Resolvent Estimates and Maximal Regularity in Weighted L^q-spaces of the Stokes Operator in an Infinite Cylinder

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Abstract

Let $\Omega = \Sigma \times \mathbb{R}$ be an infinite cylinder of $\mathbb{R}^n, n \geq 3$, with a bounded cross-section $\Sigma \subset \mathbb{R}^{n-1}$ of $C^{1,1}$-class. We study resolvent estimates and maximal regularity of the Stokes operator in $L^q(\mathbb{R}; L^r_\omega(\Sigma))$ for $1 < q, r < \infty$ and for arbitrary Muckenhoupt weights $\omega \in A_r$ with respect to $x' \in \Sigma$. The proofs use an operator-valued Fourier multiplier theorem and techniques of unconditional Schauder decompositions based on the $\mathcal{R}$-boundedness of the family of solution operators for a system in Σ parametrized by the phase variable of the one-dimensional partial Fourier transform.

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1 Introduction

In this paper we show that the Stokes operator in the space $L^q(\Omega), 1 < q < \infty$, on an infinite cylinder $\Omega = \Sigma \times \mathbb{R}$ of $\mathbb{R}^n, n \geq 3$, generates a bounded and exponentially decaying analytic semigroup and has maximal L^p-regularity. We show these properties to hold even in $L^q(\mathbb{R}; L^r_\omega(\Sigma))$ for $1 < q, r < \infty$ and for arbitrary Muckenhoupt weight $\omega \in A_r(\mathbb{R}^{n-1})$ with respect to $x' \in \Sigma$ (see Section 2 for the definition). We note that the resolvent estimate gives, when $\lambda = 0$, a new result on the existence of a unique flow with zero flux for the Stokes system in $L^q(\mathbb{R}, L^r_\omega(\Sigma))$.

The proofs in this paper are mainly based on the theory of Fourier analysis. By the application of the partial Fourier transform along the axis of the cylinder Ω the

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generalized Stokes resolvent system

$$
(R_\lambda) \qquad
\begin{array}{rcll}
\lambda u - \Delta u + \nabla p &=& f & \text{in } \Omega\\
\operatorname{div} u &=& g & \text{in } \Omega\\
u &=& 0 & \text{on } \partial\Omega
\end{array}
$$

is reduced to the *parametrized Stokes system* in the cross-section Σ

$$
(R_{\lambda,\xi}) \qquad
\begin{array}{rcll}
(\lambda+\xi^2-\Delta')\hat{u}' + \nabla'\hat{p} &=& \hat{f}' & \text{in } \Sigma\\
(\lambda+\xi^2-\Delta')\hat{u}_n + i\xi\hat{p} &=& \hat{f}_n & \text{in } \Sigma\\
\operatorname{div}'\hat{u}' + i\xi\hat{u}_n &=& \hat{g} & \text{in } \Sigma\\
\hat{u}'=0, \quad \hat{u}_n &=& 0 & \text{on } \partial\Sigma
\end{array}
$$

which involves the Fourier phase variable $\xi \in \mathbb{R}$ as parameter. We will get parameter-independent estimates of solutions to $(R_{\lambda,\xi}), \xi \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$, in L^r-spaces with Muckenhoupt weights, which yield R-boundedness of the family of solution operators $a(\xi)$ for $(R_{\lambda,\xi})$ with $g = 0$ due to an extrapolation property of operators defined on L^r-spaces with Muckenhoupt weights, see Theorem 5.8. Then the solution u to (R_λ) with $g = 0$ in the whole cylinder Ω is represented by $u = \mathcal{F}^{-1}(a(\xi)\mathcal{F}f)$, and an operator-valued Fourier multiplier theorem ([31]) implies the resolvent estimate. In order to prove maximal regularity we use that maximal regularity of an operator A in a *UMD* space X is implied by the $\mathcal{R}$-boundedness of the operator family

$$\{\lambda(\lambda+A)^{-1}: \ \lambda \in i\,\mathbb{R}\} \tag{1.1}$$

in $\mathcal{L}(X)$, see [31]. We show the $\mathcal{R}$-boundedness of (1.1) for the Stokes operator $A := A_{q,r;\omega}$ in $L^q(\mathbb{R} : L^r_\omega(\Sigma))$ by virtue of Schauder decomposition techniques; to be more precise, we use the Schauder decomposition $\{\Delta_j\}_{j\in\mathbb{Z}}$ where $\Delta_j = \mathcal{F}^{-1}\chi_{[2^j,2^{j+1})}\mathcal{F}$ and again the R-boundedness of the family of solution operators for $(R_{\lambda,\xi})$.

To obtain parameter-independent estimates of the solution to $(R_{\lambda,\xi}), \xi \in \mathbb{R}^*$, we start with the case $\Sigma = \mathbb{R}^{n-1}$ using Fourier multiplier theory in spaces with Muckenhoupt weights (Theorem 3.1). Next, for $(R_{\lambda,\xi})$ on the half space $\Sigma = \mathbb{R}^{n-1}_+$ (Theorem 3.4), we first consider an estimate for $\hat{p}$; for this a result on Fourier multipliers in trace spaces of Sobolev spaces with Muckenhoupt weights is crucial, see Lemma 3.2. Then the estimate for $\hat{u}$ is obtained using the Laplace resolvent equation. The result for the case of bent half spaces $\Sigma = H_\sigma$ (Theorem 3.5; see (3.2) for the definition of H_σ) is obtained by Kato's perturbation argument. For bounded domains Σ, using cut-off functions and the results for the whole, half and bent half spaces, we start with a preliminary *a priori* estimate in weighted spaces for $(R_{\lambda,\xi})$ (Lemma 4.2) and are finally led to weighted estimates of the solution to $(R_{\lambda,\xi})$ by a contradiction argument (Lemma 4.3).

There are many papers dealing with resolvent estimates ([6], [7], [13], [14], [18]; see Introduction of [9] for more details) or maximal regularity (see e.g. [1], [12], [14]) of Stokes operators for domains with compact boundaries as well as for domains with noncompact boundaries. General unbounded domains are considered in [5] by replacing the space L^q by $L^q \cap L^2$ or $L^q + L^2$. In [9], [10] the system (R_λ) was studied in $L^q(\mathbb{R}; L^2(\Sigma)), 1 < q < \infty$, and, when $g = 0$, in vector-valued homogeneous Besov space $\dot{\mathcal{B}}^s_{pq}(\mathbb{R}; L^r(\Sigma))$ for $1 \le p, q \le \infty,\ s \in \mathbb{R},\ 1 < r < \infty$. For partial results in the Bloch space of uniformly square integrable functions on a cylinder we refer to [28]. Further results on stationary and instationary Stokes and Navier-Stokes systems in unbounded cylindrical domains can be found in [2], [15], [16], [19]-[26], [28]-[30].

This paper is organized as follows. In Section 2 the main results of this paper (Theorem 2.1, Corollary 2.2 and Theorem 2.3) and preliminaries are given. In Section 3 we obtain the estimates for $(R_{\lambda,\xi})$ on the whole, half and bent half spaces. Section 4 is devoted to obtain the estimate for $(R_{\lambda,\xi})$ on bounded domains, see Theorem 4.4. In Section 5 proofs of the main results are given.

2 Main Results and Preliminaries

Let $\Omega = \Sigma \times \mathbb{R}$ be an infinite cylinder of $\mathbb{R}^n$ with bounded cross section $\Sigma \subset \mathbb{R}^{n-1}$ and with generic point $x \in \Omega$ written in the form $x = (x', x_n) \in \Omega$, where $x' \in \Sigma$ and $x_n \in \mathbb{R}$. Similarly, differential operators in $\mathbb{R}^n$ are split, in particular, $\Delta = \Delta' + \partial_n^2$ and $\nabla = (\nabla', \partial_n)$.

For $q \in (1, \infty)$ we use the standard notation $L^q(\Omega) = L^q(\mathbb{R}; L^q(\Sigma))$ for classical Lebesgue spaces with norm $\|\cdot\|_q = \|\cdot\|_{q;\Omega}$ and $W^{k,q}(\Omega), k \in \mathbb{N}$, for the usual Sobolev spaces with norm $\|\cdot\|_{k,q;\Omega}$. We do not distinguish between spaces of scalar functions and vector-valued functions as long as no confusion arises. In particular, we use the short notation $\|u, v\|_r$ for $\|u\|_r + \|v\|_r$, even if u and v are tensors of different order.

Let $1 < r < \infty$. A function $0 \le \omega \in L^1_{\rm loc}(\mathbb{R}^{n-1})$ is called A_r-*weight* (*Muckenhoupt weight*) on $\mathbb{R}^{n-1}$ iff

$$\mathcal{A}_r(\omega) := \sup_Q \left(\frac{1}{|Q|}\int_Q \omega\, dx'\right) \cdot \left(\frac{1}{|Q|}\int_Q \omega^{-1/(r-1)}\, dx'\right)^{r-1} < \infty$$

where the supremum is taken over all cubes of $\mathbb{R}^{n-1}$ and $|Q|$ denotes the $(n-1)$-dimensional Lebesgue measure of Q. We call $\mathcal{A}_r(\omega)$ the A_r-constant of ω and denote the set of all A_r-weights on $\mathbb{R}^{n-1}$ by $A_r = A_r(\mathbb{R}^{n-1})$. Note that

$$\omega \in A_r \quad \text{iff} \quad \omega' := \omega^{-1/(r-1)} \in A_{r'}, \quad r' = r/(r-1)$$

and $\mathcal{A}_{r'}(\omega') = \mathcal{A}_r(\omega)^{r'/r}$. A constant $C = C(\omega)$ is called A_r-*consistent* if for every $d > 0$

$$\sup\{C(\omega) :\ \omega \in A_r,\ \mathcal{A}_r(\omega) < d\} < \infty.$$

We write $\omega(Q)$ for $\int_Q \omega\, dx'$.

Given $\omega \in A_r, r \in (1, \infty)$, and an arbitrary domain $\Sigma \subset \mathbb{R}^{n-1}$ let

$$L^r_\omega(\Sigma) = \Big\{u \in L^1_{\rm loc}(\bar{\Sigma}) : \|u\|_{r,\omega} = \|u\|_{r,\omega;\Sigma} = \Big(\int_\Sigma |u|^r \omega\, dx'\Big)^{1/r} < \infty\Big\}.$$

For short we will write L^r_ω for $L^r_\omega(\Sigma)$ provided that the underlying domain Σ is known from the context. It is well-known that L^r_ω is a separable reflexive Banach space with dense subspace $C_0^\infty(\Sigma)$. In particular $(L^r_\omega)^* = L^{r'}_{\omega'}$. As usual, $W^{k,r}_\omega(\Sigma)$, $k \in \mathbb{N}$, denotes the weighted Sobolev space with norm

$$\|u\|_{k,r,\omega} = \Big(\sum_{|\alpha|\le k} \|D^\alpha u\|^r_{r,\omega}\Big)^{1/r},$$

where $|\alpha| = \alpha_1+\cdots+\alpha_{n-1}$ is the length of the multi-index $\alpha = (\alpha_1,\ldots,\alpha_{n-1}) \in \mathbb{N}_0^{n-1}$ and $D^\alpha = \partial_1^{\alpha_1}\cdot\ldots\cdot\partial_{n-1}^{\alpha_{n-1}}$; moreover, $W^{k,r}_{0,\omega}(\Sigma) := \overline{C_0^\infty(\Sigma)}^{\|\cdot\|_{k,r,\omega}}$ and $W^{-k,r}_{0,\omega}(\Sigma) := (W^{k,r'}_{0,\omega'}(\Sigma))^*$, where $r' = r/(r-1)$. We introduce the weighted homogeneous Sobolev space

$$\widehat{W}^{1,r}_\omega(\Sigma) = \big\{u \in L^1_{\mathrm{loc}}(\bar\Sigma)/\mathbb{R} :\ \nabla' u \in L^r_\omega(\Sigma)\big\}$$

with norm $\|\nabla' u\|_{r,\omega}$ and its dual space $\widehat{W}^{-1,r'}_{\omega'} := (\widehat{W}^{1,r}_\omega)^*$ with norm $\|\cdot\|_{-1,r',\omega'} = \|\cdot\|_{-1,r',\omega';\Sigma}$.

Let $q,r \in (1,\infty)$. On an infinite cylinder $\Omega = \Sigma\times\mathbb{R}$, where Σ is a bounded $C^{1,1}$-domain of $\mathbb{R}^{n-1}$, we introduce the function space $L^q(L^r_\omega) := L^q(\mathbb{R}; L^r_\omega(\Sigma))$ with norm

$$\|u\|_{L^q(L^r_\omega)} = \bigg(\int_{\mathbb{R}} \Big(\int_\Sigma |u(x',x_n)|^r \omega(x')\,dx'\Big)^{q/r} dx_n\bigg)^{1/q}.$$

Furthermore, $W^{k;q,r}_\omega(\Omega), k \in \mathbb{N}$, denotes the Banach space of all functions in Ω whose derivatives of order up to k belong to $L^q(L^r_\omega)$ with norm $\|u\|_{W^{k;q,r}_\omega} = (\sum_{|\alpha|\le k}\|D^\alpha u\|^2_{L^q(L^r_\omega)})^{1/2}$, where $\alpha \in \mathbb{N}_0^n$, and let $W^{1;q,r}_{0,\omega}(\Omega)$ be the completion of the set $C_0^\infty(\Omega)$ in $W^{1;q,r}_\omega(\Omega)$. The weighted homogeneous Sobolev space $\widehat{W}^{1;q,r}_\omega(\Omega)$ is defined by

$$\widehat{W}^{1;q,r}_\omega(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega)/\mathbb{R} : \nabla u \in L^q(L^r_\omega)\}$$

with norm $\|\nabla u\|_{L^q(L^r_\omega)}$. Finally, $L^q(L^r_\omega)_\sigma$ is the completion in the space $L^q(L^r_\omega)$ of the set

$$C^\infty_{0,\sigma}(\Omega) = \{u \in C_0^\infty(\Omega)^n; \quad \operatorname{div} u = 0\}.$$

The Fourier transform in the variable x_n is denoted by $\mathcal{F}$ or $\hat{\ }$ and the inverse Fourier transform by $\mathcal{F}^{-1}$ or $^\vee$. For $\varepsilon \in (0,\frac{\pi}{2})$ we define the complex sector

$$S_\varepsilon = \{\lambda \in \mathbb{C}; \lambda \ne 0, |\mathrm{arg}\lambda| < \frac{\pi}{2}+\varepsilon\}.$$

The first main theorem of this paper is as follows.

Theorem 2.1 (Weighted Resolvent Estimates) *Let Σ be a bounded domain of $C^{1,1}$-class with $\alpha_0 > 0$ being the least eigenvalue of the Dirichlet Laplacian in Σ, and let $0<\varepsilon<\frac{\pi}{2}$, $1<q,r<\infty$ and $\omega \in A_r$. Then for every $f \in L^q(\mathbb{R}; L^r_\omega(\Sigma))$, every $\alpha \in (0,\alpha_0)$ and $\lambda \in -\alpha + S_\varepsilon$ there exists a unique solution*

$$(u,p) \in \big(W^{2;q,r}_\omega(\Omega)\cap W^{1;q,r}_{0,\omega}(\Omega)\big)\times \widehat{W}^{1;q,r}_\omega(\Omega)$$

to (R_λ) (with $g=0$) satisfying the estimate

$$\|(\lambda+\alpha)u, \nabla^2 u, \nabla p\|_{L^q(L^r_\omega)} \le C\|f\|_{L^q(L^r_\omega)} \tag{2.1}$$

with an A_r-consistent constant $C = C(q,r,\alpha,\varepsilon,\Sigma,\mathcal{A}_r(\omega))$ independent of λ.

In particular we obtain from Theorem 2.1 the following corollary on resolvent estimates of the Stokes operator in the cylinder Ω.

Corollary 2.2 (Stokes Operator and Stokes Semigroup) *Let $1 < q, r < \infty$, $\omega \in A_r(\mathbb{R}^{n-1})$ and define the Stokes operator $A = A_{q,r;\omega}$ on Ω by*

$$D(A) = W^{2;q,r}_{\omega}(\Omega) \cap W^{1;q,r}_{0,\omega}(\Omega) \cap L^q(L^r_\omega)_\sigma \subset L^q(L^r_\omega)_\sigma,\ Au = -P_{q,r;\omega}\Delta u, \tag{2.2}$$

where $P_{q,r;\omega}$ is the Helmholtz projection in $L^q(\mathbb{R}; L^r_\omega(\Sigma))$ (see [8]). Then, for every $\varepsilon \in (0, \frac{\pi}{2})$ and $\alpha \in (0, \alpha_0)$, $-\alpha + S_\varepsilon$ is contained in the resolvent set of $-A$, and the estimate

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(L^q(L^r_\omega)_\sigma)} \le \frac{C}{|\lambda + \alpha|} \quad \forall \lambda \in -\alpha + S_\varepsilon \tag{2.3}$$

holds with an A_r-consistent constant $C = C(\Sigma, q, r, \alpha, \varepsilon, \mathcal{A}_r(\omega))$.

As a consequence, the Stokes operator generates a bounded analytic semigroup $\{e^{-tA_{q,r;\omega}}; t \ge 0\}$ on $L^q(L^r_\omega)_\sigma$ satisfying the estimate

$$\|e^{-tA_{q,r;\omega}}\|_{\mathcal{L}(L^q(L^r_\omega)_\sigma)} \le C\, e^{-\alpha t} \quad \forall \alpha \in (0, \alpha_0), \forall t > 0 \tag{2.4}$$

with a constant $C = C(q, r, \alpha, \varepsilon, \Sigma, \mathcal{A}_r(\omega))$.

The second important result of this paper is the *maximal regularity* of the Stokes operator in an infinite straight cylinder.

Theorem 2.3 (Maximal Regularity) *Let $1 < p, q, r < \infty$ and $\omega \in A_r(\mathbb{R}^{n-1})$. Then the Stokes operator $A = A_{q,r;\omega}$ has maximal regularity in $L^q(L^r_\omega)_\sigma$. To be more precise, for each $f \in L^p(\mathbb{R}_+; L^q(L^r_\omega)_\sigma)$ the instationary system*

$$u_t + Au = f, \quad u(0) = 0 \tag{2.5}$$

has a unique solution $u \in W^{1,p}(\mathbb{R}_+; L^q(L^r_\omega)_\sigma) \cap L^p(\mathbb{R}_+; D(A))$ such that

$$\|u, u_t, Au\|_{L^p(\mathbb{R}_+; L^q(L^r_\omega)_\sigma)} \le C\|f\|_{L^p(\mathbb{R}_+; L^q(L^r_\omega)_\sigma)}. \tag{2.6}$$

Analogously, for every $f \in L^p(\mathbb{R}_+; L^q(L^r_\omega))$, the instationary system

$$u_t - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0, \quad u(0) = 0$$

has a unique solution $(u, \nabla p) \in \big(W^{1,p}(\mathbb{R}_+; L^q(L^r_\omega)_\sigma) \cap L^p(\mathbb{R}_+; D(A))\big) \times L^p(\mathbb{R}_+; L^q(L^r_\omega))$ satisfying the a priori estimate

$$\|u_t, u, \nabla u, \nabla^2 u, \nabla p\|_{L^p(\mathbb{R}_+; L^q(L^r_\omega))} \le C\|f\|_{L^p(\mathbb{R}_+; L^q(L^r_\omega))}. \tag{2.7}$$

Moreover, if $e^{\alpha t} f \in L^p(\mathbb{R}_+; L^q(L^r_\omega)_\sigma)$ for some $\alpha \in (0, \alpha_0)$, then the solution u satisfies the estimate

$$\|e^{\alpha t} u, e^{\alpha t} u_t, e^{\alpha t} Au\|_{L^p(\mathbb{R}_+; L^q(L^r_\omega)_\sigma)} \le C\|e^{\alpha t} f\|_{L^p(\mathbb{R}_+; L^q(L^r_\omega)_\sigma)}. \tag{2.8}$$

In each estimate $C = C(\Sigma, q, r, \mathcal{A}_r(\omega))$ and $C = C(\Sigma, q, r, \mathcal{A}_r(\omega), \alpha)$, respectively.

Remark 2.4 (1) We note that in (2.5) we may take nonzero initial values $u(0) = u_0$ in the interpolation space $(L^q(L^r_\omega)_\sigma, D(A_{q,r;\omega}))_{1-1/p,p}$.

(2) By [1], Theorem 1.3, maximal regularity in $L^q(\Omega)$ of $cI + A_q$ with some $c > 0$, where A_q is the Stokes operator in $L^q(\Omega)$, will follow; this result is weaker than the particular case $q = r$ and $\omega \equiv 1$ in Theorem 2.3.

For the proofs in Section 3 and Section 4, we need some preliminary results for Muckenhoupt weights.

Proposition 2.5 ([8], Lemma 2.4) *Let $1 < r < \infty$ and $\omega \in A_r(\mathbb{R}^{n-1})$.*

(1) Let $T : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be a bijective, bi-Lipschitz vector field. Then also $\omega \circ T \in A_r(\mathbb{R}^{n-1})$ and $\mathcal{A}_r(\omega \circ T) \le c\,\mathcal{A}_r(\omega)$ with a constant $c = c(T, r) > 0$ independent of ω.

(2) Define the weight $\tilde\omega(x') = \omega(|x_1|, x'')$ for $x' = (x_1, x'') \in \mathbb{R}^{n-1}$. Then $\tilde\omega \in A_r$ and $\mathcal{A}_r(\tilde\omega) \le 2^r\,\mathcal{A}_r(\omega)$.

(3) Let $\Sigma \subset \mathbb{R}^{n-1}$ be a bounded domain. Then there exist $\tilde s, s \in (1, \infty)$ satisfying

$$L^{\tilde s}(\Sigma) \hookrightarrow L^r_\omega(\Sigma) \hookrightarrow L^s(\Sigma).$$

Here $\tilde s$ and $\frac{1}{s}$ are A_r-consistent. Moreover, the embedding constants can be chosen uniformly on a set $W \subset A_r$ provided that

$$\sup_{\omega \in W} \mathcal{A}_r(\omega) < \infty, \qquad \int_Q \omega\, dx' = 1 \quad \text{for all } \omega \in W, \tag{2.9}$$

for a cube $Q \subset \mathbb{R}^{n-1}$ with $\bar\Sigma \subset Q$.

Proposition 2.6 ([8], Proposition 2.5) *Let $\Sigma \subset \mathbb{R}^{n-1}$ be a bounded Lipschitz domain and let $1 < r < \infty$.*

(1) For every $\omega \in A_r$ the continuous embedding $W^{1,r}_\omega(\Sigma) \hookrightarrow L^r_\omega(\Sigma)$ is compact.

(2) Consider a sequence of weights $(\omega_j) \subset A_r$ satisfying (2.9) for $W = \{\omega_j : j \in \mathbb{N}\}$ and a fixed cube $Q \subset \mathbb{R}^{n-1}$ with $\bar\Sigma \subset Q$. Further let (u_j) be a sequence of functions on Σ satisfying

$$\sup_j \|u_j\|_{1,r,\omega_j} < \infty \quad \text{and} \quad u_j \rightharpoonup 0 \quad \text{in } W^{1,s}(\Sigma)$$

for $j \to \infty$ where s is given by Proposition 2.5 (3). Then

$$\|u_j\|_{r,\omega_j} \to 0 \quad \text{for } j \to \infty.$$

(3) Under the same assumptions on $(\omega_j) \subset A_r$ as in (2) consider a sequence of functions (v_j) on Σ satisfying

$$\sup_j \|v_j\|_{r,\omega_j} < \infty \quad \text{and} \quad v_j \rightharpoonup 0 \quad \text{in } L^s(\Sigma)$$

for $j \to \infty$. Then considering v_j as functionals on $W^{1,r'}_{\omega'_j}(\Sigma)$

$$\|v_j\|_{(W^{1,r'}_{\omega'_j}(\Sigma))^*} \to 0 \quad \text{for } j \to \infty.$$

Proposition 2.7 *Let $r \in (1,\infty)$, $\omega \in A_r$ and $\Sigma \subset \mathbb{R}^{n-1}$ be a bounded Lipschitz domain. Then there exists an A_r-consistent constant $c = c(r,\Sigma,\mathcal{A}_r(\omega)) > 0$ such that*

$$\|u\|_{r,\omega} \le c\|\nabla' u\|_{r,\omega}$$

for all $u \in W^{1,r}_\omega(\Sigma)$ with vanishing integral mean $\int_\Sigma u\,dx' = 0$.

Proof: See the proof of [14], Corollary 2.1 and its conclusions; checking the proof, one sees that the constant $c = c(r,\Sigma,\mathcal{A}_r(\omega))$ is A_r-consistent. ■

Finally we cite the Fourier multiplier theorem in weighted spaces.

Theorem 2.8 ([17], Ch. IV, Theorem 3.9) *Let $m \in C^k(\mathbb{R}^k \setminus \{0\}), k \in \mathbb{N}$, admit a constant $M \in \mathbb{R}$ such that*

$$|\eta|^\gamma |D^\gamma m(\eta)| \le M \quad \textit{for all} \quad \eta \in \mathbb{R}^k \setminus \{0\}$$

and multi-indices $\gamma \in \mathbb{N}_0^k$ with $|\gamma| \le k$. Then for all $1 < r < \infty$ and $\omega \in A_r(\mathbb{R}^k)$ the multiplier operator $Tf = \mathcal{F}^{-1}m(\cdot)\mathcal{F}$ defined for all rapidly decreasing functions $f \in \mathcal{S}(\mathbb{R}^k)$ can be uniquely extended to a bounded linear operator from $L^r_\omega(\mathbb{R}^k)$ to $L^r_\omega(\mathbb{R}^k)$. Moreover, there exists an A_r-consistent constant $C = C(r,\mathcal{A}_r(\omega))$ such that

$$\|Tf\|_{r,\omega} \le CM\|f\|_{r,\omega}\,, \quad f \in L^r_\omega(\mathbb{R}^k)\,.$$

3 The Problem $(R_{\lambda,\xi})$ in Half Spaces

Consider the parametrized resolvent problem $(R_{\lambda,\xi})$ for all $\xi \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $\lambda \in S_\varepsilon, 0 < \varepsilon < \frac{\pi}{2}$. In this section Σ denotes either $\mathbb{R}^{n-1}$ or the half space

$$\Sigma = \mathbb{R}^{n-1}_+ = \{x' = (x_1, x'') :\ x'' \in \mathbb{R}^{n-2}, x_1 > 0\}, \tag{3.1}$$

or a bent half space

$$H_\sigma = \{x' = (x_1, x'') :\ x_1 > \sigma(x''), x'' \in \mathbb{R}^{n-2}\}, \tag{3.2}$$

where σ is a $C^{1,1}$-function. For notational convenience we omit the symbol $\hat{}$ for the one-dimensional Fourier transform; thus

$$u = (u', u_n), p, f, g \quad \text{stand for} \quad \hat{u} = (\hat{u'}, \hat{u_n}), \hat{p}, \hat{f}, \hat{g}.$$

Let $\omega \in A_r(\mathbb{R}^{n-1})$ be an arbitrary Muckenhoupt weight. For the divergence g $(\hat{=}\hat{g})$, by the same argument as in Section 2 of [9], we may define, for $r \in (1,\infty)$ and $\xi \in \mathbb{R}^*$, the spaces

$$\widehat{W}^{1,r}_\omega(\Sigma) \cap L^r_{\omega,\xi}(\Sigma) \cong W^{1,r}_\omega(\Sigma) \quad \text{with norm} \quad \max\{\|\nabla' u, \xi u\|_{r,\omega}\}$$

and

$$\widehat{W}^{-1,r}_\omega + L^r_{\omega,1/\xi} := (\hat{W}^{1,r'}_{\omega'} \cap L^{r'}_{\omega',\xi})^* \cong (W^{1,r'}_{\omega'})^*, \quad r' = r/(r-1),$$

with $\xi-$dependent norm

$$\|h;\widehat{W}_{\omega}^{-1,r}+L^r_{\omega,1/\xi}\| = \inf\{\|h_0\|_{-1,r,\omega}+\|h_1/\xi\|_{r,\omega} : \, h=h_0+h_1, h_0\in \hat{W}_{\omega}^{-1,r}, h_1\in L^r_{\omega}\}.$$

Assume that

$$f\in L^r_\omega(\Sigma), \quad g\in W^{1,r}_\omega(\Sigma).$$

Note that $W^{1,r}_\omega(\Sigma)$ is obviously contained in the sum $\widehat{W}_{\omega}^{-1,r}(\Sigma)+L^r_{\omega,1/\xi}(\Sigma)$.

Now we start with the case $\Sigma=\mathbb{R}^{n-1}$. Since $C_0^\infty(\mathbb{R}^{n-1})$ is dense in $\widehat{W}^{1,r'}_{\omega'}(\mathbb{R}^{n-1})$, if $g=g_0+g_1$, $g_0\in\widehat{W}_{\omega}^{-1,r}$ and $g_1\in L^r_{\omega,1/\xi}$, is any splitting of g, Hahn-Banach's theorem implies the existence of a vector field $h\in L^r_\omega$ such that

$$g_0=\operatorname{div}' h, \quad \|g_0\|_{-1,r,\omega}=\|h\|_{r,\omega}.$$

An elementary calculation shows that p in $(R_{\lambda,\xi})$ satisfies the equation

$$(\xi^2-\Delta')p=(\lambda+\xi^2-\Delta')g-(\operatorname{div}' f'+i\xi f_n). \tag{3.3}$$

Introducing the $(n-1)$-dimensional Fourier transform $\tilde{}$ with respect to x' and with phase variable $s\in\mathbb{R}^{n-1}$ we get

$$\begin{aligned}\tilde{p} &= \tilde{g}+\frac{\lambda}{\xi^2+|s|^2}\tilde{g}-\frac{is}{\xi^2+|s|^2}\cdot\tilde{f}'-\frac{i\xi}{\xi^2+|s|^2}\tilde{f}_n\\ &= \tilde{g}+\frac{\lambda is}{\xi^2+|s|^2}\cdot\tilde{h}+\frac{\lambda\xi}{\xi^2+|s|^2}(\tilde{g}_1/\xi)-\frac{is}{\xi^2+|s|^2}\cdot\tilde{f}'-\frac{i\xi}{\xi^2+|s|^2}\tilde{f}_n.\end{aligned}$$

Obviously the functions

$$m_\xi(s)=\frac{s_js_k}{\xi^2+|s|^2}, \quad \frac{s_j\xi}{\xi^2+|s|^2}, \quad \frac{\xi^2}{\xi^2+|s|^2}, \quad 1\le j,k\le n-1,$$

are classical multiplier functions satisfying the pointwise Hörmander-Michlin condition

$$|s|^\alpha|\nabla_s^\alpha m_\xi(s)|\le c_\alpha, \quad 0\ne s\in\mathbb{R}^{n-1},\, \alpha\in\mathbb{N}_0^{n-1}, |\alpha|\le n-1, \tag{3.4}$$

uniformly with respect to $\xi\in\mathbb{R}^*$. Then Theorem 2.7 applied to $\nabla' p$ and to ξp yields the estimate

$$\begin{aligned}\|\nabla' p,\xi p\|_{r,\omega} &\le c(\|f,\nabla' g,\xi g\|_{r,\omega}+\|\lambda h,\lambda g_1/\xi\|_{r,\omega})\\ &\le c(\|f,\nabla' g,\xi g\|_{r,\omega}+\|\lambda g_0\|_{-1,r,\omega}+\|\lambda g_1/\xi\|_{r,\omega}).\end{aligned} \tag{3.5}$$

Next consider the Laplace resolvent equations for u' and u_n, i.e.,

$$\begin{aligned}(\lambda+\xi^2-\Delta')u' &= F' \quad \text{in} \quad \mathbb{R}^{n-1},\\ (\lambda+\xi^2-\Delta')u_n &= F_n \quad \text{in} \quad \mathbb{R}^{n-1}\end{aligned} \tag{3.6}$$

with resolvent parameters $\lambda+\xi^2$, where $F':=f'-\nabla' p, F_n:=f_n-i\xi p$ and p is the solution to (3.3) satisfying (3.5). Again applying the $(n-1)$-dimensional Fourier transform with respect to $x'\in\mathbb{R}^{n-1}$ to (3.6), we get

$$\tilde{u}'=\frac{\tilde{F}'}{\lambda+\xi^2+|s|^2}, \quad \tilde{u}_n=\frac{\tilde{F}_n}{\lambda+\xi^2+|s|^2}.$$

Therefore, using the fact that

$$\frac{\lambda+\xi^2}{\lambda+\xi^2+|s|^2}, \quad \frac{\sqrt{\lambda+\xi^2}s_j}{\lambda+\xi^2+|s|^2}, \quad \frac{s_j s_k}{\lambda+\xi^2+|s|^2}, \quad j,k=1,\dots,n-1,$$

are Fourier multipliers satisfying (3.4), we get the existence of a solution $u=(u',u_n)$ to (3.6) satisfying

$$\begin{aligned} \|(\lambda+\xi^2)u, \sqrt{\lambda+\xi^2}\nabla' u, \nabla'^2 u\|_{r,\omega} &\le c\|f, \nabla' p, \xi p\|_{r,\omega} \\ &\le c(\|f,\nabla' g, \xi g\|_{r,\omega} + \|\lambda g_0\|_{-1,r,\omega} + \|\lambda g_1/\xi\|_{r,\omega}) \end{aligned} \tag{3.7}$$

with A_r-consistent constants $c=c(\varepsilon, r, \mathcal{A}_r(\omega))$.

Let $\mu = |\lambda+\xi^2|^{1/2}$. We can prove the following theorem.

Theorem 3.1 *Let $\Sigma=\mathbb{R}^{n-1}$, $1<r<\infty$ and $\omega \in A_r(\mathbb{R}^{n-1})$. If $f\in L^r_\omega(\Sigma)$ and $g \in W^{1,r}_\omega(\Sigma)$, then for every $\lambda \in S_\varepsilon, 0<\varepsilon<\frac{\pi}{2}$, and $\xi \in \mathbb{R}^*$ the problem $(R_{\lambda,\xi})$ has a unique solution $(u,p)\in W^{2,r}_\omega(\Sigma)\times W^{1,r}_\omega(\Sigma)$ satisfying*

$$\|\mu^2 u, \mu\nabla' u, \nabla'^2 u, \nabla' p, \xi p\|_{r,\omega} \le c\big(\|f, \nabla' g, \xi g\|_{r,\omega} + \|\lambda g; \widehat{W}^{-1,r}_\omega + L^r_{\omega,1/\xi}\|\big) \tag{3.8}$$

with an A_r-consistent constant $c=c(\varepsilon, r, \mathcal{A}_r(\omega))$.

Proof: Let u be a solution to (3.6) where p is a solution to (3.3). We have already seen that $(u,p)\in W^{2,r}_\omega(\Sigma)\times W^{1,r}_\omega(\Sigma)$ satisfies the estimate (3.8) since $g=g_0+g_1$ in the estimate (3.5), (3.7) is an arbitrary splitting of $g\in \widehat{W}^{-1,r}_\omega + L^r_{\omega,1/\xi}$. Therefore, for the proof of the existence of a solution, it is enough to show that (u,p) solves the divergence equation of $(R_{\lambda,\xi})$. A simple calculation with (3.3) and (3.6) yields

$$(\lambda+\xi^2-\Delta')(\operatorname{div}' u' + i\xi u_n - g) = 0 \quad \text{in} \quad \mathbb{R}^{n-1}.$$

Hence standard arguments from Fourier analysis show that $\operatorname{div}' u' + i\xi u_n = g$. The uniqueness of the solution is obvious from the above Fourier multiplier technique, i.e., if (u,p) is a solution to $(R_{\lambda,\xi})$ with $f=0, g=0$, then u satisfies (3.6) with $f=0$ and $(\xi^2-\Delta')p=0$ yielding $p=0$, and hence $u=0$. ■

In the next main step we consider the case $\Sigma=\mathbb{R}^{n-1}_+$, see (3.1). Just as for $x'=(x_1,x'')$ we write $u'=(u_1,u'')$, $f'=(f_1,f'')$. For a function $h:\Sigma\to\mathbb{R}$ define the even extension h_e by

$$h_e(x_1,x'') = \begin{cases} h(x_1,x'') & \text{for} \quad x_1>0 \\ h(-x_1,x'') & \text{for} \quad x_1<0, \end{cases}$$

while the odd extension h_o of h is defined by

$$h_o(x_1,x'') = -h(-x_1,x'') \quad \text{for } x_1<0.$$

Given $(R_{\lambda,\xi})$ in (Σ), take the even extension f''_e of f'', f_{ne} of f_n and g_e of g, but the odd extension f_{1o} of f_1. Then obviously

$$(f_{1o}, f''_e, f_{ne}) \in L^r_{\tilde\omega}(\mathbb{R}^{n-1}), \quad g_e \in W^{1,r}_{\tilde\omega}(\mathbb{R}^{n-1}),$$

where $\tilde\omega(x_1, x'') = \omega(|x_1|, x'')$. Note that $\mathcal{A}_r(\tilde\omega) \le 2^r\,\mathcal{A}_r(\omega)$, see Proposition 2.5 (2). It is clear that

$$\|h_o, h_e\|_{r,\tilde\omega;\mathbb{R}^{n-1}} \le c(r)\,\|h\|_{r,\omega;\Sigma}; \tag{3.9}$$

moreover, for a function $h \in L^r_\omega(\mathbb{R}^{n-1}_+) \cap \widehat{W}^{-1,r}_\omega(\mathbb{R}^{n-1}_+)$ we get

$$\begin{aligned} \|h_e\|_{\widehat{W}^{-1,r}_{\tilde\omega}(\mathbb{R}^{n-1})} &= \sup_\varphi \Big| \int_{\mathbb{R}^{n-1}} h_e\,\varphi\,dx' \Big| \\ &= \sup_\varphi \Big| \int_\Sigma h\,\varphi\,dx' + \int_\Sigma h\,\varphi(-x_1,x'')\,dx' \Big| \\ &\le 2\|h\|_{\widehat{W}^{-1,r}_\omega(\Sigma)}, \end{aligned} \tag{3.10}$$

where the supremum is taken over all $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$ with $\|\nabla'\varphi\|_{r',\omega';\mathbb{R}^{n-1}} \le 1$.

Now we will solve $(R_{\lambda,\xi})$ in the whole space $\mathbb{R}^{n-1}$ with right-hand side $(f_{1o}, f''_e, f_{ne}), g_e$. By the uniqueness assertion it is easily seen that the solution (U,P) of this extended problem is even with respect to x_1 except for the component U_1 which is odd with respect to x_1. In particular $U_1 = 0$ for $x_1 = 0$ and, due to (3.8),

$$\begin{aligned} &\|\mu^2 U, \mu\nabla' U, \nabla'^2 U, \nabla' P, \xi P\|_{r,\omega;\Sigma} \\ &\le c\big(\|f_{1o}, f''_e, f_{ne}, \nabla' g_e, \xi g_e\|_{r,\tilde\omega;\mathbb{R}^{n-1}} + \|\lambda g_e; \widehat{W}^{-1,r}_{\tilde\omega}(\mathbb{R}^{n-1}) + L^r_{\tilde\omega,1/\xi}(\mathbb{R}^{n-1})\|\big) \end{aligned} \tag{3.11}$$

where $\mu = |\lambda + \xi^2|^{1/2}$ and the constant c is A_r-consistent due to Proposition 2.5. Thus, from (3.9)–(3.11), we get

$$\begin{aligned} &\|\mu^2 U, \mu\nabla' U, \nabla'^2 U, \nabla' P, \xi P\|_{r,\omega;\Sigma} \\ &\qquad \le c\big(\|f, \nabla' g, \xi g\|_{r,\omega;\Sigma} + \|\lambda g; \widehat{W}^{-1,r}_\omega + L^r_{\omega,1/\xi}\|\big) \end{aligned} \tag{3.12}$$

with an A_r-consistent constant $c = c(\varepsilon, r, \mathcal{A}_r(\omega))$.

Subtracting (U,P) in $(R_{\lambda,\xi})$, the parametrized resolvent problem $(R_{\lambda,\xi})$ is reduced to the homogeneous system

$$\begin{aligned} (\lambda + \xi^2 - \Delta')u' + \nabla' p &= 0 \quad \text{in} \quad \Sigma = \mathbb{R}^{n-1}_+ \\ (\lambda + \xi^2 - \Delta')u_n + i\xi p &= 0 \quad \text{in} \quad \Sigma \\ \operatorname{div}' u' + i\xi u_n &= 0 \quad \text{in} \quad \Sigma \end{aligned} \tag{3.13}$$

with inhomogeneous boundary values

$$u = \Phi := U|_{\partial\Sigma} \quad \text{on} \quad \partial\Sigma. \tag{3.14}$$

With the splittings $\Delta' = \partial_1^2 + \Delta''$, $\operatorname{div}' u' = \partial_1 u_1 + \operatorname{div}'' u''$ and $\nabla' = (\partial_1, \nabla'')$ elementary operations with (3.13), (3.14) yield the fourth order equation

$$\begin{aligned} (\lambda + \xi^2 - \Delta')(\xi^2 - \Delta')u_1 &= 0 && \text{in} \quad \Sigma \\ u_1 &= 0 && \text{on} \quad \partial\Sigma \\ \partial_1 u_1 &= -\operatorname{div}''\Phi'' - i\xi\Phi_n && \text{on} \quad \partial\Sigma. \end{aligned} \tag{3.15}$$

Let us introduce the additional partial Fourier transform $\mathcal{F}_\sigma = \tilde{}$ with respect to the variable $x'' \in \mathbb{R}^{n-2}$ and with phase variable $\sigma \in \mathbb{R}^{n-2}$. Applying $\tilde{}$ to (3.15), we get the fourth order ordinary differential equation ($s = |\sigma|$)

$$\begin{aligned}(\lambda + \xi^2 + s^2 - \partial_1^2)(\xi^2 + s^2 - \partial_1^2)\tilde{u}_1 &= 0 && \text{for} \quad x_1 > 0\\ \tilde{u}_1 &= 0 && \text{at} \quad x_1 = 0\\ \partial_1 \tilde{u}_1 &= -i\sigma \cdot \tilde{\Phi}'' - i\xi\tilde{\Phi}_n && \text{at} \quad x_1 = 0.\end{aligned} \tag{3.16}$$

For fixed $\lambda \in S_\varepsilon, \xi \in \mathbb{R}^*$ and $\sigma \in \mathbb{R}^{n-2}$ (3.16) has a unique bounded solution $\tilde{u}_1$ in $(0, \infty)$, namely

$$\tilde{u}_1(x_1, \sigma, \xi) = \frac{e^{-\sqrt{\lambda+\xi^2+s^2}x_1} - e^{-\sqrt{\xi^2+s^2}x_1}}{\sqrt{\lambda + \xi^2 + s^2} - \sqrt{\xi^2 + s^2}}\,(i\sigma \cdot \tilde{\Phi}'' + i\xi\tilde{\Phi}_n)|_{\partial\Sigma}. \tag{3.17}$$

Furthermore (3.13), (3.17) yield after some elementary calculations

$$\begin{aligned}p(x', \xi) &= -\mathcal{F}_\sigma^{-1}\big(\tfrac{1}{\xi^2+s^2}(\lambda + \xi^2 + s^2 - \partial_1^2)\partial_1\tilde{u}_1\big)\\ &= -\mathcal{F}_\sigma^{-1}\Big(\frac{\sqrt{\lambda+\xi^2+s^2}+\sqrt{\xi^2+s^2}}{\sqrt{\xi^2+s^2}}e^{-\sqrt{\xi^2+s^2}x_1}(i\sigma \cdot \tilde{\Phi}'' + i\xi\tilde{\Phi}_n)\Big)\\ &= \mathcal{F}_\sigma^{-1}\big((1 + \frac{\sqrt{\lambda+\xi^2+s^2}}{\sqrt{\xi^2+s^2}})\tilde{v}\big),\end{aligned} \tag{3.18}$$

where

$$v = \mathcal{F}_\sigma^{-1}\big(- e^{-\sqrt{\xi^2+s^2}x_1}(i\sigma \cdot \tilde{\Phi}'' + i\xi\tilde{\Phi}_n)\big). \tag{3.19}$$

For every nonzero complex number μ and $k = 1, 2$ let $W^{k,r}_{\omega,\mu}(\mathbb{R}^{n-1})$ denote the weighted Sobolev space $W^{k,r}_\omega(\mathbb{R}^{n-1})$ endowed with the norm

$$\|u\|_{W^{k,r}_{\omega,\mu}(\mathbb{R}^{n-1})} = \|\nabla'^k u, \mu u\|_{r,\omega;\mathbb{R}^{n-1}}, \quad k = 1, 2.$$

Similarly we define the space $W^{k,r}_{\omega,\mu}(\mathbb{R}^{n-1}_+), k = 1, 2$, on the half space $\mathbb{R}^{n-1}_+$. Using the trace operator γ, well-defined for functions from $W^{k,r}_{\text{loc}}(\mathbb{R}^{n-1}_+)$, we may define the trace space $T^{k,r}_{\omega,\mu}(\mathbb{R}^{n-2}), k = 1, 2$, by

$$T^{k,r}_{\omega,\mu}(\mathbb{R}^{n-2}) := \gamma W^{k,r}_{\omega,\mu}(\mathbb{R}^{n-1}_+), \quad \|\phi\|_{T^{k,r}_{\omega,\mu}(\mathbb{R}^{n-2})} = \inf_{\gamma u = \phi} \|u\|_{W^{k,r}_{\omega,\mu}(\mathbb{R}^{n-1}_+)}.$$

Obviously the set $C_0^\infty(\mathbb{R}^{n-1})$ is dense in the Banach space $T^{k,r}_{\omega,\mu}(\mathbb{R}^{n-2}), k = 1, 2$. We note that for $\phi \in T^{2,r}_{\omega,\mu}(\mathbb{R}^{n-2})$ and $\mu \in S_\varepsilon$ the function $R_\mu\phi := \mathcal{F}_\sigma^{-1}(e^{-\sqrt{\mu+s^2}x_1}\tilde{\phi}) \in W^{2,r}_\omega(\mathbb{R}^{n-1}_+)$ is the unique solution to the Laplace resolvent equation

$$(\mu - \Delta')q = 0 \quad \text{in } \mathbb{R}^{n-1}_+, \quad q|_{\mathbb{R}^{n-2}} = \phi \tag{3.20}$$

(see [13], Theorem 4.5). Furthermore, by standard techniques using Fourier multiplier theory one can easily see that $R_\mu\phi$ satisfies the estimates

$$\|R_\mu\phi\|_{W^{2,r}_{\omega,\mu}(\mathbb{R}^{n-1}_+)} \le c(r, \varepsilon, \mathcal{A}_r(\omega))\|\phi\|_{T^{2,r}_{\omega,\mu}(\mathbb{R}^{n-2})}, \tag{3.21}$$

$$\|R_\mu\phi\|_{W^{1,r}_{\omega,\sqrt{\mu}}(\mathbb{R}^{n-1}_+)} \le c(r, \varepsilon, \mathcal{A}_r(\omega))\|\phi\|_{T^{1,r}_{\omega,\sqrt{\mu}}(\mathbb{R}^{n-2})}. \tag{3.22}$$

Lemma 3.2 *Let $m \in C^{n-2}(\mathbb{R}^{n-2}\setminus\{0\})$. If $m(\sigma)$ as well as $\frac{\sqrt{\xi^2+s^2}}{s} m(\sigma)$, $\xi \in \mathbb{R}^*$, are $(n-2)$-dimensional multiplier functions satisfying the pointwise Hörmander-Michlin condition, see Theorem 2.8, with a constant $K > 0$ independent of $\xi \in \mathbb{R}^*$, then the operator $M : \mathcal{S}(\mathbb{R}^{n-2}) \to \mathcal{S}'(\mathbb{R}^{n-2})$ defined by*

$$M\phi = \mathcal{F}_\sigma^{-1}(m(\sigma)\tilde{\phi})$$

is a bounded operator in $\mathcal{L}(T^{1,r}_{\omega,\xi}(\mathbb{R}^{n-2}))$ with $\|M\|_{\mathcal{L}(T^{1,r}_{\omega,\xi}(\mathbb{R}^{n-2}))} \le c(r,\varepsilon,\mathcal{A}_r(\omega))K$.

Proof: Let $\phi \in \mathcal{S}(\mathbb{R}^{n-2})$, let τ be the Fourier phase variable for the partial Fourier transform with respect to x_1, and let $\eta = (\tau,\sigma)$. Note that $\mathcal{F}_{x_1}\big(e^{-\sqrt{\xi^2+s^2}|x_1|}\big) = \frac{2\sqrt{\xi^2+s^2}}{\xi^2+s^2+\tau^2}$ and $\mathcal{F}_\tau^{-1}\big(\frac{\sqrt{\xi^2+s^2}+s}{s}\frac{s^2}{s^2+\tau^2}\mathcal{F}_{x_1}e^{-\sqrt{\xi^2+s^2}|x_1|}\big)\big|_{x_1=0} = 1$. Hence, by the definition of the space $T^{1,r}_{\omega,\xi}(\mathbb{R}^{n-2})$, we get

$$\begin{aligned}
&\|M\phi\|_{T^{1,r}_{\omega,\xi}(\mathbb{R}^{n-2})}\\
&\qquad\le \Big\|\mathcal{F}_\sigma^{-1}\big(m(\sigma)\mathcal{F}_\tau^{-1}\big(\tfrac{\sqrt{\xi^2+s^2}+s}{s}\tfrac{s^2}{s^2+\tau^2}\mathcal{F}_{x_1}e^{-\sqrt{\xi^2+s^2}|x_1|}\big)\widetilde{\phi}\big)\Big\|_{W^{1,r}_{\omega,\xi}(\mathbb{R}^{n-1}_+)} \qquad (3.23)\\
&\qquad\le \Big\|\mathcal{F}_\eta^{-1}\Big(m(\sigma)\big(\tfrac{\sqrt{\xi^2+s^2}+s}{s}\tfrac{s^2}{s^2+\tau^2}\mathcal{F}_{x_1}e^{-\sqrt{\xi^2+s^2}|x_1|}\big)\widetilde{\phi}\Big)\Big\|_{W^{1,r}_{\omega,\xi}(\mathbb{R}^{n-1})}.
\end{aligned}$$

Since $m(\sigma)\frac{\sqrt{\xi^2+s^2}+s}{s}\frac{s^2}{s^2+\tau^2}$ is easily seen to be an $(n-1)$-dimensional Fourier multiplier by the assumptions on m, we get from (3.23), (3.22) that

$$\begin{aligned}
\|M\phi\|_{T^{1,r}_{\omega,\xi}(\mathbb{R}^{n-2})} &\le c(\mathcal{A}_r(\omega))K\|\mathcal{F}_\sigma^{-1}(e^{-\sqrt{\xi^2+s^2}|x_1|}\tilde{\phi})\|_{W^{1,r}_{\omega,\xi}(\mathbb{R}^{n-1})}\\
&\le c(\mathcal{A}_r(\omega))K\|\mathcal{F}_\sigma^{-1}(e^{-\sqrt{\xi^2+s^2}x_1}\tilde{\phi})\|_{W^{1,r}_{\omega,\xi}(\mathbb{R}^{n-1}_+)}\\
&\le c(r,\varepsilon,\mathcal{A}_r(\omega))K\|\phi\|_{T^{1,r}_{\omega,\xi}(\mathbb{R}^{n-2})}.
\end{aligned}$$

The proof of the lemma is complete. ■

Lemma 3.3 *The function p defined by (3.18) satisfies the estimate*

$$\|\nabla' p, \xi p\|_{r,\omega;\Sigma} \le c\big(\|f,\nabla' g,\xi g\|_{r,\omega;\Sigma} + \|\lambda g; \widehat{W}_\omega^{-1,r}(\Sigma) + L^r_{\omega,1/\xi}(\Sigma)\|\big)$$

with an A_r-consistent constant $c = c(r,\varepsilon,\mathcal{A}_r(\omega))$.

Proof: First we shall show for the function v in (3.19) the estimate

$$\|\nabla' v, \xi v\|_{r,\omega;\Sigma} \le c\big(\|f,\nabla' g,\xi g\|_{r,\omega;\Sigma} + \|\lambda g; \widehat{W}_\omega^{-1,r}(\Sigma) + L^r_{\omega,1/\xi}(\Sigma)\|\big), \qquad (3.24)$$

with an A_r-consistent constant $c = c(r,\varepsilon,\mathcal{A}_r(\omega))$. Since v solves the equation $(\xi^2-\Delta')v = 0$ in $\mathbb{R}^{n-1}_+$ with boundary condition $v|_{\partial\Sigma} = -\mathrm{div}''\Phi'' - i\xi\Phi_n$, standard techniques (see [13], Theorem 4.4) and a scaling argument yield a constant $c = c(r,\mathcal{A}_r(\omega)) > 0$ independent of $\xi \in \mathbb{R}^*$ such that

$$\|\nabla' v, \xi v\|_{r,\omega;\Sigma} \le c\|\nabla'(\mathrm{div}''U'' + i\xi U_n), \xi(\mathrm{div}''U'' + i\xi U_n)\|_{r,\omega;\Sigma}.$$

Hence (3.12) yields (3.24).

Now let $\mu = \lambda + \xi^2$. We shall show the auxiliary estimate

$$\begin{aligned} &\|\mathcal{F}_\sigma^{-1}\big(\sqrt{\mu+s^2}e^{-\sqrt{\xi^2+s^2}x_1}(\sigma\cdot\tilde{\Phi}''+\xi\tilde{\Phi}_n)\big)\|_{r,\omega;\Sigma} \\ &\qquad \le c(r,\varepsilon,\mathcal{A}_r(\omega))\big(\|f,\nabla' g,\xi g\|_{r,\omega;\Sigma}+\|\lambda g;\widehat{W}_\omega^{-1,r}(\Sigma)+L^r_{\omega,1/\xi}(\Sigma)\|\big). \end{aligned} \tag{3.25}$$

By (3.22) we get

$$\begin{aligned} &\big\|\mathcal{F}_\sigma^{-1}\big(\sqrt{\mu+s^2}e^{-\sqrt{\xi^2+s^2}x_1}(\sigma\cdot\tilde{\Phi}''+\xi\tilde{\Phi}_n)\big)\big\|_{r,\omega;\Sigma} \\ &\quad = \big\|\partial_1\mathcal{F}_\sigma^{-1}\big(e^{-\sqrt{\xi^2+s^2}x_1}\sqrt{\mu+s^2}\big(\tfrac{\sigma}{\sqrt{\xi^2+s^2}}\cdot\tilde{\Phi}''+\tfrac{\xi}{\sqrt{\xi^2+s^2}}\tilde{\Phi}_n\big)\big)\big\|_{r,\omega;\Sigma} \\ &\quad \le c\big\|\mathcal{F}_\sigma^{-1}\big(\sqrt{\mu+s^2}\big(\tfrac{\sigma}{\sqrt{\xi^2+s^2}}\cdot\tilde{\Phi}''+\tfrac{\xi}{\sqrt{\xi^2+s^2}}\tilde{\Phi}_n\big)\big)\big\|_{T^{1,r}_{\omega,\xi}} \end{aligned} \tag{3.26}$$

where $c = c(r,\varepsilon,\mathcal{A}_r(\omega)) > 0$. Note that $\frac{\sigma_k}{\sqrt{\xi^2+s^2}}$, $k = 2,\dots,n-1$, and $1-\frac{\xi}{\sqrt{\xi^2+s^2}}$ satisfy the assumption of Lemma 3.2 with a constant $K > 0$ independent of $\xi \in \mathbb{R}^*$. Hence Lemma 3.2 and the fact that $\|\varphi\|_{T^{1,r}_{\omega,\xi}} \le c(\varepsilon)\|\varphi\|_{T^{1,r}_{\omega,\sqrt{\mu}}}$ for $\varphi \in T^{1,r}_{\omega,\xi}(\mathbb{R}^{n-2}_+)$ yield

$$\begin{aligned} &\big\|\mathcal{F}_\sigma^{-1}\big(\sqrt{\mu+s^2}e^{-\sqrt{\xi^2+s^2}x_1}(\sigma\cdot\tilde{\Phi}''+\xi\tilde{\Phi}_n)\big)\big\|_{r,\omega;\Sigma} \\ &\quad \le c\big\|\mathcal{F}_\sigma^{-1}\big(\big(\tfrac{\sigma}{\sqrt{\xi^2+s^2}}\cdot\sqrt{\mu+s^2}\,\tilde{\Phi}''+\big(1-\tfrac{\xi}{\sqrt{\xi^2+s^2}}\big)\sqrt{\mu+s^2}\,\tilde{\Phi}_n\big)\big)\big\|_{T^{1,r}_{\omega,\xi}} \\ &\qquad +\|\mathcal{F}_\sigma^{-1}(\sqrt{\mu+s^2}\,\tilde{\Phi}_n)\|_{T^{1,r}_{\omega,\xi}} \\ &\quad \le c\|\mathcal{F}_\sigma^{-1}(\sqrt{\mu+s^2}\,\tilde{\Phi})\|_{T^{1,r}_{\omega,\xi}} \qquad \le c\|\mathcal{F}_\sigma^{-1}(\sqrt{\mu+s^2}\,\tilde{\Phi})\|_{T^{1,r}_{\omega,\sqrt{\mu}}} \\ &\quad \le c\|\mathcal{F}_\sigma^{-1}(\sqrt{\mu+s^2}e^{-\sqrt{\mu+s^2}x_1}\,\tilde{\Phi})\|_{W^{1,r}_{\omega,\sqrt{\mu}}} \;=\; c\|\partial_1 R_\mu\Phi\|_{W^{1,r}_{\omega,\sqrt{\mu}}} \end{aligned} \tag{3.27}$$

where $c = c(r,\varepsilon,\mathcal{A}_r(\omega)) > 0$. Then, by interpolation and (3.21), we get

$$\|\partial_1 R_\mu\Phi\|_{W^{1,r}_{\omega,\sqrt{\mu}}} \le c\|R_\mu\Phi\|_{W^{2,r}_{\omega,\mu}} \le c\|\Phi\|_{T^{2,r}_{\omega,\mu}} \le c\|\mu U,\nabla'^2 U\|_{r,\omega;\Sigma}$$

where $c = c(r,\varepsilon,\mathcal{A}_r(\omega)) > 0$. Hence, from (3.12), (3.27) we get (3.25).

To complete the proof, we must obtain an estimate for $h := \mathcal{F}_\sigma^{-1}\big(\frac{\sqrt{\mu+s^2}}{\sqrt{\xi^2+s^2}}\tilde{v}\big)$; see (3.18), (3.19). Note that $\partial_1 h$ is just the left-hand side of (3.25). Moreover, $\nabla'' h, \xi h$ are represented by the left-hand side of (3.25) with Φ replaced by $\mathcal{F}_\sigma^{-1}\big(\frac{\sigma\tilde{\Phi}}{\sqrt{\xi^2+s^2}}\big), \mathcal{F}_\sigma^{-1}\big(\frac{\xi\tilde{\Phi}}{\sqrt{\xi^2+s^2}}\big)$, respectively. Therefore, using that $\frac{\sigma_j\sigma_k}{\xi^2+s^2}, \frac{\sigma_k\xi}{\xi^2+s^2}, j,k = 2,\dots,n-1$, and $1-\frac{\xi^2}{\xi^2+s^2}$ satisfy the assumptions of Lemma 3.2, we get by the same technique as before that

$$\|\nabla'' h,\xi h\|_{r,\omega;\Sigma} \le c\big(\|f,\nabla' g,\xi g\|_{r,\omega;\Sigma}+\|\lambda g;\widehat{W}_\omega^{-1,r}(\Sigma)+L^r_{\omega,1/\xi}(\Sigma)\|\big)$$

with an A_r-consistent constant $c = c(r,\varepsilon,\mathcal{A}_r(\omega))$.

The proof of the lemma is complete. ■

Now we can prove the following theorem.

Theorem 3.4 *With $\Sigma = \mathbb{R}^{n-1}_+$ the assertions of Theorem 3.1 remain true. In particular the a priori estimate* (3.8) *holds.*

Proof: Consider the system

$$\begin{aligned} (\mu - \Delta')u' &= -\nabla' p && \text{in } \Sigma \\ (\mu - \Delta')u_n &= -i\xi p && \text{in } \Sigma \\ u &= U && \text{on } \partial\Sigma \end{aligned} \tag{3.28}$$

for (u', u_n) where p is defined by (3.18). By standard techniques, cf. [13], §4.2, and a scaling argument we get that (3.28) has a unique solution $u := (u', u_n) \in W^{2,r}_\omega(\Sigma) \cap W^{1,r}_{0,\omega}(\Sigma)$ satisfying

$$\|\mu u, \sqrt{\mu}\nabla' u, \nabla'^2 u\|_{r,\omega;\Sigma} \le c\|\nabla' p, \xi p, \mu U, \nabla'^2 U\|_{r,\omega;\Sigma}$$

with an A_r-consistent constant $c = c(r, \mathcal{A}_r(\omega))$. Thus, by Lemma 3.3 it follows that the functions u, p satisfy (3.8) with $\Sigma = \mathbb{R}^{n-1}_+$.

Now, for the proof of existence, it remains to show that u satisfies the divergence equation. From the expression for p one can infer that

$$(-\Delta' + \xi^2)p = 0. \tag{3.29}$$

Hence, from (3.28) we get

$$(\mu - \Delta')(\operatorname{div}' u' + i\xi u_n) = 0 \quad \text{in} \quad \Sigma.$$

Furthermore (3.28), (3.29) imply (3.17), (3.18) with $(i\sigma \cdot \tilde{U}'' + i\xi\tilde{U}_n)|_{\partial\Sigma}$ replaced by $(-\partial_1\tilde{u}_1)|_{\partial\Sigma}$. Therefore we have $(i\sigma\cdot\tilde{U}''+i\xi\tilde{U}_n)|_{\partial\Sigma} = (-\partial_1\tilde{u}_1)|_{\partial\Sigma}$, i.e., $\operatorname{div}'u'+i\xi u_n = 0$ on $\partial\Sigma$. Thus $\operatorname{div}'u' + i\xi u_n = 0$ in Σ.

For the proof of uniqueness let $(u, p) \in \big(W^{2,r}_\omega(\mathbb{R}^{n-1}_+) \cap W^{1,r}_{0,\omega}(\mathbb{R}^{n-1}_+)\big) \times W^{1,r}_\omega(\mathbb{R}^{n-1}_+)$ be a solution to $(R_{\lambda,\xi})$ with right-hand side 0. Then Proposition 2.5 (3) yields $(u, p) \in \big(W^{2,s}(\mathbb{R}^{n-1}_+) \cap W^{1,s}_0(\mathbb{R}^{n-1}_+)\big) \times W^{1,s}(\mathbb{R}^{n-1}_+)$ with some $s \in (1, r)$. Therefore, from the uniqueness result for $(R_{\lambda,\xi})$ in spaces without weight we get $(u, p) = 0$, see [9], Theorem 2.2.

Now the proof of this theorem is complete. ■

The third main step of this section concerns $(R_{\lambda,\xi})$ in a bent half space $\Sigma = H_\sigma$, see (3.2). Note that as before u, p etc. stand for the Fourier transforms $\hat{u}, \hat{p}$ etc.

Theorem 3.5 *Let $n \ge 3$, $1 < r < \infty$, $\omega \in A_r(\mathbb{R}^{n-1})$, $0 < \varepsilon < \pi/2$ and*

$$\Sigma = H_\sigma = \{x' = (x_1, x''); \, x_1 > \sigma(x''), x'' \in \mathbb{R}^{n-2}\}$$

for a given function $\sigma \in C^{1,1}(\mathbb{R}^{n-2})$. Then there are A_r-consistent constants $K_0 = K_0(r, \varepsilon, \mathcal{A}_r(\omega)) > 0$ and $\lambda_0 = \lambda_0(r, \varepsilon, \mathcal{A}_r(\omega)) > 0$ such that, provided $\|\nabla'\sigma\|_\infty \le K_0$, for every $\lambda \in S_\varepsilon, |\lambda| \ge \lambda_0$, every $\xi \in \mathbb{R}^*$ and

$$f \in L^r_\omega(\Sigma), \quad g \in W^{1,r}_\omega(\Sigma), \tag{3.30}$$

the parametrized resolvent problem $(R_{\lambda,\xi})$ *has a unique solution* $(u,p) \in (W^{2,r}_\omega(\Sigma) \cap W^{1,r}_{0,\omega}(\Sigma)) \times W^{1,r}_\omega(\Sigma)$*. This solution satisfies the estimate* $(\mu = |\lambda + \xi^2|^{1/2})$

$$\begin{aligned} &\|\mu^2 u, \mu\nabla' u, \nabla'^2 u, \nabla' p, \xi p\|_{r,\omega} \\ &\qquad \le c\big(\|f, \nabla' g, \xi g\|_{r,\omega} + \|\lambda g; \widehat{W}^{-1,r}_\omega(\Sigma) + L^r_{\omega,1/\xi}(\Sigma)\|\big) \end{aligned} \tag{3.31}$$

with an A_r*-consistent constant* $c = c(r, \varepsilon, \mathcal{A}_r(\omega))$*. If* (3.30) *is satisfied for an additional exponent* $s \in (1,\infty)$ *and weight* $\nu \in A_r(\mathbb{R}^{n-1})$ *and if* $\|\nabla'\sigma\|_\infty \le K_0$ *for some constant* $K_0 = K_0(r, s, \varepsilon, \mathcal{A}_r(\omega), \mathcal{A}_s(\nu)) > 0$*, then the assertion* (3.31) *holds true with* L^s_ν*-norms for all* $\lambda \in S_\varepsilon, |\lambda| \ge \lambda_0$*, for some* $\lambda_0 = \lambda_0(r, s, \varepsilon, \mathcal{A}_r(\omega), \mathcal{A}_s(\nu)) > 0$ *as well.*

Proof: By the transformation

$$\Phi : H_\sigma \to \mathbb{R}^{n-1}_+, \quad x' \mapsto \tilde{x}' = (\tilde{x}_1, \tilde{x}'') = \Phi(x') = (x_1 - \sigma(x''), x''),$$

the problem $(R_{\lambda,\xi})$ in H_σ is reduced to a modified version of $(R_{\lambda,\xi})$ in the half space $H = \mathbb{R}^{n-1}_+$. Note that Φ is a bijection with Jacobian equal to 1. For a function u on H_σ define $\tilde{u}$ on H by

$$\tilde{u}(\tilde{x}') = u(\Phi^{-1}(\tilde{x}') = u(x').$$

Further let $\tilde{\partial}_i = \partial/\partial\tilde{x}_i, i = 1, \cdots, n-1, \tilde{\nabla}' = (\tilde{\partial}_1, \tilde{\nabla}'')$ etc. denote the standard differential operators acting on the variable $\tilde{x} \in H$.

Since $\partial_i u = (\tilde{\partial}_i - (\partial_i\sigma)\tilde{\partial}_1)\tilde{u}$ for $i = 1, \cdots, n-1$, we easily get

$$\begin{aligned} \Delta' u(x',\xi) &= \big(\tilde{\Delta}' + |\nabla'\sigma|^2\tilde{\partial}_1^2 - 2\nabla'\sigma\cdot(\tilde{\nabla}'\tilde{\partial}_1) - (\Delta''\sigma)\tilde{\partial}_1\big)\tilde{u}(\tilde{x}',\xi) \\ \nabla' p(x',\xi) &= \big(\tilde{\nabla}' - (\nabla'\sigma)\tilde{\partial}_1\big)\tilde{p}(\tilde{x}',\xi) \\ \operatorname{div}' u'(x',\xi) &= \big(\widetilde{\operatorname{div}'} - \nabla'\sigma\cdot\tilde{\partial}_1\big)\tilde{u}'(\tilde{x}',\xi) \end{aligned} \tag{3.32}$$

and a similar formula for $\nabla'^2 u(x',\xi)$. Note that by the change of variable $\tilde{x}' = \Phi(x'), x' \in \mathbb{R}^{n-1}$, the Muckenhoupt weight $\omega \in A_r(\mathbb{R}^{n-1})$ is mapped to $\tilde{\omega} \in A_r(\mathbb{R}^{n-1})$ satisfying

$$c^{-1}\mathcal{A}_r(\tilde{\omega}) \le \mathcal{A}_r(\omega) \le c\,\mathcal{A}_r(\tilde{\omega}) \tag{3.33}$$

with c independent of ω, cf. Proposition 2.5 (1). Therefore, it follows from (3.32) that for $u \in W^{2,r}(\Sigma)$

$$\begin{aligned} \|u\|_{r,\omega;H_\sigma} &= \|\tilde{u}\|_{r,\tilde{\omega};H} \\ \|\nabla' u\|_{r,\omega;H_\sigma} &\le c(1+K)\|\tilde{\nabla}'\tilde{u}\|_{r,\tilde{\omega};H} \\ \|\nabla'^2 u\|_{r,\omega;H_\sigma} &\le c(1+K^2)\|\tilde{\nabla}'^2\tilde{u}\|_{r,\tilde{\omega};H} + cL\|\tilde{\partial}_1\tilde{u}\|_{r,\tilde{\omega};H}, \end{aligned} \tag{3.34}$$

where $K = \|\nabla'\sigma\|_\infty, L = \|\nabla'^2\sigma\|_\infty$ and c is independent of the weight ω. Furthermore, $\|f, \xi g\|_{r,\omega;H_\sigma} = \|\tilde{f}, \xi\tilde{g}\|_{r,\tilde{\omega};H}$ and $\|\nabla' g\|_{r,\omega;H_\sigma} \le c(1+K)\|\tilde{\nabla}'\tilde{g}\|_{r,\tilde{\omega};H}$ with $c > 0$ independent of ω. Concerning the norm of g in $\widehat{W}^{-1,r}_\omega(H_\sigma) + L^r_{\omega,1/\xi}(H_\sigma)$ note that for a function $g_0 \in \widehat{W}^{-1,r}_\omega(H_\sigma) \cap L^r_\omega(H_\sigma)$ and all test functions $\varphi \in C_0^\infty(\bar{H}_\sigma)$

$$\begin{aligned} \int_{H_\sigma} g_0\varphi\,dx' &= \int_H \tilde{g}_0\tilde{\varphi}\,d\tilde{x}' \\ &\le \|\tilde{g}_0\|_{-1,r,\tilde{\omega};H}\|\tilde{\nabla}'\tilde{\varphi}\|_{r',(\tilde{\omega})';H} \\ &\le c(1+\|\nabla'\sigma\|_\infty)\|\tilde{g}_0\|_{-1,r,\tilde{\omega};H}\|\nabla'\varphi\|_{r',\omega';H_\sigma} \end{aligned}$$

with a constant c independent of ω; here we used that $(\tilde\omega)' = \widetilde{(\omega')}$, $\omega' = \omega^{-\frac{1}{r-1}}$. Since $C_0^\infty(\bar H_\sigma)$ is dense in $\widehat{W}^{1,r'}_{\tilde\omega'}(H_\sigma)$ (see e.g. [13], Corollary 4.1), we get

$$\|g_0\|_{-1,r,\omega;H_\sigma} \le c(1+K)\|\tilde g_0\|_{-1,r,\tilde\omega;H}.$$

Then for every $\xi \in \mathbb{R}^*$ and every decomposition of g into $g = g_0 + g_1$ with $g_0 \in \widehat{W}^{-1,r}_\omega(H_\sigma), g_1 \in L^r_\omega(H_\sigma)$

$$\|g_0\|_{-1,r,\omega;H_\sigma} + \|g_1/\xi\|_{r,\omega;H_\sigma} \le c(1+K)(\|\tilde g_0\|_{-1,r,\tilde\omega;H} + \|\tilde g_1/\xi\|_{r,\tilde\omega;H}),$$

where $c > 0$ is independent of ω; note that $\tilde g = \tilde g_0 + \tilde g_1$ gives all admissible decompositions of $\tilde g \in \widehat{W}^{-1,r}_{\tilde\omega}(H) + L^r_{\tilde\omega,1/\xi}(H)$. Consequently

$$\|g; \widehat{W}^{-1,r}_\omega(H_\sigma) + L^r_{\omega,1/\xi}(H_\sigma)\| \le c(1+K)\,\|\tilde g; \widehat{W}^{-1,r}_{\tilde\omega}(H) + L^r_{\tilde\omega,1/\xi}(H)\|. \tag{3.35}$$

To apply Kato's perturbation theorem we introduce for every $\xi \in \mathbb{R}^*$ on H_σ the ξ-dependent Banach spaces ($\mu = |\lambda + \xi^2|^{1/2}$)

$$\begin{aligned}
&\mathcal{X} = (W^{2,r}_\omega \cap W^{1,r}_{0,\omega})^n \times W^{1,r}_\omega, \quad \|u,p\|_{\mathcal{X}} = \|\mu^2 u, \mu\nabla' u, \nabla'^2 u, \nabla' p, \xi p\|_{r,\omega;H_\sigma},\\
&\mathcal{Y} = (L^r_\omega)^n \times W^{1,r}_\omega, \quad \|f,g\|_{\mathcal{Y}} = \|f, \nabla' g, \xi g\|_{r,\omega;H_\sigma} + \|\lambda g; \widehat{W}^{-1,r}_\omega(H_\sigma) + L^r_{\omega,1/\xi}(H_\sigma)\|,
\end{aligned}$$

and on H similar spaces $(\tilde{\mathcal{X}}, \|\cdot\|_{\tilde{\mathcal{X}}}), (\tilde{\mathcal{Y}}, \|\cdot\|_{\tilde{\mathcal{Y}}})$ with the weight $\tilde\omega$ instead of ω. Then it follows from (3.34), (3.35) that

$$\|(u,p)\|_{\mathcal{X}} \le c(1+K+K^2+L/\mu)\|(\tilde u, \tilde p)\|_{\tilde{\mathcal{X}}}, \quad \|(f,g)\|_{\mathcal{Y}} \le c(1+K)\|(\tilde f, \tilde g)\|_{\tilde{\mathcal{Y}}}, \tag{3.36}$$

and exchanging the role of the variables x' and $\tilde x'$, we get

$$\|(\tilde u, \tilde p)\|_{\tilde{\mathcal{X}}} \le c(1+K+K^2+L/\mu)\|(u,p)\|_{\mathcal{X}}, \quad \|(\tilde f, \tilde g)\|_{\tilde{\mathcal{Y}}} \le c(1+K)\|(f,g)\|_{\mathcal{Y}}, \tag{3.37}$$

with constants $c > 0$ not depending on ω, λ and ξ. Further define the operators

$$\mathcal{S} : \mathcal{X} \to \mathcal{Y}, \quad \mathcal{S}(u,p) = \begin{pmatrix} (\lambda + \xi^2 - \Delta')u' + \nabla' p \\ (\lambda + \xi^2 - \Delta')u_n + i\xi p \\ \operatorname{div}' u' + i\xi u_n \end{pmatrix},$$

and analogously $\tilde{\mathcal{S}} : \tilde{\mathcal{X}} \to \tilde{\mathcal{Y}}$. By (3.32) we get the decomposition

$$\mathcal{S}(u,p) = \tilde{\mathcal{S}}(\tilde u, \tilde p) + \mathcal{R}(\tilde u, \tilde p)$$

with a remainder term $\mathcal{R} : \tilde{\mathcal{X}} \to \tilde{\mathcal{Y}}$,

$$\begin{aligned}
\mathcal{R}(\tilde u, \tilde p)(\tilde x', \xi) \quad &= \quad \begin{pmatrix} -(\nabla'\sigma)\tilde\partial_1 \tilde p \\ 0 \\ -(\nabla'\sigma) \cdot \tilde\partial_1 \tilde u' \end{pmatrix} \\
&\quad + \begin{pmatrix} -|\nabla'\sigma|^2 \tilde\partial_1^2 \tilde u + 2\nabla'\sigma \cdot \tilde\nabla' \tilde\partial_1 \tilde u + (\Delta''\sigma)\tilde\partial_1 \tilde u \\ 0 \end{pmatrix}
\end{aligned}$$

not depending explicitly on λ and ξ. Since $\tilde u|_{\partial H} = 0$ and $\tilde\partial_1(\nabla'\sigma) = 0$, we have

$$\int_H -(\nabla'\sigma)\cdot\tilde\partial_1\tilde u'\,\varphi\,d\tilde x' = \int_H (\nabla'\sigma)\cdot\tilde u'\,\tilde\partial_1\varphi\,d\tilde x'$$

for all $\varphi \in C_0^\infty(\bar H)$; consequently

$$\|-(\nabla'\sigma)\cdot\tilde\partial_1\tilde u'; \widehat W_{\tilde\omega}^{-1,r}(H) + L^r_{\tilde\omega,1/\xi}(H)\| \le \|-(\nabla'\sigma)\cdot\tilde\partial_1\tilde u'\|_{-1,r,\tilde\omega;H} \le K\|\tilde u\|_{r,\tilde\omega;H}.$$

Hence

$$\begin{aligned}\|\mathcal{R}(\tilde u,\tilde p)\|_{\tilde{\mathcal{Y}}} &\le c(K+K^2)\|\lambda\tilde u, \xi\tilde\nabla'\tilde u, \tilde\nabla'^2\tilde u, \tilde\nabla'\tilde p\|_{r,\tilde\omega;H} + L\|\tilde\nabla'\tilde u\|_{r,\tilde\omega;H}\\ &\le c_\varepsilon(K+K^2+\tfrac{L}{\mu})\|(\tilde u,\tilde p)\|_{\tilde{\mathcal{X}}}\\ &\le c_\varepsilon(K+K^2+\tfrac{L}{\sqrt{|\lambda|}})\|(\tilde u,\tilde p)\|_{\tilde{\mathcal{X}}},\end{aligned} \tag{3.38}$$

where $c, c_\varepsilon > 0$ are independent of $\omega, \tilde\omega$; note that $|\lambda| < \frac{\mu^2}{\cos\varepsilon}$ and $|\xi| < \mu(1+\frac{1}{\cos\varepsilon})^{1/2}$ for all $\lambda \in S_\varepsilon$.

Due to Theorem 3.2 and (3.33) $\tilde{\mathcal{S}} : \tilde X \to \tilde Y$ is an isomorphism such that $\|(\tilde u,\tilde p)\|_{\tilde{\mathcal{X}}} \le C_1\|\tilde{\mathcal{S}}(\tilde u,\tilde p)\|_{\tilde{\mathcal{Y}}}$ with an A_r-consistent constant $C_1 = C_1(r,\varepsilon,\mathcal{A}_r(\omega))$ independent of $\lambda \in S_\varepsilon, \xi \in \mathbb{R}^*$. Therefore, it follows from (3.38) that there exist A_r-consistent constants $\delta_0 = \delta(\varepsilon, r, \mathcal{A}_r(\omega)), \lambda_0 = \lambda(\varepsilon, r, \mathcal{A}_r(\omega))$ such that, if $K \le \delta_0$ and $\lambda \in S_\varepsilon, |\lambda| \ge \lambda_0$, then

$$\|\mathcal{R}(\tilde u,\tilde p)\|_{\tilde{\mathcal{Y}}} \le \frac12\|\mathcal{S}(\tilde u,\tilde p)\|_{\tilde{\mathcal{Y}}} \quad \text{for all } (\tilde u,\tilde p)\in\tilde{\mathcal{X}}.$$

Hence $\tilde{\mathcal{S}} + \mathcal{R}$ is an isomorphism from $\tilde{\mathcal{X}}$ to $\tilde{\mathcal{Y}}$ satisfying

$$\|(\tilde u,\tilde p)\|_{\tilde{\mathcal{X}}} \le 2C_1\|(\tilde{\mathcal{S}}+\mathcal{R})(\tilde u,\tilde p)\|_{\tilde{\mathcal{Y}}}.$$

Thus, considering (3.32), (3.36) and (3.37), if $\|\nabla''\sigma\|_\infty \le \delta_0$ and $\lambda \in S_\varepsilon, |\lambda| \ge \lambda_0$, we get

$$\begin{aligned}\|(u,p)\|_{\mathcal{X}} &\le C_2\|(\tilde u,\tilde p)\|_{\tilde{\mathcal{X}}}\\ &\le 2C_1C_2\|\tilde{\mathcal{S}}(\tilde u,\tilde p)\|_{\tilde{\mathcal{Y}}}\\ &\le C_3\|\mathcal{S}(u,p)\|_{\mathcal{Y}},\end{aligned}$$

where the constants $C_i = C_i(\varepsilon, r, \mathcal{A}_r(\omega)), i = 1,2,3$, are A_r-consistent and independent of $\lambda \in S_\varepsilon, |\lambda| \ge \lambda_0$ and $\xi \in \mathbb{R}^*$. Thus, existence of a unique solution to $(R_{\lambda,\xi})$ in H_σ has been proved.

Assume that (3.30) is satisfied for an additional exponent $s \ne r$ and weight $\nu \in A_s(\mathbb{R}^{n-1})$. Repeating the above argument for the index s, we see $\mathcal{S}$ to be an isomorphism from $\mathcal{X}_s \cap \mathcal{X}_r$ to $\mathcal{Y}_s \cap \mathcal{Y}_r$ for $|\lambda| \ge \lambda_0 = \lambda_0(r, s, \varepsilon, \mathcal{A}_r(\omega), \mathcal{A}_s(\nu))$ under the given smallness condition $\|\nabla''\sigma\|_\infty \le \delta_0(r, s, \varepsilon, \mathcal{A}_r(\omega), \mathcal{A}_s(\nu))$. Now the proof of Theorem 3.3 is complete. ∎

4 The Problem $(R_{\lambda,\xi})$ in Bounded Domains

For a bounded domain the definition of the space for the divergence g has to be modified since it is impossible to think of the sum of $\widehat{W}^{-1,r}(\Sigma)$ and $L^r(\Sigma)$. On the bounded domain $\Sigma \subset \mathbb{R}^{n-1}$ of $C^{1,1}$-class let α_0 denote the smallest eigenvalue of the Laplacian, i.e.

$$0 < \alpha_0 = \inf\{\|\nabla u\|_2^2 : \ u \in W_0^{1,2}(\Sigma), \|u\|_2 = 1\}.$$

For fixed $\lambda \in \mathbb{C} \setminus (-\infty, -\alpha_0], \xi \in \mathbb{R}$ and $\omega \in A_r$ we introduce the *parametrized Stokes operator* $S = S^\omega_{r,\lambda,\xi}$ by

$$S(u,p) = \begin{pmatrix} (\lambda + \xi^2 - \Delta')u' + \nabla' p \\ (\lambda + \xi^2 - \Delta')u_n + i\xi p \\ -\operatorname{div}_\xi u \end{pmatrix}$$

defined on $\mathcal{D}(S) = \mathcal{D}(\Delta'_{r,\omega}) \times W^{1,r}_\omega(\Sigma)$, where $\mathcal{D}(\Delta'_{r,\omega}) = W^{2,r}_\omega(\Sigma) \cap W^{1,r}_{0,\omega}(\Sigma)$ and

$$\operatorname{div}_\xi u = \operatorname{div}' u' + i\xi u_n.$$

For $\omega \equiv 1$ the operator $S^\omega_{r,\lambda,\xi}$ will be denoted by $S_{r,\lambda,\xi}$. Note that the image of $\mathcal{D}(S)$ by div_ξ is included in $W^{1,r}_\omega(\Sigma)$ and $W^{1,r}_\omega(\Sigma) \subset L^r_{m,\omega}(\Sigma) + L^r_\omega(\Sigma)$, where

$$L^r_{m,\omega}(\Sigma) := \Big\{u \in L^r_\omega(\Sigma) : \ \int_\Sigma u \, dx' = 0\Big\}.$$

Using Poincaré's inequality in weighted spaces, see Proposition 2.7, one can easily check the continuous embedding $L^r_{m,\omega}(\Sigma) \hookrightarrow \widehat{W}^{-1,r}_\omega(\Sigma)$; more precisely,

$$\|u\|_{-1,r,\omega} \le c\|u\|_{r,\omega}\,, \quad u \in L^r_{m,\omega}(\Sigma),$$

with an A_r-consistent constant $c > 0$. For convenience we use the notation

$$\|g; L^r_{m,\omega} + L^r_{\omega,1/\xi}\|_0 := \inf\{\|g_0\|_{-1,r,\omega} + \|g_1/\xi\|_{r,\omega} : \ g = g_0 + g_1, g_0 \in L^r_{m,\omega}, g_1 \in L^r_\omega\};$$

note that this norm is equivalent to the norm $\|\cdot\|_{(W^{1,r'}_{\omega',\xi})^*}$ where $W^{1,r'}_{\omega',\xi}$ is the usual weighted Sobolev space on Σ with norm $\|\nabla' u, \xi u\|_{r',\omega'}$.

In the following, we consider the resolvent problem $(R_{\lambda,\xi})$ for arbitrary $\lambda \in -\alpha_0 + S_\varepsilon$, $0 < \varepsilon < \pi/2$.

Lemma 4.1 *For every $\lambda \in -\alpha_0 + S_\varepsilon$, $0 < \varepsilon < \pi/2$, $\xi \in \mathbb{R}^*$ and $\omega \in A_r$ the operator $S = S^\omega_{r,\lambda,\xi}$ is injective and the range $\mathcal{R}(S)$ of S is dense in $L^r_\omega(\Sigma) \times W^{1,r}_\omega(\Sigma)$.*

Proof: Since, by Proposition 2.5 (3), there is an $s \in (1, r)$ such that $L^r_\omega(\Sigma) \subset L^s(\Sigma)$, one sees immediately that $\mathcal{D}(S^\omega_{r,\lambda,\xi}) \subset \mathcal{D}(S_{s,\lambda,\xi})$. Therefore, $S^\omega_{r,\lambda,\xi}(u,p) = 0$ for some $(u,p) \in \mathcal{D}(S^\omega_{r,\lambda,\xi})$ yields $(u,p) \in \mathcal{D}(S_{s,\lambda,\xi})$ and $S_{s,\lambda,\xi}(u,p) = 0$. Hence, by [9], Lemma 3.2, $u = 0, p = 0$.

On the other hand, by Proposition 2.5 (3), there is an $\tilde{s} \in (r,\infty)$ such that $S_{\tilde{s},\lambda,\xi} \subset S^{\omega}_{r,\lambda,\xi}$. Therefore, by [9], Theorem 3.4,

$$L^{\tilde{s}}(\Sigma) \times W^{1,\tilde{s}}(\Sigma) = \mathcal{R}(S_{\tilde{s},\lambda,\xi}) \subset \mathcal{R}(S^{\omega}_{r,\lambda,\xi}) \subset L^r_{\omega}(\Sigma) \times W^{1,r}_{\omega}(\Sigma),$$

which proves the assertion on the denseness of $\mathcal{R}(S)$. ■

The following lemma gives a preliminary *a priori* estimate for a solution (u,p) of $S(u,p) = (f,-g)$.

Lemma 4.2 *Let $1 < r < \infty$, $\omega \in A_r$ and $\varepsilon \in (0,\pi/2)$. Then there exists an A_r-consistent constant $c = c(\varepsilon, r, \Sigma, \mathcal{A}_r(\omega)) > 0$ such that for every $\lambda \in -\alpha_0 + S_\varepsilon, \xi \in \mathbb{R}^*$ and every $(u,p) \in \mathcal{D}(S^{\omega}_{r,\lambda,\xi})$,*

$$\begin{aligned}\|\mu_+^2 u, \mu_+\nabla' u, \nabla'^2 u, \nabla' p, \xi p\|_{r,\omega} \le c\big(&\|f, \nabla' g, g, \xi g\|_{r,\omega} + |\lambda| \|g; L^r_{m,\omega} + L^r_{\omega,1/\xi}\|_0 \\ &+\|\nabla' u, \xi u, p\|_{r,\omega} + |\lambda| \|u\|_{(W^{1,r'}_{\omega'})^*}\big),\end{aligned} \tag{4.1}$$

where $\mu_+ = |\lambda + \alpha_0 + \xi^2|^{1/2}, (f,-g) = S(u,p)$ and $(W^{1,r'}_{\omega'})^*$ denotes the dual space of $W^{1,r'}_{\omega'}(\Sigma)$.

Proof: The proof is based on a partition of unity in Σ and on the localization procedure reducing the problem to a finite number of problems of type $(R_{\lambda,\xi})$ in bent half spaces and in the whole space $\mathbb{R}^{n-1}$. Since $\partial\Sigma \in C^{1,1}$, we can cover $\partial\Sigma$ by a finite number of balls $B_j, j \ge 1$, such that, after a translation and rotation of coordinates, $\Sigma \cap B_j$ locally coincides with a bent half space $\Sigma_j = \Sigma_{\sigma_j}$ where $\sigma_j \in C^{1,1}(\mathbb{R}^{n-1})$ has a compact support, $\sigma_j(0) = 0$ and $\nabla''\sigma_j(0) = 0$. Choosing the balls B_j small enough (and its number large enough) we may assume that $\|\nabla''\sigma_j\|_\infty \le K_0(\varepsilon, r, \Sigma, \mathcal{A}_r(\omega))$ for all $j \ge 1$ where K_0 was introduced in Theorem 3.3. According to the covering $\partial\Sigma \subset \bigcup_{j\ge1} B_j$ there are cut-off functions $0 \le \varphi_0, \varphi_j \in C^\infty(\mathbb{R}^{n-1})$ such that

$$\varphi_0 + \sum_{j\ge1} \varphi_j \equiv 1 \text{ in } \Sigma, \quad \operatorname{supp}\varphi_j \subset B_j \quad \text{and} \quad \operatorname{supp}\varphi_0 \subset \Sigma.$$

Given $(u,p) \in \mathcal{D}(S)$ and $(f,-g) = S(u,p)$, we get for each φ_j, $j \ge 0$, the local $(R_{\lambda,\xi})$-problems

$$\begin{aligned}(\lambda + \xi^2 - \Delta')(\varphi_j u') + \nabla'(\varphi_j p) &= f_j' \\ (\lambda + \xi^2 - \Delta')(\varphi_j u_n) + i\xi(\varphi_j p) &= f_{jn} \\ \operatorname{div}_\xi(\varphi_j u) &= g_j\end{aligned} \tag{4.2}$$

for $(\varphi_j u, \varphi_j p), j \ge 0$, in $\mathbb{R}^{n-1}$ or Σ_j; here

$$\begin{aligned}f_j' &= \varphi_j f' - 2\nabla'\varphi_j \cdot \nabla' u' - (\Delta'\varphi_j)u' + (\nabla'\varphi_j)p \\ f_{jn} &= \varphi_j f_n - 2\nabla'\varphi_j \cdot \nabla' u_n - (\Delta'\varphi_j)u_n \\ g_j &= \varphi_j g + \nabla'\varphi_j \cdot u'.\end{aligned} \tag{4.3}$$

To control f_j and g_j note that $u = 0$ on $\partial\Sigma$; hence Poincaré's inequality for Muckenhoupt weighted space yields for all $j \geq 0$ the estimate

$$\|f_j, \nabla' g_j, \xi g_j\|_{r,\omega;\Sigma_j} \leq c(\|f, \nabla' g, g, \xi g\|_{r,\omega;\Sigma} + \|\nabla' u, \xi u, p\|_{r,\omega;\Sigma}), \tag{4.4}$$

where $\Sigma_0 \equiv \mathbb{R}^{n-1}$ and $c > 0$ is A_r-consistent. Moreover, let $g = g_0 + g_1$ denote any splitting of $g \in L^r_{m,\omega} + L^r_{\omega,1/\xi}$. Defining the characteristic function χ_j of $\Sigma \cap \Sigma_j$ and the scalar

$$\begin{aligned} m_j &= \frac{1}{|\Sigma \cap \Sigma_j|} \int_{\Sigma\cap\Sigma_j} (\varphi_j g_0 + u' \cdot \nabla' \varphi_j) dx' \\ &= \frac{1}{|\Sigma \cap \Sigma_j|} \int_{\Sigma\cap\Sigma_j} (i\xi u_n - g_1)\varphi_j dx', \end{aligned}$$

we split g_j in the form

$$g_j = g_{j0} + g_{j1} := (\varphi_j g_0 + u' \cdot \nabla' \varphi_j - m_j \chi_j) + (\varphi_j g_1 + m_j \chi_j).$$

Concerning g_{j1} we get

$$\begin{aligned} \|g_{j1}\|^r_{r,\omega;\Sigma_j} &= \int_{\Sigma\cap\Sigma_j} |\varphi_j g_1 + m_j|^r \omega \, dx' \\ &\leq c(r)\big(\|g_1\|^r_{r,\omega;\Sigma} + |m_j|^r \omega(\Sigma \cap \Sigma_j)\big) \\ &\leq c(r)\Big(\|g_1\|^r_{r,\omega;\Sigma} + \frac{\omega(\Sigma\cap\Sigma_j) \cdot \omega'(\Sigma\cap\Sigma_j)^{r/r'}}{|\Sigma\cap\Sigma_j|^r} (\|\xi u_n\|^r_{(W^{1,r'}_{\omega'})^*} + \|g_1\|^r_{r,\omega;\Sigma})\Big) \end{aligned}$$

with $c(r) > 0$ independent of ω. Since we chose the balls B_j for $j \geq 1$ small enough, for each $j \geq 0$ there is a cube Q_j with $\Sigma \cap \Sigma_j \subset Q_j$ and $|Q_j| < c(n)|\Sigma \cap \Sigma_j|$ where the constant $c(n) > 0$ is independent of j. Therefore

$$\begin{aligned} \|g_{j1}\|_{r,\omega;\Sigma_j} &\leq c(r)\Big(\|g_1\|_{r,\omega} + \tfrac{c(n)\omega(Q_j)^{1/r} \cdot \omega'(Q_j)^{1/r'}}{|Q_j|} (\|\xi u_n\|_{(W^{1,r'}_{\omega'})^*} + \|g_1\|_{r,\omega})\Big) \\ &\leq c(r)(1 + \mathcal{A}_r(\omega)^{1/r})\big(\|\xi u_n\|_{(W^{1,r'}_{\omega'})^*} + \|g_1\|_{r,\omega;\Sigma}\big) \end{aligned} \tag{4.5}$$

for $j \geq 0$. Furthermore, for every test function $\Psi \in C_0^\infty(\bar{\Sigma}_j)$ let

$$\tilde{\Psi} = \Psi - \frac{1}{|\Sigma \cap \Sigma_j|} \int_{\Sigma\cap\Sigma_j} \Psi dx'.$$

By the definition of $m_j \chi_j$ we have $\int_{\Sigma_j} g_{j0} \, dx' = 0$; hence by Poincaré's inequality (see Proposition 2.7)

$$\begin{aligned} \textstyle\int_{\Sigma_j} g_{j0} \Psi dx' &= \textstyle\int_{\Sigma_j} g_{j0} \tilde{\Psi} dx' \\ &= \textstyle\int_\Sigma g_0 (\varphi_j \tilde{\Psi}) dx' + \int_\Sigma u' \cdot (\nabla' \varphi_j) \tilde{\Psi} dx' \\ &\leq \|g_0\|_{-1,r,\omega} \|\nabla'(\varphi_j \tilde{\Psi})\|_{r',\omega'} + \|u'\|_{(W^{1,r'}_{\omega'})^*} \|(\nabla' \varphi_j)\tilde{\Psi}\|_{1,r',\omega'} \\ &\leq c(\|g_0\|_{-1,r,\omega} + \|u'\|_{(W^{1,r'}_{\omega'})^*}) \|\nabla' \Psi\|_{r',\omega';\Sigma_j}, \end{aligned}$$

where $c > 0$ is A_r-consistent. Thus

$$\|g_{j0}\|_{-1,r,\omega;\Sigma_j} \le c\big(\|g_0\|_{-1,r,\omega} + \|u'\|_{(W^{1,r'}_{\omega'})^*}\big) \quad \text{for } j \ge 0. \tag{4.6}$$

Summarizing (4.5) and (4.6), we get for $j \ge 0$

$$\|g_j; \widehat{W}^{-1,r}_{\omega}(\Sigma_j) + L^r_{\omega,1/\xi}(\Sigma_j)\| \le c\big(\|u'\|_{(W^{1,r'}_{\omega'})^*} + \|g; L^r_{m,\omega} + L^r_{\omega,1/\xi}\|_0\big) \tag{4.7}$$

with an A_r-consistent $c = c(r, \mathcal{A}_r(\omega)) > 0$.

To complete the proof, apply Theorem 3.1 to (4.2), (4.3) when $j = 0$. Further use Theorem 3.3 in (4.2), (4.3) for $j \ge 1$, but with λ replaced by $\lambda + M$ with $M = \lambda_0 + \alpha_0$, where $\lambda_0 = \lambda_0(\varepsilon, r, \mathcal{A}_r(\omega))$ is the A_r-consistent constant indicated in Theorem 3.3. This shift in λ implies that f_j has to be replaced by $f_j + M\varphi_j u$ and that (3.31) will be used with λ replaced by $\lambda + M$. Summarizing (3.8), (3.31) as well as (4.4), (4.7) and summing over all j we arrive at (4.1) with the additional terms

$$I = \|Mu\|_{r,\omega} + \|Mu'\|_{(W^{1,r'}_{\omega'})^*} + \|Mg; L^r_{m,\omega} + L^r_{\omega,1/\xi}\|_0$$

on the right-hand side of the inequality. Note that $M = M(\varepsilon, r, \mathcal{A}_r(\omega))$ is A_r-consistent and that $g = \operatorname{div}' u' + i\xi u_n$ defines a natural splitting of $g \in L^r_{m,\omega}(\Sigma) + L^r_{\omega}(\Sigma)$. Hence Poincaré's inequality yields

$$\begin{aligned} I &\le M\big(\|u\|_{r,\omega;\Sigma} + \|\operatorname{div}' u'\|_{-1,r,\omega} + \|u_n\|_{r,\omega;\Sigma}\big) \\ &\le c_1\|u\|_{r,\omega;\Sigma} \le c_2\|\nabla' u\|_{r,\omega;\Sigma} \end{aligned}$$

with A_r-consistent constants $c_i = c_i(\varepsilon, r, \Sigma, \mathcal{A}_r(\omega)) > 0$, $i = 1, 2$. Thus (4.1) is proved. ■

Lemma 4.3 *Let $1 < r < \infty$, $\omega \in A_r$ and $\lambda \in -\alpha + S_\varepsilon$, $\varepsilon \in (0, \frac{\pi}{2})$ with $\alpha \in (0, \alpha_0)$. Then there is an A_r-consistent constant $c = c(\alpha, \varepsilon, r, \mathcal{A}_r(\omega))$ such that for every $(u,p) \in \mathcal{D}(S)$ and $(f, -g) = S(u,p)$ the estimate*

$$\begin{aligned} &\|\mu_+^2 u, \mu_+\nabla' u, \nabla'^2 u, \nabla' p, \xi p\|_{r,\omega} \\ &\qquad \le c\big(\|f, \nabla' g, g, \xi g\|_{r,\omega} + (|\lambda| + 1)\|g; L^r_{m,\omega} + L^r_{\omega,1/\xi}\|_0\big) \end{aligned} \tag{4.8}$$

holds; here $\mu_+ = |\lambda + \alpha + \xi^2|^{1/2}$.

Proof of Lemma 4.3: Assume that this lemma is wrong. Then there is a constant $c_0 > 0$, a sequence $\{\omega_j\}_{j=1}^\infty \subset A_r$ with $\mathcal{A}_r(\omega_j) \le c_0$ for all j, sequences $\{\lambda_j\}_{j=1}^\infty \subset -\alpha + S_\varepsilon$, $\{\xi_j\}_{j=1}^\infty \subset \mathbb{R}^*$ and $(u_j, p_j) \in \mathcal{D}(S^{\omega_j}_{r,\lambda_j,\xi_j})$ for all $j \in \mathbb{N}$ such that

$$\begin{aligned} &\|(\lambda_j + \alpha + \xi_j^2)u_j, (\lambda_j + \alpha + \xi_j^2)^{1/2}\nabla' u_j, \nabla'^2 u_j, \nabla' p_j, \xi_j p_j\|_{r,\omega_j} \\ &\qquad \ge j\big(\|f_j, \nabla' g_j, g_j, \xi_j g_j\|_{r,\omega_j} + (|\lambda_j| + 1)\|g_j; L^r_{m,\omega_j} + L^r_{\omega_j,1/\xi_j}\|_0 \end{aligned} \tag{4.9}$$

where $(f_j, -g_j) = S^{\omega_j}_{r,\lambda_j,\xi_j}(u_j, p_j)$. Fix an arbitrary cube Q containing Σ. We may assume without loss of generality that

$$\mathcal{A}_r(\omega_j) \le c_0, \quad \omega_j(Q) = 1 \quad \forall j \in \mathbb{N}, \tag{4.10}$$

by using the A_r-weight $\tilde{\omega}_j := \omega_j(Q)^{-1}\omega_j$ instead of ω_j if necessary. Note that (4.10) also holds for $r', \{\omega_j'\}$ in the following form: $\mathcal{A}_r(\omega_j) \le c_0^{r'/r}$, $\omega_j'(Q) \le c_0^{r'/r}|Q|^{r'}$. Therefore, by a minor modification of Proposition 2.5 (3), there exist numbers s, s_1 such that

$$L^r_{\omega_j}(\Sigma) \hookrightarrow L^s(\Sigma), \quad L^{s_1}(\Sigma) \hookrightarrow L^{r'}_{\omega_j'}, \quad j \in \mathbb{N}, \tag{4.11}$$

with embedding constants independent of $j \in \mathbb{N}$. Furthermore, we may assume without loss of generality that

$$\|(\lambda_j + \alpha + \xi_j^2)u_j, (\lambda_j + \alpha + \xi_j^2)^{1/2}\nabla' u_j, \nabla'^2 u_j, \nabla' p_j, \xi_j p_j\|_{r,\omega_j} = 1 \tag{4.12}$$

and consequently that

$$\|f_j, \nabla' g_j, g_j, \xi_j g_j\|_{r,\omega_j} + (|\lambda_j| + 1)\|g_j; L^r_{m,\omega_j} + L^r_{\omega_j,1/\xi_j}\|_0 \to 0 \quad \text{as} \quad j \to \infty. \tag{4.13}$$

From (4.11), (4.12) we have

$$\|(\lambda_j + \alpha + \xi_j^2)u_j, (\lambda_j + \alpha + \xi_j^2)^{1/2}\nabla' u_j, \nabla'^2 u_j, \nabla' p_j, \xi_j p_j\|_s \le K, \tag{4.14}$$

with some $K > 0$ for all $j \in \mathbb{N}$ and

$$\|f_j, \nabla' g_j, g_j, \xi_j g_j\|_s \to 0 \quad \text{as} \quad j \to \infty. \tag{4.15}$$

Without loss of generality let us suppose that as $j \to \infty$,

$$\begin{array}{l} \lambda_j \to \lambda \in -\alpha + \bar{S}_\varepsilon \quad \text{or} \quad |\lambda_j| \to \infty \\ \xi_j \to 0 \quad \text{or} \quad \xi_j \to \xi \ne 0 \quad \text{or} \quad |\xi_j| \to \infty. \end{array}$$

Thus we have to consider six possibilities.

(i) The case $\lambda_j \to \lambda \in -\alpha + \bar{S}_\varepsilon, \quad \xi_j \to \xi \ne 0.$

Due to (4.14) $\{u_j\} \subset W^{2,s}$ and $\{p_j\} \subset W^{1,s}$ are bounded sequences. In virtue of the compactness of the embedding $W^{1,s}(\Sigma) \hookrightarrow L^s(\Sigma)$ for the bounded domain Σ, we may assume (suppressing indices for subsequences) that

$$\begin{array}{lll} u_j \to u, \nabla' u_j \to \nabla' u & \text{in } L^s & \text{(strong convergence)} \\ \nabla'^2 u_j \rightharpoonup \nabla'^2 u & \text{in } L^s & \text{(weak convergence)} \\ p_j \to p & \text{in } L^s & \text{(strong convergence)} \\ \nabla' p_j \rightharpoonup \nabla' p & \text{in } L^s & \text{(weak convergence)} \end{array} \tag{4.16}$$

for some $(u, p) \in \mathcal{D}(S_{s,\lambda,\xi})$ as $j \to \infty$. Therefore, $S_{s,\lambda,\xi}(u,p) = 0$ and, consequently, $u = 0$, $p = 0$ by Lemma 4.1. On the other hand we get from (4.12) that $\sup_{j\in\mathbb{N}} \|u_j\|_{2,r,\omega_j} < \infty$ and $\sup_{j\in\mathbb{N}} \|p_j\|_{1,r,\omega_j} < \infty$ which, together with the weak convergences $u_j \rightharpoonup 0$ in $W^{2,s}(\Sigma)$, $p_j \rightharpoonup 0$ in $W^{1,s}(\Sigma)$, yields

$$\|u_j\|_{1,r,\omega_j} \to 0, \quad \|p_j\|_{r,\omega_j} \to 0$$

due to Proposition 2.6 (2). Moreover, since $\sup_{j\in\mathbb{N}} \|\lambda_j u_j\|_{r,\omega_j} < \infty$ and $\lambda_j u_j \rightharpoonup \lambda u = 0$ in $L^s(\Sigma)$, Proposition 2.6 (3) implies that

$$\|\lambda_j u_j\|_{(W^{1,r'}_{\omega_j'})^*} \to 0. \tag{4.17}$$

Thus (4.1), (4.12) and (4.13) yield the contradiction $1 \leq 0$.

(ii) The case $\lambda_j \to \lambda \in -\alpha + \bar{S}_\varepsilon, \quad \xi_j \to 0$.
Since $u_j|_{\partial\Sigma} = 0, \|\nabla'^2 u_j\|_s \leq K$, we have the convergence (4.16) for some $u \in W^{2,s}(\Sigma) \cap W_0^{1,s}(\Sigma)$, but concerning p we get the existence of $p \in \widehat{W}^{1,s}$ and $q \in L^s$ such that

$$\nabla' p_j \rightharpoonup \nabla' p, \quad \xi_j p_j \rightharpoonup q \quad \text{ in } L^s$$

as $j \to \infty$. Looking at (R_{λ_j,ξ_j}), the convergence of $\{u_j\}$, $\{p_j\}$ yields

$$\begin{aligned} (\lambda - \Delta')u' + \nabla' p &= 0 \\ (\lambda - \Delta')u_n + iq &= 0 \\ \operatorname{div}' u' &= 0 \end{aligned}$$

in Σ. Thus $(u', \nabla' p) = (0, 0)$, see [9], Lemma 3.3 (ii), or [6]. Obviously, q is a constant, since $\xi_j \to 0$, and $u_n \in W^{2,2}(\Sigma) \cap W_0^{1,2}(\Sigma)$ due to elliptic regularity theory.

By (4.13), for all $j \in \mathbb{N}$ there is a splitting $g_j = g_{j0} + g_{j1}$ such that

$$g_{j0} \in L^r_{m,\omega_j},\ g_{j1} \in L^r_{\omega_j} \quad \text{and} \quad (|\lambda_j| + 1)\big(\|g_{j0}\|_{-1,r,\omega_j} + \|g_{j1}/\xi_j\|_{r,\omega_j}\big) \to 0. \tag{4.18}$$

Therefore, from the divergence equation $\operatorname{div}_{\xi_j} u_j = g_j$ we get

$$(|\lambda_j| + 1) \left| \int_\Sigma u_{jn}\, dx' \right| = \frac{|\lambda_j| + 1}{|\xi_j|} \left| \int_\Sigma g_{j1}\, dx' \right| \to 0 \quad \text{ as } j \to \infty,$$

and consequently $\int_\Sigma u_n\, dx' = 0$. Now, testing the equation $(\lambda - \Delta')u_n + iq = 0$ in Σ with u_n, we see that $\lambda \int_\Sigma |u_n|^2\, dx' + \int_\Sigma |\nabla' u_n|^2\, dx' = 0$ yielding $u_n = 0$ and also $q = 0$. Thus $u_j \rightharpoonup 0$ in $W^{2,s}(\Sigma)$ which, together with $\sup_{j\in\mathbb{N}} \|u_j\|_{2,r,\omega_j} < \infty$, yields

$$\|u_j\|_{1,r,\omega_j} \to 0 \quad \text{as} \quad j \to \infty \tag{4.19}$$

due to Proposition 2.6 (2).

To come to a contradiction consider the equivalent equation $S^{\omega_j}_{r,\lambda_j,\xi_j}(u_j, p_j - p_{jm}) = (f_j - i\xi_j p_{jm} e_n, -g_j)$ with $p_{jm} = \frac{1}{|\Sigma|}\int_\Sigma p_j\, dx'$. Due to Lemma 4.2

$$\begin{aligned} &\|(\lambda_j + \alpha + \xi_j^2)u_j, (\lambda_j + \alpha + \xi_j^2)^{1/2}\nabla' u_j, \nabla'^2 u_j, \nabla' p_j, \xi_j(p_j - p_{jm})\|_{r,\omega_j} \\ &\quad \leq c\big(\|f_j, \nabla' g_j, g_j, \xi_j g_j\|_{r,\omega_j} + (|\lambda_j| + 1)\|g_j; L^r_{m,\omega_j} + L^r_{\omega_j,1/\xi}\|_0 \\ &\qquad + \|\xi_j p_{jm}\|_{r,\omega_j} + \|\nabla' u_j, \xi_j u_j, p_j - p_{jm}\|_{r,\omega_j} + \|\lambda_j u_j\|_{(W^{1,r'}_{\omega'_j})^*}\big) \end{aligned} \tag{4.20}$$

where $c > 0$ is independent of $j \in \mathbb{N}$ due to $\mathcal{A}_r(\omega_j) \leq c_0$, $j \in \mathbb{N}$. Since $\xi_j p_j \rightharpoonup q = 0$ in L^s, we have $\xi_j p_{jm} \to 0$ and, considering (4.10),

$$\|\xi_j p_{jm}\|_{r,\omega_j} = |\xi_j p_{jm}|\omega_j(\Sigma)^{1/r} \leq |\xi_j p_{jm}| \to 0. \tag{4.21}$$

From Poincaré's inequality (Proposition 2.7) and (4.12), we conclude that $\sup_j \|p_j - p_{jm}\|_{1,r,\omega_j} < \infty$, which, together with $p_j - p_{jm} \rightharpoonup 0$ in $W^{1,s}(\Sigma)$, yields

$$\|p_j - p_{jm}\|_{r,\omega_j} \to 0 \quad \text{as} \quad j \to \infty, \tag{4.22}$$

cf. Proposition 2.6 (2). Now, (4.12), (4.13), (4.17), (4.19), (4.21) and (4.22) lead in (4.20) to the contradiction $1 \le 0$.

(iii) The case $\lambda_j \to \lambda \in -\alpha + \bar{S}_\varepsilon, \quad |\xi_j| \to \infty$.
From (4.12) we get $\|\nabla' u_j, \xi_j u_j, p_j\|_{r,\omega_j} \to 0$. On the other hand, since $\|u_j\|_{r,\omega_j} \to 0$ and $u_j \to 0$ in L^s as $j \to \infty$, Proposition 2.6 (3) implies (4.17). Thus, from (4.1), (4.12) and (4.13) we get the contradiction $1 \le 0$.

(iv) The case $|\lambda_j| \to \infty, \quad \xi_j \to \xi \ne 0$.
By (4.12)

$$\|\nabla' u_j, \xi_j u_j\|_{r,\omega_j} \to 0 \quad \text{as} \quad j \to \infty. \tag{4.23}$$

Further, (4.14) yields the convergence

$$\begin{array}{lcl} u_j \to 0, \nabla' u_j \to 0 & \text{and} & \nabla'^2 u_j \rightharpoonup 0, \lambda_j u_j \rightharpoonup v, \\ p_j \to p & \text{and} & \nabla' p_j \rightharpoonup \nabla' p, \end{array}$$

in L^s, which, together with (4.15), leads to

$$v' + \nabla' p = 0, \quad v_n + i\xi p = 0. \tag{4.24}$$

From (4.11), (4.18) we see that

$$\begin{aligned} |\langle \lambda_j g_j, \varphi\rangle| &= |\langle \lambda_j g_{j0}, \varphi\rangle + \langle \lambda_j g_{j1}, \varphi\rangle| \\ &\le \|\lambda_j g_{j0}\|_{-1,r,\omega_j} \|\nabla' \varphi\|_{r',\omega_j'} + \|\lambda_j g_{j1}\|_{r,\omega_j} \|\varphi\|_{r',\omega_j'} \\ &\le c\big(\|\lambda_j g_{j0}\|_{-1,r,\omega_j}\| + \|\lambda_j g_{j1}\|_{r,\omega_j}\big)\|\varphi\|_{W^{1,s_1}(\Sigma)}. \end{aligned}$$

Consequently,

$$\lambda_j g_j \in (W^{1,s_1}(\Sigma))^* \quad \text{and} \quad \|\lambda_j g_j\|_{(W^{1,s_1}(\Sigma))^*} \to 0 \quad \text{as } j \to \infty. \tag{4.25}$$

Therefore, it follows from the divergence equation $\operatorname{div}'_{\xi_j} u_j = g_j$ that for all $\varphi \in C^\infty(\bar{\Sigma})$

$$\begin{aligned} \langle v', -\nabla'\varphi\rangle + \langle i\xi v_n, \varphi\rangle &= \lim_{j\to\infty} \langle \operatorname{div}' \lambda_j u_j' + i\lambda_j \xi_j u_{jn}, \varphi\rangle \\ &= \lim_{j\to\infty} \langle \lambda_j g_j, \varphi\rangle = 0, \end{aligned}$$

yielding $\operatorname{div}' v' = -i\xi v_n,\ v' \cdot N|_{\partial\Sigma} = 0$. Therefore (4.24) implies

$$-\Delta' p + \xi^2 p = 0 \text{ in } \Sigma, \quad \frac{\partial p}{\partial N} = 0 \text{ on } \partial\Sigma;$$

hence $p \equiv 0$ and also $v \equiv 0$. Now, due to Proposition 2.6 (2), (3), we get (4.17) and the convergence $\|p_j\|_{r,\omega_j} \to 0$, since $\lambda_j u_j \rightharpoonup 0$ in L^s, $p_j \rightharpoonup 0$ in $W^{1,s}$ and $\sup_{j\in\mathbb{N}} \|\lambda_j u_j\|_{r,\omega_j} < \infty$, $\sup_{j\in\mathbb{N}} \|p_j\|_{1,r,\omega_j} < \infty$. Thus (4.1), (4.12), (4.13) and (4.23) lead to the contradiction $1 \le 0$.

(v) The case $|\lambda_j| \to \infty, \quad \xi_j \to 0$.
It follows from (4.12) that in L^s

$$\begin{array}{l} u_j \to 0, \nabla' u_j \to 0 \quad \text{and} \quad \nabla'^2 u_j \rightharpoonup 0, \lambda_j u_j \rightharpoonup v, \\ \nabla' p_j \rightharpoonup \nabla' p, \quad \xi_j p_j \rightharpoonup q, \end{array}$$

which, looking at $(R_{\lambda,\xi})$, yields in the weak limit

$$v' + \nabla' p = 0, \quad v_n + iq = 0;$$

moreover, q is a constant. Note that (4.25) holds true in this case as well. Therefore, using (4.25), for any function φ in $C^\infty(\bar{\Sigma})$

$$0 = -\lim_{j\to\infty} \langle \lambda_j g_j, \varphi\rangle = \lim_{j\to\infty} \big(\langle \lambda_j u_j', \nabla'\varphi\rangle - \langle i\lambda_j \xi_j u_{jn}, \varphi\rangle\big) = \int_\Sigma v' \cdot \overline{\nabla'\varphi}\, dx'$$

yielding $\operatorname{div}' v' = 0, v' \cdot N|_{\partial\Sigma} = 0$. Thus the equation $v' + \nabla' p = 0$ is just the Helmholtz decomposition of the null vector field; therefore, $v' \equiv 0, \nabla' p \equiv 0$.

On the other hand, looking at (4.18) we get from the divergence equation and (4.11) that

$$\int_\Sigma \lambda_j u_{jn}\, dx' = \int_\Sigma \frac{\lambda_j}{\xi_j}(g_{j0} + g_{j1} - \operatorname{div}' u_j')\, dx' = \int_\Sigma \frac{\lambda_j g_{j1}}{\xi_j}\, dx' \to 0.$$

Consequently, the weak convergence $\lambda_j u_{jn} \rightharpoonup v_n$ in L^s yields $\int_\Sigma v_n\, dx' = 0$; since q is a constant, we get $v_n = 0$, $q = 0$. Then Proposition 2.6 (3) implies (4.17).

Now we repeat the argument as in the case (ii) to get (4.20), (4.21) and (4.22), and are finally led to the contradiction $1 \le 0$.

(vi) The case $|\lambda_j| \to \infty, \quad |\xi_j| \to \infty$.
To come to a contradiction, it is enough to prove (4.17) since $\|\nabla' u_j, \xi_j u_j, p_j\|_{r,\omega_j} \to 0$ as $j \to \infty$. From (4.12) we get the convergence

$$\begin{array}{lll} u_j \to 0, \nabla' u_j \to 0 & \text{and} & \nabla'^2 u_j \rightharpoonup 0, (\lambda_j + \xi_j^2) u_j \rightharpoonup v, \\ p_j \to 0 & \text{and} & \nabla' p_j \rightharpoonup 0, \quad \xi_j p_j \rightharpoonup q \end{array}$$

in L^s with some $v, q \in L^s$. Therefore, (4.15) and (R_{λ_j,ξ_j}) yield

$$v' = 0, \quad v_n + iq = 0.$$

Since $\|\lambda_j u_j\|_s \le c_\varepsilon \|(\lambda_j + \xi_j^2) u_j\|_s$, there exists $w = (w', w_n) \in L^s$ such that, for a suitable subsequence, $\lambda_j u_j \rightharpoonup w$. Let $g_j = g_{j0} + g_{j1}$, $j \in \mathbb{N}$, be a sequence of splittings satisfying (4.18). By (4.11) we get for all $\varphi \in C^\infty(\bar{\Sigma})$

$$|\langle \lambda_j g_{j0}, \varphi\rangle| + \Big|\langle \frac{\lambda_j g_{j1}}{\xi_j}, \varphi\rangle\Big| \to 0 \quad \text{as} \quad j \to \infty,$$

cf. (4.25) and (4.25). Hence, the divergence equation implies that for $j \to \infty$

$$\langle \lambda_j u_{jn}, \varphi\rangle = \frac{1}{i\xi_j}\langle \lambda_j g_{j0}, \varphi\rangle + \langle \frac{\lambda_j g_{j1}}{i\xi_j}, \varphi\rangle + \frac{1}{i\xi_j}\langle \lambda_j u_j', \nabla'\varphi\rangle \to 0$$

for all $\varphi \in C^\infty(\bar{\Sigma})$ yielding $\langle w_n, \varphi\rangle = 0$ and consequently $w_n = 0$.

Obviously, $\xi_j u_j \to 0$ in L^s as $j \to \infty$. Therefore, by (4.15) and the boundedness of the sequence $\big\{\|\xi_j \nabla u_j\|_{r,\omega_j}\big\}$, we get from the identity $\operatorname{div}'(\xi_j u_j') + i\xi_j^2 u_{jn} = \xi_j g_j$ that

$$\xi_j^2 u_{jn} \rightharpoonup 0 \quad \text{in } L^s \text{ as } j \to \infty.$$

Thus we proved $v_n = 0$. Now $v = 0$ together with the estimate $\|(\lambda_j + \xi_j^2)u_j\|_{r,\omega_j} \le 1$ imply due to Proposition 2.6 (3) that $\|(\lambda_j + \xi_j^2)u_j\| \to 0$ in $(W^{1,r'}_{\omega'_j})^*$ as $j \to \infty$. Hence also (4.17) is proved.

Now the proof of this lemma is complete. ■

Theorem 4.4 *Let $\Sigma \subset \mathbb{R}^{n-1}$ be a bounded domain of $C^{1,1}$-class, $1 < r < \infty$, $\omega \in A_r(\mathbb{R}^{n-1})$ and $\alpha \in (0, \alpha_0)$, $0 < \varepsilon < \frac{\pi}{2}$. Then for every $\lambda \in -\alpha + S_\varepsilon$, $\xi \in \mathbb{R}^*$ and $f \in L^r_\omega(\Sigma)$, $g \in W^{1,r}_\omega(\Sigma)$ the parametrized resolvent problem $(R_{\lambda,\xi})$ has a unique solution $(u, p) \in \big(W^{2,r}_\omega(\Sigma) \cap W^{1,r}_{0,\omega}(\Sigma)\big) \times W^{1,r}_\omega(\Sigma)$. Moreover, this solution satisfies the estimate (4.8) with an A_r-consistent constant $c = c(\alpha, \varepsilon, r, \Sigma, \mathcal{A}_r(\omega)) > 0$.*

Proof: The existence is obvious since, for every $\lambda \in -\alpha + S_\varepsilon, \xi \in \mathbb{R}^*$ and $\omega \in A_r(\mathbb{R}^{n-1})$, the range $\mathcal{R}(S^\omega_{r,\lambda,\xi})$ is closed and dense in $L^r_\omega(\Sigma) \times W^{1,r}_\omega(\Sigma)$ by Lemma 4.3 and by Lemma 4.1, respectively. Here note that for fixed $\lambda \in \mathbb{C}$, $\xi \in \mathbb{R}^*$ the norm $\|\nabla' g, g, \xi g\|_{1,r,\omega} + (1 + |\lambda|)\|g; L^r_{m,\omega} + L^r_{\omega,1/\xi}\|_0$ is equivalent to the norm of $W^{1,r}_\omega(\Sigma)$. The uniqueness of solutions is obvious from Lemma 4.1. ■

Now, for fixed $\omega \in A_r, 1 < r < \infty$, define the operator-valued functions

$$
\begin{aligned}
a_1 &: \mathbb{R}^* \to \mathcal{L}(L^r_\omega(\Sigma); W^{2,r}_{0,\omega}(\Sigma) \cap W^{1,r}_\omega(\Sigma)),\\
b_1 &: \mathbb{R}^* \to \mathcal{L}(L^r_\omega(\Sigma); W^{1,r}_\omega(\Sigma))
\end{aligned}
$$

by

$$
a_1(\xi)f := u_1(\xi), \quad b_1(\xi)f := p_1(\xi), \tag{4.26}
$$

where $(u_1(\xi), p_1(\xi))$ is the solution to $(R_{\lambda,\xi})$ corresponding to $f \in L^r_\omega(\Sigma)$ and $g = 0$. Further, define

$$
\begin{aligned}
a_2 &: \mathbb{R}^* \to \mathcal{L}(W^{1,r}_\omega(\Sigma); W^{2,r}_{0,\omega}(\Sigma) \cap W^{1,r}_\omega(\Sigma)),\\
b_2 &: \mathbb{R}^* \to \mathcal{L}(W^{1,r}_\omega(\Sigma); W^{1,r}_\omega(\Sigma))
\end{aligned}
$$

by

$$
a_2(\xi)g := u_2(\xi), \quad b_2(\xi)g := p_2(\xi). \tag{4.27}
$$

with $(u_2(\xi), p_2(\xi))$ the solution to $(R_{\lambda,\xi})$ corresponding to $f = 0$ and $g \in W^{1,r}_\omega(\Sigma)$.

Corollary 4.5 *For every $\alpha \in (0, \alpha_0)$ and $\lambda \in -\alpha + S_\varepsilon$ the operator-valued functions a_1, b_1 and a_2, b_2 defined by (4.26), (4.27) are Fréchet differentiable in $\xi \in \mathbb{R}^*$. Furthermore, their derivatives $w_1 = \frac{d}{d\xi}a_1(\xi)f$, $q_1 = \frac{d}{d\xi}b_1(\xi)f$ for fixed $f \in L^r_\omega(\Sigma)$ and $w_2 = \frac{d}{d\xi}a_2(\xi)g$, $q_2 = \frac{d}{d\xi}b_2(\xi)g$ for fixed $g \in W^{1,r}_\omega(\Sigma)$ satisfy the estimates*

$$
\|(\lambda + \alpha)\xi w_1, \xi \nabla'^2 w_1, \xi^3 w_1, \xi \nabla' q_1, \xi^2 q_1\|_{r,\omega} \le c\|f\|_{r,\omega} \tag{4.28}
$$

and

$$
\begin{aligned}
&\|(\lambda + \alpha)\xi w_2, \xi \nabla'^2 w_2, \xi^3 w_2, \xi \nabla' q_2, \xi^2 q_2\|_{r,\omega}\\
&\qquad \le c\big(\|\nabla' g, g, \xi g\|_{r,\omega} + (|\lambda| + 1)\|g; L^r_{m,\omega} + L^r_{\omega,1/\xi}\|_0\big),
\end{aligned} \tag{4.29}
$$

with an A_r-consistent constant $c = c(\alpha, r, \varepsilon, \Sigma, \mathcal{A}_r(\omega))$ independent of $\lambda \in -\alpha + S_\varepsilon$ and $\xi \in \mathbb{R}^*$.

Proof: Since ξ enters in $(R_{\lambda,\xi})$ in a polynomial way, it is easy to prove that $a_j(\xi), b_j(\xi), j = 1, 2$, are Fréchet differentiable and their derivatives w_j, q_j solve the system

$$\begin{array}{rcl}(\lambda + \xi^2 - \Delta')w_j' + \nabla' q_j &=& -2\xi u_j' \\ (\lambda + \xi^2 - \Delta')w_{jn} + i\xi q_j &=& -2\xi u_{jn} - ip_j \\ \operatorname{div}' w_j' + i\xi w_{jn} &=& -iu_{jn},\end{array} \tag{4.30}$$

where $(u_1, p_1), (u_2, p_2)$ are the solutions to $(R_{\lambda,\xi})$ for $f \in L^r_\omega(\Sigma)$, $g = 0$ and $f = 0$, $g \in W^{1,r}_\omega(\Sigma)$, respectively.

We get from (4.30) and Theorem 4.4 for $j = 1, 2$,

$$\begin{aligned} &\|(\lambda + \alpha)\xi w_j, \xi\nabla'^2 w_j, \xi^3 w_j, \xi\nabla' q_j, \xi^2 q_j\|_{r,\omega} \\ &\quad \le c\big(\|\xi^2 u_j', \xi p_j, \nabla'\xi u_{jn}, \xi^2 u_{jn}\|_{r,\omega} + (|\lambda| + 1)\|i\xi u_{jn}; L^r_{m,\omega} + L^r_{\omega,1/\xi}\|_0\big) \\ &\quad \le c\big(\|\xi^2 u_j, \xi p_j, \nabla'\xi u_j\|_{r,\omega} + (|\lambda| + 1)\|u_j\|_{r,\omega}\big) \\ &\quad \le c\|u_j, (\lambda + \alpha + \xi^2)u_j, \sqrt{\lambda + \alpha + \xi^2}\nabla' u_j, \xi p_j\|_{r,\omega} \\ &\quad \le c\|(\lambda + \alpha + \xi^2)u_j, \sqrt{\lambda + \alpha + \xi^2}\nabla' u_j, \nabla'^2 u_j, \xi p_j\|_{r,\omega}, \end{aligned} \tag{4.31}$$

with an A_r-consistent constant $c = c(\alpha, r, \varepsilon, \Sigma, \mathcal{A}_r(\omega))$; here we used the fact that $\xi^2 + |\lambda + \alpha| \le c(\varepsilon, \alpha)|\lambda + \alpha + \xi^2|$ for all $\lambda \in -\alpha + S_\varepsilon, \xi \in \mathbb{R}$ and $\|u_j\|_{r,\omega} \le c(\mathcal{A}_r(\omega))\|\nabla'^2 u_j\|_{r,\omega}$ (see [14], Corollary 2.2). Thus Theorem 4.4 and (4.31) prove (4.28), (4.29). ■

Remark 4.6 The estimates (4.29) for the operator-valued multipliers a_2, b_2 will be used in a forthcoming paper [11] to obtain estimates for the generalized Stokes resolvent systems in an infinite cylinder of $\mathbb{R}^n$ with application to the Stokes resolvent systems on unbounded cylindrical domains with several outlets to infinity.

5 Proof of the Main Results

The proof of Theorem 2.1 is based on the theory of operator-valued Fourier multipliers. The classical Hörmander-Michlin theorem for scalar-valued multipliers for $L^q(\mathbb{R}^k)$, $q \in (1, \infty)$, $k \in \mathbb{N}$, extends to an operator-valued version for Bochner spaces $L^q(\mathbb{R}^k; X)$ provided that X is a *UMD space* and that the boundedness condition for the derivatives of the multipliers is strengthened to *$\mathcal{R}$-boundedness.*

Definition 5.1 *A Banach space X is called a UMD space if the Hilbert transform*

$$Hf(t) = -\frac{1}{\pi}\, PV \int \frac{f(s)}{t - s}\, ds \qquad \text{for } f \in \mathcal{S}(\mathbb{R}; X),$$

where $\mathcal{S}(\mathbb{R}; X)$ is the Schwartz space of all rapidly decreasing X-valued functions, extends to a bounded linear operator in $L^q(\mathbb{R}; X)$ for some $q \in (1, \infty)$.

It is well known that, if X is a *UMD* space, then the Hilbert transform is bounded in $L^q(\mathbb{R}; X)$ *for all* $q \in (1, \infty)$ (see e.g. [27], Theorem 1.3) and that weighted Lebesgue spaces $L^r_\omega(\Sigma), 1 < r < \infty$, $\omega \in A_r$, are *UMD* spaces.

Definition 5.2 *Let X, Y be Banach spaces. An operator family $\mathcal{T} \subset \mathcal{L}(X;Y)$ is called $\mathcal{R}$-bounded if there is a constant $c > 0$ such that for all $T_1, \ldots, T_N \in \mathcal{T}$, $x_1, \ldots, x_N \in X$ and $N \in \mathbb{N}$*

$$\Big\| \sum_{j=1}^{N} \varepsilon_j(s) T_j x_j \Big\|_{L^q(0,1;Y)} \le c \Big\| \sum_{j=1}^{N} \varepsilon_j(s) x_j \Big\|_{L^q(0,1;X)} \tag{5.1}$$

for some $q \in [1, \infty)$, where (ε_j) is any sequence of independent, symmetric $\{-1,1\}$-valued random variables on $[0,1]$. The smallest constant c for which (5.1) holds is denoted by $R_q(\mathcal{T})$, the $\mathcal{R}$-bound of $\mathcal{T}$.

Remark 5.3 (1) Due to *Kahane's inequality* ([4])

$$\Big\| \sum_{j=1}^{N} \varepsilon_j(s) x_j \Big\|_{L^{q_1}(0,1;X)} \le c(q_1, q_2, X) \Big\| \sum_{j=1}^{N} \varepsilon_j(s) x_j \Big\|_{L^{q_2}(0,1;X)}, \ \ 1 \le q_1, q_2 < \infty, \tag{5.2}$$

the inequality (5.1) holds *for all* $q \in [1, \infty)$ if it holds for some $q \in [1, \infty)$.

(2) If an operator family $\mathcal{T} \subset \mathcal{L}(L^r_\omega(\Sigma))$, $1 < r < \infty$, $\omega \in A_r(\mathbb{R}^{n-1})$, is $\mathcal{R}$-bounded, then $\mathcal{R}_{q_1}(\mathcal{T}) \le C \mathcal{R}_{q_2}(\mathcal{T})$ for all $q_1, q_2 \in [1, \infty)$ with a constant $C = C(q_1, q_2) > 0$ independent of ω. In fact, introducing the isometric isomorphism

$$I_\omega : L^r_\omega(\Sigma) \to L^r(\Sigma), \quad I_\omega f = f \omega^{1/r},$$

for all $T \in \mathcal{L}(L^r_\omega(\Sigma))$ we have $\tilde{T}_\omega = I_\omega T I_\omega^{-1} \in \mathcal{L}(L^r(\Sigma))$ and $\|T\|_{\mathcal{L}(L^r_\omega(\Sigma))} = \|\tilde{T}_\omega\|_{\mathcal{L}(L^r(\Sigma))}$. Then it is easily seen that $\tilde{\mathcal{T}}_\omega := \{I_\omega T I_\omega^{-1} : T \in \mathcal{T}\} \subset \mathcal{L}(L^r(\Sigma))$ is $\mathcal{R}$-bounded and $\mathcal{R}_q(\tilde{\mathcal{T}}_\omega) = \mathcal{R}_q(\mathcal{T})$ for all $q \in [1, \infty)$. Thus the assertion follows.

Definition 5.4 *(1) Let X be a Banach space and $(x_n)_{n=1}^\infty \subset X$. A series $\sum_{n=1}^\infty x_n$ is called unconditionally convergent if $\sum_{n=1}^\infty x_{\sigma(n)}$ is convergent in norm for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$.*

(2) A sequence of projections $(\Delta_j)_{j \in \mathbb{N}} \subset \mathcal{L}(X)$ is called a Schauder decomposition of a Banach space X if

$$\Delta_i \Delta_j = 0 \ \text{for all} \ i \ne j, \quad \sum_{j=1}^{\infty} \Delta_j x = x \ \text{for each} \ x \in X.$$

A Schauder decomposition $(\Delta_j)_{j \in \mathbb{N}}$ is called unconditional if the series $\sum_{j=1}^\infty \Delta_j x$ converges unconditionally for each $x \in X$.

Remark 5.5 (1) If $(\Delta_j)_{j \in \mathbb{N}}$ is an unconditional Schauder decomposition of a Banach space Y, then for each $p \in [1, \infty)$ there is a constant $c_\Delta = c_\Delta(p) > 0$ such that for all x_j in the range $\mathcal{R}(\Delta_j)$ of Δ_j the inequalities

$$c_\Delta^{-1} \Big\| \sum_{j=l}^{k} x_j \Big\|_Y \ \ \le \ \ \Big\| \sum_{j=l}^{k} \varepsilon_j(s) x_j \Big\|_{L^p(0,1;Y)} \le c_\Delta \Big\| \sum_{j=l}^{k} x_j \Big\|_Y \tag{5.3}$$

are valid for any sequence $(\varepsilon_j(s))$ of independent, symmetric $\{-1,1\}$-valued random variables defined on $(0,1)$ and for all $l \le k \in \mathbb{Z}$, see e.g. [3], (3.8).

(2) Let $Y = L^q(\mathbb{R}; L^r_\omega(\Sigma))$ and assume that each Δ_j commutes with the isomorphism I_ω introduced in Remark 5.3 (2). Then the constant c_Δ is easily seen to be independent of the weight ω.

(3) In the previous definitions and results the set of indices $\mathbb{N}$ may be replaced by $\mathbb{Z}$ without any further changes.

(4) Let X be a *UMD* space and $\chi_{[a,b)}$ denote the characteristic function for the interval $[a,b)$. Let $R_s = \mathcal{F}^{-1}\chi_{[s,\infty)}\mathcal{F}$ and

$$\Delta_j := R_{2^j} - R_{2^{j+1}},\ j \in \mathbb{Z}.$$

It is well known that the Riesz projection R_0 is bounded in $L^q(\mathbb{R}; X)$ and that the set $\{R_s - R_t :\ s,t \in \mathbb{R}\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^q(\mathbb{R}; X))$ for each $q \in (1,\infty)$. In particular, $\{\Delta_j :\ j \in \mathbb{Z}\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^q(\mathbb{R}; X))$ and an unconditional Schauder decomposition of $R_0L^q(\mathbb{R}; X)$, the image of $L^q(\mathbb{R}; X)$ by the Riesz projection R_0, see [3], proof of Theorem 3.19.

We recall an operator-valued Fourier multiplier theorem in Banach spaces. Let $\mathcal{D}_0(\mathbb{R}; X)$ denote the set of C^∞-functions $f : \mathbb{R} \to X$ with compact support in $\mathbb{R}^*$.

Theorem 5.6 ([3], Theorem 3.19, [31], Theorem 3.4) *Let X and Y be UMD spaces and $1 < q < \infty$. Let $M : \mathbb{R}^* \to \mathcal{L}(X,Y)$ be a differentiable function such that*

$$\mathcal{R}_q\big(\{M(t),\, tM'(t) :\ t \in \mathbb{R}^*\}\big) \le A.$$

Then the operator

$$Tf = \big(M(\cdot)\hat{f}(\cdot)\big)^\vee, \quad f \in \mathcal{D}_0(X),$$

extends to a bounded operator $T : L^q(\mathbb{R}; X) \to L^q(\mathbb{R}; Y)$ with operator norm $\|T\|_{\mathcal{L}(L^q(\mathbb{R};X);L^q(\mathbb{R};Y))} \le CA$ where $C > 0$ depends only on q, X and Y.

Remark 5.7 Let $\mathcal{X}$ be a *UMD*-space and $X = Y = L^q(\mathbb{R}; \mathcal{X})$. Checking the proof of [3], Theorem 3.19, one can see that the constant C in Theorem 5.6 equals

$$C = \mathcal{R}(\mathcal{P}) \cdot (c_\Delta)^2$$

where $\mathcal{R}(\mathcal{P})$ is the $\mathcal{R}$-bound of the operator family $\mathcal{P} = \{R_s - R_t :\ s,t \in \mathbb{R}\}$ in $\mathcal{L}(L^q(\mathbb{R}; \mathcal{X}))$ and c_Δ is the *unconditional constant* of the *Schauder decomposition* $\{\Delta_j :\ j \in \mathbb{Z}\}$ of the space $R_0L^q(\mathbb{R}; \mathcal{X})$; see [3], Section 3, for details. In particular, for $\mathcal{X} = L^r_\omega(\Sigma)$, $1 < r < \infty$, $\omega \in A_r$, using the isometry I_ω of Remark 5.3 (2), we get that the constants $\mathcal{R}(\mathcal{P})$, see Remark 5.3 (2), and c_Δ do not depend on the weight ω; concerning c_Δ we again use that I_ω commutes with each Δ_j.

Theorem 5.8 (Extrapolation Theorem) *Let $1 < r, s < \infty$, $\omega \in A_r(\mathbb{R}^{n-1})$ and $\Sigma \subset \mathbb{R}^{n-1}$ be an open set. Moreover let $\mathcal{T}$ be a family of linear operators with the property that there exists an A_s-consistent constant $C_\mathcal{T} = C_\mathcal{T}(\mathcal{A}_s(\nu)) > 0$ such that for all $\nu \in A_s$*

$$\|Tf\|_{s,\nu} \le C_\mathcal{T}\|f\|_{s,\nu}$$

for all $T \in \mathcal{T}$ and all $f \in L^s_\nu(\Sigma)$. Then every $T \in \mathcal{T}$ can be extended to $L^r_\omega(\Sigma)$ and $\mathcal{T}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^r_\omega(\Sigma))$ with an A_r-consistent $\mathcal{R}$-bound $c_{\mathcal{T}}(q,r,\mathcal{A}_r(\omega))$, i.e.,

$$\mathcal{R}_q(\mathcal{T}) \le c_{\mathcal{T}}(q,r,\mathcal{A}_r(\omega)) \quad \textit{for all} \quad q \in (1,\infty). \tag{5.4}$$

Proof: From the proof of [14], Theorem 4.3, it can be deduced that $\mathcal{T}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^r_\omega(\Sigma))$ and that (5.4) is satisfied for $q = r$. Then Remark 5.3 yields (5.4) for every $1 < q < \infty$. ■

Now we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1: Let us define u, p in the cylinder $\Omega = \Sigma \times \mathbb{R}$ by

$$u(x) = \mathcal{F}^{-1}(a_1 \hat{f})(x), \quad p(x) = \mathcal{F}^{-1}(b_1 \hat{f})(x),$$

where a_1, b_1 are the operator-valued multiplier functions defined in (4.26). We will show that (u,p) is the unique solution to (R_λ) with $g = 0$ satisfying

$$(u,p) \in \big(W^{2;q,r}_\omega(\Omega) \cap W^{1;q,r}_{0,\omega}(\Omega)\big) \times \widehat{W}^{1;q,r}_\omega(\Omega) \tag{5.5}$$

and the estimate (2.1). Obviously, (u,p) solves the resolvent problem (R_λ) with $g = 0$. For $\xi \in \mathbb{R}^*$ define $m_\lambda(\xi) : L^r_\omega(\Sigma) \to L^r_\omega(\Sigma)$ by

$$m_\lambda(\xi)f := \big((\lambda+\alpha)a_1(\xi)\hat{f}, \xi\nabla' a_1(\xi)\hat{f}, \nabla'^2 a_1(\xi)\hat{f}, \xi^2 a_1(\xi)\hat{f}, \nabla' b_1(\xi)\hat{f}, \xi b_1(\xi)\hat{f}\big).$$

Theorem 4.4 and Corollary 4.5 show that the operator family $\{m_\lambda(\xi), \xi m'_\lambda(\xi) : \xi \in \mathbb{R}^*\}$ satisfies the assumptions of Theorem 5.8, e.g., with $s = r$. Therefore, this operator family is $\mathcal{R}$-bounded in $\mathcal{L}(L^r_\omega(\Sigma))$; to be more precise,

$$\mathcal{R}_q\big(\{m_\lambda(\xi), \xi m'_\lambda(\xi) : \xi \in \mathbb{R}^*\}\big) \le c(q,r,\alpha,\varepsilon,\Sigma,\mathcal{A}_r(\omega)) < \infty.$$

Hence Theorem 5.6 and Remark 5.7 imply that

$$\|(m_\lambda \hat{f})^\vee\|_{L^q(L^r_\omega)} \le C\|f\|_{L^q(L^r_\omega)}$$

with an A_r-consistent constant $C = C(q,r,\alpha,\varepsilon,\Sigma,\mathcal{A}_r(\omega)) > 0$ independent of the resolvent parameter $\lambda \in -\alpha + S_\varepsilon$. Note that, due to the definition of the multiplier $m_\lambda(\xi)$, we have $(\lambda+\alpha)u, \nabla^2 u, \nabla p \in L^q(L^r_\omega)$ and

$$\|(\lambda+\alpha)u, \nabla^2 u, \nabla p\|_{L^q(L^r_\omega)} \le \|(m\hat{f})^\vee\|_{L^q(L^r_\omega)}.$$

Thus the existence of a solution satisfying (2.1) is proved.

For the uniqueness of solutions let $(u,p) \in \big(W^{2;q,r}_\omega(\Omega) \cap W^{1;q,r}_{0,\omega}(\Omega)\big) \times \widehat{W}^{1;q,r}_\omega(\Omega)$ satisfy (R_λ) with $f = 0$, $g = 0$. Fix $h \in L^{q'}(L^{r'}_{\omega'})$ arbitrarily and let $(v,z) \in \big(W^{2;q',r'}_{\omega'}(\Omega) \cap W^{1;q',r'}_{0,\omega'}(\Omega) \cap L^{q'}(L^{r'}_{\omega'})_\sigma\big) \times \widehat{W}^{1;q',r'}_{\omega'}(\Omega)$ be a solution to $(R_{\bar{\lambda}})$ with right-hand side h. Then using the denseness of $C^\infty_{0,\sigma}(\Omega)$ in $W^{1;q',r'}_{0,\omega}(\Omega) \cap L^{q'}(L^{r'}_{\omega'})_\sigma$ we get

$$0 = (\lambda u - \Delta u + \nabla p, v) = (u, \bar{\lambda}v - \Delta v + \nabla z) = (u,h)_{L^q(L^r_\omega), L^{q'}(L^{r'}_{\omega'})}$$

yielding $u = 0$, and consequently, $\nabla p = 0$. Now the proof of Theorem 2.1 is complete. ■

Proof of Corollary 2.2: Defining the Stokes operator $A = A_{q,r;\omega}$ by (2.2), due to the Helmholtz decomposition of the space $L^q(L^r_\omega)$ on the cylinder Ω, see [8], we get that for $F \in L^q(L^r_\omega)_\sigma$ the solvability of the equation

$$(\lambda + A)u = F \quad \text{in} \quad L^q(L^r_\omega)_\sigma \tag{5.6}$$

is equivalent to the solvability of (R_λ) with right-hand side $f \equiv F, g \equiv 0$. By virtue of Theorem 2.1 for every $\lambda \in -\alpha + S_\varepsilon$ there exists a unique solution $u = (\lambda + A)^{-1}F \in D(A)$ to (5.6) satisfying the estimate

$$\|(\lambda + \alpha)u\|_{L^q(L^r_\omega)_\sigma} \le C\|F\|_{L^q(L^r_\omega)_\sigma}$$

with $C = C(q, r, \alpha, \varepsilon, \Sigma, \mathcal{A}_r(\omega))$ independent of λ; hence (2.3) is proved. Then (2.4) is a direct consequence of (2.3) using semigroup theory. ■

Proof of Theorem 2.3: We shall show that the operator family

$$\mathcal{T} = \{\lambda(\lambda + A_{q,r;\omega})^{-1} : \ \lambda \in \mathrm{i}\mathbb{R}\}$$

is $\mathcal{R}$-bounded in $\mathcal{L}(L^q(L^r_\omega))$. To this end, for $\xi \in \mathbb{R}^*$ and $\lambda \in S_\varepsilon$, let $m_\lambda(\xi) := \lambda a_1(\xi)$ where $a_1(\xi)$ is the solution operator for $(R_{\lambda,\xi})$ with $g = 0$ defined by (4.26). Then $\lambda(\lambda + A_{q,r;\omega})^{-1}f = (m_\lambda(\xi)\hat{f})^\vee$ for $f \in \mathcal{S}(\mathbb{R}; L^r_\omega(\Sigma)_\sigma)$. In view of Definition 5.2 and the denseness of $\mathcal{S}(\mathbb{R}; L^r_\omega(\Sigma)_\sigma)$ in $L^q(\mathbb{R}; L^r_\omega(\Sigma)_\sigma)$ we will prove that there is a constant $C > 0$ such that

$$\|\sum_{i=1}^N \varepsilon_i (m_{\lambda_i}\hat{f}_i)^\vee\|_{L^q(0,1;L^q(\mathbb{R}:L^r_\omega(\Sigma)))} \le C\|\sum_{i=1}^N \varepsilon_i f_i\|_{L^q(0,1;L^q(\mathbb{R}:L^r_\omega(\Sigma)))} \tag{5.7}$$

for any independent, symmetric and $\{-1, 1\}$-valued random variables $(\varepsilon_i(s))$ defined on $(0, 1)$, for all $(\lambda_i) \subset \mathrm{i}\mathbb{R}$ and $(f_i) \subset \mathcal{S}(\mathbb{R}; L^r_\omega(\Sigma)_\sigma)$. Without loss of generality we may assume that $(f_i) \subset Y := R_0 L^q(\mathbb{R}; L^r_\omega(\Sigma)_\sigma)$, since R_0 is continuous in $L^q(\mathbb{R}; L^r_\omega(\Sigma)_\sigma)$ and

$$f_i(x', x_n) = (\chi_{[0,\infty)}\hat{f}_i(\xi))^\vee(x', x_n) + (\chi_{[0,\infty)}\hat{f}_i(-\xi))^\vee(x', -x_n).$$

Therefore, we shall show that $\mathcal{T}$ is $\mathcal{R}$-bounded in $\mathcal{L}(Y)$; note that, if $\mathrm{supp}\hat{f} \subset [0, \infty)$, then $\mathrm{supp}(m_\lambda \hat{f}) \subset [0, \infty)$ as well.

Obviously $m_\lambda(\xi) = m_\lambda(2^j) + \int_{2^j}^\xi m'_\lambda(\tau)\,d\tau$ for $\xi \in [2^j, 2^{j+1})$, $j \in \mathbb{Z}$, and $\big(m_\lambda(2^j)\widehat{\Delta_j f}\big)^\vee = m_\lambda(2^j)\Delta_j f$ for $f \in \mathcal{S}(\mathbb{R}; L^r_\omega(\Sigma)_\sigma)$. Furthermore,

$$\begin{aligned}\Big(\int_{2^j}^\xi m'_\lambda(\tau)\,d\tau\,\widehat{\Delta_j f}(\xi)\Big)^\vee &= \Big(\int_{2^j}^{2^{j+1}} m'_\lambda(\tau)\chi_{[2^j,\xi)}(\tau)\widehat{\Delta_j f}(\xi)\,d\tau\Big)^\vee \\ &= \Big(\int_0^1 2^j m'_\lambda(2^j(1+t))\chi_{[2^j,\xi)}(2^j(1+t))\chi_{[2^j,2^{j+1})}(\xi)\hat{f}(\xi)\,dt\Big)^\vee \\ &= \int_0^1 2^j m'_\lambda(2^j(1+t))B_{j,t}\Delta_j f\,dt.\end{aligned}$$

where $B_{j,t} = R_{2^j(1+t)} - R_{2^{j+1}}$. Thus we get

$$\begin{aligned}
\big(m_\lambda(\xi)\hat{f}(\xi)\big)^\vee &= \sum_{j\in\mathbb{Z}} \Big(\big(m_\lambda(2^j) + \int_{2^j}^{\xi} m_\lambda'(\tau)\,d\tau\big)\,\widehat{\Delta_j f}\Big)^\vee \\
&= \sum_{j\in\mathbb{Z}} \big(m_\lambda(2^j)\widehat{\Delta_j f}\big)^\vee + \sum_{j\in\mathbb{Z}} \Big(\int_{2^j}^{\xi} m_\lambda'(\tau)\,d\tau\,\widehat{\Delta_j f}\Big)^\vee \\
&= \sum_{j\in\mathbb{Z}} m_\lambda(2^j)\Delta_j f + \sum_{j\in\mathbb{Z}} \int_0^1 2^j m_\lambda'(2^j(1+t))B_{j,t}\Delta_j f\,dt.
\end{aligned} \tag{5.8}$$

First let us prove

$$\Big\|\sum_{i=1}^N \varepsilon_i(s)\sum_{j\in\mathbb{Z}} m_{\lambda_i}(2^j)\Delta_j f_i\Big\|_{L^q(0,1;Y)} \le C\Big\|\sum_{i=1}^N \varepsilon_i(s) f_i\Big\|_{L^q(0,1;Y)}. \tag{5.9}$$

Note that the operator $m_{\lambda_i}(2^j)$ commutes with Δ_j, $j\in\mathbb{Z}$; hence, for almost all $s\in(0,1)$, the sum $\sum_{i=1}^N \varepsilon_i(s)m_{\lambda_i}(2^j)\Delta_j f_i$ belongs to the range of Δ_j. Therefore, for any $l,k\in\mathbb{Z}$ we get by (5.3) that

$$\begin{aligned}
&\Big\|\sum_{i=1}^N \varepsilon_i \sum_{j=l}^k m_{\lambda_i}(2^j)\Delta_j f_i\Big\|_{L^q(0,1;Y)} \\
&\qquad = \Big(\int_0^1 \Big\|\sum_{j=l}^k\sum_{i=1}^N \varepsilon_i(s)m_{\lambda_i}(2^j)\Delta_j f_i\Big\|_Y^q\,ds\Big)^{1/q} \\
&\qquad \le c_\Delta\Big(\int_0^1\int_0^1 \Big\|\sum_{j=l}^k \varepsilon_j(\tau)\sum_{i=1}^N \varepsilon_i(s)m_{\lambda_i}(2^j)\Delta_j f_i\Big\|_Y^q\,d\tau\,ds\Big)^{1/q} \\
&\qquad = c_\Delta\Big\|\sum_{i=1}^N\sum_{j=l}^k \varepsilon_{ij}(s,\tau)m_{\lambda_i}(2^j)\Delta_j f_i\Big\|_{L^q((0,1)^2;Y)}
\end{aligned} \tag{5.10}$$

where $\varepsilon_{ij}(s,\tau)=\varepsilon_i(s)\varepsilon_j(\tau)$; note that $(\varepsilon_{ij})_{i,j\in\mathbb{Z}}$ is a sequence of independent, symmetric and $\{-1,1\}$-valued random variables defined on $(0,1)\times(0,1)$. Furthermore, due to Theorem 4.4, the operator family $\{m_\lambda(\xi) : \lambda\in i\mathbb{R},\ \xi\in\mathbb{R}^*\}\subset\mathcal{L}(L^r_\omega(\Sigma))$ is uniformly bounded by an A_r-consistent constant, and hence it is $\mathcal{R}$-bounded by Theorem 5.8. Therefore, using Fubini's theorem and (5.3), we proceed in (5.10) as follows:

$$\begin{aligned}
&= c_\Delta\Big\|\sum_{i=1}^N\sum_{j=l}^k \varepsilon_{ij}(s,\tau)m_{\lambda_i}(2^j)\Delta_j f_i\Big\|_{L^q(\mathbb{R};L^q((0,1)^2;L^r_\omega(\Sigma)))} \\
&\le Cc_\Delta\Big\|\sum_{i=1}^N\sum_{j=l}^k \varepsilon_{ij}(s,\tau)\Delta_j f_i\Big\|_{L^q(\mathbb{R};L^q((0,1)^2;L^r_\omega(\Sigma)))} \\
&= Cc_\Delta\Big\|\sum_{i=1}^N\sum_{j=l}^k \varepsilon_{ij}(s,\tau)\Delta_j f_i\Big\|_{L^q((0,1)^2;Y)} \le Cc_\Delta^2\Big\|\sum_{i=1}^N \varepsilon_i\sum_{j=l}^k \Delta_j f_i\Big\|_{L^q(0,1;Y)}.
\end{aligned} \tag{5.11}$$

Since $\{\sum_{j=l}^k \Delta_j \;:\; l,k \in \mathbb{Z}\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(Y)$ and (Δ_j) is a Schauder decomposition of Y, we see by Lebesgue's theorem that the right-hand side of (5.11) converges to 0 as either $l,k \to \infty$ or $l,k \to -\infty$. Thus, by (5.10), (5.11), the series $\sum_{i=1}^N \varepsilon_i(s) \sum_{j\in\mathbb{Z}} m_{\lambda_i}(2^j)\Delta_j f_i$ converges in $L^q(0,1;Y)$, and (5.9) holds.

Next let us show that

$$\Big\| \sum_{i=1}^N \varepsilon_i(s) \sum_{j\in\mathbb{Z}} \int_0^1 2^j m'_{\lambda_i}(2^j(1+t)) B_{j,t}\Delta_j f_i \, dt\Big\|_{L^q(0,1;Y)} \le C\Big\| \sum_{i=1}^N \varepsilon_i(s) f_i\Big\|_{L^q(0,1;Y)}. \tag{5.12}$$

Using the same argument as in the proof of (5.9) and the $\mathcal{R}$-boundedness of the operator families $\{B_{j,t} :\; j\in\mathbb{Z}, t\in(0,1)\} \subset \mathcal{L}(Y)$ and $\{2^j(1+t)m'_\lambda(2^j(1+t)) :\; \lambda \in \mathrm{i}\mathbb{R}, j\in\mathbb{Z}, t\in(0,1)\} \subset \mathcal{L}(L^r_\omega(\Sigma))$, see Corollary 4.5, we have

$$\begin{aligned}
&\Big\| \sum_{i=1}^N \varepsilon_i(s) \sum_{j=l}^k \int_0^1 2^j m'_{\lambda_i}(2^j(1+t)) B_{j,t}\Delta_j f_i \, dt\Big\|_{L^q(0,1;Y)} \\
&\qquad \le \int_0^1 \Big\| \sum_{i=1}^N \varepsilon_i(s) \sum_{j=l}^k 2^j m'_{\lambda_i}(2^j(1+t)) B_{j,t}\Delta_j f_i\Big\|_{L^q(0,1;Y)} \, dt \\
&\qquad \le c_\Delta \int_0^1 \Big\| \sum_{i=1}^N \sum_{j=l}^k \varepsilon_{ij}(s,\tau) 2^j m'_{\lambda_i}(2^j(1+t)) B_{j,t}\Delta_j f_i\Big\|_{L^q((0,1)^2;Y)} \, dt \\
&\qquad \le c_\Delta \int_0^1 \Big\| \sum_{i=1}^N \sum_{j=l}^k \varepsilon_{ij}(s,\tau) 2^j (1+t) m'_{\lambda_i}(2^j(1+t)) \Delta_j f_i\Big\|_{L^q((0,1)^2;Y)} \, dt \\
&\qquad \le C c_\Delta^2 \Big\| \sum_{i=1}^N \varepsilon_i(s) \sum_{j=l}^k \Delta_j f_i\Big\|_{L^q((0,1);Y)}
\end{aligned}$$

for all $l,k\in\mathbb{Z}$. Thus (5.12) is proved.

By (5.9), (5.12) we conclude that the operator family $\mathcal{T} = \{\lambda(\lambda + A_{q,r;\omega})^{-1} :\; \lambda \in \mathrm{i}\mathbb{R}\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^q(L^r_\omega))$. Then, by [31], Corollary 4.4, for each $f \in L^p(\mathbb{R}_+; L^q(L^r_\omega)_\sigma)$, $1<p<\infty$, the mild solution u to the system

$$u_t + A_{q,r;\omega} u = f, \quad u(0) = 0 \tag{5.13}$$

belongs to $L^p(\mathbb{R}_+; L^q(L^r_\omega)_\sigma) \cap L^p(\mathbb{R}_+; D(A_{q,r;\omega}))$ and satisfies the estimate

$$\|u_t, A_{q,r;\omega}u\|_{L^p(\mathbb{R}_+;L^q(L^r_\omega)_\sigma)} \le C\|f\|_{L^p(\mathbb{R}_+;L^q(L^r_\omega)_\sigma)}.$$

Furthermore, (2.3) with $\lambda = 0$ implies that even u satisfies this inequality. If $f \in L^p(\mathbb{R}_+; L^q(L^r_\omega))$, let u be the solution of (5.13) with f replaced by Pf, where $P = P_{q,r;\omega}$ denotes the Helmholtz projection in $L^p(\mathbb{R}_+; L^q(L^r_\omega))$, and define p by $\nabla p = (I-P)(f - u_t + \Delta u)$. By (2.1) with $\lambda = 0$ and the boundedness of P we get (2.7). Finally, assume $e^{\alpha t} f \in L^p(\mathbb{R}_+; L^q(L^r_\omega)_\sigma)$ for some $\alpha \in (0,\alpha_0)$ and let v be the solution of the system $v_t + (A-\alpha)v = e^{\alpha t} f$, $v(0) = 0$. Obviously, replacing A by $A - \alpha$ in the previous arguments, v is easily seen to satisfy estimate (2.6). Then $u(t) = e^{-\alpha t} v(t)$ solves (5.13) and satisfies (2.8). In each case the constant C depends only on $\mathcal{A}_r(\omega)$ due to Remark 5.7.

The proof of Theorem 2.3 is complete. ∎

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