A Weighted L^q-approach to Stokes Flow Around a Rotating Body

R. Farwig[1], M. Krbec[2,3] and Š. Nečasová[2,4]

Abstract

Considering time-periodic Stokes flow around a rotating body in $\mathbb{R}^2$ or $\mathbb{R}^3$ we prove weighted *a priori* estimates in L^q-spaces for the whole space problem. After a time-dependent change of coordinates the problem is reduced to a stationary Stokes equation with the additional term $(\omega \times x) \cdot \nabla u$ in the equation of momentum, where ω denotes the angular velocity. In cylindrical coordinates attached to the rotating body we allow for Muckenhoupt weights which may be anisotropic or even depend on the angular variable and prove weighted L^q-estimates using the weighted theory of Littlewood-Paley decomposition and of maximal operators.

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1 Introduction

The problem of the motion of a rigid body in a liquid has attracted the attention of scientists since more than a century. The first systematic study of this subject was initiated by the pioneering works by Kirchhoff [13] and

[1]Department of Mathematics, Darmstadt University of Technology, 64289 Darmstadt, Germany, e-mail: farwig@mathematik.tu-darmstadt.de
[2]Mathematical Institute of Academy of Sciences, Žitná 25, 11567 Prague 1, Czech Republic
[3]e-mail: krbecm@math.cas.cz
[4]e-mail: matus@math.cas.cz

Lord Kelvin [26], regarding the motion of one or more bodies in an inviscid liquid. After that many mathematicians have furnished significant contributions to this fascinating field under different assumptions on the body and on the fluid. We would like to quote the work of Brenner [3] concerning the steady motion of one or more bodies in a linear viscous liquid in the Stokes approximation as well as Weinberger [27], [28], Serre [23] regarding the fall of a body in an incompressible Navier-Stokes fluid under the action of gravity and Borchers [2] for the existence of weak solutions. Among more recent articles we refer to Farwig, Hishida and Müller [7], Farwig [5], [6], Galdi [8], [9], Gunther, Hudspeth, Thomann [11], Martin, Starovoitov and Tucsnak [17] and references in these papers.

In this paper we consider a (two- or) three-dimensional rigid body rotating with angular velocity $\omega = (0,0,1)^T$ and assume that the complement is filled with a viscous incompressible fluid modelled by the Navier-Stokes equations. Given the coefficient of viscosity $\nu > 0$ and an external force $\tilde{f} = \tilde{f}(y,t)$, we are looking for the velocity $v = v(y,t)$ and the pressure $q = q(y,t)$ solving the nonlinear system

$$\begin{aligned} v_t - \nu\Delta v + v\cdot\nabla v + \nabla q &= \tilde{f} && \text{in } \Omega(t),\ t>0,\\ \operatorname{div} v &= 0 && \text{in } \Omega(t),\ t>0,\\ v(y,t) &= \omega\wedge y && \text{on } \partial\Omega(t),\ t>0,\\ v(y,t) &\to 0 && \text{as } |y|\to\infty\,. \end{aligned} \tag{1.1}$$

Here the time-dependent exterior domain $\Omega(t)$ is given, due to the rotation with the angular velocity ω, by

$$\Omega(t) = O(t)\Omega,$$

where $\Omega \subset \mathbb{R}^3$ is a fixed exterior domain and $O(t)$ denotes the orthogonal matrix

$$O(t) = \begin{pmatrix} \cos t & -\sin t & 0\\ \sin t & \cos t & 0\\ 0 & 0 & 1 \end{pmatrix}. \tag{1.2}$$

In the two-dimensional case $\Omega \subset \mathbb{R}^2$ is a fixed exterior planar domain and

$$O(t) = \begin{pmatrix} \cos t & -\sin t\\ \sin t & \cos t \end{pmatrix}. \tag{1.3}$$

After the change of variables

$$x = O(t)^T y \tag{1.4}$$

and passing to the new functions

$$u(x,t) = O(t)^T v(y,t), \quad p(x,t) = q(y,t), \tag{1.5}$$

as well as to the force term $f(x,t) = O(t)^T \tilde f(y,t)$ we arrive at the modified Navier-Stokes system

$$\begin{aligned} u_t - \nu\Delta u + u\cdot\nabla u - (\omega\wedge x)\cdot\nabla u + \omega\wedge u + \nabla p = f \quad & \text{in } \Omega,\\ \operatorname{div} u = 0 \quad & \text{in } \Omega,\\ u(x,t) = \omega\wedge x \quad & \text{on } \partial\Omega,\\ u(x,t) \to 0 \quad & \text{at } \infty, \end{aligned} \tag{1.6}$$

for all $t > 0$ in the exterior time-independent domain Ω. Note that because of the new coordinate system attached to the rotating body (1.6) contains two new linear terms, the classical Coriolis force term $\omega\wedge u$ (up to a multiplicative constant) and the term $(\omega\wedge x)\cdot\nabla u$ which is not subordinate to the Laplacean in unbounded domains. Linearizing (1.6) in u at $u\equiv 0$ and considering only the stationary whole space problem we arrive at the modified Stokes system

$$\begin{aligned} -\nu\Delta u - (\omega\wedge x)\cdot\nabla u + \omega\wedge u + \nabla p = f \quad & \text{in } \mathbb{R}^3,\\ \operatorname{div} u = 0 \quad & \text{in } \mathbb{R}^3,\\ u \to 0 \quad & \text{as } |x|\to\infty, \end{aligned} \tag{1.7}$$

where $n = 2$ or $n = 3$; in the two-dimensional case plainly $\omega\wedge x \mathrel{\widehat=} (-x_2, x_1)$ for $x = (x_1, x_2)$ and $\omega\wedge u \mathrel{\widehat=} (-u_2, u_1)$ for $u = (u_1, u_2)$.

The linear system (1.7) has been analyzed in L^q-spaces, $1 < q < \infty$, in [7], proving the *a priori*-estimate

$$\|\nu\nabla^2 u\|_q + \|(\omega\wedge x)\cdot u - \omega\wedge u\|_q + \|\nabla p\|_q \le c\|f\|_q. \tag{1.8}$$

Similar results were obtained in the case of a rotating body with constant translational velocity u_∞ parallel to ω, leading to an Oseen system like (1.7) in which the additional term $u_\infty\cdot\nabla u$ has to be added in the equation of the balance of momentum, see [5], [6]. For related L^q-results on weak solutions we refer to [12], for the investigation of several auxiliary linear problems to [19], [20], and for weak solutions to an Oseen system of type (1.7) in L^2 with anisotropic weights see [15].

The aim of this paper is to generalize the *a priori*-estimate (1.8) to weighted L^q-spaces. For this reason we shall consider the weighted Lebesgue space

$$L^q_w(\mathbb{R}^n) = L^q_w = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) : \|u\|_{q,w} = \left(\int_{\mathbb{R}^n} |u(x)|^q w(x)\, dx \right)^{1/q} < \infty \right\},$$

where $w \in L^1_{\text{loc}}$ is a weight function. A *weight* or a *weight function* will be always an a.e. nonnegative and locally integrable function. In order to apply estimates for singular integral operators, multiplier operators and maximal operators, the weight function w will be supposed to satisfy the Muckenhoupt A_p-condition.

Definition 1.1. A weight function $0 \le w \in L^1_{\text{loc}}$ belongs to the *Muckenhoupt class* A_q, $1 \le q < \infty$, if there exists a constant $C > 0$ such that

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q w^{-1/(q-1)}\, dx \right)^{q-1} \le C < +\infty,$$

if $1 < q < \infty$, and

$$\sup_Q \frac{1}{|Q|} \int_Q w(x)\, dx \le C w(x_0), \quad \text{for a.a. } x_0 \in \mathbb{R}^N,$$

if $q = 1$, respectively. In the first case, the supremum is taken over all cubes Q in $\mathbb{R}^n$ with edges parallel to the coordinate axes, in the second case over those of such cubes which contain x_0; here $|Q|$ denotes the n-dimensional Lebesgue measure of Q. The least constant C above is called the A_q-*constant* of the weight. We note that in the second case we may restrict ourselves to cubes Q centered at x. Since the A_q weights satisfy the doubling property, that is, $w(2Q) \le cw(Q)$, where $w(Q) = \int_Q w(x)\, dx$ and $2Q$ is the cube with the same center as Q but with double side length, one can consider balls instead of cubes; the observation on centres of cubes applies to balls as well (see e.g. [10]).

Theorem 1.2. *(i) Let w be a weight in $\mathbb{R}^n$, $n = 2$ or $n = 3$, and assume that w is independent of the angular variable θ in a cylindrical coordinate system attached to the axis of revolution $(0, 0, 1)^T$. Moreover, let w satisfy the following condition depending on $q \in (1, \infty)$: either*

$$2 < q < \infty \text{ and } w^r \in A_{rq/2} \quad \text{for some } r \in [1, \infty) \tag{1.9}$$

or

$$q = 2 \text{ and } w^r \in A_r \quad \text{or} \quad w^{-r} \in A_r \quad \text{for some } r \in [1, \infty) \tag{1.10}$$

or

$$1 < q < 2 \text{ and } w^r \in A_{rq/2} \quad \text{for some } r \in \left(\frac{2}{q}, \frac{2}{2-q}\right]. \tag{1.11}$$

Given $f \in L^q_w(\mathbb{R}^n)^n$ *there exists a solution* $(u, p) \in L^1_{\text{loc}}(\mathbb{R}^n)^n \times L^1_{\text{loc}}(\mathbb{R}^n)$ *of* (1.7) *satisfying the estimate*

$$\|\nu \nabla^2 u\|_{q,w} + \|(\omega \wedge x) \cdot u - \omega \wedge u\|_{q,w} + \|\nabla p\|_{q,w} \le c\|f\|_{q,w} \tag{1.12}$$

with a constant $c = c(q, w) > 0$.
(ii) Let $f \in L^{q_1}_{w_1}(\mathbb{R}^n)^n \cap L^{q_2}_{w_2}(\mathbb{R}^n)^n$, $n = 2$ *or* $n = 3$, *such that both* (q_1, w_1) *and* (q_2, w_2) *satisfy the conditions* (1.9) – (1.11), *and let* $u_1, u_2 \in L^1_{\text{loc}}(\mathbb{R}^n)^n$ *together with corresponding pressure functions* $p_1, p_2 \in L^1_{\text{loc}}(\mathbb{R}^n)$ *be solutions of* (1.7) *satisfying* (1.12) *for* (q_1, w_1) *and* (q_2, w_2), *respectively. Then there are* $\alpha, \beta, \gamma \in \mathbb{R}$ *such that* u_1 *coincides with* u_2 *up to an affine linear field* $\alpha\omega + \beta\omega \wedge x + \gamma(x_1, x_2, -2x_3)^T$. *For* $n = 2$ *obviously* $u_1 = u_2$ *up to the field* $\beta(-x_2, x_1)^T$.

As an example we consider power weights of the type $|x|^\alpha$ and $(1 + |x|)^\alpha$, $\alpha \in \mathbb{R}$, as well as weights of the type $(1 + |x|)^\alpha (1 + r)^\beta$, $\alpha, \beta \in \mathbb{R}$, where $r = \sqrt{x_1^2 + x_2^2}$ is the radial distance of $x = (x_1, x_2, x_3)$ from the axis of revolution.

Corollary 1.3. *(i) The a priori estimate* (1.12) *holds for power weights*

$$w(x) = |x|^\alpha \quad \text{and} \quad w(x) = (1 + |x|)^\alpha,$$

where

$$\begin{aligned} 2 < q < \infty &: \quad -n < \alpha < \tfrac{nq}{2} \\ q = 2 &: \quad -n < \alpha < n \\ 1 < q < 2 &: \quad -\tfrac{nq}{2} < \alpha < n(q-1). \end{aligned}$$

(ii) The a priori estimate (1.12) *holds for the anisotropic, axially symmetric weight* $w(x) = (1 + |x|)^\alpha (1 + r)^\beta$ *on* $\mathbb{R}^3$ *provided that*

$$\begin{aligned} 2 < q < \infty &: \quad -2 < \beta < q && \text{and} \quad -3 < \alpha + \beta < \tfrac{3q}{2} \\ q = 2 &: \quad -2 < \beta < 2 && \text{and} \quad -3 < \alpha + \beta < 3 \\ 1 < q < 2 &: \quad -q < \beta < 2(q-1) && \text{and} \quad -\tfrac{3q}{2} < \alpha + \beta < 3(q-1). \end{aligned}$$

Actually Theorem 1.2 holds also for θ-dependent weights w provided that $w(x) = w(r, x_3, \theta)$ satisfies an additional one-dimensional Muckenhoupt condition with respect to the angular variable $\theta \in [0, 2\pi)$ and with an $A_q(\theta)$-constant independent of (r, x_3), see Corollary 1.4 and its proof in §3 below. As an example we consider the anisotropic weight functions

$$w(x) = \eta^\alpha_\beta(x) = (1 + |x|)^\alpha (1 + s(x))^\beta, \quad s(x) = r - x_1,$$

introduced in [4] to analyze the Oseen equations.

Corollary 1.4. *The a priori estimate* (1.12) *holds for the anisotropic, θ-dependent weights*

$$w(x) = \sigma^\alpha_\beta(x) = |x|^\alpha s(x)^\beta$$

and

$$w(x) = \eta^\alpha_\beta(x) = (1 + |x|)^\alpha (1 + s(x))^\beta$$

provided that

$$\begin{array}{rcll} 2 < q < \infty & : & -\frac{1}{2} < \beta < \frac{q}{4} & \text{and} \quad -n < \alpha + \beta < \frac{nq}{2} \\ q = 2 & : & -\frac{1}{2} < \beta < \frac{1}{2} & \text{and} \quad -n < \alpha + \beta < n \\ 1 < q < 2 & : & -\frac{q}{4} < \beta < \frac{q-1}{2} & \text{and} \quad -\frac{nq}{2} < \alpha + \beta < n(q-1). \end{array}$$

2 Preliminaries

To prove Theorem 1.2 we need several properties of Muckenhoupt weights.

Lemma 2.1. *(1) Let μ be a nonnegative regular Borel measure such that the centered Hardy-Littlewood maximal operator*

$$\mathcal{M}\mu(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q d\mu$$

is finite for almost all $x \in \mathbb{R}^n$; here Q runs through the set of all cubes in $\mathbb{R}^n$ centered at x (with edges parallel to the coordinate axes) and $|Q|$ denotes the Lebesgue measure of Q. Then $(\mathcal{M}\mu)^\gamma \in A_1$ for all $\gamma \in [0, 1)$.

(2) Let $w_1, w_2 \in A_1$ and $0 < \theta < 1$. Then $w_1^{1-\theta} w_2^\theta \in A_1$.

(3) For all $1 < q < r$,

$$A_1 \subset A_q \subset A_r.$$

(4) Let $1 < q < \infty$ and $w \in A_q$. Then there are $w_1, w_2 \in A_1$ such that

$$w = \frac{w_1}{w_2^{q-1}}.$$

Conversely, a weight $w = w_1 w_2^{1-q}$ belongs to A_q if $w_1, w_2 \in A_1$.

Proof. (1) See [10, Theorem II 3.4].

The claims (2), (3) and the second part of (4) are simple consequences of Hölder's inequality. The first part of (4), the factorization of A_q-weights, can be found e.g. in [25, V 5.3, Proposition 9]. □

For a rapidly decreasing function $u \in \mathcal{S}(\mathbb{R}^n)$ let

$$\mathcal{F}u(\xi) = \widehat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx, \quad \xi \in \mathbb{R}^n,$$

be the Fourier transform of u. Its inverse will be denoted by $\mathcal{F}^{-1}$. Moreover, we shall use the centered Hardy-Littlewood maximal operator

$$\mathcal{M}u(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |u(y)|\, dy, \quad x \in \mathbb{R}^n,$$

for $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where again Q runs through the set of all cubes centered at x.

Theorem 2.2. *Let $1 < q < \infty$ and $w \in A_q$. Then the following statements hold true:*

(i) The operator $\mathcal{M}$, defined e.g. on $\mathcal{S}(\mathbb{R}^n)$, can be extended to a bounded operator from L^q_w to L^q_w.

(ii) Let $m \in C^n(\mathbb{R}^n \setminus \{0\})$ satisfy the pointwise Hörmander-Mikhlin multiplier condition

$$|\xi|^{|\alpha|}\, |D^\alpha m(\xi)| \le c_\alpha \quad \text{for all} \quad \xi \in \mathbb{R}^n \setminus \{0\}$$

and all multi-indices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le n$. Then the multiplier operator $u \mapsto \mathcal{F}^{-1}(m\widehat{u})$, $u \in \mathcal{S}(\mathbb{R}^n)$, can be extended to a bounded linear operator from L^q_w to L^q_w.

Proof. (i) See [10, Theorem IV 2.8].

(ii) See [10, Theorem IV 3.9] or [16, Theorem 4]. Note that the pointwise condition on m implies the integral condition in [10], [16]. □

Due to the geometry of the problem it is reasonable to introduce (polar or) cylindrical coordinates $(r, x_3, \theta) \in (0, \infty) \times \mathbb{R} \times [0, 2\pi)$. Then the term $(\omega \wedge x) \cdot \nabla u = -x_2 \partial_1 u + x_1 \partial_2 u$ may be rewritten in the form

$$(w \wedge x) \cdot \nabla u = \partial_\theta u,$$

using the angular derivative ∂_θ applied to $u(r, x_3, \theta)$.

Now we will solve (1.7) explicitly using Fourier transforms and multiplier operators. Working first of all formally or in the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions we apply the Fourier transform $\mathcal{F}$, denoted by $\widehat{\ }$, to (1.7). With the Fourier variable $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ and $s = |\xi|$ we get from (1.7)

$$\nu s^2 \widehat{u} - \partial_\phi \widehat{u} + \omega \times \widehat{u} + i\xi \widehat{p} = \widehat{f}, \quad i\xi \cdot \widehat{u} = 0. \tag{2.1}$$

Here $(\omega \times \xi) \cdot \nabla_\xi = -\xi_2 \partial/\partial\xi_1 + \xi_1 \partial/\partial\xi_2 = \partial_\phi$ is the angular derivative in Fourier space when using (polar or) cylindrical coordinates for $\xi \in \mathbb{R}^n$. Since $i\xi \cdot \widehat{u} = 0$ implies $i\xi \cdot \big(\partial_\phi \widehat{u} - \omega \times \widehat{u}\big) = 0$, the unknown pressure p is given by $-|\xi|^2 \widehat{p} = i\xi \cdot \widehat{f}$, i.e.,

$$\widehat{\nabla p}(\xi) = i\xi \cdot \widehat{p} = \frac{(\xi \cdot \widehat{f})\widehat{f}}{|\xi|^2}\,.$$

Then the Hörmander-Mikhlin multiplier theorem in weighted L^q-spaces (Theorem 2.2 (ii)) yields for every weight $w \in A_q(\mathbb{R}^n)$ the estimate

$$\|\nabla p\|_{q,w} \le c \|f\|_{q,w}\,, \tag{2.2}$$

where $c = c(q, w) > 0$; in particular $\nabla p \in L^q_w$. Hence u may be considered as a (solenoidal) solution of the reduced problem

$$-\nu \Delta u - \partial_\theta u + \omega \wedge u = F := f - \nabla p \quad \text{in} \quad \mathbb{R}^n, \tag{2.3}$$

or – in Fourier space – as a solution of the second order ordinary differential equation

$$-\partial_\phi \widehat{u} + w \wedge \widehat{u} + \nu s^2 \widehat{u} = \widehat{F}$$

with respect to ϕ. As deduced in [7] this equation has the unique solution

$$\widehat{u}(\phi) = \int_0^\infty e^{-\nu s^2 t} O(t)^T \widehat{F}(\phi + t)\, dt,$$

using the 2π-periodicity of $\widehat{u}$ with respect to ϕ. In this way we arrive at the solution

$$\widehat{u}(\xi) = \int_0^\infty e^{-\nu s^2 t} O(t)^T \widehat{F}(O(t)\xi)\, dt \tag{2.4}$$

of (1.7). Note that $F = f - \nabla p$ is solenoidal so that the identity $i\xi \cdot \widehat{F} = 0$ implies that also u is solenoidal. Since $e^{-\nu|\xi|^2 t}$ multiplied by $(2\pi)^{-n/2}$ is the Fourier transform of the heat kernel

$$E_t(x) = \frac{1}{(4\pi\nu t)^{n/2}} e^{-\frac{|x|^2}{4\nu t}}$$

we get the formal solution

$$\begin{aligned} u(x) &= \int_0^\infty O(t)^T E_t * F(O(t)\cdot)(x)\, dt \\ &= \int_0^\infty O(t)^T (E_t * F)(O(t)x)\, dt\,. \end{aligned} \tag{2.5}$$

Remarks 2.3. (i) In the case $n = 3$ and $F \in \mathcal{S}(\mathbb{R}^3)^3$, the integrals in (2.4) and (2.5) converge absolutely and define a solution $u \in \mathcal{S}'(\mathbb{R}^3)^3$ of (2.3).

(ii) In the case $n = 2$ both integrals (2.4), (2.5) do not converge in $\mathcal{S}'(\mathbb{R}^2)^2$. Therefore, we modify the solution formula (2.5) by means of the convergent integrals

$$\begin{aligned} \langle u, \phi\rangle = \langle \widehat{u}, \check{\phi}\rangle = \int_{|\xi|\geq 1} \int_0^\infty e^{-\nu s^2 t} O(t)^T \widehat{F}(O(t)\xi)^T \xi) \cdot \check{\phi}(\xi)\, dt\, d\xi \\ + \int_{|\xi|<1} \int_0^\infty e^{-\nu s^2 t} O(t)^T \widehat{F}(O(t)\xi)^T \cdot (\check{\phi}(\xi) - \check{\phi}(0))\, dt\, d\xi \end{aligned}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^2)^2$, here $\check{}$ denotes the inverse Fourier transform $\mathcal{F}^{-1}$ on $\mathbb{R}^2$.

(iii) In the case $n = 3$ we arrive at the identity

$$u(x) = \int_{\mathbb{R}^3} \Gamma(x, y) f(y)\, dy$$

with the fundamental solution

$$\Gamma(x, y) = \int_0^\infty O(t)^T E_t(O(t)x - y)\, dt.$$

Further, $-\Delta u(x)$ can be expressed as $\int_{\mathbb{R}^3} K(x, y) F(y)\, dy$, where

$$\begin{aligned} &K(x, y) = -\Delta_x \Gamma(x, y) \\ &= \int_0^\infty O(t)^T \frac{1}{(4\pi\nu t)^{3/2}} \Big(\frac{3}{2\nu t} - \frac{|O(t)x - y|^2}{(2\nu t)^2}\Big) \exp\Big(\frac{-|O(t)x - y|^2}{4\nu t}\Big)\, dt \\ &= K_1(x, y) + K_2(x, y). \end{aligned} \tag{2.6}$$

Proposition 2.1 in [7] indicates that Δu is not defined by a classical Calderón-Zygmund integral operator, since both kernels K_1 and K_2 fail to be bounded by $\frac{C}{|x-y|}$; actually, for certain $x, y \in \mathbb{R}^3$ with $|x|$, $|y| \to \infty$

$$|K_1(x,y)|,\ |K_2(x,y)| \geq \frac{\alpha}{|x-y|}\,.$$

The main ingredients of the proof of Theorem 1.2 are a weighted version of the Littlewood-Paley theory and a decomposition of the integral operator

$$Tf(x) = \int_0^\infty (-\Delta)O(t)^T(E_t * f)(O(t)x)\,dt = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy \qquad (2.7)$$

in the Fourier space. Note that by virtue of the formula $\widehat{\partial_j\partial_k u}(\xi) = -\xi_j\xi_k\widehat{u} = \frac{\xi_j\xi_k}{|\xi|^2}\big(\widehat{-\Delta u}(\xi)\big)$, $1 \leq j,\, k \leq n$, and Theorem 1.2 it suffices to find an estimate of $\|\Delta u\|_{q,w}$ in order to estimate arbitrary second order derivatives $\partial_j\partial_k u$ of u. Since

$$\mathcal{F}\big((-\Delta)O(t)^T(E_t * f)(O(t)\cdot)\big)(\xi) = O(t)^T(t|\xi|^2)\,e^{-\nu|\xi|^2 t}\widehat{f}(O(t)\xi)\frac{1}{t}\,,$$

we define $\psi \in \mathcal{S}(\mathbb{R}^n)$ by its Fourier transform

$$\widehat{\psi}(\xi) = (2\pi)^{-n/2}|\xi|^2 e^{-\nu|\xi|^2} = (\widehat{-\Delta E_1})(\xi)$$

and for all $t > 0$,

$$\psi_t(x) = t^{-n/2}\psi\left(\frac{x}{\sqrt{t}}\right), \quad \widehat{\psi_t}(\xi) = \widehat{\psi}(\sqrt{t}\xi) = (2\pi)^{-n/2}t|\xi|^2 e^{\nu t|\xi|^2}. \qquad (2.8)$$

Thus the kernel $K(x,y)$ in (2.6), (2.7) may be rewritten in the form

$$K(x,y) = \int_0^\infty O(t)^T\psi_t(O(t)x - y)\,\frac{dt}{t}\,.$$

To decompose $\widehat{\psi_t}$ choose $\widetilde{\chi} \in C_0^\infty(\frac{1}{2}, 2)$ satisfying $0 \leq \widetilde{\chi} \leq 1$ and

$$\sum_{j=-\infty}^{\infty} \widetilde{\chi}(2^{-j}s) = 1 \quad \text{for all} \quad s > 0.$$

Then define χ_j for $\xi \in \mathbb{R}^n$ and $j \in \mathbb{Z}$ by its Fourier transform

$$\widehat{\chi_j}(\xi) = \widetilde{\chi}(2^{-j}|\xi|), \quad \xi \in \mathbb{R}^n,$$

yielding $\sum_{j=-\infty}^{\infty} \widehat{\chi}_j = 1$ on $\mathbb{R}^n \setminus \{0\}$ and

$$\operatorname{supp} \widehat{\chi}_j \subset A(2^{j-1}, 2^{j+1}) := \{\xi \in \mathbb{R}^n : \, 2^{j-1} < |\xi| < 2^{j+1}\}.$$

Using χ_j we define for $j \in \mathbb{Z}$

$$\psi^j = \frac{1}{(2\pi)^{n/2}} \chi_j * \psi \quad \big(\widehat{\psi^j} = \widehat{\chi}_j \cdot \widehat{\psi}\,\big).$$

Obviously, $\sum_{j=-\infty}^{\infty} \psi^j = \psi$ on $\mathbb{R}^n$. Moreover, define the kernel

$$K_j(x,y) = \int_0^\infty O(t)^T \psi_t^j(O(t)x - y)\, \frac{dt}{t}\,, \quad j \in \mathbb{Z},$$

where ψ_t^j, $t > 0$, is defined by ψ^j applying the scaling transform as in (2.8). Then the kernels K_j define the linear operators

$$T_j f(x) = \int_{\mathbb{R}^n} K_j(x,y) f(y)\, dy = \int_0^\infty O(t)^T (\psi_t^j * f)(O(t)x)\, \frac{dt}{t}\,.$$

Since formally $T = \sum_{j=-\infty}^{\infty} T_j$, we wish to prove that this infinite series converges even in the operator norm on L^q_w.

For later use we recall the following lemma, see [7].

Lemma 2.4. *The functions ψ^j, ψ_t^j, $j \in \mathbb{Z}$, $t > 0$, have the following properties:*

(i) $\operatorname{supp} \widehat{\psi_t^j} \subset A\left(\frac{2^{j-1}}{\sqrt{t}}, \frac{2^{j+1}}{\sqrt{t}}\right)$.

(ii) For $m > \frac{n}{2}$ let $h(x) = (1 + |x|^2)^{-m}$ and $h_t(x) = t^{-n/2} h\left(\frac{x}{\sqrt{t}}\right)$, $t > 0$. Then there exists a constant $c > 0$ independent of $j \in \mathbb{Z}$ such that

$$\begin{aligned} |\psi^j(x)| &\leq c 2^{-2|j|}\, h_{2^{-2j}}(x), \quad x \in \mathbb{R}^n, \\ \|\psi^j\|_1 &\leq c 2^{-2|j|}\,. \end{aligned}$$

To obtain a weighted Littlewood-Paley decomposition of L^q_w let us choose $\widetilde{\phi} \in C_0^\infty(\frac{1}{2}, 2)$ such that $0 \leq \widetilde{\phi} \leq 1$ and

$$\int_0^\infty \widetilde{\phi}(s)^2\, \frac{ds}{s} = \frac{1}{2}\,.$$

Then define $\phi \in \mathcal{S}(\mathbb{R}^n)$ by its Fourier transform $\widehat{\phi}(\xi) = \widetilde{\phi}(|\xi|)$ yielding for every $s > 0$

$$\widehat{\phi}_s(\xi) = \widetilde{\phi}(\sqrt{s}|\xi|), \quad \operatorname{supp} \widehat{\phi}_s \subset A\left(\frac{1}{2\sqrt{2}}, \frac{2}{\sqrt{2}}\right) \tag{2.9}$$

and the normalization $\int_0^\infty \widehat{\phi}_s(\xi)^2 \frac{ds}{s} = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

Theorem 2.5. *Let $1 < q < \infty$ and $w \in A_q(\mathbb{R}^n)$. Then there are constants $c_1, c_2 > 0$ depending on q, w and ϕ such that for all $f \in L^q_w$,*

$$c_1\|f\|_{q,w} \le \left\| \Big(\int_0^\infty |\phi_s * f(\cdot)|^2 \frac{ds}{s} \Big)^{1/2} \right\|_{q,w} \le c_2\|f\|_{q,w},$$

where $\phi_s \in \mathcal{S}(\mathbb{R}^n)$ is defined by (2.9).

Proof. See [21], Proposition 1.9, Theorem 1.10, and also [16], [24]. □

3 Proofs

As a preliminary version of Theorem 1.2 we prove the following proposition. The extension to more general weights based on complex interpolation of L^q_w-spaces will be postponed to the end of this section.

Proposition 3.1. *Assume that the weight w satisfies*

$$\begin{aligned} &w \in A_{q/2} && \text{if } q > 2, \\ &w \in A_1 \text{ or } w^{-1} \in A_1 && \text{if } q = 2, \\ &w^{2/(2-q)} \in A_{q/(2-q)} && \text{if } 1 < q < 2. \end{aligned} \tag{3.1}$$

Then the linear operator T defined by (2.7) satisfies the estimate

$$\|Tf\|_{q,w} \le c\|f\|_{q,w}, \qquad f \in L^q_w,$$

with a constant $c = c(q, w) > 0$ independent of f.

Proof. First we consider the case $q > 2$ and $w \in A_{q/2} \subset A_q$ and define the sublinear operator $\mathcal{M}^j$, a modified maximal operator, by

$$\mathcal{M}^j g(x) = \sup_{s>0} \int_{A_s} \big(|\psi_t^j| * |g|\big)(O(t)^T x)\, \frac{dt}{t},$$

where $A_s = [\frac{s}{16}, 16s]$. Then we will prove the auxiliary estimate

$$\|T_j f\|_{q,w} \le c\|\psi^j\|_1^{1/2} \|\mathcal{M}^j\|_{L_v^{(q/2)'}}^{1/2} \|f\|_{q,w}\,, \quad j \in \mathbb{Z}, \tag{3.2}$$

where v denotes the weight

$$v = w^{-\left(\frac{q}{2}\right)'/\left(\frac{q}{2}\right)} = w^{-\frac{2}{q-2}} \in A_{(q/2)'} = A_{q/(q-2)}. \tag{3.3}$$

To prove (3.2) we use the Littlewood-Paley decomposition of L^q_w, see Theorem 2.5, that is,

$$c_1^2\|f\|_{q,w}^2 \le \left\| \left(\int_0^\infty |\phi_s * f(\cdot)|^2 \, \frac{ds}{s} \right) \right\|_{q/2,w} \le c_2^2 \|f\|_{q,w}^2. \tag{3.4}$$

By a duality argument there exists a function $0 \le g \in L_v^{(q/2)'} = \left(L_w^{(q/2)}\right)^*$ with $\|g\|_{(q/2)',v} = 1$ such that

$$\left\| \int_0^\infty |\phi_s * T_j f(\cdot)|^2 \, \frac{ds}{s} \right\|_{q/2,w} = \int_0^\infty \int_{\mathbb{R}^n} |\phi_s * T_j f(x)|^2 g(x) \, dx \, \frac{ds}{s}\,. \tag{3.5}$$

To estimate the right-hand side of (3.5) note that

$$\phi_s * T_j f(x) = \int_0^\infty O(t)^T (\phi_s * \psi_t^j * f)(O(t)x) \, \frac{dt}{t}\,,$$

where $\phi_s * \psi_t^j = 0$ unless $t \in A(s,j) := [2^{2j-4}s, 2^{2j+4}s]$. Since $\int_{t \in A(s,j)} \frac{dt}{t} = \log 2^8$ for every $j \in \mathbb{Z}$, $s > 0$, we get by the inequality of Cauchy-Schwarz and the associativity of convolutions that

$$\begin{aligned} |\phi_s * T_j f(x)|^2 &\le c \int_{A(s,j)} \left| \left(\psi_t^j * (\phi_s * f)\right)(O(t)x)\right|^2 \frac{dt}{t} \\ &\le c\|\psi^j\|_1 \int_{A(s,j)} \left(|\psi_t^j| * |\phi_s * f|^2\right)(O(t)x) \, \frac{dt}{t}\,; \end{aligned}$$

here we used the estimate $|(\psi_t^j * (\phi_s * f))(y)|^2 \le \|\psi_t^j\|_1 \left(|\psi_t^j| * |\phi_s * f|^2\right)(y)$ and the identity $\|\psi_t^j\|_1 = \|\psi^j\|_1$, see (2.8). Thus

$$\begin{aligned} \|T_j f\|_{q,w}^2 &\le c\|\psi^j\|_1 \int_0^\infty \int_{A(s,j)} \int_{\mathbb{R}^n} (|\psi_t^j| * |\phi_s * f|^2)(x) g(O(t)^T x) \, dx \, \frac{dt}{t} \frac{ds}{s} \\ &\le c\|\psi^j\|_1 \int_{\mathbb{R}^n} \int_0^\infty |\phi_s * f|^2(x) \int_{A(s,j)} \left(|\psi_t^j| * g\right)(O(t)^T x) \, \frac{dt}{t} \, \frac{ds}{s} \, dx, \end{aligned} \tag{3.6}$$

since ψ_t^j is radially symmetric. By definition of $\mathcal{M}^j$ the innermost integral is bounded by $\mathcal{M}^j g(x)$ uniformly in $s > 0$. Hence we may proceed in (3.6) using Hölder's inequality as follows:

$$\begin{aligned}\|T_j f\|_{q,w}^2 &\le c\|\psi^j\|_1 \int_{\mathbb{R}^n} \left(\int_0^\infty |\phi_s * f|^2(x) \, \frac{ds}{s} \right) \mathcal{M}^j g(x) \, dx \\ &\le c\|\psi^j\|_1 \left(\int_{\mathbb{R}^n} \Big(\int_0^\infty |\phi_s * f|^2(x) \, \frac{ds}{s} \Big)^{q/2} w(x) \, dx \right)^{2/q} \|\mathcal{M}^j g\|_{(q/2)',v}.\end{aligned}$$

Now (3.4) and the normalization $\|g\|_{(q/2)',v} = 1$ complete the proof of (3.2).

In the next step we estimate $\|\mathcal{M}^j g\|_{(q/2)',v}$. Since $\frac{q}{2} \in (1,\infty)$ is arbitrary, we have to consider $\|\mathcal{M}^j\|_{L^p_\rho}$ for arbitrary $p \in (1,\infty)$ and θ-independent weights $\rho \in A_p$. For this reason we define the "angular" maximal operator

$$\mathcal{M}_\theta g(x) = \sup_{s>0} \int_{A_s} |g(O(t)^T x)| \, \frac{dt}{t},$$

where $A_s = \left[\frac{s}{16}, 16s\right]$. Then we claim that

$$\mathcal{M}^j g(x) \le c 2^{-2|j|} \mathcal{M}(\mathcal{M}_\theta g)(x) \qquad \text{for a.a. } x \in \mathbb{R}^n, \tag{3.7}$$

$$\|\mathcal{M}^j g\|_{p,\rho} \le c 2^{-2|j|} \|g\|_{p,\rho} \qquad \text{for } 1 < p < \infty, \ \rho \in A_p. \tag{3.8}$$

To prove (3.7) invoke the pointwise estimate $|\psi_t^j(x)| \le c 2^{-2|j|} h_{t2^{-2j}}(x)$, see Lemma 2.4 (ii) and (2.8). We get

$$\mathcal{M}^j g(x) \le c 2^{-2|j|} \sup_{s>0} \int_{A_s} (h_{t2^{-2j}} * |g|)(O(t)^T x) \, \frac{dt}{t}.$$

Moreover, there exists a constant $c > 0$ independent of $s > 0$, $j \in \mathbb{Z}$, such that $h_{t2^{-2j}} \le c h_{s2^{-2j}}$ for all $t \in A_s$. Consequently

$$\begin{aligned}\mathcal{M}^j g(x) &\le c 2^{-2|j|} \sup_{s>0} h_{s2^{-2j}} * \int_{A_s} |g|(O(t)^T x) \, \frac{dt}{t} \\ &\le c 2^{-2|j|} \sup_{t>0} h_t * \mathcal{M}_\theta g(x).\end{aligned}$$

Since h is nonnegative, radially decreasing, and $\|h_t\|_1 = \|h\|_1 = c_0 > 0$ for all $t > 0$, a well-known convolution estimate, see [25, II §2.1], yields the pointwise estimate (3.7). Note that up to now we have not used any special properties of the weight $\rho \in A_p$.

Concerning (3.8) note that for arbitrary but fixed radial distance $r = (x_1^2 + x_2^2)^{1/2}$, and $x_3 \in \mathbb{R}$ for $n = 3$,

$$\begin{aligned}\mathcal{M}_\theta g(r, x_3, \theta) &\le \sup_{s>0} \frac{16}{s} \int_{-16s}^{16s} |g|(r, x_3, \theta)|\, dt \\ &\le c\mathcal{M}_1 g(r, x_3, \theta)\,,\end{aligned}$$

where $\mathcal{M}_1$ is the classical maximal operator on $L^1_{\mathrm{loc}}(\mathbb{R})$. Since $\rho(x) = \rho(r, x_3, \theta)$ is independent of θ, Fubini's theorem and the boundedness of $\mathcal{M}_1$ on L^p-spaces of 2π-periodic functions imply that

$$\begin{aligned}\|\mathcal{M}_\theta g\|_{p,\rho}^p &\le c \int_{\mathbb{R}} \int_0^\infty \|\mathcal{M}_1 g(r, x_3, \theta)\|^p_{L_{\rho(r,x_3,\cdot)}(0,2\pi)}\, r^2\, dr\, dx_3 \\ &\le c \int_{\mathbb{R}} \int_0^\infty \|g(r, x_3, \theta)\|^p_{L_{\rho(r,x_3,\cdot)}(0,2\pi)}\, r^2\, dr\, dx_3 \\ &= c\|g\|^p_{p,\rho}\,.\end{aligned}$$

Finally, since $\mathcal{M}$ is bounded on L^p_ρ, see Theorem 2.2 (i), (3.7) yields (3.8).

Summarizing (3.2), Lemma 2.4 (ii) and (3.8) we get that

$$\|T_j f\|_{q,w} \le c 2^{-2|j|} \|f\|_{q,w} \quad \text{for} \quad q > 2,\ w \in A_{q/2}.$$

Hence $T = \sum_{j=-\infty}^{\infty} T_j$ converges on L^q_w and defines a bounded linear operator thus proving Proposition 3.1 for $q > 2$.

Now let $q = 2$ and $w \in A_1$. In this case the Littlewood-Paley decomposition (3.4) in L^2_w implies that

$$\|T_j f\|^2_{2,w} \le c \int_0^\infty \int_{\mathbb{R}^n} |\phi_s * T_j f|^2(x) g(x)\, dx\, \frac{ds}{s}\,,$$

where

$$g \in L^\infty_v,\ v = \frac{1}{w} \quad \text{and} \quad \|g\|_{\infty,v} = \operatorname{ess\,sup}_{\mathbb{R}^n} |g\, v| = 1,$$

cf. (3.3)–(3.5). By the same reasoning as before we arrive at (3.2), that is,

$$\|T_j f\|_{2,w} \le c 2^{-|j|} \|\mathcal{M}^j g\|^{1/2}_{\infty,v} \|f\|_{2,w}. \tag{3.9}$$

Since $|g(x)| \le \frac{1}{v(x)} = w(x)$ for a.a. $x \in \mathbb{R}^n$ and ψ^j_t is radially symmetric, the operator $\mathcal{M}^j$ satisfies the pointwise estimate

$$\begin{aligned}\mathcal{M}^j g(x) \le \mathcal{M}^j w(x) &= \sup_{s>0} \int_{A_s} \big(|\psi_t^j| * w(O(t)^T\cdot)\big)(x)\,\frac{dt}{t}\\ &\le c2^{-2|j|} \sup_{t>0} h_t * \mathcal{M}_\theta w(x)\\ &\le c2^{-2|j|}\mathcal{M}\big(\mathcal{M}_1 w\big)(x).\end{aligned}$$

Exploiting as before that w is θ-independent, we know that $\mathcal{M}_1 w(r,x_3) = w(r,x_3)$. Moreover, the assumption $w \in A_1$ implies that $\mathcal{M}w \le cw$. Hence $\mathcal{M}^j g(x) \le c2^{-2|j|}w(x)$ and consequently

$$\|\mathcal{M}^j g(x)\|_{\infty,v} \le c2^{-2|j|}. \tag{3.10}$$

Now (3.9) and (3.10) lead to the operator bound $\|T_j\|_{L^2_w} \le c2^{-2|j|}$ and the boundedness of T on L^2_w.

The remaining estimates are proved by duality arguments. Obviously the dual operator to T is defined by

$$T^*f(x) = \int_0^\infty (-\Delta)O(t)E_t * f(O(t)^T x)\,dt = \int_{\mathbb{R}^n} K^*(x,y)f(y)\,dy,$$

where the kernel K^* has the same structure as K. Hence T^* is bounded on L^q_w for $q \ge 2$ and $w \in A_{q/2}$. Now let $1 < q < 2$ and $w^{2/(2-q)} \in A_{q/(2-q)} = A_{(q'/2)'}$ or equivalently $w' = w^{-q'/q} \in A_{(q'/2)}$. Then

$$|\langle Tf, g\rangle| = |\langle f, T^*g\rangle| \le \|f\|_{q,w}\|T^*g\|_{q',w'} \le c\|f\|_{q,w}\|Tg\|_{q',w'}$$

since $q' > 2$ and $w' = w^{-q'/q} \in A_{(q'/2)}$.

Finally let $q = 2$ and $\frac{1}{w} \in A_1$. As before, since $\frac{1}{w} \in A_1$,

$$|\langle Tf, g\rangle| \le \|f\|_{2,w}\|T^*g\|_{2,1/w} \le c\|f\|_{2,w}\|g\|_{2,1/w}\,.$$

Now Proposition 3.1 is completely proved. □

To extend the results of Proposition 3.1 to further weight functions as described in Theorem 1.2 we recall a well-known theorem on complex interpolation of L^q_w-spaces, see [1]. Note that we use a different definition of the weighted space L^q_w than in [1].

Lemma 3.2. *Let* $1 \le p_1, p_2 < \infty$*, let* $0 < w_1, w_2$ *be weight functions,* $\delta \in (0,1)$*, and*

$$\frac{1}{p} = \frac{1-\delta}{p_1} + \frac{\delta}{p_2}, \qquad w^{\frac{1}{p}} = w_1^{\frac{1-\delta}{p_1}} \cdot w_2^{\frac{\delta}{p_2}}.$$

Then

$$\left[L^{p_1}_{w_1}, L^{p_2}_{w_2}\right]_\delta = L^p_w$$

in the sense of complex interpolation.

Proof of Theorem 1.2 (i). Let $q \in (1,\infty)$ and $w \in A_q$ such that the L^q_w-estimate of ∇p holds, see (2.2). Hence it suffices to consider u defined by (2.3)–(2.5). Choose arbitrary q_1, q_2 with

$$1 < q_1 < q < q_2 < \infty \quad \text{and} \quad q_1 \le 2 \le q_2 \tag{3.11}$$

and $\delta \in (0,1)$ satisfying

$$\frac{1}{q} = \frac{1-\delta}{q_1} + \frac{\delta}{q_2} \tag{3.12}$$

as well as weights w_1, w_2 such that

$$w_1^{2/(2-q_1)} \in A_{q_1/(2-q_1)} \quad \text{and} \quad w_2 \in A_{q_2/2}\,,$$

cf. (3.1) in Proposition 3.1. By Lemma 2.1 there exist $u_1, v_1, u_2, v_2 \in A_1$ with

$$w_1^{2/(2-q_1)} = \frac{u_1}{v_1^{\frac{2(q_1-1)}{2-q_1}}} \quad \text{and} \quad w_2 = \frac{u_2}{v_2^{\frac{q_2-2}{2}}}.$$

Since the linear operator T is bounded on both $L^{q_1}_{w_1}$ and $L^{q_2}_{w_2}$, Lemma 3.2 shows that it is bounded on $L^q_{\widetilde{w}}$ as well, where

$$\widetilde{w} = w_1^{\frac{q(1-\delta)}{q_1}} \cdot w_2^{\frac{q\delta}{q_2}} = \frac{u_1^{q(1-\delta)\frac{2-q_1}{2q_1}}}{v_1^{q(1-\delta)\frac{q_1-1}{q_1}}} \cdot \frac{u_2^{\frac{q\delta}{q_2}}}{v_2^{q\delta\frac{q_2-2}{2q_2}}}.$$

Note that the sum of the exponents of the numerators equals $1 - \frac{q(1-\delta)}{2} = \frac{2-q(1-\delta)}{2}$. Therefore, taking the $\frac{2-q(1-\delta)}{2}$th root of the previous identity, and choosing $u_1 = u_2$ and $v_1 = v_2$, we arrive with an elementary calculation at

$$\widetilde{w}^{\,\frac{2}{2-q(1-\delta)}} = \frac{u_1}{v_1^{\frac{q}{2-q(1-\delta)}-1}}$$

which by Lemma 2.1 is a weight in $A_{q/(2-q(1-\delta))}$. Since $u_1, v_1 \in A_1$ are arbitrary, we proved the boundedness of T on L^q_w for arbitrary $w = \widetilde{w}$ if

$$w^r \in A_{rq/2}, \quad r = \frac{2}{2-q(1-\delta)}.$$

Now we have to find all admissible r subject to the restrictions given by (3.11), (3.12). For this reason consider the easier term

$$\tau = 2\left(1 - \frac{1}{r}\right) = q(1-\delta) = q\,\frac{\frac{1}{q} - \frac{1}{q_2}}{\frac{1}{q_1} - \frac{1}{q_2}}.$$

First Case $1 < q < 2$, in which $1 < q_1 < q$ and $q_2 \ge 2$. Due to monotonicity properties of τ as a function of $\frac{1}{q_1}$ and of $\frac{1}{q_2}$ it suffices to check τ at the corners of the rectangle $(\frac{1}{q}, 1) \times (0, \frac{1}{2}]$. The corresponding function values are $q, 1$ and $2-q$. Hence the range of τ equals the interval $(2-q, q)$ yielding for $r = \frac{2}{2-\tau}$ the condition

$$\frac{2}{q} < r < \frac{2}{2-q}.$$

Note that the limiting value $r = \frac{2}{2-q}$ is allowed due to Proposition 3.1. Finally the condition $w^r \in A_{rq/2}$, $\frac{2}{q} < r \le \frac{2}{2-q}$, easily implies that $w \in A_q$: By Lemma 2.1 there exist $v_1, v_2 \in A_1$ such that

$$w = \frac{v_1^{\frac{1}{r}}}{v_2^{\frac{q}{2}-\frac{1}{r}}}, \tag{3.13}$$

where $v_1^{\frac{1}{r}} \in A_1$ and $\frac{q}{2} - \frac{1}{r} \le q-1$ yielding $v_2^{(\frac{q}{2}-\frac{1}{r})/(q-1)} \in A_1$.

Second Case $q > 2$, in which $1 < q_1 \le 2$ and $q_2 > q$. In this case the values of τ at the corners of the rectangle $[\frac{1}{2}, 1) \times (0, \frac{1}{q})$ in the $\left(\frac{1}{q_1}, \frac{1}{q_2}\right)$-plane are $0, 1$ and 2. Hence

$$1 < r < \infty,$$

and we observe that $r = 1$ is admissible due to Proposition 3.1. Finally note that the condition $w^r \in A_{rq/2}$ implies also $w \in A_q$: There exist $v_1, v_2 \in A_1$ such that w satisfies (3.13), where again $\frac{q}{2} - \frac{1}{r} + 1 \le q$ for all $r \in (1, \infty)$.

Third Case $q = 2$. In this case it suffices to interpolate between $L^2_{w_1}$ and $L^2_{w_2}$, where $w_1 \in A_1$ and $\frac{1}{w_2} \in A_1$, see Proposition 3.1. Then T is bounded on L^2_w for all

$$w = \frac{w_1^{1-\delta}}{w_2^{\delta}},\ 0 < \delta < 1.$$

Then $w^{1/(1-\delta)} = w_1/w_2^{\delta/(1-\delta)}$, or with $r = \frac{1}{1-\delta} \in (1, \infty)$,

$$w^r = \frac{w_1}{w_2^{r-1}} \in A_r = A_{rq/2}.$$

(ii) Note that $L^{q_i}_{w_i}(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$, $i = 1, 2$; indeed, $w_i \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\int_{|x|\geq 1} w_i(x)|x|^{-nq_i}\, dx < \infty$, see [10, IV.3 (3.2)]. Since the equation (1.7) is linear, it suffices to consider $f = 0$ and a solution $u \in S'(\mathbb{R}^n)^n$ of (1.7). In the proof of [7], Theorem 1.1 (2), (3), it was shown that under these assumptions u is a polynomial and that $u(x) = \alpha\omega + \beta\omega \wedge x + \gamma(x_1, x_2, -2x_3)^T$, $\alpha, \beta, \gamma \in \mathbb{R}$ ($u(x) = \beta(-x_2, x_1)$ if $n = 2$). □

Proof of Corollary 1.3. (i) Let $w(x) = |x|^\alpha$ or $(1 + |x|)^\alpha$. As is well-known, see e.g. [25, p. 218], $w \in A_q$ iff $-n < \alpha < n(q - 1)$, $1 < q < \infty$, and $w \in A_1$ iff $-n < \alpha \leq 0$. If $q > 2$, then the condition $w^r \in A_{rq/2}$ in (1.9) implies that $-n < \alpha r < n\big(\frac{rq}{2} - 1\big)$, $r \in [1, \infty)$, or equivalently that $-n < \alpha < nq/2$ by choosing $r = 1$ or r as large as possible. If $1 < q < 2$, then $w^r \in A_{rq/2}$ in (1.11) is equivalent to $-n < \alpha r < n(\frac{rq}{2} - 1)$ for $r \in (\frac{2}{q}, \frac{2}{2-q}]$ yielding $-\frac{nq}{2} < \alpha < n(q - 1)$ by choosing $r = \frac{2}{2-q}$ or $r \to \frac{2}{q}$. Finally, for $q = 2$, (1.10) immediately admits all weights w satisfying $-n < \alpha < n$.

(ii) By Lemma 3.3 below $w(x) = (1 + |x|)^\alpha(1 + |(x_1, x_2)|)^\beta \in A_q$, $1 < q < \infty$, if $-2 < \beta < 2(q - 1)$ and $-3 < \alpha + \beta < 3(q - 1)$. If $q > 2$, the condition $w^r \in A_{rq/2}$, $r \in [1, \infty)$, implies that $-2 < \beta < q$ and $-3 < \alpha + \beta < \frac{3q}{2}$; for the proof choose $r = 1$ or $r \to \infty$. For $q < 2$ the condition $w^r \in A_{rq/2}$, $r \in (\frac{2}{q}, \frac{2}{2-q}]$, yields the restrictions $-q < \beta < 2(q - 1)$ and $\frac{-3q}{2} < \alpha + \beta < 3(q - 1)$ for α, β by choosing $r = \frac{2}{2-q}$ or $r \to \frac{2}{q}$. For $q = 2$ we obviously get the bounds $-2 < \beta < 2$ and $-3 < \alpha + \beta < 3$. □

Lemma 3.3. *Let w be the weight $w(x) = (1 + |x|)^\alpha(1 + r)^\beta$, where $r = |x'| = \sqrt{x_1^2 + x_2^2}$ denotes the distance of $x = (x_1, x_2, x_3)$ to the axis of revolution. Then the following statements hold true:*

(i) For all $\beta \in (-2, 0]$ and $-3 < \alpha + \beta \leq 0$ we have $w \in A_1$.

(ii) Given $1 < q < \infty$ the weight w lies in A_q if $-2 < \beta < 2(q - 1)$ and $-3 < \alpha + \beta < 3(q - 1)$.

Proof. (i) For $b \in (-1, 2]$ define the regular Borel measure

$$\mu(A) = \int_{A \cap \mathbb{R}} |y_3|^b \, dy$$

on $\mathbb{R}^3$, where $A \cap \mathbb{R}$ stands for $A \cap \{(0, 0, y_3) \in \mathbb{R}^3 : \, y_3 \in \mathbb{R}\}$; for a similar ansatz see [4]. We claim that for the maximal operator $\mathcal{M}$, see Lemma 2.1,

$$\mathcal{M}\mu(x) = \sup_{Q \ni x} \frac{\mu(Q)}{|Q|} \sim \frac{|x|^b}{r^2}. \tag{3.14}$$

To prove the equivalence in (3.14) we consider an arbitrary $x \in \mathbb{R}^n$ with $x_3 > 0$; let $R > 0$ denote the half of the side length of the cube Q centered at x in the definition of $\mathcal{M}\mu(x)$. For simplicity assume that Q is closed. Then we consider three cases:

First Case, $0 < R < \frac{x_3}{2}$. Then $y_3 \sim x_3$ for all $(0, 0, y_3) \in Q \cap \mathbb{R}$ and

$$\frac{\mu(Q)}{|Q|} = \frac{1}{8R^3} \int_{Q \cap \mathbb{R}} y_3^b \, dy_3 \sim \frac{x_3^b}{R^2} \le \frac{x_3^b}{r^2}$$

since $r = |x'| \le R$ such that $Q \cap \mathbb{R} \neq \emptyset$ is possible.

Second Case, $\frac{x_3}{2} \le R \le x_3$. Then

$$\frac{\mu(Q)}{|Q|} \sim \frac{1}{R^3} \int_{x_3 - R}^{x_3 + R} |y_3|^b \, dy_3 \sim R^{b-2} \sim x_3^{b-2}.$$

Third Case, $R > x_3$. Now

$$\frac{\mu(Q)}{|Q|} \sim \frac{1}{R^3} \int_{-R}^{R} |y_3|^b \, dy_3 \sim R^{b-2} \le x_3^{b-2}$$

since $b > -1$. This case also shows that $b \le 2$ is needed to get $\mathcal{M}\mu(x)$ finite for a.a. $x \in \mathbb{R}^3$.

Summarizing the previous three cases we now consider $x \in \mathbb{R}^3$ with either $r < x_3$ or $r \ge x_3$. If $r < x_3$, then $x_3 \sim |x|$ and due to the first case

$$\mathcal{M}\mu(x) \sim \frac{x_3^b}{r^2} \sim \frac{|x|^b}{r^2}.$$

Finally, if $r \geq x_3$, then the third case applies with $R = r \sim |x|$. Hence

$$\mathcal{M}\mu(x) \sim r^{b-2} \sim \frac{|x|^b}{r^2}.$$

Now (3.14) is proved.

By Lemma 2.1 $w(x) = (\frac{|x|^b}{r^2})^\gamma$, $0 \leq \gamma < 1$, $-1 < b \leq 2$, is an A_1-weight. In other words, $w(x) = |x|^\alpha r^\beta \in A_1$ for all $\beta \in (-2, 0]$ and $\alpha \in (\frac{\beta}{2}, -\beta]$. This set of admissible (α, β) defines a half open triangle in the (α, β)-plane with vertices $(0,0), (-1,-2)$ and $(2,-2)$. Eventually, since also $|x|^\gamma \in A_1$ for $-3 < \gamma \leq 0$, Lemma 2.1 implies that $w \in A_1$ for all (α, β) in the open parallelogram

$$P : -2 < \beta < 0,\ -3 < \alpha + \beta < 0,$$

plus the line segments $-2 < \beta \leq 0,\ \alpha = -\beta$ and $\beta = 0,\ -3 < \alpha + \beta \leq 0$.

To prove the same result for nondegenerate weights, note that $\alpha = 0$ is allowed, i.e., $r^\beta \in A_1$ for all $\beta \in (-2, 0]$. Moreover, since the sum of two A_1-weights and also the minimum of two A_1-weights is an A_1-weight as well, we conclude that $w(x) = (1 + |x|)^\alpha r^\beta \in A_1$ for the same α, β as before. Note that the same result will hold when the axis of revolution is parallel to the third unit vector, but passes through $(x_0', 0)$, $|x_0'| = r \leq 1$. Obviously the corresponding A_1-constant is independent of x_0'. Hence for all cubes Q centered at $x \in \mathbb{R}^3$,

$$\begin{aligned}
&\frac{1}{|Q|} \int_Q \left(\int_{|x_0'| \leq 1} (1 + |y|)^\alpha |y' - x_0'|^\beta \, dx_0' \right) dy \\
&\qquad = \int_{|x_0'| \leq 1} \left(\frac{1}{|Q|} \int_Q (1 + |y|)^\alpha |y' - x_0'|^\beta \, dy \right) dx_0' \\
&\qquad \leq \int_{|x_0'| \leq 1} (1 + |x|)^\alpha \, |x' - x_0'|^\beta \, dx_0'.
\end{aligned}$$

Since $\beta > -2$, we conclude that

$$\int_{|x_0'| \leq 1} (1 + |x|)^\alpha \, |x' - x_0'|^\beta \, dx_0' \sim (1 + |x|)^\alpha (1 + |x'|)^\beta$$

is an A_1-weight for all $\beta \in (-2, 0],\ -3 < \alpha + \beta \leq 0$.

(ii) Consider $w_j(x) = (1 + |x|)^{\alpha_j} (1 + r)^{\beta_j} \in A_1$, $j = 1, 2$, where α_j, β_j run through all of the parallelogram P. By Lemma 2.1

$$\frac{w_1(x)}{w_2(x)^{q-1}} = (1 + |x|)^{\alpha_1 - (q-1)\alpha_2} (1 + r)^{\beta_1 - (q-1)\beta_2} \in A_q.$$

Now it can easily be seen that $w(x) = (1+|x|)^\alpha(1+r)^\beta \in A_q$ for all α, β satisfying $-2 < \beta < 2(q-1)$, $-3 < \alpha+\beta < 3(q-1)$. □

Proof of Corollary 1.4. As to the weight $w = \eta_\beta^\alpha$ we may proceed as in the proof of Theorem 1.2 based on Proposition 3.1. Hence the operators T_j satisfy the estimate (3.9) and the maximal operators $\mathcal{M}^j$ satisfy the pointwise estimate (3.7) just as before. However, we have to modify the proof of (3.8) by analyzing $\mathcal{M}_\theta$ more carefully.

For the moment let $\rho(x) := \rho_1(r)\rho_2(r,\theta)$ be an A_p-weight on $\mathbb{R}^2$, $1 < p < \infty$, such that $\rho_2(r,\theta)$ is 2π-periodic with respect to θ and satisfies a modified Muckenhoupt condition for 2π-periodic weights, i.e.,

$$\sup_{r>0} \sup_{0<|b-a|\le 2\pi} I_{a,b}(\rho_2) < \infty, \tag{3.15}$$

where

$$I_{a,b}(\rho_2) = \left(\frac{1}{b-a}\int_a^b \rho_2(r,\theta)\,d\theta\right)\cdot\left(\frac{1}{b-a}\int_a^b \rho_2(r,\theta)^{-1/(p-1)}\,d\theta\right)^{p-1},$$

see [18, Corollary 4], without the additional parameter r. Then consider for given $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ the restriction $g_r(\theta) = g(r,\theta)$. Obviously, $\mathcal{M}_\theta$ is estimated by the one-dimensional Hardy-Littlewood maximal operator $\mathcal{M}_1^{\mathrm{per}}$ for 2π-periodic functions, see [18], i.e., for all $\theta \in (0,2\pi)$

$$\mathcal{M}_\theta\, g(r,\theta) \le c(\mathcal{M}_1^{\mathrm{per}} g_r)(\theta), \quad r > 0.$$

Then by [18, Corollary 4], using polar coordinates and Fubini's theorem,

$$\begin{aligned}
||\mathcal{M}_\theta g||_{p,\rho}^p &\le c\int_0^\infty r\rho_1(r)||\mathcal{M}_1^{\mathrm{per}} g_r||^p_{L^p(0,2\pi,\rho_2(r,\cdot))}\,dr \\
&\le c\,B_p \int_0^\infty r\rho_1(r)||g_r||^p_{L^p(0,2\pi,\rho_2(r,\cdot))}\,dr \\
&= c\,||g||^p_{p,\rho}.
\end{aligned}$$

In the case $n = 3$ we proceed analogously for a weight $\rho(x) := \rho_1(r,x_3)\rho_2(r,x_3,\theta)$, inserting the additional variable x_3 in condition (3.15) and by performing an additional integration with respect to $x_3 \in \mathbb{R}$ in the last estimate.

Applying this general procedure we have to check for which $\alpha, \beta \in \mathbb{R}$ the weights $\rho = \sigma_\beta^\alpha$ and $\rho = \eta_\beta^\alpha$ satisfy the condition (3.15). First let $\rho(x) =$

$\sigma_\beta^\alpha(x) = r^{\alpha+\beta}(1-\cos\theta)^\beta$ yielding $\rho_2(r,\theta) = \rho_2(\theta) = (1-\cos\theta)^\beta$. Since $\rho_2(\theta) \sim \theta^{2\beta}$ near the origin and $\rho_2(\theta) \sim (2k\pi - \theta)^{2\beta}$ near $2k\pi$, $k \in \mathbb{Z}$, we may restrict ourselves to small $|a|, |b|$, $b > 0$, in (3.15). If $0 = a < b$, then

$$I_{0,b}(\rho_2) \sim \frac{1}{b}\int_0^b \theta^{2\beta}\, d\theta \left(\frac{1}{b}\int_0^b \theta^{-2\beta/(p-1)}\, d\theta\right)^{p-1} = c_\beta$$

provided $-\frac{1}{2} < \beta < \frac{1}{2}(p-1)$. For $0 < a < \frac{b}{2}$, the term $I_{a,b}(\rho_2)$ may be compared with $I_{0,b}(\rho_2)$, and for $0 < \frac{b}{2} \le a < b$ both integrands in $I_{a,b}(\rho_2)$ may be compared with the constants $b^{2\beta}$ and $b^{-2\beta/(p-1)}$, respectively. Finally, if $a < 0 < b$, $|a| \le b$, then $I_{a,b}(\rho_2) \sim I_{0,b}(\rho_2)$. Hence we proved

$$\rho = \sigma_\beta^\alpha : \sup_{\substack{0<|b-a|\le 2\pi \\ r>0}} I_{a,b}(\rho_2) < \infty \quad \text{for} \quad -\frac{1}{2} < \beta < \frac{1}{2}(p-1).$$

Next we consider the weight $\rho = \eta_\beta^\alpha$ on $\mathbb{R}^2$. In this case

$$\rho_2(r,\theta) = (1 + r(1-\cos\theta))^\beta \sim \begin{cases} (1+r)^\beta, & \theta \sim (2k+1)\pi, \\ (1 + r(2k\pi - \theta)^2)^\beta, & \theta \sim 2k\pi, \end{cases}$$

is r-dependent. Then, if $0 = a < b < \pi$, using the change of variables $\vartheta = \sqrt{r}\theta$ and the notation $B = b\sqrt{r}$,

$$\begin{aligned} I_{0,b}(\rho_2) &\sim \frac{1}{b}\int_0^b (1 + r\theta^2)^\beta\, d\theta \left(\frac{1}{b}\int_0^b (1 + r\theta^2)^{-\beta/(p-1)}\, d\theta\right)^{p-1} \\ &\sim \frac{1}{B}\int_0^B (1+\vartheta)^{2\beta}\, d\vartheta \left(\frac{1}{B}\int_0^B (1+\vartheta)^{-2\beta/(p-1)}\, d\vartheta\right)^{p-1} \\ &\sim \frac{1}{B^p}((1+B)^{2\beta+1} - 1)\big(1 - (1+B)^{1-2\beta/(p-1)}\big)^{p-1}. \end{aligned} \tag{3.16}$$

Now it is easy to see that the last term is uniformly bounded in $B \in (0,\infty)$ provided that $-\frac{1}{2} < \beta < \frac{1}{2}(p-1)$. In this way, omitting further cases, we proved

$$\rho = \eta_\beta^\alpha, n = 2 : \sup_{\substack{0<|b-a|\le 2\pi \\ r>0}} I_{a,b}(\rho_2(r,\cdot)) < \infty \quad \text{for} \quad -\frac{1}{2} < \beta < \frac{1}{2}(p-1).$$

Finally we investigate the weight $\rho = \eta_\beta^\alpha$ on $\mathbb{R}^3$. Since $|x| = \sqrt{r^2 + x_3^2}$, we get $\rho(x) = \rho_1(r)\rho_2(r, x_3, \theta)$, where

$$\begin{aligned}\rho_2(r, x_3, \theta) &= 1 + \sqrt{r^2 + x_3^2} - r + r(1 - \cos\theta) \\ &\sim 1 + \sqrt{r^2 + x_3^2} - r + r \begin{cases} 1, & \theta \sim (2k+1)\pi, \\ (2k\pi - \theta)^2, & \theta \sim 2k\pi. \end{cases}\end{aligned}$$

If $0 = a < b < \pi$, we proceed as in the previous case and get with $R = 1 + \sqrt{r^2 + x_3^2} - r \geq 1$ and $B = b\sqrt{r/R}$ that

$$\begin{aligned}I_{0,b}(\rho_2) &\sim \frac{1}{b\sqrt{r}} \int_0^{b\sqrt{r}} (\sqrt{R} + \theta)^{2\beta}\, d\theta \cdot \left(\frac{1}{b\sqrt{r}} \int_0^{b\sqrt{r}} (\sqrt{R} + \theta)^{-2\beta/(p-1)}\, d\theta \right)^{p-1} \\ &= \frac{1}{B} \int_0^B (1 + \theta)^{2\beta}\, d\theta \left(\frac{1}{B} \int_0^B (1 + \theta)^{-2\beta/(p-1)}\, d\theta \right)^{p-1}.\end{aligned}$$

Comparing with (3.16) we see that we proved

$$\rho = \eta_\beta^\alpha,\ n = 3 : \sup_{\substack{0<|b-a|\leq 2\pi \\ r>0,\, x_3 \in \mathbb{R}}} I_{a,b}(\rho_2(r, x_3, \cdot)) < \infty \quad \text{for } -\frac{1}{2} < \beta < \frac{1}{2}(p-1).$$

Even in this final case (3.15) is satisfied (with an obvious modification for $x_3 \in \mathbb{R}$).

To complete the proof we look at Proposition 3.1 and its proof. Up to now we showed that the linear operator T, see (2.7), is bounded if (3.1) is satisfied—with an obvious modification of the class A_p. To be more precise, A_p is replaced by the class of weights

$$\widetilde{A_p} = \left\{ w = \sigma_\beta^\alpha :\ -n < \alpha + \beta < n(p-1),\ -\frac{1}{2} < \beta < \frac{1}{2}(p-1) \right\}$$

or

$$\widetilde{A_p} = \left\{ w = \eta_\beta^\alpha :\ -n < \alpha + \beta < n(p-1),\ -\frac{1}{2} < \beta < \frac{1}{2}(p-1) \right\};$$

for the bound on $\alpha + \beta$ we used the well-known result

$$\sigma_\beta^\alpha, \eta_\beta^\alpha \in A_p \quad \text{iff} \quad -\frac{n-1}{2} < \beta < \frac{n-1}{2}(p-1),\ -n < \alpha + \beta < n(p-1),$$

see [4] for the case $n = 3$, $p = 2$ and [14] for the general case. If $p = 1$, then $\alpha + \beta = 0$ and $\beta = 0$ are allowed as well in the previous characterizations. Now we use Lemma 3.2 on interpolation within each class $\tilde{A}_p$. Looking at the proof of Theorem 1.2 we get a result analogous to (1.9)–(1.11), with A_p replaced by $\tilde{A}_p$. Finally we repeat the proof of Corollary 1.3 (ii). The only differences are due to the new upper and lower bounds on β and on $\alpha+\beta$. □

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