}$*. Then the map*

$$Z^2_c(\mathfrak{g}, \mathfrak{a}) \to \mathrm{Ext}_{\rho_{\mathfrak{a}}}(\mathfrak{g}, \mathfrak{a}), \quad \omega \mapsto [\mathfrak{a} \oplus_\omega \mathfrak{g}]$$

factors through a bijection

$$H^2_c(\mathfrak{g}, \mathfrak{a}) \to \mathrm{Ext}_{\rho_{\mathfrak{a}}}(\mathfrak{g}, \mathfrak{a}), \quad [\omega] \mapsto [\mathfrak{a} \oplus_\omega \mathfrak{g}].$$

Proof. Suppose that $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ is an $\mathfrak{a}$-extension of $\mathfrak{g}$ for which the induced $\mathfrak{g}$-module structure on $\mathfrak{a}$ coincides with $\rho_{\mathfrak{a}}$. Let $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ be a continuous linear section, so that $q \circ \sigma = \mathrm{id}_{\mathfrak{g}}$. Then

$$\omega(x, y) := [\sigma(x), \sigma(y)] - \sigma([x, y])$$

has values in the subspace $\mathfrak{a} = \ker q$ of $\widehat{\mathfrak{g}}$ and the map

$$\mathfrak{a} \times \mathfrak{g} \to \widehat{\mathfrak{g}}, \quad (a, x) \mapsto a + \sigma(x)$$

defines an isomorphism of topological Lie algebras $\mathfrak{a} \oplus_\omega \mathfrak{g} \to \widehat{\mathfrak{g}}$.

It is easy to verify that $\mathfrak{a} \oplus_\omega \mathfrak{g} \sim \mathfrak{a} \oplus_\eta \mathfrak{g}$ if and only if $\omega - \eta \in B^2_c(\mathfrak{g}, \mathfrak{a})$. Therefore the quotient space $H^2_c(\mathfrak{g}, \mathfrak{a})$ classifies the equivalence classes of $\mathfrak{a}$-extensions of $\mathfrak{g}$ by the assignment $[\omega] \mapsto [\mathfrak{a} \oplus_\omega \mathfrak{g}]$. ■

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Karl-Hermann Neeb
Technische Universität Darmstadt
Schlossgartenstrasse 7
D-64289 Darmstadt
Deutschland
neeb@mathematik.tu-darmstadt.de