KOLMOGOROV KERNEL ESTIMATES FOR THE ORNSTEIN-UHLENBECK OPERATOR

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Abstract. Replacing the Gaussian semigroup in the heat kernel estimates by the Ornstein-Uhlenbeck semigroup on $\mathbb{R}^d$, we define the notion of Kolmogorov kernel estimates. This allows us to show that under Dirichlet boundary conditions Ornstein-Uhlenbeck operators are generators of consistent, positive, (quasi-)contractive C_0-semigroups on $L^p(\Omega)$ for all $1 \leq p < \infty$ and for every domain $\Omega \subseteq \mathbb{R}^d$. For exterior domains with sufficiently smooth boundary a result on the location of the spectrum of these operators is also given.

1. Introduction

Heat kernel estimates have proved to be a powerful tool for the analysis of elliptic differential operators. Beside many other things, they allow the extension of a given semigroup on L^{p_0} for some p_0 to the whole scale of L^p-spaces for $1 \leq p < \infty$ in a consistent way, transfering certain nice properties to all these semigroups, such as analyticity, and yielding p-invariance of the spectrum.

Dealing with Ornstein-Uhlenbeck operators on $L^p(\Omega)$ for unbounded domains Ω we evidently cannot expect to get heat kernel estimates, as the spectrum of these operators in $L^p(\mathbb{R}^d)$ already depends heavily on p. Nevertheless, the well-known representation for the Ornstein-Uhlenbeck semigroup on $L^p(\mathbb{R}^d)$ that is due to A. N. Kolmogorov and given by

$$T(t)f(x) = \int_{\mathbb{R}^d} k_t(\mathrm{e}^{tB}x - y)f(y)\,\mathrm{d}y = (k_t * f)(\mathrm{e}^{tB}x),$$

where the Kolmogorov kernel

$$k_t(x) = \frac{1}{(4\pi)^{\frac{d}{2}}(\det Q_t)^{\frac{1}{2}}} \exp\left(-\frac{1}{4}Q_t^{-1}x \cdot x\right) \quad \text{with} \quad Q_t = \int_0^t \mathrm{e}^{sB}Q\mathrm{e}^{sB^*}\,\mathrm{d}s$$

looks very similar to the heat semigroup. So it is a natural idea to consider Kolmogorov kernel estimates, by replacing the Gaussian semigroup by the Kolmogorov semigroup $(T(t))_{t\geq 0}$, thus getting a majorising semigroup that is well adapted to operators of the Ornstein-Uhlenbeck type. These estimates then allow us to extend semigroups on L^{p_0} to the whole scale of L^p for $1 \leq p < \infty$, analogously to the case of heat kernel bounds.

Having established this idea, in the sequel we apply it to Ornstein-Uhlenbeck operators with Dirichlet boundary conditions in $L^p(\Omega)$, where $1 \leq p < \infty$ and Ω is a domain in $\mathbb{R}^d$. Ornstein-Uhlenbeck operators are differential operators, formally given by

$$(\mathcal{A}u)(x) = \sum_{i,j=1}^d q_{ij}D_iD_ju(x) + Bx \cdot \nabla u(x), \tag{1}$$

where $Q = (q_{ij})_{i,j=1}^d \in \mathbb{R}^{d\times d}$ is a symmetric and positive definite matrix and $B = (b_{ij})_{i,j=1}^d \in \mathbb{R}^{d\times d} \setminus \{0\}$. They first appeared in stochastical analysis, describing a Brownian motion with an additional drift. In this context one usually works in

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spaces of continuous functions on $\mathbb{R}^d$ or in the spaces $L^p(\mathbb{R}^d, \mu)$, where μ is the invariant measure of the underlying process.

Recently, it became clear that an analytic treatment of these operators is of great interest. For instance, looking at the Stokes equation in the exterior of a rotating obstacle leads to operators of the Ornstein-Uhlenbeck type, see [13], [12] and [8]. Thus one is interested in their behaviour on $L^p(\Omega)$ with respect to the Lebesgue measure for domains and especially exterior domains.

Passing from the invariant measure to the Lebesgue measure changes the properties of the operator completely. The spectrum is no longer contained in the negative real axis, instead it contains a vertical line (cf. [14]), so the semigroup $(T(t))_{t\geq 0}$ on $L^p(\mathbb{R}^d)$ is not analytic, as it is in $L^p(\mathbb{R}^d, \mu)$ (cf. [15]), and it even fails to be eventually norm-continuous.

Whereas Ornstein-Uhlenbeck operators are well understood in $L^p(\mathbb{R}^d)$ (cf. [14], [17], [16]) and for bounded domains, where they can be viewed as a perturbation of lower order of the elliptic diffusion part, there are very few results for unbounded domains. M. Geissert, H. Heck, M. Hieber and I. Wood showed in [9] that in the case of an exterior domain Ω with sufficiently smooth boundary, a realisation of $\mathcal{A}$ on $L^p(\Omega)$ generates a C_0-semigroup and G. da Prato and A. Lunardi treat the case of L^2-spaces of convex sets with respect to Neumann boundary conditions and infinitesimally invariant measures in [4]. The Dirichlet problem in spaces of bounded continuous functions on smooth domains is treated by S. Fornaro, G. Metafune and E. Priola in [7]. An overview may be found in [3].

In this paper we show that for arbitrary domains $\Omega \subseteq \mathbb{R}^d$ a realisation of $\mathcal{A}$ in $L^2(\Omega)$ generates a (quasi-)contractive, positive C_0-semigroup, that has a Kolmogorov kernel estimate, see Theorem 3.2. In the sequel this allows us to define consistent Ornstein-Uhlenbeck semigroups on $L^p(\Omega)$ for $1 \leq p < \infty$ that have the same contractivity, positivity and domination properties. This immediately gives an upper bound on the growth bound of the semigroups and implies that the Ornstein-Uhlenbeck operator admits a bounded H^∞-calculus on $L^p(\Omega)$ (Proposition 4.5).

In the special case of an exterior domain with sufficiently smooth boundary, it turns out that the domain is

$$W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \cap \{f \in L^p(\Omega) : Bx \cdot \nabla f \in L^p(\Omega)\}$$

and we even deduce in Theorem 5.2 that the same vertical line as in the case of the whole space is contained in the spectrum of the operator, so the spectral behaviour is the same as for the case $\Omega = \mathbb{R}^d$. This means that also in this case the semigroup is not eventually norm-continuous. Nevertheless we can show that its growth bound and the spectral bound of its generator coincide, which is no longer clear by standard spectral theory for semigroups.

The paper is organised as follows. In section 2 we introduce the notion of Kolmogorov kernel estimates and prove their main implications. The generation result for $\mathcal{A}$ in $L^2(\Omega)$ is contained in section 3 and in section 4 we show that this semigroup is positive and admits a Kolmogorov kernel estimate. Section 5 finally contains the results for exterior domains.

Notations. Throughout this paper we use the following notation.

For a closed operator $(A, D(A))$ on some Banach space X we denote by $\sigma(A)$ the spectrum, by $\varrho(A)$ the resolvent set and by $R(\lambda, A) = (\lambda - A)^{-1}$, $\lambda \in \varrho(A)$, the resolvent of A. Furthermore, the space of all bounded linear operators on X is denoted by $\mathcal{L}(X)$.

As usual, for $\Omega \subseteq \mathbb{R}^d$ open, $\|\cdot\|_p$ stands for the norm of the Lebesgue spaces $L^p(\Omega)$ whenever the set Ω is clear from the context. We write $W^{k,p}(\Omega)$, or $H^k(\Omega)$ in the case $p = 2$, for the Sobolev spaces, $C_c^\infty(\Omega)$ for the space of all smooth functions having compact support in Ω and $W_0^{1,p}(\Omega)$, or $H_0^1(\Omega)$, is the closure of $C_c^\infty(\Omega)$ in the norm of $W^{1,p}(\Omega)$ or $H^1(\Omega)$, respectively. Furthermore, if X is a function space, X_+ stands for the cone of all positive functions in X.

Finally, $B_r(x_0)$ is the open ball of radius r with centre x_0 and, given a matrix $B \in \mathbb{R}^{d\times d}$, we write $\operatorname{tr}(B) = \sum_{j=1}^d b_{jj}$ for its trace.

2. Kolmogorov kernel estimates

Given a matrix $B \in \mathbb{R}^{d\times d} \setminus \{0\}$ and a positive definite matrix $Q \in \mathbb{R}^{d\times d}$, we define the Kolmogorov semigroup $(K_p(t))_{t\geq 0}$, on $L^p(\mathbb{R}^d)$ by

$$\big(K_p(t)f\big)(x) = \int_{\mathbb{R}^d} k_t(\mathrm{e}^{tB}x - y)f(y)\,\mathrm{d}y, \qquad f \in L^p(\mathbb{R}^d),$$

where the Kolmogorov kernel k_t is given by

$$k_t(x) = \frac{1}{(4\pi)^{\frac{d}{2}}(\det Q_t)^{\frac{1}{2}}} \exp\left(-\frac{1}{4}Q_t^{-1}x\cdot x\right) \quad \text{with} \quad Q_t = \int_0^t \mathrm{e}^{sB}Q\mathrm{e}^{sB^*}\,\mathrm{d}s.$$

It is well known (cf. [14]) that $(K_p(t))_{t\geq 0}$ is a positive C_0-semigroup on $L^p(\mathbb{R}^d)$ for every $1 \leq p < \infty$ and it is straightforward by substitution and Young's inequality that for every $f \in L^p(\mathbb{R}^d)$

$$\|K_p(t)f\|_p = \left(\int_{\mathbb{R}^d} \left|(k_t * f)(\mathrm{e}^{tB}x)\right|^p \mathrm{d}x\right)^{\frac{1}{p}} = \mathrm{e}^{-\frac{\operatorname{tr}(B)}{p}t}\|k_t * f\|_p \leq \mathrm{e}^{-\frac{\operatorname{tr}(B)}{p}t}\|f\|_p, \quad (2)$$

since $\|k_t\|_1 = 1$. The generator $A_{\mathbb{R}^d,p}$ of this semigroup is the Ornstein-Uhlenbeck operator $\mathcal{A}$ given in (1) and G. Metafune, J. Prüss, A. Rhandi, and R. Schnaubelt showed in [17] that its domain is

$$D(A_{\mathbb{R}^d,p}) = W^{2,p}(\mathbb{R}^d) \cap \{f \in L^p(\mathbb{R}^d) : Bx\cdot\nabla f \in L^p(\mathbb{R}^d)\} \quad (3)$$

for $1 < p < \infty$.

Now, let $\Omega \subseteq \mathbb{R}^d$ be an open set, $1 \leq p < \infty$ and let $(T(t))_{t\geq 0}$ be a C_0-semigroup on $L^p(\Omega)$ with generator A.

Definition 2.1. *We say, that the semigroup $(T(t))_{t\geq 0}$ on $L^p(\Omega)$ satisfies a* Kolmogorov kernel estimate, *if there exist a matrix $B \in \mathbb{R}^{d\times d} \setminus \{0\}$, a positive definite matrix $Q \in \mathbb{R}^{d\times d}$, $M \geq 0$ and $\omega \in \mathbb{R}$, such that for all $f \in L^p(\Omega)$ and all $t \geq 0$ we have the pointwise estimate*

$$|T(t)f| \leq M\mathrm{e}^{\omega t}K_p(t)|\tilde{f}|,$$

where $\tilde{f}$ denotes the trivial extension of f to $\mathbb{R}^d$.

For such semigroups we have the following result.

Proposition 2.2. *Let $\Omega \subseteq \mathbb{R}^d$ be open, $1 \leq p < \infty$ and let $(T(t))_{t\geq 0}$ be a C_0-semigroup on $L^p(\Omega)$ that satisfies a Kolmogorov kernel estimate for some matrix B, a positive definite matrix Q and constants M and ω. Then for $1 \leq q < \infty$ there exist consistent C_0-semigroups $(T_q(t))_{t\geq 0}$ on $L^q(\Omega)$, such that $T_p = T$ and*

$$\|T_q(t)\|_{\mathcal{L}(L^q(\Omega))} \leq M\mathrm{e}^{(\omega - \frac{\operatorname{tr}(B)}{q})t}$$

for every $t \geq 0$. Furthermore, T_q satisfies the same Kolmogorov kernel estimate and if T is a positive semigroup, T_q is also positive.

Proof. Let $g \in L^p(\Omega) \cap L^q(\Omega)$ and $t \geq 0$. Then, by consistency of the semigroups K_q for $1 \leq q < \infty$ and (2), we have

$$\begin{aligned}\|T(t)g\|_q &= \left(\int_\Omega |T(t)g|^q\right)^{\frac{1}{q}} \leq M\mathrm{e}^{\omega t}\left(\int_{\mathbb{R}^d}(K_p(t)|\tilde{g}|)^q\right)^{\frac{1}{q}} \\ &= M\mathrm{e}^{\omega t}\|K_q(t)|\tilde{g}|\|_q \leq M\mathrm{e}^{(\omega - \frac{\mathrm{tr}(B)}{q})t}\|g\|_q.\end{aligned}$$

Thus we can extend the operator $T(t)$ continuously to an operator $T_q(t)$ on $L^q(\Omega)$, obtaining C_0-semigroups that are consistent by construction and that obey the stated norm estimate by the above calculation.

For $f \in L^q(\Omega) \cap L^p(\Omega)$ the inequality

$$|T_q(t)f| \leq M\mathrm{e}^{\omega t}K_q(t)|\tilde{f}|$$

is immediate by consistency. Thus $M\mathrm{e}^{\omega t}K_q(t)|\tilde{f}| - |T_q(t)f|$ is positive for all these functions and the Kolmogorov kernel estimates follow for arbitrary $f \in L^q(\Omega)$, since $L^q(\Omega)_+$ is closed in $L^q(\Omega)$.

Finally, the same closedness argument yields the positivity of T_q, whenever T is positive. □

For every $1 \leq q < \infty$ we denote the generator of $(T_q(t))_{t\geq 0}$ by A_q. The Kolmogorov kernel estimates also provide consistency results for these operators and their resolvents. We collect them in the next proposition.

Proposition 2.3. *Let $1 \leq q < \infty$. Then*

(1) *$R(\lambda, A_q)f = R(\lambda, A_r)f$ for all $f \in L^q(\Omega) \cap L^r(\Omega)$, all $1 \leq r < \infty$ and all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \lambda_0 := \max(\omega - \frac{\mathrm{tr}(B)}{q}, \omega - \frac{\mathrm{tr}(B)}{r})$.*

(2) *The set $\{f \in D(A) \cap L^q(\Omega) : Af \in L^q(\Omega)\}$ is contained in $D(A_q)$ and $A_q f = Af$ for all such f.*

(3) *$|R(\lambda, A_q)f| \leq MR(\lambda, \omega + A_{\mathbb{R}^d,q})|\tilde{f}|$ for all $f \in L^q(\Omega)$ and all $\lambda > \omega - \frac{\mathrm{tr}(B)}{q}$.*

Proof. (1) By Proposition 2.2 we know, that the growth bounds of T_q and T_r are at most λ_0. Taking $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \lambda_0$, this allows us to conclude with the help of the Laplace transform

$$R(\lambda, A_q)f = \int_0^\infty \mathrm{e}^{-\lambda t}T_q(t)f \,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-\lambda t}T_r(t)f \,\mathrm{d}t = R(\lambda, A_r)f$$

for every $f \in L^q(\Omega) \cap L^r(\Omega)$ by the consistency of the semigroups.

(2) Let $f \in D(A) \cap L^q(\Omega)$ with $Af \in L^q(\Omega)$. Then, choosing $\lambda > \omega + |\mathrm{tr}(B)|$, we have $\lambda \in \varrho(A) \cap \varrho(A_q)$ and since $(\lambda - A)f \in L^p(\Omega) \cap L^q(\Omega)$, we get

$$f = R(\lambda, A)(\lambda - A)f = R(\lambda, A_q)(\lambda - A)f \in D(A_q)$$

by part (1) of this proof. The above equality also yields $A_q f = Af$.

(3) Let $\lambda > \omega - \frac{\mathrm{tr}(B)}{q}$. Then we may again use Laplace transform for $R(\lambda, A_q)$ and we obtain for $f \in L^q(\Omega)$

$$\begin{aligned}|R(\lambda, A_q)f| &= \left|\int_0^\infty \mathrm{e}^{-\lambda t}T_q(t)f \,\mathrm{d}t\right| \\ &\leq M\int_0^\infty \mathrm{e}^{-\lambda t}\mathrm{e}^{\omega t}K_q(t)|\tilde{f}| \,\mathrm{d}t = MR(\lambda, \omega + A_{\mathbb{R}^d,q})|\tilde{f}|,\end{aligned}$$

since T_q also has a Kolmogorov kernel estimate by Proposition 2.2.

□

In the special case that $M = 1$ in the kernel estimates, the operators $A_q - \omega + \operatorname{tr}(B)/q$ generate contraction semigroups for all $q \in (1,\infty)$. Since $|\operatorname{tr}(B)/q| \leq |\operatorname{tr}(B)|$ for all $q \in (1,\infty)$, the amount of the shift is bounded. Thus $A_q - \omega - |\operatorname{tr}(B)|$, $1 < q < \infty$, is a family of generators of contraction semigroups on $L^q(\Omega)$. If in addition the semigroup $(T(t))_{t\geq 0}$ is positive, by Proposition 2.2 all the semigroups are positive. This immediately yields a bounded H^∞-calculus for their generators, see [11].

Proposition 2.4. *The operators $A_q - \omega - |\operatorname{tr}(B)|$ admit a bounded H^∞-calculus for every $1 < q < \infty$, whenever $(T(t))_{t\geq 0}$ is a positive semigroup satisfying a Kolmogorov kernel estimate with $M = 1$.*

3. The Ornstein-Uhlenbeck semigroup on $L^2(\Omega)$

In the following we want to use Kolmogorov kernel estimates to show that the Ornstein-Uhlenbeck operator $\mathcal{A}$ is the generator of a positive C_0-semigroup on $L^p(\Omega)$ for every open and connected subset Ω of $\mathbb{R}^d$. As we already mentioned in the introduction, the key is a generation result for the case $p = 2$ and Kolmogorov kernel estimates. We will prove these two items in this and the following section.

We define the realisation of the Ornstein-Uhlenbeck operator in $L^2(\Omega)$ by

$$\begin{aligned}
D(A_{\Omega,2}) &= H_0^1(\Omega) \cap \{u \in H^2_{\mathrm{loc}}(\Omega) : \mathcal{A}u \in L^2(\Omega)\},\\
(\mathcal{A}u)(x) &= \sum_{i,j=1}^d q_{ij} D_i D_j u(x) + Bx \cdot \nabla u(x)\\
&= \sum_{i,j=1}^d q_{ij} D_i D_j u(x) + \sum_{i,j=1}^d b_{ij} x_j D_i u(x), \qquad x \in \Omega \subset \mathbb{R}^d,
\end{aligned}$$

where $Q = (q_{ij})_{i,j=1}^d \in \mathbb{R}^{d\times d}$ is a symmetric and positive definite matrix and $B = (b_{ij})_{i,j=1}^d \in \mathbb{R}^{d\times d} \setminus \{0\}$. Note that this notation is consistent with the definition of $A_{\mathbb{R}^d,2}$ in (3). In fact, the domain in (3) is clearly contained in the domain given here, so when we have shown in Proposition 3.5 that $A_{\Omega,2}$ is dissipative, the equality of the two domains follows by the following general observation.

Remark 3.1. Let B be the generator of a C_0-semigroup on some Banach space X and let $A \supseteq B$ be dissipative. Then we already have $A = B$. In fact, there exists $\lambda > 0$, such that $\lambda - B$ is surjective and $\lambda - A$ is injective by dissipativity. Thus the claim follows by [6, IV. 1.21. (5)].

Now we can formulate our result for the case $p = 2$.

Theorem 3.2. *Let $\Omega \subseteq \mathbb{R}^d$ be a domain. Then the operator $A_{\Omega,2}$ generates a positive C_0-semigroup $(T_{\Omega,2}(t))_{t\geq 0}$ on $L^2(\Omega)$ with $\|T_{\Omega,2}(t)\|_{\mathcal{L}(L^2(\Omega))} \leq \mathrm{e}^{-\frac{\operatorname{tr}(B)}{2}t}$ for all $t \geq 0$. Moreover for every $\lambda > -\operatorname{tr}(B)/2$ and every $t \geq 0$ we have the domination properties*

$$\begin{aligned}
|R(\lambda, A_{\Omega,2})f| &\leq R(\lambda, A_{\mathbb{R}^d,2})|\tilde{f}|, && f \in L^2(\Omega),\\
|T_{\Omega,2}(t)f| &\leq T_{\mathbb{R}^d,2}(t)|\tilde{f}| = K_2(t)|\tilde{f}|, && f \in L^2(\Omega),
\end{aligned}$$

where $\tilde{f}$ denotes the extension of f by 0. In particular $T_{\Omega,2}$ fulfills a Kolmogorov kernel estimate with $M = 1$ and $\omega = 0$.

In the following we use the notation

$$(\mathcal{A}_0 u)(x) = \sum_{i,j=1}^d q_{ij} D_i D_j u(x) \qquad \text{and} \qquad (\mathcal{L}u)(x) = Bx \cdot \nabla u(x)$$

for the diffusion and the drift part of $\mathcal{A}$, respectively. We start with a simple lemma, that will be useful for many proofs.

Lemma 3.3. *Let $G \subseteq \mathbb{R}^d$ be open and $u \in H_0^1(G)$ be a real-valued function. For any $\varphi \in C_c^\infty(\mathbb{R}^d)$ we have*

$$\int_G (\mathcal{L}u)u\varphi = -\frac{\operatorname{tr}(B)}{2}\int_\Omega u^2\varphi - \frac{1}{2}\int_\Omega u^2\mathcal{L}\varphi.$$

Proof. Since $u \in H_0^1(G)$, there is a sequence $(u_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(G)$ converging to u in $H^1(G)$. Therefore, we have $\int_G (\mathcal{L}u)u\varphi = \lim_{n\to\infty}\int_G \mathcal{L}u_n u_n\varphi$, since Bx is bounded on the support of φ. We get

$$\begin{aligned}\int_G \mathcal{L}u_n u_n\varphi &= \int_G Bx\cdot\nabla u_n u_n\varphi = -\int_G \operatorname{div}\big(u_n\varphi Bx\big)u_n \\ &= -\int_G \mathcal{L}u_n u_n\varphi - \operatorname{tr}(B)\int_G u_n^2\varphi - \int_G u_n^2\mathcal{L}\varphi.\end{aligned}$$

Letting n tend to ∞, we derive

$$\int_G (\mathcal{L}u)u\varphi = -\frac{\operatorname{tr}(B)}{2}\int_G u^2\varphi - \frac{1}{2}\int_G u^2\mathcal{L}\varphi.$$

□

Remark 3.4. To be precise, one has to check that integration by parts is allowed in the proof of Lemma 3.3. This will be used again later on, so it might be useful to note the following generalisation. For any open set G, integration by parts is possible if $u \in C_c^\infty(G)$ and $v \in H^1_{\mathrm{loc}}(G)$.

In this case, there is a compact set $K \subseteq G$ with $\operatorname{supp}(u) \subseteq K^\circ$. Then $v \in H^1(K)$ and by the definition of weak derivatives one gets for all $1 \le i \le d$

$$\int_G u(D_i v) = \int_K u(D_i v) = -\int_K (D_i u)v = -\int_G (D_i u)v.$$

In order to show that $A_{\Omega,2}$ is a generator of a C_0-semigroup we will apply the Lumer-Phillips theorem. So we first need dissipativity.

Proposition 3.5. *The operator*

$$A := A_{\Omega,2} + \frac{\operatorname{tr}(B)}{2}$$

is dissipative in $L^2(\Omega)$.

Proof. Let $f \in D(A_{\Omega,2}) = D(A)$. Writing $f = u + iv$ for suitable real $u, v \in L^2(\Omega)$, we get

$$\operatorname{Re}\int_\Omega (Af)\overline{f} = \int_\Omega (Au)u + \int_\Omega (Av)v,$$

so it suffices to show $\int_\Omega (Au)u \le 0$ for real-valued functions u. Note that $u, v \in D(A_{\Omega,2})$, since the coefficients of $\mathcal{A}$ are real.

We choose $\eta \in C_c^\infty(\mathbb{R}^d)$ vanishing outside of $B_2(0)$ with $\eta|_{B_1(0)} = \mathbf{1}$ and define $\eta_m(x) = \eta(\frac{x}{m})$ for $m \in \mathbb{N}$. Since $(Au)u \in L^1(\Omega)$ we derive $\lim_{m\to\infty}\int_\Omega (Au)u\eta_m = \int_\Omega (Au)u$ by the pointwise convergence of η_m to $\mathbf{1}$ and Lebesgue's Theorem.

Next we choose a sequence $(u_n)_{n\in\mathbb{N}}$ of C_c^∞-functions converging to u with respect to the H^1-norm. Now, partial integration (as in Remark 3.4) yields

$$\begin{aligned}
&\int_\Omega (A_{\Omega,2}u)(x)u_n(x)\eta_m(x)\,\mathrm{d}x\\
&= \int_\Omega \sum_{i,j=1}^d q_{ij}D_iD_ju(x)u_n(x)\eta_m(x)\,\mathrm{d}x + \int_\Omega \sum_{i,j=1}^d b_{ij}x_jD_iu(x)u_n(x)\eta_m(x)\,\mathrm{d}x\\
&= -\int_\Omega \sum_{i,j=1}^d q_{ij}D_iu(x)D_ju_n(x)\eta_m(x)\,\mathrm{d}x - \int_\Omega \sum_{i,j=1}^d q_{ij}D_iu(x)u_n(x)D_j\eta_m(x)\,\mathrm{d}x\\
&\quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^d b_{ij}x_jD_iu(x)u_n(x)\eta_m(x)\,\mathrm{d}x - \frac{1}{2}\int_\Omega \sum_{i,j=1}^d b_{ij}x_ju(x)D_iu_n(x)\eta_m(x)\,\mathrm{d}x\\
&\quad - \frac{1}{2}\int_\Omega \sum_{i,j=1}^d b_{ij}x_ju_n(x)u(x)D_i\eta_m(x)\,\mathrm{d}x - \frac{\mathrm{tr}(B)}{2}\int_\Omega u(x)u_n(x)\eta_m(x)\,\mathrm{d}x.
\end{aligned}$$

Note that, thanks to the bounded supports of the functions η_m and u_n, all integrals in the above calculations are well defined.

The H^1-convergence of the sequence $(u_n)_{n\in\mathbb{N}}$ yields

$$\lim_{n\to\infty}\int_\Omega \sum_{i,j=1}^d q_{ij}D_iu(x)D_ju_n(x)\eta_m(x)\,\mathrm{d}x = \int_\Omega \sum_{i,j=1}^d q_{ij}D_iu(x)D_ju(x)\eta_m(x)\,\mathrm{d}x$$

and

$$\lim_{n\to\infty}\int_\Omega \sum_{i,j=1}^d b_{ij}x_jD_iu_n(x)u(x)\eta_m(x)\,\mathrm{d}x = \int_\Omega \sum_{i,j=1}^d b_{ij}x_jD_iu(x)u(x)\eta_m(x)\,\mathrm{d}x,$$

since $x_j\eta_m(x)$ is bounded. The other summands can be treated analogously, so we derive

$$\begin{aligned}
&\int_\Omega Au(x)u(x)\eta_m(x)\,\mathrm{d}x = \lim_{n\to\infty}\int_\Omega (Au)(x)u_n(x)\eta_m(x)\,\mathrm{d}x\\
&= -\int_\Omega \sum_{i,j=1}^d q_{ij}D_iu(x)D_ju(x)\eta_m(x)\,\mathrm{d}x - \int_\Omega \sum_{i,j=1}^d q_{ij}D_iu(x)u(x)D_j\eta_m(x)\,\mathrm{d}x\\
&\quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^d b_{ij}x_jD_iu(x)u(x)\eta_m(x)\,\mathrm{d}x - \frac{1}{2}\int_\Omega \sum_{i,j=1}^d b_{ij}x_jD_iu(x)u(x)\eta_m(x)\,\mathrm{d}x\\
&\quad - \frac{1}{2}\int_\Omega \sum_{i,j=1}^d b_{ij}x_ju(x)u(x)D_i\eta_m(x)\,\mathrm{d}x.
\end{aligned}$$

Next, we want to pass to the limit m to ∞, so we have to consider the terms containing derivatives of η_m. The equality $(D_i\eta_m)(x) = \frac{1}{m}(D_i\eta)(\frac{x}{m})$ implies

$$x_jD_i\eta_m(x) = \frac{x_j}{m}D_i\eta\left(\frac{x}{m}\right) = 0 \quad \text{for} \quad \frac{|x|}{m} > 2,$$

and therefore all functions $x_jD_i\eta_m(x)$, $1 \le j \le d$, are bounded with

$$|x_jD_i\eta_m(x)| = \left|\frac{x_j}{m}D_i\eta\left(\frac{x}{m}\right)\right| < 2\|\nabla\eta\|_\infty$$

and have support inside $\{x \in \mathbb{R}^d : m \le |x| \le 2m\}$.

Now let $\varepsilon > 0$. Since $u^2 \in L^1(\Omega)$, there is a compact subset $K_\varepsilon \subseteq \Omega$ with

$$\int_{\Omega \setminus K_\varepsilon} u^2 \leq \frac{\varepsilon}{2\|\nabla\eta\|_\infty}.$$

If we choose m_0 large enough, we have $K_\varepsilon \cap \operatorname{supp}(x_j D_i \eta_m) = \emptyset$ for all $m \geq m_0$ and therefore

$$\left| \int_\Omega u^2(x) x_j D_i \eta_m(x) \, \mathrm{d}x \right| = \left| \int_{\Omega \setminus K_\varepsilon} u^2(x) x_j D_i \eta_m(x) \, \mathrm{d}x \right| \leq \varepsilon.$$

This proves

$$\lim_{m\to\infty} \int_\Omega \sum_{i,j=1}^d b_{ij} x_j u(x) u(x) D_i \eta_m(x) \, \mathrm{d}x = 0.$$

Since $\|D_i \eta_m\|_\infty \leq \frac{1}{m} \|\nabla\eta\|_\infty$, we also have

$$\lim_{m\to\infty} \int_\Omega \sum_{i,j=1}^d q_{ij} D_i u(x) u(x) D_j \eta_m(x) \, \mathrm{d}x = 0,$$

Finally we obtain dissipativity of A by

$$\int_\Omega (Au)u = \lim_{m\to\infty} \int_\Omega (Au) u \eta_m = -\int_\Omega \sum_{i,j=1}^d q_{ij} D_i u(x) D_j u(x) \, \mathrm{d}x \leq 0,$$

as Q is positive definite. □

In order to show that $A_{\Omega,2}$ is a generator, it remains to be proven that $\lambda - A$ is surjective for some fixed $\lambda > 0$. This will be done by approximating the solution u of the resolvent problem $(\lambda - A)u = f$, $f \in L^2(\Omega)$, with solutions of the same problem on bounded and regular subdomains of Ω.

By [5, II.4, Lemma 1], there exists an increasing sequence $(\Omega_n)_{n\in\mathbb{N}}$ of bounded subdomains of Ω, that have a C^2-boundary, such that $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$. Note that the specific choice of this sequence is not important, since by dissipativity of A a solution of the resolvent problem is unique, whenever it exists.

Since the coefficients of $\mathcal{L}$ are bounded on bounded sets, by standard perturbation theory, the operator $\mathcal{A} + \operatorname{tr}(B)/2$ generates a C_0-semigroup on $L^2(\Omega_n)$ for every $n \in \mathbb{N}$, when we equip it with the domain $D = H_0^1(\Omega_n) \cap H^2(\Omega_n)$. As $D(A_{\Omega_n,2})$ contains D, we get $D(A_{\Omega,2}) = D$ again by Remark 3.1.

This coinciding of the domains even gives us some more precious information. Since the generator $A_{\Omega_n,2} + \operatorname{tr}(B)/2$ is dissipative it even generates a contraction semigroup on $L^2(\Omega_n)$ for every $n \in \mathbb{N}$. Thus the bounds on the resolvent do not depend on n, which will be important in the following.

Fixing $\lambda > 0$ and $f \in L^2(\Omega)$ that means we find a unique solution $u_n \in H_0^1(\Omega_n) \cap H^2(\Omega_n)$ for the problem $\lambda u_n - A_{\Omega_n,2} u_n - \frac{\operatorname{tr}(B)}{2} u_n = f|_{\Omega_n}$ for every $n \in \mathbb{N}$ with

$$\|u_n\|_{L^2(\Omega_n)} = \left\| R\left(\lambda, A_{\Omega_n,2} + \frac{\operatorname{tr}(B)}{2}\right) f|_{\Omega_n} \right\|_{L^2(\Omega_n)} \leq \frac{1}{\lambda} \|f\|_{L^2(\Omega)}, \tag{4}$$

independently of the domain Ω_n.

As by [1, Lemma 3.22], the trivial extension of u_n is an element of $H^1(\mathbb{R}^d)$ and hence of $H_0^1(\Omega)$, we may regard $(u_n)_{n\in\mathbb{N}}$ as a sequence in $H_0^1(\Omega)$. The next lemma will show that even the gradients of the functions u_n are bounded independently of the chosen domain, which does not follow by elliptic regularity alone.

Lemma 3.6. *Let $G \subseteq \mathbb{R}^d$ be a bounded domain with C^2-boundary and $u \in H_0^1(G) \cap H^2(G)$ be a solution of $\lambda u - A_{G,2}u - \frac{\operatorname{tr}(B)}{2}u = g$ for some $\lambda > 0$. Then*

$$\|\nabla u\|_2 \leq \sqrt{\frac{2}{\lambda}}\|Q^{-\frac{1}{2}}\|\|g\|_2.$$

Proof. Let $u = v + iw$ with real-valued $v, w \in H_0^1(G) \cap H^2(G)$. Then

$$\|D^\alpha u\|_2^2 = \int_G D^\alpha u \overline{D^\alpha u} = \int_G (D^\alpha v + iD^\alpha w)(D^\alpha v - iD^\alpha w) = \|D^\alpha v\|_2^2 + \|D^\alpha w\|_2^2$$

for any multiindex α with $|\alpha| \leq 2$. For the function v we calculate by Lemma 3.3

$$\begin{aligned}
&\int_G Q^{\frac{1}{2}}\nabla v \overline{Q^{\frac{1}{2}}\nabla v} = \int_G Q\nabla v \nabla v = -\int_G (\mathcal{A}_0 v)v \\
&= \int_G \left(\lambda v - A_{G,2}v - \frac{\operatorname{tr}(B)}{2}v\right) v - \lambda\int_G vv + \int_G (\mathcal{L}v)v + \frac{\operatorname{tr}(B)}{2}\int_G vv \\
&= \int_G \left(\lambda v - A_{G,2}v - \frac{\operatorname{tr}(B)}{2}v\right) v - \lambda\int_G vv \\
&= \int_G \operatorname{Re}(g) R\left(\lambda, A_{G,2} + \frac{\operatorname{tr}(B)}{2}\right)\operatorname{Re}(g) - \lambda\left\|R\left(\lambda, A_{G,2} + \frac{\operatorname{tr}(B)}{2}\right)\operatorname{Re}(g)\right\|_2^2.
\end{aligned}$$

We conclude by (4)

$$\|\nabla v\|_2^2 = \|Q^{-\frac{1}{2}}(Q^{\frac{1}{2}}\nabla v)\|_2^2 \leq \|Q^{-\frac{1}{2}}\|^2\frac{2}{\lambda}\|\operatorname{Re}(g)\|_2^2.$$

Repeating the same calculations for w, we obtain

$$\|\nabla u\|_2^2 \leq \|Q^{-\frac{1}{2}}\|^2\frac{2}{\lambda}\|\operatorname{Re}(g)\|_2^2 + \|Q^{-\frac{1}{2}}\|^2\frac{2}{\lambda}\|\operatorname{Im}(g)\|_2^2 = \|Q^{-\frac{1}{2}}\|^2\frac{2}{\lambda}\|g\|_2^2.$$

□

Now we can prove the main result of this section.

Proposition 3.7. *Let $\Omega \subseteq \mathbb{R}^d$ be a domain. Then $A_{\Omega,2}$ is the generator of a C_0-semigroup $(T_{\Omega,2}(t))_{t\geq 0}$ on $L^2(\Omega)$ with $\|T_{\Omega,2}(t)\| \leq e^{-\frac{\operatorname{tr}(B)}{2}t}$.*

Proof. It only remains to be shown that for a fixed $\lambda > 0$ and for every $f \in L^2(\Omega)$ there exists a function $u \in H_0^1(\Omega) \cap \{v \in H^2_{\mathrm{loc}}(\Omega) : \mathcal{A}v \in L^2(\Omega)\}$ with $\lambda u - Au = f$.

We consider the sequence $(u_n)_{n\in\mathbb{N}}$ of trivial extensions of the solutions on Ω_n mentioned above. In view of Lemma 3.6, it is bounded in $H_0^1(\Omega)$, so there exists a weakly convergent subsequence $(u_{n_k})_{k\in\mathbb{N}}$. We denote its limit by u and show in the following that u is the desired solution.

As a first step, we will prove that $u \in H^2_{\mathrm{loc}}(\Omega)$. Fix two compact sets $K_1, K_2 \subseteq \Omega$ with $\overline{K_1} \subseteq K_2^\circ$. Then, by construction, $K_2 \subseteq \Omega_{n_k}$ for sufficiently large k. The coefficients of $\mathcal{A}$ are bounded on K_2, so [10, Theorem 9.11], implies that there is a constant C depending on K_1, K_2 and the bound of the coefficients on K_2, such that $\|u_{n_k}\|_{H^2(K_1)} \leq C(\|u_{n_k}\|_2 + \|f\|_2)$, so there exists a weakly convergent subsequence $(u_{n_{k_l}})_{l\in\mathbb{N}}$ in $H^2(K_1)$. Let v denote its limit. Since the sequence $(u_{n_{k_l}}|_{K_1})_{l\in\mathbb{N}}$ also converges weakly to $u|_{K_1}$ in $L^2(K_1)$, we derive $v = u|_{K_1}$ from the uniqueness of weak limits in $L^2(K_1)$. This shows $u \in H^2_{\mathrm{loc}}(\Omega)$.

In order to finish the proof, it remains to show $\lambda u - Au = f$. Let $g \in C_c^\infty(\Omega)$ and fix a compactum K with $\operatorname{supp}(g) \subseteq K^\circ$. Then $K \subseteq \Omega_{n_k}$ for k large enough. Let $(u_{n_{k_l}})_{l\in\mathbb{N}}$ be again a weakly convergent subsequence of $(u_{n_k})_{k\in\mathbb{N}}$ on $H^2(K)$ and

$A^\star$ be the formal adjoint of A. Then we conclude

$$\int_\Omega (\lambda u - Au - f)g = \int_K (\lambda u - Au - f)g = \int_K (\lambda ug - uA^\star g - fg)$$
$$= \lim_{l\to\infty} \int_K (\lambda u_{n_{k_l}} g - u_{n_{k_l}} A^\star g - fg) = \lim_{l\to\infty} \int_K (\lambda u_{n_{k_l}} - Au_{n_{k_l}} - f)g = 0.$$

Thus the assertion follows by the fundamental theorem of variational calculus. $\square$

4. Domination and positivity of the semigroup

The aim of this section is to prove Kolmogorov kernel estimates for the semigroup $(T_{\Omega,2}(t))_{t\geq 0}$ obtained in Proposition 3.7. While doing so, we also obtain positivity of the semigroup. Our method is inspired by the proof of the heat kernel estimates for the Dirichlet Laplacian, cf. [2].

Lemma 4.1. *Let $\Omega \subseteq \mathbb{R}^d$ be a domain, $\lambda > -\mathrm{tr}(B)/2$ and let $u \in D(A_{\Omega,2})$, $v \in H^1(\Omega) \cap \{f \in H^2_{\mathrm{loc}}(\Omega) : \mathcal{A}f \in L^2(\Omega)\}$ be real-valued functions, such that $v \geq 0$. Then the inequality $(\lambda - \mathcal{A})u \leq (\lambda - \mathcal{A})v$ a.e. implies $u \leq v$ a.e.*

Proof. As in the proof of Proposition 3.5, we choose a positive $\eta \in C_c^\infty(\mathbb{R}^d)$ vanishing outside of $B_2(0)$ with $\eta|_{B_1(0)} = \mathbf{1}$ and define $\eta_m(x) = \eta(\frac{x}{m})$ for $m \in \mathbb{N}$.

By hypotheses, we have

$$\lambda(u-v) - \mathcal{A}_0(u-v) - \mathcal{L}(u-v) \leq 0, \quad \text{a.e.}$$

so we obtain

$$\lambda \int_\Omega (u-v)\varphi\eta_m - \sum_{i,j=1}^d \int_\Omega q_{ij} D_i D_j (u-v)\varphi\eta_m - \int_\Omega Bx \cdot \nabla(u-v)\eta_m\varphi \leq 0$$

for all $m \in \mathbb{N}$ and all positive $\varphi \in C_c^\infty(\Omega)$. By integration by parts, cf. Remark 3.4, we conclude that

$$\lambda \int_\Omega (u-v)\varphi\eta_m + \sum_{i,j=1}^d \int_\Omega q_{ij} D_j(u-v) D_i\varphi\eta_m + \sum_{i,j=1}^d \int_\Omega q_{ij} D_j(u-v)\varphi D_i\eta_m$$
$$- \int_\Omega Bx \cdot \nabla(u-v)\eta_m\varphi \leq 0.$$

for all $m \in \mathbb{N}$. Now, this last inequality is even valid for all $\varphi \in H_0^1(\Omega)_+$ by density.

In the following we show, that $(u-v)^+ \in H_0^1(\Omega)_+$. In order to do so, we choose a sequence $(u_n) \subseteq C_c^\infty(\Omega)$, that converges to u in $H^1(\Omega)$. Then the function $(u_n - v)^+$ is in $H^1(\Omega)_+$ for every $n \in \mathbb{N}$ and since $v \geq 0$, we get $\mathrm{supp}((u_n - v)^+) \subseteq \mathrm{supp}(u_n)$. Thus $(u_n - v)^+$ has compact support in Ω, which implies $(u_n - v)^+ \in H_0^1(\Omega)_+$. This finally yields $(u-v)^+ \in H_0^1(\Omega)_+$, as $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$.

Putting $\varphi = (u-v)^+$ in the above inequality and observing that then all the integrals vanish on the set $\{u \leq v\}$, we get

$$\lambda \int_\Omega \big((u-v)^+\big)^2 \eta_m + \int_\Omega \nabla(u-v)^+ Q \nabla(u-v)^+ \eta_m$$
$$- \int_\Omega Bx \cdot \nabla(u-v)^+ (u-v)^+ \eta_m + \sum_{i,j=1}^n \int_\Omega q_{ij} D_j (u-v)^+ (u-v)^+ D_i\eta_m \leq 0.$$

Now Lemma 3.3 yields for the third integral

$$\int_\Omega Bx \cdot \nabla(u-v)^+ (u-v)^+ \eta_m = -\frac{\mathrm{tr}(B)}{2} \int_\Omega \big((u-v)^+\big)^2 \eta_m - \int_\Omega Bx \cdot \nabla\eta_m \big((u-v)^+\big)^2.$$

As all the limits for $m \to \infty$ on the left hand side exist (cf. proof of Proposition 3.5), this implies

$$-\int_\Omega Bx \cdot \nabla(u-v)^+(u-v)^+ = \frac{\operatorname{tr}(B)}{2}\int_\Omega \left((u-v)^+\right)^2.$$

We derive the inequality

$$\left(\lambda + \frac{\operatorname{tr}(B)}{2}\right)\int_\Omega \left((u-v)^+\right)^2 + \int_\Omega \nabla(u-v)^+ Q\nabla(u-v)^+ \le 0.$$

Since Q is positive definite and $\lambda > -\operatorname{tr}(B)/2$, this can only be true, if both integrals are zero, in particular, we get $\|(u-v)^+\|_{L^2(\Omega)} = 0$, which implies $u \le v$ a.e. □

Setting $v = 0$, as an immediate corollary, we get the following positivity result for $\mathcal{A}$.

Corollary 4.2. *Let $\lambda > -\operatorname{tr}(B)/2$ and let $u \in D(A_{\Omega,2})$ be a real-valued function. Then*

$$(\lambda - \mathcal{A})u \le 0 \text{ a.e.} \implies u \le 0 \text{ a.e. and}$$
$$(\lambda - \mathcal{A})u \ge 0 \text{ a.e.} \implies u \ge 0 \text{ a.e.}$$

The next step is to show that the resolvent and the semigroup react monotonely, when the domain Ω is enlarged. As a byproduct of the proof, we also deduce the positivity of the semigroup and the resolvent.

Proposition 4.3. *Let $\Omega_1, \Omega_2 \subseteq \mathbb{R}^d$ be domains and let $\Omega_1 \subseteq \Omega_2$. Then for every $f \in L^2(\Omega_1)_+$, for every $\lambda > -\operatorname{tr}(B)/2$ and every $t \ge 0$, we have*

1. *$0 \le R(\lambda, A_{\Omega_1,2})f \le R(\lambda, A_{\Omega_2,2})\tilde{f}$,*
2. *$0 \le T_{\Omega_1,2}(t)f \le T_{\Omega_2,2}(t)\tilde{f}$*

almost everywhere. Here $\tilde{f}$ denotes the extension of f by 0.

Proof. (1) Put $u := R(\lambda, A_{\Omega_1,2})f$ and $v := R(\lambda, A_{\Omega_2,2})\tilde{f}$. Then, both u and v are real-valued. Indeed, for $u = u_1 + iu_2$, we have $f = (\lambda - A_{\Omega_1,2})u = (\lambda - A_{\Omega_1,2})u_1 + i(\lambda - A_{\Omega_1,2})u_2$, where $(\lambda - A_{\Omega_1,2})u_j$, $j = 1, 2$, are real-valued functions. This implies $(\lambda - A_{\Omega_1,2})u_2 = 0$, and, by injectivity of $(\lambda - A_{\Omega_1,2})$, even $u_2 = 0$. The argument for v is the same.

Since $u \in D(A_{\Omega_1,2})$, $v \in D(A_{\Omega_2,2})$ and f and $\tilde{f}$ are positive we get $u, v \ge 0$ a.e. by Corollary 4.2. It remains to show $u \le v$. As $(\lambda - \mathcal{A})u = f = (\lambda - \mathcal{A})v$ a .e. in Ω_1, we have in particular $(\lambda - \mathcal{A})u \le (\lambda - \mathcal{A})v$. Furthermore, we know that $u \in D(A_{\Omega_1,2})$ and the restriction of v to Ω_1 is in $H^1(\Omega_1) \cap \{f \in H^2_{\mathrm{loc}}(\Omega_1) : \mathcal{A}f \in L^2(\Omega_1)\}$. Thus we may apply Lemma 4.1, in order to get $u \le v$ a.e. in Ω_1.

(2) Since $R(\lambda, A_{\Omega_j,2}) = \frac{1}{\lambda}(I - \frac{1}{\lambda}A_{\Omega_j,2})^{-1}$, $j = 1, 2$, the first part of the proof yields

$$0 \le (I - sA_{\Omega_1,2})^{-1}f \le (I - sA_{\Omega_2,2})^{-1}\tilde{f}$$

for all $0 < s < 2/|\operatorname{tr}(B)|$ or for every $s > 0$ if $\operatorname{tr}(B) = 0$. Thus we have the same for every power $k \in \mathbb{N}$:

$$0 \le (I - sA_{\Omega_1,2})^{-k}f \le (I - sA_{\Omega_2,2})^{-k}\tilde{f}.$$

Now let $t > 0$. Then, if k is large enough, we have $t/k < 2/|\operatorname{tr}(B)|$ and

$$0 \le (I - \frac{t}{k}A_{\Omega_1,2})^{-k}f \le (I - \frac{t}{k}A_{\Omega_2,2})^{-k}\tilde{f}.$$

Passing to the limit for $k \to \infty$, the claim follows (cf. [6, Corollary III.5.5]). □

For arbitrary $f \in L^2(\Omega_1)$, we get:

Corollary 4.4. *If Ω_1 and Ω_2 are as in Proposition 4.3, we have for every $f \in L^2(\Omega_1)$, for every $\lambda > -\mathrm{tr}(B)/2$ and every $t \geq 0$*

(1) $|T_{\Omega_1,2}(t)f| \leq T_{\Omega_2,2}(t)|\tilde{f}|$,
(2) $|R(\lambda, A_{\Omega_1,2})f| \leq R(\lambda, A_{\Omega_2,2})|\tilde{f}|$.

Proof. Looking at the special case $\Omega_2 = \mathbb{R}^d$ we see that the semigroup $(T_{\Omega_1,2}(t))_{t\geq 0}$ is dominated by the semigroup $(T_{\mathbb{R}^d,2}(t))_{t\geq 0}$, which is given by positive, regular integral kernel operators. It is a classical result (cf. [18, IV. §9]), that in this situation also the operators $T_{\Omega_1,2}(t)$, $t > 0$, are given by regular integral kernels K_t, that are positive, since $T_{\Omega_1,2}(t)$ are positive operators. Thus

$$\begin{aligned} |T_{\Omega_1,2}(t)f| &= \left| \int_{\Omega_1} K_t(\cdot, y) f(y) \, \mathrm{d}y \right| \leq \int_{\Omega_1} K_t(\cdot, y)|f(y)| \, \mathrm{d}y = T_{\Omega_1,2}(t)|f| \\ &\leq T_{\Omega_2,2}(t)|\tilde{f}| \end{aligned}$$

by Proposition 4.3. Now, the resolvent estimate follows as in the proof of Proposition 2.3 (3). □

Setting $\Omega_2 = \mathbb{R}^d$ in Corollary 4.4, we have finally completed the proof of Theorem 3.2.

Now we have shown that $A_{\Omega,2}$ satisfies a Kolmogorov kernel estimate with $M = 1$ and $\omega = 0$, so the results stated in Chapter 2 hold true. We collect them in the following theorem.

Theorem 4.5. *Let $\Omega \subseteq \mathbb{R}^d$ be a domain. Then $(T_{\Omega,p}(t))_{t\geq 0}$, $1 \leq p < \infty$, is a family of consistent, positive C_0-semigroups on $L^p(\Omega)$ with $\|T_{\Omega,p}(t)\|_{\mathcal{L}(L^p(\Omega))} \leq \mathrm{e}^{-\frac{\mathrm{tr}(B)}{p}t}$ for all $1 \leq p < \infty$ and all $t \geq 0$. For every $\lambda > -\mathrm{tr}(B)/p$ and every $t > 0$ we have the domination properties*

$$\begin{aligned} |R(\lambda, A_{\Omega,p})f| &\leq R(\lambda, A_{\mathbb{R}^d,p})|\tilde{f}|, & f \in L^p(\Omega), \\ |T_{\Omega,p}(t)f| &\leq T_{\mathbb{R}^d,p}(t)|\tilde{f}| = (k_t * |\tilde{f}|)(e^{tB}\cdot), & f \in L^p(\Omega), \end{aligned}$$

where $\tilde{f}$ denotes the extension of f by 0. Moreover, the operators $A_{\Omega,p} - |\mathrm{tr}(B)|$ admit a bounded H^∞-calculus for every $1 < p < \infty$ and we have

(1) *$R(\lambda, A_{\Omega,p})f = R(\lambda, A_{\Omega,q})f$ for all $f \in L^p(\Omega) \cap L^q(\Omega)$, all $1 \leq p, q < \infty$ and all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \max(-\frac{\mathrm{tr}(B)}{p}, -\frac{\mathrm{tr}(B)}{q})$.*
(2) *The set $\{f \in D(A_{\Omega,2}) \cap L^p(\Omega) : A_{\Omega,2}f \in L^p(\Omega)\}$ is contained in $D(A_{\Omega,p})$ and $A_{\Omega,p}f = A_{\Omega,2}f$ for all such f and $1 \leq p < \infty$.*

5. The spectrum of $A_{\Omega,p}$ on exterior domains

We finally turn our attention to the special case of an exterior domain, i.e. $\Omega = \mathbb{R}^d \setminus K$ for some compact set $K \subseteq \mathbb{R}^d$ with $C^{1,1}$-boundary. Scaling the set K down to $\{0\}$, we show that the spectral behaviour of $A_{\Omega,p}$ is the same as in the case $\Omega = \mathbb{R}^d$. That means that the spectrum of the drift operator $\mathcal{L}$ on $\mathbb{R}^d$, i.e. the whole vertical line $-\mathrm{tr}(B)/p + i\mathbb{R}$ (or in the case $\mathrm{tr}(B) = 0$ at least an unbounded subgroup of it) is contained in $\sigma(A_{\Omega,p})$. This implies that the semigroup $(T_{\Omega,p}(t))_{t\geq 0}$ is not eventually norm-continuous. Nevertheless, we have that the spectral bound of $A_{\Omega,p}$ and the growth bound of $T_{\Omega,p}$ coincide and $s(A_{\Omega,p}) = \omega_0(T_{\Omega,p}) = -\mathrm{tr}(B)/p$. Before we can formulate this theorem, we have to introduce the realisation $L_{\mathbb{R}^d,p}$ of $\mathcal{L}$ in $L^p(\mathbb{R}^d)$:

$$D(L_{\mathbb{R}^d,p}) = \{u \in L^p(\mathbb{R}^d) : \mathcal{L}u \in L^p(\mathbb{R}^d)\}, \quad L_{\mathbb{R}^d,p}u = \mathcal{L}u,$$

where $\mathcal{L}u$ is understood in the sense of distributions.

We collect the information on the operators $L_{\mathbb{R}^d,p}$ that we need in the following in the next proposition. For proofs, see [14].

Proposition 5.1. *Let $1 \leq p < \infty$. Then the following holds:*

(1) *The operator $L_{\mathbb{R}^d,p}$ is the generator of a C_0-semigroup $(S(t))_{t\geq 0}$ on $L^p(\mathbb{R}^d)$, given by $(S(t)f)(x) = f(e^{tB}x)$, $x \in \mathbb{R}^d$, for every $f \in L^p(\mathbb{R}^d)$.*
(2) *If $\operatorname{tr}(B) \neq 0$, then $\sigma(L_{\mathbb{R}^d,p}) = -\operatorname{tr}(B)/p + i\mathbb{R}$.*
(3) *If $\operatorname{tr}(B) = 0$, then $\sigma(L_{\mathbb{R}^d,p})$ is an additive subgroup of $i\mathbb{R}$, that is not $\{0\}$.*

Now the result that we want to prove in this section can be formulated as follows.

Theorem 5.2. *Let $K \subset \mathbb{R}^d$ be compact with a $C^{1,1}$-boundary and $\Omega = \mathbb{R}^d \setminus K$. Then for every $1 < p < \infty$ we have*

$$D(A_{\Omega,p}) = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \cap \{f \in L^p(\Omega) : \mathcal{L}f \in L^p(\Omega)\}. \tag{5}$$

Furthermore we have the inclusion $\sigma(L_{\mathbb{R}^d,p}) \subseteq \sigma(A_{\Omega,p})$ and $s(A_{\Omega,p}) = \omega_0(T_{\Omega,p}) = -\operatorname{tr}(B)/p$.

Note, that the whole picture changes completely for bounded domains as the semigroup then becomes analytic. It remains however an open question, whether Theorem 5.2 is still valid for more general unbounded domains.

Remark 5.3. If Ω is an exterior domain with $C^{1,1}$-boundary, we already know from [9] that the Ornstein-Uhlenbeck operator equipped with the domain $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \cap \{f \in L^p(\Omega) : \mathcal{L}f \in L^p(\Omega)\}$ generates a C_0-semigroup for every $1 < p < \infty$. For the following results, it is necessary to check that $A_{\Omega,p}$ coincides with this operator, which we will denote by $\tilde{A}_p$. For $p = 2$, the inclusion $D(\tilde{A}_2) \subseteq D(A_{\Omega,2})$ is clear, so we get $D(\tilde{A}_2) = D(A_{\Omega,2})$ by Remark 3.1.

The construction of the semigroups in [9] now immediately yields consistency. From this we deduce that the semigroups generated by $\tilde{A}_p$ and $A_{\Omega,p}$ and hence their generators coincide for $1 < p < \infty$.

In the following we often identify functions $f \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ with their extension by $f(0) = 0$ and thus view $C_c^\infty(\mathbb{R}^d \setminus \{0\})$ as a subspace of $C_c^\infty(\mathbb{R}^d)$. The importance of the space $C_c^\infty(\mathbb{R}^d \setminus \{0\})$ for the following is due to the following lemma.

Lemma 5.4. *The subspace $C_c^\infty(\mathbb{R}^d \setminus \{0\}) \subseteq D(L_{\mathbb{R}^d,p})$ is a core for $L_{\mathbb{R}^d,p}$.*

Proof. Let $f \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ and $t > 0$. Then $0 \notin \operatorname{supp}(f)$. In view of the linearity, bijectivity and continuity of the map $x \mapsto e^{-tB}x$, we conclude that $\operatorname{supp}(S(t)f) = e^{-tB}\operatorname{supp}(f)$ is a compact set not containing 0, hence $S(t)f \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ for every $t \geq 0$. Moreover, $C_c^\infty(\mathbb{R}^d \setminus \{0\})$ is dense in $L^p(\mathbb{R}^d)$, so the lemma follows from [6, Proposition I.1.7]. ☐

Now, if $\Omega = \mathbb{R}^d \setminus K$ for some compact set $K \subseteq \mathbb{R}^d$ with $C^{1,1}$-boundary, for every $k \in \mathbb{N}$ we set

$$\Omega_k := \{x \in \mathbb{R}^d : kx \in \Omega\} = \mathbb{R}^d \setminus \frac{1}{k}K$$

and consider the operator

$$A_k = \frac{1}{k^2} \sum_{i,j=1}^{d} q_{ij} D_i D_j + \mathcal{L},$$

with domain

$$D(A_k) = W_0^{1,p}(\Omega_k) \cap W^{2,p}(\Omega_k) \cap \{f \in L^p(\Omega_k) : \mathcal{L}f \in L^p(\Omega_k)\}$$

for some given $p \in (1, \infty)$.

By Remark 5.3 we have $D(A_k) = D(A_{\Omega_k,p})$, so Theorem 4.5 implies that $\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > -\frac{\operatorname{tr}(B)}{p}\} \subseteq \varrho(A_k)$ for all $k \in \mathbb{N}$. Thus we may compare the resolvents of A_k with the resolvent of $A_{\Omega,p} = A_1$ for all these λ. This yields the following result.

Lemma 5.5. *We have* $\|R(\lambda, A_k)\|_{\mathcal{L}(L^p(\Omega_k))} = \|R(\lambda, A_{\Omega,p})\|_{\mathcal{L}(L^p(\Omega))}$ *for every* $k \in \mathbb{N}$ *and every* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda) > -\frac{\mathrm{tr}(B)}{p}$.

Proof. We consider the map

$$V_k : L^p(\Omega) \to L^p(\Omega_k), \quad V_k(f)(x) = k^{\frac{d}{p}} f(kx).$$

The transformation formula yields that V_k is an isometry, whose inverse is given by

$$V_k^{-1}(f)(x) = k^{-\frac{d}{p}} f\left(\frac{1}{k}x\right).$$

For $f \in W^{2,p}(\Omega)$ we get by the chain rule $V_k(f) \in W^{2,p}(\Omega_k)$ and

$$\begin{aligned} V_k^{-1} A_k V_k f &= k^{\frac{d}{p}} V_k^{-1} A_k f(kx) \\ &= V_k^{-1}\left[k^{\frac{d}{p}}\left(\frac{1}{k^2}\sum_{i,j=1}^d k^2 q_{ij} D_i D_j f(kx) + \sum_{i,j=1}^d b_{ij} k x_j D_i f(kx)\right)\right] \\ &= V_k^{-1}\left[k^{\frac{d}{p}}\left(\sum_{i,j=1}^d q_{ij} D_i D_j f(kx) + \sum_{i,j=1}^d b_{ij} k x_j D_i f(kx)\right)\right] = Af. \end{aligned}$$

If $f \in W_0^{1,p}(\Omega)$, there is a sequence $f_n \in C_c^\infty(\Omega)$, which converges to f in $W^{1,p}(\Omega)$. Then the sequence $(V_k f_n)_{n\in\mathbb{N}}$ is in $C_c^\infty(\Omega_k)$ with $\mathrm{supp}(V_k f_n) = \frac{1}{k}\mathrm{supp}(f_n)$ and it converges to $V_k f$ in $W^{1,p}(\Omega_k)$. Again, the transformation formula implies $\mathcal{L}V_k f \in L^p(\Omega_k)$. For V_k^{-1} one can argue analogously. So V_k induces a bijection from $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \cap \{f \in L^p(\Omega) : \mathcal{L}f \in L^p(\Omega)\}$ to $W^{2,p}(\Omega_k) \cap W_0^{1,p}(\Omega_k) \cap \{f \in L^p(\Omega_k) : \mathcal{L}f \in L^p(\Omega_k)\}$. In fact, we have $A_{\Omega,p} f = V_k^{-1} A_k V_k f$ for all $f \in D(A_{\Omega,p})$, and $(\lambda - A_{\Omega,p})f = (\lambda - V_k^{-1} A_k V_k) f = V_k^{-1}(\lambda - A_k) V_k f$. Now we get $R(\lambda, A_{\Omega,p}) = V_k^{-1}(\lambda - A_k)^{-1} V_k$. We conclude

$$\begin{aligned} \|R(\lambda, A_{\Omega,p})f\|_p &= \|V_k^{-1}(\lambda - A_k)^{-1} V_k f\|_p = \|V_k^{-1}(\lambda - A_k)^{-1} V_k f\|_p \\ &\le \|V_k^{-1}\|_{\mathcal{L}(L^p(\Omega), L^p(\Omega_k))} \|(\lambda - A_k)^{-1}\|_{\mathcal{L}(L^p(\Omega_k))} \|V_k f\|_p \\ &\le \|R(\lambda, A_k)\|_{\mathcal{L}(L^p(\Omega_k))} \|f\|_p. \end{aligned}$$

The reverse inequality follows analogously. □

Having this equality in hand, we may finally show that the resolvents of $A_{\Omega,p}$ obey the inequality $\|R(\lambda, L_{\mathbb{R}^d,p})\|_{\mathcal{L}(L_p(\mathbb{R}^d))} \le \|R(\lambda, A_{\Omega,p})\|_{\mathcal{L}(L_p(\Omega))}$ for $\mathrm{Re}(\lambda) > -\frac{\mathrm{tr}(B)}{p}$. Since $\sigma(L_{\mathbb{R}^d,p}) \subseteq -\mathrm{tr}(B)/p + i\mathbb{R}$ by Proposition 5.1, we can approximate every $\mu \in \sigma(L_{\mathbb{R}^d,p})$ by $\lambda \in \varrho(A_{\Omega,p})$. The above inequality then implies the divergence of $\|R(\lambda, A_{\Omega,p})\|_{\mathcal{L}(L^p(\Omega))}$ for $\lambda \to \mu$. This yields $\sigma(L_{\mathbb{R}^d,p}) \subseteq \sigma(A_{\Omega,p})$. So we have shown

$$-\frac{\mathrm{tr}(B)}{p} \le s(A_{\Omega,p}) \le \omega_0(T_{\Omega,p}) \le -\frac{\mathrm{tr}(B)}{p},$$

which implies the equality stated in Theorem 5.2.

Proposition 5.6. *For* $\mathrm{Re}(\lambda) > -\frac{\mathrm{tr}(B)}{p}$ *we have*

$$\|R(\lambda, L_{\mathbb{R}^d,p})\|_{\mathcal{L}(L^p(\mathbb{R}^d))} \le \|R(\lambda, A_{\Omega,p})\|_{\mathcal{L}(L^p(\Omega))}.$$

Proof. Let $g \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$. Then $g \in D(A_k)$ for k large enough. It follows

$$\|A_k g - L_{\mathbb{R}^d,p} g\|_p = \left\|\frac{1}{k^2}\sum_{i,j=1}^d q_{ij} D_i D_j g\right\|_p \longrightarrow 0$$

for $k \to \infty$. Now we consider $f = (\lambda - \mathcal{L})g$, $g \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$. For k large enough we have $f \in D(A_k)$. Then for $n \geq k$ we get with the previous Lemma

$$\begin{aligned}
&\|R(\lambda, A_n)f - R(\lambda, L_{\mathbb{R}^d,p})\tilde{f}\|_p = \|R(\lambda, A_n)(\lambda - L_{\mathbb{R}^d,p})g - g\|_p \\
=&\|R(\lambda, A_n)\big((\lambda - A_n)g - (\lambda - A_n)g + (\lambda - L_{\mathbb{R}^d,p})g\big) - g\|_p \\
=&\|R(\lambda, A_n)(A_n - L_{\mathbb{R}^d,p})g\|_p \leq \|R(\lambda, A_{\Omega,p})\|_{\mathcal{L}(L^p(\Omega))}\|(A_n - L_{\mathbb{R}^d,p})g\|_p,
\end{aligned}$$

so $R(\lambda, A_n)f - R(\lambda, L_{\mathbb{R}^d,p})f$ converges to 0 in $L^p(\Omega)$. Then for each $\varepsilon > 0$ and $f \in (\lambda - L_{\mathbb{R}^d,p})C_c^\infty(\mathbb{R}^d \setminus \{0\})$ there exists a k with

$$\|R(\lambda, L_{\mathbb{R}^d,p})f\|_p \leq \|R(\lambda, A_k)f\|_p + \varepsilon \leq \|R(\lambda, A_{\Omega,p})\|_{\mathcal{L}(L^p(\Omega))}\|f\|_p + \varepsilon.$$

Thus we have

$$\|R(\lambda, L_{\mathbb{R}^d,p})f\|_p \leq \|R(\lambda, A_{\Omega,p})\|_{\mathcal{L}(L^p(\Omega))}\|f\|_p$$

for all $f \in (\lambda - L_{\mathbb{R}^d,p})C_c^\infty(\mathbb{R}^d \setminus \{0\})$. Since $C_c^\infty(\mathbb{R}^d \setminus \{0\})$ is a core for $L_{\mathbb{R}^d,p}$ by Lemma 5.4, the assertion follows. □

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