An $L^q(L^2)$-Theory of the Generalized Stokes Resolvent System in Infinite Cylinders

Reinhard Farwig[1] and Ri Myong-Hwan[2,3]

Abstract

Estimates of the generalized Stokes resolvent system, i.e. with prescribed divergence, in an infinite cylinder $\Omega = \Sigma \times \mathbb{R}$ with $\Sigma \subset \mathbb{R}^{n-1}$, a bounded domain of $C^{1,1}$-class, are obtained in the space $L^q(\mathbb{R}; L^2(\Sigma))$, $q \in (1, \infty)$. For the preparation, spectral decompositions of vector-valued homogeneous Sobolev spaces are studied. The main theorem is proved using the techniques of Schauder decompositions, operator-valued multiplier functions and R-boundedness of operator families.

2000 Mathematical Subject Classification: 35Q30, 76D07
Keywords: generalized Stokes resolvent system; unbounded cylindrical domains; Schauder decompositions; operator-valued multiplier functions; R-boundedness

1 Introduction

In this paper we study the *generalized Stokes resolvent system*

$$(R_\lambda) \qquad \begin{aligned} \lambda u - \Delta u + \nabla p &= f \quad \text{in } \Omega \\ \operatorname{div} u &= g \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega = \Sigma \times \mathbb{R}$ is an infinite straight cylinder with cross-section $\Sigma \subset \mathbb{R}^{n-1}, n \geq 3$, a bounded domain of $C^{1,1}$-class. This system is a key problem for the study of instationary Stokes and Navier-Stokes equations. The case of $g = 0$ in (R_λ) was studied in [16]. In this paper the general case $g \neq 0$, i.e. generalized Stokes resolvent systems in an infinite cylinder, is studied to deal with Stokes systems in more general unbounded cylindrical domains such as cylindrical domains with several outlets to infinity using a cut-off procedure.

[1]Department of Mathematics, Darmstadt University of Technology, 64289 Darmstadt, Germany, email: farwig@mathematik.tu-darmstadt.de
[2]Institute of Mathematics, Academy of Sciences, DPR Korea, email: ri@mathematik.tu-darmstadt.de
[3]Supported by the Gottlieb Daimler- und Karl Benz-Stiftung, grant no. S 025/02-10/03.

There are many papers dealing with generalized Stokes resolvent systems for half spaces, bounded and exterior domains, aperture domains and layer-like domains (see e.g. [1] - [5], [12] - [14], [17], [18] and the Introduction of [16] for more details), but no result for unbounded cylindrical domains has been known up to now. Here we study the solvability of the system (R_λ) in the space $L^q(\mathbb{R}; L^2(\Sigma))$ for $1 < q < \infty$. The main result of this paper is the following Theorem.

Theorem 1.1 *Let $\Sigma \subset \mathbb{R}^{n-1}, n \geq 3$, be a bounded domain of $C^{1,1}$-class, $\alpha_0 > 0$ the smallest eigenvalue of the Dirichlet Laplacian in Σ, let $0 < \varepsilon < \frac{\pi}{2}$ and $1 < q < \infty$. If $f \in L^q(\mathbb{R}; L^2(\Sigma))$ and $g \in W^{1;q,2}(\Omega) \cap \widehat{W}^{-1;q,2}(\Omega)$, then for every $\alpha \in (0, \alpha_0)$ and $\lambda \in -\alpha + S_\varepsilon$ there exists a unique solution (u, p) to (R_λ) satisfying $u, \nabla^2 u, \nabla p \in L^q(\mathbb{R}; L^2(\Sigma))$ and the estimate*

$$\begin{aligned} &\|(\lambda+\alpha)u, \nabla^2 u, \nabla p\|_{L^q(\mathbb{R};L^2(\Sigma))} \\ &\qquad \leq C(\|f\|_{L^q(\mathbb{R};L^2(\Sigma))} + \|g\|_{W^{1;q,2}(\Omega)} + (|\lambda|+1)\|g\|_{\widehat{W}^{-1;q,2}(\Omega)}), \end{aligned} \tag{1.1}$$

where the constant C is independent of λ and depending only on α, ε, q and Σ. In particular, if $\int_\Sigma g(x', x_n)\, dx' = 0$ for almost all $x_n \in \mathbb{R}$, a stronger estimate

$$\|(\lambda+\alpha)u, \nabla^2 u, \nabla p\|_{L^q(\mathbb{R};L^2(\Sigma))} \leq C(\|f\|_{L^q(\mathbb{R};L^2(\Sigma))} + \|g\|_{W^{1;q,2}(\Omega)} + |\lambda|\|g\|_{\widehat{W}^{-1;q,2}(\Omega)}) \tag{1.2}$$

holds with $C = C(\alpha, \varepsilon, q, \Sigma)$.

We use the following notations. For $\varepsilon \in (0, \frac{\pi}{2})$, let S_ε denote the sector of the complex plane

$$\{\lambda \in \mathbb{C}; \lambda \neq 0, |\arg\lambda| < \tfrac{\pi}{2} + \varepsilon\}.$$

We do not distinguish among spaces of scalar and vector-valued functions as long as no confusion arises. In particular, given a norm in some Banach function space, we use the short notation $\|u, v\|$ for $\|u\| + \|v\|$, even if u and v are tensors of different order. For a Banach space X let X^* denote its dual space and $L^q(\mathbb{R}; X), 1 < q < \infty$, the Bochner space of all X-valued measurable functions with finite norm

$$\|u\|_{L^q(\mathbb{R};X)} = \Big(\int_{\mathbb{R}} \|u(t)\|_X^q dt\Big)^{1/q}.$$

Let $\Omega = \Sigma \times \mathbb{R}$ be an infinite cylinder of $\mathbb{R}^n$ with bounded cross section $\Sigma \subset \mathbb{R}^{n-1}$ and with generic point $x \in \Omega$ written in the form $x = (x', x_n) \in \Omega$, where $x' \in \Sigma$ and $x_n \in \mathbb{R}$. Similarly, differential operators in $\mathbb{R}^n$ are splitted, in particular, $\Delta = \Delta' + \partial_n^2$ and $\nabla = (\nabla', \partial_n)$.

Let $r \in (1, \infty)$ and $s \in (0, \infty)$. Then, $L^r(\Sigma)$ and $W^{s,r}(\Sigma)$ are the usual Lebesgue and Sobolev spaces with norm $\|\cdot\|_{r;\Sigma}$ and $\|\cdot\|_{s,r;\Sigma}$, respectively. Moreover, $\widehat{W}^{1,r}(\Sigma)$ is the homogeneous Sobolev space, i.e.,

$$\widehat{W}^{1,r}(\Sigma) = \{u \in L^1_{\text{loc}}(\bar{\Sigma})/\mathbb{R}; \nabla' u \in L^r(\Sigma)\}, \quad \|u\|_{\widehat{W}^{1,r}(\Sigma)} = \|\nabla' u\|_{r;\Sigma},$$

and $\widehat{W}^{-1,r}(\Sigma) = (\widehat{W}^{1,r'}(\Sigma))^*$ is the dual space of $\widehat{W}^{1,r'}(\Sigma)$, $r' = \frac{r}{r-1}$, with norm $\|\cdot\|_{\widehat{W}^{-1,r}(\Sigma)}$. We denote by $W^{k;q,r}(\Omega), k \in \mathbb{N}, q \in (1, \infty)$, the Banach space of all functions on Ω whose derivatives of order up to k belong to $L^q(\mathbb{R}; L^r(\Sigma))$ with norm

$$\|u\|_{W^{k;q,r}(\Omega)} = (\sum_{|\alpha|\le k} \|D^\alpha u\|^q_{L^q(\mathbb{R};L^r(\Sigma))})^{1/q};$$

here $D^\alpha u = \partial_1^{\alpha_1} \cdot \ldots \cdot \partial_n^{\alpha_n} u$ for a multi-index $\alpha \in \mathbb{N}_0^n$ of order $|\alpha| \le k$. Moreover, $W_0^{1;q,r}(\Omega)$ is the completion of the set $C_0^\infty(\Omega)^n$ in $W^{1;q,r}(\Omega)$. Finally, let $\widehat{W}^{1;q,r}(\Omega)$ be the Banach space defined by

$$\widehat{W}^{1;q,r}(\Omega) = \{u \in L^1_{\mathrm{loc}}(\bar{\Omega})/\mathbb{R}; \nabla u \in L^q(\mathbb{R}; L^r(\Sigma))\}$$

endowed with the norm $\|u\|_{\widehat{W}^{1;q,r}(\Omega)} = \|\nabla u\|_{L^q(\mathbb{R};L^r(\Sigma))}$; its dual space is denoted by $\widehat{W}^{-1;q',r'}(\Omega) = (\widehat{W}^{1;q,r}(\Omega))^*$, where $q' = q/(q-1), r' = r/(r-1)$. For notational convenience, as long as no confusion arises, we denote constants $c, C, \ldots$ appearing in the proofs by the same symbol even though they may be different line by line.

In an n-dimensional *infinite layer* the Stokes resolvent system is reduced by the $(n-1)$-dimensional partial Fourier transform to a system of ordinary differential equations with the Fourier phase variable as a parameter; in [2], [3] and [5] the authors applied Fourier multiplier theorems to the explicit solution of the reduced system of ordinary differential equations to get the final Stokes resolvent estimates.

However, in an n-dimensional *infinite cylinder* $\Omega = \Sigma \times \mathbb{R}$ the Stokes resolvent system (R_λ) is reduced by the application of the one dimensional partial Fourier transform $\mathcal{F} \equiv \hat{}$ along the axis of Ω to the parametrized Stokes system $(R_{\lambda,\xi})$ on the cross-section Σ

$$(R_{\lambda,\xi}) \qquad \begin{aligned} (\lambda + \xi^2 - \Delta')U' + \nabla' P &= F' && \text{in } \Sigma \\ (\lambda + \xi^2 - \Delta')U_n + i\xi P &= F_n && \text{in } \Sigma \\ \operatorname{div}' U' + i\xi U_n &= G && \text{in } \Sigma \\ U' = 0, \quad U_n &= 0 && \text{on } \partial\Sigma, \end{aligned}$$

which is elliptic in the sense of Agmon, Douglis and Nirenberg [6]; here $U = \hat{u}$, $P = \hat{p}$, and $U = (U', U_n)$, $F = (F', F_n)$ etc. In [16] the authors obtained the estimate

$$\|(\lambda + \alpha)U, \xi^2 U, \xi\nabla' U, \nabla'^2 U, \xi P, \nabla' P\|_{2;\Sigma} \le c\|F, \nabla' G, G, \xi G\|_{2;\Sigma} + \ldots$$

of the solution $\{U(\xi), P(\xi)\}$ to $(R_{\lambda,\xi})$ where some terms for G have been omitted; see (3.3) – (3.5) below and [16], Theorem 3.4, for details. Then Fourier multiplier techniques are used to get the final estimate of (u, p) when $g = 0$. However, the estimate of $\{U(\xi), P(\xi)\}$ for $(R_{\lambda,\xi})$ involves the function G with ξ-dependent parameters as well as with norms in the sum and intersection of several Sobolev spaces. Therefore, the Fourier multiplier technique cannot directly be applied to the case $g \ne 0$.

To get an estimate for (R_λ) from the estimate for $(R_{\lambda,\xi})$, we use the unconditionality of dyadic Schauder decompositions of $L^q(\mathbb{R}; L^2(\Sigma))$ for $1 < q < \infty$, vector-valued homogeneous Sobolev spaces and the R-boundedness of operator families. Having obtained Stokes resolvent estimates in the straight cylinder $\Omega = \Sigma \times \mathbb{R}$, one can get resolvent estimates in unbounded cylindrical domains with several outlets to infinity; at the end of the paper, we briefly mention the main idea using the method of cut-off functions.

This paper is organized as follows. Section 2 is devoted to some preliminaries for the proof of the main theorem including dyadic spectral decompositions of vector-valued homogeneous Sobolev spaces. Section 3 includes the proof of Theorem 1.1 and a remark concerning the application to unbounded cylindrical domains with several outlets to infinity (Remark 3.1).

2 Preliminaries

First let us consider *vector-valued homogeneous Sobolev spaces*. Let X be a reflexive Banach space and $1 < q < \infty$. We define the space $\widehat{W}^{1,q}(\mathbb{R}; X)$ by

$$\widehat{W}^{1,q}(\mathbb{R}; X) := \{u \in L^1_{\mathrm{loc}}(\mathbb{R}; X); Du \in L^q(\mathbb{R}; X)\}$$

endowed with the (semi–)norm

$$\|u\|_{\widehat{W}^{1,q}(\mathbb{R};X)} = \|Du\|_{L^q(\mathbb{R};X)},$$

where D is the derivative of first order; here we neglect the technicality that $\widehat{W}^{1,q}(\mathbb{R}; X)$ should be defined as a quotient space (of functions modulo constants). Using the one-dimensional Fourier transform $\mathcal{F} \equiv \hat{}$ the space $\widehat{W}^{1,q}(\mathbb{R}; X)$ may be rewritten as

$$\widehat{W}^{1,q}(\mathbb{R}; X) = \{u \in L^1_{\mathrm{loc}}(\mathbb{R}; X); \mathcal{F}^{-1}(\xi\hat{u}) \in L^q(\mathbb{R}; X)\}$$

with norm

$$\|u\|_{\widehat{W}^{1,q}(\mathbb{R};X)} = \|\mathcal{F}^{-1}(\xi\hat{u})\|_{L^q(\mathbb{R};X)},$$

where ξ is the phase variable of the Fourier transform. It is easy to see that $\widehat{W}^{1,q}(\mathbb{R}; X), 1 < q < \infty$, is a reflexive Banach space.

Let $\mathcal{D}(\mathbb{R}; X)$ be the space of all compactly supported and infinitely differentiable X-valued functions and $\mathcal{D}'(\mathbb{R}; X^*)$ the space of X^*-valued distributions. Moreover, $\mathcal{S}(\mathbb{R}; X)$ is the Schwartz space of all rapidly decreasing X-valued functions, with dual space $\mathcal{S}'(\mathbb{R}; X^*)$, the space of tempered X^*-valued distributions.

Lemma 2.1 *(i)* $\mathcal{D}(\mathbb{R}; X)$ *is dense in* $\widehat{W}^{1,q}(\mathbb{R}; X)$ *for each* $q \in (1, \infty)$.
(ii) $C_0^\infty(\overline{\Omega})$ *is dense in* $\widehat{W}^{1;q,r}(\Omega)$ *for each* $q, r \in (1, \infty)$.

Proof: (i) Let $f \in (\widehat{W}^{1,q}(\mathbb{R}; X))^*$ vanish on $\mathcal{D}(\mathbb{R}; X)$. Then, due to the Hahn-Banach theorem, there exists $h \in L^{q'}(\mathbb{R}; X^*)$, $q' = q/(q-1)$, such that

$$0 = \langle f, \phi\rangle = \langle h, D\phi\rangle \quad \forall\phi \in \mathcal{D}(\mathbb{R}; X).$$

In particular, for all $\varphi \in \mathcal{D}(\mathbb{R})$ and $x \in X$, we have

$$0 = \langle h, D\varphi \cdot x\rangle = \langle\, \langle h(\cdot), x\rangle_{X^*,X}, D\varphi\rangle_{\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R})}$$

which together with $\langle h(\cdot), x\rangle_{X^*,X} \in L^{q'}(\mathbb{R})$ yields

$$\langle h(\cdot), x\rangle_{X^*,X} = \text{const} = 0 \quad \text{for all } x \in X.$$

Hence $h = 0$, and $f = 0$.

(ii) Given $u \in \widehat{W}^{1;q,r}(\Omega)$ define $u_0(x_n) = \frac{1}{|\Sigma|}\int_\Sigma u(x', x_n)dx'$ where $|\Sigma|$ denotes the $(n-1)$-dimensional Lebesgue measure of Σ. Since $u_0 \in \widehat{W}^{1,q}(\mathbb{R};\mathbb{R})$, we may apply part (i) and assume that $u \in \widehat{W}^{1;q,r}(\Omega)$ has vanishing means on Σ for almost all $x_n \in \mathbb{R}$. Then by Poincaré's inequality applied to $u(\cdot, x_n)$ on Σ it is easy to see that u may be approximated by elements of the space $\{v \in \widehat{W}^{1;q,r}(\Omega);\ \operatorname{supp} v \text{ is compact in } \overline{\Omega}\}$. Finally, a standard approximation argument proves that $C_0^\infty(\overline{\Omega})$ is dense in the latter space with respect to the norm $\|\cdot\|_{\widehat{W}^{1;q,r}(\Omega)}$. ■

By the Hahn-Banach theorem, for every $f \in (\widehat{W}^{1,q}(\mathbb{R};X))^*$ there is some $h \in L^{q'}(\mathbb{R};X^*)$ such that

$$f = Dh \quad \text{and} \quad \|f\|_{(\widehat{W}^{1,q}(\mathbb{R};X))^*} = \|h\|_{L^{q'}(\mathbb{R};X^*)},$$

cf. Lemma 2.1. Conversely, it is obvious from Lemma 2.1 (i) that, if $h \in L^{q'}(\mathbb{R};X^*)$, then $Dh \in (\widehat{W}^{1,q}(\mathbb{R};X))^*$. Thus we conclude that

$$\begin{aligned} (\widehat{W}^{1,q}(\mathbb{R};X))^* &= \{f \in \mathcal{S}'(\mathbb{R};X^*);\ \mathcal{F}^{-1}(\tfrac{1}{\xi}\hat{f}) \in L^{q'}(\mathbb{R};X^*)\}, \\ \|f\|_{(\widehat{W}^{1,q}(\mathbb{R};X))^*} &= \|\mathcal{F}^{-1}(\tfrac{1}{\xi}\hat{f})\|_{L^{q'}(\mathbb{R};X^*)}. \end{aligned} \tag{2.1}$$

In consideration of (2.1) we shall denote the space $(\widehat{W}^{1,q}(\mathbb{R};X))^*$ by $\widehat{W}^{-1,q'}(\mathbb{R};X^*)$ for $1 < q < \infty$.

Now we introduce the notions of UMD-spaces, Schauder decompositions of Banach spaces and R-boundedness of operator families.

Definition 2.2 *A Banach space X is called a UMD-space if the Hilbert transform*

$$Hf(t) = -\frac{1}{\pi}\ PV \int_{\mathbb{R}} \frac{f(s)}{t-s}\,ds\,, \quad f \in \mathcal{S}(\mathbb{R};X),$$

extends to a bounded linear operator in $L^q(\mathbb{R};X)$ for some $q \in (1,\infty)$.

It is well known that, if X is a UMD space, then X is reflexive (see e.g. [9]) and the Hilbert transform is bounded in $L^q(\mathbb{R};X)$ for *all* $q \in (1,\infty)$ (see e.g. [23], Theorem 1.3, [21], Proposition 2.3). Closed subspaces of, the dual of, and the quotient of UMD spaces are UMD spaces as well. If X is a UMD space, then $L^q(G;X)$ for $1 < q < \infty$ and for any open set G of $\mathbb{R}^d, d \in \mathbb{N}$, is also a UMD space.

Definition 2.3 *Let X be a Banach space and $(x_n)_{n=1}^\infty \subset X$. A series $\sum_{n=1}^\infty x_n$ is called unconditionally convergent if $\sum_{n=1}^\infty x_{\sigma(n)}$ is convergent in norm for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$.*

Note that if $\sum_{n=1}^\infty x_n$ is unconditionally convergent, then the sum $\sum_{n=1}^\infty x_{\sigma(n)}$ is independent of the permutation σ, see e.g. [10], §3.2.

Definition 2.4 *A sequence of projections* $(\Delta_j)_{j\in\mathbb{N}} \subset \mathcal{L}(\mathcal{X})$ *is called a Schauder decomposition of a Banach space* $\mathcal{X}$ *if*

$$\Delta_i\Delta_j = 0 \qquad \text{for all} \quad i \neq j$$

and

$$\sum_{j=1}^{\infty} \Delta_j x = x \qquad \text{for each} \quad x \in \mathcal{X}.$$

A Schauder decomposition $(\Delta_j)_{j\in\mathbb{N}}$ *is called unconditional if the series* $\sum_{j=1}^{\infty} \Delta_j x$ *converges unconditionally for each* $x \in \mathcal{X}$*.*

If $(\Delta_j)_{j\in\mathbb{N}}$ is an unconditional Schauder decomposition of a Banach space $\mathcal{X}$, then there is a constant $c > 0$ such that

$$\Big\| \sum_{j=1}^{N} \varepsilon_j \Delta_j x \Big\|_{\mathcal{X}} \le c \Big\| \sum_{j=1}^{N} \Delta_j x \Big\|_{\mathcal{X}} \quad \text{for all } N \in \mathbb{N},\, x \in \mathcal{X},\ \varepsilon_j \in \{-1, 1\}, \tag{2.2}$$

see e.g. [10], Proposition 3.14. Moreover, there is a constant $c_\Delta > 0$ such that for all x_j in the range $\mathcal{R}(\Delta_j)$ of Δ_j the inequalities

$$c_\Delta^{-1} \Big\| \sum_{j=l}^{k} x_j \Big\|_{\mathcal{X}} \quad \le \quad \Big\| \sum_{j=l}^{k} \varepsilon_j(s) x_j \Big\|_{L^p(0,1;\mathcal{X})} \le c_\Delta \Big\| \sum_{j=l}^{k} x_j \Big\|_{\mathcal{X}}, \tag{2.3}$$

are valid for any sequence $(\varepsilon_j(s))$ of independent, symmetric $\{-1, 1\}$-valued random variables defined on (0,1), for all $l \le k \in \mathbb{Z}$ and for each $p \in [1, \infty)$, see e.g. [10], (3.8). Given an interpolation couple $\mathcal{X}_1$, $\mathcal{X}_2$ of Banach spaces, it is easily seen that a Schauder decomposition of both $\mathcal{X}_1$ and $\mathcal{X}_2$ is a Schauder decomposition of $\mathcal{X}_1 \cap \mathcal{X}_2$ and $\mathcal{X}_1 + \mathcal{X}_2$ as well. We note that in the previous definitions and results the set of indices $\mathbb{N}$ may be replaced by $\mathbb{Z}$ without any further changes.

Let X be a UMD space and $\chi_{[a,b)}$ denote the characteristic function for the interval $[a, b)$. Let R be the Riesz projection, i.e.

$$R := \mathcal{F}^{-1}\chi_{[0,\infty)}\mathcal{F},$$

and define

$$\Delta_j := \mathcal{F}^{-1}\chi_{[2^j,2^{j+1})}\mathcal{F},\ j \in \mathbb{Z}.$$

It is well known that R and $\Delta_j, j \in \mathbb{Z}$, are bounded in $L^q(\mathbb{R}; X)$ for each $q \in (1, \infty)$ and that $\{\Delta_j; j \in \mathbb{Z}\}$ is an unconditional Schauder decomposition of $RL^q(\mathbb{R}; X)$, the image of $L^q(\mathbb{R}; X)$ by the Riesz projection R, see [10], proof of Theorem 3.19. Furthermore, $\{\Delta_j; j \in \mathbb{Z}\}$ is an unconditional Schauder decomposition of both $R\widehat{W}^{1,q}(\mathbb{R}; X)$ and $R\widehat{W}^{-1,q}(\mathbb{R}; X)$ for each $q \in (1, \infty)$ since for every permutation σ of $\mathbb{N}$, every $l < k \in \mathbb{Z}$ and any $u \in R\widehat{W}^{1,q}(\mathbb{R}; X)$

$$\Big\| u - \sum_{j=l}^{k} \Delta_{\sigma(j)} u \Big\|_{\widehat{W}^{1,q}(\mathbb{R};X)} = \Big\| Du - \sum_{j=l}^{k} \Delta_{\sigma(j)} Du \Big\|_{L^q(\mathbb{R};X)},$$

as well as for any $v \in R\widehat{W}^{-1,q}(\mathbb{R}; X)$

$$\Big\| v - \sum_{j=l}^{k} \Delta_{\sigma(j)} v \Big\|_{\widehat{W}^{-1,q}(\mathbb{R};X)} = \Big\| \mathcal{F}^{-1}(\xi^{-1}\hat{v}) - \sum_{j=l}^{k} \Delta_{\sigma(j)} \mathcal{F}^{-1}(\xi^{-1}\hat{v}) \Big\|_{L^q(\mathbb{R};X)}.$$

Definition 2.5 *Let X, Y be Banach spaces. An operator family $\mathcal{T} \subset \mathcal{L}(X;Y)$ is called R-bounded if there is a constant $c > 0$ such that for all $T_1, \cdots, T_N \in \mathcal{T}$, all $x_1, \cdots, x_N \in X$ and $N \in \mathbb{N}$*

$$\Big\| \sum_{j=1}^{N} \varepsilon_j(s) T_j x_j \Big\|_{L^p(0,1;Y)} \le c \Big\| \sum_{j=1}^{N} \varepsilon_j(s) x_j \Big\|_{L^p(0,1;X)} \tag{2.4}$$

for some $p \in [1, \infty)$; here $(\varepsilon_j(s))$ is a sequence of independent, symmetric $\{-1, 1\}$-valued random variables on $[0, 1]$, e.g. the Rademacher functions

$$r_j(s) = \mathrm{sign}\ \sin(2^j \pi s), \quad j \in \mathbb{N}.$$

The smallest constant c for which (2.4) holds is denoted by $R_p(\mathcal{T})$.

Due to *Kahane's inequality* ([11]) for all $p_1, p_2 \in [1, \infty)$ and for any Banach space X there exists a constant $c = c(p_1, p_2, X) > 0$ such that for all $x_1, \dots, x_N \in X$, $N \in \mathbb{N}$,

$$\Big\| \sum_{j=1}^{N} \varepsilon_j(s) x_j \Big\|_{L^{p_1}(0,1;X)} \le c \Big\| \sum_{j=1}^{N} \varepsilon_j(s) x_j \Big\|_{L^{p_2}(0,1;X)}; \tag{2.5}$$

hence, if (2.4) holds for some $p \in [1, \infty)$, then it does for all $p \in [1, \infty)$.

Lemma 2.6 *Let $(H, (\cdot, \cdot), \| \cdot \|_H)$ be a Hilbert space and let $1 < q < \infty$. Then there is a constant $c > 0$ such that for all $x_j = \Delta_j x_j \in L^q(\mathbb{R}; H)$ the inequalities*

$$\frac{1}{c} \big\| \big(\sum_{j=l}^{k} \|x_j\|_H^2 \big)^{1/2} \big\|_{q;\mathbb{R}} \le \big\| \sum_{j=l}^{k} x_j \big\|_{L^q(\mathbb{R};H)} \le c \big\| \big(\sum_{j=l}^{k} \|x_j\|_H^2 \big)^{1/2} \big\|_{q;\mathbb{R}} \tag{2.6}$$

hold for all $l < k \in \mathbb{Z}$.

Proof: Choose a sequence $(\varepsilon_j(s))$ of $\{-1, 1\}-$valued symmetric, independent random variables on $[0, 1]$. Then by (2.3), Fubini's Theorem and Kahane's inequality (2.5)

$$\begin{aligned} \big\| \sum_{j=l}^{k} x_j \big\|_{L^q(\mathbb{R};H)} &\le c_\Delta \big\| \sum_{j=l}^{k} \varepsilon_j(s) x_j \big\|_{L^q(0,1;L^q(\mathbb{R};H))} \\ &= c_\Delta \big\| \sum_{j=l}^{k} \varepsilon_j(s) x_j \big\|_{L^q(\mathbb{R};L^q(0,1;H))} \\ &\le c_\Delta \cdot c \big\| \sum_{j=l}^{k} \varepsilon_j(s) x_j \big\|_{L^q(\mathbb{R};L^2(0,1;H))}. \end{aligned} \tag{2.7}$$

Since $\int_0^1 \varepsilon_j(s)\varepsilon_i(s)\,ds = \delta_{ji}$ by the assumption on $(\varepsilon_j(s))$, we get due to the Hilbert space structure of H

$$\big\|\sum_{j=l}^{k} \varepsilon_j(s)x_j\big\|_{L^2(0,1;H)} = \big(\sum_{j=l}^{k}\|x_j\|_H^2\big)^{1/2}.$$

Therefore (2.7) leads to the estimate

$$\big\|\sum_{j=l}^{k} x_j\big\|_{L^q(\mathbb{R};H)} \leq c\,\big\|\big(\sum_{j=l}^{k}\|x_j\|_H^2\big)^{1/2}\big\|_{q;\mathbb{R}}. \tag{2.8}$$

Since in (2.7) the reversed inequality holds as well, (2.6) is proved. ■

Lemma 2.7 *Let X be a UMD space, $1<q<\infty$ and $R_{a,b} := \mathcal{F}^{-1}\chi_{[a,b)}\mathcal{F}$ for $-\infty < a < b < \infty$. If $g \in \widehat{W}^{-1,q}(\mathbb{R};X)$, then $R_{a,b}g \in L^q(\mathbb{R};X)$ and there exists a constant $c(q,X)>0$ such that*

$$\|R_{a,b}g\|_{L^q(\mathbb{R};X)} \leq c(q,X)\max\{|a|,|b|\}\|R_{a,b}g\|_{\widehat{W}^{-1,q}(\mathbb{R};X)}.$$

In particular, if $a>0$, then

$$\frac{1}{b\,c(q,X)}\|R_{a,b}g\|_{L^q(\mathbb{R};X)} \leq \|R_{a,b}g\|_{\widehat{W}^{-1,q}(\mathbb{R};X)} \leq \frac{c(q,X)}{a}\|R_{a,b}g\|_{L^q(\mathbb{R};X)}.$$

Proof: Let $m_1(\xi)$ be a continuously differentiable function on $\mathbb{R}$ such that $m_1(\xi)=\xi$ in (a,b) and

$$\sup_{\xi\in\mathbb{R}}\{|m_1(\xi)|,|\xi m_1'(\xi)|\} \leq 2\max\{|a|,|b|\}.$$

Then, by [26], Proposition 3, m_1 is a Fourier multiplier in $L^q(\mathbb{R};X)$, and we get

$$\begin{aligned}\|R_{a,b}g\|_{L^q(\mathbb{R};X)} &= \|\mathcal{F}^{-1}(m_1(\xi)\xi^{-1}\chi_{[a,b)}\hat{g})\|_{L^q(\mathbb{R};X)}\\ &\leq c(q,X)\max\{|a|,|b|\}\|R_{a,b}g\|_{\widehat{W}^{-1,q}(\mathbb{R};X)}.\end{aligned}$$

If $a>0$, we define a C^1-function $m_2(\xi)$ on $\mathbb{R}$ such that $m_2(\xi)=\frac{1}{\xi}$ in (a,b) and

$$\sup_{\xi\in\mathbb{R}}\{|m_2(\xi)|,|\xi m_2'(\xi)|\} \leq \frac{2}{a}.$$

Then we get for $g\in L^q(\mathbb{R};X)$

$$\begin{aligned}\|R_{a,b}g\|_{\widehat{W}^{-1,q}(\mathbb{R};X)} &= \|\mathcal{F}^{-1}(\xi^{-1}\chi_{[a,b)}\hat{g})\|_{L^q(\mathbb{R};X)}\\ &= \|\mathcal{F}^{-1}(m_2(\xi)\chi_{[a,b)}\hat{g})\|_{L^q(\mathbb{R};X)}\\ &\leq \tfrac{c(q,X)}{a}\|R_{a,b}g\|_{L^q(\mathbb{R};X)}.\end{aligned}$$

■

Lemma 2.8 *Let X be a UMD space and $q \in (1,\infty)$. There is a constant $c > 0$ such that for all $g \in L^q(\mathbb{R};X)$ and for any $l \le k \in \mathbb{Z}$ the following two formulae hold:*

$$c^{-1}\|\sum_{j=l}^k 2^j\Delta_j g\|_{L^q(\mathbb{R};X)} \le \|\sum_{j=l}^k \Delta_j g\|_{\widehat{W}^{1,q}(\mathbb{R};X)} \le c\|\sum_{j=l}^k 2^j\Delta_j g\|_{L^q(\mathbb{R};X)} \tag{2.9}$$

$$c^{-1}\|\sum_{j=l}^k 2^{-j}\Delta_j g\|_{L^q(\mathbb{R};X)} \le \|\sum_{j=l}^k \Delta_j g\|_{\widehat{W}^{-1,q}(\mathbb{R};X)} \le c\|\sum_{j=l}^k 2^{-j}\Delta_j g\|_{L^q(\mathbb{R};X)}. \tag{2.10}$$

Proof: Define the functions m_1, m_2 by

$$m_1(\xi) = \sum_{j\in\mathbb{Z}} \frac{2^j}{\xi}\chi_{[2^j,2^{j+1})}(\xi), \quad m_2(\xi) = \sum_{j\in\mathbb{Z}} \frac{\xi}{2^j}\chi_{[2^j,2^{j+1})}(\xi).$$

Obviously $\sup_{j\in\mathbb{Z}} \mathrm{Var}(\chi_{[2^j,2^{j+1})}m_i) < \infty$ for $i = 1,2$, where 'Var' means the total variation on $\mathbb{R}$. Note that for $i = 1,2$,

$$m_i(\xi) = \sum_{j\in\mathbb{Z}} \chi_{[2^j,2^{j+1})}(\xi)m_i(\xi) \quad \forall \xi \in \mathbb{R} \quad \text{and} \quad m_i(\xi) = 0 \quad \text{for } \xi < 0.$$

Then by [25], Theorem 3.2, $m_i, i = 1,2$, is a Marcinkiewicz type multiplier in $L^q(\mathbb{R};X)$, that is, there is a constant $c > 0$ satisfying

$$\|\mathcal{F}^{-1}(m_i\hat{f})\|_{L^q(\mathbb{R};X)} \le c\|f\|_{L^q(\mathbb{R};X)} \quad \text{for all } f \in L^q(\mathbb{R};X).$$

Consequently, we get for each $g \in L^q(\mathbb{R};X)$

$$\begin{aligned} \|\sum_{j=l}^k 2^j\Delta_j g\|_{L^q(\mathbb{R};X)} &= \|\mathcal{F}^{-1}\big(\sum_{j=l}^k \tfrac{2^j}{\xi}\chi_{[2^j,2^{j+1})}(\xi)\widehat{Dg}(\xi)\big)\|_{L^q(\mathbb{R};X)} \\ &= \|\mathcal{F}^{-1}\big(m_1\mathcal{F}(D(\sum_{j=l}^k \Delta_j g))\big)\|_{L^q(\mathbb{R};X)} \\ &\le c\|\sum_{j=l}^k \Delta_j g\|_{\widehat{W}^{1,q}(\mathbb{R};X)}. \end{aligned}$$

The second inequality of (2.9) is proved using the multiplier m_2, that is, we have

$$\begin{aligned} \|\sum_{j=l}^k \Delta_j g\|_{\widehat{W}^{1,q}(\mathbb{R};X)} &= \|\sum_{j=l}^k \mathcal{F}^{-1}\big(\xi\,\chi_{[2^j,2^{j+1})}(\xi)\hat{g}(\xi)\big)\|_{L^q(\mathbb{R};X)} \\ &= \|\mathcal{F}^{-1}\big(m_2\mathcal{F}(\sum_{j=l}^k 2^j\Delta_j g)\big)\|_{L^q(\mathbb{R};X)} \\ &\le c\|\sum_{j=l}^k 2^j\Delta_j g\|_{L^q(\mathbb{R};X)}. \end{aligned}$$

The formula (2.10) is proved similarly. ∎

Now let Σ be a bounded Lipschitz domain of $\mathbb{R}^{n-1}$. Then, $L^r(\Sigma)$ and $\widehat{W}^{1,r}(\Sigma)$ for all $r \in (1,\infty)$ are UMD spaces, see e.g. [7], Theorem III.4.5.2.

Lemma 2.9 *Suppose $m : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ satisfies*

$$\sup_{\xi\in\mathbb{R}\setminus\{0\}} |m(\xi)| \le c_0, \qquad \sup_{\xi\in\mathbb{R}\setminus\{0\}} |\xi m'(\xi)| \le c_0.$$

Then the multiplier operator defined by

$$Mf := \mathcal{F}^{-1}(m\hat{f})$$

is bounded in $L^q(\mathbb{R};\widehat{W}^{-1,r}(\Sigma))$ and $\widehat{W}^{-1,q}(\mathbb{R};L^r(\Sigma))$, respectively, with bound $c = c(q,r,\Sigma)c_0$ for $q,r \in (1,\infty)$.

Proof: It is trivial from [26], Proposition 3, to show that M is bounded in $L^q(\mathbb{R};\widehat{W}^{-1,r}(\Sigma))$ since $\widehat{W}^{-1,r}(\Sigma)$ is a UMD space. Moreover, considering (2.1), we get for $f\in\widehat{W}^{-1,q}(\mathbb{R};L^r(\Sigma))$

$$\|Mf\|_{\widehat{W}^{-1,q}(\mathbb{R};L^r(\Sigma))}=\|M\mathcal{F}^{-1}(\xi^{-1}\hat{f})\|_{L^q(\mathbb{R};L^r(\Sigma))}\le c\|f\|_{\widehat{W}^{-1,q}(\mathbb{R};L^r(\Sigma))},$$

which completes the proof of this lemma. ∎

Lemma 2.10 *Let $1<q,r<\infty$. Then the operator family $\{R_{a,b};\,-\infty<a<b<\infty\}$ is R-bounded in $L^q(\mathbb{R};L^r(\Sigma))$.*

Proof: In the proof of [10], Theorem 3.19, the R-boundedness of the operator family $\{R_{a,b};a,b\in\mathbb{R}\}$ in $L^q(\mathbb{R};X)$ for UMD spaces X is shown. ∎

3 Generalized Resolvent Estimate

In this section we study the Stokes resolvent system (R_λ) on Ω (see Introduction), where $\Omega=\Sigma\times\mathbb{R}$ is an infinite straight cylinder with cross-section $\Sigma\subset\mathbb{R}^{n-1}, n\ge 3$, a bounded domain of $C^{1,1}$-class. Let a generic point $x\in\Omega$ be written in the form $x=(x',x_n)\in\Omega$, where $x'\in\Sigma$ and $x_n\in\mathbb{R}$. Similarly, differential operators in $\mathbb{R}^n$ are split, in particular, $\Delta=\Delta'+\partial_n^2$ and $\nabla=(\nabla',\partial_n)$. The Fourier transform in the variable x_n is denoted by $\mathcal{F}$ or $\hat{}$ and the inverse Fourier transform by $\mathcal{F}^{-1}$ or $^\vee$.

First, we consider the spaces concerning the divergence equation. If $u\in W^{2;q,r}(\Omega)\cap W_0^{1;q,r}(\Omega)$ for some $q,r\in(1,\infty)$ solves the divergence equation of (R_λ), then

$$g\in W^{1;q,r}(\Omega)\cap\widehat{W}^{-1;q,r}(\Omega). \tag{3.1}$$

In fact, given $\varphi\in\widehat{W}^{1;q',r'}(\Omega)$ and a sequence $(\varphi_k)\subset C^\infty(\overline{\Omega})$ converging to φ in $\widehat{W}^{1;q',r'}(\Omega)$, see Lemma 2.1 (ii), we have for all $k\in\mathbb{N}$

$$\langle g,\varphi_k\rangle=\int_\Omega \operatorname{div}u\;\varphi_k\,dx=-\int_\Omega u\cdot\nabla\varphi_k\,dx.$$

Hence $\langle g,\varphi\rangle$ is well defined and $\|g\|_{\widehat{W}^{-1;q,r}(\Sigma)}\le\|u\|_{L^q(\mathbb{R};L^r(\Sigma))}$.

Moreover, we shall show that

$$\widehat{W}^{-1;q,r}(\Omega)=L^q(\mathbb{R};\widehat{W}^{-1,r}(\Sigma))+\widehat{W}^{-1,q}(\mathbb{R};L^r(\Sigma)) \tag{3.2}$$

with equivalent norms. In fact, if $g\in\widehat{W}^{-1;q,r}(\Omega)$, then there exist functions $f_1,f_2\in L^q(\mathbb{R};L^r(\Sigma))$ such that for all $\varphi\in\widehat{W}^{1;q',r'}(\Omega)$

$$\langle g,\varphi\rangle=\int_\Omega f_1\cdot\nabla'\varphi\,dx+\int_\Omega f_2\partial_n\varphi\,dx\quad\text{and}\quad\|g\|_{-1;q,r}=\|f_1,f_2\|_{L^q(\mathbb{R};L^r(\Sigma))},$$

where $\langle\cdot\,,\,\cdot\rangle$ denotes the duality product between $\widehat{W}^{-1;q,r}(\Omega)$ and $\widehat{W}^{1;q',r'}(\Omega)$. Now, defining g_1,g_2 by

$$\langle g_1,\varphi\rangle=\int_\Omega f_1\cdot\nabla'\varphi\,dx,\quad\langle g_2,\varphi\rangle=\int_\Omega f_2\partial_n\varphi\,dx,$$

we get $g = g_1 + g_2, g_1 \in L^q(\mathbb{R}; \widehat{W}^{-1,r}(\Sigma)), g_2 \in \widehat{W}^{-1,q}(\mathbb{R}; L^r(\Sigma))$ and

$$\|g_1\|_{L^q(\mathbb{R};\widehat{W}^{-1,r}(\Sigma))} \le \|f_1\|_{L^q(\mathbb{R};L^r(\Sigma))}, \quad \|g_2\|_{\widehat{W}^{-1,q}(\mathbb{R};L^r(\Sigma))} \le \|f_2\|_{L^q(\mathbb{R};L^r(\Sigma))}.$$

Hence $\widehat{W}^{-1;q,r}(\Omega)$ is continuously embedded in $L^q(\mathbb{R}; \widehat{W}^{-1,r}(\Sigma)) + \widehat{W}^{-1,q}(\mathbb{R}; L^r(\Sigma))$. The continuity of the other embedding is trivial.

Proof of Theorem 1.1: To prove the existence of a solution, it is enough to consider the case $f = 0$, $g \in \mathcal{S}(\mathbb{R}; W^{1,2}(\Sigma)) \cap \widehat{W}^{-1;q,2}(\Omega)$. Actually, the theorem is already proved for the case $f \ne 0$, $g = 0$, see [16], Theorem 1.1. Moreover, we mention that $\mathcal{S}(\mathbb{R}; W^{1,2}(\Sigma)) \cap \widehat{W}^{-1;q,2}(\Omega)$ is dense in $W^{1;q,2}(\Omega) \cap \widehat{W}^{-1;q,2}(\Omega)$; for the proof standard techniques as in [24], Ch. I, 1.2, may be used.

By [16], Theorem 3.4, for every $\xi \in \mathbb{R}^*$ and $\lambda \in -\alpha + S_\varepsilon$ the parametrized Stokes system $(R_{\lambda,\xi})$ with $F = \hat{f} = 0$ and $G = \hat{g} \in W^{1,2}(\Sigma)$, see the Introduction, has a unique solution

$$(U_G, P_G) := (U_G(\xi), P_G(\xi)) \in (W^{2,2}(\Sigma) \cap W_0^{1,2}(\Sigma)) \times W^{1,2}(\Sigma)$$

such that

$$\begin{aligned} &\|(\lambda + \alpha)U_G, \xi^2 U_G, \xi\nabla' U_G, \nabla'^2 U_G, \xi P_G, \nabla' P_G\|_{2;\Sigma} \\ &\qquad \le c\big(\|\nabla' G, G, \xi G\|_{2;\Sigma} + (|\lambda| + 1)\|G; L^2_m + L^2_{1/\xi}\|_0\big), \end{aligned} \tag{3.3}$$

and, by [16], Corollary 3.6,

$$\begin{aligned} &\big\|\xi \tfrac{d}{d\xi}\left((\lambda + \alpha)U_G, \xi^2 U_G, \xi\nabla' U_G, \nabla'^2 U_G, \xi P_G, \nabla' P_G\big)\right\|_{2;\Sigma} \\ &\qquad \le c\big(\|\nabla' G, G, \xi G\|_{2;\Sigma} + (|\lambda| + 1)\|G; L^2_m + L^2_{1/\xi}\|_0\big); \end{aligned} \tag{3.4}$$

here the constant $c = c(\alpha, \varepsilon, \Sigma) > 0$ is independent of $\lambda \in -\alpha + S_\varepsilon, \xi \in \mathbb{R}^*$, and

$$\begin{aligned} &\|G; L^2_m + L^2_{1/\xi}\|_0 \\ &:= \inf\big\{\|G_0\|_{\widehat{W}^{-1,2}(\Sigma)} + \|G_1/\xi\|_{2;\Sigma};\ G = G_0 + G_1, G_0 \in L^2_m(\Sigma), G_1 \in L^2(\Sigma)\big\}. \end{aligned} \tag{3.5}$$

Moreover, if $\int_\Sigma G\, dx' = 0$, on the right-hand sides of (3.3) and (3.4) the factor $|\lambda|+1$ may be replaced by $|\lambda|$. Therefore, the operator $M(\xi) : W^{1,2}(\Sigma) \to L^2(\Sigma)$, defined for $\xi \in \mathbb{R}^*$ by

$$M(\xi)G := ((\lambda + \alpha)U_G, \xi^2 U_G, \xi\nabla' U_G, \nabla'^2 U_G, \xi P_G, \nabla' P_G),$$

is Frechét differentiable in $\xi \in \mathbb{R}^*$ and satisfies the estimates

$$\|M(\xi)G, \xi M'(\xi)G\|_{2,\Sigma} \le c(\alpha, \varepsilon, \Sigma)\big(\|\nabla' G, G, \xi G\|_{2;\Sigma} + (|\lambda| + 1)\|G; L^2_m + L^2_{1/\xi}\|_0\big) \tag{3.6a}$$

and, if $\int_\Sigma G\, dx' = 0$,

$$\|M(\xi)G, \xi M'(\xi)G\|_{2,\Sigma} \le c(\alpha, \varepsilon, \Sigma)\big(\|\nabla' G, G, \xi G\|_{2;\Sigma} + |\lambda|\|G; L^2_m + L^2_{1/\xi}\|_0\big). \tag{3.6b}$$

Let

$$u := \mathcal{F}_\xi^{-1} U_{\hat{g}(\xi)}, \quad p := \mathcal{F}_\xi^{-1} P_{\hat{g}(\xi)}. \tag{3.7}$$

We shall show that $\{u, p\}$ is the unique solution to (R_λ) satisfying (1.1). Obviously $\{u, p\}$ solves (R_λ) with right-hand side $(0, g)$ in the sense of distributions. For the proof of (1.1), we may assume without loss of generality that $\operatorname{supp} \hat{g} \subset [0, \infty)$ due to the relation

$$\begin{aligned} g(x', x_n) &= (\chi_{[0,\infty)}\hat{g}(\xi))^\vee(x', x_n) + (\chi_{(-\infty,0]}\hat{g}(\xi))^\vee(x', x_n) \\ &= (\chi_{[0,\infty)}\hat{g}(\xi))^\vee(x', x_n) + (\chi_{[0,\infty)}\hat{g}(-\xi))^\vee(x', -x_n) \end{aligned}$$

and due to the linearity of the problem (R_λ). Since $((\lambda + \alpha)u, \nabla^2 u, \nabla p) = (M(\xi)\hat{g}(\xi))^\vee$, our aim is to estimate $\|(M(\xi)\hat{g}(\xi))^\vee\|_{L^q(\mathbb{R}, L^2(\Sigma))}$.

For notational convenience, we introduce the space

$$\begin{aligned} \mathcal{X} &= W^{1;q,2}(\Omega) \cap \widehat{W}^{-1;q,2}(\Omega) \\ &= \big(W^{1,q}(\mathbb{R}; L^2(\Sigma)) \cap L^q(\mathbb{R}; W^{1,2}(\Sigma))\big) \cap \big(\widehat{W}^{-1,q}(\mathbb{R}; L^2(\Sigma)) + L^q(\mathbb{R}; \widehat{W}^{-1,2}(\Sigma))\big). \end{aligned}$$

As mentioned in Section 2 the operator family $\{\Delta_j = \mathcal{F}^{-1}\chi_{[2^j,2^{j+1})}(\xi)\mathcal{F}; j \in \mathbb{Z}\}$ is easily seen to be a Schauder decomposition of $R\mathcal{X}$, the image of $\mathcal{X}$ by the Riesz projection R; hence $g = \sum_{j\in\mathbb{Z}} \Delta_j g$ in $\mathcal{X}$. Moreover, for $s \in \mathbb{R}$ we define

$$R_s = \mathcal{F}^{-1}\chi_{[s,\infty)}\mathcal{F}.$$

Note that $M(\xi) = M(2^j) + \int_{2^j}^{\xi} M'(\tau)\, d\tau$ for $\xi \in [2^j, 2^{j+1}), j \in \mathbb{Z}$, and that obviously $(M(2^j)\widehat{\Delta_j g})^\vee = M(2^j)\Delta_j g$; furthermore,

$$\begin{aligned} \Big(\int_{2^j}^{\xi} M'(\tau)\, d\tau\, \widehat{\Delta_j g}(\xi)\Big)^\vee &= \Big(\int_{2^j}^{2^{j+1}} M'(\tau)\chi_{[2^j,\xi)}(\tau)\widehat{\Delta_j g}(\xi)\, d\tau\Big)^\vee \\ &= \Big(\int_0^1 2^j M'(2^j(1+t))\chi_{[2^j,\xi)}(2^j(1+t))\chi_{[2^j,2^{j+1})}(\xi)\hat{g}(\xi)\, dt\Big)^\vee \\ &= \int_0^1 2^j M'(2^j(1+t)) \int_{2^j(1+t)}^{2^{j+1}} \hat{g}(\xi)e^{ix_n\xi}\, d\xi\, dt \\ &= \int_0^1 2^j M'(2^j(1+t))(R_{2^j(1+t)} - R_{2^{j+1}})\Delta_j g\, dt. \end{aligned}$$

Thus we get

$$\begin{aligned} \big(M(\xi)\hat{g}(\xi)\big)^\vee &= \Big(\sum_{j\in\mathbb{Z}} \chi_{[2^j,2^{j+1})}(\xi)M(\xi)\widehat{\Delta_j g}\Big)^\vee \\ &= \sum_{j\in\mathbb{Z}} \Big(\big(M(2^j) + \int_{2^j}^{\xi} M'(\tau)\, d\tau\big)\, \widehat{\Delta_j g}\Big)^\vee \\ &= \sum_{j\in\mathbb{Z}} \big(M(2^j)\widehat{\Delta_j g}\big)^\vee + \sum_{j\in\mathbb{Z}} \Big(\int_{2^j}^{\xi} M'(\tau)\, d\tau\, \widehat{\Delta_j g}\Big)^\vee \\ &= \sum_{j\in\mathbb{Z}} M(2^j)\Delta_j g + \sum_{j\in\mathbb{Z}} \int_0^1 2^j M'(2^j(1+t))(R_{2^j(1+t)} - R_{2^{j+1}})\Delta_j g\, dt. \end{aligned} \tag{3.8}$$

To estimate the first term on the right-hand side of (3.8) in $L^q(\mathbb{R};L^2(\Sigma))$, note that for each $j\in\mathbb{Z}$ the operator $M(2^j)$ commutes with Δ_j and that $\{\Delta_j; j\in\mathbb{Z}\}$ is a Schauder decomposition of $RL^q(\mathbb{R};L^2(\Sigma))$. Then Lemma 2.6 and (3.3) yield the estimate

$$\begin{aligned}
&\big\|\sum_{j=l}^{k} M(2^j)\Delta_j g\big\|_{L^q(\mathbb{R};L^2(\Sigma))}\\
&\qquad\le c\,\big\|\big(\sum_{j=l}^{k}\|M(2^j)\Delta_j g\|_{2;\Sigma}^2\big)^{1/2}\|_{q;\mathbb{R}}\\
&\qquad\le c\,\Big(\big\|\big(\sum_{j=l}^{k}\|\Delta_j g\|_{1,2;\Sigma}^2\big)^{1/2}\big\|_{q,\mathbb{R}}+\big\|\big(\sum_{j=l}^{k}2^{2j}\|\Delta_j g\|_{2;\Sigma}^2\big)^{1/2}\big\|_{q,\mathbb{R}}\\
&\qquad\quad+(|\lambda|+1)\big\|\big(\sum_{j=l}^{k}\|\Delta_j g;L^2_m+L^2_{1/2^j}\|_0^2\big)^{1/2}\big\|_{q,\mathbb{R}}\Big)
\end{aligned}\tag{3.9}$$

with $c=c(\alpha,\varepsilon,q,\Sigma)$.

Now, let us estimate each term on the right-hand side of (3.9). Again, using Lemma 2.6, we get

$$\big\|\big(\sum_{j=l}^{k}\|\Delta_j g\|_{1,2;\Sigma}^2\big)^{1/2}\big\|_{q,\mathbb{R}}\le c(q,\Sigma)\big\|\sum_{j=l}^{k}\Delta_j g\big\|_{L^q(\mathbb{R};W^{1,2}(\Sigma))}.\tag{3.10}$$

By analogy, exploiting also Lemma 2.8,

$$\begin{aligned}
\big\|\big(\textstyle\sum_{j=l}^{k}2^{2j}\|\Delta_j g\|_{2;\Sigma}^2\big)^{1/2}\big\|_{q,\mathbb{R}} &\le c(q,\Sigma)\big\|\textstyle\sum_{j=l}^{k}2^j\Delta_j g\big\|_{L^q(\mathbb{R};L^2(\Sigma))}\\
&\le c(q,\Sigma)\big\|\textstyle\sum_{j=l}^{k}\Delta_j g\big\|_{\widehat{W}^{1,q}(\mathbb{R};L^2(\Sigma))}.
\end{aligned}\tag{3.11}$$

In order to get an estimate of the last term on the right-hand side of (3.9), let

$$\sum_{j=l}^{k}\Delta_j g=g_0+g_1,\quad g_0\in L^q(\mathbb{R};\widehat{W}^{-1,2}(\Sigma)),\ g_1\in\widehat{W}^{-1,q}(\mathbb{R};L^2(\Sigma)),$$

be any splitting of $\sum_{j=l}^{k}\Delta_j g$. Note that $\Delta_j g=\Delta_j g_0+\Delta_j g_1$ for all $j=l,\dots,k$, and moreover, by Lemma 2.7, that $\Delta_j g_1\in L^q(\mathbb{R};L^2(\Sigma))$ and consequently even $\Delta_j g_0\in L^q(\mathbb{R};\widehat{W}^{-1,2}(\Sigma)\cap L^2(\Sigma))=L^q(\mathbb{R};L^2_m(\Sigma))$. By the triangle inequality and Lemma 2.6 applied also in the Hilbert space $\widehat{W}^{-1,2}(\Sigma)$ we get that

$$\begin{aligned}
&\big\|\big(\sum_{j=l}^{k}\|\Delta_j g;L^2_m+L^2_{1/2^j}\|_0^2\big)^{1/2}\big\|_{q,\mathbb{R}}\\
&\qquad\le\big\|\big(\sum_{j=l}^{k}\|\Delta_j g_0\|_{-1,2;\Sigma}^2\big)^{1/2}\big\|_{q,\mathbb{R}}+\big\|\big(\sum_{j=l}^{k}2^{-2j}\|\Delta_j g_1\|_{2;\Sigma}^2\big)^{1/2}\big\|_{q,\mathbb{R}}\\
&\qquad\le c\Big(\big\|\sum_{j=l}^{k}\Delta_j g_0\big\|_{L^q(\mathbb{R};\widehat{W}^{-1,2}(\Sigma))}+\big\|\sum_{j=l}^{k}2^{-j}\Delta_j g_1\big\|_{L^q(\mathbb{R};L^2(\Sigma))}\Big).
\end{aligned}$$

Then Lemma 2.8, Lemma 2.10 and (3.2) imply the estimate

$$\begin{aligned}
&\big\|\big(\sum_{j=l}^{k}\|\Delta_j g; L^2_m + L^2_{1/2^j}\|_0^2\big)^{1/2}\big\|_{q,\mathbb{R}}\\
&\le c\Big(\big\|\sum_{j=l}^{k}\Delta_j g_0\big\|_{L^q(\mathbb{R};\widehat{W}^{-1,2}(\Sigma))} + \big\|\sum_{j=l}^{k}\Delta_j g_1\big\|_{\widehat{W}^{-1,q}(\mathbb{R};L^2(\Sigma))}\Big)\\
&\le c\big(\|g_0\|_{L^q(\mathbb{R};\widehat{W}^{-1,2}(\Sigma))} + \|g_1\|_{\widehat{W}^{-1,q}(\mathbb{R};L^2(\Sigma))}\big)\\
&\le c(q,\Sigma)\big\|\sum_{j=l}^{k}\Delta_j g\big\|_{\widehat{W}^{-1;q,2}(\Omega)}.
\end{aligned} \tag{3.12}$$

with $c = c(q,\Sigma)$ independent of $l, k \in \mathbb{Z}$. Summarizing (3.9)-(3.12), we get that

$$\begin{aligned}
&\big\|\textstyle\sum_{j=l}^{k} M(2^j)\Delta_j g\big\|_{L^q(\mathbb{R};L^2(\Sigma))}\\
&\qquad\le c\big(\big\|\textstyle\sum_{j=l}^{k}\Delta_j g\big\|_{W^{1;q,2}(\Omega)} + (|\lambda|+1)\big\|\textstyle\sum_{j=l}^{k}\Delta_j g\big\|_{\widehat{W}^{-1;q,2}(\Omega)}\big)
\end{aligned} \tag{3.13}$$

with $c = c(\alpha,\varepsilon,q,\Sigma)$ for all $l, k \in \mathbb{Z}$ and for all $\lambda \in -\alpha + S_\varepsilon$. Since $(\Delta_j)_{j\in\mathbb{Z}}$ defines unconditional Schauder decompositions of $RW^{1;q,2}(\Omega)$ and of $R\widehat{W}^{-1;q,2}(\Omega)$, (3.13) implies that the series $\sum_{j\in\mathbb{Z}} M(2^j)\Delta_j g$ converges in $L^q(\mathbb{R};L^2(\Sigma))$ and

$$\Big\|\sum_{j\in\mathbb{Z}} M(2^j)\Delta_j g\Big\|_{L^q(\mathbb{R};L^2(\Sigma))} \le c\big(\|g\|_{W^{1;q,2}(\Omega)} + (|\lambda|+1)\|g\|_{\widehat{W}^{-1;q,2}(\Omega)}\big)$$

with $c = c(\alpha,\varepsilon,q,\Sigma)$. This is the desired estimate of the first term on the right-hand side of (3.8).

Next let us estimate the second term on the right-hand side of (3.8). Note that the operator family

$$\{R_{2^j(1+t)} - R_{2^{j+1}} : j \in \mathbb{N}, t \in (0,1)\} \subset \mathcal{L}(L^q(\mathbb{R};L^2(\Sigma)))$$

is R-bounded, cf. Lemma 2.10. Moreover, for $t \in (0,1)$, the operator $M(2^j(1+t))$ commutes with the operator $B_{j,t} := R_{2^j(1+t)} - R_{2^{j+1}}$ and the range of $B_{j,t}$ is contained in the range of Δ_j. Hence it follows from (2.3), (2.4) that for any independent symmetric $\{-1,1\}$-valued random variables $\{\varepsilon_j(s)\}$ on $(0,1)$

$$\begin{aligned}
&\big\|\sum_{j=l}^{k}\int_0^1 2^j M'(2^j(1+t))B_{j,t}\Delta_j g\,dt\big\|_{L^q(\mathbb{R};L^2(\Sigma))}\\
&\le \int_0^1 \big\|\sum_{j=l}^{k} 2^j B_{j,t}M'(2^j(1+t))\Delta_j g\big\|_{L^q(\mathbb{R};L^2(\Sigma))}\,dt\\
&\le c_\Delta \int_0^1 \big\|\sum_{j=l}^{k}\varepsilon_j(s)2^j B_{j,t}M'(2^j(1+t))\Delta_j g\big\|_{L^q(0,1;L^q(\mathbb{R};L^2(\Sigma)))}\,dt\\
&\le c\int_0^1 \big\|\sum_{j=l}^{k}\varepsilon_j(s)2^j M'(2^j(1+t))\Delta_j g\big\|_{L^q(0,1;L^q(\mathbb{R};L^2(\Sigma)))}\,dt.
\end{aligned} \tag{3.14}$$

By similar arguments as in the proof of Lemma 2.6 we proceed in (3.14) as follows:

$$\begin{aligned} &\le c\int_0^1 \Big\| \sum_{j=l}^k \varepsilon_j(s) 2^j M'(2^j(1+t))\Delta_j g\Big\|_{L^q(\mathbb{R};L^2(0,1;L^2(\Sigma)))}\,dt \\ &\le c\int_0^1 \Big\| \Big(\sum_{j=l}^k \|2^j(1+t)M'(2^j(1+t))\Delta_j g\|^2_{2,\Sigma}\Big)^{1/2}\Big\|_{q,\mathbb{R}}\,dt \end{aligned} \tag{3.15}$$

with $c = c(q,\Sigma)$. Therefore it follows from (3.6a) and the arguments leading from (3.9) to (3.13) that

the r.h.s. of (3.15)

$$\begin{aligned} &\le c(\alpha,\varepsilon,q,\Sigma)\Big(\int_0^1 \Big\|\Big\{\sum_{j=l}^k \big[\|\Delta_j g\|^2_{W^{1,2}(\Sigma)} + 2^{2j}(1+t)^2\|\Delta_j g\|^2_{2;\Sigma} \\ &\qquad\qquad + |\lambda+1|^2\|\Delta_j g; L^2_m + L^2_{2^{-j}(1+t)^{-1}}\|^2_0\big]\Big\}^{1/2}\Big\|_{q,\mathbb{R}}\,dt\Big) \\ &\le c(\alpha,\varepsilon,q,\Sigma)\Big(\Big\|\Big(\sum_{j=l}^k \|\Delta_j g\|^2_{W^{1,2}(\Sigma)}\Big)^{1/2}\Big\|_{q,\mathbb{R}} + \Big\|\Big(\sum_{j=l}^k 2^{2j}\|\Delta_j g\|^2_{2;\Sigma}\Big)^{1/2}\Big\|_{q,\mathbb{R}} \\ &\qquad\qquad + |\lambda+1|\Big\|\Big(\sum_{j=l}^k \|\Delta_j g; L^2_m + L^2_{2^{-j}}\|^2_0\Big)^{1/2}\Big\|_{q,\mathbb{R}}\Big) \\ &\le c(\alpha,\varepsilon,q,\Sigma)\big(\|g\|_{W^{1;q,2}(\Omega)} + (|\lambda|+1)\|g\|_{\widehat{W}^{-1;q,2}(\Omega)}\big). \end{aligned}$$

Thus we finally proved the existence of a solution satisfying the estimate (1.1). It is clear that, if $\int_\Sigma g(x',\cdot)\,dx' = 0$, the solution satisfies the estimate (1.2); for the proof (3.6b) is used in place of (3.6a).

The uniqueness of solution is obvious from the uniqueness result for $f \neq 0, g = 0$, see [16]. The proof of the theorem is complete. ■

Remark 3.1 Theorem 1.1 may be applied to obtain resolvent estimates of the Stokes system for more general domains, e.g. for unbounded cylindrical domains with several outlets to infinity. Let $\Omega = \bigcup_{i=0}^m \Omega_i$ be a cylindrical domain of $C^{1,1}$-class such that Ω_0 is a bounded domain and $\Omega_i, i = 1,\dots,m$, are semi-infinite straight cylinders with boundaries of $C^{1,1}$-class; to be more precise, for each $i = 1,\dots,m$, we may find orthogonal coordinates $x^i = (x^i_1,\dots,x^i_n)$ such that

$$\Omega_i = \{x^i \in \mathbb{R}^n; x^i_n > 0, (x^i_1,\dots,x^i_{n-1}) \in \Sigma_i\}$$

and $\Omega_i \cap \Omega_j = \emptyset$ for $i,j = 1,\dots,m$ with $i \neq j$. Without loss of generality we may assume that there exist cut-off functions $\{\varphi_i\}_{i=0}^m$ such that

$$\begin{aligned} &\textstyle\sum_{i=0}^m \varphi_i(x) = 1, \quad 0 \le \varphi_i(x) \le 1 \quad \text{for } x \in \Omega, \\ &\varphi_i \in C^\infty(\bar\Omega_i), \quad \operatorname{supp}\varphi_i \subset \bar\Omega_i \setminus (\partial\Omega_i \cap \Omega),\ i = 0,\dots,m. \end{aligned}$$

Now consider the resolvent system

$$(R_\lambda) \qquad \begin{aligned} \lambda u - \Delta u + \nabla p &= f \quad \text{in } \Omega \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

and let $\{u,p\}$ be a solution to (R_λ). Then we are led to a resolvent system with unknown $\{\varphi_0 u, \varphi_0 p\}$ on Ω_0,

$$(R_\lambda)_0 \qquad \begin{aligned} \lambda(\varphi_0 u) - \Delta(\varphi_0 u) + \nabla(\varphi_0 p) &= f^0 \quad \text{in } \Omega_0 \\ \operatorname{div}(\varphi_0 u) &= g^0 \quad \text{in } \Omega_0 \\ \varphi_0 u &= 0 \quad \text{on } \partial\Omega_0, \end{aligned}$$

where

$$f^0 := \varphi_0 f + (\nabla\varphi_0)p - (\Delta\varphi_0)u - 2\nabla\varphi_0 \cdot \nabla u, \quad g^0 := \nabla\varphi_0 \cdot u,$$

and a finite number of resolvent systems with unknowns $\{\widetilde{\varphi_i u}, \widetilde{\varphi_i p}\}$ on $\widetilde{\Omega}_i, i = 1, \dots, m$,

$$(R_\lambda)_i \qquad \begin{aligned} \lambda(\widetilde{\varphi_i u}) - \Delta(\widetilde{\varphi_i u}) + \nabla(\widetilde{\varphi_i p}) &= \tilde{f}^i \quad \text{in } \widetilde{\Omega}_i \\ \operatorname{div}(\widetilde{\varphi_i u}) &= \tilde{g}^i \quad \text{in } \widetilde{\Omega}_i \\ \widetilde{\varphi_i u} &= 0 \quad \text{on } \partial\widetilde{\Omega}_i, \end{aligned}$$

where $\widetilde{\Omega}_i$ is the infinite straight cylinder extending the semi-infinite cylinder Ω_i; moreover, $\widetilde{\varphi_i u}$, $\widetilde{\varphi_i p}$, $\tilde{f}^i, \tilde{g}^i$ are zero extensions onto $\widetilde{\Omega}_i$ of functions $\varphi_i u, \varphi_i p$,

$$f^i := \varphi_i f + (\nabla\varphi_i)p - (\Delta\varphi_i)u - 2\nabla\varphi_i \cdot \nabla u, \quad g^i := \nabla\varphi_i \cdot u,$$

respectively. Obviously $\int_{\Omega_0} g^0\, dx = 0, \int_{\widetilde{\Omega}_i} \tilde{g}^i\, dx = 0, i = 1, \cdots, m$. Then, under adequate assumptions on f, using the results for Stokes resolvent systems on bounded domains (see e.g. [12]) for $(R_\lambda)_0$ and Theorem 1.1 for $(R_\lambda)_i$, $i = 1, \dots, m$, we may obtain *a priori* estimates for $\{\varphi_0 u, \varphi_0 p\}$ and $\{\widetilde{\varphi_i u}, \widetilde{\varphi_i p}\}, i = 1, \cdots, m$ with norms of lower order terms on the right-hand side. Finally we get estimates for $u = \sum_{i=0}^m \varphi_i u$ and $p = \sum_{i=0}^m \varphi_i p$ using a well known contradiction argument, see [12].

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