On the Use of Hypotheses in Cumulative Type Theory (Enlarged Version)

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ABSTRACT: Given a language of ramified cumulative type theory as introduced in (Zahn 2004). We shall construct and investigate an extension, $\mathcal{L}$, of it, which is a language of the same sort, but also containes sentences which express that certain sentences of $\mathcal{L}$ are deducible from others (hypotheses) by given rules. To this we introduce 'names' of terms and formulas of $\mathcal{L}$ and include them in $\mathcal{L}$. So in $\mathcal{L}$ we can not only use but also 'speak about' sentences of that language. Especially, by means of first order sentences we can speak about higher order sentences. Despite this possibility of 'reduction' of order, all sentences of $\mathcal{L}$ are non-circular. The considered deducibility-relations of sentences from others correspond to systems of labelled modal logic of types K_4 and G.

Motivation: In everyday speech and in empirical sciences one does necessarily not only assert established facts but also uses universal hypotheses or conjectures, which often do not even get cited. If A is the conjunction of all current hypotheses, we could use (assert) any sentence B as short for $A \to B$. But as soon as A becomes rejected, it becomes obviously unserviceable to assert sentences of the form $A \to B$ (or abbreviations for them). Accordingly, if A contains (probably) untrue hypotheses (such as simplifications of conjectures) we can instead of $A \to B$ better use the statement that B has been *deduced* from A and already justly asserted sentences of a given class, K, by the rules of classical logic (e.g.). This statement reminds of *necessity*, say "B is *necessary* with respect to (A, K)". (The set K should be chosen considering particular purposes. It might be a set of physical or medical sentences, e.g., that can possibly be verified.) Then the sets S_i of all sentences that are deducible at successive times t_i $(i = 0, 1, 2, \ldots)$ form a monotonic increasing sequence $S_0, S_1, S_2, \ldots$. (The corresponding concept of necessity means that necessary is what is deducible from given hypotheses and particular established facts by rules that are admissible by linguistic conventions.)

1 A Language of Cumulative Type Theory

At first we incompletely sketch the construction of the language of a particular ramified cumulative type theory, which has been investigated in (Zahn 2004).

Assume that we already dispose of certain **elementary formulas** and terms, which are said to be **original terms** (of order 0). All variables that occur in those formulas or terms are also said to be of order 0. Let

$\mathcal{V}_0$ = set of all variables of order 0
$\mathcal{T}_{\mathrm{Or}}$ = set of all original terms, $\mathcal{V}_0 \subset \mathcal{T}_{\mathrm{Or}}$
$\mathcal{E}$ = set of all elementary formulas (to be considered).
$\mathcal{V}_0$ is permitted to contain variables of several sorts. (Of course, $\mathcal{V}_0$ is supposed to contain denumerably many variables of every of those sorts. Also $\mathcal{T}_{\mathrm{Or}}$ and $\mathcal{E}$ are supposed to satisfy certain conditions.) Let **constants / sentences** be closed terms / formulas, respectively (i.e. without free occurring variables).

In the following, certain properties and relations-in-intension will be given by constants. They are simply said to be (particular) *sets*. As such sets we shall introduce sets of order 1, whose elements are (tuples of) constants of order 0 (or objects denoted by them), sets of order 2, whose elements are (tuples of) constants of order 0 or 1, etc. So a set of order n contains only elements that have orders $< n$. However, a set of order n will also be said to have any order larger than n.

To this end we shall introduce the following sets of higher order terms and formulas:
$\mathcal{T}_n$ = set of all terms of order n,
$\mathcal{F}_n$ = set of all formulas of order n.
Here and in the following, m, n range over (signs of) ordinal numbers belonging to a given set Ω with $\mathbb{N} = \{0, 0', 0'', 0''', \ldots\} \subseteq \Omega \subseteq \mathcal{C}_0$ (defined as follows). We define
$\mathcal{C}_n \rightleftharpoons$ set of all constants of order n, i.e. belonging to $\mathcal{T}_n$,
$\overline{\mathcal{C}}_n \rightleftharpoons \bigcup_{j \in \mathbb{N}^+} \mathcal{C}_n^j$,
which is the set of all j-tuples $(c_1, \ldots, c_j)$ of constants $c_i \in \mathcal{C}_n$ with arbitrary length $j \in \mathbb{N}^+ \rightleftharpoons \mathbb{N} \setminus \{0\}$.

We presuppose that we have stipulated that every element of $\mathcal{V}_0$ is to be used as a variable for certain (or all) elements of $\mathcal{C}_0$. Let also be given two disjunct denumerable sets $\mathcal{V}$ and $\overline{\mathcal{V}}$ of 'new' variables which do not occur in elements of $\mathcal{T}_{\mathrm{Or}} \cup \mathcal{E}$. We shall use the elements of $\mathcal{V}$ as variables for elements of $\mathcal{C} \rightleftharpoons \bigcup_{n \in \Omega} \mathcal{C}_n$, i.e. for constants of arbitrary order, and the elements of $\overline{\mathcal{V}}$ as variables for elements of $\overline{\mathcal{C}} \rightleftharpoons \bigcup_{n \in \Omega} \overline{\mathcal{C}}_n$, i.e. for arbitrary tuples of constants. - Moreover, let
$\overline{\mathcal{T}}_n \rightleftharpoons \bigcup_{j \in \mathbb{N}^+} \mathcal{T}_n^j \cup \overline{\mathcal{V}}$.
So $\overline{\mathcal{C}}_n$ is the set of all closed elements of $\overline{\mathcal{T}}_n$.

We shall also use the following abbreviations: $\mathcal{F} \rightleftharpoons \bigcup_{n \in \Omega} \mathcal{F}_n$, $\mathcal{T} \rightleftharpoons \bigcup_{n \in \Omega} \mathcal{T}_n$, $\overline{\mathcal{T}} \rightleftharpoons \bigcup_{n \in \Omega} \overline{\mathcal{T}}_n$, and $\mathcal{A} \rightleftharpoons$ set of all sentences of $\mathcal{F}$.

For the present, we let the letters $x, x_1, x_2, \ldots$ range over variables of $\mathcal{V}_0 \cup \mathcal{V}$, and $\overline{x}, \overline{y}$ over variables of $\overline{\mathcal{V}}$. In the object language we shall use the signs $C_m, \overline{C}_m$, and ε instead of $\mathcal{C}_m, \overline{\mathcal{C}}_m$, and $\in$, respectively.

All elements of $\mathcal{C}_n \setminus \mathcal{C}_0$ are to be introduced as (signs of) subsets of $\bigcup_{m<n} \overline{\mathcal{C}}_m$. A constant of the form $\{\overline{x} \;\varepsilon\; \overline{C}_m : A(\overline{x})\}$ is to denote the set of all elements $\overline{c} \in \overline{\mathcal{C}}_m$ satisfying $A(\overline{c})$. A sentence of the form $\exists x \;\varepsilon\; C_m. A(x)$ is to mean that there exists a value $c \in \mathcal{C}_m$ of x satisfying $A(c)$. By this means, we can define j-ary relations ($j \in \mathbb{N}^+$) thus:

$$\{(x_1,\ldots,x_j)\ \varepsilon\ C_m^j\colon A(x_1,\ldots,x_j)\} \rightleftharpoons$$
$$\{\overline{x}\ \varepsilon\ \overline{C}_m\colon \exists x_1\ \varepsilon\ C_m.\ \ldots \exists x_j\ \varepsilon\ C_m.\ (\overline{x} =_m (x_1,\ldots,x_j) \wedge A(x_1,\ldots,x_j))\}.$$

(To this end, the sign '$=_m$' must previously be introduced suitably.) - So we at first demand that

$$\begin{array}{rll} t \in \mathcal{T}_n & \text{if} & t \in \mathcal{T}_{\mathrm{Or}} \cup \mathcal{V}, \\ \{\overline{x}\ \varepsilon\ \overline{C}_m\colon F\} \in \mathcal{T}_n & \text{if} & F \in \mathcal{F}_n,\ m < n, \\ E \in \mathcal{F}_n & \text{if} & E \in \mathcal{E}, \\ (F \wedge G), (F \vee G) \in \mathcal{F}_n & \text{if} & F, G \in \mathcal{F}_n, \\ (\neg F) \in \mathcal{F}_n & \text{if} & F \in \mathcal{F}_n, \\ (\exists x\ \varepsilon\ C_m.\ F) \in \mathcal{F}_n & \text{if} & F \in \mathcal{F}_n,\ m < n, \\ (\overline{s}\ \varepsilon\ t) \in \mathcal{F}_n & \text{if} & \overline{s} \in \overline{\mathcal{T}}_n,\ t \in \mathcal{T}_n. \end{array}$$

Note that we need not deal with complicated types that include information about 'arities' of relations. So we may simply identify types with orders.

The latter and certain subsequently adduced 'demands' can be formulated as formal rules to construct terms and formulas of order n. But we need also 'semantical' stipulations. Accordingly, in (Zahn 2004) is also introduced an assertion game, which contains certain '*primary rules*' to restrict assertions of sentences of arbitrary order. All inference rules of classical logic can be shown to be admissible in the '*classical game*' of assertion which is given by the agreement that a sentence may be asserted in this game if and only if the assertion of its double negation would not violate a primary rule. - Note that, for purposes of classical reasoning, the particles $\rightarrow$, $\leftrightarrow$, and $\forall$ can be defined by means of $\wedge, \neg$, and $\exists$.

For mathematical purposes we want also to dispose of sequences R of relations $R(0), R(1), R(2), \ldots \in \mathcal{C}_n$ satisfying

$$(\underline{c}, k)\ \varepsilon\ R(l) \leftrightarrow (\underline{c})\ \varepsilon\ \overline{C}_m\ \wedge\ k < l\ \wedge\ A((\underline{c}), k, R(k))$$

for all tuples $(\underline{c}) \equiv (c_1,\ldots,c_j)$ of constants and all $k, l \in \Omega$, if any formula $A(\overline{x}, \mu, z) \in \mathcal{F}_n$ and any ordinal $m < n$ are given. (We write '$\equiv$' to denote the literal equality of strings of symbols.) By this 'recursive characterization', $R(l)$ depends upon the relations $R(k)$ with numbers $k < l$ only. - We designate R by $(\mathrm{J}\overline{x}\ \varepsilon\ \overline{C}_m, \mu, z : A(\overline{x}, \mu, z))$. Accordingly, we demand:

$$(\mathrm{J}\overline{x}\ \varepsilon\ \overline{C}_m, \mu, z\colon F)(q) \in \mathcal{T}_n \quad \text{if} \quad F \in \mathcal{F}_n,\ q \in \mathcal{T}(\Omega),\ m < n,\ \mu \in \mathcal{V}(\Omega), z \in \mathcal{V}$$

where $\mathcal{T}(\Omega)$ ($\subseteq \mathcal{T}_{\mathrm{Or}}$) is a given set of terms whose substitution instances are elements of Ω, and $\mathcal{V}(\Omega) = \mathcal{V}_0 \cap \mathcal{T}(\Omega)$ is a set of variables for elements of Ω. ('J' is an 'induction operator') - Then it can be shown that if two formulas $A(\overline{x}), B(\overline{x}, \mu, z) \in \mathcal{F}_n$ and an order $m < n$ are given, there also exists a term $S_\nu \in \mathcal{T}_n$ such that $S_0, S_1, S_2, \ldots \in \mathcal{C}_n$ is a sequence of relations satisfying

$$\begin{array}{rcl} \overline{c}\ \varepsilon\ S_0 & \leftrightarrow & \overline{c}\ \varepsilon\ \overline{C}_m\ \wedge\ A(\overline{c}) \\ \overline{c}\ \varepsilon\ S_{k+1} & \leftrightarrow & \overline{c}\ \varepsilon\ \overline{C}_m\ \wedge\ B(\overline{c}, k, S_k) \end{array}$$

for all $\overline{c} \in \overline{\mathcal{C}}$ and all $k \in \mathbb{N}$.

We want to introduce equations $x=y$ such that all formulas $A(x)$ of arbitrary order are invariant under $(=)$, i.e. satisfy $c{=}d \wedge A(c) \rightarrow A(d)$ for all constants c, d. To this end, equal constants must especially have the same orders, and equal sets must contain the same elements:

$$\begin{aligned} c = d \quad &\rightarrow \quad \forall \mu \,\varepsilon\, C_0.\, (c \,\varepsilon\, C_\mu \;\leftrightarrow\; d \,\varepsilon\, C_\mu) \\ c = d \;\wedge\; \neg\, (c \,\varepsilon\, C_0) \quad &\rightarrow \quad c \subseteq d \;\wedge\; d \subseteq c \end{aligned}$$

where $\mu \in \mathcal{V}(\Omega)$ (again), and $c \subseteq d$ means that c is a subset of d (see below). Since the formulas $c \,\varepsilon\, C_\mu$ and $c \subseteq d$ should belong to the object language considered, we demand and define the following (where $\exists \overline{x} \,\varepsilon\, t.\, F$ is to be read as "For some $\overline{x}$, $\overline{x} \,\varepsilon\, t$ and F"):

$$\begin{aligned} (t \,\varepsilon\, C_q) \in \mathcal{F}_n \quad &\text{if} \quad t \in \mathcal{T}_n,\; q \in \mathcal{T}(\Omega) \\ (\exists \overline{x} \,\varepsilon\, t.\, F) \in \mathcal{F}_n \quad &\text{if} \quad t \in \mathcal{T}_n,\; F \in \mathcal{F}_n \\ \forall \overline{x} \,\varepsilon\, s.\, F \quad &\rightleftharpoons \quad \neg\, \exists \overline{x} \,\varepsilon\, s.\, \neg F \\ s \subseteq t \quad &\rightleftharpoons \quad \forall \overline{x} \,\varepsilon\, s.\, \overline{x} \,\varepsilon\, t \;\wedge\; \neg\, (s \,\varepsilon\, C_0) \;\wedge\; \neg\, (t \,\varepsilon\, C_0). \end{aligned}$$

Notice, however, that if q (is or) contains a variable, we do not rank C_q with the terms of $\mathcal{T}$.

Now we presuppose: Let $(=_0)$ be an equivalence relation on $\mathcal{C}_0$ (which has already been introduced and is suitable for certain purposes). Assume that all terms of $\mathcal{T}_{\mathrm{Or}}$ and all formulas of $\mathcal{E}$ are invariant under $(=_0)$. For terms s, t of any order we define

$$\begin{aligned} s \sim t \quad &\rightleftharpoons \quad \forall \mu \,\varepsilon\, C_0.\, (s \,\varepsilon\, C_\mu \;\leftrightarrow\; t \,\varepsilon\, C_\mu) \\ s = t \quad &\rightleftharpoons \quad s =_0 t \;\vee\; (s \subseteq t \;\wedge\; t \subseteq s \;\wedge\; s \sim t). \end{aligned}$$

Of course, we demand that

$$(s =_0 t) \in \mathcal{F}_n \quad \text{if} \quad s, t \in \mathcal{T}_n.$$

Then it can be shown that all formulas of $\mathcal{F}$ are invariant under $(=)$.

The 'type-free' relations $(\subseteq)$, $(\sim)$, and $(=)$ are definable in our object language, but they are neither elements of $\mathcal{C}$ nor elements of elements of $\mathcal{C}$.

Given a formula $A(x)$, a tuple $\overline{c} \equiv (c_1, \ldots, c_j) \in \overline{\mathcal{C}}_m$ of constants, and some $i = 1, \ldots, j$. Then $A(c_i)$ means that the i^{th} component of $\overline{c}$ satisfies $A(x)$. Since our object language also contains variables $\overline{y}$ for such tuples $\overline{c}$ of constants, we postulate, in addition, that the object language contains a formula expressing that the i^{th} component of any given value of $\overline{y}$ belongs to $\mathcal{C}_m$ and satisfies $A(x)$. For that formula we take $\exists x \,\varepsilon\, \pi_m(\overline{y}, i).\, A(x)$ (with π for "projection"). Generalizing we demand

$$(\exists x \,\varepsilon\, \pi_m(\overline{s}, p).\, F) \in \mathcal{F}_n \;\text{ if }\; m < n,\; \overline{s} \,\varepsilon\, \overline{\mathcal{T}}_n,\; p \in \mathcal{T}(\mathbb{N}),\; F \in \mathcal{F}_n$$

where $\mathcal{T}(\mathbb{N})$ ($\subseteq \mathcal{T}_{\rm or}$) is a given set of terms (inclusive of variables) whose substitution instances are elements of $\mathbb{N}$. Then all sentences of the form

$$\exists x \,\varepsilon\, \pi_m((c_1, \ldots, c_j), i).\, A(x) \;\leftrightarrow\; c_i \,\varepsilon\, C_m \wedge A(c_i)$$

($i = 1, \ldots, j$) may be asserted in the correspondingly stipulated classsical game.

In the above definition of j-ary relations we have already used equations $\overline{s} =_m \overline{t}$ between tuples $\overline{s}, \overline{t} \in \overline{\mathcal{T}}$. These equations can be defined by

$$\begin{aligned} \overline{s} =_m \overline{t} \;\;\rightleftharpoons\;\; & \overline{s}, \overline{t} \,\varepsilon\, \overline{C}_m \\ & \wedge \forall \kappa \,\varepsilon\, C_0.\, \forall x \,\varepsilon\, C_m.\, \big(\exists y \,\varepsilon\, \pi_m(\overline{s}, \kappa).\, x = y \;\leftrightarrow\; \exists y \,\varepsilon\, \pi_m(\overline{t}, \kappa).\, x = y \big) \end{aligned}$$

where $\kappa \in \mathcal{V}_0$ is a variable for elements of $\mathbb{N}$, and $x, y \in \mathcal{V}$ are different variables that do not occur in $\overline{s}$ or $\overline{t}$. - For all $\overline{a} \equiv (a_1, \ldots, a_j)$ and $\overline{b} \equiv (b_1, \ldots, b_j)$ we obtain:

$$\overline{a} =_m \overline{b} \;\leftrightarrow\; a_1 = b_1 \,\varepsilon\, C_m \;\wedge\; \ldots \;\wedge\; a_j = b_j \,\varepsilon\, C_m.$$

For constructing the object language, we must (among other things) previously have introduced the set $\mathbb{N}$ ($\subseteq \mathcal{C}_0$) and a set $\mathcal{V}(\mathbb{N})$ ($= \mathcal{V}_0 \cap \mathcal{T}(\mathbb{N})$) of variables that range over $\mathbb{N}$. Thereafter we may use the sign $\mathbb{N}$ also in the object language as an abbreviation of the element $\{\kappa \,\varepsilon\, C_0 \colon 0 =_0 0\}$ of $\mathcal{C}_1$ with $\kappa \in \mathcal{V}(\mathbb{N})$. The sign Ω may be used similarly in the object language.

The main results of (Zahn 2004) are these: All sentences of $\mathcal{A}$ are (with respect to the primary rules of assertion) non-circular and even well-founded. All terms of $\mathcal{T}$ and all formulas of $\mathcal{F}$ are invariant under (=).

2 Deducibility of sentences from hypotheses considered modal-logically

We have just sketched a comprehensive language of a ramified cumulative type theory. We shall construct and investigate an extension, $\mathcal{L}$, of it, which is a language of the same sort, but also contains sentences which express that certain sentences of $\mathcal{L}$ are deducible from others by given rules. To this we shall introduce 'names' of sentences of $\mathcal{L}$ and include them in $\mathcal{L}$. So in $\mathcal{L}$ we can not only use but also 'speak about' sentences of that language. But in *this* section we only deal with the deducibility of sentences from 'hypotheses' by given axioms and rules. (Later we shall show how we can formulate that deducibility in $\mathcal{L}$.)

Note. A. Tarski and others have already shown that ramified type theory can, in a certain sense, be used as a metalanguage of itself. But the type theory considered in this paper is an extension of Russell's ramified type theory. So it is appropriate to permit the use of corresponding additional axioms in deductions from hypotheses. We shall enclose such axioms in the system $\mathcal{S}$ assigned below.

Assume that $A \equiv A_1 \wedge \ldots \wedge A_j$ is the conjunction of all 'current hypotheses'. We shall introduce sentences of the form $A \rhd B$ which are to mean that B is *deducible* from A and certain additional axioms by certain rules. The system of those axioms and rules will be denoted by $\mathcal{S}$.

Let now be given a language of type theory as described in section 1. Define: $\mathcal{W} \rightleftharpoons \mathcal{V}_0 \cup \mathcal{V}$ and $\overline{\mathcal{W}} \rightleftharpoons \mathcal{W} \cup \overline{\mathcal{V}}$. We extend $\mathcal{F}$ as follows: Let $\mathcal{F}^+$ ($\supset \mathcal{F}$) be the set of all formulas constructible by the following six rules (where '$\Rightarrow$' indicates the steps of construction):

$$\begin{array}{rcll} & \Rightarrow & F, & \text{if } F \in \mathcal{F} \\ F & \Rightarrow & (\neg F),\ (\exists x\, F), & \text{if } x \in \overline{\mathcal{W}} \\ F, G & \Rightarrow & (F \wedge G),\ (F \vee G),\ (F \rhd G). & \end{array}$$

In the following, the letters x, y, z range over $\overline{\mathcal{W}}$, $\overline{x}$ over $\overline{\mathcal{V}}$, and $\underline{y}, \underline{z}$ over all lists $z_1, \ldots, z_j$ of variables $z_i \in \overline{\mathcal{W}}$ with arbitrary length $j \in \mathbb{N}$; s, t range over $\mathcal{T}$, $\overline{s}, \overline{t}$ over $\overline{\mathcal{T}}$, T over $\mathcal{T} \cup \overline{\mathcal{T}}$; m over Ω; F, G, H over $\mathcal{F}^+$; and A, B, C over $\mathcal{A}^+$, i.e. the set of all sentences belonging to $\mathcal{F}^+$. Moreover, we let $\forall \underline{y}$ and $\forall \underline{z}$ range over all prefices of the form $\forall z_1 \ldots \forall z_j$ with $j \geq 0$. (In case $j = 0$, $\forall \underline{z}\, F$ stands for F.)

Note that the quantifications in $\exists x\, F$ and in $\forall \underline{z}\, F$ are not restricted to any order, and that formulas of $\mathcal{F}^+ \setminus \mathcal{F}$ do not occur in terms of $\mathcal{T}$.

Now we assign the **axioms** of $\mathcal{S}$ under 1. - 4.:

1. Let *PL* be the 'propositional language' whose formulas are as usual composed of 'propositional variables' and $\bot$ ($\rightleftharpoons\ 0 = 1$) by means of $\wedge, \vee, \neg$, and (,). Let *TAU* be a particular finite set of tautologies that are formulated in *PL*. *TAU* with the rule of *modus ponens* is assumed to be 'complete'. As axioms of $\mathcal{S}$ we take at first all formulas of the shape $\forall \underline{z}\, F$ where F results from an element of *TAU* by replacing all occurrences of propositional variables with *formulas* of $\mathcal{F}^+$.

Notes. Also the following axioms of $\mathcal{S}$ have the shape $\forall \underline{z}\, F$. But, for convenience, they are permitted to contain free variables, i.e. we do not demand that all variables occurring free in F are bound by the prefix $\forall \underline{z}$. All substitution instances of these axioms may be asserted due to conventions of the classical game of assertion. - In the following, $\mathrm{Fr}(t, x, F)$ is to mean that t is free for x in F (which is to be defined suitably), and $\mathrm{N}(y, G)$ is to mean that y does not occur free in G.

2. Let all formulas of $\mathcal{F}^+$ of the following shapes be axioms of $\mathcal{S}$:

$$\begin{array}{rcll} & & \forall \underline{z}\, (t = t); & \\ \forall \underline{z}\, (x = t & \rightarrow & (F \leftrightarrow F_x t)) & \text{with } \mathrm{Fr}(t, x, F),\ x \in \mathcal{W}; \\ \forall \underline{z}\, (F_x T & \rightarrow & \exists x\, F x) & \text{with } \mathrm{Fr}(T, x, F); \\ \forall \underline{z}\, (\forall y\, (F_x\, y \rightarrow H) & \rightarrow & (\exists x\, F \rightarrow H)) & \text{with } \mathrm{Fr}(y, x, F),\ \mathrm{N}(y, (\exists x\, F \rightarrow H)); \end{array}$$

$$\begin{array}{rcll}\forall \underline{z}\,(\exists x \,\varepsilon\, C_m.\, F & \leftrightarrow & \exists x\,(x \,\varepsilon\, C_m \wedge F)) & \text{with}\ \ x \in \mathcal{W},\ F \in \mathcal{F};\\ \forall \underline{z}\,(\exists \overline{x} \,\varepsilon\, t.\, F & \leftrightarrow & \exists \overline{x}\,(\overline{x} \,\varepsilon\, t\ \wedge\ F)) & \text{with}\ \ F \in \mathcal{F};\\ \forall \underline{z}\,(\overline{s} \,\varepsilon\, \{\overline{x} \,\varepsilon\, \overline{C}_m \colon F\} & \leftrightarrow & \overline{s} \,\varepsilon\, \overline{C}_m\ \wedge\ F_{\overline{x}}\,\overline{s}) & \text{with}\ \ \mathrm{Fr}(\overline{s}, \overline{x}, F),\ F \in \mathcal{F};\end{array}$$

$$\forall \underline{z}\,\big(\,(\underline{s}, p) \,\varepsilon\, R(q)\ \leftrightarrow\ (\underline{s}) \,\varepsilon\, \overline{C}_m\ \wedge\ p < q\ \wedge\ F((\underline{s}), p, R(p))\,\big)$$

with $(\underline{s}) \in \overline{\mathcal{T}}$, $p, q \in \mathcal{T}(\Omega)$, $R \equiv (\mathrm{J}\overline{x} \,\varepsilon\, \overline{C}_m, \mu, z :\ F(\overline{x}, \mu, z))$, $\mu \in \mathcal{V}(\Omega)$, $z \in \mathcal{V}$, $\mathrm{Fr}((\underline{s}), \overline{x}, F(\ldots))$, and $\mathrm{Fr}(p, \mu, F(\ldots))$;

$$\begin{array}{l}\forall \underline{z}\,(\exists x \,\varepsilon\, \pi_m((t_1, \ldots, t_j), i).\, F\ \leftrightarrow\ t_i \,\varepsilon\, C_m\ \wedge\ F_x\, t_i),\\ \forall \underline{z}\,(\exists x \,\varepsilon\, \pi_m((t_1, \ldots, t_j), p).\, F\ \rightarrow\ p = 1\ \vee\ \ldots\ \vee\ p = j)\end{array}$$

with $x \in \mathcal{W}$; $t_1, \ldots, t_j \in \mathcal{T}$; $i = 1, \ldots, j$; $F \in \mathcal{F}$, and $p \in \mathcal{T}(\mathbb{N})$, where x does not occur free in $t_1, \ldots, t_j$ or p;

$$\begin{array}{ll}\forall \underline{z}\, \neg \exists x \,\varepsilon\, \pi_m(\overline{t}, p).\, F & \text{with}\ \ x\ \text{occuring free in}\ (\overline{t}, p);\\ \forall \underline{z}\,(s =_0 t\ \leftrightarrow\ s = t\ \wedge\ s \,\varepsilon\, C_0). & \end{array}$$

(These axioms can be supplemented by axioms concerning the use of the signs C_m and $\overline{C}_m$ with $m \in \Omega$. We shall deal with this topic in section 5.)

3. The following axiom schemes, which we include in $\mathcal{S}$, concern the connective $\triangleright$:

$$\begin{array}{cr}\forall \underline{z}\,(F \triangleright F) & [1]\\ \forall \underline{z}\,(F \triangleright G\ \wedge\ G \triangleright H\ \rightarrow\ F \triangleright H) & [2]\\ \forall \underline{z}\,(F \triangleright \forall \underline{y}\,(G \rightarrow H)\ \rightarrow\ (F \triangleright \forall \underline{y}\, G\ \rightarrow\ F \triangleright \forall \underline{y}\, H)) & [3]\\ \forall \underline{z}\,(F \triangleright \forall \underline{y}\, H\ \rightarrow\ F \triangleright \forall \underline{y}\,(G \triangleright H)) & [4]\\ \forall \underline{z}\,((F \wedge G) \triangleright H \rightarrow F \triangleright (G \triangleright H)) & [5].\end{array}$$

Note. [3] and [4] remind of the following axiom schemes of labelled modal logic: $\boxed{\mathrm{i}}\,(B \rightarrow C) \rightarrow (\,\boxed{\mathrm{i}}\,B \rightarrow \boxed{\mathrm{i}}\,C\,)$ and $\boxed{\mathrm{i}}\,C \rightarrow \boxed{\mathrm{i}}\,\boxed{\mathrm{j}}\,C$, respectively, which are in case $i = j$ (or without labelles i, j) usually designated by (K) and (4) (cf. (Popkorn 1994), chap. 2).

4. As axioms of $\mathcal{S}$ we can (for certain purposes) also take other formulas whose substitution instances may be asserted due to certain rules of assertion, especially formulas of the shape $\forall \underline{z}\,(E_1 \wedge \ldots \wedge E_n \rightarrow E)$ with $E_1, \ldots, E_n, E \in \mathcal{E}$, where $E_1, \ldots, E_n \Rightarrow E$ (with metavariables for certain elements of $\mathcal{C}_0$ in place of variables) is an agreed rule of assertion (cf. (Zahn 2004, section 0)).

As **rules** of $\mathcal{S}$ we take

$$\begin{array}{rcll}\forall \underline{y}\,(G \rightarrow H),\ \forall \underline{y}\, G & \Rightarrow & \forall \underline{y}\, H & (\textit{modus ponens})\\ \forall \underline{y}\, H & \Rightarrow & \forall \underline{y}\,(G \triangleright H) & (\textit{necessitation}).\end{array}$$

(Special cases of these rules are $(G \rightarrow H),\ G \Rightarrow H$ and $H \Rightarrow G \triangleright H$.)

Let $\mathcal{S} \vdash B$ be short for "B is deducible in $\mathcal{S}$ (i.e. from the axioms of $\mathcal{S}$ by the rules of $\mathcal{S}$)", and $\mathcal{S}(A) \vdash B$ for "B is deducible in $\mathcal{S}(A)$ (i.e. from A and the axioms of $\mathcal{S}$ by the rules of $\mathcal{S}$)." We now interprete $A \rhd B$ as $\mathcal{S}(A) \vdash B$, i.e. we fix the 'primary rule' (cf. section 1): Assert $A \rhd B$ only if $\mathcal{S}(A) \vdash B$ has been asserted. (This rule is invertible, since we do not restrict the assertion of $A \rhd B$ by other rules.) But all sentences of $\mathcal{A}^+$ are to be understood classically, i.e. with respect to the classical game of assertion (mentioned in section 1).

Notes. 1. We have: $\mathcal{S} \vdash \forall \underline{z}\, F$ if and only if $\mathcal{S} \vdash F$. This can be shown by induction on $\mathcal{S}$ (i.e. on the number of corresponding deduction steps). The same also holds for $\mathcal{S}(A)$ instead of $\mathcal{S}$. Moreover, we have $\mathcal{S} \vdash \forall \underline{z}\, \big(F \rhd \forall x\, G \to \forall x\, (F \rhd G)\big)$ if $\mathrm{N}(x, F)$; this reminds of the *inverse Barcan formula*, $\boxed{\mathrm{i}}\, \forall x\, G \to \forall x\, \boxed{\mathrm{i}}\, G$.
2. It can be shown that $\mathcal{S} \vdash A \rhd (B \to C) \to (A \wedge B) \rhd C$, and by axiom [5] we especially have $\mathcal{S} \vdash (A \wedge B) \rhd C \to A \rhd (B \rhd C)$. But I do not know whether $A \rhd (B \rhd C) \to A \rhd (B \to C)$ is deducible in $\mathcal{S}$ or in an expansion of $\mathcal{S}$ which also satisfies the following propositions 2.1 - 2.4 (cf. the end of section 4).

2.1 Lemma: For all $A \in \mathcal{A}^+$ and all $F \in \mathcal{F}^+$, if $\mathcal{S}(A) \vdash F$ then $\mathcal{S} \vdash A \rhd F$.

Proof, by induction on $\mathcal{S}(A)$: Let $\mathcal{S}(A) \vdash F$. If F is an axiom of $\mathcal{S}$, then $\mathcal{S} \vdash A \rhd F$ by *necessitation*. If $F \equiv A$, then $\mathcal{S} \vdash A \rhd F$ by axiom [1]. - If $F \equiv \forall \underline{y}\, H$ has been deduced in $\mathcal{S}(A)$ by applying *modus ponens* from the premises $\forall \underline{y}\, (G \to H)$ and $\forall \underline{y}\, G$, say, then we may use the induction hypotheses that $\mathcal{S} \vdash A \rhd \forall \underline{y}\, (G \to H)$ and $\mathcal{S} \vdash A \rhd \forall \underline{y}\, G$. Then, by axiom [3] and *modus ponens*, $\mathcal{S} \vdash A \rhd \forall \underline{y}\, H$. - If $F \equiv \forall \underline{y}\, (G \rhd H)$ has been deduced in $\mathcal{S}(A)$ by applying *necessitation* from the premise $\forall \underline{y}\, H$, then, by induction hypothesis, we have $\mathcal{S} \vdash A \rhd \forall \underline{y}\, H$ and so, by [4] and *modus ponens*, $\mathcal{S} \vdash A \rhd \forall \underline{y}\, (G \rhd H)$. $\square$

Similarly we also obtain:

2.2 Lemma: For all $A, B \in \mathcal{A}^+$ and all $F \in \mathcal{F}^+$, if $\mathcal{S}(A \wedge B) \vdash F$ then $\mathcal{S}(A) \vdash B \rhd F$.

2.3 Proposition: If $\mathcal{S} \vdash F$, then all substitution instances of F are true (assertible).

Proof (by a well-known model, see (Smullyan 1987), chap. 26, proof of Theorem 1, e.g.): At first we show that all substitution instances of the axioms [1] - [5] are true. To this, we consider any substitution instance A, B, C of the triple F, G, H.
Ad [1]: Let A be said to be deducible from itself. So $\mathcal{S}(A) \vdash A$, i.e. $A \rhd A$ is true.
Ad [2]: If $\mathcal{S}(A) \vdash B$ and $\mathcal{S}(B) \vdash C$, then $\mathcal{S}(A) \vdash C$.
Ad [3]: Let $A \rhd \forall \underline{y}\, (B\underline{y} \to C\underline{y})$ be a substitution instance of $F \rhd \forall \underline{y}\, (G \to H)$.
If $\mathcal{S}(A) \vdash \forall \underline{y}\, (B\underline{y} \to C\underline{y})$ and $\mathcal{S}(A) \vdash \forall \underline{y}\, B\underline{y}$, then, by *modus ponens*, $\mathcal{S}(A) \vdash \forall \underline{y}\, C\underline{y}$.
Ad [4]: If $\mathcal{S}(A) \vdash \forall \underline{y}\, C\underline{y}$, then, by *necessitation*, $\mathcal{S}(A) \vdash \forall \underline{y}\, (B\underline{y} \rhd C\underline{y})$.
Ad [5]: By 2.2, every sentence of the shape [5] is true.

Also all substitution instances of the residual axioms of $\mathcal{S}$ are true. Now we easily obtain 2.3 by induction on $\mathcal{S}$. To this note the following: To obtain $\mathcal{S} \vdash \forall \underline{y}\,(G \triangleright H)$ by *necessitation*, we must previously have $\mathcal{S} \vdash \forall \underline{y}\, H$. But then, for any substitution instance $B \triangleright C$ of $G \triangleright H$, C is deducible in $\mathcal{S}$ and so in $\mathcal{S}(B)$ so that $B \triangleright C$ is true. Thus every substitution instance of $\forall \underline{y}\,(G \triangleright H)$ is true. $\square$

From 2.1 and 2.3 we obtain

2.4 Corollary: For all $A, B \in \mathcal{A}^+$, $\mathcal{S}(A) \vdash B$ if and only if $\mathcal{S} \vdash A \triangleright B$.

3 A language of ramified cumulative type theory containing names of terms and formulas

In the following we construct a language of that sort. By means of first order sentences of it we can also speak about higher order sentences. Despite this possibility of 'reduction' of order, all sentences of that language are non-circular.

In the context of section 1 we say that the 'language' $\mathcal{T}, \mathcal{F}$ **results from** $\mathcal{T}_{\mathrm{or}}, \mathcal{E}, \mathcal{V}_0, \mathcal{V}, \overline{\mathcal{V}}$ [namely by the rules of construction ('demands') fixed in section 1]. Now we presuppose that a given language $\mathcal{T}^\circ, \mathcal{F}^\circ$ results from $\mathcal{T}^\circ_{\mathrm{or}}, \mathcal{E}^\circ, \mathcal{V}^\circ_0, \mathcal{V}, \overline{\mathcal{V}}$. We shall define extensions $\mathcal{T}_{\mathrm{or}} \supset \mathcal{T}^\circ_{\mathrm{or}}$, $\mathcal{E} \supseteq \mathcal{E}^\circ$, and $\mathcal{V}_0 \supset \mathcal{V}^\circ_0$ such that the language $\mathcal{T}, \mathcal{F}$, say, which results from $\mathcal{T}_{\mathrm{or}}, \mathcal{E}, \mathcal{V}_0, \mathcal{V}, \overline{\mathcal{V}}$ contains sentences expressing that $\mathcal{S}(A) \vdash B$, for $\mathcal{S}$ as above and any sentences A, B of $\mathcal{A}^+$. *Regard that this language is also a language of ramified cumulative type theory.*

Note. Instead of constructing the extensions $\mathcal{T} \supset \mathcal{T}^\circ$ and $\mathcal{F} \supset \mathcal{F}^\circ$, and using certain names of terms an formulas, we could apply arithmetization - on the condition that $\mathbb{N} \cup \mathcal{V}(\mathbb{N}) \subset \mathcal{T}(\mathbb{N}) \subseteq \mathcal{T}^\circ_{\mathrm{or}}$ (cf. section 1) where $\mathbb{N} = \{0, 0', 0'', 0''', \ldots\}$ and $\kappa', \kappa'', \kappa''', \ldots \in \mathcal{T}(\mathbb{N})$ for all $\kappa \in \mathcal{V}(\mathbb{N})$, and $(s =_0 t) \in \mathcal{E}^\circ$ for all $s, t \in \mathcal{T}(\mathbb{N})$. Here, for all $m, n \in \mathbb{N}$, $m =_0 n$ is to mean that m is literally equal to n. - But for practical use it is more convenient to take something of names of terms and formulas as substitutes for their Gödel numbers.

For the definitions of $\mathcal{T}_{\mathrm{or}}$ and $\mathcal{E}$ we need some preparations: Let $\mathcal{V}_{\mathcal{N}}$ be a denumerable set of 'new variables' that do not occur in the elements of $\mathcal{T}^\circ \cup \mathcal{F}^\circ$. ($\mathcal{N}$ will be defined below.) Let the set Σ° contain all atomic symbols occurring in elements of $\mathcal{T}^\circ \cup \mathcal{F}^\circ \cup \mathcal{V}_{\mathcal{N}}$, and the additional symbols $\triangleright$, N, Fr, and Sub. Let the symbol set Σ result from Σ° by adding the new symbols $\lceil\alpha\rceil, \lceil\lceil\alpha\rceil\rceil, \lceil\lceil\lceil\alpha\rceil\rceil\rceil, \ldots$, for every $\alpha \in \Sigma^\circ$. The symbols $\lceil, \rceil$ are supposed not to belong to Σ°. We do also *not* include them in Σ. So we may consider all elements of Σ as *atomic* symbols.

Let Σ^* be the set of all strings $\alpha_1 \ldots \alpha_j$ ($j \geq 0$) of symbols $\alpha_i \in \Sigma$. So, especially, the 'empty word' belongs to Σ^* (case $j = 0$). For $\alpha_1, \ldots, \alpha_j \in \Sigma$ ($j \geq 0$) we define

$$\lceil\alpha_1\alpha_2 \ldots \alpha_j\rceil \rightleftharpoons \lceil\alpha_1\rceil\lceil\alpha_2\rceil \ldots \lceil\alpha_j\rceil.$$

Let this figure be said to be the **name** of $\alpha_1\alpha_2\ldots\alpha_j$. Especially $\ulcorner\urcorner$ stands for the empty word, which is its own name. Let $\mathcal{N}$ be the set of all such names of elements of Σ^*. We shall use the elements of $\mathcal{V}_{\mathcal{N}}$ as variables for (all or particular) elements of $\mathcal{N}$. All variables occurring in an element of $\mathcal{N}$ are considered to be *bound* (by the 'quotation marks' $\ulcorner, \urcorner$).

For our purpose we have also to define a set of terms, whose substitution instances are names: Let $\mathcal{T}(\mathcal{N})$ be the set of all figures $t_1t_2\ldots t_j$ with $t_i \in \mathcal{N} \cup \mathcal{V}_{\mathcal{N}}$ and $j \geq 0$.

Now we can introduce the extensions $\mathcal{T} \supset \mathcal{T}^\circ$ and $\mathcal{F} \supset \mathcal{F}^\circ$:
Let $\mathcal{V}_0 \rightleftharpoons \mathcal{V}_0^\circ \cup \mathcal{V}_{\mathcal{N}}$, $\mathcal{T}_{\mathrm{Or}} \rightleftharpoons \mathcal{T}_{\mathrm{Or}}^\circ \cup \mathcal{T}(\mathcal{N})$, and let $\mathcal{E}$ contain all elements of $\mathcal{E}^\circ$, all equations $(\sigma =_0 \tau)$ with $\sigma, \tau \in \mathcal{T}(\mathcal{N})$, and certain further formulas, which we shall specify below. Let, as announced, $\mathcal{T}, \mathcal{F}$ result from $\mathcal{T}_{\mathrm{Or}}, \mathcal{E}, \mathcal{V}_0, \mathcal{V}, \overline{\mathcal{V}}$ by the rules of construction fixed in section 1.

We define the meaning of equations $a =_0 b$ between *names* a, b thus: For $\beta, \gamma \in \Sigma^*$ let $\ulcorner\beta\urcorner =_0 \ulcorner\gamma\urcorner$ mean that $(\beta, \gamma \in \mathcal{C}_0$ and $\beta =_0 \gamma)$ or $(\beta, \gamma \notin \mathcal{C}_0$ and $\beta \equiv \gamma)$.

Examples of tuples containing the empty word are: $(\,)$, $(t,\,)$, $(\,,\,)$, $(\,,\,,t)$, for any $t \in \mathcal{T}$. Such tuples are particular elements of $\overline{\mathcal{T}}$.

Given a system $\mathcal{S}$ of axioms and rules as indicated in section 2. We want to define sets $R_0, R_1, R_2, \ldots \in \mathcal{C}_1$ such that, for all $n \in \mathbb{N}$, R_n is the set of all names of formulas that are deducible in $\mathcal{S}$ by $\leq n$ steps of deduction.

So long we have used the letters $F, G, H, x, y, \overline{x}, \overline{s}, t, T, m, \ldots$ as metavariables. However, to make the following definitions easier to understand, we now write these and some other letters for particular variables of $\mathcal{V}_{\mathcal{N}}$ that range over certain subsets of $\mathcal{N}$. (This will be specified below.) By this means we at first define a formula 'Axiom(X)' of $\mathcal{F}_1$, which can be read as "X is the name of an axiom of $\mathcal{S}$" and will be explained below:

$$
\begin{aligned}
\mathrm{Axiom}(X) \;\rightleftharpoons\;& \exists\, F, G, H, P, Q, w, x, y, \overline{x}, \eta, \zeta, \overline{s}, t, T, m \;\varepsilon\; C_0.\, \big(X =_0 \zeta \ulcorner(\check{F} \triangleright \check{F})\urcorner \\
&\vee\; X =_0 \zeta \ulcorner(\check{F} \triangleright \check{G} \;\wedge\; \check{G} \triangleright \check{H} \;\rightarrow\; \check{F} \triangleright \check{H})\urcorner \\
&\vee\; X =_0 \zeta \ulcorner(\check{F} \triangleright \check{\eta}\,(\check{G} \rightarrow \check{H}) \;\rightarrow\; (\check{F} \triangleright \check{\eta}\,\check{G} \rightarrow \check{F} \triangleright \check{\eta}\,\check{H}))\urcorner \\
&\vee\; \ldots \\
&\vee\; \big(X =_0 \zeta \ulcorner(\check{G} \rightarrow \exists\check{x}\,\check{F})\urcorner \;\wedge\; \mathrm{Sub}(G, F, T, x) \;\wedge\; \mathrm{Fr}(T, x, F)\big) \\
&\vee\; \big(X =_0 \zeta \ulcorner(\forall\check{y}\,(\check{G} \rightarrow \check{H}) \rightarrow (\exists\check{x}\,\check{F} \rightarrow \check{H}))\urcorner \\
&\qquad \wedge\, \mathrm{Sub}(G, F, y, x) \;\wedge\; \mathrm{Fr}(y, x, F) \;\wedge\; \mathrm{N}(y, \ulcorner\exists\check{x}\,\check{F} \rightarrow \check{H}\urcorner)\big) \\
&\vee\; X =_0 \zeta \ulcorner(\exists\check{w} \;\varepsilon\; C_{\check{m}}.\, \check{P} \;\leftrightarrow\; \exists\check{w}\,(\check{w} \;\varepsilon\; C_{\check{m}} \wedge \check{P}))\urcorner \\
&\vee\; X =_0 \zeta \ulcorner(\exists\check{\overline{x}} \;\varepsilon\; \check{t}.\, \check{P} \;\leftrightarrow\; \exists\check{\overline{x}}\,(\check{\overline{x}} \;\varepsilon\; \check{t} \wedge \check{P}))\urcorner \\
&\vee\; \big(X =_0 \zeta \ulcorner(\check{\overline{s}} \;\varepsilon\; \{\check{\overline{x}} \;\varepsilon\; \overline{C}_{\check{m}} \colon \check{P}\} \;\leftrightarrow\; \check{\overline{s}} \;\varepsilon\; \overline{C}_{\check{m}} \;\wedge\; \check{Q})\urcorner \\
&\qquad \wedge\, \mathrm{Sub}(Q, P, \overline{s}, \overline{x}) \;\wedge\; \mathrm{Fr}(\overline{s}, \overline{x}, P)\big) \\
&\vee\; \ldots\big).
\end{aligned}
$$

To explain this definition, we need some preliminary definitions and conventions. For any $\mathcal{U} \subseteq \Sigma^*$ let $\mathcal{U}^{\ulcorner\urcorner}$ be the set of all names $\ulcorner u\urcorner$ of elements u of $\mathcal{U}$. (So we have

$\mathcal{U}^{\ulcorner\urcorner} \subseteq \Sigma^{*\ulcorner\urcorner} = \mathcal{N}$.) In the defintion of 'Axiom(X)', let w ($\in \mathcal{V}_{\mathcal{N}}$) range over $\mathcal{W}^{\ulcorner\urcorner}$, x, y over $\overline{\mathcal{W}}^{\ulcorner\urcorner}$, $\overline{x}$ over $\overline{\mathcal{V}}^{\ulcorner\urcorner}$, t over $\mathcal{T}^{\ulcorner\urcorner}$, $\overline{s}$ over $\overline{\mathcal{T}}^{\ulcorner\urcorner}$, T over $\mathcal{T}^{\ulcorner\urcorner} \cup \overline{\mathcal{T}}^{\ulcorner\urcorner}$, m over $\Omega^{\ulcorner\urcorner}$, $F, G,$ H, X over $\mathcal{F}^{+\ulcorner\urcorner}$, P, Q over $\mathcal{F}^{\ulcorner\urcorner}$ only, and η, ζ over names $\ulcorner\forall z_1 \ldots \forall z_j\urcorner$ of prefices with variables $z_i \in \overline{\mathcal{W}}$ and length $j \geq 0$. . (Of course, we presuppose that $\mathcal{V}_{\mathcal{N}}$ contains denumerably many variables of each of those sorts.)

Moreover, $\ulcorner \ldots \check{F}$ - - -$\urcorner$ stands for $\ulcorner \ldots \urcorner F \ulcorner$- - -$\urcorner$ (wherein F occurs free), and $\ulcorner \ldots \check{\eta}\, \check{G}$- - -$\urcorner$ for $\ulcorner \ldots \urcorner \eta\, G \ulcorner$- - -$\urcorner$, e.g.

Of course, a *sentence* of the shape $\mathrm{Fr}(T, x, F)$ with *names* T, x, F (in place of variables) is to mean that T' is free for x' in F' where T' is the term denoted by T, x' is the variable denoted by x, and F' is the formula denoted by F. Similarly, $\mathrm{N}(y, G)$ is to mean that y' does not occur free in G', and $\mathrm{Sub}(G, F, T, x)$ is to mean that G' results from F' by substituting T' for x'. We include all *formulas* of those shapes in $\mathcal{E}$. To formulate this in more detail, we at first define: For $\mathcal{U} \subseteq \Sigma^*$ let $\mathcal{T}(\mathcal{U}^{\ulcorner\urcorner})$ be the set of all elements of $\mathcal{T}(\mathcal{N})$ whose substitution instances are elements of $\mathcal{U}^{\ulcorner\urcorner}$. Now let $\mathcal{E}$ also contain all formulas $\mathrm{Fr}(T, x, F)$, $\mathrm{N}(y, G)$, and $\mathrm{Sub}(G, F, T, x)$ with $x, y \in \mathcal{T}(\overline{\mathcal{W}}^{\ulcorner\urcorner})$; $F, G \in \mathcal{T}(\mathcal{F}^{+\ulcorner\urcorner})$, and $T \in \mathcal{T}(\mathcal{T}^{\ulcorner\urcorner} \cup \overline{\mathcal{T}}^{\ulcorner\urcorner})$. (We have ommited several brackets in the definition of 'Axiom(X)'.)

Regard that, by certain appropriate requirements on $\mathcal{T}^{\circ}_{\mathrm{Or}}$ and $\mathcal{E}^{\circ}$, we have $\mathcal{T}_{\mathrm{Or}} \cap \mathcal{F}^{+} = \emptyset$ so that the sign '$=_0$' occurring in 'Axiom(X)' means the literal equality of formulas.

As announced, we now recursively define sets $R_0, R_1, R_2, \ldots$ such that R_n is the set of all names of formulas that are deducible in $\mathcal{S}$ by $\leq n$ steps:

$$\begin{aligned} R_0 &= \big\{X \,\varepsilon\, C_0\colon \mathrm{Axiom}(X)\big\} \\ R_{n+1} &= \big\{X \,\varepsilon\, C_0\colon\ X \,\varepsilon\, R_n \\ &\qquad \vee\ \exists\, G, H, \eta \,\varepsilon\, C_0.\, (\eta \ulcorner(\check{G} \to \check{H})\urcorner \,\varepsilon\, R_n \ \wedge\ \eta\, G \,\varepsilon\, R_n \ \wedge\ X =_0 \eta\, H) \\ &\qquad \vee\ \exists\, G, H, \eta \,\varepsilon\, C_0.\, (\eta\, H \,\varepsilon\, R_n \ \wedge\ X =_0 \eta \ulcorner(\check{G} \rhd \check{H})\urcorner)\,\big\}. \end{aligned}$$

For a variable $\nu \in \mathcal{V}(\Omega)$, R_ν can also be defined as a first order term, i.e. as an element of $\mathcal{T}_1$ (cf. section 1). Let $R \rightleftharpoons \bigcup_{\nu\,\varepsilon\,\mathrm{I\!N}} R_\nu$. So $R \in \mathcal{C}_1$, and for any $B \in \mathcal{A}^+$, the sentence $\ulcorner B\urcorner \,\varepsilon\, R$ belongs to $\mathcal{A}_1$ and means that B is deducible in $\mathcal{S}$. So, by 2.4, $\ulcorner A \rhd B\urcorner \,\varepsilon\, R$ means that B is deducible in $\mathcal{S}(A)$ (i.e. 'from A in $\mathcal{S}$').

Result: Deducibility from hypotheses can be formulated within the object language. So this language can serve as its own metalanguage, to some extend.

Remarks: In the above definition of $R_0, R_1, R_2, \ldots$ it has been convenient to use several sorts of variables belonging to $\mathcal{V}_{\mathcal{N}}$, namely for every set $\mathcal{U} \in \{\mathcal{W}, \overline{\mathcal{V}}, \overline{\mathcal{W}}, \mathcal{T}, \overline{\mathcal{T}}, \mathcal{T} \cup \overline{\mathcal{T}}, \Omega, \mathcal{F}, \mathcal{F}^{+}\}$ variables ranging over $\mathcal{U}^{\ulcorner\urcorner}$, and variables ranging over names $\ulcorner\forall z_1 \ldots \forall z_j\urcorner$ of prefices with $z_i \in \overline{\mathcal{W}}$ and $j \geq 0$. To this use we must previously have introduced such variables. But instead of them we need only *one* sort of variables, namely variables ranging over $\mathcal{N}$. Then we have to reformulate Axiom(X) thus:

$$\exists F, \ldots, \zeta, \ldots \;\varepsilon\; C_0. \big(F \;\varepsilon\; \mathcal{F}^{+\ulcorner\urcorner} \;\wedge\; \ldots \;\wedge\; \zeta \;\varepsilon\; \Pi^{\ulcorner\urcorner} \;\wedge\; \ldots \;\wedge\; (X =_0 \zeta \,\ulcorner(\check{F} \triangleright \check{F})\urcorner \;\vee\; \ldots)\big)$$

where $X, F, \zeta, \ldots$ are elements of $\mathcal{V}_{\mathcal{N}}$ that range over $\mathcal{N}$. Here we have added the clauses $F \;\varepsilon\; \mathcal{F}^{+\ulcorner\urcorner}$, $\zeta \;\varepsilon\; \Pi^{\ulcorner\urcorner}, \ldots$, where Π denotes the set of the above mentioned prefices. (To avoid misunderstandings we can replace '$\mathcal{F}^{+}$' by a new sign in this context.) We can effect that the latter clauses are in $\mathcal{F}_1$ - provided that $\mathcal{T}^{\circ}_{\mathrm{Or}}, \mathcal{E}^{\circ}, \mathcal{V}^{\circ}_0, \mathcal{V}$, and $\overline{\mathcal{V}}$ are recursively enumerable (i.e. constructible by formal rules). Indeed, in this case also the sets $\mathcal{T}, \overline{\mathcal{T}}, \mathcal{F}, \mathcal{F}^{+}, \Pi, \ldots$ are recursively enumerable so that the corresponding sets of names for elements of those sets can be introduced as elements of $\mathcal{C}_1$ (namely on the model of the above introduction of $R \rightleftharpoons \bigcup_{\nu \,\varepsilon\, \mathbb{N}} R_\nu$). - Complete reformulations of 'Axiom(X)' and the definition of R_{n+1} are left to the reader.

The predicates $\mathrm{N}(\cdot,\cdot)$, $\mathrm{Fr}(\cdot,\cdot,\cdot)$, and $\mathrm{Sub}(.,.,.)$ have recursively enumerable extents and can, therefore, be defined to be elements of $\mathcal{C}_1$. So it suffices to take $\mathcal{E}$ to be the union of $\mathcal{E}^{\circ}$ and the set of all equations $(\sigma =_0 \tau)$ with $\sigma, \tau \in \mathcal{T}(\mathcal{N})$. So we may also omit the signs N, Fr, and Sub from Σ°. - We shall, however, not employ these reductions of basic means of the object language.

When we say that a sentence B is deducible in $\mathcal{S}(A)$, we do not *use* the sentences A and B, we only *refer* to them. To indicate this fact we can put them in quotation marks. Accordingly, it would be adequate to understand $A \triangleright B$ as a shorthand of $\ulcorner A\urcorner \triangleright \ulcorner B\urcorner$. But then the definiens of 'Axiom(X)' turns in

$$\exists F, \ldots, \zeta, \ldots \;\varepsilon\; C_0. \big(X =_0 \zeta \,\ulcorner(F \triangleright F)\urcorner \;\vee\; \ldots \big),$$

where several occurrences of F are bound by $\ulcorner, \urcorner$, which misses the intended meaning. We do no further discuss that matter, but we shall deal with similar problems in section 5.

4 A version of the Theorem of Löb

Modifying an idea of Craig (see (Smullyan 1987), chap. 26, e.g.) we now extend $\mathcal{F}^{+}$ by the following rule: For all $F, G \in \mathcal{F}^{+}$ let $\Delta(F, G)$ be a formula of $\mathcal{E}$ $(\subset \mathcal{F} \subset \mathcal{F}^{+})$. For all $A, B \in \mathcal{A}^{+}$ let

$$\Delta(A, B) \text{ mean that } \mathcal{S}(A) \vdash (\Delta(A, B) \rightarrow B).$$

(Note that the latter deducibility relation does not depend on the meaning of $\Delta(A, B)$.) So for all $F, G \in \mathcal{F}^{+}$, all substitution instances of

$$\forall \underline{z} \big(\Delta(F, G) \;\leftrightarrow\; F \triangleright (\Delta(F, G) \rightarrow G) \big)$$

are true. We now take all formulas of this form as additional axioms of $\mathcal{S}$. (These axioms can also easily be enclosed in 'Axiom(X)'.) So all formulas of the following form are deducible in $\mathcal{S}$:

$$\forall \underline{z} \,\big\{ (\Delta(F, G) \rightarrow G) \;\leftrightarrow\; [\, F \triangleright (\Delta(F, G) \rightarrow G) \rightarrow G] \,\big\}.$$

Writing H for $(\Delta(F,G) \to G)$ we obtain this version of the

Diagonal Lemma: For all $F, G \in \mathcal{F}^+$ there is an $H \in \mathcal{F}^+$ satisfying $\mathcal{S} \vdash \forall \underline{z} \left\{ H \;\leftrightarrow\; (F \triangleright H \to G) \right\}$.

The special case with $\mathcal{A}^+$ instead of $\mathcal{F}^+$ implies the following version of the

Theorem of Löb: For all $B, C \in \mathcal{A}^+$, if $\mathcal{S} \vdash (B \triangleright C \to C)$, then $\mathcal{S} \vdash C$.

The proof given in (Boolos 1989), p.187, can easily be transformed into a proof of this version of Löb's theorem. By another well known theorem of modal logic, this version yields

$$\mathcal{S} \vdash (B \triangleright (B \triangleright C \to C) \to B \triangleright C),$$

which reminds of the modal scheme $\boxed{\mathrm{i}}(\boxed{\mathrm{i}}C \to C) \to \boxed{\mathrm{i}}C$. Obviously, all results of this section also hold for $\mathcal{S}(A)$ instead of $\mathcal{S}$.

Notes: Let $\top \rightleftharpoons \neg\bot$, e.g. Because of $\mathcal{S} \nvdash \bot$, Löb's theorem especially implies $\mathcal{S} \nvdash \neg\,(\top \triangleright \bot)$ (cf. Gödel's second incompleteness theorem). So, for some sentences A, B, we have $\mathcal{S} \nvdash \neg(A \triangleright B)$ but $\mathcal{S} \vdash \neg(A \to B)$, and thus $\mathcal{S} \nvdash (A \triangleright B) \to (A \to B)$.

5 Supplementary axiom schemes

The axioms of $\mathcal{S}$ can be supplemented in several respects. So it may be useful for certain purposes to expand $\mathcal{S}$ by axioms concerning $\mathbb{N}, \Omega$, the order relation $(<)$ between elements of Ω, and the basic equality relation $(=_0)$. Examples are the following axioms of induction on $\mathbb{N}$ and Ω, respectively:

$$\forall \underline{z} \left(F_\kappa 0 \wedge \forall \kappa\, (F \to F_\kappa \kappa') \to \forall \kappa\, F\right),$$
$$\forall \underline{z} \left(\forall \nu\, (\forall \mu < \nu.\, F_\nu \mu \to F) \to \forall \nu F\right),$$

where $\underline{z}$ is any list of variables of $\overline{\mathcal{W}}$, $F \in \mathcal{F}^+$, $\kappa \in \mathcal{V}(\mathbb{N})$, and $\mu, \nu \in \mathcal{V}(\Omega)$ with $\mathrm{N}(\mu, F)$ and $\mathrm{Fr}(\mu, \nu, F)$. But as long as we have only *presupposed* that a set Ω and relations $(<)$ and $(=_0)$ with certain properties are given, we could only outline these *properties* by axioms concerning $\Omega, (<)$, and $(=_0)$, .

The grammar of our object language is essentially determined by the sets $\mathcal{F}_n, \mathcal{T}_n$ and $\overline{\mathcal{T}}_n$ with $n \in \Omega$ (which depend on each other). But that language contains the signs C_n and $\overline{C}_n$ for the sets $\mathcal{C}_n$ and $\overline{\mathcal{C}}_n$, which are particular subsets of $\mathcal{T}_n$ and $\overline{\mathcal{T}}_n$, respectively. Moreover, we have

$$\begin{aligned} \overline{b}\, \varepsilon\, \{\overline{x}\, \varepsilon\, \overline{C}_n \colon A(\overline{x})\} \;&\leftrightarrow\; \overline{b}\, \varepsilon\, \overline{C}_n \wedge A(\overline{b}), \\ \exists x\, \varepsilon\, C_n.\, A(x) \;&\leftrightarrow\; \exists x\, (x\, \varepsilon\, C_n \wedge A(x)). \end{aligned}$$

So the semantics (the assertibility of sentences) of the object language is also determined by the constructions of $\mathcal{T}_n$ and $\mathcal{F}_n$ (and thus by the grammar of that language). This shows that the axiom system $\mathcal{S}$ is incomplete in a corresponding manner.

First we can partially characterize $\mathcal{T}_n^{\ulcorner\urcorner}$ (in place of $\mathcal{T}_n$) with $n \in \Omega$ by the following axioms (with any prefices $\forall \underline{z}$ as above), which we can add to $\mathcal{S}$; we write them in an abbreviated shape and shall explain them subsequently:

$$\begin{aligned}\forall \underline{z}\, \{t \,\varepsilon\, \mathcal{T}_\nu^{\ulcorner\urcorner} \quad \leftrightarrow \quad & t \,\varepsilon\, \mathcal{T}_{\mathrm{or}}^{\ulcorner\urcorner} \cup \mathcal{V}^{\ulcorner\urcorner} \;\vee \\ & \exists \overline{x} \,\varepsilon\, \overline{\mathcal{V}}^{\ulcorner\urcorner}.\, \exists m \,\varepsilon\, \Omega^{\ulcorner\urcorner}.\, \exists F \,\varepsilon\, \mathcal{F}_\nu^{\ulcorner\urcorner}.\, \exists \lambda \,\varepsilon\, \mathcal{V}(\Omega)^{\ulcorner\urcorner}.\, \exists z \,\varepsilon\, \mathcal{V}^{\ulcorner\urcorner}.\, \exists q \,\varepsilon\, \mathcal{T}(\Omega)^{\ulcorner\urcorner}. \\ & \big((t =_0 \ulcorner\{\check{\overline{x}} \,\varepsilon\, \overline{C}_{\check{m}} \colon \check{F}\}\urcorner \;\vee\; t =_0 \ulcorner(\mathrm{J}\check{\overline{x}} \,\varepsilon\, \overline{C}_{\check{m}}, \check{\lambda}, \check{z}.\, \check{F})(\check{q})\urcorner) \\ & \wedge\, m < \nu\big)\}\end{aligned}$$

where $\nu \in \mathcal{V}(\Omega)$, and $t, \overline{x}, m, F, \lambda, z, q \in \mathcal{V}_{\mathcal{N}}$ are particular variables that range over $\mathcal{N}$. (Essential is the case that $\underline{z}$ consists of t and ν.) At the end of these axioms the ineqation $m < \nu$ occurs where m 'stands for' the *name* of an element of Ω, and ν for any element of Ω. Of course, this inequation is to mean that the ordinal denoted by m is smaller than ν. We shall define this ineqation as an abbreviation of a compound formula.

In the latter axioms we have used the signs '$\mathcal{T}_{\mathrm{or}}$', '$\mathcal{V}$', etc., which we have introduced in the metalanguage. To avoid misunderstandings, they can be replaced here by new signs. Notice, for instance, that if the set $\mathcal{T}_{\mathrm{or}}$ is recursively enumerable, we can introduce first order formulas $(t \,\varepsilon\, \mathcal{T}_{\mathrm{or}}^{\ulcorner\urcorner})$ with $t \in \mathcal{T}(\mathcal{N})$ such that, for any $a \in \mathcal{N}$, the sentence $(a \,\varepsilon\, \mathcal{T}_{\mathrm{or}}^{\ulcorner\urcorner})$ means that $a \in \mathcal{T}_{\mathrm{or}}^{\ulcorner\urcorner}$.

The sets $\overline{\mathcal{T}}_n^{\ulcorner\urcorner}, \mathcal{C}_n^{\ulcorner\urcorner}, \overline{\mathcal{C}}_n^{\ulcorner\urcorner}$, and $\mathcal{F}_n^{\ulcorner\urcorner}$ can also partially be characterized by axioms, which we can add to $\mathcal{S}$. Desirable are also axioms for characterizing $\mathcal{T}_{\mathrm{or}}^{\ulcorner\urcorner}, \mathcal{E}^{\ulcorner\urcorner}$, etc. But we need, above all, axioms for characterizing C_n instead of $\mathcal{C}_n^{\ulcorner\urcorner}$, namely general axioms that sum up the sentences

$$c \,\varepsilon\, C_n \leftrightarrow \ulcorner c\urcorner \,\varepsilon\, \mathcal{C}_n^{\ulcorner\urcorner}$$

with arbitrary constants $c \in \mathcal{C}$ and $n \in \Omega$, and corresponding axioms for $\overline{C}_n$ in place of C_n. Axioms of the shape

$$\forall \underline{z}\, (x \,\varepsilon\, C_\nu \leftrightarrow \ulcorner x\urcorner \,\varepsilon\, \mathcal{C}_\nu^{\ulcorner\urcorner})$$

are not suitable for that purpose since the latter occurrence of x is bound by the quotation marks. So we take, instead, as additional axioms of $\mathcal{S}$

$$\forall \underline{z}\, (x \,\varepsilon\, C_\nu \leftrightarrow \lfloor x\rfloor \,\varepsilon\, \mathcal{C}_\nu^{\ulcorner\urcorner}),$$
$$\forall \underline{z}\, (\overline{x} \,\varepsilon\, \overline{C}_\nu \leftrightarrow \lfloor \overline{x}\rfloor \,\varepsilon\, \overline{\mathcal{C}}_\nu^{\ulcorner\urcorner}).$$

To this we stipulate that any variable y ($\in \overline{\mathcal{W}}$) occurs *free* in $\lfloor y\rfloor$, and that in any term or formula in which a constant c ($\in \mathcal{C} \cup \overline{\mathcal{C}}$) is to be substituted for y, its name $\ulcorner c\urcorner$ is to be substituted for $\lfloor y\rfloor$.

As further axioms of $\mathcal{S}$ we also take, for distinct variables $x, y \in \mathcal{V}_0$:

$$\forall \underline{z}\,(\lfloor x \rfloor =_0 \lfloor y \rfloor \leftrightarrow x = y).$$

The inequation $m < \nu$ occurring in the above adduced axiom concerning $\mathcal{T}_\nu^{\lceil\rceil}$ can now be defined by

$$m < \nu \;\rightleftharpoons\; \exists \mu\,(m =_0 \lfloor \mu \rfloor \;\wedge\; \mu < \nu).$$

Since the latter axioms contain the new signs $\lfloor$ and $\rfloor$, we have to expand the object language. To this end we add these signs to Σ and stipulate the following: For any term $T \in \mathcal{T} \cup \overline{\mathcal{T}}$ let $\lfloor T \rfloor$ result from $\lceil T \rceil$ by substituting $\lfloor y \rfloor$ for $\lceil y \rceil$, for every free occurrence of a variable y ($\in \overline{\mathcal{W}}$) in T. This means that if

$$T \equiv \beta_0 y_1 \beta_1 y_2 \beta_2 \ldots y_k \beta_k$$

(where $y_1, \ldots, y_k$ is the list of *all* free occurrences of variables in T, and β_i may also be empty ($i = 0, \ldots, k$)), then

$$\lfloor T \rfloor \equiv \lceil \beta_0 \rceil \lfloor y_1 \rfloor \lceil \beta_1 \rceil \lfloor y_2 \rfloor \lceil \beta_2 \rfloor \ldots \lfloor y_k \rfloor \lceil \beta_k \rceil.$$

Especially for constants c we have $\lfloor c \rfloor \equiv \lceil c \rceil$.

Now we can expand $\mathcal{F}^+$ by including the (new) formulas $(\lfloor t \rfloor \;\varepsilon\; C_q^{\lceil\rceil})$ and $(\lfloor \overline{s} \rfloor \;\varepsilon\; \overline{C}_q^{\lceil\rceil})$ with $t \in \mathcal{T}$, $\overline{s} \in \overline{\mathcal{T}}$, and $q \in \mathcal{T}(\Omega)$, and the formulas $(\lfloor s_0 \rfloor =_0 \lfloor t_0 \rfloor)$, $(\sigma =_0 \lfloor t_0 \rfloor)$, and $(\lfloor s_0 \rfloor =_0 \tau)$ with $s_0, t_0 \in \mathcal{T}_{\mathrm{Or}}$ and $\sigma, \tau \in \mathcal{T}(\mathcal{N})$. Of course, we stipulate that (as in section 2) further formulas of $\mathcal{F}^+$ may be composed of others. But we do not include $\lfloor T \rfloor$ in $\mathcal{T}$ for any $T \in \mathcal{T} \cup \overline{\mathcal{T}}$.

Correspondingly, we supplement the primary rules of assertion by the following rule, e.g.: Assert $\exists x\, A(x, \lfloor x \rfloor)$ only if we have asserted $A(c, \lceil c \rceil)$ for some value c of x. This rule may also be inverted.

Of course, we also expand the system $\mathcal{S}$ by admitting that its axioms are formulas of the expanded set $\mathcal{F}^+$. In view of 2.3 we have to show that all axioms of the system expanded so are true.

Proof: At first we deal with axioms of the shape

$$\forall \underline{z}\,(F(T, \lfloor T \rfloor) \rightarrow \exists x\, F(x, \lfloor x \rfloor))$$

with $x \in \overline{\mathcal{W}}$ and $\mathrm{Fr}(T, x, F)$. Let $T \equiv \beta_0 y_1 \beta_1 y_2 \beta_2 \ldots y_k \beta_k$ as above. We assume that a_i is a value of y_i, and that $a_i \equiv a_j$ if $y_i \equiv y_j$ ($i, j = 1, \ldots, k$). Let $\lfloor T \rfloor_{\underline{y}}\, \underline{a}$ result from $\lfloor T \rfloor$ by substituting $\lceil a_i \rceil$ for $\lfloor y_i \rfloor$ ($i = 1, \ldots, k$). Then we have

$$\lfloor T \rfloor_{\underline{y}}\, \underline{a} \equiv \lceil \beta_0 a_1 \beta_1 a_2 \beta_2 \ldots a_k \beta_k \rceil \equiv \lceil T_{\underline{y}}\, \underline{a} \rceil.$$

Let T be free for x in a formula $B(x, \lfloor x \rfloor)$. Then we have, for all values $\underline{a}$ of $\underline{y}$,

$$B(T, \lfloor T \rfloor)_{\underline{y}}\, \underline{a} \equiv B(T_{\underline{y}}\, \underline{a}, \lceil T_{\underline{y}}\, \underline{a} \rceil)$$

so that we may assert: $B(T, \lfloor T \rfloor)\,{}_y\,\underline{a} \to \exists x\, B(x, \lfloor x \rfloor)$. Thus all substitution instances of axioms of the shape $\forall \underline{z}\,(F(T, \lfloor T \rfloor) \to \exists x\, F(x, \lfloor x \rfloor))$ are true.

Also all substitution instances of axioms of the shape

$$\forall \underline{z}\,(x = y \to (F \leftrightarrow F_x\, y))$$

are true. The simple proof by induction on $\mathcal{F}^+$ begins as follows: 1. We have

$$a = b \;\to\; (t_x a \,\varepsilon\, C_n \leftrightarrow t_x b \,\varepsilon\, C_n) \;\to\; (\lceil t_x a \rceil \,\varepsilon\, \mathcal{C}_n^{\ulcorner\urcorner} \leftrightarrow \lceil t_x b \rceil \,\varepsilon\, \mathcal{C}_n^{\ulcorner\urcorner}),$$

if t is a term in which only the variable x occurs free, and a, b are values of x.
2. If especially $t \in \mathcal{T}_{\mathrm{or}}$, we have $x \in \mathcal{V}_0$; $a, b \in \mathcal{C}_0$; $t_x a, t_x b \in \mathcal{C}_0$ (by a requirement on $\mathcal{T}^{\circ}_{\mathrm{or}}$), and so

$$a = b \;\to\; a =_0 b \;\to\; t_x a =_0 t_x b \;\to\; \lceil t_x a \rceil =_0 \lceil t_x b \rceil .$$

The truth of the residual axioms of $\mathcal{S}$ is not problematic.

Remark: For constants $a, b \notin \mathcal{C}_0$, the sentence $a = b \to \lceil a \rceil = \lceil b \rceil$ is generally not true. But equations of the shape $(\lfloor x \rfloor = \lfloor y \rfloor)$ with $x, y \in \mathcal{V}$ do not belong to $\mathcal{F}^+$.

By the mentioned and similar means we cannot complete $\mathcal{S}$, not even with respect to $\mathcal{F}$ (instead of $\mathcal{F}^+$). We can at best take axioms which are sufficient for particular purposes. - All axioms which we have considered in this section can also be enclosed in the formula ‘Axiom(X)’.

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