The second cohomology of current algebras of general Lie algebras

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November 10, 2005

Abstract

Let A be a unital commutative associative algebra over a field of characteristic zero, $\mathfrak{k}$ be a Lie algebra, and $\mathfrak{z}$ a vector space, considered as a trivial module of the Lie algebra $\mathfrak{g} := A \otimes \mathfrak{k}$. In this paper we give a description of the cohomology space $H^2(\mathfrak{g}, \mathfrak{z})$ in terms of well accessible data associated to A and $\mathfrak{k}$. We also discuss the topological situation, where A and $\mathfrak{k}$ are locally convex algebras.

Keywords: current algebra, Lie algebra cohomology, Lie algebra homology, invariant bilinear form, central extension

AMS-Class: 17B56, 17B65

Introduction

Let A be a unital commutative associative algebra over a field $\mathbb{K}$ of characteristic zero and $\mathfrak{k}$ be a $\mathbb{K}$-Lie algebra. Then the tensor product $\mathfrak{g} := A \otimes \mathfrak{k}$ is a Lie algebra with respect to the bracket

$$[a \otimes x, a' \otimes x'] := aa' \otimes [x, x'].$$

Let $\mathfrak{z}$ be a vector space, considered as a trivial $\mathfrak{g}$-module. The main point of the present paper is to give a description of the set $H^2(\mathfrak{g}, \mathfrak{z})$ of cohomology classes of $\mathfrak{z}$-valued 2-cocycles on the Lie algebra $\mathfrak{g}$ in terms of data associated to A and $\mathfrak{k}$ which is as explicit as possible.

We consider $\mathfrak{z}$-valued 2-cochains on $\mathfrak{g}$ as linear functions $f\colon \Lambda^2(\mathfrak{g}) \to \mathfrak{z}$. Such a function is a 2-*cocycle* if and only if it vanishes on the subspace $B_2(\mathfrak{g})$ of 2-boundaries, which is the image of the linear map

$$\partial\colon \Lambda^3(\mathfrak{g}) \to \Lambda^2(\mathfrak{g}), \quad x \wedge y \wedge z \mapsto [x, y] \wedge z + [y, z] \wedge x + [z, x] \wedge y.$$

In view of the Jacobi identity, $B_2(\mathfrak{g})$ is contained in the subspace $Z_2(\mathfrak{g})$ of 2-cycles, i.e., the kernel of the linear map $b_{\mathfrak{g}}\colon \Lambda^2(\mathfrak{g}) \to \mathfrak{g}, x \wedge y \mapsto [x, y]$. The quotient space

$$H_2(\mathfrak{g}) := Z_2(\mathfrak{g})/B_2(\mathfrak{g})$$

is the *second homology space of* $\mathfrak{g}$.

A 2-cocycle f is a *coboundary* if it is of the form $f(x,y) = d_{\mathfrak{g}}\ell(x,y) := -\ell([x,y])$ for some linear map $\ell\colon \mathfrak{g} \to \mathfrak{z}$. We write $B^2(\mathfrak{g},\mathfrak{z})$ for the set of 2-coboundaries and $Z^2(\mathfrak{g},\mathfrak{z})$ for the set of 2-cocycles. This means that a coboundary vanishes on $Z_2(\mathfrak{g})$. If, conversely, a 2-cocycle vanishes on $Z_2(\mathfrak{g})$, then there exists a linear map $\alpha\colon \operatorname{im}(b_{\mathfrak{g}}) = [\mathfrak{g},\mathfrak{g}] \to \mathfrak{z}$ with $f = -b_{\mathfrak{g}}^*\alpha$, and any linear extension ℓ of α to all of $\mathfrak{g}$ yields $f = d_{\mathfrak{g}}\ell$. This leads to the following description of the second $\mathfrak{z}$-valued cohomology group

$$H^2(\mathfrak{g},\mathfrak{z}) := Z^2(\mathfrak{g},\mathfrak{z})/B^2(\mathfrak{g},\mathfrak{z}) \cong \operatorname{Lin}(H_2(\mathfrak{g}),\mathfrak{z}) \hookrightarrow \operatorname{Lin}(Z_2(\mathfrak{g}),\mathfrak{z}).$$

From this picture it is clear that we obtain a good description of $H^2(\mathfrak{g},\mathfrak{z})$ if we have an accessible description of the space $Z_2(\mathfrak{g})$ and its subspace $B_2(\mathfrak{g})$, hence of the quotient space $H_2(\mathfrak{g})$. Our goal is a description of this space and the cocycles in terms of accessible data attached to the commutative algebra A and the Lie algebra $\mathfrak{k}$. Indeed, the first step, carried out in Section 2, is to show that the direct sum decomposition

$$\Lambda^2(\mathfrak{g}) \cong (\Lambda^2(A) \otimes S^2(\mathfrak{k})) \oplus (A \otimes \Lambda^2(\mathfrak{k})) \oplus (I_A \otimes \Lambda^2(\mathfrak{k})), \tag{1}$$

where $I_A \subseteq S^2(A)$ is the kernel of the multiplication map, induces a corresponding decomposition of the space of 2-cycles

$$Z_2(\mathfrak{g}) \cong (\Lambda^2(A) \otimes S^2(\mathfrak{k})) \oplus (A \otimes Z_2(\mathfrak{k})) \oplus (I_A \otimes \Lambda^2(\mathfrak{k})). \tag{2}$$

We think of a cocycle $f \in Z^2(\mathfrak{g},\mathfrak{z})$ as represented by three linear maps

$$f_1 : \Lambda^2(A) \otimes S^2(\mathfrak{k}) \to \mathfrak{z}, \quad f_2 : A \otimes \Lambda^2(\mathfrak{k}) \to \mathfrak{z}, \quad \text{and} \quad f_3 : I_A \otimes \Lambda^2(\mathfrak{k}) \to \mathfrak{z}, \tag{3}$$

satisfying $f = f_1 \oplus f_2 \oplus f_3$ on $Z_2(\mathfrak{g})$ in the sense of (1). Since two cocycles define the same cohomology class if and only if they coincide on the subspace $Z_2(\mathfrak{g})$ of $\Lambda^2(\mathfrak{g})$, the cohomology class $[f] \in H^2(\mathfrak{g},\mathfrak{z})$ is represented by the triple (f_1, f_2', f_3), where $f_2' := f_2|_{A\otimes Z_2(\mathfrak{k})}$. Conversely, three linear maps f_1, f_2 and f_3 as in (3) define a cocycle if and only if $f := f_1 \oplus f_2 \oplus f_3$ vanishes on $B_2(\mathfrak{g})$. The main result of the present paper is Theorem 3.1 which makes this condition more explicit as follows:

(a) The alternating linear map $\widetilde{f}_1 : A \times A \to \operatorname{Sym}^2(\mathfrak{k},\mathfrak{z})$ defined by $\widetilde{f}_1(a,b)(x,y) := f_1(a \wedge b \otimes x \vee y)$ has values in the set $\operatorname{Sym}^2(\mathfrak{k},\mathfrak{z})^{\mathfrak{k}}$ of invariant symmetric bilinear maps and f_1 vanishes on $T_0(A) \otimes (\mathfrak{k} \vee \mathfrak{k}')$, where

$$T_0(A) := \operatorname{span}\{ab \wedge c + bc \wedge a + ca \wedge b - abc \wedge \mathbf{1} \colon a,b,c \in A\}$$

and $\mathfrak{k}' := [\mathfrak{k},\mathfrak{k}]$ denotes the commutator algebra of $\mathfrak{k}$.

(b) For the map $\widetilde{f}_2 \colon A \to \operatorname{Alt}^2(\mathfrak{k},\mathfrak{z})$ defined by $\widetilde{f}_2(a)(x,y) := f_2(a \vee \mathbf{1} \otimes x \wedge y)$, we have

$$d_{\mathfrak{k}}(\widetilde{f}_2(a))(x,y,z) = -\widetilde{f}_2(a)(\partial(x \wedge y \wedge z)) = \widetilde{f}_1(a,\mathbf{1})([x,y],z) \quad \text{for all} \quad a \in A, x,y,z \in \mathfrak{k},$$

with the Lie algebra differential $d_{\mathfrak{k}} \colon C^2(\mathfrak{k},\mathfrak{z}) = \operatorname{Alt}^2(\mathfrak{k},\mathfrak{z}) \to Z^3(\mathfrak{k},\mathfrak{z})$.

(c) f_3 vanishes on $I_A \otimes (\mathfrak{k} \times \mathfrak{k}')$.

Note that these conditions imply that the two maps $f_1 \oplus f_2$ and f_3 are also cocycles, whereas f_1 and f_2 are cocycles if and only if f_1 vanishes on $(A \wedge \mathbf{1}) \otimes (\mathfrak{k} \vee \mathfrak{k}')$, which, in view of (b), means that $\widetilde{f}_2(A)$ vanishes on $B_2(\mathfrak{k})$, i.e., $\widetilde{f}_2$ has values in the space $Z^2(\mathfrak{k},\mathfrak{z})$ of $\mathfrak{z}$-valued 2-cocycles on $\mathfrak{k}$. Cocycles of the form $f_1 \oplus f_2$, where f_1 and f_2 are not cocycles, are called *coupled*. All coboundaries are of the form $f = f_2$ ($f_1 = f_3 = 0$), so that the cohomology class of a coupled cocycle contains only coupled cocycles.

We show that $\mathfrak{g}$ posesses non-zero coupled 2-cocycles if and only if the image of the universal derivation $d_A\colon A \to \Omega^1(A)$ is non-trivial and $\mathfrak{k}$ possesses a symmetric invariant bilinear form κ for which the 3-cocycle $\Gamma(\kappa)(x,y,z) := \kappa([x,y],z)$ is a non-zero coboundary. The map $\Gamma\colon \mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}} \to Z^3(\mathfrak{k})$ is called the *Cartan map*. Calling an invariant symmetric bilinear form $\kappa \in \mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}}$ *exact* if $\Gamma(\kappa)$ is a coboundary, this means that $\mathfrak{k}$ possesses exact invariant bilinear forms κ with $\Gamma(\kappa)$ non-zero. Note that this is not the case if $\mathfrak{k}$ is finite-dimensional semisimple, so that there are no coupled cocycles in this case.

Our approach leads us to an exact sequence of the form

$$\{0\} \to H^2(\mathfrak{g}/\mathfrak{g}') \oplus \big(A \otimes H^2(\mathfrak{k})\big) \xrightarrow{\Phi} H^2(\mathfrak{g}) \xrightarrow{\Psi} \mathrm{Lin}((\Omega^1(A), d_A(A)), (Z^3(\mathfrak{k})_\Gamma, B^3(\mathfrak{k})_\Gamma)) \to \{0\},$$

which is the main result of Section 4. Here we use for two pairs (X, X') and (Y, Y') of linear spaces with $X \subseteq X'$ and $Y \subseteq Y'$ the notation

$$\mathrm{Lin}((X,X'),(Y,Y')) := \{f \in \mathrm{Lin}(X',Y')\colon f(X) \subseteq Y\},$$

and put $Z^3(\mathfrak{k})_\Gamma := \mathrm{im}(\Gamma) \subseteq Z^3(\mathfrak{k})$ and $B^3(\mathfrak{k})_\Gamma := B^3(\mathfrak{k}) \cap \mathrm{im}(\Gamma)$. From the exact sequence it follows that a crucial part of the description of $H^2(\mathfrak{g})$ lies in an understanding of the spaces $Z^3(\mathfrak{k})_\Gamma$ and $B^3(\mathfrak{k})_\Gamma$. In an appendix we show that the Cartan map Γ induces a map $\gamma\colon \mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}} \to H^3(\mathfrak{k})$, which is part of an exact sequence

$$\{0\} \to H^2(\mathfrak{k}) \to H^1(\mathfrak{k},\mathfrak{k}^*) \to \mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}} \xrightarrow{\gamma} H^3(\mathfrak{k}) \to H^2(\mathfrak{k},\mathfrak{k}^*) \to H^1(\mathfrak{k},\mathrm{Sym}^2(\mathfrak{k})), \tag{4}$$

which implies that for the space $\mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}}_{\mathrm{ex}} = \ker\gamma$ of exact invariant forms we have

$$\mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}}_{\mathrm{ex}} \cong H^1(\mathfrak{k},\mathfrak{k}^*)/H^2(\mathfrak{k}) \quad \text{and} \quad \mathrm{im}(\gamma) \cong \mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}}/\mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}}_{\mathrm{ex}} \cong Z^3(\mathfrak{k})_\Gamma/B^3(\mathfrak{k})_\Gamma.$$

In Section 5, we give an example of a non-trivial coupled 2-cocycle and in Section 6 we explain how our results can be used for the analysis of continuous cocycles if $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ and A and $\mathfrak{k}$ are locally convex spaces with continuous algebra structures. Then $\mathfrak{g} = A \otimes \mathfrak{k}$ carries the structure of a locally convex Lie algebra, and we are interested in the space $H^2_c(\mathfrak{g},\mathfrak{z})$ of cohomology classes of continuous 2-cocycles with values in a locally convex space $\mathfrak{z}$ modulo those coboundaries coming from continuous linear maps $\mathfrak{g} \to \mathfrak{z}$. The main difficulty in applying the algebraic results in the topological context with an infinite dimensional Lie algebra $\mathfrak{k}$ is the possible discontinuity of a linear map $h : \mathfrak{g} \to \mathfrak{z}$ bounding an algebraically trivial 2-cocycle.

If $\mathfrak{k}$ is a finite dimensional semi-simple Lie algebra and A a topological algebra, then the continuous second cohomology space $H^2_c(\mathfrak{g},\mathbb{K})$ has been determined in [Ma02] as $Z^1_c(A,\mathbb{K}) \otimes \mathrm{Sym}^2(\mathfrak{k},\mathbb{K})^{\mathfrak{k}}$, where $Z^1_c(A,\mathbb{K})$ denotes the space of continuous $\mathbb{K}$-valued cyclic 1-cocycles on A. As any exact form vanishes on a semi-simple Lie algebra, there are no coupled cocycles in this case.

The main previous contributions to the investigations of $H_2(\mathfrak{g})$ for $\mathfrak{g} = A \otimes \mathfrak{k}$ and arbitrary $\mathfrak{k}$ and A are the articles by Haddi [Ha92] and Zusmanovich [Zus94]. Both offer a description of $H_2(\mathfrak{g})$ in terms of (sub- or quotient) spaces. Haddi [Ha92] uses the projection $s_2 : H_2(\mathfrak{g}) \to H_2((\Lambda^*(\mathfrak{g})_{\mathfrak{k}},\partial))$ of $H_2(\mathfrak{g})$ to the homology of the quotient complex of $\mathfrak{k}$-coinvariants and computes kernel and cokernel of this map. The cokernel of s_2 is isomorphic to $\mathfrak{g}'/[\mathfrak{g},\mathfrak{g}']$, and the kernel is isomorphic to $(A \otimes \overline{H}_2(\mathfrak{k},\mathfrak{k}')) \oplus D(A,\mathfrak{k},\mathfrak{k}')$, $\overline{H}_2(\mathfrak{k},\mathfrak{k}')$ is the kernel of the projection $p_2 : H_2(\mathfrak{k}) \to H_2(\mathfrak{k}/\mathfrak{k}')$ (the subspace of *essential homology*), and $D(A,\mathfrak{k},\mathfrak{k}')$ is the subspace of $H_2(\mathfrak{g})$ generated by cycles of the form $ax \wedge y + ay \wedge x$ for x or $y \in \mathfrak{k}'$ and $a \in A$, which lies in $\Lambda^2(A) \otimes S^2(\mathfrak{k})$ (in our notation). Furthermore he uses a non-canonical splitting to identify the homology of the coinvariants $H_2((\Lambda^*(\mathfrak{g})_{\mathfrak{k}},\partial))$ with $(\Omega^1(A)/d_A(A) \otimes B_{\mathfrak{k},\mathfrak{k}'}) \oplus \Lambda^2(\mathfrak{g}/\mathfrak{g}')$, where

$B_{\mathfrak{k},\mathfrak{k}'}$ is the image of $\mathfrak{k} \vee \mathfrak{k}'$ in the space of $\mathfrak{k}$-coinvariants of symmetric 2-tensors on $\mathfrak{k}$. He thus obtains an exact sequence

$$0 \to A \otimes \overline{H}_2(\mathfrak{k},\mathfrak{k}') \oplus D(A,\mathfrak{k},\mathfrak{k}') \to H_2(\mathfrak{g}) \xrightarrow{s_2} \Omega^1(A)/d_A(A) \otimes B_{\mathfrak{k},\mathfrak{k}'} \oplus \Lambda^2(\mathfrak{g}/\mathfrak{g}') \to \mathfrak{g}'/[\mathfrak{g},\mathfrak{g}'] \to 0.$$

It is instructive to compare this sequence with our exact cohomology sequence describe above.

Zusmanovich [Zus94] uses as an extra data a free presentation of $\mathfrak{k}$ and deduces one of $\mathfrak{g}$. He describes the subspace of essential homology $H_2^{\mathrm{ess}}(\mathfrak{g}) = \ker(p_2 : H_2(\mathfrak{g}) \to H_2(\mathfrak{g}/\mathfrak{g}'))$ by the Hopf formula in terms of the presentation. In this way he identifies the different terms in the exact sequence given by the 5-term exact sequence of the Hochschild–Serre spectral sequence for the subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ (using non-canonical splittings). His description yields

$$H_2(\mathfrak{g}) \simeq (A \otimes H_2(\mathfrak{k})) \oplus (\Omega^1(A)/d_A(A) \otimes B(\mathfrak{k})) \oplus (\Lambda^2(\mathfrak{k}/\mathfrak{k}') \otimes I_A) \oplus (S^2(\mathfrak{k}/\mathfrak{k}') \otimes T(A)),$$

where $B(\mathfrak{k})$ is the space of $\mathfrak{k}$-coinvariants in $S^2(\mathfrak{k})$, and $T(A) \subset \Lambda^2(A)$ is spanned by the elements $ab \wedge c + ca \wedge b + bc \wedge a$ for $a,b,c \in A$.

The main advantage of our approach is that is does not require any auxiliary data and provides a quite explicit description of cocycles representing the different types of cohomology classes. In particular, this direct approach lead us to the interesting new class of coupled cocycles. In subsequent work, we plan to use the methods developed in [Ne02] to study global central extensions of Lie groups G whose Lie algebras are of the form $\mathfrak{g} = A \otimes \mathfrak{k}$ defined by coupled Lie algebra cocycles. For algebras of the type $A = C^\infty_c(M,\mathbb{R})$, i.e., compactly supported smooth functions on a manifold M, this has been carried out in [MN03] and [Ne04].

Thanks: We are greatful to M. Bordemann for a stimulating email exchange and for pointing out the relation to the exact sequence (4), part of which is due to him.

Notation

In the following we write elements of $\mathfrak{g} = A \otimes \mathfrak{k}$ simply as $ax := a \otimes x$ to simplify notation. Elements of A are mostly denoted $a, b, c, \ldots$ or $a, a', a'', \ldots$ and elements of $\mathfrak{k}$ are denoted $x, y, z, \ldots$ or $x, x', x'', \ldots$. We write $\mathfrak{k}' := [\mathfrak{k},\mathfrak{k}]$ for the commutator algebra of $\mathfrak{k}$ and observe that $\mathfrak{g}' = A \otimes \mathfrak{k}'$ is the commutator algebra of $\mathfrak{g}$.

We also write $C^p(\mathfrak{g}) := C^p(\mathfrak{g},\mathbb{K})$, $Z^p(\mathfrak{g}) := Z^p(\mathfrak{g},\mathbb{K})$, $B^p(\mathfrak{g}) := B^p(\mathfrak{g},\mathbb{K})$, and $H^p(\mathfrak{g}) := H^p(\mathfrak{g},\mathbb{K})$ for the spaces of Lie algebra p-cochains, cocycles, coboundaries and cohomology classes with values in the trivial module $\mathbb{K}$. We likewise write $\mathrm{Sym}^2(\mathfrak{k}) := \mathrm{Sym}^2(\mathfrak{k},\mathbb{K})$ for the space of $\mathbb{K}$-valued symmetric bilinear forms on $\mathfrak{k}$.

1 Several approaches to the universal differential module of A

In this section we review different constructions of the universal differential module $\Omega^1(A)$. The relationship between these constructions will play a crucial role in the following.

An important object attached to the algebra A is its universal differential module $\Omega^1(A)$. This is an A-module with a derivation $d_A \colon A \to \Omega^1(A)$ which is universal in the sense that for any other A-module M and any derivation $D \colon A \to M$, there exists a unique module morphism $\alpha \colon \Omega^1(A) \to M$ of A-modules with $D = \alpha \circ d_A$. From its universal property it is easy to derive that the universal differential module is unique up to isomorphism, but there are many realizations, looking at first sight quite different.

Let $\mu_A \colon A \otimes A \to A, a \otimes b \mapsto ab$ denote the multiplication of A. Then μ_A is an algebra morphism, so that $J_A := \ker \mu_A$ is an ideal of the commutative algebra $A \otimes A$. From the A-module structure on $A \otimes A$ given by $a.(b \otimes c) := ab \otimes c$, we thus derive an A-module structure on the quotient space J_A/J_A^2, which also is a (non-unital) commutative algebra. Let $[x]$ denote the image of $x \in J_A$ in J_A/J_A^2. Then

$$D \colon A \to J_A/J_A^2, \quad a \mapsto [\mathbf{1} \otimes a - a \otimes \mathbf{1}]$$

is a derivation and it is not hard to verify that $(J_A/J_A^2, D)$ has the universal property of $(\Omega^1(A), d_A)$ (cf. [Ma02]). We obviously have the direct decomposition $A \otimes A \cong (A \otimes \mathbf{1}) \oplus J_A$, where the projection onto the subspace J_A is given by

$$p \colon A \otimes A \to J_A, \quad a \otimes b \mapsto a \otimes b - ab \otimes \mathbf{1}.$$

This implies that

$$J_A = \operatorname{span}\{a \otimes b - ab \otimes \mathbf{1} \colon a, b \in A\} = (A \otimes \mathbf{1}).\operatorname{span}\{\mathbf{1} \otimes b - b \otimes \mathbf{1} \colon b \in A\},$$

and thus

$$\begin{aligned} J_A^2 &= \operatorname{span}\{(a \otimes \mathbf{1})(\mathbf{1} \otimes b - b \otimes \mathbf{1})(\mathbf{1} \otimes c - c \otimes \mathbf{1}) \colon a, b, c \in A\} \\ &= \operatorname{span}\{a \otimes bc - ab \otimes c - ac \otimes b + abc \otimes \mathbf{1} \colon a, b, c \in A\}. \end{aligned} \tag{5}$$

Another way to construct $\Omega^1(A)$ is by observing that each linear map $D \colon A \to M$ leads to a linear map $\widetilde{D} \colon A \otimes A \to M, a \otimes b \mapsto aDb$, and that D is a derivation if and only if

$$\ker \widetilde{D} \supseteq \{\mathbf{1} \otimes ab - a \otimes b - b \otimes a \colon a, b \in A\},$$

which implies that $\ker \widetilde{D}$ contains the A-submodule

$$B_1(A) := \operatorname{span}\{c \otimes ab - ca \otimes b - cb \otimes a \colon a, b, c \in A\} = \operatorname{span}\{ab \otimes c + ac \otimes b - a \otimes bc \colon a, b, c \in A\},$$

of $A \otimes A$. The quotient

$$HH_1(A) := (A \otimes A)/B_1(A)$$

is called the *first Hochschild homology space of* A. From the preceding discussion it follows that the map

$$HH_1(A) \to \Omega^1(A), \quad [a \otimes b] \mapsto a d_A(b) \tag{6}$$

is an isomorphism of A-modules because the map $D \colon A \to HH_1(A), a \mapsto [\mathbf{1} \otimes a]$ is a derivation with the universal property (cf. [Lo98], Prop. 1.1.10). The link between the description of $\Omega^1(A)$ as $HH_1(A)$ and J_A/J_A^2 is given by the commutative diagram

$$\begin{array}{ccc} A \otimes A & \xrightarrow{p} & J_A \\ \downarrow & & \downarrow \\ HH_1(A) & \xrightarrow{\varphi} & J_A/J_A^2 \end{array}$$

with the isomorphism $\varphi([a \otimes b]) = aD(b) = [a \otimes b - ab \otimes \mathbf{1}]$. Note that the commutativity of the diagram implies that

$$J_A^2 = p(B_1(A)). \tag{7}$$

Let

$$T(A) := \operatorname{span}\{ab \wedge c + bc \wedge a + ca \wedge b \in \Lambda^2(A) \colon a, b, c \in A\}$$

denote the image of the subspace $B_1(A) \subseteq A \otimes A$ under the quotient map $A \otimes A \to \Lambda^2(A), a \otimes b \mapsto a \wedge b$. In view of $ad_A(b) + bd_A(a) = d_A(ab)$, the image of the subspace of symmetric tensors, which we identify with $S^2(A)$, in $\Omega^1(A)$ coincides with $d_A(A)$, so that equation (6) immediately shows that the map

$$\Lambda^2(A)/T(A) \cong (A \otimes A)/(S^2(A) + B_1(A)) \to \Omega^1(A)/d_A(A), \quad [a \wedge b] \mapsto [ad_A(b)]$$

induces a linear isomorphism. We shall see below that the *first cyclic homology space*

$$HC_1(A) := \Omega^1(A)/d_A(A) \cong HH_1(A)/[\mathbf{1} \otimes A] = \Lambda^2(A)/T(A)$$

is of central importance for Lie algebra 2-cocycles on Lie algebras of the form $A \otimes \mathfrak{k}$.

Alternating bilinear maps $f \colon A \times A \to \mathfrak{z}$ for which the corresponding map $\Lambda^2(A) \to \mathfrak{z}$ vanishes on $T(A)$ are called *cyclic* 1-*cocycles*, which means that

$$f(a, bc) + f(b, ca) + f(c, ab) = 0 \quad \text{for} \quad a, b, c \in A.$$

From the above, it follows that the space $Z^1(A, \mathfrak{z})$ of $\mathfrak{z}$-valued cyclic 1-cocycles can be identified with

$$\operatorname{Lin}(HC_1(A), \mathfrak{z}) \cong \{L \in \operatorname{Lin}(\Omega^1(A), \mathfrak{z}) \colon d_A(A) \subseteq \ker L\}.$$

We define two trilinear maps

$$T \colon A^3 \to \Lambda^2(A), \quad (a, b, c) \mapsto \sum_{\text{cyc.}} ab \wedge c := ab \wedge c + bc \wedge a + ca \wedge b$$

and

$$T_0 \colon A^3 \to \Lambda^2(A), \quad T_0(a, b, c) := T(a, b, c) - abc \wedge \mathbf{1}.$$

We also put $T_0(A) := \operatorname{span}(\operatorname{im}(T_0))$.

Lemma 1.1 *The map*

$$\gamma_A \colon \Lambda^2(A) \to \Omega^1(A), \quad a \wedge b \mapsto ad_A(b) - bd_A(a)$$

is surjective and $\ker \gamma_A = T_0(A)$.

Proof. That γ_A is surjective follows from

$$\gamma_A(a \wedge b + \mathbf{1} \wedge ab) = ad_A(b) - bd_A(a) + d_A(ab) = 2ad_A(b).$$

For the determination of the kernel of γ_A, we use the realization of $\Omega^1(A)$ as J_A/J_A^2. In this case $d_A(a) = [\mathbf{1} \otimes a - a \otimes \mathbf{1}]$, so that

$$\gamma_A(a \wedge b) = [a \otimes b - ab \otimes \mathbf{1} - b \otimes a + ba \otimes \mathbf{1}] = [a \otimes b - b \otimes a].$$

Therefore the kernel of γ_A is the intersection $\Lambda^2(A) \cap J_A^2$, where we consider $\Lambda^2(A)$ as the subspace of skew-symmetric tensors in $A \otimes A$.

Writing $A \otimes A$ as $\Lambda^2(A) \oplus S^2(A)$, the commutativity of the multiplication of A shows that

$$J_A = \Lambda^2(A) \oplus I_A \quad \text{holds for} \quad I_A := J_A \cap S^2(A).$$

Since the flip involution is an algebra isomorphism of $A \otimes A$, we have

$$\Lambda^2(A)\Lambda^2(A) + S^2(A)S^2(A) \subseteq S^2(A) \quad \text{and} \quad \Lambda^2(A)S^2(A) \subseteq \Lambda^2(A).$$

This implies that

$$\ker \gamma_A = \Lambda^2(A) \cap J_A^2 = I_A \cdot \Lambda^2(A),$$

and that this subspace coincides with the image of J_A^2 under the projection

$$\alpha\colon A \otimes A \to \Lambda^2(A), \quad a \otimes b \mapsto a \wedge b = \frac{1}{2}(a \otimes b - b \otimes a).$$

Finally, this leads with (5) to

$$\ker \gamma_A = \alpha(J_A^2) = \operatorname{span}\{a \wedge bc - ab \wedge c - ac \wedge b + abc \wedge \mathbf{1} \colon a, b, c \in A\} = T_0(A).$$

■

2 A decomposition of $\Lambda^2(\mathfrak{g})$

In this section we turn to the identification of the space $B_2(\mathfrak{g})$ of 2-coboundaries in $Z_2(\mathfrak{g})$ in terms of our threefold direct sum decomposition (2).

From the universal property of $\Lambda^2(\mathfrak{g})$ we immediately obtain linear maps

$$p_+ \colon \Lambda^2(\mathfrak{g}) \to \Lambda^2(A) \otimes S^2(\mathfrak{k}), \quad ax \wedge by \mapsto a \wedge b \otimes x \vee y$$

and

$$p_- \colon \Lambda^2(\mathfrak{g}) \to S^2(A) \otimes \Lambda^2(\mathfrak{k}), \quad ax \wedge by \mapsto a \vee b \otimes x \wedge y.$$

We likewise have linear maps

$$\sigma_+ \colon \Lambda^2(A) \otimes S^2(\mathfrak{k}) \to \Lambda^2(\mathfrak{g}), \quad a \wedge b \otimes x \vee y \to \frac{1}{2}(ax \wedge by + ay \wedge bx)$$

and

$$\sigma_- \colon S^2(A) \otimes \Lambda^2(\mathfrak{k}) \to \Lambda^2(\mathfrak{g}), \quad a \vee b \otimes x \wedge y \to \frac{1}{2}(ax \wedge by - ay \wedge bx)$$

satisfying

$$p_+ \circ \sigma_+ = \operatorname{id}, \quad p_- \circ \sigma_- = \operatorname{id} \quad \text{and} \quad \sigma_+ p_+ + \sigma_- p_- = \operatorname{id}_{\Lambda^2(\mathfrak{g})}.$$

In this sense we have

$$\Lambda^2(\mathfrak{g}) \cong \big(\Lambda^2(A) \otimes S^2(\mathfrak{k})\big) \oplus \big(S^2(A) \otimes \Lambda^2(\mathfrak{k})\big),$$

and the projections on the two summands are given by $p_\pm$.

Recall the kernel J_A of the multiplication map $\mu_A \colon A \otimes A \to A$. Since A is commutative, we then have $\Lambda^2(A) \subseteq J_A$, so that

$$J_A = \Lambda^2(A) \oplus I_A \quad \text{with} \quad I_A := J_A \cap S^2(A),$$

according to the decomposition $A \otimes A \cong \Lambda^2(A) \oplus S^2(A)$. The map

$$\sigma_A \colon A \to S^2(A), \quad a \mapsto a \vee \mathbf{1}$$

is a section of the multiplication map μ_A, so that we obtain a direct sum decomposition

$$S^2(A) = (A \vee \mathbf{1}) \oplus I_A \cong A \oplus I_A.$$

In view of this decomposition, we obtain a direct sum decomposition of $\Lambda^2(\mathfrak{g})$:

$$\Lambda^2(\mathfrak{g}) \cong (\Lambda^2(A) \otimes S^2(\mathfrak{k})) \oplus (A \otimes \Lambda^2(\mathfrak{k})) \oplus (I_A \otimes \Lambda^2(\mathfrak{k})),$$

where the projections p_1, p_2, p_3 on the three summands are given by

$$p_1(ax \wedge by) = p_+(ax \wedge by) = a \wedge b \otimes x \vee y,$$

$$p_2(ax \wedge by) = ab \otimes x \wedge y, \quad \text{and} \quad p_3(ax \wedge by) = (a \vee b - ab \vee \mathbf{1}) \otimes x \wedge y.$$

The following lemma provides the decomposition of $Z_2(\mathfrak{g})$ which is a central tool in the following.

Lemma 2.1 *The space $Z_2(\mathfrak{g})$ is adapted to the direct sum decomposition of $\Lambda^2(\mathfrak{g})$:*

$$Z_2(\mathfrak{g}) = (\Lambda^2(A) \otimes S^2(\mathfrak{k})) \oplus (A \otimes Z_2(\mathfrak{k})) \oplus (I_A \otimes \Lambda^2(\mathfrak{k})).$$

Proof. Since $b_{\mathfrak{g}}(ax \wedge by) = ab[x,y]$ is symmetric in a, b and alternating in x, y, its kernel contains $\Lambda^2(A) \otimes S^2(\mathfrak{k})$. The formula for $b_{\mathfrak{g}}$ also shows immediately that $I_A \otimes \Lambda^2(\mathfrak{k}) \subseteq \ker b_{\mathfrak{g}}$, so that it remains to observe that

$$Z_2(\mathfrak{g}) \cap (A \otimes \Lambda^2(\mathfrak{k})) = \ker b_{\mathfrak{g}} \cap (A \otimes \Lambda^2(\mathfrak{k})) = A \otimes Z_2(\mathfrak{k})$$

because $b_{\mathfrak{g}}(a \vee \mathbf{1} \otimes x \wedge y) = \frac{1}{2} b_{\mathfrak{g}}(ax \wedge y + x \wedge ay) = a[x,y] = ab_{\mathfrak{k}}(x \wedge y)$. ∎

In the following we write $\equiv \mod B_2(\mathfrak{g})$ for congruence of elements of $\Lambda^2(\mathfrak{g})$ modulo $B_2(\mathfrak{g})$.

Lemma 2.2 *For $a, b, c \in A$ and $x, y, z \in \mathfrak{k}$ we have*

$$p_1(\partial(ax \wedge by \wedge cz)) \equiv p_1(abcx \wedge [y,z]) \equiv -p_2(\partial(ax \wedge by \wedge cz)) \mod B_2(\mathfrak{g}).$$

In particular $(p_1 + p_2)(B_2(\mathfrak{g})) \subseteq B_2(\mathfrak{g})$.

Proof. From

$$\begin{aligned}
\partial(ax \wedge by \wedge cz) &= ab[x,y] \wedge cz + bc[y,z] \wedge ax + ac[z,x] \wedge by \\
\partial(cx \wedge ay \wedge bz) &= ac[x,y] \wedge bz + ab[y,z] \wedge cx + bc[z,x] \wedge ay \\
\partial(x \wedge acy \wedge bz) &= ac[x,y] \wedge bz + abc[y,z] \wedge x + b[z,x] \wedge acy \\
\partial(cx \wedge y \wedge abz) &= c[x,y] \wedge abz + ab[y,z] \wedge cx + abc[z,x] \wedge y \\
\partial(bcx \wedge ay \wedge z) &= abc[x,y] \wedge z + a[y,z] \wedge bcx + bc[z,x] \wedge ay \\
\partial(abcx \wedge y \wedge z) &= abc[x,y] \wedge z + [y,z] \wedge abcx + abc[z,x] \wedge y
\end{aligned}$$

we derive

$$\begin{aligned}
& \partial(ax \wedge by \wedge cz) + \partial(cx \wedge ay \wedge bz) - \partial(x \wedge acy \wedge bz) - \partial(cx \wedge y \wedge abz) \\
& -\partial(bcx \wedge ay \wedge z) + \partial(abcx \wedge y \wedge z) \\
= \; & ab[x,y] \wedge cz + bc[y,z] \wedge ax + ac[z,x] \wedge by - b[z,x] \wedge acy - c[x,y] \wedge abz - a[y,z] \wedge bcx \\
& -abc[y,z] \wedge x + [y,z] \wedge abcx \\
= \; & 2ab \wedge c \otimes [x,y] \vee z + 2bc \wedge a \otimes [y,z] \vee x + 2ac \wedge b \otimes [z,x] \vee y - 2abc \wedge 1 \otimes x \vee [y,z] \\
= \; & 2p_1(\partial(ax \wedge by \wedge cz)) - 2p_1(abcx \wedge [y,z]).
\end{aligned}$$

This proves the first congruence.

Note that for $a \in A$ and $x, y, z \in \mathfrak{k}$ we have

$$\partial(ax \wedge y \wedge z) = a[x,y] \wedge z + [y,z] \wedge ax + a[z,x] \wedge y,$$

which implies that

$$a[x,y] \wedge z + a[z,x] \wedge y \equiv ax \wedge [y,z] \mod B_2(\mathfrak{g}). \tag{8}$$

Summing over all cyclic permutations of (x, y, z), leads to

$$2\sum_{\text{cyc.}} a[x,y] \wedge z = \sum_{\text{cyc.}} ax \wedge [y,z] \mod B_2(\mathfrak{g}). \tag{9}$$

From the relation (8) we get

$$2p_1(abcx \wedge [y,z]) = abc[y,z] \wedge x + abcx \wedge [y,z] \equiv \sum_{\text{cyc.}} abc[y,z] \wedge x = \sum_{\text{cyc.}} abc[x,y] \wedge z.$$

In view of

$$\begin{aligned} p_2(\partial(ax \wedge by \wedge cz)) &= p_2(ab[x,y] \wedge cz + bc[y,z] \wedge ax + ca[z,x] \wedge by) \\ &= abc \otimes ([x,y] \wedge z + [y,z] \wedge x + [z,x] \wedge y) \\ &= abc \otimes \partial(x \wedge y \wedge z), \end{aligned} \tag{10}$$

relation (9) yields

$$\begin{aligned} 2p_2(\partial(ax \wedge by \wedge cz)) &= \sum_{\text{cyc.}} abc[x,y] \wedge z - \sum_{\text{cyc.}} abcz \wedge [x,y] \equiv \sum_{\text{cyc.}} abc[x,y] \wedge z - 2\sum_{\text{cyc.}} abc[x,y] \wedge z \\ &= -\sum_{\text{cyc.}} abc[x,y] \wedge z \equiv -2p_1(\partial(ax \wedge by \wedge cz)). \end{aligned}$$

■

In view of the preceding lemma, the projection $p_1 + p_2$ of $\Lambda^2(\mathfrak{g})$ onto the subspace

$$\Lambda^2(A) \otimes S^2(\mathfrak{k}) \oplus A \otimes \Lambda^2(\mathfrak{k})$$

preserves $B_2(\mathfrak{g})$. This also implies that $\mathrm{id} - p_1 - p_2 = p_3$ preserves $B_2(\mathfrak{g})$, and we derive that

$$B_2(\mathfrak{g}) = B_2(\mathfrak{g}) \cap \left(\Lambda^2(A) \otimes S^2(\mathfrak{k}) \oplus A \otimes \Lambda^2(\mathfrak{k})\right) \oplus B_2(\mathfrak{g}) \cap (I_A \otimes \Lambda^2(\mathfrak{k})).$$

The following lemma provides refined information.

Lemma 2.3 **(1)** $\Lambda^2(A) \otimes \mathfrak{k}.S^2(\mathfrak{k}) + T_0(A) \otimes \mathfrak{k} \vee \mathfrak{k}' \subseteq B_2(\mathfrak{g})$ *and* $p_1(B_2(\mathfrak{g})) = \Lambda^2(A) \otimes \mathfrak{k}.S^2(\mathfrak{k}) + T(A) \otimes \mathfrak{k} \vee \mathfrak{k}'$.

(2) $p_2(B_2(\mathfrak{g})) = A \otimes B_2(\mathfrak{k})$.

(3) $I_A \otimes (\mathfrak{k} \wedge \mathfrak{k}') = p_3(B_2(\mathfrak{g})) \subseteq B_2(\mathfrak{g})$.

Proof. (2) follows immediately from formula (10).

(1) That $\Lambda^2(A) \otimes \mathfrak{k}.S^2(\mathfrak{k})$ is contained in $B_2(\mathfrak{g})$ follows immediately from

$$\begin{aligned} & \partial(ax \wedge by \wedge z) - \partial(bx \wedge ay \wedge z) \\ = \;& ab[x,y] \wedge z + b[y,z] \wedge ax + a[z,x] \wedge by - ab[x,y] \wedge z - a[y,z] \wedge bx - b[z,x] \wedge ay \\ = \;& b[y,z] \wedge ax + a[z,x] \wedge by - a[y,z] \wedge bx - b[z,x] \wedge ay \\ = \;& 2b \wedge a \otimes [y,z] \vee x + 2a \wedge b \otimes [z,x] \vee y \\ = \;& 2a \wedge b \otimes ([z,x] \vee y - [y,z] \vee x) \\ = \;& 2a \wedge b \otimes z.(x \vee y). \end{aligned}$$

Therefore the description of $p_1(B_2(\mathfrak{g})) = \mathrm{im}(p_1 \circ \partial)$ follows from

$$\begin{aligned} p_1(\partial(ax \wedge by \wedge cz)) &= p_1(ab[x,y] \wedge cz + bc[y,z] \wedge ax + ca[z,x] \wedge by) \\ &= ab \wedge c \otimes [x,y] \vee z + bc \wedge a \otimes [y,z] \vee x + ca \wedge b \otimes [z,x] \vee y \\ &\equiv (ab \wedge c + bc \wedge a + ca \wedge b) \otimes [x,y] \vee z \qquad \mathrm{mod}\ \Lambda^2(A) \otimes \mathfrak{k}.S^2(\mathfrak{k}) \\ &= T(a,b,c) \otimes [x,y] \vee z. \end{aligned}$$

In (10), we have seen that

$$p_2(\partial(ax \wedge by \wedge cz)) = abc \otimes \partial(x \wedge y \wedge z),$$

and this implies that

$$p_2(\partial(abx \wedge y \wedge cz)) = abc \otimes \partial(x \wedge y \wedge z),$$

which leads to

$$\partial(ax \wedge by \wedge cz) - \partial(abx \wedge y \wedge cz) \in \ker p_2.$$

In view of

$$T(a,b,c) - T(ab,\mathbf{1},c) = T(a,b,c) - (ab \wedge c + c \wedge ab + abc \wedge \mathbf{1}) = T(a,b,c) - abc \wedge \mathbf{1} = T_0(a,b,c)$$

and Lemma 2.2, the following element is contained in $B_2(\mathfrak{g})$:

$$\begin{aligned} p_1(\partial(ax \wedge by \wedge cz) - \partial(abx \wedge y \wedge cz)) &\in \big(T(a,b,c) - T(ab,\mathbf{1},c)\big) \otimes [x,y] \vee z + \Lambda^2(A) \otimes \mathfrak{k}.S^2(\mathfrak{k}) \\ &\subseteq T_0(a,b,c) \otimes [x,y] \vee z + B_2(\mathfrak{g}), \end{aligned}$$

and now Lemma 2.2 implies that $T_0(A) \otimes \mathfrak{k}' \vee \mathfrak{k} \subseteq B_2(\mathfrak{g})$.

(3) First we note that

$$\begin{aligned} & p_3(\partial(ax \wedge by \wedge z)) = p_3(ab[x,y] \wedge z + b[y,z] \wedge ax + a[z,x] \wedge by) \\ = \;& (ab \vee \mathbf{1} - ab \vee \mathbf{1}) \otimes [x,y] \wedge z + (b \vee a - ab \vee \mathbf{1}) \otimes [y,z] \wedge x + (a \vee b - ab \vee \mathbf{1}) \otimes [z,x] \wedge y \\ = \;& (a \vee b - ab \vee \mathbf{1}) \otimes ([y,z] \wedge x + [z,x] \wedge y). \end{aligned} \tag{11}$$

Since p_3 preserves $B_2(\mathfrak{g})$ (Lemma 2.2), this expressions lies in $B_2(\mathfrak{g})$. Using the same formula for all cyclic permutations of x, y, z and adding all three terms, we see that

$$2(a \vee b - ab \vee \mathbf{1}) \otimes \sum_{\mathrm{cyc.}} [x,y] \wedge z \in B_2(\mathfrak{g}).$$

This also implies that

$$(a \vee b - ab \vee \mathbf{1}) \otimes [x,y] \wedge z = (a \vee b - ab \vee \mathbf{1}) \otimes \Big(\sum_{\text{cyc.}} [x,y] \wedge z - ([y,z] \wedge x + [z,x] \wedge y)\Big) \in B_2(\mathfrak{g}).$$

Next we note that I_A is spanned by elements of the form $a \vee b - ab \vee \mathbf{1}$, because $a \vee b \mapsto a \vee b - ab \vee \mathbf{1}$ is the projection of $S^2(A)$ onto I_A with kernel $A \cong A \vee \mathbf{1}$. Therefore $B_2(\mathfrak{g})$ contains $I_A \otimes \mathfrak{k}' \wedge \mathfrak{k}$. On the other hand, (11) shows that $p_3(B_2(\mathfrak{g}))$ is clearly contained in $I_A \otimes \mathfrak{k}' \wedge \mathfrak{k}$. ■

Theorem 2.4 *With the linear map*

$$F \colon A \otimes \Lambda^3(\mathfrak{k}) \to \Lambda^2(\mathfrak{g}), \quad a \otimes (x \wedge y \wedge z) \mapsto (a \wedge \mathbf{1} \otimes [x,y] \vee z) + a \otimes \partial(x \wedge y \wedge z)$$

we get the following description of $B_2(\mathfrak{g})$:

$$B_2(\mathfrak{g}) = \Lambda^2(A) \otimes \mathfrak{k}.S^2(\mathfrak{k}) + T_0(A) \otimes \mathfrak{k} \vee \mathfrak{k}' + \mathrm{im}(F) + I_A \otimes (\mathfrak{k} \wedge \mathfrak{k}').$$

Proof. The description of the position of $B_2(\mathfrak{g})$ given in Lemma 2.3 is already quite detailed. It shows in particular that the part of $B_2(\mathfrak{g})$ lying in $\Lambda^2(A) \otimes S^2(\mathfrak{k}) \oplus A \otimes Z_2(\mathfrak{k})$ contains

$$T_0(A) \otimes \mathfrak{k} \vee \mathfrak{k}' + \Lambda^2(A) \otimes \mathfrak{k}.S^2(\mathfrak{k})$$

and is contained in

$$A \otimes B_2(\mathfrak{k}) \oplus T(A) \otimes \mathfrak{k} \vee \mathfrak{k}' + \Lambda^2(A) \otimes \mathfrak{k}.S^2(\mathfrak{k}) = A \otimes B_2(\mathfrak{k}) \oplus (A \wedge \mathbf{1}) \otimes \mathfrak{k} \vee \mathfrak{k}' + T_0(A) \otimes \mathfrak{k} \vee \mathfrak{k}' + \Lambda^2(A) \otimes \mathfrak{k}.S^2(\mathfrak{k}).$$

Further Lemma 2.3 (1) implies that

$$F(a, x \wedge y \wedge z) \in (p_1 + p_2)(\partial(ax \wedge y \wedge z)) + (\Lambda^2(A) \otimes \mathfrak{k}.S^2(\mathfrak{k})),$$

because

$$p_1(\partial(ax \wedge y \wedge z)) \equiv a \wedge \mathbf{1} \otimes [x,y] \vee z \mod \Lambda^2(A) \otimes \mathfrak{k}.S^2(\mathfrak{k})$$

and

$$p_2(\partial(ax \wedge y \wedge z)) = a \otimes \partial(x \wedge y \wedge z).$$

We thus deduce that $\mathrm{Im}(F) \subset B_2(\mathfrak{g})$. Now the theorem follows. ■

3 The description of the 2-cocycles

As explained in the introduction, we think of the elements of $H^2(\mathfrak{g}, \mathfrak{z})$ as linear maps $f \colon Z_2(\mathfrak{g}) \to \mathfrak{z}$ vanishing on the subspace $B_2(\mathfrak{g})$. We further write 2-cocycles as $f = f_1 + f_2 + f_3$, according to the decomposition in Lemma 2.1, where

$$f_1 \colon \Lambda^2(A) \otimes S^2(\mathfrak{k}) \to \mathfrak{z}, \quad f_2 \colon A \otimes \Lambda^2(\mathfrak{k}) \to \mathfrak{z} \quad \text{and} \quad f_3 \colon I_A \otimes \Lambda^2(\mathfrak{k}) \to \mathfrak{z}.$$

We then think of f_1 as an alternating bilinear map $\widetilde{f}_1 \colon A \times A \to \mathrm{Sym}^2(\mathfrak{k}, \mathfrak{z})$, of f_2 as a linear map $\widetilde{f}_2 \colon A \to \mathrm{Lin}(\Lambda^2(\mathfrak{k}), \mathfrak{z})$, and of f_3 as a symmetric bilinear map $\widetilde{f}_3 \colon I_A \to \mathrm{Alt}^2(\mathfrak{k}, \mathfrak{z})$.

The condition, that three such maps $\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3$ combine to a 2-cocycle

$$f\colon Z_2(\mathfrak{g}) \to \mathfrak{z}, \quad a \wedge a' \otimes x \vee x' + (b \vee \mathbf{1}) \otimes y \wedge y' + c \otimes (z \wedge z') \mapsto \widetilde{f}_1(a,a')(x,x') + \widetilde{f}_2(b)(y,y') + \widetilde{f}_3(c)(z,z'),$$

is that f vanishes on $B_2(\mathfrak{g})$. To make this condition more explicit, we define the *Cartan map*

$$\Gamma\colon \operatorname{Lin}(\mathfrak{k} \vee \mathfrak{k}', \mathfrak{z})^{\mathfrak{k}} \to Z^3(\mathfrak{k}, \mathfrak{z}), \quad \Gamma(\kappa)(x,y,z) := \kappa([x,y],z).$$

That $\Gamma(\kappa)$ is alternating follows from

$$\Gamma(\kappa)(x,z,y) = \kappa([x,z],y) = \kappa(y,[x,z]) = \kappa([y,x],z) = -\Gamma(\kappa)(x,y,z)$$

and the fact that the symmetric group S_3 is generated by the transpositions $(1\ 2)$ and $(2\ 3)$. That the image of Γ consists of 3-cocycles follows from

$$\begin{aligned}
(d_{\mathfrak{k}}\Gamma(\kappa))(x_0,x_1,x_2,x_3) &= -\kappa([[x_0,x_1],x_2],x_3) + \kappa([[x_0,x_2],x_1],x_3) - \kappa([[x_0,x_3],x_1],x_2) \\
&\quad -\kappa([[x_1,x_2],x_0],x_3) + \kappa([[x_1,x_3],x_0],x_2) - \kappa([[x_2,x_3],x_0],x_1) \\
&= \kappa(-[[x_0,x_1],x_2] + [[x_0,x_2],x_1] - [[x_1,x_2],x_0],x_3) \\
&\quad +\kappa(-[[x_0,x_3],x_1] + [[x_1,x_3],x_0],x_2) - \kappa(x_2,[x_3,[x_0,x_1]]) \\
&= \kappa(-[[x_0,x_3],x_1] + [[x_1,x_3],x_0] - [x_3,[x_0,x_1]],x_2) = 0.
\end{aligned}$$

For the following theorem we observe that the Lie algebra differential $d_{\mathfrak{k}}\colon C^2(\mathfrak{k},\mathfrak{z}) = \operatorname{Alt}^2(\mathfrak{k},\mathfrak{z}) \to Z^3(\mathfrak{k},\mathfrak{z})$ factors through the surjective map $\operatorname{Alt}^2(\mathfrak{k},\mathfrak{z}) \twoheadrightarrow \operatorname{Lin}(Z_2(\mathfrak{k}),\mathfrak{z})$, whose kernel are the 2-coboundaries.

Theorem 3.1 (Description of cocycles) *The function $f = f_1 + f_2 + f_3$ as above is a 2-cocycle if and only if the following conditions are satisfied:*

(a) $\operatorname{im}(\widetilde{f}_1) \subseteq \operatorname{Sym}^2(\mathfrak{k},\mathfrak{z})^{\mathfrak{k}}$.

(b) $\widetilde{f}_1(T_0(A))$ *vanishes on* $\mathfrak{k} \times \mathfrak{k}'$.

(c) $d_{\mathfrak{k}}(\widetilde{f}_2(a)) = \Gamma(\widetilde{f}_1(a,\mathbf{1}))$ *for each* $a \in A$.

(d) $\widetilde{f}_3(I_A)$ *vanishes on* $\mathfrak{k} \times \mathfrak{k}'$.

Proof. The linear map f is a 2-cocycle if and only if it vanishes on $B_2(\mathfrak{g})$. In view of Theorem 2.4, $B_2(\mathfrak{g})$ is the sum of four subspaces, so that we get four conditions.

Condition (a) means that f vanishes on $\Lambda^2(A) \otimes \mathfrak{k}.S^2(\mathfrak{k})$, and condition (b) that it vanishes on the subspace $T_0(A) \otimes \mathfrak{k} \vee \mathfrak{k}'$.

That f vanishes on the image of F, means that

$$\Gamma(\widetilde{f}_1(a,\mathbf{1}))(x,y,z) = \widetilde{f}_1(a,\mathbf{1})([x,y],z) = -\widetilde{f}_2(a)(\partial(x \wedge y \wedge z)) = (d_{\mathfrak{k}}\widetilde{f}_2(a))(x,y,z)$$

for $a \in A$ and $x,y,z \in \mathfrak{k}$, which is (c).

Finally, (d) means that f vanishes on $I_A \otimes \mathfrak{k} \wedge \mathfrak{k}'$. ■

Corollary 3.2 *$f = f_1 + f_2 + f_3$ is a cocycle if and only if $f_1 + f_2$ and f_3 are cocycles.*

Corollary 3.3 *A function of one of the three types $f = f_i$, $i = 1,2,3$, is a 2-cocycle if and only if the following conditions are satisfied:*

($i = 1$) $\mathrm{im}(\widetilde{f}_1) \subseteq \mathrm{Sym}^2(\mathfrak{k}, \mathfrak{z})^{\mathfrak{k}}$ *and the induced map $A \times A \to \mathrm{Lin}(\mathfrak{k} \vee \mathfrak{k}', \mathfrak{z})^{\mathfrak{k}}$ is a cyclic 1-cocycle.*

($i = 2$) $\widetilde{f}_2(A) \subseteq Z^2(\mathfrak{k}, \mathfrak{z})$.

($i = 3$) $\widetilde{f}_3(I_A)$ *vanishes on $\mathfrak{k} \times \mathfrak{k}'$.*

Proof. That $f = f_i$ is a 2-cocycle is equivalent to f vanishing on $p_i(B_2(\mathfrak{g}))$, so that Lemma 2.3 leads to the stated characterizations. ∎

Remark 3.4 A special class of cocycles are those of the form $f = f_1$, vanishing on $\mathfrak{g} \times \mathfrak{g}'$. The cocycles of the form $f = f_3$ also vanish on the commutator algebra, and the sums of these two types exhaust the image of the injective pull-back map $H^2(\mathfrak{g}/\mathfrak{g}', \mathfrak{z}) \cong \mathrm{Alt}^2(\mathfrak{g}/\mathfrak{g}', \mathfrak{z}) \to H^2(\mathfrak{g}, \mathfrak{z})$.

Corollary 3.5 *For each cocycle $f = f_1 + f_2 + f_3$ there exists a decomposition $f_1 = f_1^0 + f_1^1$, where*

$$f_1^0(\mathfrak{g}, \mathfrak{g}') = \{0\}, \quad \mathrm{im}(\widetilde{f}_1^1) \subseteq \mathrm{Sym}^2(\mathfrak{k}, \mathfrak{z})^{\mathfrak{k}} \quad \textit{and} \quad T_0(A) \subseteq \ker \widetilde{f}_1^1.$$

Proof. Conditions (a) and (b) in Theorem 3.1 only refer to the restriction $\overline{f}_1^1$ of f_1 to the subspace $\Lambda^2(A) \otimes (\mathfrak{k} \vee \mathfrak{k}')$ of $\Lambda^2(A) \otimes S^2(\mathfrak{k})$. This has the following interesting consequence. We have a short exact sequence

$$\{0\} \to \mathrm{Sym}^2(\mathfrak{k}/\mathfrak{k}', \mathfrak{z}) \to \mathrm{Sym}^2(\mathfrak{k}, \mathfrak{z})^{\mathfrak{k}} \to \mathrm{Lin}(\mathfrak{k} \vee \mathfrak{k}', \mathfrak{z})^{\mathfrak{k}} \to \{0\},$$

where the surjectivity of the map $\mathrm{Sym}^2(\mathfrak{k}, \mathfrak{z})^{\mathfrak{k}} \to \mathrm{Lin}(\mathfrak{k} \vee \mathfrak{k}', \mathfrak{z})^{\mathfrak{k}}$ follows from the fact that any symmetric bilinear extension of an element of $\mathrm{Lin}(\mathfrak{k} \vee \mathfrak{k}', \mathfrak{z})^{\mathfrak{k}}$ is invariant. Any splitting of this sequence extends $\overline{f}_1^1$ to an alternating bilinear map $\widetilde{f}_1^1 \colon A \times A \to \mathrm{Sym}^2(\mathfrak{k}, \mathfrak{z})^{\mathfrak{k}}$ with

$$\widetilde{f}_1^1(a,b)(x,y) = \widetilde{f}_1(a,b)(x,y) \quad \text{for} \quad a, b \in A, x \in \mathfrak{k}, y \in \mathfrak{k}'$$

and such that $T_0(A) \subseteq \ker \widetilde{f}_1^1$. Then

$$\Gamma(\widetilde{f}_1^1(a, \mathbf{1})) = \Gamma(\widetilde{f}_1(a, \mathbf{1})) \quad \text{for} \quad a \in A,$$

so that $f_1^1 + f_2 + f_3$ also is a cocycle by Theorem 3.1. We conclude that $f_1^0 := f_1 - f_1^1$ is a cocycle vanishing on $\mathfrak{g} \times \mathfrak{g}'$. This proves the assertion. ∎

Proposition 3.6 (Description of coboundaries) *A cocycle $f = f_1 + f_2 + f_3$ is a coboundary if and only if $f_1 = f_3 = 0$ and there exists a linear map $\ell \colon A \to \mathrm{Lin}(\mathfrak{k}, \mathfrak{z})$ with $d_{\mathfrak{g}}\ell = f_2$, i.e.,*

$$\widetilde{f}_2(a) = d_{\mathfrak{k}}(\ell(a)) \quad \textit{for all} \quad a \in A.$$

Proof. That f is a coboundary means that it vanishes on $Z_2(\mathfrak{g})$. According to Lemma 2.1, this implies that $f_1 = f_3 = 0$. Since the bracket map $b_{\mathfrak{g}} \colon \Lambda^2(\mathfrak{g}) \to \mathfrak{g}$ is alternating in $\mathfrak{k}$ and symmetric in A, all coboundaries are of the form $f = f_2$. ∎

We call cocycles of the form $f_1 + f_2$ for which f_1 is not a cocycle *coupled*. The following theorem characterizes the pairs $(A, \mathfrak{k})$ for which $A \otimes \mathfrak{k}$ possesses coupled cocycles. In Section 5 below we shall also give a concrete example of a Lie algebra $\mathfrak{k}$ satisfying this condition.

Theorem 3.7 *The Lie algebra $\mathfrak{g} = A \otimes \mathfrak{k}$ possesses coupled cocycles if and only if $d_A(A) \neq \{0\}$ and $\mathfrak{k}$ possesses a symmetric invariant bilinear form κ for which $\Gamma(\kappa) \in Z^3(\mathfrak{k})$ is a non-zero coboundary.*

If this is not the case, then each cocycle $f \in Z^2(\mathfrak{g})$ is a sum

$$f = f_1 + f_2 + f_3 = f_1^0 + f_1^1 + f_2 + f_3.$$

of four cocycles, where

(a) *f_1^0 vanishes on $\mathfrak{g} \times \mathfrak{g}'$.*

(b) *$\widetilde{f}_1^1 \in Z^1(A, \mathrm{Sym}^2(\mathfrak{k}, \mathfrak{z})^{\mathfrak{k}})$ is a cyclic 1-cocycle.*

(c) *$\widetilde{f}_2(A) \subseteq Z^2(\mathfrak{k}, \mathfrak{z})$.*

(d) *f_3 vanishes on $\mathfrak{g} \times \mathfrak{g}'$.*

Proof. First let $f = f_1 + f_2$ be a coupled cocycle on $\mathfrak{g}$. Then we have $\Gamma(\widetilde{f}_1(A, \mathbf{1})) \neq \{0\}$. Composing with a suitable linear functional $\chi \colon \mathfrak{z} \to \mathbb{K}$ with

$$\chi \circ \Gamma(\widetilde{f}_1(A, \mathbf{1})) = \Gamma((\chi \circ f_1)^{\sim}(A, \mathbf{1})) \neq \{0\},$$

we may w.l.o.g. assume that $\mathfrak{z} = \mathbb{K}$. Then there exists an $a \in A$ with

$$d_{\mathfrak{k}}(\widetilde{f}_2(a)) = \Gamma(\widetilde{f}_1(a, \mathbf{1})) \neq 0.$$

Now $\kappa := \widetilde{f}_1(a, \mathbf{1}) \in \mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}}$ is an invariant symmetric bilinear form for which $\Gamma(\kappa)$ is exact and non-zero. Then $a \in T(A) \setminus T_0(A)$, so that $0 \neq d_A(a)$ in $\Omega^1(A)$ (Theorem 3.1).

If, conversely, $d_A(A) \neq \{0\}$ and κ is an invariant symmetric bilinear form on $\mathfrak{k}$ for which $\Gamma(\kappa)$ is a non-zero coboundary, then we pick $\eta \in C^2(\mathfrak{k}) = \mathrm{Alt}^2(\mathfrak{k})$ with $d_{\mathfrak{k}}\eta = \Gamma(\kappa)$. We now define linear maps

$$\widetilde{f}_1 := \gamma_A \otimes \kappa \colon \Lambda^2(A) \to \mathrm{Sym}^2(\mathfrak{k}, \Omega^1(A))^{\mathfrak{k}}, \quad \widetilde{f}_1(a \wedge b)(x, y) := \kappa(x, y) \cdot (a d_A(b) - b d_A(a))$$

and

$$\widetilde{f}_2 := -d_A \otimes \eta \colon A \to C^2(\mathfrak{k}, \Omega^1(A)), \quad \widetilde{f}_2(a)(x, y) := -\eta(x, y) \cdot d_A(a).$$

We claim that the corresponding map $f = f_1 + f_2$ is a 2-cocycle by verifying the conditions in Theorem 3.1. Condition (a) is obviously satisfied, and (b) follows from $T_0(A) = \ker \gamma_A$ (Lemma 1.1). Further $f_3 = 0$, and (c) follows from

$$d_{\mathfrak{k}} \widetilde{f}_2(a) = -(d_{\mathfrak{k}}\eta) \cdot d_A(a) = -\Gamma(\kappa) d_A(a) = \Gamma(\widetilde{f}_1(a, \mathbf{1})).$$

That f_1 is not a cocycle, i.e., that f is coupled, means that $\widetilde{f}_1(A \wedge \mathbf{1})(\mathfrak{k} \times \mathfrak{k}') \neq \{0\}$, which is equivalent to $d_A(A) \neq \{0\}$ and $\Gamma(\kappa) = \eta \neq 0$. This completes the proof of the first part of the theorem.

For the second part we assume that either $d_A(A) \cong T_0(A)/T(A)$ vanishes, which means that $T_0(A) = T(A)$, or that for each exact invariant bilinear form κ on $\mathfrak{k}$ we have $\Gamma(\kappa) = 0$. Then for each cocycle $f = f_1^0 + f_1^1 + f_2 + f_3$ as in Corollary 3.5, either $\widetilde{f}_1^1$ vanishes on $T(A)$ (if $d_A(A)$ vanishes) or $\widetilde{f}_2(A) \subseteq Z^2(\mathfrak{k}, \mathfrak{z})$ (if for all exact forms on $\mathfrak{k}$ the 3-cocycle $\Gamma(\kappa)$ vanishes). Both conditions imply that f_1^1 and f_2 are cocycles. Hence the assertion follows from Corollary 3.3. ■

Corollary 3.8 *If $H^1(\mathfrak{k}, \mathfrak{k}^*) = \{0\}$, then $\mathfrak{g} = A \otimes \mathfrak{k}$ has no coupled cocycles.*

Proof. From the exact sequence in Proposition 7.2 below, it follows that the Cartan map $\gamma\colon \mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}} \to H^3(\mathfrak{k}), \kappa \mapsto [\Gamma(\kappa)]$ is injective, and this implies that each exact invariant form vanishes. ∎

The following proposition describes the universal cocycle for $\mathfrak{g}$ in terms of our threefold direct sum decomposition.

Proposition 3.9 (A universal cocycle) *Let $p_{\mathfrak{k}}\colon \Lambda^2(\mathfrak{k}) \to Z_2(\mathfrak{k})$ denote a linear projection onto $Z_2(\mathfrak{k})$. Then the linear map*

$$\widetilde{f}^u := p_1 \oplus (\mathrm{id}_A \otimes p_{\mathfrak{k}}) \oplus p_3 \colon \Lambda^2(\mathfrak{g}) \to Z_2(\mathfrak{g})$$

maps $B_2(\mathfrak{g})$ into itself, hence induces a 2-cocycle

$$f^u\colon \Lambda^2(\mathfrak{g}) \to H_2(\mathfrak{g}) = Z_2(\mathfrak{g})/B_2(\mathfrak{g}).$$

It is universal in the sense that the map for each space $\mathfrak{z}$ the map

$$\mathrm{Lin}(H_2(\mathfrak{g}), \mathfrak{z}) \to H^2(\mathfrak{g}, \mathfrak{z}), \quad \varphi \mapsto \varphi \circ f^u$$

is a linear bijection.

Proof. That $\widetilde{f}^u$ is a linear projection onto $Z_2(\mathfrak{g})$ follows from Lemma 2.1. The remainder follows from the fact that $H^2(\mathfrak{g}, \mathfrak{z}) \to \mathrm{Lin}(Z_2(\mathfrak{g}), \mathfrak{z}), [f] \mapsto f|_{Z_2(\mathfrak{g})}$ is injective onto the set of all maps vanishing on $B_2(\mathfrak{g})$. ∎

4 The structure of the second cohomology space

In this section we use the results of the present section to give a quite explicit description of the space $H^2(\mathfrak{g})$ in terms of data associated directly to $\mathfrak{g}$ and A.

Lemma 4.1 *Associating with each linear map $\widetilde{f}_2\colon A \to Z^2(\mathfrak{k})$ the corresponding cocycle $f_2 \in Z^2(\mathfrak{g})$, we obtain, together with the natural pullback map $H^2(\mathfrak{g}/\mathfrak{g}') \to H^2(\mathfrak{g})$, an injection*

$$H^2(\mathfrak{g}/\mathfrak{g}') \oplus \mathrm{Lin}(A, H^2(\mathfrak{k})) \xrightarrow{\Phi} H^2(\mathfrak{g})$$

whose image consists of all classes of cocycles of the form $f_1^0 + f_2 + f_3$.

Proof. The image of the pullback map consists of those cohomology classes represented by cocycles vanishing on $\mathfrak{g} \times \mathfrak{g}'$, which are the cocycles of the form $f_1^0 + f_3$. Since the space of these cocycles intersects $B^2(\mathfrak{g})$ trivially, the space $H^2(\mathfrak{g}/\mathfrak{g}')$ injects into $H^2(\mathfrak{g})$ (Remark 3.4 and Prop. 3.6).

Next we recall that the cocycles of the form $f = f_2\colon A \otimes \Lambda^2(\mathfrak{k}) \to \mathbb{K}$ correspond to linear maps $\widetilde{f}_2\colon A \to Z^2(\mathfrak{k})$ (which means that f_2 vanishes on $A \otimes B_2(\mathfrak{k})$), and that such a map is a coboundary if and only if $\mathrm{im}(\widetilde{f}_2)(A) \subseteq B^2(\mathfrak{k})$, because this implies the existence of a linear map $\ell\colon A \to \mathrm{Lin}(\mathfrak{k})$ with $\widetilde{f}_2(a) = d_{\mathfrak{k}}(\ell(a))$ for all $a \in A$. The latter condition means that f_2 vanishes on $A \otimes Z_2(\mathfrak{k})$, so that the cohomology classes correspond to elements in

$$\mathrm{Lin}(A \otimes Z_2(\mathfrak{k})/(A \otimes B_2(\mathfrak{k})), \mathbb{K}) \cong \mathrm{Lin}(A \otimes (Z_2(\mathfrak{k})/B_2(\mathfrak{k})), \mathbb{K}) \cong \mathrm{Lin}(A \otimes H_2(\mathfrak{k}), \mathbb{K}) \cong \mathrm{Lin}(A, H^2(\mathfrak{k})).$$

∎

Given a cocycle $f = f_1 + f_2 + f_3$ in $Z^2(\mathfrak{g})$, we obtain the map $\Gamma \circ \widetilde{f}_1 \colon \Lambda^2(A) \to Z^3(\mathfrak{k})$, whose kernel contains $T_0(A)$, so that it induces a linear map

$$f^\flat \colon \Omega^1(A) \cong \Lambda^2(A)/T_0(A) \to Z^3(\mathfrak{k}), \quad a \cdot d_A(b) - b \cdot d_A(a) \mapsto \Gamma(\widetilde{f}_1(a,b)),$$

mapping the subspace $d_A(A) \subseteq \Omega^1(A)$ into the subspace $B^3(\mathfrak{k})$ (Theorem 3.1). In view of

$$f^\flat(d_A(a)) = -\Gamma(\widetilde{f}_1(a,\mathbf{1})) = -d_{\mathfrak{k}}(\widetilde{f}_2(a)), \tag{12}$$

the range of each map $\Gamma \circ \widetilde{f}_1$ lies in the subspace $Z^3(\mathfrak{k})_\Gamma := \mathrm{im}(\Gamma) \subseteq Z^3(\mathfrak{k})$ and

$$f^\flat(d_A(A)) \subseteq B^3(\mathfrak{k})_\Gamma := B^3(\mathfrak{k}) \cap \mathrm{im}(\Gamma).$$

We thus obtain a map

$$\Psi \colon H^2(\mathfrak{g}) \to \mathrm{Lin}((\Omega^1(A), d_A(A)), (Z^3(\mathfrak{k})_\Gamma, B^3(\mathfrak{k})_\Gamma)), \quad [f] \mapsto [\Gamma \circ \widetilde{f}_1],$$

where for pairs (X, X') and (Y, Y') of linear spaces with $X \subseteq X'$ and $Y \subseteq Y'$ we write

$$\mathrm{Lin}((X,X'),(Y,Y')) := \{f \in \mathrm{Lin}(X',Y') \colon f(X) \subseteq Y\}.$$

Theorem 4.2 *The sequence*

$$\{0\} \to H^2(\mathfrak{g}/\mathfrak{g}') \oplus \big(A \otimes H^2(\mathfrak{k})\big) \xrightarrow{\ \Phi\ } H^2(\mathfrak{g}) \xrightarrow{\ \Psi\ } \mathrm{Lin}((\Omega^1(A), d_A(A)), (Z^3(\mathfrak{k})_\Gamma, B^3(\mathfrak{k})_\Gamma)) \to \{0\}$$

is exact.

Proof. We have already seen in Lemma 4.1 that Φ is injective.

The kernel of Ψ consists of all cocycles $f = f_1 + f_2 + f_3$ for which $\Gamma \circ \widetilde{f}_1 = 0$. This is equivalent to $\widetilde{f}_1(\Lambda^2(A))$ vanishing on $\mathfrak{k} \vee \mathfrak{k}'$, which means that f_1 vanishes on $\mathfrak{g} \times \mathfrak{g}'$, i.e., $f_1 = f_1^0$. This shows that $\ker \Psi = \mathrm{im}\, \Phi$.

To see that Ψ is surjective, let $\alpha \in \mathrm{Lin}((\Omega^1(A), d_A(A)), (Z^3(\mathfrak{k})_\Gamma, B^3(\mathfrak{k})_\Gamma))$ and observe that there exists a linear map

$$f^\flat \colon \Omega^1(A) \to \mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}} \quad \text{with} \quad \Gamma \circ f^\flat = \alpha,$$

and a linear map $\beta \colon d_A(A) \to C^2(\mathfrak{k})$ with

$$d_{\mathfrak{k}}(\beta(d_A(a))) = \alpha(d_A(a)) \quad \text{for all} \quad a \in A.$$

For

$$\widetilde{f}_1 \colon \Lambda^2(A) \to \mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}}, \quad \widetilde{f}_1(a,b) := f^\flat(a \cdot d_A(b) - b \cdot d_A(a)) \quad \text{and} \quad \widetilde{f}_2 \colon A \to C^2(\mathfrak{k}), \quad a \mapsto -\beta(d_A(a))$$

we then have

$$d_{\mathfrak{k}}(\widetilde{f}_2(a)) = -d_{\mathfrak{k}}(\beta(d_A(a))) = -\alpha(d_A(a)) = -\Gamma(f^\flat(d_A(a))) = \Gamma(\widetilde{f}_1(a,\mathbf{1})),$$

so that the corresponding maps f_1 and f_2 sum up to a 2-cocycle $f := f_1 + f_2$ satisfying $\Psi([f]) = \alpha$. ■

The quotient $Z^3(\mathfrak{k})_\Gamma / B^3(\mathfrak{k})_\Gamma$ can be identified with the image of the map

$$\gamma\colon \operatorname{Sym}^2(\mathfrak{k})^{\mathfrak{k}} \to H^3(\mathfrak{k}), \quad \kappa \mapsto [\Gamma(\kappa)]$$

discussed in the appendix below. From the exactness of the sequence in Proposition 7.2, it follows that the space $\operatorname{Sym}^2(\mathfrak{k})^{\mathfrak{k}}_{\mathrm{ex}} := \ker\gamma$ of exact invariant bilinear forms satisfies

$$\operatorname{Sym}^2(\mathfrak{k})^{\mathfrak{k}}_{\mathrm{ex}} \cong H^1(\mathfrak{k}, \mathfrak{k}^*)/H^2(\mathfrak{k}). \tag{13}$$

We also note that for a *quadratic Lie algebra*, i.e., a finite-dimensional Lie algebra $\mathfrak{k}$ with an invariant non-degenerate symmetric bilinear form κ_0, the space $\operatorname{out}(\mathfrak{k}) := \operatorname{der}(\mathfrak{k})/\operatorname{ad}\mathfrak{k}$ of *outer derivations* satisfies

$$H^1(\mathfrak{k}, \mathfrak{k}^*) \cong H^1(\mathfrak{k}, \mathfrak{k}) \cong \operatorname{der}(\mathfrak{k})/\operatorname{ad}\mathfrak{k} = \operatorname{out}(\mathfrak{k}),$$

and that the subspace $H^2(\mathfrak{k}) \subseteq H^1(\mathfrak{k}, \mathfrak{k}^*)$ consists of those classes $[D]$ of derivations D which are skew-symmetric with respect to κ_0.

We further have $\ker\Gamma \cong \operatorname{Sym}^2(\mathfrak{k}/\mathfrak{k}')$, so that

$$B^3(\mathfrak{k})_\Gamma \cong \operatorname{Sym}^2(\mathfrak{k})^{\mathfrak{k}}_{\mathrm{ex}}/\operatorname{Sym}^2(\mathfrak{k}/\mathfrak{k}') \quad \text{and} \quad Z^3(\mathfrak{k})_\Gamma \cong \operatorname{Sym}^2(\mathfrak{k})^{\mathfrak{k}}/\operatorname{Sym}^2(\mathfrak{k}/\mathfrak{k}').$$

To obtain an explicit description of $H^2(\mathfrak{g})$, it is therefore necessary to have a good description of the space $\operatorname{Sym}^2(\mathfrak{k})^{\mathfrak{k}}$ of invariant quadratic forms on $\mathfrak{k}$ and its subspace of exact forms.

Problem 4.3 Let $\mathfrak{k}$ be a finite-dimensional $\mathbb{K}$-Lie algebra. We consider the space $S := \operatorname{Sym}^2(\mathfrak{k})^{\mathfrak{k}}$ of invariant symmetric bilinear forms on $\mathfrak{k}$.

Let $\mathfrak{n} := \bigcap\{\operatorname{rad}(\kappa)\colon \kappa \in S\}$ denote the common radical of all invariant symmetric bilinear forms on $\mathfrak{k}$. Fix an element $\kappa \in S$ of maximal rank. Then $\mathfrak{n} \subseteq \operatorname{rad}(\kappa)$, but is there some κ for which we have equality?

In the following remark we collect some information that is useful to determine the space $Z^3(\mathfrak{k})_\Gamma$.

Remark 4.4 Suppose that $(\mathfrak{k}, \kappa_0)$ is a quadratic Lie algebra, i.e., κ_0 is a non-degenerate invariant symmetric bilinear form on $\mathfrak{k}$. Then there exists for each invariant symmetric bilinear form $\kappa \in \operatorname{Sym}(\mathfrak{k})^{\mathfrak{k}}$ a uniquely determined endomorphism $A_\kappa \in \operatorname{End}(\mathfrak{k})$ with

$$\kappa(x, y) = \kappa_0(A_\kappa.x, y) \quad \text{for} \quad x, y \in \mathfrak{k}.$$

Now the invariance of κ implies that A_κ is contained in the *centroid*

$$\operatorname{Cent}(\mathfrak{k}) := \{A \in \operatorname{End}(\mathfrak{k})\colon (\forall x \in \mathfrak{k})\ [A, \operatorname{ad} x] = 0\}.$$

The centroid of $\mathfrak{k}$ is an associative subalgebra of $\operatorname{End}(\mathfrak{k})$ on which transposition $A \mapsto A^\top$ with respect to κ_0 induces a linear antiautomorphism, satisfying

$$\kappa_0(A.x, y) = \kappa_0(x, A^\top.y) = \kappa_0(A^\top.y, x) \quad \text{for all} \quad x, y \in \mathfrak{k}.$$

It follows in particular that for $A \in \operatorname{Cent}(\mathfrak{k})$ the invariant bilinear form $\kappa_A(x, y) := \kappa_0(A.x, y)$ is symmetric if and only if $A^\top = A$. This leads to a linear bijection

$$\operatorname{Cent}(\mathfrak{k})_+ := \{A \in \operatorname{Cent}(\mathfrak{k})\colon A^\top = A\} \to \operatorname{Sym}^2(\mathfrak{k})^{\mathfrak{k}}, \quad A \mapsto \kappa_A.$$

For $A^\top = -A$ the invariant form κ_A is alternating, which implies that κ_A vanishes on $\mathfrak{k} \times \mathfrak{k}'$, and this implies that

$$A(\mathfrak{k}) \subseteq (\mathfrak{k}')^\perp = \mathfrak{z}(\mathfrak{k}) \quad \text{and} \quad A(\mathfrak{k}') = \{0\}.$$

Conversely, any $A \in \mathrm{End}(\mathfrak{k})$ with $\mathfrak{k}' \subseteq \ker A$ and $\mathrm{im}(A) \subseteq \mathfrak{z}(\mathfrak{k})$ satisfies $A \circ \mathrm{ad}\, x = \mathrm{ad}\, x \circ A = 0$ for all $x \in \mathfrak{k}$, hence is contained in the centroid. We put

$$\mathrm{Cent}_0(\mathfrak{k}) := \{A \in \mathrm{End}(\mathfrak{k}) \colon \mathfrak{k}' \subseteq \ker A,\ \mathrm{im}(A) \subseteq \mathfrak{z}(\mathfrak{k})\}$$

and observe that $\mathrm{Cent}_0(\mathfrak{k}) \trianglelefteq \mathrm{Cent}(\mathfrak{k})$ is an ideal of the associative algebra $\mathrm{Cent}(\mathfrak{k})$ because

$$\mathrm{Cent}_0(\mathfrak{k}) = \{A \in \mathrm{Cent}(\mathfrak{k}) \colon A|_{\mathfrak{k}'} = 0\}$$

is the kernel of the restriction homomorphism $\mathrm{Cent}(\mathfrak{k}) \to \mathrm{End}(\mathfrak{k}')$.

If $A \in \mathrm{Cent}_0(\mathfrak{k})$, then

$$\kappa_0(A^\top.[\mathfrak{k},\mathfrak{k}],\mathfrak{k}) = \kappa_0([\mathfrak{k},\mathfrak{k}], A.\mathfrak{k}) \subseteq \kappa_0(\mathfrak{k}', \mathfrak{z}(\mathfrak{k})) = \{0\},$$

so that $A^\top \in \mathrm{Cent}_0(\mathfrak{k})$. Hence the ideal $\mathrm{Cent}_0(\mathfrak{k})$ is invariant under transposition. We have already seen that $\mathrm{Cent}_0(\mathfrak{k})$ contains all skew-symmetric elements of $\mathrm{Cent}(\mathfrak{k})$, so that the involution induced on the quotient algebra

$$\mathrm{Cent}_{\mathrm{red}}(\mathfrak{k}) := \mathrm{Cent}(\mathfrak{k})/\mathrm{Cent}_0(\mathfrak{k}) \hookrightarrow \mathrm{End}(\mathfrak{k}')$$

is trivial, which implies that this algebra is commutative.

We conclude that

$$\mathrm{Cent}(\mathfrak{k})_+ := \{A \in \mathrm{Cent}(\mathfrak{k}) \colon A^\top = A\} \cong \mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}}$$

and that

$$\mathrm{Cent}_0(\mathfrak{k})_+ := \{A \in \mathrm{Cent}_0(\mathfrak{k}) \colon A^\top = A\} \cong \mathrm{Sym}^2(\mathfrak{k}/\mathfrak{k}').$$

Therefore

$$Z^3(\mathfrak{k})_\Gamma = \mathrm{im}(\Gamma) \cong \mathrm{Cent}(\mathfrak{k})_+/\mathrm{Cent}_0(\mathfrak{k})_+ \cong \mathrm{Cent}_{\mathrm{red}}(\mathfrak{k})$$

carries the structure of an associative commutative algebra.

In [MR93], Th. 2.3, Medina and Revoy describe the structure of the associative algebra $\mathrm{Cent}(\mathfrak{k})$ for a Lie algebra $\mathfrak{k}$ whose center $Z(\mathfrak{k})$ is contained in $\mathfrak{k}'$: The algebra $\mathrm{Cent}(\mathfrak{k})$ has a decomposition with respect to orthogonal indecomposable idempotents $e_1, \ldots, e_r$ with $\sum_i e_i = \mathrm{id}_{\mathfrak{k}}$, so that $\mathfrak{k}$ is the direct product of the ideals $\mathfrak{k}_i := e_i\mathfrak{k}$. Moreover, the algebra $\mathrm{Cent}(\mathfrak{k}_i) \simeq e_i \,\mathrm{Cent}(\mathfrak{k}) e_i$ is a local ring, and we have

$$\mathrm{Cent}(\mathfrak{k}) = \bigoplus_{i,j=1}^r \mathrm{Cent}_{ij}, \quad \text{where} \quad \mathrm{Cent}_{ij} := e_i \,\mathrm{Cent}(\mathfrak{k}) e_j \cong \mathrm{Lin}(\mathfrak{k}_j/\mathfrak{k}_j', Z(\mathfrak{k}_i))$$

as linear spaces, and

$$\mathrm{Cent}_0(\mathfrak{k}) = \Big(\bigoplus_{i=1}^r \mathrm{Cent}_0(\mathfrak{k}_i)\Big) \oplus \Big(\bigoplus_{i\neq j} \mathrm{Cent}_{ij}\Big).$$

If, in addition, $\mathfrak{k}$ carries a non-degenerate quadratic from κ_0, then Th. 2.5 *loc.cit.* implies that the decomposition of $\mathfrak{k}$ as a direct sum of ideals $\mathfrak{k}_i$ is orthogonal and the idempotents e_i are symmetric with respect to κ_0. We conclude in particular that

$$\mathrm{Cent}_{\mathrm{red}}(\mathfrak{k}) = \mathrm{Cent}(\mathfrak{k})/\mathrm{Cent}_0(\mathfrak{k}) \cong \bigoplus_{i=1}^r \mathrm{Cent}_{\mathrm{red}}(\mathfrak{k}_i).$$

5 An example

This section is devoted to a concrete example of a Lie algebra $\mathfrak{g} = A \otimes \mathfrak{k}$ which has coupled cocycles.

Let $\mathfrak{h}$ be the 3-dimensional Heisenberg algebra $\mathfrak{h}$ with generators x, y and c and the only non-trivial relation $[x,y] = c$. Then pass to the extension $\mathfrak{k} = \mathfrak{h} \rtimes \mathbb{K}D$ of $\mathfrak{h}$ by a derivation D like for affine algebras. Explicitly, we take $D(x) = x$, $D(y) = -y$ and $D(c) = 0$ (cf. [MP95], p.98, Ex. 6). The Lie algebra $\mathfrak{k}$ is 4-dimensional, and has an invariant bilinear symmetric form κ, as any Lie algebra with symmetrizable Cartan matrix (cf. [MP95], Prop. 4, p. 362). We call $\mathfrak{k}$ the *split oscillator algebra over* $\mathbb{K}$.

Remark 5.1 Let us compute the dimensions of the spaces of cochains, cocycles and cohomology spaces:

degree p	0	1	2	3	4
$\dim C^p(\mathfrak{k})$	1	4	6	4	1
$\dim H^p(\mathfrak{k})$	1	1	0	1	1
$\dim B^p(\mathfrak{k})$	0	0	3	3	0
$\dim Z^p(\mathfrak{k})$	1	1	3	4	1

In the preceding table the dimension of the cohomology spaces is computed as follows: $\dim H^0(\mathfrak{k}) = 1$ by definition. As $\mathfrak{k}/[\mathfrak{k},\mathfrak{k}] = \mathbb{K}D$, $\dim H^1(\mathfrak{k}) = 1$. By unimodularity (cf. [Mi04], Definition 4.3), $\mathfrak{k}$ satisfies Poincaré duality, so that the dimensions in degree 3 and 4 follow. But the Euler characteristic of a finite dimensional Lie algebra vanishes [Go55], which implies that $H^2(\mathfrak{k}) = \{0\}$.

The dimensions of the boundary spaces are clear in degree 0 and 1. In degree 2, there remain 3 dimensions as the difference of $\dim C^1(\mathfrak{k})$ and $\dim Z^1(\mathfrak{k})$. In the same way, we get the dimensions of $B^p(\mathfrak{k})$ for $p = 3, 4$. Finally, $\dim Z^p(\mathfrak{k})$ is the sum of $\dim B^p(\mathfrak{k})$ and $\dim H^p(\mathfrak{k})$.

Observe that $[\mathfrak{k},\mathfrak{k}] = \mathfrak{h}$ and $[\mathfrak{h},\mathfrak{h}] = \mathbb{R}c$, so that $\mathfrak{k}$ is solvable, but $[\mathfrak{k},\mathfrak{h}] = \mathfrak{h}$, so that $\mathfrak{k}$ is not nilpotent.

We claim that each invariant bilinear form κ is exact, which gives rise to coupled cocycles (in the sense of Section 3): If $0 \neq \mu \in C^4(\mathfrak{k})$, then the fact that $\mathfrak{k}$ is unimodular implies that all 3-cochains $i_h\mu$, $h \in \mathfrak{k}$, are 3-cocycles. If $h \in [\mathfrak{k},\mathfrak{k}] = \mathfrak{h}$, then $i_h\mu$ is exact, so that $i_D\mu$ yields a basis of the one-dimensional space $H^3(\mathfrak{k})$. Since $0 \neq (i_D\mu)(x,y,c) = \mu(D,x,y,c)$ and for each invariant symmetric bilinear form κ we have $\kappa([x,y],c) = \kappa(x,[y,c]) = 0$, we see that $\Gamma(\kappa) \in \operatorname{span}\{i_h\mu \colon h \in \mathfrak{h}\} = B^3(\mathfrak{k})$. Hence each invariant symmetric bilinear form is exact.

Remark 5.2 We now turn to the space $\operatorname{Sym}^2(\mathfrak{k})^{\mathfrak{k}}$: Any invariant symmetric bilinear form κ satisfies

$$\kappa(c,x) = \kappa([x,y],x) = -\kappa([y,x],x) = -\kappa(y,[x,x]) = 0.$$

$$\kappa(c,y) = \kappa([x,y],y) = \kappa(x,[y,y]) = 0.$$

$$\kappa(c,c) = \kappa([x,y],c) = \kappa(x,[y,c]) = 0.$$

$$\kappa(d,x) = \kappa(d,[d,x]) = \kappa([d,d],x) = 0.$$

$$\kappa(d,y) = -\kappa(d,[d,y]) = -\kappa([d,d],y) = 0.$$

$$\kappa(d,c) = \kappa(d,[x,y]) = \kappa([d,x],y) = \kappa(x,y).$$

$$\kappa(x,x) = \kappa(x,[d,x]) = -\kappa(x,[x,d]) = -\kappa([x,x],d) = 0.$$

$$\kappa(y,y) = -\kappa(y,[d,y]) = \kappa(y,[y,d]) = \kappa([y,y],d) = 0.$$

We immediately conclude that the space of invariant symmetric bilinear forms is at most 2-dimensional and that each such form κ is determined by $\kappa(d,c) = \kappa(x,y)$ and $\kappa(d,d)$ (note that d is not a commutator). Let us denote by κ_1 the (invariant symmetric bilinear) with $\kappa_1(d,d) = 1$ and $\kappa_1(x,y) = 0$ and κ_2 the invariant symmetric bilinear form with $\kappa_2(x,y) = \kappa_2(d,c) = 1$ and $\kappa_2(d,d) = 0$. Then κ_2 coincides with the invariant form κ introduced above and κ_1, κ_2 form basis of $\mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}}$. Combining with the observation in the preceding remark and Section 4, we get

$$\mathbb{K}^2 \cong \mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}} \cong \mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}}_{\mathrm{ex}} \cong H^1(\mathfrak{k},\mathfrak{k}^*) \cong H^1(\mathfrak{k},\mathfrak{k}) \cong \mathrm{out}(\mathfrak{k}).$$

For the reduced centroid we thus get

$$\mathrm{Cent}_{\mathrm{red}}(\mathfrak{k}) = \mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}} / \mathrm{Sym}^2(\mathfrak{k}/[\mathfrak{k},\mathfrak{k}]) = \mathbb{K}[\kappa_2].$$

We further get $Z^3(\mathfrak{k})_\Gamma = B^3(\mathfrak{k})_\Gamma \cong \mathbb{K}$.

For any algebra A and $\mathfrak{g} = A \otimes \mathfrak{k}$ the exact sequence in Theorem 4.2 now turns into a sequence of the form

$$\{0\} \to H^2(\mathfrak{g}/\mathfrak{g}') \cong \Lambda^2(A) \xrightarrow{\Phi} H^2(\mathfrak{g}) \to \mathrm{Lin}(\Omega^1(A), Z^3(\mathfrak{k})_\Gamma) \cong \Omega^1(A)^* \to \{0\}.$$

Therefore the essential part of $H^2(\mathfrak{g})$ is isomorphic to the dual space of $\Omega^1(A)$. From the construction in the proof of Theorem 4.2 it follows that the coupled cocycles correspond to the elements of $\Omega^1(A)^*$ not vanishing on the subspace $d_A(A)$.

6 The topological setting

In this section we explain how the algebraic results from the preceding section can be used in the topological setting. Actually these applications were our original motivation to study the work of Haddi and Zusmanovich.

We now assume that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let A be a unital commutative locally convex associative $\mathbb{K}$-algebra and $\mathfrak{k}$ a locally convex $\mathbb{K}$-Lie algebra. We endow $\mathfrak{g} = A \otimes \mathfrak{k}$ with the projective tensor product topology, turning it into a locally convex space with the universal property that for each locally convex space $\mathfrak{z}$ a bilinear map $\varphi\colon A \times \mathfrak{k} \to \mathfrak{z}$ is continuous if and only if the corresponding linear map $\widetilde{\varphi}\colon A \otimes \mathfrak{k} \to \mathfrak{z}$ is continuous. Then the Lie bracket on $\mathfrak{g}$ is continuous because the quadrilinear map

$$A \times \mathfrak{k} \times A \times \mathfrak{k} \to A \otimes \mathfrak{k}, \quad (a, x, a', x') \mapsto aa' \otimes [x, x']$$

is continuous and the continuous quadrilinear maps correspond to the continuous linear maps on $(A \otimes \mathfrak{k}) \otimes (A \otimes \mathfrak{k})$.

In the topological context, we consider for a locally convex space $\mathfrak{z}$ the space $Z^2_c(\mathfrak{g},\mathfrak{z})$ of continuous cocycles and the subspace $B^2_c(\mathfrak{g},\mathfrak{z})$ of all coboundaries of the form $d_{\mathfrak{g}}\ell$, where $\ell\colon \mathfrak{g} \to \mathfrak{z}$ is a continuous linear map. In the topological context, the relation between the space

$$H^2_c(\mathfrak{g},\mathfrak{z}) := Z^2_c(\mathfrak{g},\mathfrak{z})/B^2_c(\mathfrak{g},\mathfrak{z})$$

and the space of all linear maps from

$$H_{2,c}(\mathfrak{g}) := Z_2(\mathfrak{g})/\overline{B_2(\mathfrak{g})} \to \mathfrak{z}$$

is more complicated than in the algebraic setup ([Ne02b]). To define the topological version of $H_2(\mathfrak{g})$, we have to use the closure of $B_2(\mathfrak{g})$ to obtain a Hausdorff topology on the quotient space. We always have a natural map

$$H^2_c(\mathfrak{g},\mathfrak{z}) \to \mathrm{Lin}(H_{2,c}(\mathfrak{g}),\mathfrak{z}),$$

but in general there is no reason for this map to be injective or surjective. Therefore the homology space is much less interesting in the topological setting, and it often is easier to work directly with cocycles and coboundaries which is made possible by our results in Section III.

The flip involution on $\mathfrak{g}\otimes\mathfrak{g}$, endowed with the projective tensor product topology, is continuous, so that the kernel of the quotient map $\mathfrak{g}\otimes\mathfrak{g}\to\Lambda^2(\mathfrak{g}), x\otimes y\mapsto x\wedge y$ is closed, which leads to a locally convex topology on $\Lambda^2(\mathfrak{g})$. Further the bracket map $b_{\mathfrak{g}}\colon \Lambda^2(\mathfrak{g})\to\mathfrak{g}$ is continuous because it is induced from the continuous bracket map, which shows that its kernel $Z_2(\mathfrak{g})$ is closed.

One easily verifies that the maps $p_\pm$ and $\sigma_\pm$ from Section 2 are continuous, and likewise that the maps

$$A\to A\otimes A,\quad a\mapsto a\otimes\mathbf{1}\quad\text{ and }\quad A\to S^2(A),\quad a\mapsto a\vee\mathbf{1}$$

are continuous. Therefore Lemma 2.1 yields a topological decomposition of the closed subspace $Z_2(\mathfrak{g})$ of $\Lambda^2(\mathfrak{g})$:

$$Z_2(\mathfrak{g}) = (\Lambda^2(A)\otimes S^2(\mathfrak{k}))\oplus(A\otimes Z_2(\mathfrak{k}))\oplus(I_A\otimes\Lambda^2(\mathfrak{k})).$$

This implies that any continuous cocycle $f\colon\mathfrak{g}\times\mathfrak{g}\to\mathfrak{z}$ defines three continuous maps

$$f_1\colon \Lambda^2(A)\otimes S^2(\mathfrak{k})\to\mathfrak{z},\quad f_2\colon A\otimes\Lambda^2(\mathfrak{k})\to\mathfrak{z}\quad\text{ and }\quad f_3\colon I_A\otimes\Lambda^2(\mathfrak{k})\to\mathfrak{z}.$$

Conversely, three such continuous linear maps combine to a continuous 2-cocycle of $\mathfrak{g}$ if and only if they satisfy the conditions from Theorem 3.1.

If a continuous cocycle $f = f_1+f_2+f_3$ is in $B^2_c(\mathfrak{g},\mathfrak{z})$, then it vanishes on $Z_2(\mathfrak{g})$, which implies $f_1=f_3=0$ and that f_2 is a continuous coboundary, i.e., there exists a continuous linear map $\ell\colon\mathfrak{g}\to\mathfrak{z}$ with

$$f(ax,by) = f_2(ax,by) = \widetilde{f}_2(ab)(x,y) = \ell(ab[x,y])\quad\text{ for all }\quad a,b\in A, x,y\in\mathfrak{k}.$$

Clearly, this implies that $\widetilde{f}_2(A)\subseteq B^2_c(\mathfrak{k},\mathfrak{z})$. If, conversely, $\widetilde{f}_2(A)\subseteq B^2_c(\mathfrak{k},\mathfrak{z})$, then there exists a linear map $h\colon A\to\mathrm{Lin}(\mathfrak{k},\mathfrak{z})$ with $d_{\mathfrak{k}}h(a)=\widetilde{f}_2(a)$ for all $a\in A$, but it is not clear whether the corresponding map $\widetilde{h}\colon A\times\mathfrak{k}\to\mathfrak{z}$ will be continuous. Therefore the exactness condition is quite subtle.

If $\mathfrak{k}$ is finite-dimensional, then the situation simplifies significantly. Then $B^2_c(\mathfrak{k},\mathfrak{z}) = B^2(\mathfrak{k},\mathfrak{z})$ and if $\mathrm{Lin}(\cdot,\cdot)$ stands for "continuous linear maps", then

$$\mathrm{Lin}(A\otimes\Lambda^2(\mathfrak{k}),\mathfrak{z})\cong\Lambda^2(\mathfrak{k})^*\otimes\mathrm{Lin}(A,\mathfrak{z}),$$

so that we may consider f_2 as a 2-cocycle in $Z^2(\mathfrak{k},\mathrm{Lin}(A,\mathfrak{z}))$. If this map vanishes on $B_2(\mathfrak{k})$, then there exists a linear map $h\colon\mathfrak{k}\to\mathrm{Lin}(A,\mathfrak{z})$ with

$$f_2(x,y)(a) = h([x,y])(a),\quad x,y\in\mathfrak{k}, a\in A.$$

Then the map $\ell\colon A\times\mathfrak{k}\to\mathfrak{z}, (a,x)\mapsto h(x)(a)$ is continuous and satisfies $f_2 = -d_{\mathfrak{g}}\ell$. We thus get

$$B^2(\mathfrak{k},\mathrm{Lin}(A,\mathfrak{z}))\cong B^2_c(\mathfrak{g},\mathfrak{z}).$$

We collect the previous remarks in the following theorem which is analoguous to Theorem 4.2. It determines the structure of the second continuous cohomology space for current algebras. Let us denote by $\Omega^1_c(A) = J_A\,/\,\overline{J_A^2}$ the locally convex module of Kähler differentials for the locally convex commutative associative algebra A.

Theorem 6.1 *Let $\mathfrak{k}$ be a finite dimensional Lie algebra over $\mathbb{K}$. Then the sequence*

$$\{0\} \to H^2_c(\mathfrak{g}/\mathfrak{g}') \oplus \big(A \otimes H^2(\mathfrak{k})\big) \xrightarrow{\quad\Phi\quad} H^2_c(\mathfrak{g}) \xrightarrow{\quad\Psi\quad} \mathrm{Lin}((\Omega^1_c(A), \overline{d_A(A)}), (Z^3(\mathfrak{k})_\Gamma, B^3(\mathfrak{k})_\Gamma)) \to \{0\}$$

is exact.

Proof. First we note that the short exact sequence $\mathbf{0} \to \mathfrak{k}' \to \mathfrak{k} \to \mathfrak{k}/\mathfrak{k}' \to \mathbf{0}$ of finite-dimensional vector spaces splits. Since $\mathfrak{g}' = A \otimes \mathfrak{k}'$ is closed in $\mathfrak{g}$, it follows that the short exact sequence $\mathbf{0} \to \mathfrak{g}' \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}' \to \mathbf{0}$ also splits topologically. As we have observed above, Theorem 3.7 and its corollaries remain true in the topological setting. For Corollary 3.5 we use the topological splitting of $\mathfrak{k}'$ in $\mathfrak{k}$. We have also seen above that the corresponding description of the coboundaries remains valid, because $\mathfrak{k}$ is finite dimensional. Further, the topological splitting of $\mathfrak{g}'$ implies that Lemma 4.1 remains true. This implies the injectivity of Φ.

That $\ker \Psi = \operatorname{im} \Phi$ is shown as in the proof of Theorem 4.2. Finally, the surjectivity of Ψ follows from the fact that $f^\flat$ and β can be chosen as continuous maps, because of the existence of (continuous) linear right inverses of surjective linear maps to finite dimensional vector spaces. ■

Example 6.2 We consider the special case where M is a compact manifold and $A = C^\infty(M, \mathbb{R})$ the Fréchet algebra of all smooth real-valued functions on M. According to [Ma02] or [Co85], the universal topological differential module of A is given by $\Omega^1_c(A) \cong \Omega^1(M, \mathbb{R})$, the space of smooth $\mathbb{R}$-valued 1-forms on M, and the de Rham-differential $d\colon C^\infty(M, \mathbb{R}) \to \Omega^1(M, \mathbb{R})$ is a universal continuous derivation. It follows in particular that the space $d_A(A)$ is the space of exact 1-forms, which is non-zero.

Now let $\mathfrak{k}$ be a finite-dimensional real Lie algebra and

$$\mathfrak{g} := A \otimes \mathfrak{k} \cong C^\infty(M, \mathfrak{k}).$$

Up to cocycles vanishing on $\mathfrak{g} \times \mathfrak{g}'$, all continuous cohomology classes in $H^2_c(\mathfrak{g})$ are then represented by sums $f = f_1 + f_2$, where

$$f_1 \colon A \times A \to \mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}}$$

is an alternating continuous linear map for which there is a continuous linear map

$$f_1^\flat \colon \Omega^1(M, \mathbb{R}) \to \mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}} \quad \text{with} \quad \widetilde{f}_1(a, b) = f_1^\flat(a \cdot d(b) - b \cdot d(a)),$$

and

$$d_{\mathfrak{k}}(\widetilde{f}_2(a)) = -\Gamma(f_1^\flat(da)) \quad \text{for all} \quad a \in A.$$

We interprete the continuous linear map $f_1^\flat$ as a $\mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}}$-valued *current on M*. It is a closed current if and only if it vanishes on exact forms. Typical examples of such currents arise from pairs (ξ, κ), where $\xi\colon [0,1] \to M$ is a piecewise smooth path and $\kappa \in \mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}}_{\mathrm{ex}}$ via

$$f_1^\flat(\alpha) := \Big(\int_\xi \alpha\Big) \cdot \kappa,$$

but these examples satisfy $\gamma \circ f_1^\flat = 0$.

7 Appendix: A useful exact sequence

The following section is very much based on information and hints we got from M. Bordemann ([Bo97]).

Definition 7.1 Let $\mathfrak{k}$ be a Lie algebra and $\mathfrak{a}$ a $\mathfrak{k}$-module. We denote the action as $\mathfrak{k} \times \mathfrak{a} \to \mathfrak{a}$ by $(x,a) \mapsto x.a$. On the space $C^p(\mathfrak{k},\mathfrak{a})$ of $\mathfrak{a}$-valued Lie algebra cochains we have a natural action of $\mathfrak{k}$ denoted by

$$(\mathcal{L}_x\omega)(x_1,\ldots,x_p) = x.\omega(x_1,\ldots,x_p) - \sum_{i=1}^{p} \omega(x_1,\ldots,x_{i-1},[x,x_i],x_{i+1},\ldots,x_p).$$

The Lie algebra differential $d_{\mathfrak{k}} \colon C^p(\mathfrak{k},\mathfrak{a}) \to C^{p+1}(\mathfrak{k},\mathfrak{a})$ is given by

$$\begin{aligned}(d_{\mathfrak{k}}\omega)(x_0,\ldots,x_p) \quad &:= \sum_{j=0}^{p}(-1)^j x_j.\omega(x_0,\ldots,\widehat{x}_j,\ldots,x_p)\\ &+\sum_{i<j}(-1)^{i+j}\omega([x_i,x_j],x_0,\ldots,\widehat{x}_i,\ldots,\widehat{x}_j,\ldots,x_p),\end{aligned} \tag{14}$$

where $\widehat{x}_j$ indicates omission of x_j.

For $p,q \in \mathbb{N}_0$ we consider the injection

$$\widetilde{T}_p \colon C^{p+q}(\mathfrak{k},\mathfrak{a}) \to C^p(\mathfrak{k}, C^q(\mathfrak{k},\mathfrak{a})), \quad (\widetilde{T}_p f)(x_1,\ldots,x_p)(y_1,\ldots,y_q) := f(x_1,\ldots,x_p,y_1,\ldots,y_q).$$

From the action of $\mathfrak{k}$ on the spaces $C^q(\mathfrak{k},\mathfrak{a})$ we obtain Lie algebra differentials

$$d'_{\mathfrak{k}} \colon C^p(\mathfrak{k}, C^q(\mathfrak{k},\mathfrak{a})) \to C^{p+1}(\mathfrak{k}, C^q(\mathfrak{k},\mathfrak{a}))$$

and we also have

$$d''_{\mathfrak{k}} \colon C^p(\mathfrak{k}, C^q(\mathfrak{k},\mathfrak{a})) \to C^p(\mathfrak{k}, C^{q+1}(\mathfrak{k},\mathfrak{a})), \quad \omega \mapsto d_{\mathfrak{k}} \circ \omega$$

satisfying on $C^{p+q}(\mathfrak{k},\mathfrak{a})$ the identity

$$\widetilde{T}_{p+1} \circ d_{\mathfrak{k}} = d'_{\mathfrak{k}} \circ \widetilde{T}_p + (-1)^{p+1} d''_{\mathfrak{k}} \circ \widetilde{T}_{p+1}. \tag{15}$$

(cf. [Ne05b], Lemma A.1).

Specializing to the trivial module $\mathfrak{a} = \mathbb{K}$, we obtain in particular the maps

$$\widetilde{T}_{p-1} \colon C^p(\mathfrak{k}) \to C^{p-1}(\mathfrak{k}, C^1(\mathfrak{k})/B^1(\mathfrak{k})) = C^{p-1}(\mathfrak{k},\mathfrak{k}^*),$$

which, in view of equation (15), commute with the respective Lie algebra differentials because $d''_{\mathfrak{k}} \circ \widetilde{T}_p$ vanishes on $C^p(\mathfrak{k},\mathbb{K})$. Hence they induce linear maps

$$\alpha_p \colon H^p(\mathfrak{k}) \to H^{p-1}(\mathfrak{k},\mathfrak{k}^*), \quad [\omega] \mapsto [\widetilde{T}_{p-1}(\omega)].$$

For the $\mathfrak{k}$-module $\mathfrak{k}^*$ the subspace $d_{\mathfrak{k}}\mathfrak{k}^*$ of $C^1(\mathfrak{k},\mathfrak{k}^*)$ consists of maps whose associated bilinear map is alternating. We thus have a well-defined map

$$S \colon C^1(\mathfrak{k},\mathfrak{k}^*)/B^1(\mathfrak{k},\mathfrak{k}^*) \to \mathrm{Sym}^2(\mathfrak{k})$$

which is a morphism of $\mathfrak{k}$-modules. We now obtain maps

$$\widetilde{\beta}_p = S \circ \widetilde{T}_{p-1} \colon C^p(\mathfrak{k}, \mathfrak{k}^*) \to C^{p-1}(\mathfrak{k}, \mathrm{Sym}^2(\mathfrak{k}))$$

satisfying

$$\widetilde{\beta}_p \circ d_{\mathfrak{k}} \quad = \quad S \circ \widetilde{T}_{p-1} \circ d_{\mathfrak{k}} = S \circ (d'_{\mathfrak{k}} \circ \widetilde{T}_{p-2}) = d'_{\mathfrak{k}} \circ S \circ \widetilde{T}_{p-2} = d'_{\mathfrak{k}} \circ \widetilde{\beta}_{p-1}. \tag{16}$$

Hence $\widetilde{\beta}_p$ induces a linear map

$$\beta_p \colon H^p(\mathfrak{k}, \mathfrak{k}^*) \to H^{p-1}(\mathfrak{k}, \mathrm{Sym}^2(\mathfrak{k})).$$

From the construction we immediately get $\widetilde{\beta}_p \circ \widetilde{\alpha}_{p+1} = 0$, which leads to $\beta_p \circ \alpha_{p+1} = 0$.

Proposition 7.2 *For any Lie algebra $\mathfrak{k}$ we obtain with $\gamma(\kappa) := [\Gamma(\kappa)]$ an exact sequence*

$$\{0\} \to H^2(\mathfrak{k}) \xrightarrow{\alpha_2} H^1(\mathfrak{k}, \mathfrak{k}^*) \xrightarrow{\beta_1} \mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}} \xrightarrow{\gamma} H^3(\mathfrak{k}) \xrightarrow{\alpha_3} H^2(\mathfrak{k}, \mathfrak{k}^*) \xrightarrow{\beta_2} H^1(\mathfrak{k}, \mathrm{Sym}^2(\mathfrak{k})).$$

Proof. To see that for each cocycle $\omega \in Z^1(\mathfrak{k}, \mathfrak{k}^*)$ the symmetric bilinear form $\beta_1(\omega)$ is invariant, we note that

$$\beta_1(\omega)([x,y],z) = \omega([x,y])(z) + \omega(z)([x,y]),$$

and if ω is a cocycle, this can be written as

$$\beta_1(\omega)([x,y],z) = (x.\omega(y))(z) - (y.\omega(x))(z) + \omega(z)([x,y]) = \omega(y)([z,x]) + \omega(x)([y,z]) + \omega(z)([x,y]),$$

showing that this trilinear form is alternating, and hence that $\beta_1(\omega)$ is invariant.

Exactness in $H^2(\mathfrak{k})$: We only have to show that α_2 is injective. If $\omega \in Z^2(\mathfrak{k})$ satisfies $\alpha_2(\omega) = d_{\mathfrak{k}}\eta$ for some $\eta \in \mathfrak{k}^*$, then

$$\omega(x,y) = (d_{\mathfrak{k}}\alpha)(x)(y) = (x.\alpha)(y) = -\alpha([x,y]),$$

which implies that ω is a 2-coboundary.

Exactness in $H^1(\mathfrak{k}, \mathfrak{k}^*)$: Clearly $\beta_1 \circ \alpha_2 = 0$. If, conversely, $\beta_1([\omega]) = 0$, then $\omega \colon \mathfrak{k} \to \mathfrak{k}^*$ is a linear map whose associated bilinear form $\widetilde{\omega}(x,y) := \omega(x)(y)$ is alternating. In this situation we have

$$\begin{aligned} d\widetilde{\omega}(x,y,z) \quad &= \quad -\omega([x,y])(z) - \omega([y,z])(x) - \omega([z,x])(y) = -\omega([x,y])(z) + \omega(x)([y,z]) + \omega(y)([z,x]) \\ &= \quad \big(-\omega([x,y]) - y.\omega(x) + x.\omega(y)\big)(z) = (d_{\mathfrak{k}}\omega)(x,y)(z). \end{aligned} \tag{17}$$

We conclude that $\widetilde{\omega}$ is a cocycle if and only if ω is one, and from that we derive that $\ker \beta_1 = \mathrm{im}\, \alpha_2$.

Exactness in $\mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}}$: Next we show that $\gamma \circ \beta_1 = 0$. So let $\omega \in Z^1(\mathfrak{k}, \mathfrak{k}^*)$ and write $\widetilde{\omega} = \omega_+ + \omega_-$, where ω_+ is symmetric and ω_- is alternating. Then

$$\gamma(\beta_1(\omega))(x,y,z) \quad = \quad \omega([x,y])(z) + \omega(z)([x,y]) = \omega(y)([z,x]) + \omega(x)([y,z]) + \omega(z)([x,y]), \tag{18}$$

and the closedness of ω also shows that $\sum_{\mathrm{cyc.}} \omega([x,y])(z) = 2\sum_{\mathrm{cyc.}} \omega(x)([y,z])$, which leads to

$$\gamma(\beta_1(\omega))(x,y,z) = \sum_{\mathrm{cyc.}} \omega([x,y])(z) - \omega(z)([x,y]) = 2\sum_{\mathrm{cyc.}} \omega_-([x,y],z) = -2d_{\mathfrak{k}}\omega_-(x,y,z).$$

Hence $\gamma(\beta_1(\omega))$ is always exact, so that $\gamma \circ \beta_1$ vanishes on the level of cohomology spaces.

To see that $\ker\gamma \subseteq \operatorname{im}\beta_1$, suppose that κ is an exact invariant bilinear form and $\eta \in C^2(\mathfrak{k})$ satisfies $d\eta = -\Gamma(\kappa) = -\gamma(\kappa)$. Then $\omega(x)(y) := \kappa(x,y) + \eta(x,y)$ defines a linear map $\omega\colon \mathfrak{k} \to \mathfrak{k}^*$ with

$$\begin{aligned}(d_{\mathfrak{k}}\omega)(x,y)(z) &= (x.\omega(y) - y.\omega(x) - \omega([x,y]))(z)\\ &= \omega(y)([z,x]) + \omega(x)([y,z]) - \omega([x,y])(z)\\ &= \kappa([x,y],z) + \eta(y,[z,x]) + \eta(x,[y,z]) - \eta([x,y],z)\\ &= \kappa([x,y],z) + d_{\mathfrak{k}}\eta(x,y,z) = (\Gamma(\kappa) + d_{\mathfrak{k}}\eta)(x,y,z) = 0. \end{aligned} \tag{19}$$

From the preceding calculation we also see by putting $\eta = 0$, that the linear map $\widetilde{\kappa}\colon \mathfrak{k} \to \mathfrak{k}^*$ defined by an invariant symmetric bilinear form κ is a 1-cocycle if and only if $\Gamma(\kappa)$ vanishes.

Exactness in $H^3(\mathfrak{k})$: Since $B^1(\mathfrak{k}) \subseteq \mathfrak{k}^*$ vanishes, the transfer formula for differentials implies that an alternating trilinear form ω on $\mathfrak{k}$ is a 3-cocycle if and only if the corresponding alternating bilinear form $\alpha_3(\omega)$ is a 2-cocycle. Therefore the image of α_3 consists of those cohomology classes having a representing cocycle whose associated trilinear form is alternating.

For $\kappa \in \operatorname{Sym}^2(\mathfrak{k})^{\mathfrak{k}}$, the corresponding 3-cocycle $\Gamma(\kappa)$ and the corresponding linear map $\widetilde{\kappa}\colon \mathfrak{k} \to \mathfrak{k}^*$, we have

$$\alpha_3(\Gamma(\kappa))(x,y) = \kappa([x,y],\cdot) = -(d\widetilde{\kappa})(x)(y)$$

because

$$\begin{aligned} d\widetilde{\kappa}(x,y)(z) &= (x.\widetilde{\kappa}(y))(z) - (y.\widetilde{\kappa}(x))(z) - \widetilde{\kappa}([x,y])(z)\\ &= -\kappa(y,[x,z]) + \kappa(x,[y,z]) - \kappa([x,y],z) = \kappa([x,y],z). \end{aligned} \tag{20}$$

We conclude that $\alpha_3(\Gamma(\kappa))$ is exact, so that $\alpha_3 \circ \gamma$ induces the trivial map $\operatorname{Sym}^2(\mathfrak{k})^{\mathfrak{k}} \to H^2(\mathfrak{k},\mathfrak{k}^*)$.

Let $f \in C^1(\mathfrak{k},\mathfrak{k}^*)$ and write $\widetilde{f}(a,b) = f(a)(b)$. We then have

$$\begin{aligned} df(x,y)(z) &= (x.f(y) - y.f(x) - f([x,y]))(z) = f(y)([z,x]) + f(x)([y,z]) - f([x,y])(z)\\ &= f(y)([z,x]) - (y.\widetilde{f})(x,z). \end{aligned} \tag{21}$$

This map is alternating in (x,y), and it is alternating in (x,z) if and only if $y.\widetilde{f}$ is alternating. Writing $\widetilde{f} = \widetilde{f}_+ + \widetilde{f}_-$ for the decomposition of $\widetilde{f}$ into symmetric and alternating components, this is equivalent to $y.\widetilde{f}_+ = 0$. We conclude that $df(x,y)(z)$ is alternating if and only if $\widetilde{f}_+$ is invariant.

To verify the exactness in $H^3(\mathfrak{k})$, we now assume that $\omega \in Z^3(\mathfrak{k})$ satisfies $\alpha_3(\omega) \in B^2(\mathfrak{k},\mathfrak{k}^*)$, i.e., $\alpha_3([\omega]) = 0$. Then there exists an $f \in C^1(\mathfrak{k},\mathfrak{k}^*)$ with $\alpha_3(\omega) = d_{\mathfrak{k}}f$, and the preceding paragraph implies that $\widetilde{f}_+$ is an invariant symmetric bilinear form on $\mathfrak{k}$ satisfying

$$\alpha_3(\omega) = d_{\mathfrak{k}}f = -\Gamma(\widetilde{f}_+) + d_{\mathfrak{k}}f_-,$$

where $f = f_+ + f_-$ corresponds to the decomposition $\widetilde{f} = \widetilde{f}_+ + \widetilde{f}_-$. We conclude that $[\alpha_3(\omega)] = -[\Gamma(\widetilde{f}_+)]$, which implies exactness in $H^3(\mathfrak{k})$.

Exactness in $H^2(\mathfrak{k},\mathfrak{k}^*)$: We claim that $\ker\beta_2 = \operatorname{im}\alpha_3$. To verify this claim, pick $\omega \in Z^2(\mathfrak{k},\mathfrak{k}^*)$ for which $\beta_2(\omega)$ is exact, i.e., there exists a symmetric bilinear form $\kappa \in \operatorname{Sym}^2(\mathfrak{k})$ with $\beta_2(\omega) = d_{\mathfrak{k}}\kappa$, i.e., for $x,y,z \in \mathfrak{k}$ we have

$$\omega(x,y)(z) + \omega(x,z)(y) = (x.\kappa)(y,z) = -\kappa([x,y],z) - \kappa(y,[x,z]).$$

Let $\widetilde{\eta} \in C^1(\mathfrak{k},\mathfrak{k}^*)$ and write η for the corresponding bilinear map on $\mathfrak{k}$ with $\eta(x,y) = \widetilde{\eta}(x)(y)$. Then

$$d\widetilde{\eta}(x,y)(z) = (x.\widetilde{\eta}(y) - y.\widetilde{\eta}(x) - \widetilde{\eta}([x,y]))(z) = -\eta(y,[x,z]) + \eta(x,[y,z]) - \eta([x,y],z).$$

Therefore

$$\begin{aligned}\beta_2(d_{\mathfrak{k}}\widetilde{\eta})(x)(y,z) &= \widetilde{\eta}(y)([z,x]) - \widetilde{\eta}([x,y])(z) + \widetilde{\eta}(z)([y,x]) - \widetilde{\eta}([x,z],y) \\ &= 2(\widetilde{\eta}_+([y,x])(z) + \widetilde{\eta}_+([z,x])(y)) = 2(x.\eta_+)(y,z),\end{aligned} \tag{22}$$

and this leads to

$$\beta_2(d_{\mathfrak{k}}\widetilde{\eta}) = 2d_{\mathfrak{k}}\eta_+ = d_{\mathfrak{k}}(\beta_1(\eta)).$$

Since $\beta_1(C^1(\mathfrak{k},\mathfrak{k}^*)) = \mathrm{Sym}^2(\mathfrak{k})$, we find some $\widetilde{\eta} \in C^1(\mathfrak{k},\mathfrak{k}^*)$ with $\beta_1(\widetilde{\eta}) = \kappa$, and then

$$\beta_2(\omega - d_{\mathfrak{k}}\eta) = d_{\mathfrak{k}}\kappa - d_{\mathfrak{k}}\eta_+ = 0,$$

so that for $\omega' := \omega - d_{\mathfrak{k}}\eta \in Z^2(\mathfrak{k},\mathfrak{k}^*)$ the corresponding trilinear map is alternating. This means that $[\omega] = [\omega'] \in \mathrm{im}\,\alpha_3$. ■

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