Well-posedness of dynamic Cosserat plasticity.

Patrizio Neff and Krzysztof Chełmiński *

Department of Mathematics, University of Technology Darmstadt

and

Faculty of Mathematics and Information Science, Warsaw University of Technology

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Abstract

We investigate the regularizing properties of generalized continua of micropolar type for dynamic elasto-plasticity. To this end we propose an extension of classical infinitesimal elasto-plasticity to include consistently non-dissipative micropolar effects and we show that the dynamic model allows for unique, global in-time solution of the corresponding rate-independent initial boundary value problem. The method of choice are the Yosida-approximation and a passage to the limit.

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Contents

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1 Introduction

This article is a sequel to work begun in [21]. There we have established the regularizing power and well-posedness of a certain **geometrically linear Cosserat model** [4] in conjunction with quasistatic rate-independent elasto-plasticity. In this contribution we extend these results to cover also the fully dynamic case.

For the relevance of the Cosserat model we refer to the introduction in [21]. Readers may also consult [9, 8, 3] for the general micropolar approach or [26, 20, 2, 5, 6, 19, 22] for its application to elasto-plasticity. Recently, Cosserat elasto-plasticity has been applied in [16, 23, 24, 25, 27, 15, 10] and references therein.

Micropolar models are characterized by an additional independent field of (infinitesimal) microrotations $\overline{A} \in \mathfrak{so}(3,\mathbb{R})$, coupled in some way to the displacement u. These new degrees of freedom introduce in a natural way length scale effects into the model which are a convenient way to regularize non-wellposed situations without compromising the physical relevance of the model.

In order to be sufficiently self-contained we recapitulate briefly the static elastic micropolar model and its quasistatic elasto-plastic extension as treated in [21]. The quasistatic model is then extended in a straightforward manner to include the dynamic effects for both the standard displacement u and the new microrotation $\overline{A}$ by writing down an appropriate Lagrangian function.

Subsequently, we mathematically study the obtained dynamic rate-independent case and show, by means of the Yosida approximation and a passage to the limit, that the rate-independent problem admits a unique, global in-time solution for displacements and microrotations in standard Sobolev spaces under fairly mild assumptions on the data. The notation is found in the appendix.

2 The Cosserat model

2.1 The infinitesimal elastic Cosserat model

We begin by recalling the infinitesimal Cosserat approach. First, in the purely elastic case, an infinitesimal Cosserat theory can be obtained by introducing the additive decomposition of the macroscopic displacement gradient ∇u into infinitesimal **microrotation** $\overline{A} \in \mathfrak{so}(3,\mathbb{R})$ (infinitesimal Cosserat rotation tensor) and infinitesimal **micropolar stretch tensor** (or first Cosserat deformation tensor) $\overline{\varepsilon} \in \mathbb{M}^{3\times 3}$ with

$$\nabla u = \overline{\varepsilon} + \overline{A}\,, \tag{2.1}$$

where $\overline{\varepsilon}$ is **not necessarily symmetric**, such that (2.1) is in general not the decomposition of ∇u into infinitesimal continuum stretch $\operatorname{sym}(\nabla u)$ and infinitesimal continuum rotation $\operatorname{skew}(\nabla u)$.

In the quasistatic case, the Cosserat theory is then obtained from a variational principle [25, p.51] or [28] for the infinitesimal displacement $u : [0,T]\times\overline{\Omega} \mapsto \mathbb{R}^3$ and the independent infinitesimal microrotation $\overline{A} : \overline{\Omega} \mapsto \mathfrak{so}(3, \mathbb{R})$:

$$\begin{aligned}\mathcal{E}(u, \overline{A}) = &\int_\Omega W(\nabla u, \overline{A}, \mathrm{D_x}\overline{A}) - \langle f, u\rangle - \langle M, \overline{A}\rangle \,\mathrm{dx} \\ &\qquad - \int_{\Gamma_S} \langle N, u\rangle \,\mathrm{dS} - \int_{\Gamma_C} \langle M_c, \overline{A}\rangle \,\mathrm{dS} \mapsto \min .\ \text{w.r.t.}\ (u, \overline{A}), \\ \overline{A}_{|\Gamma} = \overline{A}_d\,, &\quad u_{|\Gamma} = u_d(t, x)\,.\end{aligned} \tag{2.2}$$

Here W represents the elastic energy density and $\Omega \subset \mathbb{R}^3$ is a domain with boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ is that part of the boundary, where Dirichlet conditions $g_d, \overline{A}_d$ for infinitesimal displacements and rotations, respectively, are prescribed while $\Gamma_S \subset \partial\Omega$ is a part of the boundary, where traction boundary conditions N are applied with $\Gamma \cap \Gamma_S = \emptyset$. In addition, $\Gamma_C \subset \partial\Omega$ is the part of the boundary where external surface couples M_c are applied with $\Gamma \cap \Gamma_C = \emptyset$. The classical volume force is denoted by f and the additional volume couple by M. Variation of the action $\mathcal{E}$ with respect to u yields the equation for linearized balance of linear momentum and variation of $\mathcal{E}$ with respect to $\overline{A}$ yields the linearized version of balance of angular momentum.

It remains to specify the analytic form of the energy density W. A linearized version of material frame-indifference implies the reduction $W(\nabla u, \overline{A}, \mathrm{D_x}\overline{A}) = W(\overline{\varepsilon}, \mathrm{D_x}\overline{A})$, and for infinitesimal displacements u and small curvature $\mathrm{D_x}\overline{A}$ a quadratic ansatz is appropriate: $W(\overline{\varepsilon}, \mathrm{D_x}\overline{A}) = W_{\mathrm{mp}}^{\mathrm{infin}}(\overline{\varepsilon}) + W_{\mathrm{curv}}^{\mathrm{small}}(\mathrm{D_x}\overline{A})$ with an additive decomposition of the energy density into microstretch $\overline{\varepsilon}$ and curvature parts. In the isotropic case it is standard to assume for the stretch energy

$$W_{\mathrm{mp}}^{\mathrm{infin}}(\overline{\varepsilon}) = \mu\, \|\operatorname{sym}\nabla u\|^2 + \mu_c\, \|\operatorname{skew}(\nabla u) - \overline{A}\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}(\nabla u)\right]^2, \tag{2.3}$$

where the **Cosserat couple modulus** $\mu_c \geq 0\,[\mathrm{MPa}]$ is an additional parameter, complementing the two Lamè constants $\mu, \lambda > 0\,[\mathrm{MPa}]$. For the curvature term we assume

$$W_{\mathrm{curv}}^{\mathrm{small}}(\mathrm{D_x}\overline{A}) = \mu\frac{L_c^2}{2}\left(\alpha_5\, \|\operatorname{sym}\mathrm{D_x}\overline{A}\|^2 + \alpha_6\, \|\operatorname{skew}\mathrm{D_x}\overline{A}\|^2 + \alpha_7 \operatorname{tr}\left[\mathrm{D_x}\overline{A}\right]^2\right). \tag{2.4}$$

Here, $L_c > 0$ with units of length introduces a specific **internal characteristic length** into the elastic formulation. In general one assumes $\alpha_5 > 0$, $\alpha_6, \alpha_7 \geq 0$.

We observe that if $\mu_c = 0$, the infinitesimal minimization problem (2.3) completely decouples - the infinitesimal microrotations $\overline{A}$ have no influence at all on the macroscopic behaviour of the infinitesimal displacements and classical infinitesimal elasticity results.[1]

[1] Note that $\operatorname{axl}\overline{A} \times \xi = \overline{A}.\xi$ for all $\xi \in \mathbb{R}^3$, such that

$$\operatorname{axl}\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} := \begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix}, \quad \overline{A}_{ij} = \varepsilon_{ijk} \cdot \operatorname{axl}(\overline{A})_k\,, \tag{2.5}$$

where ε_{ijk} is the totally antisymmetric permutation tensor. Here, $\overline{A}.\xi$ denotes the application of the matrix $\overline{A}$ to the vector ξ and $a \times b$ is the usual cross-product. This induces the **canonical identification** of skew-symmetric matrices $\mathfrak{so}(3, \mathbb{R})$ with $\mathbb{R}^3$.

In the limit of zero internal length scale $L_c = 0$ and for $\mu_c > 0$,[2] balance of angular momentum reads

$$D_{\overline{A}} W_{\mathrm{mp}}(\nabla u, \overline{A}) \in \mathrm{Sym} \Leftrightarrow D_{\overline{A}} W_{\mathrm{mp}}(\nabla u, \overline{A}) = 0\,, \tag{2.6}$$

and implies already that infinitesimal continuum rotations and infinitesimal microrotations coincide: $\mathrm{skew}(\nabla u) = \overline{A}$, and this in turn is equivalent to the symmetry of the infinitesimal Cauchy stress σ or the so called **Boltzmann axiom**.

If we consider $\mu_c > 0$, it is standard to prove that the corresponding minimization problem admits a unique minimizing pair $(u, \overline{A}) \in H^1(\Omega, \mathbb{R}^3) \times H^1(\Omega, \mathfrak{so}(3, \mathbb{R}))$. Existence results of this type have been obtained e.g. in [7, 17, 12, 13] and in [21].

2.2 Non-dissipative extension to micropolar elasto-plasticity

Now we extend the formulation of micropolar elasticity to cover infinitesimal elasto-plasticity as well. It is clear that there exists various ways of obtaining such an extension, for an overview of the competing models we refer to the instructive survey article [11]. Incidentally, the Cosserats themselves [4, p.5] already envisaged the application of their general theory to plasticity and fracture. Without restricting generality we base the following considerations on a simplified curvature expression by setting $\alpha_5 = \alpha_6 = 1$, $\alpha_7 = 0$.

The idea of a **non-dissipative** extension is simple. Consider the additive decomposition of the total micropolar stretch into elastic and plastic parts

$$\overline{\varepsilon} = \overline{\varepsilon}_e + \overline{\varepsilon}_p\,, \tag{2.7}$$

and assume that microrotational effects remain purely elastic: $\overline{A}_e := \overline{A}$. Now we replace formally $\overline{\varepsilon}$ in (2.3) with $\overline{\varepsilon}_e$ which yields (note that $\|\mathrm{D_x}\overline{A}_e\|^2 = 2\|\nabla\,\mathrm{axl}(\overline{A}_e)\|^2$)

$$\mathcal{E}(\overline{\varepsilon}_e, \overline{A}_e) = \int_\Omega \mu\,\|\,\mathrm{sym}\,\overline{\varepsilon}_e\|^2 + \mu_c\,\|\,\mathrm{skew}(\overline{\varepsilon}_e)\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\,[\overline{\varepsilon}_e]^2 + \mu\,L_c^2\,\|\nabla\,\mathrm{axl}(\overline{A}_e)\|^2\,\mathrm{dx} \tag{2.8}$$

$$= \int_\Omega \mu\,\|\varepsilon - \mathrm{sym}\,\overline{\varepsilon}_p\|^2 + \mu_c\,\|\,\mathrm{skew}(\nabla u - \overline{A}_e - \overline{\varepsilon}_p)\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\,[\varepsilon - \overline{\varepsilon}_p]^2 + \mu\,L_c^2\,\|\nabla\,\mathrm{axl}(\overline{A}_e)\|^2\,\mathrm{dx}$$

as **thermodynamic potential** $\mathcal{E}$, where $\varepsilon = \mathrm{sym}\,\nabla u$ is the symmetric part of the displacement gradient. We need to supply a consistent flow rule for $\overline{\varepsilon}_p$ (note again that $\overline{A}_e$ acts solely elastically). By choosing

$$\dot{\overline{\varepsilon}}_p(t) \in \mathfrak{f}(T_E), \quad T_E := -\partial_{\overline{\varepsilon}_p} W^{\mathrm{infin}}_{\mathrm{mp}}(\overline{\varepsilon}_e) = \partial_{\overline{\varepsilon}_e} W^{\mathrm{infin}}_{\mathrm{mp}}(\overline{\varepsilon}_e)\,, \qquad \overline{\varepsilon}_e = \overline{\varepsilon} - \overline{\varepsilon}_p\,, \tag{2.9}$$

$$W^{\mathrm{infin}}_{\mathrm{mp}}(\overline{\varepsilon}_e) = \mu\,\|\,\mathrm{sym}\,\overline{\varepsilon}_e\|^2 + \mu_c\,\|\,\mathrm{skew}(\overline{\varepsilon}_e)\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\,[\overline{\varepsilon}_e]^2\,,$$

with a constitutive multifunction $\mathfrak{f}$ such that $\langle \mathfrak{f}(\Sigma), \Sigma\rangle \geq 0$, $\forall\,\Sigma \neq 0$, the **reduced dissipation inequality**

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}(\varepsilon, \overline{A}_e, \overline{\varepsilon}_p) \leq 0 \tag{2.10}$$

[2] Corresponding as well to the limit of arbitrary large samples, which can be seen by a simple scaling argument.

at fixed in time $(\nabla u, \overline{A}_e)$ is automatically satisfied, thus ensuring the second law of thermodynamics.

We assume that the multifunction $\mathfrak{f}$ takes **trace free symmetric values** only, i.e. $\mathfrak{f}(T_E) \in \mathrm{Sym}(3) \cap \mathfrak{sl}(3,\mathbb{R})$. This sets the **infinitesimal plastic spin** $\mathrm{skew}(\overline{\varepsilon}_p)$ to **zero** and restricts attention to incompressible plasticity as in classical formulations of ideal plasticity. Since then $\overline{\varepsilon}_p \in \mathrm{Sym}(3)$ we may identify $\overline{\varepsilon}_p = \mathrm{sym}(\overline{\varepsilon}_p) = \varepsilon_p$, formally as in classical ideal plasticity. In [21] we have shown that the ensuing quasistatic elasto-plastic model is well-posed.

2.3 Infinitesimal dynamic elasto-plastic Cosserat model

The dynamic infinitesimal-strain system with non-dissipative Cosserat effects can be obtained by augmenting the previous strain energy with suitable inertia terms for both the displacement u and the elastic microrotation $\overline{A}_e$. Without loss of generality we assume henceforth for the density $\varrho(x) \equiv 1$. We assume the **Lagrangian** expression

$$\begin{aligned}\int_0^T \int_\Omega & \frac{1}{2}\|u_t\|^2 + 2\|\,\mathrm{axl}\,\overline{A}_{e_t}\|^2 + \langle f, u\rangle + \langle M, \overline{A}_e\rangle \\ & - \left(\mu\|\varepsilon - \varepsilon_p\|^2 + \mu_c\,\|\,\mathrm{skew}(\nabla u - \overline{A}_e)\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\,[\varepsilon]^2 + \mu\, L_c^2\,\|\nabla\,\mathrm{axl}(\overline{A}_e)\|^2\right)\,\mathrm{dx} \\ & + \int_{\Gamma_S} \langle N, u\rangle\,\mathrm{dS} + \int_{\Gamma_C} \langle M_c, \overline{A}_e\rangle\,\mathrm{dS}\,\mathrm{ds} \mapsto \mathrm{stat}\,. \quad \text{w.r.t. } (u, \overline{A}_e) \text{ at fixed } \varepsilon_p\,,\end{aligned} \tag{2.11}$$

together with the flow rule

$$\dot{\varepsilon_p}(t) \in \mathfrak{f}(T_E), \qquad T_E = 2\mu\,(\varepsilon - \varepsilon_p)\,, \tag{2.12}$$

and suitable initial and boundary values.

The **corresponding system of dynamic partial differential equations** coupled with the flow rule is given by (use that $\|\overline{A}_e\|^2 = 2\|\,\mathrm{axl}(\overline{A}_e)\|^2$ for $\overline{A}_e \in \mathfrak{so}(3,\mathbb{R})$)

$$\begin{aligned}\mathrm{Div}\,\sigma &= u_{tt} - f, \quad x \in \Omega\,, \\ \sigma &= 2\mu\,(\varepsilon - \varepsilon_p) + 2\,\mu_c\,(\mathrm{skew}(\nabla u) - \overline{A}_e) + \lambda\,\mathrm{tr}\,[\varepsilon]\cdot 1\!\!1\,, \qquad (2.13)\\ \overline{A}_{e_{tt}} - \mu\frac{L_c^2}{2}\,\Delta\,\mathrm{axl}(\overline{A}_e) &= \mu_c\,\mathrm{axl}(\mathrm{skew}(\nabla u) - \overline{A}_e) + \frac{1}{2}\,\mathrm{axl}(\mathrm{skew}(M))\,, \\ \dot{\varepsilon_p}(t) &\in \mathfrak{f}(T_E), \quad T_E = 2\mu\,(\varepsilon - \varepsilon_p)\,, \\ u_{|\Gamma}(t,x) &= u_d(t,x)\,, \quad \overline{A}_{e|_\Gamma} = \overline{A}_d(t,x)_{|_\Gamma}\,, \\ u(x,0) &= u^0(x)\,, \quad \dot{u}(x,0) = u^1(x)\,, \\ \overline{A}_e(x,0) &= A^0(x)\,, \quad \dot{\overline{A}}_e(x,0) = A^1(x)\,, \quad \varepsilon_p(0) = \varepsilon_p^0\,,\end{aligned}$$

$$\sigma.\vec{n}|_{\Gamma_S}(t,x) = N, \quad \sigma.\vec{n}|_{\partial\Omega\setminus\{\Gamma\cup\Gamma_S\}}(t,x) = 0\,,$$

$$\mu\, L_c^2 \nabla\, \mathrm{axl}(\overline{A}_e).\vec{n}|_{\Gamma_C}(t,x) = \mathrm{axl}(\mathrm{skew}(M_c)), \quad \mu\, L_c^2 \nabla\, \mathrm{axl}(\overline{A}_e).\vec{n}|_{\partial\Omega\setminus\{\Gamma\cup\Gamma_C\}}(t,x) = 0\,,$$

$$\mathrm{tr}\,[\varepsilon_p(0)] = 0\,, \quad \varepsilon_p(0) \in \mathrm{Sym}(3)\,.$$

3 Mathematical analysis of the dynamic model

For brevity of notation we write in this part A instead of $\overline{A}_e$ and l_c instead of the positive constant $\mu\,\frac{L_c^2}{2}$. Moreover, we study pure Dirichlet boundary conditions, i.e. $\Gamma = \partial\Omega$. Second time derivatives are written as $\ddot{u}$. Thus we consider the well-posedness of the following nonlinear initial boundary-value problem

$$\begin{aligned}
\ddot{u} - \mathrm{Div}\,\sigma &= f\,, \\
\sigma &= 2\mu\,(\varepsilon - \varepsilon_p) + 2\,\mu_c\,(\mathrm{skew}(\nabla u) - A) + \lambda\,\mathrm{tr}\,[\varepsilon]\cdot 1\!\!1\,, \\
\ddot{A} - l_c\,\Delta\,\mathrm{axl}(A) &= \mu_c\,\mathrm{axl}(\mathrm{skew}(\nabla u) - A) + g\,, \\
\dot{\varepsilon}_p &\in \mathfrak{f}(T_E)\,, \quad T_E = 2\mu\,(\varepsilon - \varepsilon_p)\,, \\
u_{|\partial\Omega} &= u_d\,, \quad A_{|\partial\Omega} = A_d\,, \\
u(0) = u^0\,, \quad \dot{u}(0) = u^1\,, \quad A(0) &= A^0\,, \quad \dot{A}(0) = A^1\,, \quad \varepsilon_p(0) = \varepsilon_p^0\,,
\end{aligned} \tag{3.14}$$

where f, g are given volume force and volume couple and u_d, A_d are given boundary data and $u^0, u^1, A^0, A^1, \varepsilon_p^0$ are given initial data. Moreover, we assume that the inelastic constitutive multifunction $\mathfrak{f} : D(\mathfrak{f}) \subset \mathrm{Sym}(3) \to \mathcal{P}(\mathrm{Sym}(3))$ is a **maximal monotone** mapping ([1, Definition 1 p. 140]) satisfying $0 \in \mathfrak{f}(0)$. Here, for any set X the symbol $\mathcal{P}(X)$ denotes the family of all subsets of X. The monotonicity assumption for $\mathfrak{f}$ yields that the considered model is thermodynamical admissible. Note, that the flow function corresponding to classical ideal plasticity possesses the same properties.

To prove that system (3.14) possesses global in time L^2-solutions we approximate the flow function $\mathfrak{f}$ by single-valued, global Lipschitz functions $\mathfrak{f}_\eta$, called in the literature the **Yosida approximation** (see for example [1, Theorem 2, page 144]). Thus, we first consider system (3.14) with $\mathfrak{f}_\eta$ instead of $\mathfrak{f}$ and try to pass to the limit $\eta \to 0^+$. Following this idea, for all $\eta > 0$ we study the approximated initial boundary-value problem in the form

$$\begin{aligned}
\ddot{u}^\eta - \mathrm{Div}\,\sigma^\eta &= f\,, \\
\sigma^\eta &= 2\mu\,(\varepsilon^\eta - \varepsilon_p^\eta) + 2\,\mu_c\,(\mathrm{skew}(\nabla u^\eta) - A^\eta) + \lambda\,\mathrm{tr}\,[\varepsilon^\eta]\cdot 1\!\!1\,, \\
\ddot{A}^\eta - l_c\,\Delta\,\mathrm{axl}(A^\eta) &= -\mu_c\,\mathrm{axl}(A^\eta) + \mu_c\,\mathrm{axl}(\mathrm{skew}(\nabla u^\eta)) + g\,, \\
\dot{\varepsilon}_p^\eta &= \mathfrak{f}_\eta(T_E^\eta)\,, \quad T_E^\eta = 2\mu\,(\varepsilon^\eta - \varepsilon_p^\eta)\,, \\
u^\eta_{|\partial\Omega} &= u_d\,, \quad A^\eta_{|\partial\Omega} = A_d\,, \\
u^\eta(0) = u^0\,, \quad \dot{u}^\eta(0) = u^1\,, \quad A^\eta(0) &= A^0\,, \quad \dot{A}^\eta(0) = A^1\,, \quad \varepsilon_p^\eta(0) = \varepsilon_p^0\,.
\end{aligned} \tag{3.15}$$

The system (3.15) contains only global Lipschitz nonlinearities, hence using the standard fixed point method we obtain the following existence and uniqueness result:

Theorem 3.1 (Global existence and uniqueness for approximated problem) *Let us assume that the given data possess the following regularity: for all times $T > 0$*

$$f \in C^1([0,T], L^2(\Omega,\mathbb{R}^3))\,,\ g \in C^1([0,T], L^2(\Omega, \mathfrak{so}(3,\mathbb{R})))$$

$$u_d \in C^1([0,T], H^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3))\,,\ A_d \in C^1([0,T], H^{\frac{3}{2}}(\partial\Omega, \mathfrak{so}(3,\mathbb{R})))$$

and the initial data satisfy

$$u^0, u^1 \in H^1(\Omega,\mathbb{R}^3)\,, A^0 \in H^2(\Omega, \mathfrak{so}(3,\mathbb{R}))\,,\ A^1 \in H^1(\Omega, \mathfrak{so}(3,\mathbb{R}))\,,\ \varepsilon_p^0 \in L^2(\Omega, \mathrm{Sym}(3))\,.$$

Moreover, suppose that the following compatibility condition holds

$$u^0(x) = u_d(x,0)\,, u^1(x) = \dot{u}_d(x,0)\,, A^0(x) = A_d(x,0)\,, A^1(x) = \dot{A}_d(x,0)\ \text{ for } x \in \partial\Omega\,.$$

Then the approximated problem has a global in time, unique solution $(u^\eta, \varepsilon_p^\eta, A^\eta)$ with the regularity

$$u^\eta \in C^1([0,T], H^1(\Omega,\mathbb{R}^3))\,, \ddot{u}^\eta \in C([0,T], L^2(\Omega,\mathbb{R}^3))\,, \varepsilon_p^\eta \in C^1([0,T], L^2(\Omega, \mathrm{Sym}(3)))\,,$$

$$A^\eta \in C([0,T], H^2(\Omega, \mathfrak{so}(3,\mathbb{R})))\,,\ \ddot{A}^\eta \in C([0,T], L^2(\Omega, \mathfrak{so}(3,\mathbb{R})))\,.$$

Proof. The proof is a standard application of the Banach Fixed Point Theorem and can therefore be omitted. For similar results the reader may consult [18]. ■

Next, we are going to obtain some estimates for the approximated sequence $(u^\eta, \varepsilon_p^\eta, A^\eta)$. To do this we use the **energy** associated with the dynamic problem (not the Lagrangian) which is defined by

$$\mathcal{E}(u,\varepsilon,\varepsilon_p,A)(t) := \int_\Omega \Big(\frac{1}{2}\|\dot{u}\|^2 + 2\|\,\mathrm{axl}(\dot{A})\,\|^2$$
$$+\mu\,\|\varepsilon-\varepsilon_p\|^2 + \frac{\lambda}{2}\mathrm{tr}\,[\varepsilon]^2 + \mu_c\|\,\mathrm{skew}(\nabla u) - A\|^2 + 2l_c\,\|\nabla\,\mathrm{axl}(A)\|^2\Big)\,\mathrm{dx}\,.$$

For $\lambda > 0$ the energy function is **elastically coercive** which means that

$$\mathcal{E}(u,\varepsilon,\varepsilon_p,A) + C_d \geq C_E(\|u\|^2_{H^1(\Omega\times(0,T))} + \|A\|^2_{H^1(\Omega\times(0,T))})\,, \tag{3.16}$$

where the constant C_E does not depend on u and A and the constant C_d depends on boundary data of u and A only. The proof of this important property is based on the fact that the operators curl and Div together control the total gradient, see [14, p.36], i.e. the inequality

$$\exists\, C > 0\ \forall\, \phi \in C_0^\infty(\Omega,\mathbb{R}^3) : \int_\Omega \|\,\mathrm{curl}\,\phi(x)\|^2_{\mathbb{R}^3} + (\mathrm{Div}\,\phi(x))^2\ \mathrm{dx} \geq C\,\|\phi\|^2_{H^{1,2}(\Omega,\mathbb{R}^3)}\,, \tag{3.17}$$

holds for smooth functions with compact support $C_0^\infty(\Omega,\mathbb{R}^3)$. Observe that $(\mathrm{Div}\,u)^2 = \mathrm{tr}\,[\varepsilon]^2$ and $\|\,\mathrm{curl}\,u\|^2_{\mathbb{R}^3} = 4\,\|\,\mathrm{axl}\,\mathrm{skew}(\nabla u)\|^2$.

Let us denote by v_d the time derivative of u_d and by B_d the time derivative of A_d. In contrast to the quasistatic case we prove first energy estimates for the time derivatives and from this result we conclude the boundedness of the energy on finite time intervals.

Theorem 3.2 (Energy estimate for time derivatives)
Suppose that the given data possess more time regularity than in the last theorem and satisfy additionally: for all times $T > 0$

$$\ddot{v}_d \in L^2((0,T); H^{\frac{1}{2}}(\partial\Omega, \mathbb{R}^3)\,, \ddot{B}_d \in L^2((0,T); H^{\frac{1}{2}}(\partial\Omega, \mathfrak{so}(3,\mathbb{R}))\,. \tag{3.18}$$

Moreover, assume that the initial data $u^0, u^1, A^0, A^1, \varepsilon_p^0$ have the regularity required in Theorem 3.1 and assume that the initial value of the reduced Eshelby tensor $T_E(0) = 2\mu\,(\frac{1}{2}(\nabla u^0 + \nabla^T u^0) - \varepsilon_p^0)$ belongs to the domain of the maximal monotone operator $\mathfrak{f}$. Then there exists a positive constant $C(T)$, independent of η, such that

$$\mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta)(t) \le C(T) \quad \text{for all} \ \ t \in [0,T)\,.$$

Proof. For $h > 0$ let us denote by $(u_h^\eta(t), \varepsilon_h^\eta(t), \varepsilon_{p,h}^\eta(t), A_h^\eta(t))$ the shifted functions $(u^\eta(t+h), \varepsilon^\eta(t+h), \varepsilon_p^\eta(t+h), A^\eta(t+h))$ and calculate the energy evaluated on the differences $(u_h^\eta - u^\eta, \ldots)$. Then for the time derivative we have

$$\begin{aligned}
\dot{\mathcal{E}}(u_h^\eta - u^\eta, \varepsilon_h^\eta - \varepsilon^\eta, \varepsilon_{p,h}^\eta - \varepsilon_p^\eta, A_h^\eta - A^\eta)(t) &= \int_\Omega \langle \dot{u}_h^\eta - \dot{u}^\eta\,,\, \ddot{u}_h^\eta - \ddot{u}^\eta\rangle \,\mathrm{dx} \\
+4\int_\Omega \langle \mathrm{axl}(\dot{A}_h^\eta - \dot{A}^\eta)\,,\, \mathrm{axl}(\ddot{A}_h^\eta - \ddot{A}^\eta)\rangle \,\mathrm{dx} &+ \int_\Omega 2\mu\,\langle \varepsilon_h^\eta - \varepsilon^\eta - \varepsilon_{p,h}^\eta + \varepsilon_p^\eta, \dot{\varepsilon}_h^\eta - \dot{\varepsilon}^\eta - \dot{\varepsilon}_{p,h}^\eta + \dot{\varepsilon}_p^\eta\rangle \,\mathrm{dx} \\
&+2\mu_c \int_\Omega \langle \mathrm{skew}(\nabla u_h^\eta - \nabla u^\eta) - A_h^\eta + A^\eta, \mathrm{skew}(\nabla \dot{u}_h^\eta - \nabla \dot{u}^\eta) - \dot{A}_h^\eta + \dot{A}^\eta\rangle \,\mathrm{dx} \\
&+\lambda \int_\Omega \mathrm{tr}\,[\varepsilon_h^\eta - \varepsilon^\eta]\mathrm{tr}\,[\dot{\varepsilon}_h^\eta - \dot{\varepsilon}^\eta] \,\mathrm{dx} + 4l_c \int_\Omega \langle \nabla\,\mathrm{axl}(A_h^\eta - A^\eta), \nabla\,\mathrm{axl}(\dot{A}_h^\eta - \dot{A}^\eta)\rangle \,\mathrm{dx} \qquad (3.19) \\
&= -\int_\Omega \langle T_{E,h}^\eta - T_E^\eta, \dot{\varepsilon}_{p,h}^\eta - \dot{\varepsilon}_p^\eta\rangle \,\mathrm{dx} + \int_\Omega \langle \sigma_h^\eta - \sigma^\eta, \nabla \dot{u}_h^\eta - \nabla \dot{u}^\eta\rangle \,\mathrm{dx} \\
&+ \int_\Omega \langle \dot{u}_h^\eta - \dot{u}^\eta\,,\, \ddot{u}_h^\eta - \ddot{u}^\eta\rangle \,\mathrm{dx} + 4\int_\Omega \langle \mathrm{axl}(\dot{A}_h^\eta - \dot{A}^\eta)\,,\, \mathrm{axl}(\ddot{A}_h^\eta - \ddot{A}^\eta)\rangle \,\mathrm{dx} \\
&-4\mu_c \int_\Omega \langle \mathrm{axl}\,\mathrm{skew}(\nabla u_h^\eta - \nabla u^\eta) - \mathrm{axl}(A_h^\eta - A^\eta), \mathrm{axl}(\dot{A}_h^\eta - \dot{A}^\eta)\rangle \,\mathrm{dx} \\
&+4l_c \int_\Omega \langle \nabla\,\mathrm{axl}(A_h^\eta - A^\eta), \nabla\,\mathrm{axl}(\dot{A}_h^\eta - \dot{A}^\eta)\rangle \,\mathrm{dx}\,,
\end{aligned}$$

where $T_{E,h}^\eta(t) = T_E^\eta(t+h)$ and $\sigma_h^\eta(t) = \sigma^\eta(t+h)$. Using the monotonicity of $\mathfrak{f}_\eta$ we have that the first integral on the right hand side of (3.19) is non-positive. Next, we integrate by parts in the second and in the last integral and use the equation of motion and the equation for microrotations. Hence, we conclude that

$$\begin{aligned}
&\dot{\mathcal{E}}(u^\eta_h - u^\eta, \varepsilon^\eta_h - \varepsilon^\eta, \varepsilon^\eta_{p,h} - \varepsilon^\eta_p, A^\eta_h - A^\eta)(t) \\
&\quad \le \int_\Omega \langle f_h - f, \dot u^\eta_h - \dot u^\eta\rangle \,\mathrm{dx} + 4\int_\Omega \langle g_h - g, \operatorname{axl}\dot A^\eta_h - \operatorname{axl}\dot A^\eta\rangle \,\mathrm{dx} \\
&+ \int_{\partial\Omega} \langle (\sigma^\eta_h - \sigma^\eta).n, \dot u_{d,h} - \dot u_d\rangle \,\mathrm{ds} + 4l_c \int_{\partial\Omega} \langle \nabla \operatorname{axl}(A^\eta_h - A^\eta).n, \operatorname{axl}(\dot A_{d,h} - \dot A_d\rangle \,\mathrm{ds}\,, \quad (3.20)
\end{aligned}$$

where $f_h(t) = f(t+h)$, $g_h(t) = g(t+h)$, $u_{d,h}(t) = u_d(t+h)$ and $A_{d,h}(t) = A_d(t+h)$. Next, we integrate (3.20) in time and obtain

$$\begin{aligned}
&\mathcal{E}(u^\eta_h - u^\eta, \varepsilon^\eta_h - \varepsilon^\eta, \varepsilon^\eta_{p,h} - \varepsilon^\eta_p, A^\eta_h - A^\eta)(t) \le \mathcal{E}(u^\eta_h - u^\eta, \varepsilon^\eta_h - \varepsilon^\eta, \varepsilon^\eta_{p,h} - \varepsilon^\eta_p, A^\eta_h - A^\eta)(0) \\
&\quad + \int_0^t\int_\Omega \langle f_h - f, \dot u^\eta_h - \dot u^\eta\rangle \,\mathrm{dx}\, d\tau + 4\int_0^t\int_\Omega \langle g_h - g, \operatorname{axl}\dot A^\eta_h - \operatorname{axl}\dot A^\eta\rangle \,\mathrm{dx}\, d\tau \\
&\quad + \int_0^t\int_{\partial\Omega} \langle (\sigma^\eta_h - \sigma^\eta).n, \dot u_{d,h} - \dot u_d\rangle \,\mathrm{ds}\, d\tau + 4l_c \int_0^t\int_{\partial\Omega} \langle \nabla \operatorname{axl}(A^\eta_h - A^\eta).n, \operatorname{axl}(\dot A_{d,h} - \dot A_d\rangle \,\mathrm{ds}\, d\tau\,.
\end{aligned}$$

In the last two integrals on the right hand side of (3.21) we shift the difference operator onto the given data. Next, we divide by h^2 and pass to the limit $h \to 0^+$. Hence, we arrive at the following inequality

$$\begin{aligned}
&\mathcal{E}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon^\eta_p, \dot A^\eta)(t) \le \mathcal{E}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon^\eta_p, \dot A^\eta)(0) + \int_0^t\int_\Omega \langle \dot f, \ddot u^\eta\rangle \,\mathrm{dx}\, d\tau + 4\int_0^t\int_\Omega \langle \dot g, \operatorname{axl}\ddot A^\eta\rangle \,\mathrm{dx}\, d\tau \\
&\quad - \int_0^t\int_{\partial\Omega} \langle \sigma^\eta.n, \ddot v_d\rangle \,\mathrm{ds}\, d\tau - \int_{\partial\Omega} \langle \sigma^\eta(0).n, \dot v_d(0)\rangle \,\mathrm{ds} + \int_{\partial\Omega} \langle \sigma^\eta(t).n, \dot v_d(t)\rangle \,\mathrm{ds} \\
&\quad -4l_c \int_0^t\int_{\partial\Omega} \langle \nabla \operatorname{axl}(A^\eta).n, \operatorname{axl}(\ddot B_d)\rangle \,\mathrm{ds}\, d\tau - 4l_c \int_{\partial\Omega} \langle \nabla \operatorname{axl}(A^\eta)(0).n, \operatorname{axl}(\dot B_d)(0)\rangle \,\mathrm{ds} \\
&\quad +4l_c \int_{\partial\Omega} \langle \nabla \operatorname{axl}(A^\eta)(t).n, \operatorname{axl}(\dot B_d)(t)\rangle \,\mathrm{ds}\,. \qquad (3.21)
\end{aligned}$$

The boundedness of the initial energy for time derivatives follows from the assumption $T_E(0) \in \mathcal{D}(\mathfrak{f})$. This implies that the sequence $\{\mathfrak{f}_\eta(T_E(0))\}$ is bounded in $L^2(\Omega, \mathrm{Sym}(3))$. Next, we estimate all integral terms from the right hand side of (3.21).

$$\left|\int_0^t\int_\Omega \langle \dot f, \ddot u^\eta\rangle \,\mathrm{dx}\, d\tau\right| \le \int_0^t \|\dot f\|_{L^2}\|\ddot u^\eta\|_{L^2}\, d\tau \le \frac{1}{2}\int_0^t \|\dot f\|^2_{L^2}\, d\tau + \int_0^t \mathcal{E}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon^\eta_p, \dot A^\eta)(\tau)\, d\tau\,. \qquad (3.22)$$

In the same manner we obtain

$$\left|\int_0^t\int_\Omega \langle \dot g, \operatorname{axl}\ddot A^\eta\rangle \,\mathrm{dx}\, d\tau\right| \le \frac12\int_0^t \|\dot g\|^2_{L^2}\, d\tau + \frac14\int_0^t \mathcal{E}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon^\eta_p, \dot A^\eta)(\tau)\, d\tau\,. \tag{3.23}$$

To estimate the appearing boundary integrals we use the trace theorem in the space $L^2(\mathrm{Div})$.

$$\begin{aligned}\left|\int_0^t\int_{\partial\Omega} \langle \sigma^\eta.n, \ddot v_d\rangle \,\mathrm{ds}\, d\tau\right| &\le \int_0^t \|\sigma^\eta.n\|_{H^{-\frac12}}\|\ddot v_d\|_{H^{\frac12}}\, d\tau\\
\le C\int_0^t (\|\sigma^\eta\|_{L^2} + \|\operatorname{Div}\sigma^\eta\|_{L^2})\|\ddot v_d\|_{H^{\frac12}}\, d\tau &\le C\int_0^t (\|\dot\sigma^\eta\|_{L^2} + \|\sigma^\eta(0)\|_{L^2})\|\ddot v_d\|_{H^{\frac12}}\, d\tau\\
+C\int_0^t (\|f\|_{L^2} + \|\ddot u^\eta\|_{L^2})\|\ddot v_d\|_{H^{\frac12}}\, d\tau &\le C\int_0^t \mathcal{E}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon^\eta_p, \dot A^\eta)(\tau)\, d\tau + C_1\,,\end{aligned} \tag{3.24}$$

where the positive constant C does not depend on η and the constant C_1 depends on given data only. For the next boundary term we obtain

$$\begin{aligned}\left|\int_{\partial\Omega} \langle \sigma^\eta(t).n, \dot v_d(t)\rangle \,\mathrm{ds}\right| &\le \|\sigma^\eta.n\|_{H^{-\frac12}}\|\dot v_d\|_{H^{\frac12}} \le C(\|\sigma^\eta\|_{L^2} + \|\operatorname{Div}\sigma^\eta\|_{L^2})\|\dot v_d\|_{H^{\frac12}}\\
\le C(\|\dot\sigma^\eta\|_{L^2} + \|\sigma^\eta(0)\|_{L^2} + \|f\|_{L^2} + \|\ddot u^\eta\|_{L^2})\|\dot v_d\|_{H^{\frac12}} &\le C\mathcal{E}^{\frac12}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon^\eta_p, \dot A^\eta)(t) + D,\end{aligned} \tag{3.25}$$

where the positive constants C, D depend on given data only. The boundary integrals containing microrotations are estimated using the same idea. Note that by the evolution equation for microrotations we have $\operatorname{Div}\nabla\operatorname{axl}(A)^\eta = l_c^{-1}\operatorname{axl}\ddot A^\eta - \mu_c l_c^{-1}\operatorname{axl}(\operatorname{skew}(\nabla u) - A) - l_c^{-1}g$ and the two first terms on the right hand side of the last equality appear in the energy function. Hence, we conclude that

$$\begin{aligned}&\left|\int_0^t\int_{\partial\Omega} \langle \nabla\operatorname{axl}(A^\eta).n, \operatorname{axl}(\ddot B_d)\rangle \,\mathrm{ds}\, d\tau\right| + \left|\int_{\partial\Omega} \langle \nabla\operatorname{axl}(A^\eta)(t).n, \operatorname{axl}(\dot B_d)(t)\rangle \,\mathrm{ds}\right|\\
&\qquad\le C\int_0^t \mathcal{E}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon^\eta_p, \dot A^\eta)(\tau)\, d\tau + C_1\mathcal{E}^{\frac12}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon^\eta_p, \dot A^\eta)(t) + C_2\,,\end{aligned} \tag{3.26}$$

where all positive constants C, C_1, C_2 do not depend on η. Inserting (3.22), (3.23), (3.24), (3.25) and (3.26) into (3.21) we finally arrive at the following inequality

$$\mathcal{E}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon^\eta_p, \dot A^\eta)(t) \le C_1\mathcal{E}^{\frac12}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon^\eta_p, \dot A^\eta)(t) + C_2\int_0^t \mathcal{E}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon^\eta_p, \dot A^\eta)(\tau)\, d\tau + C_3\,,$$

where C_1, C_2, C_3 do not depend on η. This inequality immediately implies that

$$\mathcal{E}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon_p^\eta, \dot A^\eta)(t) \le D_1 \int_0^t \mathcal{E}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon_p^\eta, \dot A^\eta)(\tau)\, d\tau + D_2\,,$$

where again the constants D_1, D_2 do not depend on η. Gronwall's inequality completes the proof. ■

The energy estimate for time derivatives yields that the sequence $(\dot\sigma^\eta, \nabla \dot A^\eta, \ddot u^\eta, \ddot A^\eta)$ is $L^\infty(L^2)$–bounded. This implies that the sequence $(\sigma^\eta, \nabla A^\eta, \dot u^\eta, \dot A^\eta)$ is also $L^\infty(L^2)$–bounded. Note, that for example the equality $\sigma^\eta(t) = \int_0^t \dot\sigma^\eta(\tau)\, d\tau + \sigma^\eta(0)$ implies that $\|\sigma^\eta\|_{L^2} \le \int_0^t \|\dot\sigma^\eta\|_{L^2} + \|\sigma^\eta(0)\|_{L^2}$. Moreover, by the coercivity of the energy (3.16) we have that the sequences $\{\varepsilon^\eta\}$ and $\{\varepsilon_p^\eta\}$ are $L^\infty(L^2)$–bounded. Hence, for a subsequence (again denoted using the superscript η) we have: for all $T > 0$

$$\begin{array}{lll}
\sigma^\eta \stackrel{*}{\rightharpoonup} \sigma & \text{in} & L^\infty((0,T), L^2(\Omega, \mathrm{Sym}(3)))\,, \\
\dot\sigma^\eta \stackrel{*}{\rightharpoonup} \dot\sigma & \text{in} & L^\infty((0,T), L^2(\Omega, \mathrm{Sym}(3)))\,, \\
A^\eta \stackrel{*}{\rightharpoonup} A & \text{in} & L^\infty((0,T), H^1(\Omega, \mathfrak{so}(3,\mathbb{R})))\,, \\
\ddot A^\eta \stackrel{*}{\rightharpoonup} \ddot A & \text{in} & L^\infty((0,T), L^2(\Omega, \mathfrak{so}(3,\mathbb{R})))\,, \\
u^\eta \stackrel{*}{\rightharpoonup} u & \text{in} & L^\infty((0,T), H^1(\Omega, \mathbb{R}^3))\,, \\
\ddot u^\eta \stackrel{*}{\rightharpoonup} \ddot u & \text{in} & L^\infty((0,T), L^2(\Omega, \mathbb{R}^3))\,, \\
\varepsilon^\eta \stackrel{*}{\rightharpoonup} \varepsilon & \text{in} & L^\infty((0,T), L^2(\Omega, \mathrm{Sym}(3)))\,, \\
\dot\varepsilon^\eta \stackrel{*}{\rightharpoonup} \dot\varepsilon & \text{in} & L^\infty((0,T), L^2(\Omega, \mathrm{Sym}(3)))\,, \\
\varepsilon_p^\eta \stackrel{*}{\rightharpoonup} \varepsilon_p & \text{in} & L^\infty((0,T), L^2(\Omega, \mathrm{Sym}(3)))\,, \\
\dot\varepsilon_p^\eta \stackrel{*}{\rightharpoonup} \dot\varepsilon_p & \text{in} & L^\infty((0,T), L^2(\Omega, \mathrm{Sym}(3)))
\end{array}$$

and the limit functions satisfy

$$\begin{array}{rcl}
\ddot u - \mathrm{Div}\,\sigma &=& f\,, \\
\sigma &=& 2\mu\,(\varepsilon - \varepsilon_p) + 2\,\mu_c\,(\mathrm{skew}(\nabla u) - A) + \lambda\,\mathrm{tr}\,[\varepsilon]\cdot 1\!\!1\,, \\
\ddot A - l_c\,\Delta\,\mathrm{axl}(A) &=& \mu_c\ \mathrm{axl}(\mathrm{skew}(\nabla u) - A) + g\,, \\
\dot\varepsilon_p &=& \mathfrak{f}_0 = weak - \lim \mathfrak{f}_\eta(T_E^\eta)\,, \quad T_E = 2\mu\,(\varepsilon - \varepsilon_p)\,, \\
u_{|\partial\Omega} &=& u_d\,, \quad A_{|\partial\Omega} = A_d\,, \\
\multicolumn{3}{c}{u(0) = u^0\,, \quad \dot u(0) = u^1\,, \quad A(0) = A^0\,, \quad \dot A(0) = A^1\,, \quad \varepsilon_p(0) = \varepsilon_p^0\,.}
\end{array} \tag{3.27}$$

To finish the existence theory for our system we need only to prove that

$$\mathfrak{f}_0(t,x) \in \mathfrak{f}(T_E(t,x)) \quad \text{a.e. in } \ (0,T)\times\Omega\,. \tag{3.28}$$

To do this we follow the standard idea which is based on the following property: the graph of a maximal monotone operator is weakly-strongly closed. Thus, we are going to improve the weak convergence of the sequence $\{T_E^\eta\}$.

Theorem 3.3 (Strong convergence of stresses)
Let us assume that the given data satisfy all requirements of Theorem 3.2. Then $\mathcal{E}(u^\eta - u^\nu, \varepsilon^\eta - \varepsilon^\nu, \varepsilon_p^\eta - \varepsilon_p^\nu, A^\eta - A^\nu)(t) \to 0$ for $\eta, \nu \to 0^+$ uniformly on bounded time intervals.

Proof. We use the standard energy method and calculate the time derivative of the energy evaluated on the differences of two approximation steps. Hence, we obtain

$$
\begin{aligned}
\dot{\mathcal{E}}(u^\eta - u^\nu, \varepsilon^\eta - \varepsilon^\nu, \varepsilon_p^\eta - \varepsilon_p^\nu, A^\eta - A^\nu)(t) &= \int_\Omega \langle \dot{u}^\eta - \dot{u}^\nu \,,\, \ddot{u}^\eta - \ddot{u}^\nu\rangle \, \mathrm{dx} \\
&+4\int_\Omega \langle \operatorname{axl}(\dot{A}^\eta - \dot{A}^\nu)\,,\, \operatorname{axl}(\ddot{A}^\eta - \ddot{A}^\nu)\rangle \, \mathrm{dx} + 2\mu \int_\Omega \langle \varepsilon^\eta - \varepsilon^\nu - \varepsilon_p^\eta + \varepsilon_p^\nu, \dot{\varepsilon}^\eta - \dot{\varepsilon}^\nu - \dot{\varepsilon}_p^\eta + \dot{\varepsilon}_p^\nu\rangle \, \mathrm{dx} \\
&+\lambda \int_\Omega \operatorname{tr}[\varepsilon^\eta - \varepsilon^\nu] \operatorname{tr}[\dot{\varepsilon}^\eta - \dot{\varepsilon}^\nu] \, \mathrm{dx} + 4l_c \int_\Omega \langle \nabla \operatorname{axl}(A^\eta - A^\nu), \nabla \operatorname{axl}(\dot{A}^\eta - \dot{A}^\nu)\rangle \, \mathrm{dx} \\
&+2\mu_c \int_\Omega \langle \operatorname{skew}(\nabla u^\eta - \nabla u^\nu) - A^\eta + A^\nu, \operatorname{skew}(\nabla \dot{u}^\eta - \nabla \dot{u}^\nu) - \dot{A}^\eta + \dot{A}^\nu \rangle \, \mathrm{dx} \,.
\end{aligned}
$$

Using that the given data for both approximation steps are the same we conclude that

$$
\dot{\mathcal{E}}(u^\eta - u^\nu, \varepsilon^\eta - \varepsilon^\nu, \varepsilon_p^\eta - \varepsilon_p^\nu, A^\eta - A^\nu)(t) = -\int_\Omega \langle T_E^\eta - T_E^\nu, \mathfrak{f}_\eta(T_E^\eta) - \mathfrak{f}_\nu(T_E^\nu)\rangle \, \mathrm{dx} \,. \tag{3.29}
$$

Next, to estimate the right hand side of (3.29), we use the standard procedure from the theory of maximal monotone operators (compare with the proof of Theorem 1 p. 147 in [1]). This yields that

$$
\dot{\mathcal{E}}(u^\eta - u^\nu, \varepsilon^\eta - \varepsilon^\nu, \varepsilon_p^\eta - \varepsilon_p^\nu, A^\eta - A^\nu)(t) \le (\eta + \nu) C(T) \,,
$$

where the positive constant $C(T)$ does not depend on η and ν. The last inequality completes immediately the proof. ■

Theorem 3.3 implies that the sequence of stresses $\{T_E^\eta\}$ is a Cauchy sequence in the space $L^\infty((0,T); L^2(\Omega; \mathrm{Sym}(3)))$. Hence, $\{T_E^\eta\}$ converges strongly to T_E. Moreover, by the definition of the Yosida approximation we have $\mathfrak{f}_\eta(T_E^\eta) \in \mathfrak{f}(J_\eta(T_E^\eta))$, where $J_\eta(T_E^\eta) = T_E^\eta - \eta \mathfrak{f}_\eta(T_E^\eta)$ is the resolvent operator. We see that J_η is a global Lipschitz operator and therefore the sequence $\{J_\eta(T_E^\eta)\}$ converges strongly to T_E. Consequently, the sequence $(J_\eta(T_E^\eta), \mathfrak{f}_\eta(T_E^\eta))$ is contained in the graph of the maximal monotone operator $\mathfrak{f}$ and converges strongly-weakly to $(T_e, \mathfrak{f}_0)$. Hence, the maximality of $\mathfrak{f}$ yields that $\mathfrak{f}_0$ belongs to the set $\mathfrak{f}(T_E)$ and the limit functions $(u, \varepsilon, \varepsilon_p, A)$ satisfy (3.14). This finishes the existence part. ■

Next, we study the uniqueness of solutions for system (3.14).

Theorem 3.4 (Uniqueness of solutions)
Let us assume that the given data $f, u_d, A_d, \varepsilon_p^0$ satisfy all requirements of Theorem 3.2 Then the system (3.14) possesses a unique, global in time solution $(u, \varepsilon, \varepsilon_p, A)$.

Proof. The proof is based on the energy method. Assume that $(u^1, \varepsilon^1, \varepsilon_p^1, A^1)$ and $(u^2, \varepsilon^2, \varepsilon_p^2, A^2)$ are two solutions of (3.14) for the same given data. Then for the energy function evaluated on differences of these solutions we have

$$\dot{\mathcal{E}}(u^1-u^2, \varepsilon^1-\varepsilon^2, \varepsilon_p^1-\varepsilon_p^2, A^1-A^2)(t) = \int_\Omega \langle \dot u^1 - \dot u^2\,,\, \ddot u^1 - \ddot u^2\rangle\,\mathrm{dx}$$

$$+4\int_\Omega \langle \mathrm{axl}(\dot A^1 - \dot A^2)\,,\, \mathrm{axl}(\ddot A^1 - \ddot A^2)\rangle\,\mathrm{dx} + 2\mu\int_\Omega \langle \varepsilon^1-\varepsilon^2-\varepsilon_p^1+\varepsilon_p^2, \dot\varepsilon^1-\dot\varepsilon^2-\dot\varepsilon_p^1+\dot\varepsilon_p^2\rangle\,\mathrm{dx}$$

$$+\lambda\int_\Omega \mathrm{tr}\,[\varepsilon^\eta - \varepsilon^\nu]\mathrm{tr}\,[\dot\varepsilon^\eta - \dot\varepsilon^\nu]\,\mathrm{dx} + 4l_c\int_\Omega \langle \nabla\,\mathrm{axl}(A^1-A^2), \nabla\,\mathrm{axl}(\dot A^1 - \dot A^2)\rangle\,\mathrm{dx}$$

$$+2\mu_c\int_\Omega \langle \mathrm{skew}(\nabla u^1 - \nabla u^2) - A^1 + A^2, \mathrm{skew}(\nabla \dot u^1 - \nabla \dot u^2) - \dot A^1 + \dot A^2\rangle\,\mathrm{dx}$$

$$= -\int_\Omega \langle T_E^1 - T_E^2, \dot\varepsilon_p^1 - \dot\varepsilon_p^2\rangle\,\mathrm{dx} \le 0\,.$$

This implies that

$$\mathcal{E}(u^1-u^2, \varepsilon^1-\varepsilon^2, \varepsilon_p^1-\varepsilon_p^2, A^1-A^2)(t) \le \mathcal{E}(u^1-u^2, \varepsilon^1-\varepsilon^2, \varepsilon_p^1-\varepsilon_p^2, A^1-A^2)(0) = 0$$

and the statement is a consequence of the coerciveness of the energy function. ■

At the end of this section we formulate the existence and uniqueness theorem, which we have proved:

Theorem 3.5 (Existence for the dynamical model)
Suppose that the given data f, g, u_d, A_d satisfy: for all times $T > 0$

$$\begin{aligned}
&f \in C^1([0,T], L^2(\Omega,\mathbb{R}^3))\,, && g \in C^1([0,T], L^2(\Omega,\mathbb{R}^3))\\
&u_d \in C^2([0,T], H^{\frac12}(\partial\Omega,\mathbb{R}^3))\,, && \partial_{ttt}u_d \in L^2((0,T); H^{\frac12}(\partial\Omega,\mathbb{R}^3)\,,\\
&A_d \in C^2([0,T], H^{\frac32}(\partial\Omega,\mathfrak{so}(3,\mathbb{R})))\,, && \partial_{ttt}A_d \in L^2((0,T); H^{\frac12}(\partial\Omega,\mathfrak{so}(3,\mathbb{R}))\,.
\end{aligned}$$

Moreover, assume that the initial data have the regularity

$$u^0, u^1 \in H^1(\Omega,\mathbb{R}^3)\,, A^0 \in H^2(\Omega,\mathfrak{so}(3,\mathbb{R}))\,,\ A^1 \in H^1(\Omega,\mathfrak{so}(3,\mathbb{R}))\,,\ \varepsilon_p^0 \in L^2(\Omega,\mathrm{Sym}(3))$$

and satisfy the compatibility condition

$$u^0(x) = u_d(x,0)\,, u^1(x) = \dot u_d(x,0)\,, A^0(x) = A_d(x,0)\,, A^1(x) = \dot A_d(x,0)\ \ \textit{for } x\in\partial\Omega\,.$$

Additionally, suppose that the initial data is chosen such that the initial value of the reduced Eshelby tensor $T_E(0) = 2\mu\,(\frac12(\nabla u^0 + \nabla^T u^0) - \varepsilon_p^0)$ belongs to the domain of the maximal monotone operator $\mathfrak{f}$. Then the system (3.14) possesses a global in time, unique solution $(u, \varepsilon, \varepsilon_p, A)$ with the regularity: for all times $T > 0$

$$\begin{aligned}
&u \in H^{1,\infty}((0,T), H^1(\Omega,\mathbb{R}^3))\,, && \ddot u \in L^\infty((0,T), L^2(\Omega,\mathbb{R}^3))\,,\\
&A \in L^\infty((0,T), H^2(\Omega,\mathfrak{so}(3,\mathbb{R})))\,, && \ddot A \in L^\infty((0,T), L^2(\Omega,\mathfrak{so}(3,\mathbb{R})))\,,\\
&\varepsilon, \varepsilon_p \in H^{1,\infty}((0,T), L^2(\Omega,\mathrm{Sym}(3)))\,.
\end{aligned}$$

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Address:

Patrizio Neff
Fachbereich Mathematik
Darmstadt University of Technology
Schlossgartenstrasse 7
64289 Darmstadt, Germany
email: neff@mathematik.tu-darmstadt.de

Krzysztof Chełmiński
Faculty of Mathematics and Information Science,
Warsaw University of Technology
Pl. Politechniki 1,
00-661 Warsaw, Poland
email: kchelmin@mini.pw.edu.pl

Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times 3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times 3}}$ (we use these symbols indifferently for tensors and vectors). The identity tensor on $\mathbb{M}^{3\times 3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[X] = \langle X, 1\!\!1\rangle$. We let Sym and PSym denote the symmetric and positive definite symmetric tensors respectively. We adopt the usual abbreviations of Lie-algebra theory, i.e. $\mathfrak{so}(3,\mathbb{R}) := \{X \in \mathbb{M}^{3\times 3} \,|X^T = -X\}$ are skew symmetric second order tensors and $\mathfrak{sl}(3,\mathbb{R}) := \{X \in \mathbb{M}^{3\times 3} \,|\operatorname{tr}[X] = 0\}$ are traceless tensors. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$. For $X \in \mathbb{M}^{3\times 3}$ we set for the deviatoric part $\operatorname{dev} X = X - \frac{1}{3}\operatorname{tr}[X]\,1\!\!1 \in \mathfrak{sl}(3,\mathbb{R})$. For a second order tensor X we let $X.e_i$ be the application of the tensor X to the column vector e_i and we define the third order tensor $\mathfrak{h} = \mathrm{D_x}X(x) = (\nabla(X(x).e_1), \nabla(X(x).e_2), \nabla(X(x).e_3)) = (\mathfrak{h}^1, \mathfrak{h}^2, \mathfrak{h}^3) \in (\mathbb{M}^{3\times 3})^3$. For $\mathfrak{h}$ we set $\|\mathfrak{h}\|^2 = \sum_{i=1}^3 \|\mathfrak{h}^i\|^2$ together with $\operatorname{sym}(\mathfrak{h}) := (\operatorname{sym}\mathfrak{h}^1, \operatorname{sym}\mathfrak{h}^2, \operatorname{sym}\mathfrak{h}^3)$ and $\operatorname{tr}[\mathfrak{h}] := (\operatorname{tr}\left[\mathfrak{h}^1\right], \operatorname{tr}\left[\mathfrak{h}^2\right], \operatorname{tr}\left[\mathfrak{h}^3\right]) \in \mathbb{R}^3$. The first and second differential of a scalar valued function $W(F)$ are written $D_F W(F).H$ and $D_F^2 W(F).(H,H)$, respectively. Sometimes we use also $\partial_X W(X)$ to denote the first derivative of W with respect to X. We employ the standard notation of Sobolev spaces, i.e. $L^2(\Omega), H^{1,2}(\Omega), H_\circ^{1,2}(\Omega)$, which we use indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions.