# Stokes Resolvent Systems in an Infinite Cylinder

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**Abstract**

In an infinite cylinder $\Omega = \Sigma \times \mathbb{R}$, where $\Sigma \subset \mathbb{R}^{n-1}, n \geq 3$, is a bounded domain of $C^{1,1}$ class, we study the unique solvability of Stokes resolvent systems in $L^q(\mathbb{R}; L^2(\Sigma))$ for $1 < q < \infty$ and in vector-valued homogeneous Besov spaces $\dot{\mathcal{B}}^s_{pq}(\mathbb{R}; L^r(\Sigma))$ for $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, $1 < r < \infty$. By a partial Fourier transform along the axis of the cylinder $\Omega$ the given system is reduced to a parametrized system on $\Sigma$, for which parameter independent estimates are proved. For further applications we obtain even parameter independent estimates in $L^r(\Sigma), 1 < r < \infty$, in the non-solenoidal case prescribing an arbitrary divergence $g = \operatorname{div} u$. Then operator-valued multiplier theorems are used for the final estimates of the Stokes resolvent systems in $\Omega$.

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## 1 Introduction and Main Results

In this paper we study the Stokes resolvent system

$$(R_\lambda) \qquad \begin{aligned} \lambda u - \Delta u + \nabla p &= f \quad \text{in } \Omega \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega = \Sigma \times \mathbb{R}$ is an infinite straight cylinder with cross-section $\Sigma \subset \mathbb{R}^{n-1}, n \geq 3$, a bounded domain of $C^{1,1}$ class.

Much efforts have been made to study Stokes and Navier-Stokes systems in unbounded cylindrical domains due to their great importance for practical application (see e.g. [7], [16], [17], [21] - [30]). Most of these papers are restricted to stationary

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systems, whereas instationary systems have been less studied. As is well known, the analytic semigroup approach to Stokes and Navier-Stokes equations is a very convenient tool to get existence, uniqueness and decay of strong solutions; to this end, resolvent estimates of the Stokes operator have to be obtained.

Up to now the Stokes resolvent system has been analyzed e.g. in [1] - [5], [9] - [11], [14] and [15]. Resolvent estimates for the Stokes operator in $L^q$-spaces in the case of $\operatorname{div} u = 0$ or $\operatorname{div} u \neq 0$ in $(R_\lambda)$ were obtained for bounded and exterior domains as well as for bent and perturbed half spaces in [9], [10] and [18] (see Introduction and References in [10] for more details); corresponding results in weighted $L^q$-spaces can be found in [11], [14], [15]. In [2], [3] and [5], $L^q$-resolvent estimates of the Stokes operator in an infinite layer $\mathbb{R}^{n-1} \times (0,1)$ are considered; the main idea is to apply the classical Fourier multiplier theorem directly to an explicit representation of solutions to a boundary value problems of ordinary differential equations in $(0,1)$ which are obtained by the application of the $(n-1)$-dimensional Fourier transform to the original resolvent system. Recently Stokes resolvent estimates in layer-like domains were obtained in [4] using the theory of pseudo-differential operators.

For Stokes resolvent estimates in cylindrical domains $\Omega = \Sigma \times \mathbb{R}$ we follow in principle the approach in [2], [3] and [5] by applying a partial Fourier transform. However, the corresponding differential equations are elliptic boundary value problems in $\Sigma$ and the Fourier multipliers are operator–valued. For a different result on resolvent estimates in the Bloch space of uniformly square integrable functions on a cylinder we refer to [28].

In this paper we use the following notations. Let $\Omega = \Sigma \times \mathbb{R}$ be an infinite cylinder of $\mathbb{R}^n$ with bounded cross section $\Sigma \subset \mathbb{R}^{n-1}$ and with a generic point $x \in \Omega$ written in the form $x = (x', x_n) \in \Omega$, where $x' \in \Sigma$ and $x_n \in \mathbb{R}$. Similarly, differential operators in $\mathbb{R}^n$ are splitted, in particular, $\Delta = \Delta' + \partial_n^2$ and $\nabla = (\nabla', \partial_n)$. For $\varepsilon \in (0, \frac{\pi}{2})$, let $S_\varepsilon$ denote the sector of the complex plane

$$\{\lambda \in \mathbb{C}; \lambda \neq 0, |\arg\lambda| < \tfrac{\pi}{2} + \varepsilon\}.$$

The partial Fourier transform in the variable $x_n$ is denoted by $\mathcal{F}$ or $\hat{}$ and its inverse by $\mathcal{F}^{-1}$ or $\check{}$.

Let $r \in (1, \infty)$ and $s \in \mathbb{R}_+$. Then $L^r(\Sigma)$ and $W^{s,r}(\Sigma)$ are the usual Lebesgue and Sobolev spaces with norm $\|\cdot\|_{r;\Sigma}$ and $\|\cdot\|_{s,r;\Sigma}$, respectively. Moreover, $\hat{W}^{1,r}(\Sigma)$ is the homogeneous Sobolev space, i.e.,

$$\hat{W}^{1,r}(\Sigma) = \{u \in L^1_{loc}(\bar{\Sigma})/\mathbb{R}; \nabla' u \in L^r(\Sigma)\}, \quad \|u\|_{\hat{W}^{1,r}(\Sigma)} = \|\nabla' u\|_{L^r(\Sigma)},$$

and $\hat{W}^{-1,r}(\Sigma) = (\hat{W}^{1,r}(\Sigma))^*$ is its dual with the norm $\|\cdot\|_{-1,r}$. We do not distinguish among spaces of scalar functions and vector-valued functions as long as no confusion arises. In particular, we use the short notation $\|u, v\|_r$ for $\|u\|_r + \|v\|_r$, even if $u$ and $v$ are tensors of different order.

For $q \in (1, \infty)$ and a Banach space $X$, let $L^q(\mathbb{R}; X)$ be the Bochner space of all $X$-valued measurable functions with finite norm

$$\|u\|_{L^q(\mathbb{R};X)} = (\textstyle\int_{\mathbb{R}} \|u(t)\|_X^q dt)^{1/q}.$$

Then $L^q(L^2)_\sigma$ is defined as the completion of the set $C^\infty_{0,\sigma}(\Omega)$ in $L^q(\mathbb{R}; L^2(\Sigma))$, where

$$C^\infty_{0,\sigma}(\Omega) = \{u \in C^\infty_0(\Omega)^n; \quad \operatorname{div} u = 0\}.$$

Finally, $W^{k;q,2}(\Omega), k \in \mathbb{N}$, denotes the Banach space of all functions in $\Omega$ whose derivatives of order up to $k$ belong to $L^q(\mathbb{R}; L^2(\Sigma))$ with norm

$$\|u\|_{W^{k;q,2}} = \big(\textstyle\sum_{|\alpha|\le k} \|D^\alpha u\|^2_{L^q(\mathbb{R};L^2(\Sigma))}\big)^{1/2},$$

where $D^\alpha u = \partial_1^{\alpha_1} \cdot \ldots \cdot \partial_n^{\alpha_n} u$ for a multi-index $\alpha \in \mathbb{N}_0^n$ of order $|\alpha| \le k$, and $W^{1;q,2}_0(\Omega)$ is the completion of the set $C^\infty_0(\Omega)^n$ in $W^{1;q,2}(\Omega)$.

For $p, q \in [1, \infty]$ and $s \in \mathbb{R}$, we use the notation $\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}; X)$ for the homogeneous Besov space of $X$-valued distributions on $\mathbb{R}$ with norm

$$\|u\|_{\dot{\mathcal{B}}^s_{p,q}(\mathbb{R};X)} = \|(2^{sk}\|\mathcal{F}^{-1}(\psi_k \mathcal{F} u)\|_{L^p(\mathbb{R};X)})_{k\in\mathbb{Z}}\|_{l_q},$$

where $(\psi_k)_{k\in\mathbb{Z}}$ is a dyadic resolution of the identity on $\mathbb{R}$ such that $\psi_k(\xi) = \varphi(2^{-k}\xi)$ for $k \in \mathbb{Z}$ with some function $\varphi \in C^\infty_0(\mathbb{R})$; here $\varphi$ satisfies

$$\operatorname{supp}\varphi \in \Big[\frac{1}{2}, 2\Big], \quad \varphi(\xi) > 0 \ \text{ for } \frac{1}{2} < |\xi| < 2 \quad \text{and} \sum_{k=-\infty}^{\infty} \varphi(2^{-k}\xi) = 1, \quad \xi \neq 0,$$

and $l_q$ is the space of scalar sequences $(a_k)_{k\in\mathbb{Z}}$ such that $\|(a_k)_{k\in\mathbb{Z}}\|_{l_q} = \big(\sum_{k=-\infty}^\infty |a_k|^q\big)^{1/q} < \infty$.

For notational convenience, as long as no confusion arises, we denote constants appearing in the proofs by the same symbol, say $c$ or $C$, even though they may be different line by line.

The main results of this paper are as follows.

**Theorem 1.1** *Let $\Sigma$ be a bounded domain of $C^{1,1}$-class, $\alpha_0 > 0$ the smallest eigenvalue of the Dirichlet Laplacian in $\Sigma$, let $0 < \varepsilon < \frac{\pi}{2}$ and $1 < q < \infty$. Then for every $f \in L^q(\mathbb{R}; L^2(\Sigma))$, every $\alpha \in (0, \alpha_0)$ and $\lambda \in -\alpha + S_\varepsilon$, there exists a unique solution $\{u, p\}$ to $(R_\lambda)$ satisfying $u, \nabla^2 u, \nabla p \in L^q(\mathbb{R}; L^2(\Sigma))$ and the estimate*

$$\|(\lambda + \alpha)u, \nabla^2 u, \nabla p\|_{L^q(\mathbb{R},L^2(\Sigma))} \le C\|f\|_{L^q(\mathbb{R},L^2(\Sigma))}, \tag{1.1}$$

*where the constant $C$ is independent of $\lambda$ and dependent only on $\alpha, \varepsilon, q$ and $\Sigma$.*

In particular we obtain from Theorem 1.1 the following corollary about resolvent estimates of the Stokes operator in the cylinder $\Omega$.

**Corollary 1.2** *Let $A = A_{q,2}, 1 < q < \infty$, be the Stokes operator on $\Omega$ defined by*

$$D(A) = W^{2;q,2}(\Omega) \cap W^{1;q,2}_0(\Omega) \cap L^q(L^2)_\sigma \subset L^q(L^2)_\sigma, \quad Au = -P\Delta u, \tag{1.2}$$

*where $P$ is the Helmholtz projection in $L^q(\mathbb{R}; L^2(\Sigma))$ (see [12]). Then, for every $\varepsilon \in (0, \frac{\pi}{2})$ and $\alpha \in (0, \alpha_0)$, $-\alpha + S_\varepsilon$ is contained in the resolvent set of $-A$, and the estimate*

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(L^q(L^2)_\sigma)} \le \frac{C}{|\lambda + \alpha|} \quad \forall \lambda \in -\alpha + S_\varepsilon \tag{1.3}$$

*holds with* $C = C(\alpha, q, \varepsilon, \Sigma)$.

*As a consequence, the Stokes operator generates a bounded analytic semigroup* $\{e^{-tA_{q,2}}; t \geq 0\}$ *on* $L^q(L^2)_\sigma$ *satisfying for all* $\alpha \in (0, \alpha_0)$ *the estimate*

$$\|e^{-tA_{q,2}}\|_{\mathcal{L}(L^q(L^2)_\sigma)} \leq C\, e^{-\alpha t}, \quad 0 \leq t < \infty, \tag{1.4}$$

*with* $C = C(\alpha, q, \varepsilon, \Sigma)$.

Moreover, we have a corresponding result in vector-valued homogeneous Besov spaces for which the space $L^2(\Sigma)$ may be replaced by $L^r(\Sigma)$ for any $1 < r < \infty$.

**Theorem 1.3** *Assume for* $\Sigma$, $\alpha_0 > 0$ *and* $0 < \varepsilon < \frac{\pi}{2}$ *the same as in Theorem 1.1; furthermore, let* $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$ *and* $1 < r < \infty$. *Given* $f \in \dot{\mathcal{B}}^s_{pq}(\mathbb{R}; L^r(\Sigma))$, $\alpha \in (0, \alpha_0)$ *and* $\lambda \in -\alpha + S_\varepsilon$, *there exists a unique solution* $\{u, p\}$ *to* $(R_\lambda)$ *satisfying* $u, \nabla^2 u, \nabla p \in \dot{\mathcal{B}}^s_{pq}(\mathbb{R}; L^r(\Sigma))$ *and the estimate*

$$\|(\lambda + \alpha)u, \nabla^2 u, \nabla p\|_{\dot{\mathcal{B}}^s_{pq}(\mathbb{R}; L^r(\Sigma))} \leq C\|f\|_{\dot{\mathcal{B}}^s_{pq}(\mathbb{R}; L^r(\Sigma))}, \tag{1.5}$$

*where the constant* $C > 0$ *is independent of* $\lambda$ *and dependent only on* $\alpha, \varepsilon, p, q, r, s$ *and* $\Sigma$.

For the proof of the above Theorems we apply to the system $(R_\lambda)$ the partial Fourier transform along the axis of the cylinder $\Omega$ to reduce the problem to a parametrized system on the cross-section $\Sigma$ with Fourier phase variable $\xi \in \mathbb{R}$ as a parameter, for which we will obtain parameter independent estimates of the solution. To this end, in principle, we follow the argument in [10] using perturbation and localization techniques. Therefore, we need to consider also the non-solenoidal case in the parametrized systems for the whole space $\mathbb{R}^{n-1}$ (Theorem 2.1), the half space $\mathbb{R}^{n-1}_+$ (Theorem 2.2) and bent half spaces $\Sigma_\omega$ (Theorem 2.3 ) (see (2.2) below for the definition of $\Sigma_\omega$). Note that even for bounded domains (Theorem 3.4) we consider the non-solenoidal case and show Fréchet differentiability of operator-valued multiplier functions concerned with $(R_{\lambda,\xi})$ (Corollary 3.6). This more general approach allows to analyze the so-called generalized Stokes resolvent system, i.e. with prescribed divergence $g$, in an infinite straight cylinder with application to unbounded cylindrical domains with several exits to infinity, see a forthcoming article [13]. Hence $(R_\lambda)$ will be replaced by the $\xi$-dependent elliptic system in the sense of Agmon, Douglis and Nirenberg [6]

$$(R_{\lambda,\xi}) \qquad \begin{array}{rcll} (\lambda + \xi^2 - \Delta')\hat{u}' + \nabla'\hat{p} &=& \hat{f}' & \text{in } \Sigma \\ (\lambda + \xi^2 - \Delta')\hat{u}_n + i\xi\hat{p} &=& \hat{f}_n & \text{in } \Sigma \\ \operatorname{div}'\hat{u}' + i\xi\hat{u}_n &=& \hat{g} & \text{in } \Sigma \\ \hat{u}' = 0, \quad \hat{u}_n &=& 0 & \text{on } \partial\Sigma. \end{array}$$

The proofs for the cases $\Sigma = \mathbb{R}^{n-1}$ and $\Sigma = \mathbb{R}^{n-1}_+$ are based on the theory of Fourier multipliers and elliptic boundary value problems. A consideration of the sum of negative homogeneous Sobolev spaces and $L^r$ spaces with weight $1/|\xi|, \xi \neq 0$, is needed concerning the divergence $\hat{g}$, and some scaling arguments are used for $\xi$-independent estimates of solutions to $(R_{\lambda,\xi})$. For bounded $\Sigma$ the Hilbert space setting of $(R_{\lambda,\xi})$ in $L^2(\Sigma) \times W^{1,2}(\Sigma)$ is studied first (Lemma 3.2); for general $r \in (1,\infty)$ mapping properties of the parametrized Stokes operator (see (3.1) for the definition) (Lemma 3.3) are shown which enable us to obtain the final estimate. Having obtained parameter independent estimates of the system $(R_{\lambda,\xi})$ in $\Sigma$, we use operator-valued multiplier theorems (see [8] and [32]) for the estimates of solutions to $(R_\lambda)$ in the whole cylinder $\Omega$.

We acknowledge our thanks to C. Kaiser for a result on operator-valued Fourier multiplier theory in vector-valued homogeneous Besov spaces [19].

This paper is organized as follows. In Section 2 we obtain the estimates for $(R_{\lambda,\xi})$ on whole, half and bent half spaces. Section 3 is devoted to obtain the estimate for $(R_{\lambda,\xi})$ on bounded domains. In Section 4 the proofs of Theorems 1.1 and 1.3 are given.

## 2 The Problem $(R_{\lambda,\xi})$ in Half Spaces

Consider the parametrized resolvent problem $(R_{\lambda,\xi})$ for all $\xi \in \mathbb{R}$ and all $\lambda \in S_\varepsilon$, $0 < \varepsilon < \frac{\pi}{2}$. In this section $\Sigma$ denotes $\mathbb{R}^{n-1}$ or the half space

$$\Sigma = \mathbb{R}^{n-1}_+ = \{x' = (x_1, x''); x'' \in \mathbb{R}^{n-2}, x_1 > 0\}, \tag{2.1}$$

or a bent half space

$$\Sigma_\omega = \{x' = (x_1, x''); x_1 > \omega(x''), x'' \in \mathbb{R}^{n-2}\}, \tag{2.2}$$

where $\omega$ is a $C^{1,1}$-function. For notational convenience we omit the symbol $\hat{}$ for the one-dimensional Fourier transform; thus

$$u = (u', u_n), p, f, g \quad \text{stand for} \quad \hat{u} = (\hat{u'}, \hat{u_n}), \hat{p}, \hat{f}, \hat{g}.$$

For the divergence $g$ ($\hat{=}\hat{g}$) we need for $r \in (1,\infty)$ the definition of the space $\hat{W}^{-1,r}(\Sigma) + L^r(\Sigma)_{1/\xi}$ parametrized by $\xi \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$. Consider the direct sum $L^r \oplus \mathbb{R}$ (we omit the symbol of the underlying domain $\Sigma$) and its quotient space

$$\hat{L}^r = (L^r \oplus \mathbb{R})/\mathbb{R}.$$

Since $\Sigma$ has unbounded measure, $\hat{L}^r$ equipped with $\|\cdot\|_r$ is isometric to $L^r$. This isomorphism allows to define the intersection of the Banach spaces $\hat{W}^{1,r}$ and $L^r$, namely,

$$\hat{W}^{1,r} \cap L^r_\xi, \quad \|h; \hat{W}^{1,r} \cap L^r_\xi\| := \max(\|\nabla' h\|_r, \|\xi h\|_r),$$

which for fixed $\xi \in \mathbb{R}^*$ is isomorphic to $W^{1,r}$. Obviously $C_0^\infty(\bar{\Sigma}) \subset \hat{W}^{1,r}(\Sigma) \cap L^r_\xi(\Sigma)$ is dense in both $\hat{W}^{1,r}(\Sigma)$ and $L^r_\xi(\Sigma)$, see e.g. [14], Corollary 4.1. This observation implies that, if $r' = r/(r-1)$,

$$\hat{W}^{-1,r} + L^r_{1/\xi} = (\hat{W}^{1,r'} \cap L^{r'}_\xi)^* \cong (W^{1,r'})^*,$$

see e.g. [8], Theorem 2.7.1. The norm of this space is given by

$$\begin{aligned} &\|h; \hat{W}^{-1,r} + L^r_{1/\xi}\| \\ &\qquad = \inf\{\|h_0\|_{-1,r} + \|h_1/\xi\|_r; h = h_0 + h_1, h_0 \in \hat{W}^{-1,r}, h_1 \in L^r\}. \end{aligned} \tag{2.3}$$

Assume that

$$f \in L^r(\Sigma), \quad g \in W^{1,r}(\Sigma).$$

Note that $W^{1,r}(\Sigma)$ is obviously contained in the sum $\hat{W}^{-1,r}(\Sigma) + L^r_{1/\xi}(\Sigma)$.

Now we start with the case $\Sigma = \mathbb{R}^{n-1}$. If $g = g_0 + g_1$, $g_0 \in \hat{W}^{-1,r}$ and $g_1 \in L^r_{1/\xi}$, is any splitting of $g$, Hahn-Banach's theorem implies the existence of a vector field $h \in L^r$ such that

$$g_0 = \operatorname{div}' h, \quad \|g_0\|_{-1,r} = \|h\|_r.$$

An elementary calculation shows that $p$ in $(R_{\lambda,\xi})$ satisfies the equation

$$(\xi^2 - \Delta')p = (\lambda + \xi^2 - \Delta')g - (\operatorname{div}' f' + i\xi f_n). \tag{2.4}$$

Introducing the $(n-1)$-dimensional Fourier transform $\tilde{}$ with respect to $x'$ and with phase variable $s \in \mathbb{R}^{n-1}$ we get that

$$\begin{aligned} \tilde{p} &= \tilde{g} + \frac{\lambda}{\xi^2 + |s|^2}\tilde{g} - \frac{is}{\xi^2 + |s|^2} \cdot \tilde{f}' - \frac{i\xi}{\xi^2 + |s|^2}\tilde{f}_n \\ &= \tilde{g} + \frac{\lambda i s}{\xi^2 + |s|^2} \cdot \tilde{h} + \frac{\lambda \xi}{\xi^2 + |s|^2}(\tilde{g_1}/\xi) - \frac{is}{\xi^2 + |s|^2} \cdot \tilde{f}' - \frac{i\xi}{\xi^2 + |s|^2}\tilde{f}_n. \end{aligned}$$

Obviously the functions

$$m_\xi(s) = \frac{s_j s_k}{\xi^2 + |s|^2}, \quad \frac{s_j \xi}{\xi^2 + |s|^2}, \quad \frac{\xi^2}{\xi^2 + |s|^2}, \quad 1 \le j, k \le n-1,$$

are classical multiplier functions satisfying the pointwise Hörmander-Michlin condition

$$|\,|s|^\alpha \nabla_s^\alpha m_\xi(s)| \le c_\alpha, \quad 0 \ne s \in \mathbb{R}^{n-1},\ \alpha \in \mathbb{N}_0^{n-1},\ |\alpha| \le n-1, \tag{2.5}$$

with constants $c_\alpha > 0$ independent of $\xi \in \mathbb{R}^*$. Then the multiplier theorem [31] applied to $\nabla' p$ and to $\xi p$ yields the estimate

$$\begin{aligned} \|\nabla' p, \xi p\|_r &\le c(\|f, \nabla' g, \xi g\|_r + \|\lambda h, \lambda g_1/\xi\|_r) \\ &\le c(\|f, \nabla' g, \xi g\|_r + \|\lambda g_0\|_{-1,r} + \|\lambda g_1/\xi\|_r). \end{aligned}$$

By $(R_{\lambda,\xi})$ $u'$ and $u_n$ solve the resolvent problems

$$(\lambda + \xi^2 - \Delta')u' = f' - \nabla' p \quad \text{in} \quad \mathbb{R}^{n-1} \tag{2.6}$$

and

$$(\lambda + \xi^2 - \Delta')u_n = f_n - i\xi p \quad \text{in} \quad \mathbb{R}^{n-1} \tag{2.7}$$

of the Laplacian $\Delta'$ with resolvent parameters $\lambda + \xi^2$, respectively. Classical resolvent estimates based on multiplier theory, see [10], Theorem 1.3, for corresponding results on the Stokes resolvent, show the existence of a solution $u = (u', u_n)$ to (2.6), (2.7) satisfying

$$\|(\lambda+\xi^2)u, \sqrt{\lambda+\xi^2}\nabla' u, \nabla'^2 u\|_r \le c\|f, \nabla' p, \xi p\|_r.$$

Defining $\mu = |\lambda+\xi^2|^{1/2}$, we get from the above estimate for $p$

$$\|\mu^2 u, \mu\nabla' u, \nabla'^2 u, \nabla' p, \xi p\|_r \le c\big(\|f, \nabla' g, \xi g\|_r + \|\lambda g_0\|_{-1,r} + \|\lambda g_1/\xi\|_r\big).$$

Now we can prove the following theorem.

**Theorem 2.1** *Let $\Sigma = \mathbb{R}^{n-1}$, $1 < r < \infty$ and*

$$f \in L^r(\Sigma), \quad g \in W^{1,r}(\Sigma). \tag{2.8}$$

*Then for every $\lambda \in S_\varepsilon, 0 < \varepsilon < \frac{\pi}{2}$, and $\xi \in \mathbb{R}^*$ $(R_{\lambda,\xi})$ has a unique solution $(u,p) \in W^{2,r}(\Sigma) \times W^{1,r}(\Sigma)$ satisfying*

$$\|\mu^2 u, \mu\nabla' u, \nabla'^2 u, \nabla' p, \xi p\|_r \le c\big(\|f, \nabla' g, \xi g\|_r + \|\lambda g; \hat{W}^{-1,r} + L^r_{1/\xi}\|\big), \tag{2.9}$$

*where $\mu = |\lambda+\xi^2|^{1/2}$. If the assumptions (2.8) on $f, g$ are satisfied for an additional exponent $s \in (1,\infty)$, then even $(u,p) \in W^{2,s}(\Sigma) \times W^{1,s}(\Sigma)$ and (2.9) holds with $s$ replacing $r$ as well.*

**Proof:** For the existence of a solution it is enough to show that $(u,p)$, the solution of (2.4), (2.6) and (2.7), which was shown to satisfy (2.9), solves the divergence equation

$$\operatorname{div}' u' + i\xi u_n = g.$$

A simple calculation with (2.4),(2.6) and (2.7) yields

$$(\lambda + \xi^2 - \Delta')(\operatorname{div}' u' + i\xi u_n - g) = 0 \quad \text{in} \quad \mathbb{R}^{n-1},$$

from which it follows by standard Fourier multiplier techniques that $\operatorname{div}' u' + i\xi u_n = g$. This technique also yields uniqueness of solutions, i.e., if $(u,p)$ is a solution to $(R_{\lambda,\xi})$ with $f = 0, g = 0$, then $u$ satisfies (2.6) and (2.7) with $f = 0$ and $(\xi^2 - \Delta')p = 0$, which imply $p = 0$ and $u = 0$. The uniqueness argument also yields the additional $L^s$-regularity when (2.8) is satisfied for an additional $s \in (1,\infty) \setminus \{r\}$. ■

The next main step concerns the half space $\Sigma = \mathbb{R}^{n-1}_+$, see (2.1). Just as for $x' = (x_1, x'')$ we write $u' = (u_1, u'')$, $f' = (f_1, f'')$. A simple symmetry argument as follows will reduce $(R_{\lambda,\xi})$ to the case $f = 0, g = 0$ but with nonzero boundary values of $u$.

For a function $h : \Sigma \to \mathbb{R}$ define the even extension $h_e$ by

$$h_e(x_1, x'') = \begin{cases} h(x_1, x'') & \text{for} \quad x_1 > 0 \\ h(-x_1, x'') & \text{for} \quad x_1 < 0, \end{cases}$$

while the odd extension $h_o$ of $h$ is defined by

$$h_o(x_1, x'') = -h(-x_1, x'') \quad \text{for } x_1 < 0.$$

Given $(R_{\lambda,\xi})$ in $\Sigma$, take the even extension $f''_e$ of $f''$, $f_{ne}$ of $f_n$ and $g_e$ of $g$, but the odd extension $f_{1o}$ of $f_1$, and solve $(R_{\lambda,\xi})$ with right-hand side $(f_{1o}, f''_e, f_{ne}), g_e$ in the whole space $\mathbb{R}^{n-1}$. By the uniqueness assertion it is easily seen that the solution $(U, P)$ of this extended problem is even with respect to $x_1$ except for the component $U_1$ which is odd with respect to $x_1$. In particular $U_1 = 0$ for $x_1 = 0$ and, due to (2.9),

$$\begin{aligned}
&\|\mu^2 U, \mu\nabla' U, \nabla'^2 U, \nabla' P, \xi P\|_{r,\Sigma} \\
&\qquad \le c\big(\|f_{1o}, f''_e, f_{ne}, \nabla' g_e, \xi g_e\|_{r,\mathbb{R}^{n-1}} + \|\lambda g_e; \hat{W}^{-1,r}(\mathbb{R}^{n-1}) + L^r(\mathbb{R}^{n-1})_{1/\xi}\|\big) \\
&\qquad \le c\big(\|f, \nabla' g, \xi g\|_r + \|\lambda g; \hat{W}^{-1,r} + L^r_{1/\xi}\|\big),
\end{aligned}$$

where $\mu = |\lambda + \xi^2|^{1/2}$. The second part of this estimate is an easy consequence of the elementary inequalities $\|h_e; \hat{W}^{-1,r}(\mathbb{R}^{n-1})\| \le c\|h; \hat{W}^{-1,r}(\Sigma)\|$ and $\|h_o, h_e\|_{r;\mathbb{R}^{n-1}} \le c\|h\|_{r;\Sigma}$.

In general $U''$ and $U_n$ do not vanish for $x_1 = 0$, but by the trace theorem we may estimate $U|_{\partial\Sigma}$ in the trace space $W^{2-1/r,r}(\partial\Sigma)$. Let $\langle\cdot\rangle_{1-1/r}$ denote the homogeneous trace seminorm, i.e.

$$\langle h\rangle_{1-1/r} = \left(\int_{\partial\Sigma}\int_{\partial\Sigma} \frac{|h(y) - h(y')|^r}{|y - y'|^{n-3+r}}\, dy\, dy'\right)^{1/r}.$$

Then, by a simple scaling argument,

$$\|\mu^{2-1/r} U, \mu^{1-1/r}\nabla'' U\|_{r;\partial\Sigma} + \langle \mu U, \nabla'' U\rangle_{1-1/r} \le c\|\mu^2 U, \mu\nabla' U, \nabla'^2 U\|_{r;\Sigma}$$

leading to

$$\begin{aligned}
&\|\mu^{2-1/r} U, \mu^{1-1/r}\nabla'' U\|_{r;\partial\Sigma} + \langle \mu U, \nabla'' U\rangle_{1-1/r} \\
&\qquad \le c\big(\|f, \nabla' g, \xi g\|_r + \|\lambda g; \hat{W}^{-1,r} + L^r_{1/\xi}\|\big)
\end{aligned} \tag{2.10}$$

with a constant $c = c_\varepsilon$ independent of $\lambda \in S_\varepsilon, \xi \in \mathbb{R}^*$. Subtracting $(U, P)$ in $(R_{\lambda,\xi})$, the parametrized resolvent problem $(R_{\lambda,\xi})$ is reduced to the homogeneous system

$$\begin{aligned}
(\lambda + \xi^2 - \Delta')u' + \nabla' p &= 0 \quad \text{in} \quad \Sigma = \mathbb{R}^{n-1}_+ \\
(\lambda + \xi^2 - \Delta')u_n + i\xi p &= 0 \quad \text{in} \quad \Sigma \\
\operatorname{div}' u' + i\xi u_n &= 0 \quad \text{in} \quad \Sigma
\end{aligned} \tag{2.11}$$

with inhomogeneous boundary values

$$u_1 = 0, \quad u'' = U'', \quad u_n = U_n \quad \text{on} \quad \partial\Sigma. \tag{2.12}$$

In the following we will prove that the problem (2.11), (2.12) has a unique solution $(u, p)$ satisfying

$$\begin{aligned}
&\|\mu^2 u, \mu\nabla' u, \nabla'^2 u, \nabla' p, \xi p\|_{r;\Sigma} \\
&\qquad \le c\big(\|\mu^{2-1/r} U, \mu^{1-1/r}\nabla'' U\|_{r;\partial\Sigma} + \langle \mu U, \nabla'' U\rangle_{1-1/r}\big).
\end{aligned} \tag{2.13}$$

Then, summarizing (2.10) and (2.13), we will get the following theorem.

**Theorem 2.2** *With $\Sigma = \mathbb{R}^{n-1}_+$ the assertions of Theorem 2.1 remain true. In particular the a priori estimate (2.9) holds.*

**Proof:** It remains only to show (2.13), the proof of which is essentially based on multiplier theory with respect to the variable $x'' \in \mathbb{R}^{n-2}$. With the splittings $\Delta' = \partial_1^2 + \Delta''$, $\operatorname{div}' u' = \partial_1 u_1 + \operatorname{div}'' u''$ and $\nabla' = (\partial_1, \nabla'')$ elementary operations, see (2.11), (2.12), yield the fourth order equation

$$\begin{array}{rcllll} (\lambda + \xi^2 - \Delta')(\xi^2 - \Delta')u_1 & = & 0 & \text{in} & \Sigma & \\ u_1 & = & 0 & \text{on} & \partial\Sigma & \quad (2.14) \\ \partial_1 u_1 & = & -\operatorname{div}'' U'' - i\xi U_n & \text{on} & \partial\Sigma. & \end{array}$$

Then we introduce the additional partial Fourier transform $\tilde{}$ with respect to the variable $x'' \in \mathbb{R}^{n-2}$ and with phase variable $\sigma \in \mathbb{R}^{n-2}$. Applying $\tilde{}$ to (2.14) we get the fourth order ordinary differential equation ($s = |\sigma|$)

$$\begin{array}{rcllll} (\lambda + \xi^2 + s^2 - \partial_1^2)(\xi^2 + s^2 - \partial_1^2)\tilde{u}_1 & = & 0 & \text{for} & x_1 > 0 & \\ \tilde{u}_1 & = & 0 & \text{at} & x_1 = 0 & \quad (2.15) \\ \partial_1 \tilde{u}_1 & = & -i\sigma \cdot \tilde{U}'' - i\xi \tilde{U}_n & \text{at} & x_1 = 0. & \end{array}$$

For fixed $\lambda \in S_\varepsilon, \xi \in \mathbb{R}^*$ and $\sigma \in \mathbb{R}^{n-2}$ (2.15) has a unique bounded solution $\tilde{u}_1$ in $(0, \infty)$, namely

$$\tilde{u}_1(x_1, \sigma, \xi) = m_0(x_1, s, \xi)(i\sigma \cdot \tilde{U}'' + i\xi \tilde{U}_n), \tag{2.16}$$

where

$$m_0(x_1, s, \xi) = \frac{e^{-\sqrt{\lambda+\xi^2+s^2}x_1} - e^{-\sqrt{\xi^2+s^2}x_1}}{\sqrt{\lambda + \xi^2 + s^2} - \sqrt{\xi^2 + s^2}}.$$

Furthermore (2.11), (2.16) yield after some elementary operations

$$\begin{aligned} \tilde{p}(x_1, \sigma, \xi) &= -\frac{1}{\xi^2 + s^2}(\lambda + \xi^2 + s^2 - \partial_1^2)\partial_1 \tilde{u}_1 \\ &= -i\frac{\sqrt{\lambda + \xi^2 + s^2} + \sqrt{\xi^2 + s^2}}{\sqrt{\xi^2 + s^2}} e^{-\sqrt{\xi^2+s^2}x_1}(\sigma \cdot \tilde{U}'' + \xi \tilde{U}_n), \end{aligned}$$

so that $p$ satisfies the boundary condition

$$p|_{\partial\Sigma} = \varphi(x'', \xi) := \mathcal{F}_\sigma^{-1}\left( -i\frac{\sqrt{\lambda + \xi^2 + s^2} + \sqrt{\xi^2 + s^2}}{\sqrt{\xi^2 + s^2}}(\sigma \cdot \tilde{U}'' + \xi \tilde{U}_n) \right). \tag{2.17}$$

On the other hand, (2.11) yields

$$(-\Delta' + \xi^2)p = 0 \quad \text{in} \quad \Sigma. \tag{2.18}$$

Now we will obtain the estimate of the solution $p$ to the elliptic boundary value problem (2.17), (2.18). Let

$$m_1(\sigma) = \frac{(\sigma, \xi)}{\sqrt{\xi^2 + s^2}}, \quad m_2(\sigma) = -i\frac{\sqrt{\lambda + \xi^2 + s^2} + \sqrt{\xi^2 + s^2}}{\sqrt{|\lambda + \xi^2| + s^2}} \quad (s = |\sigma|),$$

and define the functions $l$ and $b$ by their Fourier transforms

$$\tilde{l}(\sigma) = m_1(\sigma) \begin{pmatrix} \tilde{U}''(\sigma) \\ \tilde{U}_n(\sigma) \end{pmatrix}, \; \tilde{b}(\sigma) = \sqrt{|\lambda + \xi^2| + s^2}\, \tilde{l}(\sigma), \tag{2.19}$$

Then $\varphi$ defined by (2.17) satisfies

$$\tilde{\varphi}(\sigma) = m_2(\sigma)\, \tilde{b}(\sigma). \tag{2.20}$$

To control the dependence of the following estimates on the parameter $\mu = |\lambda + \xi^2|^{1/2}$, we introduce for functions $h(\cdot)$ on $\mathbb{R}^{n-2}$ the scaling transform $h_\mu(y) := h(\frac{y}{\mu}), y \in \mathbb{R}^{n-2}$. If $m(\sigma)$ is a multiplier function on $\mathbb{R}^{n-2}$ and $q, r$ are functions with Fourier transforms $\tilde{q}, \tilde{r}$, respectively, such that

$$\tilde{q}(\sigma) = m(\sigma)\tilde{r}(\sigma),$$

then obviously

$$\tilde{q}_\mu(\sigma) = m(\mu\sigma)\tilde{r}_\mu(\sigma).$$

Since $m_1(\mu\sigma)$ satisfies the Hörmander-Michlin condition uniformly with respect to $\mu > 0$, the Fourier multiplier theorem applied to $l_\mu$, see (2.19), yields

$$\|l_\mu\|_{1,r;\mathbb{R}^{n-2}} \le c\|U_\mu\|_{1,r;\mathbb{R}^{n-2}}, \quad \|l_\mu\|_{2,r;\mathbb{R}^{n-2}} \le c\|U_\mu\|_{2,r;\mathbb{R}^{n-2}}.$$

Thus, by real interpolation, cf. [31], Ch. 2.4.2,

$$\|l_\mu\|_{2-1/r,r;\mathbb{R}^{n-2}} \le c\|U_\mu\|_{2-1/r,r;\mathbb{R}^{n-2}} \tag{2.21}$$

with a constant $c$ independent of $\mu$. With the scaling $b_\mu(y) := b(\frac{y}{\mu})$ the equation $\tilde{b}(\sigma) = \sqrt{\mu^2 + s^2}\, \tilde{l}(\sigma)$, see (2.19), reduces to $\tilde{b}_\mu(\sigma) = \mu\sqrt{1+s^2}\, \tilde{l}_\mu(\sigma)$. Therefore, it follows from the definition of Bessel potential spaces that

$$\|b_\mu\|_{r;\mathbb{R}^{n-2}} \le c\mu\|l_\mu\|_{1,r;\mathbb{R}^{n-2}}, \quad \|b_\mu\|_{1,r;\mathbb{R}^{n-2}} \le c\mu\|l_\mu\|_{2,r;\mathbb{R}^{n-2}}$$

with $c$ independent of $\mu$. Thus by interpolation

$$\|b_\mu\|_{1-1/r,r;\mathbb{R}^{n-2}} \le c\mu\|l_\mu\|_{2-1/r,r;\mathbb{R}^{n-2}}. \tag{2.22}$$

It is also easy to see that $m_2(\mu\sigma)$ satisfies the Hörmander-Michlin condition, in particular

$$|s^k \frac{\partial^k}{\partial\sigma^k} m_2(\mu\sigma)| \le c_k(\varepsilon), \quad 0 \le k \le n-2,$$

with $c_k(\varepsilon)$ independent of $\lambda \in S_\varepsilon, \xi \in \mathbb{R}^*$. Therefore, it follows from (2.20) that

$$\|\varphi_\mu\|_{1-1/r,r;\mathbb{R}^{n-2}} \le c_\varepsilon\|b_\mu\|_{1-1/r,r;\mathbb{R}^{n-2}}.$$

Combining this inequality with (2.21), (2.22) yields

$$\|\varphi_\mu\|_{1-1/r,r;\mathbb{R}^{n-2}} \le c_\varepsilon\mu\|U_\mu\|_{2-1/r,r;\mathbb{R}^{n-2}}. \tag{2.23}$$

By the additional scaling transforms $p_\xi(z) := p(\frac{z}{|\xi|}, \xi), z \in \Sigma$, and $\varphi_\xi(z) := \varphi(\frac{z}{|\xi|}, \xi), z \in \mathbb{R}^{n-2}$, for $\xi \in \mathbb{R}^*$, the boundary problem (2.17), (2.18) is transformed to a problem independent of $\xi$, that is,

$$\begin{array}{rcl} (-\Delta' + 1)p_\xi &=& 0 \quad \text{in} \quad \Sigma \\ p_\xi|_{\partial\Sigma} &=& \varphi_\xi \in W^{1-1/r,r}(\partial\Sigma). \end{array}$$

It is well known that the above elliptic boundary value problem has a unique solution $p_\xi \in W^{1,r}(\Sigma)$ satisfying the estimate

$$\|\nabla' p_\xi, p_\xi\|_r \le c\|\varphi_\xi\|_{1-1/r,r}$$

with a constant $c$ depending only on $r, n$, see [6]. An elementary calculation shows that this estimate is equivalent to

$$\|\nabla' p, \xi p\|_r \le c(|\xi|^{1-1/r}\|\varphi\|_r + \langle\varphi\rangle_{1-1/r}), \quad \xi \in \mathbb{R}^*,$$

which with (2.23), (2.10) implies

$$\begin{array}{rcl} \|\nabla' p, \xi p\|_r &\le& c(|\xi|^{1-1/r}\|\varphi\|_r + \langle\varphi\rangle_{1-1/r}) \\ &\le& c_\varepsilon(\mu^{1-1/r}\|\varphi\|_r + \langle\varphi\rangle_{1-1/r}) \\ &=& c_\varepsilon \mu^{1-\frac{n-1}{r}}\|\varphi_\mu\|_{1-1/r,r} \\ &\le& c_\varepsilon \mu^{2-\frac{n-1}{r}}\|U_\mu\|_{2-1/r,r} \\ &=& c_\varepsilon \mu^{2-\frac{n-1}{r}}(\|U_\mu, \nabla'' U_\mu\|_r + \langle\nabla'' U_\mu\rangle_{1-1/r}) \\ &=& c_\varepsilon\big(\mu^{2-1/r}\|U\|_r + \mu^{1-1/r}\|\nabla'' U\|_r + \langle\nabla'' U\rangle_{1-1/r}\big) \\ &\le& c_\varepsilon\big(\|f, \nabla' g, \xi g\|_r + \|\lambda g; \hat{W}^{-1,r} + L^r_{1/\xi}\|\big). \end{array}$$

Now it remains to get the estimate of the velocity $u$. The transform $w = u - U$ reduces (2.11) to the system

$$\begin{array}{ll} (\mu^2 - \Delta')w' = -\nabla' p - (\mu^2 - \Delta')U' & \text{in} \quad \Sigma \\ (\mu^2 - \Delta')w_n = -i\xi p - (\mu^2 - \Delta')U_n & \text{in} \quad \Sigma \\ w' = 0, \quad w_n = 0 & \text{on} \quad \partial\Sigma, \end{array} \tag{2.24}$$

for which we have to estimate the terms $\mu^2 w, \mu\nabla' w, \nabla'^2 w$ in $L^r$-norm. It is easily seen by the reflection argument as in [10] that the Laplace resolvent equation

$$(\mu^2 - \Delta')v = F \in L^r(\mathbb{R}^{n-1}_+), \quad v|_{\mathbb{R}^{n-2}} = 0,$$

has a unique solution $v$ satisfying

$$\|\mu^2 v, \mu\nabla' v, \nabla'^2 v\|_r \le c_\varepsilon\|F\|_r.$$

Therefore, we see that

$$\begin{array}{rl} \|\mu^2 w, \mu\nabla' w, \nabla'^2 w\|_r &\le c_\varepsilon\|\nabla' p, \xi p, \mu^2 U, \nabla'^2 U\|_r \\ &\le c_\varepsilon\big(\|f, \nabla' g, \xi g\|_r + \|\lambda g; \hat{W}^{-1,r} + L^r_{1/\xi}\|\big), \end{array}$$

which completes the proof of the theorem. ■

The third main step of Section 2 concerns $(R_{\lambda,\xi})$ in a bent half space $\Sigma = \Sigma_\omega$, see (2.2). Note that $u, p$ etc. stand for the Fourier transforms $\hat{u}, \hat{p}$ etc.

**Theorem 2.3** *Let $n \geq 3, 1 < r < \infty, 0 < \varepsilon < \pi/2$ and*

$$\Sigma = \Sigma_\omega = \{x' = (x_1, x''); x_1 > \omega(x''), x'' \in \mathbb{R}^{n-2}\}$$

*for given $\omega \in C^{1,1}(\mathbb{R}^{n-2})$. Then there are constants $K_0 = K_0(r,\varepsilon) > 0$ and $\lambda_0 = \lambda_0(r,\varepsilon) > 0$ such that provided $\|\nabla'\omega\|_\infty \leq K_0$ for every $\lambda \in S_\varepsilon, |\lambda| \geq \lambda_0$, every $\xi \in \mathbb{R}^*$ and*

$$f \in L^r(\Sigma), \quad g \in W^{1,r}(\Sigma), \tag{2.25}$$

*the parametrized resolvent problem $(R_{\lambda,\xi})$ has a unique solution $(u,p) \in (W^{2,r}(\Sigma) \cap W_0^{1,r}(\Sigma)) \times W^{1,r}(\Sigma)$. This solution satisfies the estimate ($\mu = |\lambda + \xi^2|^{1/2}$)*

$$\begin{aligned} &\|\mu^2 u, \mu\nabla' u, \nabla'^2 u, \nabla' p, \xi p\|_r \\ &\qquad \leq c(r,\varepsilon)\big(\|f, \nabla' g, \xi g\|_r + \|\lambda g; \hat{W}^{-1,r}(\Sigma) + L^r(\Sigma)_{1/\xi}\|\big). \end{aligned} \tag{2.26}$$

*If (2.25) is satisfied for an additional exponent $s \in (1,\infty)$, $s \neq r$, and $\|\nabla'\omega\|_\infty \leq K_0$ for some $K_0 = K_0(r,s,\varepsilon) > 0$, then the assertion (2.26) with $L^s$-norms holds true for all $\lambda \in S_\varepsilon, |\lambda| \geq \lambda_0$ for some $\lambda_0 = \lambda_0(r,s,\varepsilon) > 0$ as well.*

**Proof:** By the transformation

$$\Phi : \Sigma_\omega \to \mathbb{R}^{n-1}_+, \quad \tilde{x}' = (\tilde{x}_1, \tilde{x}'') = \Phi(x') = (x_1 - \omega(x''), x''),$$

the problem $(R_{\lambda,\xi})$ in $\Sigma_\omega$ is reduced to a modified version of $(R_{\lambda,\xi})$ in the half space $H = \mathbb{R}^{n-1}_+$. Note that $\Phi$ is a bijection with Jacobian equal to 1. For a function $u$ on $\Sigma_\omega$ define $\tilde{u}$ on $H$ by $\tilde{u}(\tilde{x}') = u(\Phi^{-1}(\tilde{x}')) = u(x')$. Further let $\tilde{\partial}_i = \partial/\partial\tilde{x}_i, i = 1, \cdots, n-1, \tilde{\nabla}' = (\tilde{\partial}_1, \tilde{\nabla}'')$ etc. denote standard differential operators acting on the variable $\tilde{x} \in H$.

Since $\partial_i u = (\tilde{\partial}_i - (\partial_i\omega)\tilde{\partial}_1)\tilde{u}$ for $i = 1, \cdots, n-1$, we easily get

$$\begin{aligned} \Delta' u(x',\xi) &= \big(\tilde{\Delta}' + |\nabla'\omega|^2\tilde{\partial}_1^2 - 2\nabla'\omega\cdot(\tilde{\nabla}'\tilde{\partial}_1) - (\Delta''\omega)\tilde{\partial}_1\big)\tilde{u}(\tilde{x}',\xi) \\ \nabla' p(x',\xi) &= \big(\tilde{\nabla}' - (\nabla'\omega)\tilde{\partial}_1\big)\tilde{p}(\tilde{x}',\xi) \\ \operatorname{div}' u'(x',\xi) &= \big(\widetilde{\operatorname{div}}' - \nabla'\omega\cdot\tilde{\partial}_1\big)\tilde{u}'(\tilde{x}',\xi) \end{aligned} \tag{2.27}$$

and a similar formula for $\nabla'^2 u(x',\xi)$. Hence for $u \in W^{2,r}(\Sigma)$

$$\begin{aligned} \|u\|_r &= \|\tilde{u}\|_{r;H} \\ \|\nabla' u\|_r &\leq c(1+K)\|\tilde{\nabla}'\tilde{u}\|_{r;H} \\ \|\nabla'^2 u\|_r &\leq c(1+K^2)\|\tilde{\nabla}'^2\tilde{u}\|_{r;H} + cL\|\tilde{\partial}_1\tilde{u}\|_{r;H}, \end{aligned} \tag{2.28}$$

where $K = \|\nabla'\omega\|_\infty$ and $L = \|\nabla'^2\omega\|_\infty$. Furthermore $\|f, \xi g\|_r = \|\tilde{f}, \xi\tilde{g}\|_{r;H}$ and $\|\nabla' g\|_r \leq c(1+K)\|\tilde{\nabla}'\tilde{g}\|_{r;H}$. Concerning the norm of $g$ in $\hat{W}^{-1,r}(\Sigma) + L^r(\Sigma)_{1/\xi}$ note that for a function $g_0 \in \hat{W}^{-1,r}(\Sigma) \cap L^r(\Sigma)$ we have $\int_\Sigma g_0 \varphi \, dx' = \int_H \tilde{g}_0 \tilde{\varphi} \, d\tilde{x}'$ for all test functions $\varphi \in C_0^\infty(\bar{\Sigma})$. Since $C_0^\infty(\bar{\Sigma})$ is dense in $\hat{W}^{1,r'}(\Sigma)$, we get

$$\|g_0\|_{\hat{W}^{-1,r}(\Sigma)} \le c(1+K)\|\tilde{g}_0\|_{-1,r;H}.$$

Then for every $\xi \in \mathbb{R}^*$ and every decomposition of $g$ into $g = g_0 + g_1$ with $g_0 \in \hat{W}^{-1,r}(\Sigma), g_1 \in L^r(\Sigma)$

$$\|g_0\|_{-1,r} + \|g_1/\xi\|_r \le c(1+K)(\|\tilde{g}_0\|_{-1,r;H} + \|\tilde{g}_1/\xi\|_{r;H});$$

note that $\tilde{g} = \tilde{g}_0 + \tilde{g}_1$ gives all admissible decompositions of $\tilde{g} \in \hat{W}^{-1,r}(H) + L^r(H)_{1/\xi}$. Consequently

$$\|g; \hat{W}^{-1,r}(\Sigma) + L^r(\Sigma)_{1/\xi}\| \le c(1+K)\|\tilde{g}; \hat{W}^{-1,r}(H) + L^r(H)_{1/\xi}\|. \tag{2.29}$$

To apply Kato's perturbation theorem we introduce for every $\xi \in \mathbb{R}^*$ on $\Sigma$ the $\xi$-dependent Banach spaces ($\mu = |\lambda + \xi^2|^{1/2}$)

$$\begin{array}{ll} \mathcal{X} = (W^{2,r} \cap W_0^{1,r})^n \times W^{1,r}, & \|u,p\|_{\mathcal{X}} = \|\mu^2 u, \mu \nabla' u, \nabla'^2 u, \nabla' p, \xi p\|_r, \\ \mathcal{Y} = (L^r)^n \times W^{1,r}, & \|f,g\|_{\mathcal{Y}} = \|f, \nabla' g, \xi g\|_r + \|\lambda g; \hat{W}^{-1,r} + L^r_{1/\xi}\|, \end{array}$$

and on $\mathbb{R}^{n-1}_+$ similar spaces $(\tilde{\mathcal{X}}, \|\cdot\|_{\tilde{\mathcal{X}}}), (\tilde{\mathcal{Y}}, \|\cdot\|_{\tilde{\mathcal{Y}}})$. Further define the operators

$$\mathcal{S}: \mathcal{X} \to \mathcal{Y}, \quad \mathcal{S}(u,p) = \begin{pmatrix} (\lambda + \xi^2 - \Delta')u' + \nabla' p \\ (\lambda + \xi^2 - \Delta')u_n + i\xi p \\ \operatorname{div}' u' + i\xi u_n \end{pmatrix},$$

and analogously $\tilde{\mathcal{S}}: \tilde{\mathcal{X}} \to \tilde{\mathcal{Y}}$. By (2.27) we get the decomposition

$$\mathcal{S}(u,p) = \tilde{\mathcal{S}}(\tilde{u},\tilde{p}) + \mathcal{R}(\tilde{u},\tilde{p})$$

with a remainder term $\mathcal{R}: \tilde{\mathcal{X}} \to \tilde{\mathcal{Y}}$,

$$\begin{aligned} &\mathcal{R}(\tilde{u},\tilde{p})(\tilde{x}',\xi) \\ &\quad = \begin{pmatrix} -(\nabla'\omega)\tilde{\partial}_1 \tilde{p} \\ 0 \\ -(\nabla'\omega) \cdot \tilde{\partial}_1 \tilde{u}' \end{pmatrix} + \begin{pmatrix} -|\nabla'\omega|^2 \tilde{\partial}_1^2 \tilde{u} + 2\nabla'\omega \cdot \tilde{\nabla}' \tilde{\partial}_1 \tilde{u} + (\Delta''\omega)\tilde{\partial}_1 \tilde{u} \\ 0 \end{pmatrix}, \end{aligned}$$

not depending explicitly on $\lambda$ and $\xi$. Since $\tilde{u}|_{\partial H} = 0$ and $\tilde{\partial}_1(\nabla'\omega) = 0$, we have

$$\int_H -(\nabla'\omega) \cdot \tilde{\partial}_1 \tilde{u}' \, \varphi \, d\tilde{x}' = \int_H (\nabla'\omega) \cdot \tilde{u}' \, \tilde{\partial}_1 \varphi \, d\tilde{x}'$$

for all $\varphi \in C_0^\infty(\bar{H})$; consequently, we see that

$$\| -(\nabla'\omega) \cdot \tilde{\partial}_1 \tilde{u}'; \hat{W}^{-1,r}(H) + L^r(H)_{1/\xi}\| \le \| -(\nabla'\omega) \cdot \tilde{\partial}_1 \tilde{u}'\|_{-1,r;H} \le K\|\tilde{u}\|_{r;H}.$$

Hence

$$\begin{aligned}\|\mathcal{R}(\tilde{u},\tilde{p})\|_{\tilde{\mathcal{Y}}} &\leq c(K+K^2)\|\lambda\tilde{u},\xi\tilde{\nabla}'\tilde{u},\tilde{\nabla}'^2\tilde{u},\tilde{\nabla}'\tilde{p}\|_{r;H}+cL\|\tilde{\nabla}'\tilde{u}\|_{r;H}\\ &\leq c_\varepsilon(K+K^2+L/\mu)\|(\tilde{u},\tilde{p})\|_{\tilde{\mathcal{X}}}.\end{aligned}$$

Due to Theorem 2.2 $\tilde{\mathcal{S}}:\tilde{X}\to\tilde{Y}$ is an isomorphism such that $\|(\tilde{u},\tilde{p})\|_{\tilde{\mathcal{X}}}\leq c\|\tilde{\mathcal{S}}(\tilde{u},\tilde{p})\|_{\tilde{\mathcal{Y}}}$ with a constant $c=c(r,\varepsilon)$ independent of $\lambda\in S_\varepsilon,\xi\in\mathbb{R}^*$. Thus, if $K$ is sufficiently small and $\lambda\in S_\varepsilon,|\lambda|\geq\lambda_0$, with some $\lambda_0=\lambda_0(L,r,\varepsilon)$, we get that

$$\|\mathcal{R}(\tilde{u},\tilde{p})\|_{\tilde{\mathcal{Y}}}\leq\frac{1}{2}\|\mathcal{S}(\tilde{u},\tilde{p})\|_{\tilde{\mathcal{Y}}}\quad\text{ for all }(\tilde{u},\tilde{p})\in\tilde{\mathcal{X}}.$$

Hence $\tilde{\mathcal{S}}+\mathcal{R}$ is an isomorphism from $\tilde{\mathcal{X}}$ to $\tilde{\mathcal{Y}}$ satisfying

$$\|(\tilde{u},\tilde{p})\|_{\tilde{\mathcal{X}}}\leq c\|(\tilde{\mathcal{S}}+\mathcal{R})(\tilde{u},\tilde{p})\|_{\tilde{\mathcal{Y}}},$$

where $c=c(r,\varepsilon)$ is independent of $\lambda\in S_\varepsilon,|\lambda|\geq\lambda_0$, and $\xi\in\mathbb{R}^*$. Using (2.28), (2.29), we get

$$\|(u,p)\|_{\mathcal{X}}\leq c\|(\tilde{u},\tilde{p})\|_{\tilde{\mathcal{X}}}\leq c_1\|(\tilde{\mathcal{S}}+\mathcal{R})(\tilde{u},\tilde{p})\|_{\tilde{\mathcal{Y}}}\leq c_2\|\mathcal{S}(u,p)\|_{\mathcal{Y}},$$

where the constant $c_2=c_2(r,\varepsilon,K,L/\lambda_0)$ is independent of $\lambda\in S_\varepsilon,|\lambda|\geq\lambda_0$, and $\xi\in\mathbb{R}^*$.

Assume that (2.25) is satisfied for an additional $s\neq r$. Repeating the above argument for the index $s$, we see $\mathcal{S}$ to be an isomorphism from $\mathcal{X}_s\cap\mathcal{X}_r$ to $\mathcal{Y}_s\cap\mathcal{Y}_r$ for $|\lambda|\geq\lambda_0=\lambda_0(r,s,\varepsilon)$ under the given smallness conditions on $K$. Now the proof of Theorem 2.3 is complete. ■

# 3 The Problem $(R_{\lambda,\xi})$ in Bounded Domains

Let $\Sigma\subset\mathbb{R}^{n-1}$ be a bounded domain of $C^{1,1}$-class and let $\alpha_0$ denote the smallest eigenvalue of the Laplacian on $\Sigma$, i.e.

$$0<\alpha_0=\inf\{\|\nabla'u\|_2^2;u\in W_0^{1,2}(\Sigma),\|u\|_2=1\}.$$

Recall that $u=(u',u_n),p$ etc. stand for the Fourier transforms $\hat{u}=(\hat{u'},\hat{u_n}),\hat{p}$ etc.

For fixed $\lambda\in\mathbb{C}\setminus(-\infty,-\alpha_0]$ and $\xi\in\mathbb{R}$ we introduce the *parametrized Stokes operator* $S=S_{r,\lambda,\xi}$ by

$$S(u,p)=\begin{pmatrix}(\lambda+\xi^2-\Delta')u'+\nabla'p\\(\lambda+\xi^2-\Delta')u_n+i\xi p\\-\operatorname{div}_\xi u\end{pmatrix}\tag{3.1}$$

defined on $\mathcal{D}(S)=\mathcal{D}(\Delta_r')\times W^{1,r}$, where $\mathcal{D}(\Delta_r')=W^{2,r}(\Sigma)\cap W_0^{1,r}(\Sigma)$ and

$$\operatorname{div}_\xi u:=\operatorname{div}'u'+i\xi u_n\in W^{1,r}(\Sigma).$$

Further, define the restriction $S^0 = S^0_{r,\lambda,\xi}$ by

$$S^0(u,p) = \begin{pmatrix} (\lambda + \xi^2 - \Delta')u' + \nabla' p \\ (\lambda + \xi^2 - \Delta')u_n + i\xi p \end{pmatrix}$$

on $\mathcal{D}(S^0) = \{(u,p) \in \mathcal{D}(S); \operatorname{div}_\xi u = 0\}$. For $\xi \in \mathbb{R}^*$ note that $W^{1,r}(\Sigma) \subset L^r_m(\Sigma) + L^r(\Sigma)_{1/\xi}$, where and in what follows

$$L^r_m(\Sigma) := \{u \in L^r(\Sigma); \int_\Sigma u\, dx' = 0\}.$$

In the following we use the norm

$$\|g; L^r_m + L^r_{1/\xi}\|_0 := \inf\{\|g_0\|_{-1,r} + \|g_1/\xi\|_r; g = g_0 + g_1, g_0 \in L^r_m, g_1 \in L^r\};$$

note that this norm is equivalent to the operator norm $\|\cdot\|_{(W^{1,r'}_\xi(\Sigma))^*}$ where $W^{1,r'}_\xi(\Sigma)$ equals $W^{1,r'}(\Sigma)$ endowed with the equivalent norm $\|u\|_{W^{1,r'}_\xi(\Sigma)} = \|\nabla u, \xi u\|_{r'}$.

First, we need a preliminary to deal with the Hilbert space setting of $(R_{\lambda,\xi})$. For $\xi \in \mathbb{R}^*$ define the closed subspace $V_\xi$ of $W^{1,2}_0(\Sigma)$ by

$$V_\xi = \{u \in W^{1,2}_0(\Sigma); \operatorname{div}_\xi u = 0\}.$$

**Lemma 3.1** *Suppose that $\varphi = (\varphi', \varphi_n) \in W^{-1,2}(\Sigma) := (W^{1,2}_0(\Sigma))^*$ satisfies $(\varphi, v) = 0$ for all $v \in V_\xi$. Then there is some $p \in L^2(\Sigma)$ such that*

$$\varphi = (\nabla' p, i\xi p).$$

**Proof:** It follows from the assumption that $< \varphi', v' >_{W^{-1,2},W^{1,2}_0} = 0$ for all $v' \in W^{1,2}_0(\Sigma)$ satisfying $\operatorname{div}' v' = 0$. Therefore, by [16], Corollary III 5.1, we get

$$\varphi' = \nabla' p \quad \text{with some } p \in L^2(\Sigma). \tag{3.2}$$

Then, for all $v = (v', v_n) \in \mathcal{V}_\xi := \{u \in C^\infty_0(\Sigma)^n; \operatorname{div}_\xi u = 0\}$, by assumption

$$\begin{aligned} 0 &= < \nabla' p, v' >_{W^{-1,2},W^{1,2}_0} + < \varphi_n, v_n >_{W^{-1,2},W^{1,2}_0} \\ &= < \nabla' p, v' >_{W^{-1,2},W^{1,2}_0} + < \varphi_n, -\tfrac{\operatorname{div}' v'}{i\xi} >_{W^{-1,2},W^{1,2}_0} \\ &= < \nabla'(p - \tfrac{\varphi_n}{i\xi}), v' >_{\mathcal{D}'(\Sigma),\mathcal{D}(\Sigma)} . \end{aligned} \tag{3.3}$$

Since $v' \in C^\infty_0(\Sigma)$ in (3.3) can be chosen arbitrarily due to the structure of $\mathcal{V}_\xi$, we get $\nabla'(p - \frac{\varphi_n}{i\xi}) = 0$ in the sense of distributions yielding $p - \frac{\varphi_n}{i\xi} = \text{const}$ and $\varphi_n \in L^2(\Sigma)$. Thus, choosing $p$ in (3.2) such that $\int_\Sigma (p - \frac{\varphi_n}{i\xi})\, dx' = 0$, we get $p - \frac{\varphi_n}{i\xi} = 0$. The proof of this lemma is complete. ■

In the following we consider the resolvent problem $(R_{\lambda,\xi})$ for arbitrary $\lambda \in -\alpha_0 + S_\varepsilon$. We start with the case $r = 2$.

**Lemma 3.2** (i) *For every $g \in W^{1,2}(\Sigma)$ and $\xi \in \mathbb{R}^*$ the divergence problem* $\mathrm{div}_\xi u = g$ *has at least one solution $u \in W^{2,2}(\Sigma) \cap W_0^{1,2}(\Sigma)$ such that*

$$\|u\|_{2,2} \leq c \left( \|g\|_{1,2} + \frac{1}{|\xi|} \left| \int_\Sigma g \, dx' \right| \right). \tag{3.4}$$

*Here $c > 0$ is a constant independent of $\xi$ and $g$.*

(ii) *For every $f \in L^2(\Sigma)$ and $g \in W^{1,2}(\Sigma)$ and every $\lambda \in -\alpha_0 + S_\varepsilon, \xi \in \mathbb{R}^*$ ($\xi \in \mathbb{R}$ if $g \equiv 0$), there exists a unique solution $(u,p)$ of $(R_{\lambda,\xi})$ such that $(u,p) \in (W^{2,2}(\Sigma) \cap W_0^{1,2}(\Sigma)) \times W^{1,2}(\Sigma)$.*

**Proof:** (i) Choose an arbitrary, but fixed $w = (0, \cdots, 0, w_n) \in C_0^\infty(\Sigma)$ with $\int_\Sigma w_n \, dx' = 1$. Given $g \in W^{1,2}$ with $\alpha = \int_\Sigma g \, dx'$ such that consequently $g - \alpha w_n \in W^{1,2} \cap L^2_m$, there exists by [10], Theorem 1.2, a velocity field $u = (u', 0) \in W^{2,2} \cap W_0^{1,2}$ satisfying $\mathrm{div}\, u = g - \alpha w_n$ and $\|u\|_{2,2} \leq c \|\nabla'(g - \alpha w_n)\|_2 \leq c\|g\|_{1,2}$. Then $v = u + \frac{\alpha}{i\xi} w$ solves the divergence problem and satisfies the estimate (3.4).

(ii) In consideration of (i) we may assume without loss of generality that $g = 0$. Define, for $\lambda \in -\alpha_0 + S_\varepsilon$ and $\xi \in \mathbb{R}^*$, the bilinear form $a(\cdot,\cdot) : V_\xi \times V_\xi \to \mathbb{C}$ by

$$a(u,v) = \int_\Sigma ((\lambda + \xi^2) u \cdot \bar{v} + \nabla' u \cdot \nabla' \bar{v}) \, dx'.$$

Obviously $a$ is continuous and elliptic in the sense that $|a(u,u)| \geq \alpha \|u\|_{1,2}^2$ for all $\lambda \in -\alpha_0 + S_\varepsilon, \xi \in \mathbb{R}^*$ and $u \in V_\xi$ with a constant $\alpha = \alpha(\lambda, \xi) > 0$. By the Lemma of Lax-Milgram the variational problem

$$a(u,v) = \int_\Sigma f \cdot \bar{v} \, dx' \quad \forall v \in V_\xi$$

has a unique solution $u \in V_\xi$, that is,

$$< (\lambda + \xi^2 - \Delta') u - f, v >_{W^{-1,2}, W_0^{1,2}} = 0 \quad \forall v \in V_\xi.$$

Moreover, by Lemma 3.1 there is some $p \in L^2(\Sigma)$ such that

$$(\lambda + \xi^2 - \Delta') u' + \nabla' p = f', (\lambda + \xi^2 - \Delta') u_n + i\xi p = f_n.$$

Then standard regularity results for the Stokes and Poisson equation applied to the problems

$$-\Delta' u' + \nabla' p = f' - (\lambda + \xi^2) u', \quad \mathrm{div}\,' u' = -i\xi u_n \quad \text{in } \Sigma, \quad u'|_{\partial\Sigma} = 0,$$

and $-\Delta' u_n = f_n - (\lambda + \xi^2) u_n - i\xi p$ in $\Sigma$, $u_n|_{\partial\Sigma} = 0$, yield $(u,p) \in (W^{2,2}(\Sigma) \cap W_0^{1,2}(\Sigma)) \times W^{1,2}(\Sigma)$. Since the uniqueness of $(u,p)$ is obvious, the proof of the lemma is complete. ■

The next lemma gives a preliminary *a priori* estimate for a solution $(u,p)$ of $S(u,p) = (f, -g)$.

**Lemma 3.3** *Let* $\varepsilon \in (0,\pi/2)$ *and* $1<r<\infty$.

(i) *There exists a constant* $c=c(\varepsilon,r,\Sigma)>0$ *such that for every* $\alpha\in(0,\alpha_0), \lambda\in -\alpha+S_\varepsilon, \xi\in\mathbb{R}^*$ *and every* $(u,p)\in\mathcal{D}(S_{r,\lambda,\xi})$,

$$\begin{aligned}\|\mu_+^2 u,\mu_+\nabla' u,\nabla'^2 u,\nabla' p,\xi p\|_r \le c\big(&\|f,\nabla' g,g,\xi g\|_r+|\lambda|\|g;L^r_m+L^r_{1/\xi}\|_0\\ &+\|\nabla' u,\xi u,p\|_r+|\lambda|\|u\|_{(W^{1,r'})^*}\big),\end{aligned}\tag{3.5}$$

*where* $\mu_+=|\lambda+\alpha+\xi^2|^{1/2}, (f,-g)=S(u,p)$ *and* $(W^{1,r'})^*$ *denotes the dual space of* $W^{1,r'}(\Sigma)$.

(ii) *For every* $\lambda\in-\alpha_0+S_\varepsilon$, *the operators* $S=S_{r,\lambda,\xi}$ *and* $S^0=S^0_{r,\lambda,\xi}$ *are injective for* $\xi\in\mathbb{R}$, *and the ranges* $\mathcal{R}(S)$ *and* $\mathcal{R}(S^0)$ *are dense in* $L^r\times W^{1,r}$ *and in* $L^r$, *respectively, for* $\xi\in\mathbb{R}^*$.

**Proof:** The proof of (i) is based on a partition of unity in $\Sigma$ and on the localization procedure reducing the problem to a finite number of problems of type $(R_{\lambda,\xi})$ in bent half spaces and in the whole space $\mathbb{R}^{n-1}$. Since $\partial\Sigma\in C^{1,1}$, we can cover $\partial\Sigma$ by a finite number of balls $B_j, j\ge1$, such that, after a translation and rotation of coordinates, $\Sigma\cap B_j$ locally coincides with a bent half space $\Sigma_j=\Sigma_{\omega_j}$ where $\omega_j$ has a compact support, $\omega_j(0)=0$ and $\nabla''\omega_j(0)=0$. Choosing the balls $B_j$ sufficiently small (and its number sufficiently large) we may assume that $\|\nabla''\omega_j\|_\infty\le K_0=K_0(r,\varepsilon)$ for all $j\ge1$ where $K_0$ was introduced in Theorem 2.3. According to the covering $\partial\Sigma\subset\bigcup B_j$ there are cut-off functions $0\le\varphi_0,\varphi_j\in C^\infty(\mathbb{R}^{n-1})$ such that

$$\varphi_0+\sum_{j\ge1}\varphi_j\equiv1 \text{ in } \Sigma,\quad \operatorname{supp}\varphi_j\subset B_j\quad\text{and}\quad \operatorname{supp}\varphi_0\subset\Sigma.$$

Given $(u,p)\in\mathcal{D}(S)$ and $(f,-g)=S(u,p)$ we get for each $\varphi_j, j\ge0$, the local $(R_{\lambda,\xi})$-problems

$$\begin{aligned}(\lambda+\xi^2-\Delta')(\varphi_j u')+\nabla'(\varphi_j p) &= f_j'\\ (\lambda+\xi^2-\Delta')(\varphi_j u_n)+i\xi(\varphi_j p) &= f_{jn}\\ \operatorname{div}_\xi(\varphi_j u) &= g_j\end{aligned}\tag{3.6}$$

for $(\varphi_j u,\varphi_j p), j\ge0$, in $\mathbb{R}^{n-1}$ or $\Sigma_j$; here

$$\begin{aligned}f_j' &= \varphi_j f'-2\nabla'\varphi_j\cdot\nabla' u'-(\Delta'\varphi_j)u'+(\nabla'\varphi_j)p\\ f_{jn} &= \varphi_j f_n-2\nabla'\varphi_j\cdot\nabla' u_n-(\Delta'\varphi_j)u_n\\ g_j &= \varphi_j g+\nabla'\varphi_j\cdot u'.\end{aligned}\tag{3.7}$$

To control $f_j$ and $g_j$ note that $u=0$ on $\partial\Sigma$; hence Poincare's inequality yields for all $j\ge0$ the estimate

$$\|f_j,\nabla' g_j,\xi g_j\|_{r;\Sigma_j}\le c(\|f,\nabla' g,g,\xi g\|_r+\|\nabla' u,\xi u,p\|_r),\tag{3.8}$$

where $\Sigma_0=\mathbb{R}^{n-1}$. Moreover, let $g=g_0+g_1$ denote any splitting of $g\in L^r_m+L^r_{1/\xi}$. Defining the characteristic function $\chi_j$ of $\Sigma\cap\Sigma_j$ and the scalar

$$\begin{aligned}m_j &= \frac{1}{|\Sigma\cap\Sigma_j|}\int_{\Sigma\cap\Sigma_j}(\varphi_j g_0+u'\cdot\nabla'\varphi_j)\,dx'\\ &= \frac{1}{|\Sigma\cap\Sigma_j|}\int_{\Sigma\cap\Sigma_j}(i\xi u_n-g_1)\varphi_j\,dx',\end{aligned}$$

we split $g_j$ in the form

$$g_j = g_{j0} + g_{j1} := (\varphi_j g_0 + u' \cdot \nabla' \varphi_j - m_j \chi_j) + (\varphi_j g_1 + m_j \chi_j).$$

Obviously $\|g_{j1}\|_{r;\Sigma_j} \le c(\|g_1\|_r + |\xi| \|u\|_{(W^{1,r'})^*})$. Furthermore, for every test function $\Psi \in C_0^\infty(\bar{\Sigma}_j)$ let

$$\tilde{\Psi} = \Psi - \frac{1}{|\Sigma \cap \Sigma_j|} \int_{\Sigma \cap \Sigma_j} \Psi \, dx'.$$

Since by the definition of $m_j \chi_j$ we have $\int_{\Sigma_j} g_{j0} \, dx' = 0$,

$$\int_{\Sigma_j} g_{j0} \Psi \, dx' = \int_{\Sigma_j} g_{j0} \tilde{\Psi} \, dx' = \int_{\Sigma} g_0 (\varphi_j \tilde{\Psi}) \, dx' + \int_{\Sigma} u' \cdot (\nabla' \varphi_j) \tilde{\Psi} \, dx'.$$

Hence Poincare's inequality yields

$$\left| \int_{\Sigma_j} g_{j0} \Psi \, dx' \right| \le c(\|g_0\|_{-1,r} + \|u\|_{(W^{1,r'})^*}) \|\nabla' \Psi\|_{r';\Sigma_j}.$$

Summarizing the previous inequalities we get that

$$\|g_j; \hat{W}^{-1,r}(\Sigma_j) + L^r(\Sigma_j)_{1/\xi}\| \le c\big(\|u\|_{(W^{1,r'})^*} + \|g; L^r_m + L^r_{1/\xi}\|_0\big). \tag{3.9}$$

To complete the proof of (i) apply Theorem 2.1 to (3.6), (3.7) when $j = 0$. Further use Theorem 2.3 in (3.6), (3.7) for $j \ge 1$, but with $\lambda$ replaced by $\lambda + M$ such that $\lambda + M$ satisfies the assumptions of Theorem 2.3; for example we may take $M = \alpha_0 + \lambda_0$ with $\lambda_0$ in Theorem 2.3. This shift in $\lambda$ implies that $f_j$ has to be replaced by $f_j + M\varphi_j u$ and that (2.26) will be used with $\lambda$ replaced by $\lambda + M$. Summarizing (2.9), (2.26) as well as (3.8), (3.9) and summing over all $j$ we arrive at (3.5) with the additional terms

$$\|Mu\|_r + \|Mg; L^r_m + L^r_{1/\xi}\|_0$$

on the right-hand side of the inequality. However $\|Mu\|_r \le c\|\nabla' u\|_r$, and using the canonical splitting $g = \operatorname{div}' u' + i\xi u_n$, also

$$\|g; L^r_m + L^r_{1/\xi}\|_0 \le c\|\nabla' u\|_r$$

with $c$ depending only on $r, \varepsilon, \Sigma$. Thus (3.5) is proved.

(ii) To prove the injectivity of $S_{r,\lambda,\xi}$ let $S_{r,\lambda,\xi}(u,p) = 0$. By the regularity assertions in Theorem 2.1 and Theorem 2.3 it can be proved in a finite number of steps using Sobolev's embedding theorem that $(u,p) \in \mathcal{D}(S_{2,\lambda,\xi})$. We note that in order to apply Theorem 2.3 the partition of unity of $\Sigma$ has to be refined, if necessary, such that all crucial smallness assumptions on $\|\nabla' \omega_j\|_\infty$ are fulfilled. Thus by Lemma 3.2, (ii), $(u,p) = 0$.

Let us show that $\mathcal{R}(S_{r,\lambda,\xi})$ for $\xi \in \mathbb{R}^*$ is dense in $L^r \times W^{1,r}$. Note that $C_0^\infty(\Sigma) \times C^\infty(\bar{\Sigma})$ is dense in $L^r \times W^{1,r}$. By Lemma 3.2, (ii), there is a unique solution $(u,p)$ of $S_{2,\lambda,\xi}(u,p) = (f,-g)$ with $(f,g) \in C_0^\infty(\Sigma) \times C^\infty(\bar{\Sigma})$. Moreover, this solution can be shown to be in $\mathcal{D}(S_{r,\lambda,\xi})$ for every $r \in (1,\infty)$ thus proving the denseness of $\mathcal{R}(S)$ in $L^r \times W^{1,r}$. It is obvious from the above arguments that $\mathcal{R}(S_0)$ is dense in $L^r$. ■

Now we are in a position to prove the main theorem of this section.

**Theorem 3.4** *Let $1 < r < \infty$, $\varepsilon \in (0,\pi/2)$ and $\alpha \in (0,\alpha_0)$. Then for every $f \in L^r(\Sigma)$, $g \in W^{1,r}(\Sigma)$ and $\lambda \in -\alpha + S_\varepsilon$, $\xi \in \mathbb{R}^*$ ($\xi \in \mathbb{R}$ if $g \equiv 0$) $(R_{\lambda,\xi})$ has a unique solution $(u,p)$ satisfying $(u,p) \in (W^{2,r}(\Sigma) \cap W_0^{1,r}(\Sigma)) \times W^{1,r}(\Sigma)$ and the estimate*

$$\begin{aligned} &\|u, \mu_+^2 u, \mu_+ \nabla' u, \nabla'^2 u, \nabla' p, \xi p\|_r \\ &\qquad \le c\big(\|f, \nabla' g, g, \xi g\|_r + (|\lambda|+1)\|g; L_m^r + L_{1/\xi}^r\|_0\big), \end{aligned} \tag{3.10}$$

*where $\mu_+ = |\lambda + \alpha + \xi^2|^{1/2}$ and the constant $c = c(\alpha, r, \varepsilon, \Sigma) > 0$ is independent of $\lambda, \xi, f$ and $g$.*

*In particular if $g \in L_m^r(\Sigma)$, then the solution satisfies the stronger estimate*

$$\|u, \mu_+^2 u, \mu_+ \nabla' u, \nabla'^2 u, \nabla' p, \xi p\|_r \le c\big(\|f, \nabla' g, \xi g\|_r + |\lambda| \|g; L_m^r + L_{1/\xi}^r\|_0\big) \tag{3.11}$$

*with $c = c(\alpha, r, \varepsilon, \Sigma)$.*

**Remark 3.5** We note that the estimate (3.11) does not hold for $\lambda = 0$ and for any $f \in L^r, g \in W^{1,r}(\Sigma)$ with $\int_\Sigma g\,dx' \neq 0$. In fact, let us assume (3.11) to be true even for $\lambda = 0$ and for some $f \in L^r, g \in W^{1,r}$ and let $S_{r,\lambda,\xi}(u,p) = (f,-g)$. Then there exists a constant $c > 0$ such that for all $\xi \in \mathbb{R}^*$ there is an element $u_n = u_{n\xi} \in W^{2,r} \cap W_0^{1,r}$ satisfying

$$g - i\xi u_n \in L_m^r, \quad \|u_n\|_r \le c\|f, \nabla' g, \xi g\|_r.$$

Moreover,

$$\Big|\int_\Sigma g\,dx'\Big| = |\xi| \Big|\int_\Sigma u_n\,dx'\Big| \to 0 \text{ as } \xi \to 0$$

implies that $g \in L_m^r$.

**Proof of Theorem 3.4:** The existence of a unique solution $(u,p) \in \mathcal{D}(S)$ is a direct consequence of Lemma 3.3. In fact, due to [20], Chap.2, Lemma 5.1, the *a priori* estimate (3.5) of Lemma 3.3 implies the Fredholm property of the operator $S$ considering the compact embedding $\mathcal{D}(S) \subset\subset W^{1,r}(\Sigma) \times L^r(\Sigma)$. Thus $\mathcal{R}(S)$ is closed in $L^r(\Sigma) \times W^{1,r}(\Sigma)$. Then by Lemma 3.3 (ii) $\mathcal{R}(S) = L^r(\Sigma) \times W^{1,r}(\Sigma)$, and the solution is unique by the injectivity of the operator $S$.

Now let $(u,p) \in \mathcal{D}(S)$ and $S(u,p) = (f,-g)$. We shall prove the estimate (3.10) for general $g \in W^{1,r}(\Sigma)$ and (3.11) for $g \in W^{1,r}(\Sigma) \cap L_m^r(\Sigma)$.

Based on a contradiction argument assume that there are sequences $\{\lambda_j\} \subset -\alpha + S_\varepsilon, \{\xi_j\} \subset \mathbb{R}^*$ and $\{u_j, p_j\}$ with $(u_j, p_j) \in \mathcal{D}(S_{r,\lambda_j,\xi_j})$ for all $j \in \mathbb{N}$ such that

$$\|(\lambda_j + \alpha + \xi_j^2) u_j, (\lambda_j + \alpha + \xi_j^2)^{1/2} \nabla' u_j, \nabla'^2 u_j, \nabla' p_j, \xi_j p_j\|_r = 1 \tag{3.12}$$

and for $(f_j, -g_j) = S_{r,\lambda_j,\xi_j}(u_j, p_j)$

$$\|f_j, \nabla' g_j, g_j, \xi_j g_j\|_r + (|\lambda_j| + 1)\|g_j; L^r_m + L^r_{1/\xi_j}\|_0 \to 0 \quad \text{as} \quad j \to \infty, \tag{3.13a}$$

or if $\{g_j\} \subset W^{1,r}(\Sigma) \cap L^r_m(\Sigma)$,

$$\|f_j, \nabla' g_j, \xi_j g_j\|_r + |\lambda_j|\|g_j; L^r_m + L^r_{1/\xi_j}\|_0 \to 0 \quad \text{as} \quad j \to \infty. \tag{3.13b}$$

Without loss of generality we may assume that as $j \to \infty$,

$$\begin{array}{l} \lambda_j \to \lambda \in -\alpha + \bar{S}_\varepsilon \quad \text{or} \quad |\lambda_j| \to \infty \\ \xi_j \to 0 \quad \text{or} \quad \xi_j \to \xi \neq 0 \quad \text{or} \quad |\xi_j| \to \infty. \end{array}$$

Thus we have to consider six possibilities.

(i) First assume that $\lambda_j \to \lambda \in -\alpha + \bar{S}_\varepsilon$ and $\xi_j \to \xi \neq 0$. Then due to (3.12) $\{u_j\} \subset W^{2,r}$ and $\{p_j\} \subset W^{1,r}$ are bounded sequences. Instead of introducing subsequences we may assume without loss of generality that

$$\begin{array}{llll} u_j \to u, \nabla' u_j \to \nabla' u, & \text{in } L^r & \text{(strong convergence)} \\ \nabla'^2 u_j \rightharpoonup \nabla'^2 u & \text{in } L^r & \text{(weak convergence)} \\ p_j \to p & \text{in } L^r & \text{(strong convergence)} \\ \nabla' p_j \rightharpoonup \nabla' p & \text{in } L^r & \text{(weak convergence)} \end{array} \tag{3.14}$$

for some $(u, p) \in \mathcal{D}(S_{r,\lambda,\xi})$ as $j \to \infty$. Here we used several times the compact embedding $W^{1,r}(\Sigma) \subset\subset L^r(\Sigma)$ on a bounded domain $\Sigma$. By (3.13) we see that $S_{r,\lambda,\xi}(u, p) = (0, 0)$, which implies due to Lemma 3.3 that $u = 0, p = 0$. Summarizing these facts, (3.12) and (3.5) we are led to the contradiction $1 \leq 0$ since $u_j \to 0$ in $(W^{1,r'})^*$ due to the compact embedding $L^r \subset\subset (W^{1,r'})^*$.

(ii) We assume that $\lambda_j \to \lambda \in -\alpha + \bar{S}_\varepsilon$, but that $\xi_j \to 0$. Although $\lambda + \alpha$ may be equal to 0, the properties $\|\nabla'^2 u_j\|_r \leq 1$ and $u_j|_{\partial\Sigma} = 0$ yield the convergence (3.14) for some $u \in W^{2,r} \cap W^{1,r}_0$. Concerning $p$ we get the existence of $p \in \hat{W}^{1,r}$ and $q \in L^r$ such that

$$\nabla' p_j \rightharpoonup \nabla' p, \quad \xi_j p_j \rightharpoonup q \quad \text{in } L^r$$

as $j \to \infty$. Obviously $q$ is a constant and

$$\begin{array}{rcl} (\lambda - \Delta')u' + \nabla' p & = & 0 \\ (\lambda - \Delta')u_n + iq & = & 0 \\ \nabla' \text{div}\,' u' & = & 0. \end{array}$$

The last equation shows $\text{div}\,' u'$ to be a constant with vanishing mean on $\Sigma$ since $u' \in W^{1,r}_0$. Thus $(u', p)$ solves the homogeneous Stokes system

$$(\lambda - \Delta')u' + \nabla' p = 0, \quad \text{div}\,' u' = 0 \quad \text{in } \Sigma$$

yielding $(u', \nabla' p) = (0, 0)$, cf. Lemma 3.3, (ii).

For general $\{g_j\} \subset W^{1,r}(\Sigma)$ by (3.13a) there exists a sequence of splittings $g_j = g_{j0} + g_{j1}$ with $g_{j0} \in L^r_m$, $g_{j1} \in L^r$, such that $(|\lambda_j|+1)g_{j0} \to 0$ in $\hat{W}^{-1,r}$ and moreover $(|\lambda_j|+1)g_{j1}/\xi_j \to 0$ in $L^r$. Due to the divergence equation $\operatorname{div}_{\xi_j} u_j = g_j$ we get

$$(|\lambda_j|+1)\,\Big|\int_\Sigma u_{jn}\,dx'\Big| = \frac{|\lambda_j|+1}{|\xi_j|}\,\Big|\int_\Sigma g_{j1}\,dx'\Big| \to 0 \quad \text{as } j \to \infty,$$

and consequently $\int_\Sigma u_n\,dx' = 0$. Now we test the equation $(\lambda - \Delta')u_n + iq = 0$ with $u_n$ to see that $\lambda \int_\Sigma |u_n|^2\,dx' + \int_\Sigma |\nabla' u_n|^2\,dx' = 0$. Thus $u_n = 0$ and also $q = 0$. For $\{g_j\} \subset W^{1,r}(\Sigma) \cap L^r_m(\Sigma)$, it follows from the divergence equation and $\int_\Sigma g_j\,dx' = 0$ that $\int_\Sigma u_{jn}\,dx' = 0$; thus we get $u_n = 0$ and $q = 0$ as well.

To come to a contradiction we replace $p_j$ by $p_j - p_{jm}$ where $p_{jm} = \frac{1}{|\Sigma|}\int_\Sigma p_j\,dx'$. Then we use (3.5) for $S_{r,\lambda_j,\xi_j}(u_j, p_j - p_{jm}) = (f_j - i\xi_j p_{jm} e_n, -g_j)$ yielding

$$\begin{aligned} &\|\mu_{j+}^2 u_j, \mu_{j+}\nabla' u_j, \nabla'^2 u_j, \nabla' p_j, \xi_j(p_j - p_{jm})\|_r \\ &\qquad \le c(\|f_j, \nabla' g_j, g_j, \xi_j g_j\|_r + (|\lambda_j|+1)\|g_j; L^r_m + L^r_{1/\xi}\|_0 \\ &\qquad\quad + \|\xi_j p_{jm}\|_r + \|\nabla' u_j, \xi_j u_j, p_j - p_{jm}\|_r + \|\lambda_j u_j\|_{(W^{1,r'})^*}. \end{aligned} \tag{3.15}$$

Since $\xi_j p_j \rightharpoonup q = 0$, we have $\xi_j p_{jm} \to 0$. And, due to the compact embedding $\hat{W}^{1,r} \cap L^r_m \subset\subset L^r_m$, we get for a suitable subsequence $\{p_j\}$ that $p_j - p_{jm} \to 0$ in $L^r$. Thus, by (3.12), (3.13a) and (3.13b), for a suitable subsequence we are led to the contradiction $1 \le 0$ as $j \to \infty$.

(iii) Next we consider the case $\lambda_j \to \lambda \in -\alpha + \bar{S}_\varepsilon, |\xi_j| \to \infty$. Obviously, we get from (3.12) $\|\nabla' u_j, \xi_j u_j, p_j\|_r \to 0$, and further $\|\lambda_j u_j\|_{(W^{1,r'})^*} \to 0$ since $\|u_j\|_r \to 0$ as $j \to \infty$. Thus we come to a contradiction to (3.5), (3.12).

(iv) Let $|\lambda_j| \to \infty, \xi_j \to \xi \ne 0$. Then in $L^r$

$$\begin{array}{lll} u_j \to 0, \nabla' u_j \to 0 & \text{and} & \nabla'^2 u_j \rightharpoonup 0, \lambda_j u_j \rightharpoonup v, \\ p_j \to p & \text{and} & \nabla' p_j \rightharpoonup \nabla' p, \end{array}$$

yielding $v' + \nabla' p = 0, v_n + i\xi p = 0$. To discuss the divergence equation let $g_j = g_{j0} + g_{j1}, g_{j0} \in L^r_m, g_{j1} \in L^r$, be such that by the assumption (3.13a), (3.13b)

$$\|\lambda_j g_{j0}\|_{-1,r} + \|\lambda_j g_{j1}/\xi_j\|_r \to 0.$$

Testing the divergence equation with $\varphi \in C^\infty(\bar\Sigma)$ we have

$$-\int_\Sigma \lambda_j u_j' \cdot \overline{\nabla'\varphi}\,dx' + i\xi_j \int_\Sigma \lambda_j u_{jn}\bar\varphi\,dx' = <\lambda_j g_{j0}, \varphi> + \xi_j < \frac{\lambda_j g_{j1}}{\xi_j}, \varphi > .$$

Thus, in the limit $-\int_\Sigma v' \cdot \overline{\nabla'\varphi}\,dx' + i\xi \int_\Sigma v_n \bar\varphi\,dx' = 0$, in particular,

$$\operatorname{div}' v' = -i\xi v_n, \quad v' \cdot N|_{\partial\Sigma} = 0.$$

Consequently

$$-\Delta' p + \xi^2 p = 0 \text{ in } \Sigma, \quad \frac{\partial p}{\partial N} = 0 \text{ on } \partial\Sigma,$$

yielding $p \equiv 0$, and finally $v \equiv 0$. Again we are led to a contradiction to (3.5), (3.12).

(v) The case $|\lambda_j| \to \infty, \xi_j \to 0$. It follows from (3.12) that in $L^r$

$$\begin{array}{ll} u_j \to 0, \nabla' u_j \to 0 \quad \text{and} & \nabla'^2 u_j \rightharpoonup 0, \lambda_j u_j \rightharpoonup v, \\ \nabla' p_j \rightharpoonup \nabla' p, \quad \xi_j p_j \rightharpoonup q, & \end{array}$$

which, looking at $(R_{\lambda,\xi})$, yields in the weak limit

$$v' + \nabla' p = 0, \quad v_n + iq = 0.$$

By testing the divergence equation with functions in $C^\infty(\bar{\Sigma})$, as in the case (iv), we get $\int_\Sigma v' \cdot \overline{\nabla' \varphi}\, dx' = 0$ for all $\varphi \in C^\infty(\bar{\Sigma})$ yielding $\operatorname{div}' v' = 0, v' \cdot N|_{\partial\Sigma} = 0$. Thus we see that $v' + \nabla' p = 0$ is just the Helmholtz decomposition of the null vector field; therefore, $v' \equiv 0, \nabla' p \equiv 0$. On the other hand, by (3.13a), (3.13b) there is a splitting $g_j = g_{j0} + g_{j1}$ such that

$$g_{j0} \in L^r_m, \quad g_{j1} \in L^r \quad \text{and} \quad \|\lambda_j g_{j0}\|_{-1,r} + \|\frac{\lambda_j g_{j1}}{\xi_j}\|_r \to 0 \text{ as } j \to \infty.$$

Then, since $i\xi_j u_{jn} = g_j - \operatorname{div}' u'_j, u'_j \in W^{1,r}_0$, we get

$$\left| \int_\Sigma \lambda_j u_{jn}\, dx' \right| = \left| \int_\Sigma \frac{\lambda_j g_{j1}}{\xi_j}\, dx' \right| \to 0,$$

which together with $\lambda_j u_{jn} \rightharpoonup v_n$ yields $\int_\Sigma v_n\, dx' = 0$. Since $v_n + iq = 0$ and $q$ is a constant, we get $q = 0$, and finally $v_n = 0$. Now we define $p_{jm} = \frac{1}{|\Sigma|} \int_\Sigma p_j\, dx'$ and, by repeating some arguments as in the case (ii), we obtain (3.15) and are led to the contradiction $1 \le 0$.

(vi) The case $|\lambda_j| \to \infty, |\xi_j| \to \infty$. Then (3.12) yields the convergences

$$\begin{array}{lll} u_j \to 0, \nabla' u_j \to 0 & \text{and} & \nabla'^2 u_j \rightharpoonup 0, (\lambda_j + \xi_j^2) u_j \rightharpoonup v, \\ p_j \to 0 & \text{and} & \nabla' p_j \rightharpoonup 0, \quad \xi_j p_j \rightharpoonup q \end{array}$$

in $L^r$ with some $v, q \in L^r$. An inspection of $(R_{\lambda,\xi})$ and of (3.13a), (3.13b) shows that

$$v' = 0, \quad v_n + iq = 0.$$

Since $\|\lambda_j u_j\|_r \le c_\varepsilon \|(\lambda_j + \xi_j^2) u_j\|_r$, there exists $w = (w', w_n) \in L^r$ such that for a suitable subsequence $\lambda_j u_j \rightharpoonup w$ and $\xi_j u_j \to 0$ in $L^r$ as $j \to \infty$. By (3.13a), (3.13b) there is a splitting of $g_j$ such that

$$g_j = g_{j0} + g_{j1}, \quad \|\lambda_j g_{j0}\|_{-1,r} \to 0 \text{ and } \|\lambda_j g_{j1}/\xi_j\|_r \to 0.$$

Therefore, from the divergence equation, we get

$$\begin{aligned} < w_n, \phi > &= \lim_{j\to\infty} < \lambda_j u_{jn}, \phi > \\ &= \lim_{j\to\infty} \left( \frac{1}{i\xi_j} < \lambda_j g_{j0}, \phi > + < \frac{\lambda_j g_{j1}}{i\xi_j}, \phi > + \frac{1}{i\xi_j} < \lambda_j u'_j, \nabla' \phi > \right) = 0 \end{aligned}$$

for all $\phi \in C^\infty(\bar\Sigma)$ yielding $w_n = 0$. Finally, due to the compact embedding $L^r \subset\subset (W^{1,r'})^*$, we get, as $j \to \infty$, that $\lambda_j u_{jn} \to 0$ in $(W^{1,r'})^*$ and also $\|\lambda_j u_j'\|_{(W^{1,r'})^*} \le c_\varepsilon \|(\lambda_j + \xi_j^2) u_j'\|_{(W^{1,r'})^*} \to 0$, since $v' = 0$. Again (3.5) and (3.12) lead to the contradiction $1 \le 0$.

Now the proof of the theorem is complete. ■

Let us define the operator-valued functions

$$\begin{aligned} a_1 &: \mathbb{R} \to \mathcal{L}(L^r(\Sigma); W_0^{2,r}(\Sigma) \cap W^{1,r}(\Sigma)),\\ b_1 &: \mathbb{R} \to \mathcal{L}(L^r(\Sigma); W^{1,r}(\Sigma)) \end{aligned}$$

by

$$a_1(\xi) f := u_1(\xi), \quad b_1(\xi) f := p_1(\xi), \tag{3.16}$$

where $(u_1(\xi), p_1(\xi))$ is the solution to $(R_{\lambda,\xi})$ when $f \in L^r(\Sigma)$ is arbitrary and $g = 0$. Further, define

$$\begin{aligned} a_2 &: \mathbb{R}^* \to \mathcal{L}(W^{1,r}(\Sigma); W_0^{2,r}(\Sigma) \cap W^{1,r}(\Sigma)),\\ b_2 &: \mathbb{R}^* \to \mathcal{L}(W^{1,r}(\Sigma); W^{1,r}(\Sigma)) \end{aligned}$$

by

$$a_2(\xi) g := u_2(\xi), \quad b_2(\xi) g := p_2(\xi). \tag{3.17}$$

with $(u_2(\xi), p_2(\xi))$ the solution to $(R_{\lambda,\xi})$ when $f = 0$ and $g \in W^{1,r}(\Sigma)$ is arbitrary.

**Corollary 3.6** *For every $\alpha \in (0, \alpha_0)$ and $\lambda \in -\alpha + S_\varepsilon$ the operator-valued functions $a_1, b_1$ and $a_2, b_2$ defined by (3.16), (3.17) are Fréchet differentiable in $\xi \in \mathbb{R}$ and $\xi \in \mathbb{R}^*$, respectively.*

*Given $f \in L^r(\Sigma)$ and $g \in W^{1,r}(\Sigma)$, the derivatives $w_1 = \frac{d}{d\xi} a_1(\xi) f$, $q_1 = \frac{d}{d\xi} b_1(\xi) f$ and $w_2 = \frac{d}{d\xi} a_2(\xi) g$, $q_2 = \frac{d}{d\xi} b_2(\xi) g$ satisfy the estimates*

$$\|(\lambda + \alpha)\xi w_1, \xi^3 w_1, \xi \nabla'^2 w_1, \xi \nabla' q_1, \xi^2 q_1\|_{r;\Sigma} \le c\|f\|_{r;\Sigma} \tag{3.18}$$

*and*

$$\begin{aligned} &\|(\lambda + \alpha)\xi w_2, \xi^3 w_2, \xi \nabla'^2 w_2, \xi \nabla' q_2, \xi^2 q_2\|_{r;\Sigma} \\ &\qquad \le c\big(\|\nabla' g, g, \xi g\|_{r;\Sigma} + (|\lambda| + 1)\|g; L^r_m + L^r_{1/\xi}\|_0\big), \end{aligned} \tag{3.19}$$

*with constants $c = c(\alpha, r, \varepsilon, \Sigma)$ independent of $\lambda \in -\alpha + S_\varepsilon$ and of $\xi \in \mathbb{R}$ ($\xi \in \mathbb{R}^*$).*

*In particular, if $g \in W^{1,r} \cap L^r_m$, the stronger estimate for $w_2, q_2$*

$$\begin{aligned} &\|(\lambda + \alpha)\xi w_2, \xi^3 w_2, \xi \nabla'^2 w_2, \xi \nabla' q_2, \xi^2 q_2\|_{r;\Sigma} \\ &\qquad \le c\big(\|\nabla' g, \xi g\|_{r;\Sigma} + |\lambda|\|g; L^r_m + L^r_{1/\xi}\|_0\big) \end{aligned} \tag{3.20}$$

*is valid with $c = c(\alpha, r, \varepsilon, \Sigma)$ independent of $\lambda \in -\alpha + S_\varepsilon$.*

**Proof:** Since $\xi$ enters in $(R_{\lambda,\xi})$ in a polynomial way, it is easy to prove that $a_j(\xi), b_j(\xi), j = 1, 2$, are Fréchet differentiable operators and their pointwise derivatives $w_j, q_j$ solve the system

$$\begin{array}{rcl} (\lambda + \xi^2 - \Delta')w_j' + \nabla' q_j &=& -2\xi u_j' \\ (\lambda + \xi^2 - \Delta')w_{jn} + i\xi q_j &=& -2\xi u_{jn} - ip_j \\ \operatorname{div}_\xi w_j &=& -iu_{jn}, \end{array} \tag{3.21}$$

where $(u_1, p_1), (u_2, p_2)$ are the solutions to $(R_{\lambda,\xi})$ for $f \neq 0, g = 0$ and $f = 0, g \neq 0$, respectively.

We get from Theorem 3.4 for $j = 1, 2$,

$$\begin{aligned} &\|(\lambda+\alpha)\xi w_j, \xi^3 w_j, \xi\nabla'^2 w_j, \xi\nabla' q_j, \xi^2 q_j\|_{r;\Sigma} \\ &\quad \le c\big(\|\xi^2 u_j', \xi p_j, \nabla'\xi u_{jn}, \xi^2 u_{jn}\|_{r;\Sigma} + (|\lambda|+1)\|i\xi u_{jn}; L^r_m + L^r_{1/\xi}\|_0\big) \\ &\quad \le c\big(\|\xi^2 u_j, \xi p_j, \nabla'\xi u_j\|_{r;\Sigma} + (|\lambda|+1)\|u_j\|_{r;\Sigma}\big) \\ &\quad \le c\|u_j, (\lambda+\alpha+\xi^2)u_j, \sqrt{\lambda+\alpha+\xi^2}\nabla' u_j, \xi p_j\|_{r;\Sigma}, \end{aligned} \tag{3.22}$$

with $c = c(\alpha, r, \varepsilon, \Sigma)$; note here that $\xi^2 + |\lambda + \alpha| \le c_\varepsilon|\lambda + \alpha + \xi^2|$ for all $\lambda \in -\alpha + S_\varepsilon, \xi \in \mathbb{R}$. Therefore, by Theorem 3.4, we get (3.18)-(3.20). ■

**Remark 3.7** In the remainder of this paper the result corresponding to the operator-valued multipliers $a_2, b_2$ will not be used. However, in the forthcoming paper [13], using $a_2, b_2$, we will analyze the generalized Stokes resolvent system with prescribed divergence $g$ in an infinite cylinder of $\mathbb{R}^n$, which can be applied to study the Stokes resolvent system on unbounded cylindrical domains with several outlets to infinity.

## 4 The Proof of the Main Results

In Section 3 we obtained estimates of solutions to the parametrized Stokes resolvent system $(R_{\lambda,\xi})$ in Fourier space on the cross-section $\Sigma$ of the cylinder $\Sigma \times \mathbb{R}$. Based on the estimates we can prove Theorem 1.1.

**Proof of Theorem 1.1:** The proof is based on the Fourier multiplier theory. Considering the denseness of $\mathcal{S}(\mathbb{R}; L^2(\Sigma))$ in $L^q(\mathbb{R}; L^2(\Sigma))$ for $1 < q < \infty$, it suffices to prove the theorem for $f \in \mathcal{S}(\mathbb{R}; L^2(\Sigma))$. Let us define $u, p$ in the cylinder $\Omega = \Sigma \times \mathbb{R}$ by

$$u(x) = \mathcal{F}^{-1}(a_1(\xi)\hat{f}(\xi)), \quad p(x) = \mathcal{F}^{-1}(b_1(\xi)\hat{f}(\xi)), \tag{4.1}$$

where $a_1, b_1$ are the operator-valued multiplier functions defined by (3.16) with $r = 2$. We will show that the pair $\{u, p\}$ is a unique solution of $(R_\lambda)$ satisfying

$$u, \nabla^2 u, \nabla p \in L^q(\mathbb{R}; L^2(\Sigma))$$

and the estimate (1.1). It is obvious that $\{u, p\}$ satisfies $(R_\lambda)$. For $\xi \in \mathbb{R}$ define $m(\xi) : L^2(\Sigma) \to L^2(\Sigma)$ by

$$m(\xi) := ((\lambda + \alpha)a_1(\xi), \xi\nabla' a_1(\xi), \nabla'^2 a_1(\xi), \xi^2 a_1(\xi), \nabla' b_1(\xi), \xi b_1(\xi)). \tag{4.2}$$

By Theorem 3.4 and Corollary 3.6 one sees that $m$ is a $\mathcal{L}(L^2(\Sigma))$-valued Fourier multiplier functions satisfying the Hörmander-Michlin condition

$$\sup_{\xi\in\mathbb{R}}\|m(\xi), \xi m'(\xi)\|_{\mathcal{L}(L^2(\Sigma))} \le A \tag{4.3}$$

with a constant $A$ depending only on $\varepsilon, \Sigma$ and $\alpha \in (0, \alpha_0)$ and independent of $\lambda \in -\alpha + S_\varepsilon$. Since

$$((\lambda+\alpha)u, \nabla^2 u, \nabla p) = \big((m(\xi)\hat{f}\big)^\vee, \tag{4.4}$$

Michlin's multiplier theorem (see e.g. [8], Theorem 6.1.6 or [32], Theorem 3.4) yields $(\lambda+\alpha)u, \nabla^2 u, \nabla p \in L^q(\mathbb{R}; L^2(\Sigma))$ and the estimate

$$\|(\lambda+\alpha)u, \nabla^2 u, \nabla p\|_{L^q(\mathbb{R};L^2(\Sigma))} \le C\|f\|_{L^q(\mathbb{R};L^2(\Sigma))}$$

with a constant $C$ depending only on $q, \alpha, \varepsilon, \Sigma$.

For the proof of uniqueness let $\{u, p\}$ with $u, \nabla^2 u, \nabla p \in L^q(\mathbb{R}; L^2(\Sigma))$ satisfy $(R_\lambda)$ with $\lambda \in -\alpha + S_\varepsilon$, $f = 0$. Fix $h \in L^{q'}(L^2) := L^{q'}(\mathbb{R}; L^2(\Sigma))$ arbitrarily and let $(v, z) \in (W^{2;q',2}(\Omega) \cap W_0^{1;q',2}(\Omega)) \times \hat{W}^{1;q',2}(\Omega)$ be a solution to $(R_{\bar\lambda})$ with right-hand side $h$. Then, using the denseness of $C_{0,\sigma}^\infty(\Omega)$ in $W_0^{1;q,2}(\Omega) \cap L^q(L^2)_\sigma$ for $1 < q < \infty$, we get

$$0 = (\lambda u - \Delta u + \nabla p, v)_{L^q(L^2),L^{q'}(L^2)} = (u, \bar\lambda v - \Delta v + \nabla z)_{L^q(L^2),L^{q'}(L^2)} = (u, h)_{L^q(L^2),L^{q'}(L^2)}$$

yielding $u = 0$, and consequently, $\nabla p = 0$. The proof of Theorem 1.1 is complete. ■

**Proof of Corollary 1.2:** Defining the Stokes operator $A = A_{q,2}$ by (1.2), we easily see that for $F \in L^q(L^2)_\sigma$ the equation

$$(\lambda + A)u = F \quad \text{in} \quad L^q(L^2)_\sigma \tag{4.5}$$

is equivalent to $(R_\lambda)$ with right-hand side $f \equiv F, g \equiv 0$. Therefore, by Theorem 1.1, for every $\lambda \in -\alpha + S_\varepsilon$ there exists a unique solution $u = (\lambda + A)^{-1}F \in D(A)$ to the equation (4.5) satisfying the estimate

$$\|(\lambda+\alpha)u\|_{L^q(L^2)_\sigma} \le C\|F\|_{L^q(L^2)_\sigma}$$

with $C = C(q, \alpha, \varepsilon, \Sigma)$ independent of $\lambda$, which yields (1.3). Then (1.4) is a direct consequence of (1.3) due to classical semigroup theory. ■

**Proof of Theorem 1.3:** Defining multipliers in $L^r(\Sigma)$ for $r \in (1, \infty)$ by (4.2), we see that (4.3) holds with norms taken in $\mathcal{L}(L^r(\Sigma))$ in place of $\mathcal{L}(L^2(\Sigma))$. Next we note that the classical Hörmander-Michlin multiplier theorem may be generalized without any additional assumption to homogeneous Besov spaces of distributions with values in uniformly convex Banach spaces ([19]). Hence, the functions $(u, p)$ defined by (4.1) satisfies the assertion of Theorem 1.3 due to (4.4). ■

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