NEIGHBORLY CUBICAL POLYTOPES AND SPHERES

MICHAEL JOSWIG AND THILO SCHRÖDER

Abstract. We prove that the neighborly cubical polytopes studied by Günter M. Ziegler and the first named author [14] arise as a special case of the neighborly cubical spheres constructed by Babson, Billera, and Chan [4]. By relating the two constructions we obtain an explicit description of a non-polytopal neighborly cubical sphere and, further, a new proof of the fact that the cubical equivelar surfaces of McMullen, Schulz, and Wills [16] can be embedded into $\mathbb{R}^3$.

1. Introduction

Our point of departure is a paper by Babson, Billera, and Chan [4], in which the authors introduce an inductive construction of cubical d-spheres from certain sequences of simplicial $(d-1)$-balls and their boundary spheres of dimension $d-2$. It turns out that such cubical spheres reflect many properties of the simplicial spheres involved, but in a cubical disguise. In particular, this way the boundary of a neighborly simplicial $(d-1)$-polytope yields a neighborly cubical d-sphere, that is, a sphere which has the same $\lfloor (d-1)/2 \rfloor$-skeleton as some high-dimensional cube. Later it was shown by entirely different methods that there even exist neighborly cubical spheres which are polytopal [14].

Our first result, Theorem 3.2, establishes a non-recursive combinatorial description of the cubical spheres studied by Babson, Billera, and Chan. From this it can be inferred that particular sequences of pulling triangulations of cyclic polytopes yield the polytopal spheres studied in [14], see Corollary 3.7. As a further benefit of this direct description we observe in Theorem 3.9 that the simplicial spheres involved in the construction necessarily must be polytopal in order to yield polytopal cubical spheres. In Theorem 3.10 we derive that there exists a non-polytopal neighborly cubical 5-sphere on 2048 vertices. Our proof is based on explicit construction involving a certain non-polytopal 3-sphere $\mathcal{M}^{10}_{425}$. This sphere was found by Althuler [2], and its non-polytopality was verified by Bokowski and Garms [7].

A seemingly different topic is the construction of polyhedral surfaces of 'unusually large genus,' pioneered by Coxeter [9] and Ringel [20] and continued by

Date: October 31, 2005.

Both authors are supported by Deutsche Forschungsgemeinschaft, DFG Research Group "Polyhedral Surfaces."

McMullen, Schulz, and Wills [16, 17]. Yet we can show that the neighborly cubical 4-polytopes of [14] contain cubical surfaces with n vertices of genus $O(n \log n)$ in their boundary, answering a question of Günter M. Ziegler [24]. Via Schlegel diagrams this gives a simple new proof of the known fact that there are such surfaces which can be embedded into $\mathbb{R}^3$ with straight faces.

2. Cubical and Simplicial Complexes

We review the ingredients of the construction of Babson, Billera, and Chan [4] with slight generalizations.

The reader is advised to consult the monographs [12] and [25] for general information about convex polytopes and related topics.

2.1. Cubes.

Consider the $2d$ affine halfspaces

$$H_i^+ = \left\{x \in \mathbb{R}^d : \; x_i \leq 1\right\} \text{ and } H_i^- = \left\{x \in \mathbb{R}^d : \; -x_i \leq 1\right\},$$

which define the d-dimensional cube

$$C_d = H_1^+ \cap \cdots \cap H_d^+ \cap H_1^- \cap \cdots \cap H_d^- .$$

The intersections

$$F_i^\sigma = \partial H_i^\sigma \cap C_d = \{x \in C_d \, : \, \sigma x_i = 1\}, \qquad \sigma \in \{\pm 1\},$$

of their boundaries with the cube are precisely the **facets** of C_d, that is, its maximal proper faces. The **vertices** of C_d are all the 2^d vectors of length d with coordinates ± 1, and their joint convex hull is the cube C_d. Two facets F_i^+ and F_i^- are parallel, and hence they do not share any vertex of C_d.

It is a general fact about convex polytopes that each **proper face**, that is, any intersection of the polytope with a supporting hyperplane, can be written as the intersection of facets. Thus, for the special case of the d-cube, each non-empty face F can be written uniquely as $F = F_{i_1}^{\sigma_{i_1}} \cap \ldots F_{i_k}^{\sigma_{i_k}}$, where $i_j \neq i_l$ for any $j \neq l$. By letting $\sigma_j = 0$ for all $j \notin \{i_1, \ldots, i_k\}$ the non-empty face F can be identified with the ordered sequence $(\sigma_1, \ldots, \sigma_d)$ of signs $+1, 0, -1$ of length d. Conversely, each such sign vector defines a face. For ease of notation we often omit the 1's of ± 1.

The intersection of a k-dimensional cube face, or **k-face** for short, with a facet is either empty or a $(k-1)$-face. This readily implies that the dimension of the face $(\sigma_1, \ldots, \sigma_d)$ equals the number of 0-entries in its sign vector.

The reflections at the coordinate hyperplanes $\{x \in \mathbb{R}^d \colon x_i = 0\}$ generate an elementary abelian group Σ_d of order 2^d which acts sharply transitively on the vertices of C_d. The full automorphism group Γ_d of the d-cube is isomorphic to a semi-direct product of Σ_d and the stabilizer of a vertex, which is Sym_d, the symmetric group of degree d. In fact, Γ_d is the wreath product $\mathbb{Z}_2 \wr \mathrm{Sym}_d$.

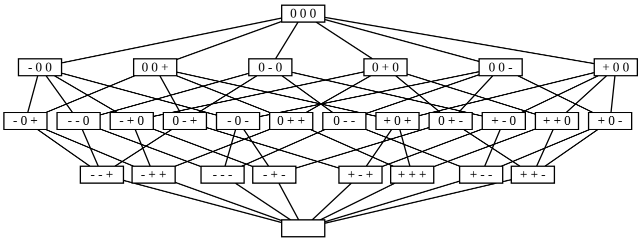


FIGURE 1. Face lattice of C_3.

2.2. **Regular Cell Complexes and Posets.** A **regular cell complex** is a family C of closed balls in a Hausdorff space X_C, such that the interiors of the balls partition X_C and the boundary of each ball is the union of balls in C. The topology of a regular cell complex is completely determined by its face poset; see Björner [6, Prop. 3.1].

An **abstract d-complex** $\mathcal{P}$ is a ranked poset of rank $d+1$ such that there exist unique lower bounds for any set of elements (that is, $\mathcal{P}$ is a meet semi-lattice) and every order ideal is combinatorially isomorphic to the face lattice of a polytope. The elements of this partially ordered set are called **faces**. The **boundary** of a finite abstract d-complex is the set of faces of corank 1 contained in only one maximal face. An abstract d-complex is a **cubical** (**simplicial**) **complex** if every face of $\mathcal{P}$ is combinatorially isomorphic to a cube (simplex). These are CW posets of polyhedral type as introduced by Björner [6]. Thus we are able to speak of topological properties of abstract d-complexes. An abstract 2-complex representing a connected 2-manifold without boundary is a **polyhedral surface**.

2.3. **Mirroring.** An abstract simplicial complex Δ on d vertices can be seen as a subcomplex of the $(d-1)$-dimensional simplex. The cube C_d is a **simple** d-polytope, that is, each of its vertex figures is a $(d-1)$-simplex. Here the **vertex figure** of a polytope P at a vertex v is the intersection of P with an affine hyperplane which separates v from the other vertices of P. We construct a (cubical) subcomplex of C_d which corresponds to a simultaneous embedding of Δ into the vertex figures of all the vertices of C_d such that these embeddings are invariant under the action of the group Σ_d.

Following Babson, Billera, and Chan [4] we encode Δ in a non-standard way: If $1, 2, \dots, d$ are the vertices of Δ we associate with a face $\varphi \in \Delta$ the characteristic function of its complement $\{1, \dots, d\} \setminus \varphi$. Using this description we define

$$\mathrm{M}(\Delta) = \left\{(\sigma_1, \dots, \sigma_d) \in \{0, \pm 1\}^d : (|\sigma_1|, \dots, |\sigma_d|) \in \Delta\right\},$$

the **mirror complex** of Δ, which is a subcomplex of the d-cube.

By construction the vertex figure of the mirror complex of a simplicial complex is isomorphic to the simplicial complex itself. Thus if the simplicial complex is a d-sphere, then its mirror complex is a $(d+1)$-manifold. The f-vector of the mirror

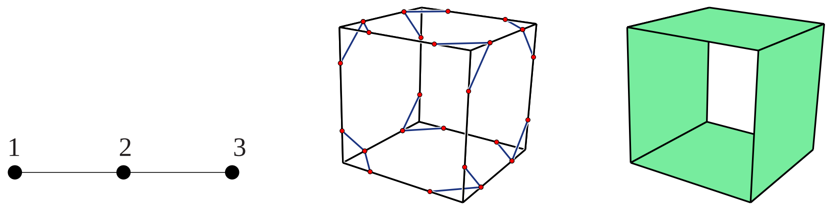


FIGURE 2. Simplicial complex Δ on three vertices, its symmetric embedding into C_d, and the mirror complex $M(\Delta)$.

complex of a simplicial complex Δ on n vertices is $f_i(M(\Delta)) = 2^{n-i} f_{i-1}(\Delta)$. Further the mirror complex of the boundary of a simplicial complex is the boundary of the mirror complex: $M(\partial\Delta) = \partial M(\Delta)$.

Proposition 2.1. *Let Δ be a simplicial complex with automorphism group Γ. Then the automorphism group of $M(\Delta)$ is isomorphic to the semi-direct product $\Sigma_d \rtimes \Gamma$.*

2.4. **Fissuring.** The cubical fissure or fissuring is an operation that produces a new cubical complex from a given one. Let C be a pure cubical d-complex, C_1 and C_2 facet-disjoint d-dimensional subcomplexes of C such that $C_1 \cup C_2 = C$. The **cubical fissure** $\mathrm{fis}_C(C_1, C_2)$ of C between C_1 and C_2 is defined by lifting C_1 to height one, dropping C_2 to height minus one and filling in the fissure with $(C_1 \cap C_2) \times [-1, 1]$. The corresponding poset is

$$\mathrm{fis}_C(C_1, C_2) = (C_1 \times \{+1\}) \cup ((C_1 \cap C_2) \times \{0\}) \cup (C_2 \times \{-1\}).$$

with $+1 < 0$ and $-1 < 0$ in the last component. If C is a subcomplex of the n-cube given as sign vectors of length n, then the cubical fissure canonically yields a subcomplex of the $(n+1)$-cube. The cubical fissure between a subcomplex C_1 and its complement C_2 yields a complex consisting of the subcomplex C_1 and its complement connected via a prism over the boundary of C_1.

Example 2.2. Consider the simplicial complex Δ of Figure 2 on three vertices. The mirror complex $C_1 = M(\Delta)$ is a subcomplex of the boundary complex of the 3-cube $C = \partial C_3$. So the cubical fissure between C_1 and its complement in C is a subcomplex of the boundary of the 4-cube, see Figure 3.

2.5. **BBC Sequences of Simplicial Balls.** Babson, Billera, and Chan [4] proved the existence of neighborly cubical spheres. Their approach is based on an inductive construction using triangulations of cyclic polytopes, mirror complexes and fissuring. A close inspection of their proof motivates the following definition. While this does look a bit technical it is one of the keys to our main results.

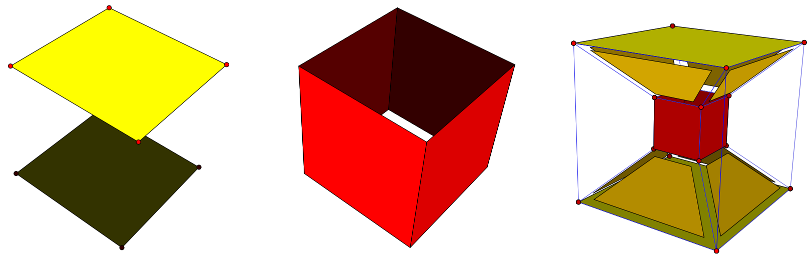


FIGURE 3. Two subcomplexes of the 3-cube and their cubical fissure in the Schlegel diagram of the 4-cube.

Definition 2.3. Let $V = \{v_1, \dots, v_n\}$ be a set of vertices and let $\mathcal{T} = (T_i)_{i=d+1}^n$ be a sequence of simplicial d-balls. We say that $\mathcal{T}$ is a **BBC sequence** if

(i) each T_i is of the form $T_i = B_{i-1} * v_i$, where B_{i-1} is a simplicial $(d-1)$-ball on vertex set $\{v_1, \dots, v_{i-1}\}$; and
(ii) $B_{i-1} \subseteq \partial T_{i-1}$ for $i = d+2, \dots, n$.

Remark 2.4. Let P be a simplicial $(d+1)$-polytope with vertices $v_1, \dots, v_n$. We may assume that the vertices of P lie in general position since P is simplicial. So every subset of at least $d+2$ of its vertices is again the vertex set of a simplicial $(d+1)$-polytope. Thus by taking an arbitrary ordering $(v_{l_1}, \dots, v_{l_n})$ of the vertices of P we obtain a sequence of $(d+1)$-balls $P_i = \mathrm{conv}(v_{l_1}, \dots, v_{l_i})$ for $i = d+2, \dots, n$. The corresponding BBC sequence $(T_i)_{i=d+2}^n$ is the sequence of pulling triangulations of P_i with respect to v_{l_i}.

Example 2.5. Consider the pentagon with its vertices labelled in cyclic order and construct a BBC sequence as in Remark 2.4. The BBC sequence consists of three elements: the triangle T_3, the pulling triangulations of the 4-gon T_4, and the pulling triangulation of the 5-gon T_5. The vector representation of T_i and ∂T_i for $i = 3, 4, 5$ are given in Table 1. The table also shows the subcomplexes $B_3 \subset \partial T_3$ and $B_4 \subset \partial T_4$ such that $T_4 = B_3 * v_4$ and $T_5 = B_4 * v_5$.

Note that $(T_i)_{i=d+1}^n$ is a BBC sequence if and only if each boundary sphere ∂T_i is **directly obtainable** from its predecessor ∂T_{i-1} in the sense of Altshuler [1]. In loc. cit. he proved that there is a 3-sphere on ten vertices which is *not* directly obtainable. Remark 2.4 implies that such a sphere cannot be polytopal.

Remark 2.6. Since each sphere of dimension at most 2 is polytopal it follows that every BBC sequence of 3-balls is a sequence of pulling triangulations of polytopes. But there exist BBC sequences of simplicial 4-balls such that the final boundary sphere is not polytopal. One such sequence is described in Section 3.5, Table 4.

TABLE 1. BBC sequence of a pentagon as described in Example 2.5. The vectors in $\{0,1\}^i$ are the characteristic functions of the complement of the vertex sets in $[i]$, for example 11001 represents $\{3,4\}$. The subcomplex B_{i-1} of ∂T_{i-1} such that T_i is the join of B_{i-1} with the vertex v_i are printed in bold.

	Facets of T_i			Facets of ∂T_i				
T_3	000			**001**	**100**	010		
T_4	0010	1000		**0011**	**1001**	**1100**	0110	
T_5	00110	10010	11000	00111	10011	11001	11001	01110

2.6. **Neighborly BBC Sequences.** A simplicial complex is **(simplicially)** k**-neighborly** if every k-subset of its vertices is a face. In other words, its $(k-1)$-skeleton is isomorphic to the $(k-1)$-skeleton of a simplex. A simplicial d-sphere is **(simplicially) neighborly** if it is simplicially $\lfloor (d+1)/2 \rfloor$-neighborly.

The next two observations are also implicit in the work of Shemer [23]. We start off with a lemma connecting the neighborliness of the cone over a simplicial ball with the neighborliness of the simplicial ball.

Lemma 2.7. *Let B be a simplicial $(d-1)$-ball on the vertex set $[i-1]$, and let $T = B * i$ its d-dimensional cone for some $d \geq 2$. Then ∂T is k-neighborly if and only if ∂B is $(k-1)$-neighborly and B is k-neighborly.*

Proof. The boundary ∂T is equal to $(\partial B * i) \cup B$ since $T = B * i$. If ∂T is k-neighborly all k-subsets of $[i]$ are faces of ∂T. Because ∂T is the union of $\partial B * i$ and B, all $(k-1)$-subsets of $[i-1]$ must be contained in ∂B and B must contain all k-subsets of $[i-1]$. This means that ∂B is $(k-1)$-neighborly and B is k-neighborly.

Conversely, if ∂B is $(k-1)$-neighborly and B is k-neighborly then $\partial T = (\partial B * i) \cup B$ obviously contains all the k-subsets of $[i]$. □

We call a BBC sequence $(T_i)_{i=d+1}^n$ **neighborly** if the final boundary sphere ∂T_n is neighborly. With Lemma 2.7 and induction we obtain a characterization of neighborly BBC sequences.

Proposition 2.8. *Let $(T_i)_{i=d+1}^n$ be a BBC sequence of d-balls for $d \geq 2$ with B_i defined as in Definition 2.3. Then the following are equivalent:*

(1) $(T_i)_{i=d+1}^n$ is a neighborly BBC sequence.
(2) ∂T_i is a neighborly $(d-1)$-sphere for all $i = d+1, \ldots, n$.
(3) For all $i = d+1, \ldots, n$ the ball B_{i-1} is $\lfloor d/2 \rfloor$-neighborly, and the sphere ∂B_{i-1} is $(\lfloor d/2 \rfloor - 1)$-neighborly.

Neighborly BBC sequences arise naturally from neighborly simplicial polytopes as in Remark 2.4. So the above definition is a generalization of the sequences of pulling triangulations of cyclic polytopes originally used by Babson, Billera, and Chan.

Corollary 2.9. *Take an arbitrary ordering of the vertices of a neighborly simplicial polytope. Then there exists a realization such that the induced pulling triangulations form a neighborly BBC sequence.*

3. Cubical Spheres from BBC sequences

In the first two sections we generalize the results of Babson, Billera, and Chan using BBC sequences. In Section 3.2 we derive a sign vector representation for the cubical sphere constructed from a BBC sequence. Then we show that the neighborly cubical spheres build from special vertex orderings of cyclic polytopes are indeed isomorphic to the neighborly cubical polytopes studied in [14]. Further we construct a non-polytopal neighborly cubical sphere based on a BBC sequence obtained from the non-polytopal Altshuler 3-sphere on 10 vertices (cf. Altshuler [2] and Bokowski and Garms [7]).

3.1. **Cubical Spheres.** Let $\mathcal{T} = (T_i)_{i=d}^{n-1}$ be a BBC sequence of simplicial $(d-1)$-balls. We inductively define cubical complexes S_{k+1} for $k = d, \ldots, n-1$ with 2^{k+1} vertices. Babson, Billera, and Chan begin their inductive definition with S_d, which is two d-cubes identified at their complete boundary. But since this does not yield a regular cell complex, we start with $S_{d+1} = \partial\, \mathrm{C}_{d+1}$ the boundary of the $(d+1)$-cube (that is, the mirror complex of the boundary of the d-simplex). Then for $k = d+1, \ldots, n-1$ we recursively define:

$$\begin{aligned} S_{k+1} &= \mathrm{fiss}_{S_k}(\mathrm{M}(T_k), S_k \setminus \mathrm{M}(T_k)) \\ &= (\mathrm{M}(T_k) \times \{+1\}) \cup (\partial\, \mathrm{M}(T_k) \times \{0\}) \cup ((S_k \setminus \mathrm{M}(T_k)) \times \{-1\}) \end{aligned}$$

We denote the final cubical complex S_n by $\mathrm{bbc}(\mathcal{T})$.

Theorem 3.1. *Let $\mathcal{T} = (T_i)_{i=d}^{n-1}$ be a BBC sequence of simplicial $(d-1)$-balls. Then the cubical complexes S_k and, in particular, $\mathrm{bbc}(\mathcal{T})$ are cubical d-spheres.*

Proof. This is part of what is proved by Babson, Billera, Chan [4, Theorem 3.1]. The essential step in the inductive argument is to see that the mirror complex of T_k is actually a subcomplex of S_k, such that fissuring is possible. □

If the simplicial balls T_i are equipped with a piecewise-linear (PL) structure, then the resulting cubical spheres are also PL [4].

3.2. **A Combinatorial Description.** The following theorem states a purely combinatorial description of the cubical spheres constructed from BBC sequences. Its characterization as a subcomplex of a high dimensional cube is close to the Cubical Gale Evenness Condition [14, Theorem 18].

Theorem 3.2. *Let $\mathcal{T} = (T_i)_{i=d}^{n-1}$ be a BBC sequence of simplicial $(d-1)$-balls with $n > d > 2$. Then the facets of the cubical d-sphere $\mathrm{bbc}(\mathcal{T})$ correspond to the following list of d-faces of the n-dimensional cube C_n; they are represented as sign vectors $\alpha \in \{0, \pm 1\}^n$ with exactly d zeros. The type t of a facet corresponds to the number of trailing non-zero entries in α:*

TABLE 2. The facets of S_{d+1} are the facets of the $(d+1)$-cube.

Type	Facets			
0	$\pm$	0	$\cdots$	0
0	0	$\pm$	0	$\vdots$
0	$\vdots$	$\ddots$	$\ddots$	0
1	0	$\cdots$	0	$\pm$

- ▹ ***type*** $t = 0$: $\alpha_n = 0$ *and* $|\alpha^{(n-1)}| := (|\alpha_1|, \ldots, |\alpha_{n-1}|) \in \partial T_{n-1}$
- ▹ ***type*** $0 < t < n - d$: $\alpha = (\alpha^{(n-t-1)}, 0, \alpha_{n-t+1} = \sigma, -1, \ldots, -1)$, *where* $\sigma \in \{\pm 1\}$, $\alpha^{(n-t-1)} \in \{0, \pm 1\}^{n-t-1}$ *with*
  - *(1)* $|\alpha^{(n-t-1)}| \in \partial T_{n-t-1}$ *and*
  - *(2) if* $\sigma = +1$, *then* $|(\alpha^{(n-t-1)}, 0)| \in T_{n-t}$;
    if $\sigma = -1$, *then* $|(\alpha^{(n-t-1)}, 0)| \notin T_{n-t}$.
- ▹ ***type*** $t = n - d$: $\alpha = (0, \ldots, 0, \sigma, -1, \ldots, -1)$ *with* $\sigma \in \{-1, +1\}$.

Proof. We prove that the vector representation of the facets of the cubical sphere given by the theorem corresponds to the facets of the inductive definition of S_k used in Theorem 3.1:

$$S_{k+1} = (\mathrm{M}(T_k) \times \{+1\}) \cup (\partial\,\mathrm{M}(T_k) \times \{0\}) \cup ((S_k \setminus \mathrm{M}(T_k)) \times \{-1\})$$

It was already shown in Theorem 3.1 that all S_k are cubical spheres. We proceed by induction on k. The sphere S_{d+1} is the boundary of the $(d+1)$-cube for $k = d+1$. The facets of S_{d+1} are the vectors $\alpha \in \{0, \pm 1\}^{d+1}$ with exactly one non-zero entry. The facets of $T_d = \Delta_{d-1}$ are the vectors in $\{0, 1\}^d$ with one 1. Thus all the facets of S_{d+1} are either of type 0 or type 1 shown in Table 2.

The inductive step is split into two claims showing the vector description and the inductive definition of S_k yield the same combinatorics.

Claim. Each facet in the vector representation given by the theorem is also a facet in the inductive definition via fissuring.

We analyze all types of facets of the vector representation of S_{k+1}.

- ▹ **type 0:** The facets of type 0 are the vectors $\alpha = (\alpha^{(k)}, 0) \in \{0, \pm 1\}^{k+1}$ with $|\alpha^{(k)}| \in \partial T_k$. This is equivalent to $\alpha^{(k)} \in \mathrm{M}(\partial T_k)$ and thus $(\alpha^{(k)}, 0) \in \mathrm{M}(\partial T_k) \times \{0\} = \partial\,\mathrm{M}(T_k) \times \{0\}$.
- ▹ **type 1:** The facets of type 1 are the vectors $\alpha = (\alpha^{(k-1)}, 0, \sigma)$ with $|\alpha^{(k-1)}| \in \partial T_{k-1}$ and
  - (1) $\sigma = +1$ if $(|\alpha^{(k-1)}|, 0) \in T_k$ or
  - (2) $\sigma = -1$ if $(|\alpha^{(k-1)}|, 0) \notin T_k$.

  By induction $(\alpha^{(k-1)}, 0)$ is a type 0 facet of S_k, and:

(1) If $(|\alpha^{(k-1)}|, 0) \in T_k$ then $(\alpha^{(k-1)}, 0) \in \mathrm{M}(T_k)$ and thus $(\alpha^{(k-1)}, 0, +1)$ is contained in $\mathrm{M}(T_k) \times \{+1\} \subseteq S_{k+1}$.
(2) Otherwise, if $(|\alpha^{(k-1)}|, 0) \notin T_k$ then $(\alpha^{(k-1)}, 0) \in S_k \setminus \mathrm{M}(T_k)$ and thus $(\alpha^{(k-1)}, 0, -1) \in (S_k \setminus \mathrm{M}(T_k)) \times \{-1\} \subseteq S_{k+1}$.

▹ **type $1 < t_0 \leq k+1-d$:** The facets of type t_0 are the vectors

$$(\alpha^{(k-t_0)}, 0, \sigma, -1, \ldots, -1) \in \{0, \pm 1\}^{k+1}$$

with $\sigma \in \{\pm 1\}$ and $\alpha^{(k-t_0)} \in \{0, \pm 1\}^{k-t_0}$. Taking only the first k entries of $\alpha = (\alpha^{(k)}, -1)$, it follows by induction that $\alpha^{(k)}$ is a type $t_0 - 1$ facet of S_k. By Definition T_k is a cone with apex k and thus all its facets contain the vertex k. Since the last entry of $\alpha^{(k)}$ is not zero, $\alpha^{(k)}$ cannot be a facet of $\mathrm{M}(T_k)$. Further it is not contained in $\mathrm{M}(\partial T_k)$ because it has d zero entries. Thus $\alpha^{(k)}$ is a facet of $S_k \setminus \mathrm{M}(T_k)$ and $(\alpha^{(k)}, -1) \in S_{k+1}$.

Claim. Each facet in the inductive definition via fissuring is also a facet in the vector representation given by the theorem.

There are three different kinds of facets of S_{k+1} according to the inductive definition. We will determine the type of each kind of facet.

▹ The facets of $\mathrm{M}(T_k) \times \{+1\}$ are the vectors $\alpha = (\alpha^{(k)}, +1) \in \{0, \pm 1\}^{k+1}$ with $|\alpha^{(k)}| \in T_k$. Since by definition of the BBC sequence $T_k = B_{k-1} * v_k$ with $B_{k-1} \subseteq \partial T_{k-1}$, we obtain $\alpha^{(k)}_k = 0$ and $|\alpha^{(k-1)}| \in \partial T_{k-1}$. Hence α is a facet of type 1 of S_{k+1}.

▹ The facets of $\mathrm{M}(\partial T_k) \times \{0\}$ are the vectors $(\alpha^{(k)}, 0) \in \{0, \pm 1\}^{k+1}$ with $|\alpha^{(k)}| \in \partial T_k$. These are the facets of type 0 of S_{k+1}.

▹ The facets of $(S_k \setminus \mathrm{M}(T_k)) \times \{-1\}$ correspond to vectors $\alpha = (\alpha^{(k)}, -1)$ with $\alpha^{(k)} \in S_k \setminus \mathrm{M}(T_k)$. Using the inductive assumptions we distinguish two kinds of facets of S_k:
(1) Either $\alpha^{(k)}$ is a facet of type 0 of S_k not contained in $\mathrm{M}(T_k)$, that is, $\alpha^{(k)} = (\alpha^{(k-1)}, 0)$ with $|\alpha^{(k-1)}| \in \partial T_{k-1}$ and $\alpha^{(k)} \notin \mathrm{M}(T_k)$,
(2) or $\alpha^{(k)}$ is a facet of type $t_0 \in \{1, \ldots, k-d\}$ of S_k not contained in $\mathrm{M}(T_k)$, that is, $\alpha^{(k)} = (\alpha^{(k-1)}, \pm 1)$.

In the first case, α is a facet of type 1 of S_{k+1} since $(|\alpha^{(k-1)}|, 0) \notin T_k$. In the second case, α is a facet of type $t_0 + 1$ of S_{k+1}.

Hence the cubical sphere $\mathsf{bbc}((T_i)_{i=d}^{n-1}) = S_n$ has the facets given by the theorem. □

If $\mathcal{T} = (T_i)_{i=d}^{n-1}$ is a neighborly BBC sequence, $n > d > 2$, then each simplicial $(d-2)$-sphere ∂T_i is neighborly. The number of facets $s(i, d-2)$ of ∂T_i is determined by the Dehn-Sommerville equations, see Grünbaum [12, §9.2]. For odd dimension d we have

$$s(i, d-2) = \binom{i - \frac{d-1}{2}}{i+1-d} + \binom{i - \frac{d+1}{2}}{i+1-d} = \frac{2i}{2i-d+1} \binom{i - \frac{d-1}{2}}{i+1-d}.$$

Adding up these equations (according to the types of facets described in Theorem 3.2) results in a formula for $f(n, d)$, the number of facets of the neighborly cubical sphere $\mathrm{bbc}(\mathcal{T})$. The even dimensional case may be treated similarly and is left to the reader. The same formula occurs in [14, Corollary 19] where, however, the even and the odd-dimensional case are erroneously exchanged:

Corollary 3.3. *Let $\mathcal{T} = (T_i)_{i=d}^{n-1}$ be a neighborly BBC sequence of simplicial $(d-1)$-balls with $n > d > 2$ and d odd. Then the number of facets of $\mathrm{bbc}(\mathcal{T})$ is given by*

$$\begin{aligned} f(n,d) &= 2(d+1) + \sum_{t=0}^{n-d-2} s(n-t-1, d-2)2^{n-t-d} \\ &= 2(d+1) + \sum_{k=d+2}^{n} \frac{2k-2}{2k-1-d}\binom{k-\frac{d+1}{2}}{k-d}2^{k-d}. \end{aligned}$$

Example 3.4. A neighborly cubical 3-sphere with the graph of the 6-cube may be constructed from a BBC sequence of the pentagon described in Example 2.5. The f-vector of the sphere is $(64, 192, 192, 64)$. The vector representations of T_i and ∂T_i for $i = 3, 4, 5$ are given in Table 1. The mirror complexes of T_i and ∂T_i are obtained by simply replacing the 1's by ±1's. So according to the inductive definition we start with S_4, the boundary of the 4-cube and fissure first along $\mathrm{M}(\partial T_4)$ and then along $\mathrm{M}(\partial T_5)$. This yields the facet description of S_6 listed in Table 3.

3.3. **Neighborly Cubical Spheres.** First we introduce the concept of cubical neighborliness. A cubical complex is **(cubically) k-neighborly** if it has the $(k-1)$-skeleton of a cube. A cubical d-sphere is **(cubically) neighborly** if it is cubically $\lfloor (d+1)/2 \rfloor$-neighborly. This is similar to the simplicial case and not as in Babson, Billera, and Chan, where a k-neighborly cubical sphere has the k-skeleton of a cube. In the following $[k]$ denotes the set of positive integers $\{1, \ldots, k\}$.

The neighborliness of a simplicial complex is preserved by mirroring, that is, the mirror complex of a k-neighborly simplicial complex is a $(k+1)$-neighborly cubical complex: In its sign vector representation a k-neighborly simplicial complex Δ on n vertices contains all vectors in $\{0,1\}^n$ with k zeros, i.e. all k-subsets of $[n]$. Thus its mirror complex $\mathrm{M}(\Delta)$ contains all vectors in $\{0,\pm1\}^n$ with k zeros. These vectors represent the k-skeleton of the n-cube, which means $\mathrm{M}(\Delta)$ is $(k+1)$-neighborly.

Cubical neighborliness is also preserved by certain fissuring operations. Given a k-neighborly pure cubical complex C and a subcomplex S containing all vertices. If the boundary of S is k-neighborly, then the cubical fissure between S and its complement in C is again a k-neighborly cubical complex.

This yields neighborly cubical spheres from any neighborly BBC sequence with the same construction as in Theorem 3.1.

Corollary 3.5. *Let $\mathcal{T}$ be a neighborly BBC sequence of simplicial $(d-1)$-balls. Then the cubical complex $\mathrm{bbc}(\mathcal{T})$ is a neighborly cubical d-sphere.*

TABLE 3. Facets of the neighborly cubical 3-sphere S_6 sorted by type; see Example 3.4. The facets occur in different multiplicities depending on the number of ± entries in the corresponding row: For example, every row of type 0 represents eight facets. The rows of the 4×4 minor printed in bold are the facets of S_4, that is, the boundary of the 4-cube. The facets of S_5 are the 8×5 minor printed in bold and plain. The horizontal lines underline the inductive structure.

Type	Facets					
3	**0**	**0**	**0**	±	−	−
2	**0**	**0**	±	**0**	+	−
2	**0**	±	**0**	**0**	−	−
2	±	**0**	**0**	**0**	+	−
1	0	0	±	±	0	+
1	±	0	0	±	0	+
1	±	±	0	0	0	+
1	0	±	±	0	0	−
0	**0**	**0**	±	±	±	**0**
0	±	**0**	**0**	±	±	**0**
0	±	±	**0**	**0**	±	**0**
0	±	±	±	**0**	**0**	**0**
0	**0**	±	±	±	**0**	**0**

3.4. **Neighborly Cubical Polytopes.** In this section we show that for very particular neighborly BBC sequences, the neighborly cubical spheres constructed therefrom are isomorphic to the boundaries of the neighborly cubical polytopes described in [14, Theorem 18] and thus polytopal. The neighborly cubical polytopes have the following sign vector representation.

Theorem 3.6 (Cubical Gale Evenness Condition)**.** *The facets of the neighborly cubical polytope* $\mathrm{ncp}_{d+1}(n)$ *are given by vectors* $\alpha \in \{0, \pm 1\}^n$ *with* d *zeros. They are classified by the number* t *of leading* ± 1*'s:*

- ▹ ***type*** $t = 0$*:* $\alpha_1 = 0$*, and* $|\alpha|$ *satisfies the simplicial Gale Evenness Condition: between any two values* $\alpha_i, \alpha_j \in \{\pm 1\}$ *there is an even number of zeros.*
- ▹ ***type*** $0 < t < n - d$*:* $\alpha = (-1, +1, \ldots, (-1)^{t-1}, \sigma, 0, \alpha^{(n-t-1)})$*, with* $\sigma \in \{\pm 1\}$*,* $\alpha^{(n-t-1)} \in \{0, \pm 1\}^{n-t-1}$ *and:*
  (1) $|\alpha^{(n-t-1)}|$ *satisfies the simplicial Gale Evenness Condition, and*
  (2) if $\sigma = (-1)^{t+1}$*, then* $\alpha^{(n-t-1)}$ *starts with an even number of zeros;*
  if $\sigma = (-1)^{t}$*, then* $\alpha^{(n-t-1)}$ *starts with an odd number of zeros.*
- ▹ ***type*** $t = n - d$*:* $\alpha = (-1, +1, \ldots, (-1)^{t-1}, \sigma, 0, \ldots, 0)$ *with* $\sigma \in \{-1, +1\}$*.*

Consider the cyclic $(d-1)$-polytope on n vertices $\mathrm{cyc}_{d-1}(n)$ given as the convex hull of the vertices $v_j = (j, j^2, \ldots, j^{d-1})$ for $j = 1, \ldots, n$. The facets of the cyclic polytope are given by Gale's Evenness Condition, cf. [10] and [25, Theorem 0.7]. Let T_i denote the pulling triangulation of $\mathrm{cyc}_{d-1}(i)$ with respect to the vertex v_i. The pulling triangulation is the cone with apex v_i over the facets of $\mathrm{cyc}_{d-1}(i)$ not containing v_i. To satisfy Gale's Evenness Condition the facets not containing v_i have to end with an even number of zeros. Thus the facets of T_i according to our vector notation for simplicial complexes are the vectors $\varphi \in \{0,1\}^i$ with d zeros, such that:

(1) $\varphi_i = 0$, i.e. all facets contain v_i,
(2) $(\varphi_1, \ldots, \varphi_{i-1})$ satisfies Gale's Evenness Condition, and
(3) φ ends with an odd number of zeros.

The facets of the neighborly cubical sphere constructed from the corresponding neighborly BBC sequence $(T_i)_{i=d}^{n-1}$ are readily derived from Theorem 3.2. These specific spheres are polytopal, since they are isomorphic to the neighborly cubical polytopes.

Corollary 3.7. *Let $d \geq 3$ and T_i be the pulling triangulation of the cyclic polytope $\mathrm{cyc}_{d-1}(i)$ with respect to the last vertex as above. Then the neighborly cubical sphere $\mathrm{bbc}((T_i)_{i=d}^{n-1})$ and the boundary of the neighborly cubical polytope $\mathrm{ncp}_{d+1}(n)$ are combinatorially isomorphic. The isomorphism is given by inverting the order and then flipping the even bits:*

$$\Phi\colon \{0,\pm 1\}^n \to \{0,\pm 1\}^n,$$
$$(\alpha_1, \alpha_2, \ldots, \alpha_n) \mapsto (\alpha_n, -\alpha_{n-1}, \ldots, (-1)^n \alpha_2, (-1)^{n+1}\alpha_1)$$

Remark 3.8. We briefly summarize some results studied in [22, Section 3.6.3]. The neighborly cubical d-spheres constructed from BBC sequences of cyclic $(d-1)$-polytopes on $d+1$ vertices are combinatorially isomorphic.

Since in **even** dimension the automorphism group of every cyclic polytope acts transitively on its vertices, its pulling triangulations are combinatorially isomorphic. Thus the neighborly cubical spheres constructed from arbitrary vertex orderings of even dimensional cyclic polytopes are combinatorially unique and thus all polytopal. This includes Example 3.4.

In **odd** dimension $d-1$, the automorphism group of a cyclic polytope with more than $d+1$ vertices does not act transitively on the vertices. Thus we were able to construct non-isomorphic neighborly cubical 4-spheres from different triangulations of the cyclic 3-polytope on six vertices.

3.5. **A Non-polytopal Neighborly Cubical Sphere.** To algorithmically decide the polytopality of spheres is generally possible but of considerable complexity, see Bokowski and Sturmfels [8], Richter-Gebert [19], and Basu et al. [5]. If standard heuristic methods fail the problem can often be approached by applying suitable ad-hoc techniques only. A good example is the 3-sphere $\mathcal{M}^{10}_{425}$ found by Altshuler in an exhaustive enumeration of all neighborly simplicial 3-manifolds on ten vertices [2]. Its polytopality at first could not be decided, and so it was up to Bokowski and Garms [7] to prove that $\mathcal{M}^{10}_{425}$ does not admit any convex realization. Below we take this very example as the starting point for a construction of a neighborly cubical sphere which cannot be polytopal in view of the following result.

Theorem 3.9. *Let $\mathcal{T} = (T_i)_{i=d}^{n-1}$ be a BBC sequence of simplicial $(d-1)$-balls, neighborly or not, such that the cubical d-sphere $\mathrm{bbc}(\mathcal{T})$ is polytopal. Then the simplicial $(d-2)$-sphere ∂T_{n-1} is necessarily polytopal.*

Proof. Suppose that $\mathrm{bbc}(\mathcal{T})$ is isomorphic to the boundary complex of a convex cubical $(d+1)$-polytope $P \subset \mathbb{R}^{d+1}$. Consider the edge e of P corresponding to the sign vector $(+1, \dots, +1, 0)$ of length n. Choose an affine hyperplane H, parallel to e, which separates the edge e from the $2^n - 2$ vertices of P not contained in e. Next we choose a second affine hyperplane H', orthogonal to e, which separates the two vertices of e. Then, since H and H' are not parallel to each other, $P/e = P \cap H \cap H'$ is a $(d-1)$-dimensional convex polytope, the **edge figure** of e with respect to P. Its face lattice is isomorphic to the filter of e, that is, the faces of P containing e, in the face lattice of P.

Now the facets of P which contain e are exactly the facets of type 0 without negative signs in their sign vector representation. From Theorem 3.2 we conclude that $\partial(P/e)$ is isomorphic to ∂T_{n-1}, and hence the claim. □

As a consequence, each BBC sequence of simplicial $(d-1)$-balls with the property that its final boundary sphere is non-polytopal yields a non-polytopal cubical sphere. Note, however, that in contrast to polytopal spheres, see Corollary 2.9, for non-polytopal spheres there is no standard procedure to obtain a corresponding BBC sequence. Moreover, it follows from work of Altshuler that there is a simplicial 3-sphere on ten vertices which does not admit any BBC sequence [1].

Theorem 3.10. *There is a non-polytopal neighborly cubical 5-sphere with $2^{11} = 2048$ vertices and $f(11,5) = 3584$ facets. Its complete f-vector is*

$$f = (2048, 11264, 28160, 33280, 17920, 3584).$$

Proof. Table 4 lists a neighborly BBC sequence $\mathcal{A} = (A_i)_{i=5}^{10}$ of simplicial 4-balls with the property that the boundary of the final ball A_{10} is isomorphic to the Altshuler 3-sphere $\mathcal{M}^{10}_{425}$.

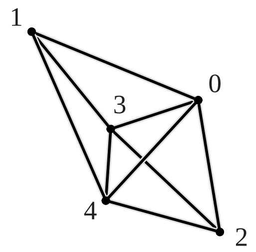
The simplicial 4-ball A_8 is a pulling triangulation of the neighborly simplicial 4-polytope on eight vertices which occurs as P^8_{36} in the list of Grünbaum and Sreedharan [13]; this is one of the two non-cyclic neighborly simplicial polytopes with these parameters.

The simplicial 4-ball A_9 is a pulling triangulation of a neighborly simplicial 4-polytope on nine vertices: The previous simplicial 3-sphere ∂A_8 is separated by the 2-sphere ∂B_8 into the 3-balls B_8 and its complement B'_8. The boundary ∂A_9 now is directly obtained from ∂A_8 by first removing B'_8 and then inserting the cone over ∂B_8 with the new vertex 9 as the apex. Since ∂A_8 is polytopal and B'_8

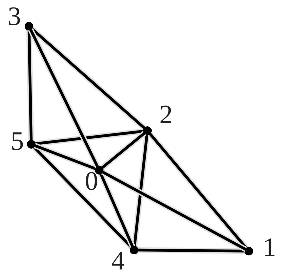
is a 3-simplex which is stacked once over each of its four facets it follows that A_9 is again polytopal: The vertex 9 can be chosen in a way such that it is **exactly beyond** all the facets of B'_8 in the sense of Shemer [23, page 301].

The sphere ∂A_{10} is directly obtained from ∂A_9 and isomorphic to $\mathcal{M}^{10}_{425}$ and thus not polytopal due to Bokowski and Garms [7]. From Theorem 3.9 it follows that the neighborly cubical sphere $\mathsf{bbc}(\mathcal{T})$ is not polytopal. □

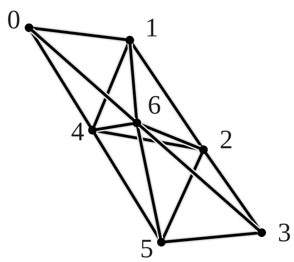
Figure 4 visualizes the complements of the simplicial 3-balls B_5, B_6, B_7, B_8, and B_9 and their boundaries.



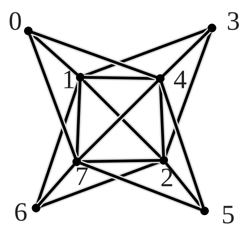
(a) Bipyramid over triangle: ∂B_5.



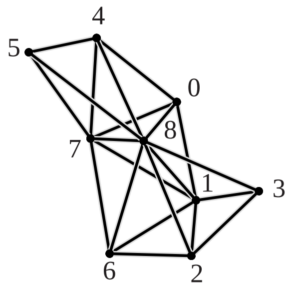
(b) 3-simplex stacked on two facets: ∂B_6.



(c) Cone over a strip of four triangles: ∂B_7.



(d) 3-simplex stacked on all its facets: ∂B_8.



(e) Cone over a strip of six triangles: ∂B_9.

FIGURE 4. The subcomplexes $B'_i = \partial A_i \setminus B_i$ of the BBC sequence of the Altshuler sphere; see Theorem 3.10. The sphere $\partial B_i = \partial B'_i$ is the link of the new vertex i in the sphere ∂A_{i+1}.

Remark 3.11. The polytopal neighborly simplicial 3-sphere ∂A_9 is the unique 3-sphere on nine vertices with Altshuler determinant equal to 103361328. It occurs as N^9_{19} in the classification of Altshuler and Steinberg [3] and as `manifold_3_9_523` in Lutz [15].

TABLE 4. Facets of the neighborly BBC sequence $\mathcal{A}$ of the non-polytopal Altshuler sphere $\mathcal{M}^{10}_{425}$. The simplicial ball A_i is the join of the vertex $i-1$ and the subcomplex B_{i-1} of the facets of ∂A_{i-1} printed in bold.

	Facets of A_i				Facets of ∂A_i				
A_5	01234				**0123**	**0124**	0134	0234	**1234**
A_6	01235	01245	12345		**0123**	0124	**0135**	**0145**	0235
					0245	**1234**	**1345**	**2345**	
A_7	01236	01356	01456	12346	**0123**	**0126**	**0135**	**0145**	0146
	13456	23456			**0236**	**0356**	**0456**	**1234**	1246
					1345	**2345**	2356	2456	
A_8	01237	01267	01357	01457	**0123**	**0126**	**0135**	**0145**	0147
	02367	03567	04567	12347	**0167**	**0236**	**0356**	**0456**	**0467**
	13457	23457			1234	1247	1267	**1345**	**2345**
					2357	**2367**	2457	**3567**	**4567**
A_9	01238	01268	01358	01458	**0123**	**0126**	**0135**	**0145**	**0148**
	01678	02368	03568	04568	**0167**	0178	**0236**	**0356**	**0456**
	04678	13458	23458	23578	**0467**	0478	1238	1268	**1345**
	23678	35678	45678		**1348**	1678	**2345**	**2348**	**2357**
					2367	**2458**	**2578**	**2678**	**3567**
					4567	4578			
A_{10}	01239	01269	01359	01459	0123	0126	0135	0145	0148
	01489	01679	02369	03569	0167	0179	0189	0236	0356
	04569	04679	13459	13489	0456	0467	0479	0489	1239
	23459	23489	23579	23679	1269	1345	1348	1389	1679
	24589	25789	26789	35679	2345	2348	2357	2367	2389
	45679				2458	2578	2678	2689	3567
					4567	4579	4589	5789	6789

4. Polyhedral Surfaces

In this section we describe a nice way to realize polyhedral surfaces of 'unusually large genus' [17] in the Schlegel diagram of neighborly cubical polytopes. The surfaces considered are cubical polyhedral surfaces where each vertex has degree q. They were first described by Coxeter [9] in 1937 in terms of reflection groups. Ringel [20] explicitly described these surfaces whilst analyzing problems concerning the graph of the n-dimensional cube. He pointed out that the surfaces are of lowest genus among all surfaces on which the graph of the n-cube may be drawn without self intersection. Further he gave an explicit combinatorial description of theses surfaces as a 2-dimensional subcomplexes of the n-cube. McMullen, Schulz, and Wills [16, 17] analyze equivelar surfaces, a much more general class of polyhedral surfaces which include the cubical surfaces of Coxeter and Ringel.

Definition 4.1. An **equivelar surface** $\mathcal{M}_{p,q}$ is a polyhedral surface such that all 2-faces are p-gons and all vertices have degree q.

McMullen, Schulz, and Wills were the first to point at the 'unusually large genus' of these surfaces. In particular, they inductively constructed embeddings for the cubical equivelar surfaces of type $\mathcal{M}_{4,q}$ in $\mathbb{R}^3$.

4.1. **Equivelar surfaces and mirror complexes.** Let Q be the boundary of a q-gon for $q > 2$ and $\mathrm{M}(Q)$ its mirror complex. With the vertices of Q labelled in cyclic order we obtain the same vector representation as Ringel [20, page 17] of $\mathrm{M}(Q)$:

$$\begin{array}{cccccccc} 0 & 0 & \pm & \pm & \cdots & \pm & \pm & \pm \\ \pm & 0 & 0 & \pm & \cdots & \pm & \pm & \pm \\ . & . & . & . & \cdots & . & . & . \\ . & . & . & . & \cdots & . & . & . \\ \pm & \pm & \pm & \pm & \cdots & \pm & 0 & 0 \\ 0 & \pm & \pm & \pm & \cdots & \pm & \pm & 0 \end{array}$$

The mirror complex of Q is an equivelar surface $\mathcal{M}_{4,q}$ since the vertex figure of every vertex of $\mathrm{M}(Q)$ is isomorphic to Q. It is embedded in the 2-skeleton of the q-cube.

The obvious way to realize a 2-dimensional subcomplex of the q-cube in $\mathbb{R}^5$ is in the Schlegel diagram of the 6-dimensional neighborly cubical polytope $\mathrm{ncp}_6(q)$ of [14]: Since the neighborly cubical polytope $\mathrm{ncp}_6(q)$ has the 2-skeleton of the q-cube, $\mathrm{M}(Q)$ is contained in the boundary of $\mathrm{ncp}_6(q)$. The Schlegel diagram of $\mathrm{ncp}_6(q)$ is embedded in $\mathbb{R}^5$, thus $\mathrm{M}(Q)$ may be realized in $\mathbb{R}^5$.

Let T_i be the pulling triangulation of the i-gon for $i = 3, \dots, q-1$. The facets of type 0 of $\mathcal{S}_3(q) = \mathrm{bbc}((T_i)_{i=3}^{q-1})$ are the mirror complex of a cone over the boundary of the $(q-1)$-gon. Since the q-gon and its pulling triangulation are subcomplexes of this cone, the mirror complex of the q-gon and the mirror complex of its pulling triangulation are both subcomplexes of $\mathcal{S}_3(q)$. This yields a cubificated embedding of the surface $\mathrm{M}(Q)$ into the neighborly cubical 3-sphere $\mathcal{S}_3(q)$. By Corollary 3.7 this sphere is isomorphic to the boundary of the neighborly cubical polytope $\mathrm{ncp}_4(q)$. Hence the mirror complex of the q-gon can be realized as a subcomplex of the Schlegel diagram of $\mathrm{ncp}_4(q)$ in $\mathbb{R}^3$. This answers a question of Günter M. Ziegler [24] whether some of the surfaces from [17] can be found as subcomplexes of the neighborly cubical polytopes.

The genus of this surface may easily be calculated from its f-vector $f(\mathrm{M}(Q)) = (2^q, 2^{q-1}q, 2^{q-2}q)$:

$$g(q) = 1 + 2^{q-3}(q-4) = O(f_0 \cdot \log f_0).$$

Thus $q = 12$ is the first parameter for which the genus $g(12) = 4097$ exceeds the number of vertices, which equals $2^{12} = 4096$.

Since the surface arising from the 12-gon is too hard to visualize we display the mirror complex of the pentagon in the Schlegel diagram of $\mathsf{ncp}_4(5)$ in Figure 5.

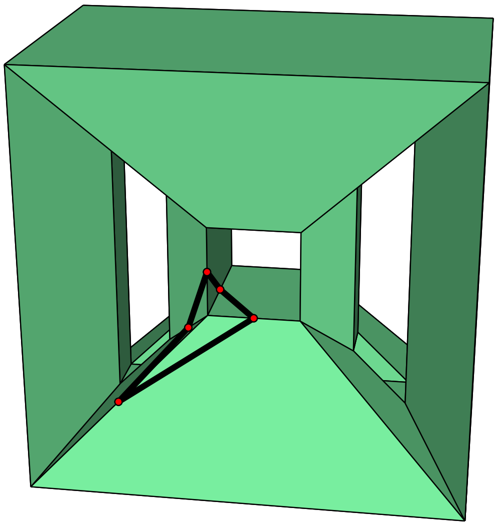


FIGURE 5. The mirror complex of the 5-gon is an equivelar surface of type $\mathcal{M}_{4,5}$ of genus 5. It is realized in the Schlegel diagram of the neighborly cubical polytope $\mathsf{ncp}_4(5)$, which is embedded in $\mathbb{R}^3$.

5. Concluding Remarks and Acknowledgments

It can be shown that if the BBC sequence $\mathcal{T}$ is polytopal then the cubical sphere $\mathsf{bbc}(\mathcal{T})$ is also polytopal [21]. In view of Theorem 3.9 this gives the complete picture of the construction as far as questions of polytopality are concerned.

For the visualization of the simplicial balls and the polyhedral surface $\mathcal{M}_{4,5}$ we used the software packages `polymake` [11] and `JavaView` [18].

We are indebted to Frank H. Lutz for his help in recognizing the sphere ∂A_9 in Table 4. Moreover, we are grateful to an anonymous referee who scrutinized the submitted version of this paper and who pointed out several minor errors.

References

1. A. Altshuler, *A peculiar triangulation of the 3-sphere*, Proc. Amer. Math. Soc. **54** (1976), 449–452. MR MR0397744 (53 #1602)
2. Amos Altshuler, *Neighborly 4-polytopes and neighborly combinatorial 3-manifolds with ten vertices*, Canad. J. Math. **29** (1977), no. 2, 400–420. MR MR0467531 (57 #7387)

3. Amos Altshuler and Leon Steinberg, *Neighborly* 4*-polytopes with* 9 *vertices*, J. Combinatorial Theory Ser. A **15** (1973), 270–287. MR MR0333981 (48 #12300)
4. Eric K. Babson, Louis J. Billera, and Clara S. Chan, *Neighborly cubical spheres and a cubical lower bound conjecture*, Israel J. Math. **102** (1997), 297–315. MR MR1489110 (99a:52011)
5. Saugata Basu, Richard Pollack, and Marie-Françoise Roy, *Algorithms in real algebraic geometry*, Algorithms and Computation in Mathematics, vol. 10, Springer-Verlag, Berlin, 2003. MR MR1998147 (2004g:14064)
6. Anders Björner, *Posets, regular CW complexes and Bruhat order*, European J. Combin. **5** (1984), no. 1, 7–16. MR MR746039 (86e:06002)
7. Jürgen Bokowski and Klaus Garms, *Altshuler's sphere* M^{10}_{425} *is not polytopal*, European J. Combin. **8** (1987), no. 3, 227–229. MR MR919873 (89a:52015)
8. Jürgen Bokowski and Bernd Sturmfels, *Computational synthetic geometry*, Lecture Notes in Mathematics, vol. 1355, Springer-Verlag, Berlin, 1989. MR MR1009366 (90i:52001)
9. H. S. M. Coxeter, *Regular skew polyhedra in three and four dimensions, and their topological analogues*, Proc. Lond. Math. Soc., II. Ser. **43** (1937), 33–62.
10. David Gale, *Neighborly and cyclic polytopes*, Proc. Sympos. Pure Math., Vol. VII, Amer. Math. Soc., Providence, R.I., 1963, pp. 225–232. MR MR0152944 (27 #2915)
11. Ewgenij Gawrilow and Michael Joswig, *polymake: a framework for analyzing convex polytopes*, Polytopes—combinatorics and computation (Oberwolfach, 1997), DMV Sem., vol. 29, Birkhäuser, Basel, 2000, pp. 43–73. MR MR1785292 (2001f:52033). `http://www.math.tu-berlin.de/polymake`
12. Branko Grünbaum, *Convex polytopes*, second ed., Graduate Texts in Mathematics, vol. 221, Springer-Verlag, New York, 2003, Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler. MR MR1976856 (2004b:52001)
13. Branko Grünbaum and V. P. Sreedharan, *An enumeration of simplicial* 4*-polytopes with* 8 *vertices*, J. Combinatorial Theory **2** (1967), 437–465. MR MR0215182 (35 #6025)
14. Michael Joswig and Günter M. Ziegler, *Neighborly cubical polytopes*, Discrete Comput. Geom. **24** (2000), no. 2-3, 325–344, The Branko Grünbaum birthday issue. MR MR1758054 (2001f:52019)
15. Frank H. Lutz, *The Manifold Page, 1999–2005*, `http://www.math.tu-berlin.de/diskregeom/stellar/`.
16. Peter McMullen, Christoph Schulz, and Jörg M. Wills, *Equivelar polyhedral manifolds in* E^3, Israel J. Math. **41** (1982), no. 4, 331–346. MR MR657865 (83h:52009)
17. ______, *Polyhedral* 2*-manifolds in* E^3 *with unusually large genus*, Israel J. Math. **46** (1983), no. 1-2, 127–144. MR MR727027 (84m:52011)
18. Konrad Polthier, Klaus Hildebrandt, Eike Preuss, and Ulrich Reitebuch, *JavaView v3.03*, `www.javaview.de`, 2004.
19. Jürgen Richter-Gebert, *Realization spaces of polytopes*, Lecture Notes in Mathematics, vol. 1643, Springer-Verlag, Berlin, 1996. MR MR1482230 (99b:52020)
20. Gerhard Ringel, *Über drei kombinatorische Probleme am n-dimensionalen Würfel und Würfelgitter*, Abh. Math. Sem. Univ. Hamburg **20** (1955), 10–19. MR MR0075586 (17,772h)
21. Raman Sanyal, Thilo Schröder, and Günter M. Ziegler, in preparation.
22. Thilo Schröder, *Neighborly cubical spheres and polytopes*, Master's thesis, Technische Universität Berlin, 2004, `http://www.math.tu-berlin.de/~thilosch/`.
23. Ido Shemer, *Neighborly polytopes*, Israel J. Math. **43** (1982), no. 4, 291–314. MR MR693351 (84k:52008)
24. Günter M. Ziegler, *personal communication, March 2004.*
25. ______, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995. MR MR1311028 (96a:52011)

Michael Joswig, Fachbereich Mathematik, AG 7, TU Darmstadt, 64289 Darmstadt, Germany
E-mail address: joswig@mathematik.tu-darmstadt.de

Thilo Schröder, Institut für Mathematik, MA 6-2, TU Berlin, 10623 Berlin, Germany
E-mail address: thilosch@math.tu-berlin.de