The Cosserat couple modulus for continuous solids is zero viz the linearized Cauchy-stress tensor is symmetric.

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Abstract

We investigate weaker than usual constitutive assumptions in linear Cosserat theory still providing for existence and uniqueness and show a continuous dependence result for Cosserat couple modulus $\mu_c \to 0$. This result is needed when using Cosserat elasticity not as a physical model but as a numerical regularization device.

Thereafter it is shown that the usually adopted material restrictions of uniform positivity for a linear Cosserat model cannot be consistent with experimental findings for continuous solids. The analytical solutions for both the torsion and the bending problem in general predict an unbounded stiffness for ever thinner samples. This unphysical behaviour can only be avoided for specific choices of parameters in the curvature energy expression. However, these choices do not satisfy the usual constitutive restrictions. We show that the possibly remaining linear elastic Cosserat problem is nevertheless well-posed but that it is impossible to determine the appearing curvature modulus independent of boundary conditions. This puts a serious doubt on the use of the linear elastic Cosserat model (or the geometrically exact model with $\mu_c > 0$) for the physically consistent description of continuous solids like polycrystals in the framework of elasto-plasticity.

The problem can be avoided in geometrically exact Cosserat models if the Cosserat couple modulus μ_c is set to zero.

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Contents

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1 Introduction

This note establishes well-posedness of the linear elastic Cosserat model in parameter ranges hitherto not considered and it reveals an inconsistency of the usually adopted uniform positivity of the free-energy for the linear elastic Cosserat model with experimental findings for continuous solids like polycrystals.

General continuum models involving **independent rotations** have been introduced by the Cosserat brothers [9] at the beginning of the last century. Their nonlinear, geometrically exact development has been largely forgotten for decades only to be rediscovered in a linearized setting in the early sixties [54, 29, 1, 21, 19, 59, 60, 31, 46, 58, 61]. At that time theoretical investigations on non-classical extended continuum theories were the main motivation [40]. Since then, the original Cosserat concept has been generalized in various directions, notably by Eringen and his coworkers who extended the Cosserat concept to include also microinertia effects and to rename it subsequently into **micropolar theory**. For an overview of these so called **microcontinuum** theories we refer to [20, 18, 6, 5, 7, 32, 45, 53].

The Cosserat model includes in a natural way **size effects**, i.e. small samples behave comparatively stiffer than large samples. These effects have recently received new attention in conjunction with nano-devices. From a computational point of view, theories with size-effect are increasingly **used to regularize non-wellposed situations**, e.g. shear-banding in elasto-plasticity without hardening [39, 15, 57, 11, 13, 12]. It has been shown by the author that infinitesimal elasto-plasticity augmented with (elastic) Cosserat effects indeed leads to a well-posed problem [51]. The mathematical analysis establishing well-posedness for the infinitesimal strain, Cosserat elastic solid is presented in [34, 16, 33, 26, 27] and in [38, 36, 37] for so called linear microstretch models. This analysis is based on the **uniform positivity** of the free energy of the Cosserat solid, which in turn **implies** that the **Cosserat couple modulus μ_c is strictly positive**. The author has extended the existence results for both the Cosserat model and the more general micromorphic models to the geometrically exact, finite-strain case, see e.g. [52, 49, 50].

The important problem of the determination of Cosserat material parameters for **continuous solids** with random microstructure must still be considered an open problem while the use of a Cosserat model for the simplified computation of man-made grid frameworks is successful in the sense that parameters for the "homogenized" Cosserat model, replacing the grid-framework, can be explicitly calculated.

Over the years, a variety of boundary value problems have been solved in terms of analytical expressions which are then used for the determination of material constants in the infinitesimal linear Cosserat model, see [35]. Notably, the solution of the pure torsion problem with prescribed torque at the end faces has been given in [24, 23, 55] and used in [24, 23, 2, 44] to determine the length scales of various materials. Similarly, an analytical solution for the pure bending problem is available [56]. These formulas will be investigated w.r.t. their behaviour for slender or thin specimens. A series of experiments with specimens of different slenderness is usually performed in order to determine the Cosserat parameters [23, 44]. We observe an unphysical **unbounded stiffening** behaviour for slender specimens which seems to make it impossible to arrive at consistent values for these parameters: the value for the parameters will depend strongly on the smallest investigated specimen size. This inconsistency may be in part responsible for the fact that 1. (linear) Cosserat parameters for continuous solids have never gained general acceptance even in the "Cosserat community" and 2. that the linear elastic Cosserat model has never been really accepted by a majority of applied scientists as a useful model to describe size effects in continuous solids.

It is of importance to note that this inconsistency is not necessarily encountered in the geometrically exact Cosserat model. Indeed, the geometrically exact Cosserat model with true rotations $\overline{R} \in \mathrm{SO}(3, \mathbb{R})$ instead of infinitesimal microrotations $\overline{A} \in \mathfrak{so}(3, \mathbb{R})$ has been shown to admit minimizers in [48] also for zero Cosserat couple modulus $\mu_c = 0$ in which case the previously discussed unbounded stiffness for ever smaller specimens does not appear. The author believes that the geometrically exact Cosserat model with $\mu_c = 0$ can do justice to the Cosserat approach as a viable physical model for a continuous solid which incorporates length scale effects. **A zero Cosserat couple modulus $\mu_c = 0$ implies that the Cauchy-stress tensor is symmetric to first order and length scale effects are of second order.** This is acceptable for a continuous solid.

In view of the far reaching consequences of our development and the certainly controversial nature of the result that μ_c must be zero, it is necessary to clearly distinguish between the different possible fields of application of the Cosserat model and to indicate precisely to which case our result relates:

1. The Cosserat model as a physically consistent description of a continuous solid with size effects. As an example we may consider a polycrystalline nano-copper wire with different grain sizes. Clearly, the body of interest is not homogeneous, but we think of the material as filling the whole space. It is possible to investigate experimentally specimens of very small size [22]. The Cosserat model should be able to reflect this situation. Whether this is true is partly the concern of this contribution and our answer will be negative for the linear elastic Cosserat model and the geometrically exact Cosserat model with $\mu_c > 0$.

2. The Cosserat model as a homogenized replacement for man made grid-structures or for materials with periodic microstructure. This is possible. The corresponding moduli can sometimes ab initio be calculated. The periodicity cell provides e.g. a lower bound for the length scale. We are not concerned with this application.

3. The Cosserat model as a homogenized replacement for man made foams [14, 41, 52] or bones [55, 2, 44]. Foams and bones are certainly not continuous solids: the smallest possible size which the Cosserat model should be able to reflect is given by the cell size. Considering the Cosserat model then for specimen-sizes below the cell size does not make sense. We do not consider this application here.

4. The Cosserat model as a homogenized model for rigid spheres in contact, e.g. sand and other granular materials [8, 3]. The Cosserat parameters are linked to properties of the spheres and notably $\mu_c > 0$ is related to the "sliding stiffness" between the particles at their contact zones while the curvature parameters are related to "rolling stiffness" and "twisting stiffness" [8]. Sliding gives rise to friction and friction is generally a dissipative mechanism. Whether there can be elastic energy storage due to inter-particle rotation is at least questionable. We do not consider this application here.

5. The Cosserat model as a localization limiter and approximation to non-wellposed situations [39, 15, 57, 11, 13, 12]. Here, one is not really interested in the introduced size effect but in the regularizing power of the Cosserat model. If applied to continuous solids, stiffening effects are not necessarily welcome. We are partly concerned with this topic in this contribution.

It should be clear that the different applications demand different sets of constitutive parameters. An investigation into this problem, however, seems not to have been done.

This contribution is organized as follows: first, we recall the linear elastic static isotropic Cosserat model in variational form and discuss weaker conditions than the uniform positivity of the strain and curvature energy which lead to a well-posed boundary value problem. We show that under these weaker conditions solutions still depend continuously on μ_c even for $\mu_c \to 0$. This result rigourously justifies the use of the Cosserat model as an approximation to classical elasticity or its use as localization limiter in elasto-plasticity.

Second, we recall one aspect of the well known analytical solution formulas, namely the comparative stiffness of the linear Cosserat response over the classical response. It is shown that in order to avoid an unbounded comparative stiffness for ever thinner samples in torsion (possible limit for continuous solids), one has to satisfy an additional condition on the coefficients of the Cosserat curvature energy. This condition is in conflict with uniform positivity. Proceeding similarly for the pure bending of a circular tube and the bending of a curved thin beam, the imposed restrictions on the curvature parameter lead to an unusual curvature expression including only the spherical part of the couple-stress tensor.[1] It is shown that this remaining Cosserat model is still well-posed in a very weak Sobolev space. However, micropolar response is not activated for any inhomogeneous boundary value problem: the determination of the (one) remaining curvature parameter can only be obtained under certain boundary conditions on the microrotations, since otherwise the classical elasticity solution is recovered. This fact excludes that this parameter can be a material parameter.[2] Altogether, this development puts a serious doubt on the use of a linear elastic Cosserat model for the physical consistent description of continuous solids. The notation is found in the appendix.

2 The linear elastic isotropic Cosserat model revisited

2.1 The linear elastic Cosserat model in variational form

For the **displacement** $u : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ and the **skew-symmetric infinitesimal microrotation** $\overline{A} : \Omega \subset \mathbb{R}^3 \mapsto \mathfrak{so}(3,\mathbb{R})$ we consider the **two-field** minimization problem

$$\begin{aligned} I(u,\overline{A}) = \int_\Omega W_{\mathrm{mp}}(\overline{\varepsilon}) + W_{\mathrm{curv}}(\nabla \operatorname{axl}(\overline{A})) - \langle f, u\rangle - \langle \overline{M}, \overline{A}\rangle \,\mathrm{dV} \\ - \int_{\Gamma_S} \langle f_S, u\rangle - \langle \overline{M}_S, \overline{A}\rangle \,\mathrm{dS} \mapsto \quad \min . \text{ w.r.t. } (u,\overline{A}), \end{aligned} \tag{2.1}$$

under the constitutive requirements and boundary conditions[3]

$$\begin{aligned} \overline{\varepsilon} &= \nabla u - \overline{A}, \quad \overline{A}_{|_\Gamma} = \overline{A}_{\mathrm{d}} \in \mathfrak{so}(3,\mathbb{R}), \quad u_{|_\Gamma} = u_{\mathrm{d}}\,, \\ W_{\mathrm{mp}}(\overline{\varepsilon}) &= \mu\,\|\operatorname{sym}\overline{\varepsilon}\|^2 + \mu_c\,\|\operatorname{skew}\overline{\varepsilon}\|^2 + \frac{\lambda}{2}\operatorname{tr}[\operatorname{sym}\overline{\varepsilon}]^2 \qquad \textbf{strain energy} \\ &= \mu\,\|\operatorname{sym}\nabla u\|^2 + \mu_c\,\|\operatorname{skew}(\nabla u - \overline{A})\|^2 + \frac{\lambda}{2}\operatorname{tr}[\operatorname{sym}\nabla u]^2 \qquad (2.3) \\ &= \mu\,\|\operatorname{dev}\operatorname{sym}\nabla u\|^2 + \mu_c\,\|\operatorname{skew}(\nabla u - \overline{A})\|^2 + \frac{2\mu+3\lambda}{6}\operatorname{tr}[\operatorname{sym}\nabla u]^2 \\ &= \mu\,\|\operatorname{sym}\nabla u\|^2 + \frac{\mu_c}{2}\,\|\operatorname{curl} u - 2\operatorname{axl}\overline{A}\|^2_{\mathbb{R}^3} + \frac{\lambda}{2}\,(\operatorname{Div} u)^2\,, \\ \phi &:= \operatorname{axl}(\overline{A}) \in \mathbb{R}^3, \quad \mathfrak{k} = \nabla\phi, \quad \|\operatorname{curl}\phi\|^2_{\mathbb{R}^3} = 4\|\operatorname{axl}\operatorname{skew}\nabla\phi\|^2_{\mathbb{R}^3} = 2\|\operatorname{skew}\nabla\phi\|^2_{\mathbb{M}^{3\times 3}}\,, \\ W_{\mathrm{curv}}(\nabla\phi) &= \frac{\gamma+\beta}{2}\|\operatorname{sym}\nabla\phi\|^2 + \frac{\gamma-\beta}{2}\|\operatorname{skew}\nabla\phi\|^2 + \frac{\alpha}{2}\operatorname{tr}[\nabla\phi]^2 \qquad \textbf{curvature energy} \\ &= \frac{\gamma+\beta}{2}\|\operatorname{dev}\operatorname{sym}\nabla\phi\|^2 + \frac{\gamma-\beta}{2}\|\operatorname{skew}\nabla\phi\|^2 + \frac{3\alpha+(\beta+\gamma)}{6}\operatorname{tr}[\nabla\phi]^2 \\ &= \frac{\gamma}{2}\|\nabla\phi\|^2 + \frac{\beta}{2}\langle\nabla\phi,\nabla\phi^T\rangle + \frac{\alpha}{2}\operatorname{tr}[\nabla\phi]^2 \end{aligned}$$

[1] which remains indeterminate in the indeterminate couple stress model of Mindlin.

[2] A material parameter is by definition independent of boundary conditions.

[3] More detailed than strictly necessary in order to accommodate the different representations in the literature. Note that $\operatorname{axl}\overline{A}\times\xi = \overline{A}.\xi$ for all $\xi\in\mathbb{R}^3$, such that

$$\operatorname{axl}\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} := \begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix}, \quad \overline{A}_{ij} = \varepsilon_{ijk}\cdot\operatorname{axl}(\overline{A})_k\,, \tag{2.2}$$

where ε_{ijk} is the totally antisymmetric permutation tensor. Here, $\overline{A}.\xi$ denotes the application of the matrix $\overline{A}$ to the vector ξ and $a\times b$ is the usual cross-product.

$$= \frac{\gamma+\beta}{2}\|\operatorname{sym}\nabla\phi\|^2 + \frac{\gamma-\beta}{4}\|\operatorname{curl}\phi\|^2_{\mathbb{R}^3} + \frac{\alpha}{2}\,(\operatorname{Div}\phi)^2 \; .$$

Here, $f, \overline{M}$ are volume force and volume couples, respectively; $f_s, \overline{M}_S$ are surface tractions and surface couples at $\Gamma_S \subset \partial\Omega$, respectively, while $u_{\mathrm{d}}, \overline{A}_{\mathrm{d}}$ are Dirichlet boundary conditions for displacement and infinitesimal microrotation at $\Gamma \subset \partial\Omega$. The strain energy W_{mp} and the curvature energy W_{curv} are the most general quadratic forms in the **non-symmetric strain** $\overline{\varepsilon} = \nabla u - \overline{A}$ and the **micropolar curvature** $\overline{\mathfrak{k}} = \nabla\operatorname{axl}(\overline{A})$ (curvature-twist tensor) in the isotropic, centro-symmetric case. The parameters μ, λ[MPa] are the classical Lamé moduli and α, β, γ are additional micropolar moduli with dimension $[\mathrm{Pa}\cdot\mathrm{m}^2] = [\mathrm{N}]$ of a force. In other contributions of the author, it is preferred to write $\alpha, \beta, \gamma \sim \mu\, L_c^2\alpha', \mu\, L_c^2\beta', \mu\, L_c^2\gamma'$ with corresponding non-dimensional parameters α', β', γ' and a material length scale $L_c > 0\,[\mathrm{m}]$.

The additional parameter $\mu_c \geq 0$[MPa] in the strain energy is the **Cosserat couple modulus**. For $\mu_c = 0$ the two fields of displacement and microrotations decouple and one is left formally with classical linear elasticity for the displacement u. The author is not aware of a rigourous mathematical study of the limit behaviour as $\mu_c \to 0$. In the case of a pure Neumann problem, the resulting system of decoupled equations may not have enough equations to provide for a unique equilibrium solution if $\mu_c = 0$. For the torsion problem, this "pathological" situation has been presented in [42]. Nevertheless, in this situation, the minimizer of the variational problem remains unique and coincides with the classical elasticity solution.

2.2 The linear elastic Cosserat balance equations: hyperelasticity

Taking free variations of the energy in (2.1) w.r.t. both displacement $u \in \mathbb{R}^3$ and infinitesimal microrotation $\overline{A} \in \mathfrak{so}(3, \mathbb{R})$, one arrives at the equilibrium system (the Euler-Lagrange equations of (2.1))

$$\begin{aligned}
&\operatorname{Div}\sigma = f\,, \qquad -\operatorname{Div} m = 2\mu_c\cdot\operatorname{axl}\operatorname{skew}\overline{\varepsilon} + \operatorname{axl}\operatorname{skew}(\overline{M})\,, \quad \overline{\varepsilon} = \nabla u - \overline{A},\\
&\quad \sigma = 2\mu\cdot\operatorname{sym}\overline{\varepsilon} + 2\mu_c\cdot\operatorname{skew}\overline{\varepsilon} + \lambda\cdot\operatorname{tr}[\overline{\varepsilon}]\cdot 1\!\!1 = (\mu+\mu_c)\cdot\overline{\varepsilon} + (\mu-\mu_c)\cdot\overline{\varepsilon}^T + \lambda\cdot\operatorname{tr}[\overline{\varepsilon}]\cdot 1\!\!1\,,\\
&\quad m = \gamma\,\nabla\phi + \beta\,\nabla\phi^T + \alpha\operatorname{tr}[\nabla\phi]\cdot 1\!\!1, \qquad \phi = \operatorname{axl}(\overline{A})\,, \qquad\qquad (2.4)\\
&\overline{A}_{|_\Gamma} = \overline{A}_{\mathrm{d}} \in \mathfrak{so}(3,\mathbb{R}), \quad u_{|_\Gamma} = u_{\mathrm{d}}\,, \quad \sigma.\vec{n}_{|_{\Gamma_S}} = f_S\,, \quad m.\vec{n}_{|_{\Gamma_S}} = \frac{1}{2}\operatorname{axl}(\operatorname{skew}(\overline{M}_S))\,,\\
&\quad \sigma.\vec{n}_{|_{\partial\Omega\setminus(\Gamma_S\cup\Gamma)}} = 0\,, \quad m.\vec{n}_{|_{\partial\Omega\setminus(\Gamma_S\cup\Gamma)}} = 0\,.
\end{aligned}$$

Here, m is the **couple stress tensor**. For comparison, in [18, p.111] or [2, 44, 24] the elastic moduli in our notation are defined to be $\mu = \mu^* + \frac{\kappa}{2}$, $\mu_c = \frac{\kappa}{2}$.[4] But in this last definition (see [10]), μ^* cannot be regarded as one of the classical Lamé constants.[5] [6]

2.3 The linear elastic Cosserat balance equations: non-variational case

By splitting the Cosserat couple modulus μ_c into different parameters μ_c^a and μ_c^b in the balance of linear and angular momentum equation, the variational character is lost and one must solve accordingly

$$\begin{aligned}
&\operatorname{Div}\sigma = f\,, \qquad -\operatorname{Div} m = 2\mu_c^b\cdot\operatorname{axl}\operatorname{skew}\overline{\varepsilon} + \operatorname{axl}\operatorname{skew}(\overline{M})\,, \quad \overline{\varepsilon} = \nabla u - \overline{A},\\
&\quad \sigma = 2\mu\cdot\operatorname{sym}\overline{\varepsilon} + 2\mu_c^a\cdot\operatorname{skew}\overline{\varepsilon} + \lambda\cdot\operatorname{tr}[\overline{\varepsilon}]\cdot 1\!\!1\,,\\
&\quad m = \gamma\,\nabla\phi + \beta\,\nabla\phi^T + \alpha\operatorname{tr}[\nabla\phi]\cdot 1\!\!1, \qquad \phi = \operatorname{axl}(\overline{A})\,, \qquad\qquad (2.5)\\
&\overline{A}_{|_\Gamma} = \overline{A}_{\mathrm{d}} \in \mathfrak{so}(3,\mathbb{R}), \quad u_{|_\Gamma} = u_{\mathrm{d}}\,, \quad \sigma.\vec{n}_{|_{\Gamma_S}} = f_S\,, \quad m.\vec{n}_{|_{\Gamma_S}} = \frac{1}{2}\operatorname{axl}(\operatorname{skew}(\overline{M}_S))\,.
\end{aligned}$$

[4] In [36, 18] the Cauchy stress tensor σ is defined as $\sigma = (\mu^*+\kappa)\,\overline{\varepsilon} + \mu^*\,\overline{\varepsilon}^T + \lambda\operatorname{tr}[\overline{\varepsilon}]\cdot 1\!\!1$ with given constants μ^*, κ, λ and one must identify $\mu^*+\kappa = \mu+\mu_c$, $\quad \mu^* = \mu - \mu_c$.

[5] A simple definition of the Lamé constants in micropolar elasticity is that they should coincide with the classical Lamé constants for symmetric situations. Equivalently, they are obtained by the classical formula $\mu = \frac{E}{2(1+\nu)}$, $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$, where E and ν are uniquely determined from uniform traction where Cosserat effects are absent.

[6] Unfortunately, while authors are consistent in their usage of material parameters, one should be careful when identifying the actually used parameters with his own usage. The different representations in (2.3) might be useful for this purpose.

This form of equation may be more suitable for certain limit problems. For example one may consider $\mu_c^a \to 0$ while $\mu_c^b = \mu = const$, especially in a geometrically exact context where $\mu_c^a = 0$ still provides for a second order coupling between microrotations and deformations..

2.4 The indeterminate couple stress model

This model is formally obtained by setting $\mu_c = \infty$, which enforces the constraint $\operatorname{curl} u = 2 \operatorname{axl} \overline{A}$ [46, 59]. For the **displacement** $u : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ we consider therefore the **one-field** minimization problem

$$I(u) = \int_\Omega W_{\mathrm{mp}}(\nabla u) + W_{\mathrm{curv}}(\nabla \operatorname{curl} u) - \langle f, u\rangle - \frac{1}{2}\langle \operatorname{axl}(\overline{M}), \operatorname{curl} u\rangle \, \mathrm{dV} \tag{2.6}$$
$$- \int_{\Gamma_S} \langle f_S, u\rangle - \frac{1}{2}\langle \operatorname{axl}(\overline{M}_S), \operatorname{curl} u\rangle \, \mathrm{dS} \mapsto \quad \min . \text{ w.r.t. } u,$$

under the constitutive requirements and boundary conditions

$$W_{\mathrm{mp}}(\overline{\varepsilon}) = \mu \, \|\operatorname{sym} \nabla u\|^2 + \frac{\lambda}{2} \operatorname{tr}[\operatorname{sym} \nabla u]^2, \quad u_{|\Gamma} = u_{\mathrm{d}}, \quad \operatorname{curl} u_{|\Gamma} = (\operatorname{curl} u)_{\mathrm{d}} \in \mathbb{R}^3,$$
$$W_{\mathrm{curv}}(\nabla \operatorname{curl} u) = \frac{\gamma+\beta}{8} \|\operatorname{sym} \nabla \operatorname{curl} u\|^2 + \frac{\gamma-\beta}{8} \|\operatorname{skew} \nabla \operatorname{curl} u\|^2 . \tag{2.7}$$

In this limit model, the curvature parameter α, related to the spherical part of the couple stress tensor m remains **indeterminate**, since $\operatorname{Div} \operatorname{axl} \overline{A} = \operatorname{Div} \frac{1}{2} \operatorname{curl} u = 0$. We remark the intricate relation between $\mu_c \to \infty$ and the indeterminacy of α.

3 Constitutive restrictions and well-posedness for Cosserat hyperelasticity

3.1 Pointwise positivity of the micropolar energy

For a mathematical treatment in the hyperelastic case we may require that for arbitrary nonzero $\overline{\varepsilon}, \overline{\mathfrak{k}} \in \mathbb{M}^{3\times 3}$ one has the **local positivity condition**

$$\forall \, \overline{\varepsilon}, \overline{\mathfrak{k}} \neq 0 : \qquad W_{\mathrm{mp}}(\overline{\varepsilon}) > 0, \qquad W_{\mathrm{curv}}(\overline{\mathfrak{k}}) > 0 . \tag{3.1}$$

This condition is most often invoked as the basis of uniqueness proofs in static micropolar elasticity, see e.g. [34, 33, 18, 17]. By splitting $\overline{\varepsilon}$ in its deviatoric and volumetric part, i.e. writing

$$\overline{\varepsilon} = \operatorname{dev} \operatorname{sym} \overline{\varepsilon} + \operatorname{skew} \overline{\varepsilon} + \frac{1}{3} \operatorname{tr}[\overline{\varepsilon}] \cdot 1\!\!1 \tag{3.2}$$

and inserting this into W_{mp} one gets

$$W_{\mathrm{mp}}(\overline{\varepsilon}) = \mu \, \|\operatorname{dev} \operatorname{sym} \overline{\varepsilon}\|^2 + \mu_c \, \|\operatorname{skew} \overline{\varepsilon}\|^2 + \frac{2\mu + 3\lambda}{6} \operatorname{tr}[\overline{\varepsilon}]^2 . \tag{3.3}$$

Since all three contributions in (3.2) can be chosen independent of each other, one obtains from (3.1) the **positive-definiteness condition**

$$\mu > 0, \quad 2\mu + 3\lambda > 0, \quad \mu_c > 0,$$
$$\gamma + \beta > 0, \quad (\gamma+\beta) + 3\alpha > 0, \quad \gamma - \beta > 0, \quad (\gamma > 0), \tag{3.4}$$

where the argument pertaining to the curvature energy W_{curv} is exactly similar, cf. [36, (2.9)]. In effect, one ensures **uniform convexity** of the integrand w.r.t $\overline{\varepsilon}, \overline{\mathfrak{k}}$. This local positivity condition **excludes**, however, **classical linear elasticity**, since $\mu_c > 0$ introduces the Cosserat effects to first order.[7]

By a **thermodynamical argument** [18] one may similarly infer the **non-negativity** of the energy (material stability), leading to

$$\mu \geq 0, \quad 2\mu + 3\lambda \geq 0, \quad \mu_c \geq 0,$$
$$\gamma + \beta \geq 0, \quad (\gamma+\beta) + 3\alpha \geq 0, \quad \gamma - \beta \geq 0, \quad (\gamma \geq 0), \tag{3.5}$$

[7] In the geometrically exact case $\mu_c = 0$ would lead to a second order coupling only.

which allows for classical linear elasticity but which condition alone is not strong enough to guarantee existence and uniqueness of the corresponding boundary value problem. Nevertheless, all constitutive restrictions on a linear Cosserat solid must at least be consistent with (3.5) from a purely physical point of view.[8]

3.2 Coercivity of the micropolar energy

What one really needs for a mathematical treatment of the mixed boundary value problem in the variational context, is, however, a **coercivity condition**, in the sense that a bounded energy I implies a bound on the displacement u and the infinitesimal microrotation $\overline{A}$ in appropriate Sobolev spaces. More precisely, for H^1-coercivity it must hold that

$$I(u,\overline{A}) \le K_1 < \infty \quad \Rightarrow \quad u \in H^{1,2}(\Omega,\mathbb{R}^3),\ \overline{A} \in H^{1,2}(\Omega,\mathfrak{so}(3,\mathbb{R}))\,. \tag{3.6}$$

In the case of Dirichlet boundary conditions for u and $\overline{A}$ on some part of the boundary $\Gamma \subset \partial\Omega$ with non-vanishing two-dimensional Hausdorff measure this **coercivity requirement may hold e.g. if one of the following four set of conditions is satisfied:**

$$\begin{aligned}&\mu > 0\,, \quad 2\mu + 3\lambda > 0\,, \quad \mu_c \ge 0\,,\\ &\gamma + \beta > 0\,, \quad (\gamma+\beta) + 3\alpha > 0\,, \quad \gamma - \beta \ge 0\,, \quad (\gamma > 0)\end{aligned} \tag{3.7}$$

$$\begin{aligned}&\mu \ge 0\,, \quad \mu_c > 0\,, \quad \lambda > 0\\ &\gamma + \beta \ge 0\,, \quad \gamma - \beta > 0\,, \quad \alpha > 0\,, \quad (\gamma > 0)\end{aligned} \tag{3.8}$$

$$\begin{aligned}&\mu > 0\,, \quad 2\mu + 3\lambda > 0\,, \quad \mu_c \ge 0\,,\\ &\gamma + \beta \ge 0\,, \quad \gamma - \beta > 0\,, \quad \alpha > 0\,, \quad (\gamma > 0)\end{aligned} \tag{3.9}$$

$$\begin{aligned}&\mu \ge 0\,, \quad \mu_c > 0\,, \quad \lambda > 0\\ &\gamma + \beta > 0\,, \quad (\gamma+\beta) + 3\alpha > 0\,, \quad \gamma - \beta \ge 0\,, \quad (\gamma > 0)\,.\end{aligned} \tag{3.10}$$

We disregard conditions (3.8) and (3.10) since they would exclude linear elasticity from the onset and they need Dirichlet conditions everywhere on the boundary $u_{|\partial\Omega}$ to provide for coercivity. Conditions (3.7) and (3.9) **do not exclude classical linear elasticity**, since the Cosserat couple modulus μ_c may be set to zero. For (3.7) and (3.9) the integrand is therefore convex in $\overline{\varepsilon}, \overline{\mathfrak{k}}$ but not uniformly convex. Nonetheless, (3.7) or (3.9) is enough to provide for uniqueness in the static case (details subsequently, (3.9) needs special boundary conditions for $\overline{A}$). The task is then to decide whether (3.7) or (3.9) is more appropriate. Since (3.7) provides for coercivity in the most general case as far as boundary conditions are concerned it seems to be appropriate. Nevertheless, the different constitutive assumptions must be confronted with experiments.

In terms of the non-dimensional **polar ratio** $\Psi = \frac{\beta+\gamma}{\alpha+\beta+\gamma}$ one has

$$\Psi := \frac{\beta+\gamma}{\alpha+\beta+\gamma} = \frac{3(\beta+\gamma)}{3(\alpha+\beta+\gamma)} = \frac{3(\beta+\gamma)}{3\alpha + (\beta+\gamma) + 2(\beta+\gamma)}\,, \tag{3.11}$$

which leads with (3.7) to the restriction $0 < \Psi < \frac{3}{2}$ while (3.9) imposes $0 \le \Psi < 1$.

3.3 Uniqueness for $\mu_c \ge 0$ and coercitive curvature in case (3.7)

Let us see why the linear Cosserat model still has unique solutions even for $\mu_c = 0$ in case of (3.7). The uniqueness under the much stronger assumption (3.4) is a well known fact.

In order to show uniqueness, it is sufficient to look at the second derivative of the energy I w.r.t. u and $\overline{A}$. It is easy to see that for increments $\tilde{u} \in C^\infty(\Omega,\mathbb{R}^3)$ and $\tilde{A} \in C^\infty(\Omega,\mathfrak{so}(3,\mathbb{R}))$,

[8] For $\nu = \frac{\lambda}{2(\mu+\lambda)} = \frac{\lambda}{2\mu+3\lambda-\lambda}$ condition (3.5) implies the well-known bound $-1 \le \nu \le \frac{1}{2}$, while (3.7) and (3.9) require $-1 < \nu < \frac{1}{2}$, but (3.8) and (3.10) would impose $0 \le \nu \le \frac{1}{2}$.

respecting the Dirichlet-boundary conditions, i.e. $\tilde{u}_{|_\Gamma} = 0$ and $\tilde{A}_{|_\Gamma} = 0$ one has

$$
\begin{aligned}
&D^2_{(u,\overline{A})} I(u,\overline{A}).[(\tilde{u},\tilde{A}),(\tilde{u},\tilde{A})] \\
&= \int_\Omega 2\mu\,\|\operatorname{sym}\nabla\tilde{u}\|^2 + 2\mu_c\,\|\operatorname{skew}\nabla\tilde{u}-\tilde{A}\|^2 + \lambda\operatorname{tr}[\operatorname{sym}\nabla\tilde{u}]^2 \\
&\qquad + (\beta+\gamma)\,\|\operatorname{sym}\nabla\operatorname{axl}\tilde{A}\|^2 + (\gamma-\beta)\,\|\operatorname{skew}\nabla\operatorname{axl}\tilde{A}\|^2 + \alpha\operatorname{tr}\left[\nabla\operatorname{axl}\tilde{A}\right]^2 \mathrm{dV}\,. \qquad (3.12)
\end{aligned}
$$

For $\mu_c = 0$, $\lambda > 0$ and coercitive curvature expression, i.e. assuming $(3.7)_2$ one obtains for some constant $c_1 > 0$ depending on $(3.7)_2$ the estimate

$$
D^2_{(u,\overline{A})} I(u,\overline{A}).[(\tilde{u},\tilde{A}),(\tilde{u},\tilde{A})] \geq \int_\Omega 2\mu\,\|\operatorname{sym}\nabla\tilde{u}\|^2 + c_1\,\|\operatorname{sym}\nabla\operatorname{axl}\tilde{A}\|^2\,\mathrm{dV}\,. \qquad (3.13)
$$

Using Korn's first inequality for both $\tilde{u}$ and $\operatorname{axl}\tilde{A}$ one gets with some positive constant $c_K > 0$, depending on the domain Ω

$$
\begin{aligned}
D^2_{(u,\overline{A})} I(u,\overline{A}).[(\tilde{u},\tilde{A}),(\tilde{u},\tilde{A})] &\geq c_K\,\left(\|\tilde{u}\|^2_{H^{1,2}(\Omega,\mathbb{R}^3)} + \|\operatorname{axl}\tilde{A}\|^2_{H^{1,2}(\Omega,\mathbb{R}^3)}\right) \\
&= c_K\,\left(\|\tilde{u}\|^2_{H^{1,2}(\Omega,\mathbb{R}^3)} + \frac{1}{2}\|\tilde{A}\|^2_{H^{1,2}(\Omega,\mathfrak{so}(3,\mathbb{R}))}\right)\,. \qquad (3.14)
\end{aligned}
$$

This shows uniform positivity of the second derivative. The energy I is strictly convex, the minimizers are unique.

3.4 Uniqueness for $\mu_c \geq 0$ and coercitive curvature in case (3.9)

Let us investigate first the case where Dirichlet conditions for $\overline{A}$ are prescribed on the entire boundary $\partial\Omega$. In case of condition (3.9) the same calculations as before lead to

$$
\begin{aligned}
D^2_{(u,\overline{A})} I(u,\overline{A}).[(\tilde{u},\tilde{A}),(\tilde{u},\tilde{A})] &\geq \int_\Omega 2\mu\,\|\operatorname{sym}\nabla\tilde{u}\|^2 + c_1\,\|\operatorname{skew}\nabla\operatorname{axl}\tilde{A}\|^2 + \alpha\operatorname{tr}\left[\nabla\operatorname{axl}\tilde{A}\right]^2 \mathrm{dV} \\
&= \int_\Omega 2\mu\,\|\operatorname{sym}\nabla\tilde{u}\|^2 + \frac{c_1}{2}\,\|\operatorname{curl}[\operatorname{axl}\tilde{A}]\|^2 + \alpha\,\left(\operatorname{Div}[\operatorname{axl}\tilde{A}]\right)^2\,\mathrm{dV} \\
&\geq c_K\,\|\tilde{u}\|^2_{H^{1,2}(\Omega,\mathbb{R}^3)} + c_{GR}\,\|\tilde{A}\|^2_{H^{1,2}(\Omega,\mathfrak{so}(3,\mathbb{R}))}\,, \qquad (3.15)
\end{aligned}
$$

where we have made use of Korn's first inequality for the first term in $\tilde{u}$ and the fact that the operators curl and Div together control the total gradient, see [28, p.36], i.e. the inequality

$$
\exists\, C > 0\;\forall\,\phi \in C_0^\infty(\Omega,\mathbb{R}^3):\; \int_\Omega \|\operatorname{curl}\phi(x)\|^2_{\mathbb{R}^3} + (\operatorname{Div}\phi(x))^2\;\mathrm{dV} \geq C\,\|\phi\|^2_{H^{1,2}(\Omega,\mathbb{R}^3)}\,, \qquad (3.16)
$$

holds for smooth functions with compact support $C_0^\infty(\Omega,\mathbb{R}^3)$. Here we see that $\mu_c = 0$ is permitted provided that $\operatorname{axl}\tilde{A} = 0$ on $\partial\Omega$ identically.

In case of condition (3.9) with $\mu_c > 0$ it is possible to relax the requirement on the boundary condition for the microrotations. It suffices to prescribe the normal component $\langle\operatorname{axl}\overline{A},\vec{n}\rangle = B(x)$ on $\partial\Omega$, while the tangential components of $\operatorname{axl}\overline{A}$ may be arbitrary. Now we compute

$$
\begin{aligned}
&D^2_{(u,\overline{A})} I(u,\overline{A}).[(\tilde{u},\tilde{A}),(\tilde{u},\tilde{A})] \\
&= \int_\Omega 2\mu\,\|\operatorname{sym}\nabla\tilde{u}\|^2 + \mu_c\,\|\operatorname{curl}u - 2\operatorname{axl}\tilde{A}\|^2 + \lambda\operatorname{tr}[\operatorname{sym}\nabla\tilde{u}]^2 \\
&\qquad + \frac{(\gamma-\beta)}{2}\,\|\operatorname{curl}\operatorname{axl}\tilde{A}\|^2 + \alpha\operatorname{tr}\left[\nabla\operatorname{axl}\tilde{A}\right]^2\mathrm{dV} \\
&= \int_\Omega 2\mu\,\|\operatorname{sym}\nabla\tilde{u}\|^2 + \mu_c\|\operatorname{curl}\tilde{u}\|^2 - 4\mu_c\langle\operatorname{curl}\tilde{u},\operatorname{axl}\tilde{A}\rangle + 4\mu_c\|\operatorname{axl}\tilde{A}\|^2 + \lambda\operatorname{tr}[\operatorname{sym}\nabla\tilde{u}]^2 \\
&\qquad + \frac{(\gamma-\beta)}{2}\,\|\operatorname{curl}\operatorname{axl}\tilde{A}\|^2 + \alpha\operatorname{tr}\left[\nabla\operatorname{axl}\tilde{A}\right]^2\mathrm{dV} \\
&\geq \int_\Omega 2\mu\,\|\operatorname{sym}\nabla\tilde{u}\|^2 - \mu_c\,|1-\frac{2}{\varepsilon}|\,\|\nabla\tilde{u}\|^2 + \mu_c\,(4-2\varepsilon)\|\operatorname{axl}\tilde{A}\|^2 + \lambda\operatorname{tr}[\operatorname{sym}\nabla\tilde{u}]^2 \\
&\qquad + \frac{(\gamma-\beta)}{2}\,\|\operatorname{curl}\operatorname{axl}\tilde{A}\|^2 + \alpha\operatorname{tr}\left[\nabla\operatorname{axl}\tilde{A}\right]^2\mathrm{dV} \\
&\geq (2\mu\,c_K - \mu_c\,|1-\frac{2}{\varepsilon}|)\,\|\tilde{u}\|^2_{H^{1,2}(\Omega,\mathbb{R}^3)} + c_{GR}\,\|\tilde{A}\|^2_{H^{1,2}(\Omega,\mathfrak{so}(3,\mathbb{R}))}\,, \qquad (3.17)
\end{aligned}
$$

where, with $\langle \operatorname{axl}\tilde{A}, \vec{n}\rangle_{|\partial\Omega} = 0$ and for some $0 < \varepsilon < 2$ we made use of the following inequality for $\operatorname{axl}\tilde{A} \in \mathbb{R}^3$

Theorem 3.1
Let $\Omega \subset \mathbb{R}^3$ be a bounded, open region with a $C^{1,1}$-boundary $\partial\Omega$. Then

$$\exists C^+ > 0 : \forall \phi \in C^\infty(\Omega, \mathbb{R}^3) :$$
$$\|\phi\|^2_{H^{1,2}(\Omega,\mathbb{R}^3)} \le C^+ \left(\|\phi\|^2_{L^2(\Omega)} + \|\operatorname{curl}\phi\|^2_{L^2(\Omega)} + \|\operatorname{Div}\phi\|^2_{L^2(\Omega)} + \|\langle\phi,\vec{n}\rangle\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \right) \quad (3.18)$$

Proof. See [28, p.56,Cor.3.7]. ∎

3.5 Continuous dependence for $\mu_c \ge 0$ and (3.7).

Apparently, there is no result in the literature which shows continuous dependence on the parameter μ_c in case of (3.7). This is, however, crucial in those cases where one wants to use the Cosserat model as a regularizing model for otherwise classical situations by letting $\mu_c \to 0$. The continuous dependence result for the case (3.4) i.e. $\mu_c \ge c^+ > 0$ is, again, well-established, even in the microstretch case [36].

In order to show continuous dependence of the solution as $\mu_c \to 0$ we consider two solutions, corresponding to $\mu_c^1 \ne \mu_c^2$ and to the same boundary data, different volume force and zero volume couples. For simplicity of exposition only we assume $\beta = \gamma > 0$ and $\lambda = \alpha = 0$. The equations satisfied, respectively, are then

$$\begin{aligned}
\operatorname{Div}\left[2\mu \operatorname{sym}\nabla u_1 + 2\mu_c^1 \operatorname{skew}(\nabla u_1 - \overline{A}_1)\right] &= f_1\,,\\
\operatorname{Div}\left[2\mu \operatorname{sym}\nabla u_2 + 2\mu_c^2 \operatorname{skew}(\nabla u_2 - \overline{A}_2)\right] &= f_2\,,\\
-\operatorname{Div}\left[2\beta \operatorname{sym}\nabla \operatorname{axl}\overline{A}_1\right] &= 2\mu_c^1 \operatorname{axl}\operatorname{skew}(\nabla u_1 - \overline{A}_1)\,,\\
-\operatorname{Div}\left[2\beta \operatorname{sym}\nabla \operatorname{axl}\overline{A}_2\right] &= 2\mu_c^2 \operatorname{axl}\operatorname{skew}(\nabla u_2 - \overline{A}_2)\,.
\end{aligned} \quad (3.19)$$

For the differences, $\hat{u} = u_1 - u_2$ and $\hat{A} = \overline{A}_1 - \overline{A}_2$ we obtain the two equations

$$\begin{aligned}
&\operatorname{Div}\Bigg[\underbrace{2\mu \operatorname{sym}\nabla\hat{u} + 2\mu_c^1 \operatorname{skew}(\nabla\hat{u} - \hat{A})}_{=:\hat{\sigma}}\Bigg] = \hat{f} + 2\,(\mu_c^2 - \mu_c^1)\operatorname{Div}\left[\operatorname{skew}(\nabla u_2 - \overline{A}_2)\right]\,,\\
&\operatorname{Div}\left[2\beta \operatorname{sym}\nabla\operatorname{axl}\hat{A}\right] = -2\mu_c^1 \operatorname{axl}\operatorname{skew}(\nabla\hat{u} - \hat{A}) + 2\,(\mu_c^2 - \mu_c^1)\operatorname{axl}\operatorname{skew}(\nabla u_2 - \overline{A}_2)\,,
\end{aligned} \quad (3.20)$$

with $\hat{f} = f_1 - f_2$. Multiplying the first equation with the difference $\hat{u}$ and using the product rule shows for $\hat{\sigma} := 2\mu \operatorname{sym}\nabla\hat{u} + 2\mu_c^1 \operatorname{skew}(\nabla\hat{u} - \hat{A})$

$$\begin{aligned}
-\langle\hat{\sigma},\nabla\hat{u}\rangle + \operatorname{Div}\left[\sigma^T.\hat{u}\right] &= \langle\hat{f},\hat{u}\rangle - 2(\mu_c^2 - \mu_c^1)\langle\operatorname{skew}(\nabla u_2 - \overline{A}_2),\nabla\hat{u}\rangle\\
&\quad + 2(\mu_c^2 - \mu_c^1)\operatorname{Div}\left(\left[\operatorname{skew}(\nabla u_2 - \overline{A}_2)\right]^T.\hat{u}\right)\,.
\end{aligned} \quad (3.21)$$

After integration, using the divergence-theorem and applying the boundary conditions (natural boundary conditions for σ on the free boundary Γ_N) one is left with

$$\begin{aligned}
-\int_\Omega \langle\hat{\sigma},\nabla\hat{u}\rangle\,\mathrm{dV} &= \int_\Omega \langle\hat{f},\hat{u}\rangle - 2(\mu_c^2 - \mu_c^1)\langle\operatorname{skew}(\nabla u_2 - \overline{A}_2),\nabla\hat{u}\rangle\,\mathrm{dV}\\
&\quad + 2(\mu_c^2 - \mu_c^1)\int_{\Gamma_N} \langle\operatorname{skew}(\nabla u_2 - \overline{A}_2).\vec{n},\hat{u}\rangle\,\mathrm{dS}\,,
\end{aligned} \quad (3.22)$$

where $\Gamma_N = \partial\Omega \setminus \Gamma$. Further on we assume that $\Gamma_N = \emptyset$. Then we have

$$\int_\Omega \langle\hat{\sigma},\nabla\hat{u}\rangle\,\mathrm{dV} = \int_\Omega -\langle\hat{f},\hat{u}\rangle + 2(\mu_c^2 - \mu_c^1)\langle\operatorname{skew}(\nabla u_2 - \overline{A}_2),\nabla\hat{u}\rangle\,\mathrm{dV}\,. \quad (3.23)$$

Because

$$\begin{aligned}
\langle\hat{\sigma},\nabla\hat{u}\rangle &= \langle 2\mu \operatorname{sym}\nabla\hat{u} + 2\mu_c^1 \operatorname{skew}(\nabla\hat{u} - \hat{A}),\nabla\hat{u}\rangle\\
&= \langle 2\mu \operatorname{sym}\nabla\hat{u} + 2\mu_c^1 \operatorname{skew}(\nabla\hat{u} - \hat{A}),\nabla\hat{u} - \hat{A}\rangle + \langle 2\mu \operatorname{sym}\nabla\hat{u} + 2\mu_c^1 \operatorname{skew}(\nabla\hat{u} - \hat{A}),\hat{A}\rangle\\
&= 2\mu\,\|\operatorname{sym}\nabla\hat{u}\|^2 + 2\mu_c^1\|\operatorname{skew}(\nabla\hat{u} - \hat{A})\|^2 + 2\mu_c^1\,\langle\operatorname{skew}(\nabla\hat{u} - \hat{A}),\hat{A}\rangle\,,
\end{aligned} \quad (3.24)$$

multiplication of $(3.20)_2$ with $\operatorname{axl}\hat{A} \in \mathbb{R}^3$ and taking into account that $\langle X, Y\rangle_{\mathbb{M}^{3\times 3}} = 2\langle \operatorname{axl} X, \operatorname{axl} Y\rangle_{\mathbb{R}^3}$ for $X, Y \in \mathfrak{so}(3,\mathbb{R})$ shows that

$$
\begin{aligned}
2\mu_c^1 \langle \operatorname{skew}(\nabla\hat{u} - \hat{A}), \hat{A}\rangle = -2\langle \operatorname{Div}\left[2\beta\ \operatorname{sym}\nabla \operatorname{axl}\hat{A}\right], \operatorname{axl}\hat{A}\rangle \\
+ 2\,(\mu_c^2 - \mu_c^1)\langle \operatorname{skew}(\nabla u_2 - \overline{A}_2), \hat{A}\rangle\,. \qquad (3.25)
\end{aligned}
$$

Inserting this relation into the former yields

$$
\begin{aligned}
\langle \hat{\sigma}, \nabla\hat{u}\rangle = 2\mu\,\|\operatorname{sym}\nabla\hat{u}\|^2 + 2\mu_c^1\|\operatorname{skew}(\nabla\hat{u} - \hat{A})\|^2 - 2\langle \operatorname{Div}\left[2\beta\ \operatorname{sym}\nabla\operatorname{axl}\hat{A}\right], \operatorname{axl}\hat{A}\rangle \\
+ 2\,(\mu_c^2 - \mu_c^1)\langle \operatorname{skew}(\nabla u_2 - \overline{A}_2), \hat{A}\rangle\,. \qquad (3.26)
\end{aligned}
$$

Hence with the boundary conditions on $\hat{A}$ we obtain from the divergence theorem that

$$
\begin{aligned}
\int_\Omega \langle \hat{\sigma}, \nabla\hat{u}\rangle = \int_\Omega 2\mu\,\|\operatorname{sym}\nabla\hat{u}\|^2 + 2\mu_c^1\|\operatorname{skew}(\nabla\hat{u} - \hat{A})\|^2 + 4\beta\,\|\operatorname{sym}\nabla\operatorname{axl}\hat{A}\|^2 \\
+ 2\,(\mu_c^2 - \mu_c^1)\langle \operatorname{skew}(\nabla u_2 - \overline{A}_2), \hat{A}\rangle\,\mathrm{dV}\,. \qquad (3.27)
\end{aligned}
$$

Combining (3.23) with (3.27) and using Hölders-inequality shows the estimate

$$
\begin{aligned}
2\mu\,\|\operatorname{sym}\nabla\hat{u}\|^2_{L^2(\Omega)} + 2\mu_c^1\|\operatorname{skew}(\nabla\hat{u} - \hat{A})\|^2_{L^2(\Omega)} + 4\beta\|\operatorname{sym}\nabla\operatorname{axl}\hat{A}\|^2_{L^2(\Omega)} \\
\le \|\hat{f}\|_{L^2(\Omega)}\,\|\hat{u}\|_{L^2(\Omega)} + 2(\mu_c^2 - \mu_c^1)\|\operatorname{skew}(\nabla u_2 - \overline{A}_2)\|_{L^2(\Omega)}\,\|\nabla\hat{u} - \hat{A}\|_{L^2(\Omega)}\,. \qquad (3.28)
\end{aligned}
$$

Korn's first inequality shows that there exists a constant $c_K = c_K(\Omega, \Gamma) > 0$ such that

$$
\begin{aligned}
2\mu\, c_K\|\hat{u}\|^2_{H^{1,2}(\Omega)} + 2\mu_c^1\|\operatorname{skew}(\nabla\hat{u} - \hat{A})\|^2_{L^2(\Omega)} + 4\beta\, c_K\|\operatorname{axl}\hat{A}\|^2_{H^{1,2}(\Omega)} \\
\le \|\hat{f}\|_{L^2(\Omega)}\,\|\hat{u}\|_{L^2(\Omega)} + 2(\mu_c^2 - \mu_c^1)\|\operatorname{skew}(\nabla u_2 - \overline{A}_2)\|_{L^2(\Omega)}\,\|\nabla\hat{u} - \hat{A}\|_{L^2(\Omega)} \qquad (3.29)
\end{aligned}
$$

Now we use Young's inequality on the right hand side to obtain for some $\varepsilon > 0$

$$
\begin{aligned}
&2\mu\, c_K\|\hat{u}\|^2_{H^{1,2}(\Omega)} + 2\mu_c^1\|\operatorname{skew}(\nabla\hat{u} - \hat{A})\|^2_{L^2(\Omega)} + 4\beta\, c_K\|\operatorname{axl}\hat{A}\|^2_{H^{1,2}(\Omega)} \\
&\quad\le \|\hat{f}\|_{L^2(\Omega)}\,\|\hat{u}\|_{L^2(\Omega)} + \frac{(\mu_c^2 - \mu_c^1)}{\varepsilon}\|\operatorname{skew}(\nabla u_2 - \overline{A}_2)\|^2_{L^2(\Omega)} + \varepsilon\|\nabla\hat{u} - \hat{A}\|^2_{L^2(\Omega)} \qquad (3.30) \\
&\quad\le \|\hat{f}\|_{L^2(\Omega)}\,\|\hat{u}\|_{L^2(\Omega)} + \frac{(\mu_c^2 - \mu_c^1)}{\varepsilon}\|\operatorname{skew}(\nabla u_2 - \overline{A}_2)\|^2_{L^2(\Omega)} + 2\varepsilon\left(\|\nabla\hat{u}\|^2_{L^2(\Omega)} + \|\hat{A}\|^2_{L^2(\Omega)}\right) \\
&\quad\le \|\hat{f}\|_{L^2(\Omega)}\,\|\hat{u}\|_{H^{1,2}(\Omega)} + \frac{(\mu_c^2 - \mu_c^1)}{\varepsilon}\|\operatorname{skew}(\nabla u_2 - \overline{A}_2)\|^2_{L^2(\Omega)} + 2\varepsilon\,\|\hat{u}\|^2_{H^{1,2}(\Omega)} + 2\varepsilon\,\|\hat{A}\|^2_{H^{1,2}(\Omega)}\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(2\mu\, c_K - \varepsilon)\|\hat{u}\|^2_{H^{1,2}(\Omega)} + 2\mu_c^1\|\operatorname{skew}(\nabla\hat{u} - \hat{A})\|^2_{L^2(\Omega)} + 2(2\beta\, c_K - \varepsilon)\,\|\operatorname{axl}\hat{A}\|^2_{H^{1,2}(\Omega)} \\
\le \|\hat{f}\|_{L^2(\Omega)}\,\|\hat{u}\|_{H^{1,2}(\Omega)} + \frac{(\mu_c^2 - \mu_c^1)}{\varepsilon}\|\operatorname{skew}(\nabla u_2 - \overline{A}_2)\|^2_{L^2(\Omega)} \\
\le \|\hat{f}\|_{L^2(\Omega)}\,\|\hat{u}\|_{H^{1,2}(\Omega)} + \frac{2(\mu_c^2 - \mu_c^1)}{\varepsilon}\|\nabla u_2\|^2_{L^2(\Omega)} + \frac{2(\mu_c^2 - \mu_c^1)}{\varepsilon}\|\overline{A}_2\|^2_{L^2(\Omega)}\,. \qquad (3.31)
\end{aligned}
$$

Now use again Korn's inequality for both terms $u_2, \overline{A}_2$ to obtain for positive K_1, K_2 the estimate (independent of $\mu_c^2 \ge 0$)

$$
\|\nabla u_2\|^2_{L^2(\Omega)} \le K_1\, I(u_2, \overline{A}_2) + K_2\,, \qquad \|\overline{A}_2\|^2_{L^2(\Omega)} \le K_1\, I(u_2, \overline{A}_2) + K_2\,. \qquad (3.32)
$$

Combining (3.31) with (3.32) we get

$$
\begin{aligned}
(2\mu\, c_K - \varepsilon)\|\hat{u}\|^2_{H^{1,2}(\Omega)} + 2\mu_c^1\|\operatorname{skew}(\nabla\hat{u} - \hat{A})\|^2_{L^2(\Omega)} + 2(2\beta\, c_K - \varepsilon)\,\|\operatorname{axl}\hat{A}\|^2_{H^{1,2}(\Omega)} \\
\le \|\hat{f}\|_{L^2(\Omega)}\,\|\hat{u}\|_{H^{1,2}(\Omega)} + \frac{4(\mu_c^2 - \mu_c^1)}{\varepsilon}\left(K_1\, I(u_2, \overline{A}_2) + K_2\right)\,. \qquad (3.33)
\end{aligned}
$$

In fact, $I(u_2, \overline{A}_2) \le K(\Omega) < \infty$, independent of $\mu_c^2 \ge 0$, therefore, we may write

$$
\begin{aligned}
(2\mu\, c_K - \varepsilon)\|\hat{u}\|^2_{H^{1,2}(\Omega)} + 2\mu_c^1\|\operatorname{skew}(\nabla\hat{u} - \hat{A})\|^2_{L^2(\Omega)} + 2(2\beta\, c_K - \varepsilon)\,\|\operatorname{axl}\hat{A}\|^2_{H^{1,2}(\Omega)} \\
\le \|\hat{f}\|_{L^2(\Omega)}\,\|\hat{u}\|_{H^{1,2}(\Omega)} + \frac{(\mu_c^2 - \mu_c^1)}{\varepsilon}K(\Omega)\,. \qquad (3.34)
\end{aligned}
$$

Now it is possible to specify $\varepsilon > 0$ such that simultaneously $2\mu\, c_k - \varepsilon > 0\,, 2\beta\, c_K - \varepsilon > 0$. The previous quadratic inequality implies then that

$$\|\hat{u}\|_{H^{1,2}(\Omega)} \leq \frac{\|\hat{f}\|_{L^2(\Omega)}}{(2\mu\, c_K - \varepsilon)} + \sqrt{\frac{\|\hat{f}\|^2_{L^2(\Omega)}}{4(2\mu\, c_K - \varepsilon)^2} + \frac{(\mu_c^2 - \mu_c^1)}{\varepsilon(2\mu\, c_K - \varepsilon)} K(\Omega)}\,. \tag{3.35}$$

Reinserting this estimate into (3.34) allows us to conclude a similar estimate for the difference $\hat{A}$. The estimate is uniform with respect to non-negative μ_c^1, μ_c^2 and can therefore be extended to include $\mu_c = 0$. Note that if $\mu_c \to 0$ this does not imply that $\overline{A}_{\mu_c} \to \operatorname{skew} \nabla u$; rather, if the boundary condition $\overline{A}_{\mathrm{d}}$ is constant, then $\overline{A}_{\mu_c} \to \overline{A}_{\mathrm{d}}$.

Altogether this shows that the linear elastic Cosserat problem is a well-posed system and provided pure Dirichlet-boundary conditions are specified, the limit $\mu_c \to 0$ exists and coincides with classical linear elasticity as far as displacements are concerned. In this sense, the Cosserat model with $\mu_c > 0$ can be viewed as an approximation to classical linear elasticity and the **Cosserat model is in itself mathematically sound, also under the weaker conditions** (3.7) **and** (3.9)**.**

4 Physical restrictions imposed by bounded stiffness

Now we turn our attention to the physical aspects of the problem of determining material parameters. We investigate the question whether the linear elastic Cosserat model can be considered to be a physically consistent description for a continuous solid showing size-effects. We assume the continuous solid to be available in any small size we can think of (this possibility is certainly included in the very definition of a continuous solid). This assumption **excludes** e.g. man made grid-structures, foams and bones but includes e.g. polycrystalline material. For the investigation we study simple boundary value problems for which analytical solutions are available. The chosen boundary conditions for the microrotations are of stress type such that stiffening behaviour due only to boundary layer phenomena can be excluded [14].

4.1 The torsion problem

In a thought experiment we subject the hypothetical continuous solid first to torsion for every slenderness we choose. Similar real experiments with metal wires of diameters in the nano-range have been performed and analyzed in [22], however, within the elasto-plastic setting. In [30] these torsional experiments have been studied numerically again in a geometrically exact context, based on the elastic moduli $\mu = 46.000\,\mathrm{MPa}$, $\mu_c = \frac{3}{46}\,\mu\,, \lambda = 69.000\,\mathrm{MPa}\,, \beta = \gamma = 0.01\,\mathrm{N}\,, \alpha = 0$.

4.1.1 An aspect of the solution for the pure torsion problem

In [24, 23] the analytical solution for pure torsion of a circular cylinder with radius $a > 0$ and length $L > 0$ is developed under the assumption of translational symmetry in axial direction (the classical solution is equally axisymmetric). For our purpose it is sufficient to look at the non-dimensional quantity $\mathbf{\Omega}_{\mathrm{t}}$, which compares the classical response with the corresponding micropolar result.

The classical relation between **torque** $Q\,[\mathrm{N}\cdot\mathrm{m}]$ and **twist per unit length** $\frac{\theta}{L}\,[1/\mathrm{m}]$ is given by

$$Q = \mu\, J\, \mathbf{\Omega}_{\mathrm{t}} \cdot \frac{\theta}{L}\,, \quad \mathbf{\Omega}_{\mathrm{t}} \equiv 1\,, \tag{4.1}$$

where $\mu > 0$ is the **classical shear modulus** coinciding with the corresponding Lamé constant while $J = \frac{\pi\, a^4}{2}$ is the polar moment of inertia of the circular cross section.

Performing the appropriate non-dimensionalization, it can be seen that in any theory without size-effects one has [22]

$$\frac{Q}{a^3}\,[\mathrm{MPa}] = h\left(\frac{\theta}{L}\, a\right)\,, \tag{4.2}$$

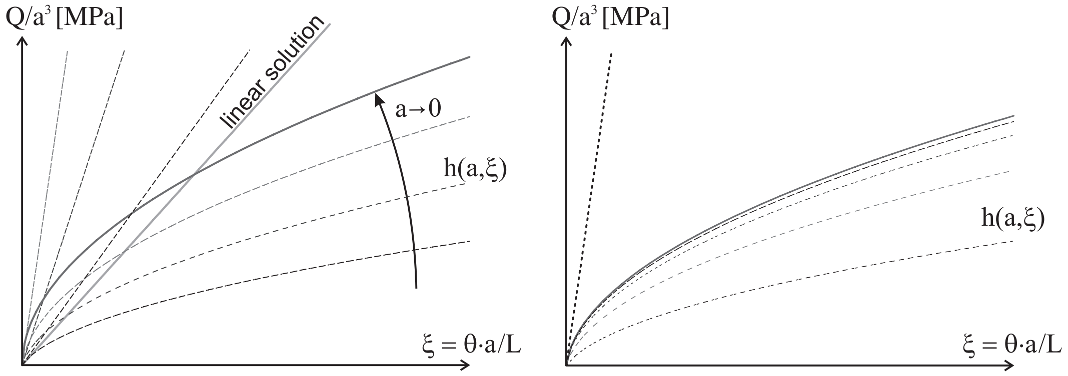

Figure 1: Idealized response curves with size-effects in torsion. Normalized torque $\frac{Q}{a^3}$ versus non-dimensional shear $\xi = \frac{\theta}{L}\,a$ at the outer radius. Left: response curves with unbounded stiffness as $a \to 0$. Bounded stiffness implies that the left picture is unphysical. Right: stiffness increase with bounded stiffness as $a \to 0$. For linear models, only tangents in 0 are relevant.

where $h : \mathbb{R} \to \mathbb{R}$ has no explicit dependence on the radius $a > 0$. $\frac{Q}{a^3}$ is a stress-like normalized torque and $\frac{\theta}{L}\,a$ is the non-dimensional shear at the outer radius. In the linear case it holds that $h(\xi) = \mu\,\frac{\pi}{2}\,\xi$.

In any experiment with size-effects, the function h will display this size effect by explicitly depending also on the radius $a > 0$ and we expect that for smaller radius a the larger $h(a, \xi)$ as a function of ξ with $h(a, 0) = 0$, $a \geq 0$. This increase of the response function is a commonplace observation for many materials. The stiffness of the material is defined as the slope of h at given $a \geq 0$ for $\xi = 0$ i.e.

$$\text{stiffness} = [\partial_\xi h(a, \xi)]_{|_{\xi=0}} \,. \tag{4.3}$$

Hence, in general, the stiffness is also a function of the radius a. In the classical linear elastic case $\partial_\xi h(a, \xi)_{|_{\xi=0}} = \mu\,\frac{\pi}{2}$ is independent of a. We expect also that stiffness increases for smaller $a > 0$, i.e. $[\partial_\xi h(a_2, \xi)]_{|_{\xi=0}} \geq [\partial_\xi h(a_1, \xi)]_{|_{\xi=0}}$ for $a_2 \leq a_1$. However, for any small dimensions we investigate, we expect **bounded stiffness** since the constitutive substructure is never rigid. This means

$$\exists K > 0 : \quad \sup_{a \geq 0}\, [\partial_\xi h(a, \xi)]_{|_{\xi=0}} \leq K \,. \tag{4.4}$$

Now we turn to the linear micropolar model with size-effects and consider the generated stiffness depending on the radius a. Since the model is linear, we need only to look at the corresponding factor Ω_{t} in (4.1).

In the micropolar case one must note that the macroscopic resultant net torque is the sum of the torque due to classical torques (the classical part) Q_{class} and the contribution of the micropolar couples Q_{cp}. According to [24, 23] it holds in the linear micropolar case

$$Q_{\mathrm{class}} + Q_{\mathrm{cp}} = Q = \mu\, J\, \boldsymbol{\Omega}_{\mathrm{t}} \cdot \frac{\theta}{L}\,, \quad \boldsymbol{\Omega}_{\mathrm{t}} = 1 + 6\left(\frac{\ell_t}{a}\right)^2 \cdot \left(\frac{1 - \frac{4}{3}\Psi \cdot \mathcal{X}(p\,a)}{1 - \Psi \cdot \mathcal{X}(p\,a)}\right)\,, \tag{4.5}$$

where

$$\begin{aligned}
\Psi &:= \frac{\beta + \gamma}{\alpha + \beta + \gamma}\,, \qquad \text{non-dimensional polar ratio}\,,\\
\ell_t^2 &:= \left(\frac{\beta + \gamma}{2\mu^* + \kappa}\right) = \frac{\beta + \gamma}{2}\,\frac{1}{\mu}\,, \qquad \text{"characteristic length for torsion"}\,, \\
\mathcal{X}(\xi) &:= \frac{I_1(\xi)}{\xi\; I_0(\xi)}\,, \qquad p^2 := \frac{2\,\kappa}{\alpha + \beta + \gamma}\,, \qquad \kappa := 2\,\mu_c\,,\\
I_1(\xi),\, I_0(\xi) & \qquad \text{modified } \textbf{Bessel functions} \text{ of the first kind}\,,\\
N^2 &:= \frac{\mu_c}{\mu + \mu_c} = \frac{\kappa}{2\,(\mu^* + \kappa)}\,, \quad \text{Cosserat coupling number}, \ 0 \leq N \leq 1\,.
\end{aligned} \tag{4.6}$$

Under condition (3.7) or (3.9) the definitions in the solution formula make sense, i.e. $\ell_t^2 \geq 0$ is ensured and since $3\alpha+\beta+\gamma \geq 0$ it follows also that $\alpha+\beta+\gamma > 0$.[9] Whether or not the model shows bounded stiffness depends solely on the factor Ω_t.

If we consider uniformly scaled specimens of the same material, i.e. all dimensions are reduced by the same factor $0 < r \leq 1$, then this induces the transformations $\ell_t \to r\,\ell_t$, $a \to r\,a$, $\Psi = const.$, $p\,a = const.$ and we see that the stiffness $\mu\,\mathbf{\Omega}_{\mathrm{t}}$ is here invariant in uniform scalings. Therefore, in the linear theory, "smaller" is not necessarily "stiffer" if boundary layer effects are not considered.

4.1.2 The response under pure torsion for small radius $a \to 0$

We are interested in what happens to $\mathbf{\Omega}_{\mathrm{t}}$ in (4.5), if we let $a \to 0$. From a physical point of view the stiffness for smaller radius should be larger than the classical one but the stiffness should certainly remain bounded since the heterogeneity of the microstructure of the physical body can never be perfectly rigid.

In order to investigate this question, we need first to examine the behaviour of the function $\mathcal{X}$ in (4.6). Let us recall a property of the modified Bessel function of the first kind. The series representation of the modified Bessel function of the first kind is given by

$$I_n(\xi) = \left(\frac{\xi}{2}\right)^n \sum_{k=0}^{\infty} \frac{\left(\frac{\xi}{2}\right)^{2k}}{k!\;\Gamma(n+k+1)}, \quad n \in \mathbb{N}_0 . \tag{4.7}$$

It holds therefore that for small $\xi > 0$

$$I_n(\xi) \sim \frac{1}{\Gamma(n+1)}\left(\frac{\xi}{2}\right)^n \quad \Rightarrow \quad I_1(\xi) \sim \frac{1}{\Gamma(2)}\frac{\xi}{2}, \qquad I_0(\xi) \sim 1, \tag{4.8}$$

where $\Gamma(x) := (x-1)!$, $x \in \mathbb{N}$ is the Gamma-function. This implies

$$\lim_{\xi\to 0} \mathcal{X}(\xi) = \lim_{\xi\to 0} \frac{I_1(\xi)}{\xi\, I_0(\xi)} \sim \frac{\frac{\xi}{2}}{\xi\cdot 1} = \frac{1}{2}. \tag{4.9}$$

More precisely, using the series representation, one obtains that for small ξ

$$\mathcal{X}(\xi) = \frac{I_1(\xi)}{\xi\, I_0(\xi)} \sim \frac{1}{2}\left(\frac{1+\left(\frac{\xi}{2}\right)^2\frac{1}{2}+\dots}{1+\left(\frac{\xi}{2}\right)^2+\dots}\right) \sim \frac{1}{2}\left(1-\frac{\xi^2}{8}\right). \tag{4.10}$$

Hence, for $a \to 0$ at fixed $p > 0$ we obtain

$$\lim_{a\to 0}\left(\frac{1-\frac{4}{3}\Psi\,\mathcal{X}(p\,a)}{1-\Psi\,\mathcal{X}(p\,a)}\right) = \frac{1-\frac{4}{3}\Psi\frac{1}{2}}{1-\Psi\,\frac{1}{2}}. \tag{4.11}$$

For $\mathbf{\Omega}_{\mathrm{t}}$ to remain bounded as $a \to 0$ under condition (3.7) (recall that then $\beta+\gamma > 0 \Rightarrow \ell_t > 0$), it is therefore **necessary and sufficient** that

$$\lim_{a\to 0}\left(\frac{1-\frac{4}{3}\Psi\mathcal{X}(p\,a)}{1-\Psi\,\mathcal{X}(p\,a)}\right) = \frac{1-\frac{4}{3}\Psi\frac{1}{2}}{1-\Psi\,\frac{1}{2}} = 0 \Leftrightarrow \Psi = \frac{3}{2}. \tag{4.12}$$

Regarding the result $\Psi = \frac{3}{2}$ we note that this value does not belong to the parameter-range permitted in (3.7) (the coercivity condition implies $\Psi < \frac{3}{2}$) leading to a well-posed boundary value problem.[10] In this sense, experimental findings regarding thinner and thinner samples in

[9] In size-experiments this solution formula is usually used to determine α **and** $\beta+\gamma$. However, information on α is only obtained if one assumes $\beta+\gamma > 0$. The author is not aware of an analytical solution to a simple boundary value problem which allows to determine α directly.

[10] Also foams and bones are not a continuous solid and the argument regarding thinner and thinner samples does therefore not strictly apply, in [41, 44] the value $\Psi = \frac{3}{2}$ has been chosen in order to accommodate bounded stiffness with experimental findings. For a syntactic foam [41] $\beta = \gamma$ has been taken for a best fit. In this case, the curvature energy looks like $W_{\mathrm{curv}}(\nabla\phi) = \gamma\,\|\operatorname{dev}\operatorname{sym}\nabla\phi\|^2$ with $\gamma > 0$. It is clear that this does not provide coercivity in $H^{1,2}(\Omega)$ for the microrotations but it would still be possible to define a Hilbert-space $H(\operatorname{dev}) \subset L^2(\Omega)$ with norm $\|\phi\|^2 + \|\operatorname{dev}\operatorname{sym}\nabla\phi\|^2$ and carry out the analysis in this space. The question of possible prescription of boundary values for ϕ is not immediately clear. For a polyurethane foam [41] $\beta \neq \gamma$ and the curvature energy looks like $W_{\mathrm{curv}}(\nabla\phi) = \frac{\beta+\gamma}{2}\,\|\operatorname{dev}\operatorname{sym}\nabla\phi\|^2 + \frac{\gamma-\beta}{4}\|\operatorname{curl}\phi\|^2$. This last curvature energy still allows to describe tangential values of microrotations, since the curl part is present [28, p.34] but H^1-coercivity is again lost.

torsion (showing bounded stiffness) cannot be consistently described as a linear elastic Cosserat solid within condition (3.7).

If, however, condition (3.9) is adopted, then $\beta+\gamma=0$ may be chosen, implying $\ell_t=0$ and $\boldsymbol{\Omega}_{\rm t}$ remains not only bounded as $a\to 0$, but coincides identically with the classical result. Therefore, bounded stiffness in torsion and the possibility to describe size-effects within the linear Cosserat model is only possible by taking the problematic value $\Psi=\frac{3}{2}$.

4.1.3 The response under pure torsion for $\mu_c\to 0$

At given $\ell_t>0$ and radius $a>0$ we now investigate the limit behaviour for Cosserat couple modulus $\mu_c\to 0$. If $\mu_c\to 0$ then also $p\to 0$ and vice-versa. Hence we need again the result for $\mathcal{X}(\xi)$ for small ξ. Based on the expansion (4.10) we obtain for small $p>0$

$$\boldsymbol{\Omega}_{\rm t}=1+6\left(\frac{\ell_t}{a}\right)^2\cdot\left(\frac{1-\frac{4}{3}\Psi\cdot\mathcal{X}(p\,a)}{1-\Psi\cdot\mathcal{X}(p\,a)}\right)\sim 1+6\left(\frac{\ell_t}{a}\right)^2\cdot\left(\frac{1-\frac{4}{3}\Psi\cdot\frac{1}{2}(1-\frac{p^2\,a^2}{8})}{1-\Psi\cdot\frac{1}{2}(1-\frac{p^2\,a^2}{8})}\right)\,,\tag{4.13}$$

which, for $\Psi=\frac{3}{2}$ shows that

$$\boldsymbol{\Omega}_{\rm t}\sim 1+3\,\ell_t^2\,p^2=1+3\,\frac{\beta+\gamma}{2\mu}\,\frac{4\mu_c}{\alpha+\beta+\gamma}=1+3\Psi\,\frac{2\mu_c}{\mu}=1+9\frac{\mu_c}{\mu}\,.\tag{4.14}$$

Hence, $\mu_c\to 0$ and $\Psi=\frac{3}{2}$ lead to $\boldsymbol{\Omega}_{\rm t}\equiv 1$ in the limit, as in classical linear elasticity.

If, however, $\Psi<\frac{3}{2}$ then we consider $\Psi=\frac{3}{2}-\theta$. In terms of the difference $\theta>0$ we obtain for small p to leading order

$$\boldsymbol{\Omega}_{\rm t}\sim 1+\left(\frac{\ell_t}{a}\right)^2\cdot\frac{16\,\theta}{1+2\,\theta}\,.\tag{4.15}$$

Hence, as $\mu_c\to 0$ but $\ell_t>0$, $a>0$ we observe that $\boldsymbol{\Omega}_{\rm t}\to\boldsymbol{\Omega}_{\rm t}^0>1$. This shows a departure from classical elasticity. The result is not in conflict with the development in section (3.5) for $\mu_c\to 0$ since there other boundary conditions have been investigated than those used in the derivation of the analytical solution.

4.1.4 The response under pure torsion for $\mu_c>0$, $a>0$ but $\alpha,\beta,\gamma\to 0$

Finally, for torsion we investigate the case of vanishing internal length, i.e. $\alpha,\beta,\gamma\to 0$ at the same rate. This implies that $\Psi=const.$ and $\ell_t\to 0$. Moreover, $p\to 0$. It is easy to see that $\mathcal{X}(\xi)\to 0$ as $\xi\to\infty$. Hence in this limit the leading order behaviour is given by

$$\boldsymbol{\Omega}_{\rm t}\sim 1+\left(\frac{\ell_t}{a}\right)^2\cdot 1\,,\tag{4.16}$$

and for $\ell_t\to 0$ we recover classical linear elasticity without further restrictions.

4.2 The pure bending problem of a cylinder with circular cross-section

4.2.1 The analytical solution

An analytical solution formula for the bending of a micropolar circular cylinder under opposite compressive axial loads with radius $a>0$ and length $L>0$ has been obtained in [56]. Similarly as in the torsion case, we focus on the relative stiffness factor compared to classical elasticity.

According to [56] it holds that

$$
\begin{aligned}
\mathbf{\Omega}_{\mathrm{b}} &= 1 + \frac{8\,N^2}{\nu+1}\left(\frac{1-\left(\frac{\beta}{\gamma}\right)^2}{(\delta\,a)^2} + \frac{\left(\left(\frac{\beta}{\gamma}\right)+\nu\right)^2}{\zeta(\delta\,a) + 8\,N^2\,(1-\nu)}\right)\,,\\
\zeta(\xi) &= \xi^2\,\frac{\xi\,I_0(\xi) - I_1(\xi)}{\xi\,I_0(\xi) - 2\,I_1(\xi)}\,,\\
\delta^2 &= \frac{\kappa\,(2\mu^*+\kappa)}{\gamma\,(\mu^*+\mu)} = \frac{4\,\mu_c\,\mu}{\gamma\,(\mu+\mu_c)} = \frac{4\,N^2\,\mu}{\gamma} = \frac{N^2}{\ell_b^2}\,,\\
\ell_b^2 &= \frac{\gamma}{2\,(2\mu^*+\kappa)} = \frac{\gamma}{4\mu}\,, \qquad \text{"characteristic length for bending"}\,,\\
N^2 &:= \frac{\mu_c}{\mu+\mu_c} = \frac{\kappa}{2\,(\mu^*+\kappa)}\,, \qquad \text{Cosserat coupling number, } 0\le N\le 1\,,\\
&I_1(\xi),\, I_0(\xi) \qquad \text{modified } \textbf{Bessel functions} \text{ of the first kind}\,,\\
\nu &= \frac{\lambda}{2\mu^*+2\lambda+\kappa} = \frac{\lambda}{2\,(\mu+\lambda)}\,, \qquad \text{classical Poisson ratio.}
\end{aligned}
\tag{4.17}
$$

We note that under condition (3.7) and (3.9) the terms in the solution formula still make sense, as $\gamma > 0$.[11]

4.2.2 The response under pure bending for small radius $a \to 0$

For positive Cosserat couple modulus $\mu_c > 0$ and non-vanishing length scale $\ell_b > 0$ we are interested in the behaviour of $\mathbf{\Omega}_{\mathrm{b}}$ as $a \to 0$. Since $\delta = const.$ we consider $\xi := \delta\,a$ and clarify first the behaviour of $\zeta(\xi)$ as $\xi \to 0$. It holds that

$$
\begin{aligned}
\xi\,I_0(\xi) - I_1(\xi) &= \xi\left[1 + \left(\frac{\xi}{2}\right)^2 + \ldots\right] - \frac{\xi}{2}\left[1 + \left(\frac{\xi}{2}\right)^2\frac{1}{2} + \ldots\right]\\
&= \xi + \frac{\xi^3}{4} + \ldots - \frac{\xi}{2} - \frac{\xi}{4}\frac{\xi^2}{4} + \ldots \quad \sim \frac{\xi}{2}\,,\\
\xi\,I_0(\xi) - 2I_1(\xi) &= \xi\left[1 + \left(\frac{\xi}{2}\right)^2 + \ldots\right] - 2\frac{\xi}{2}\left[1 + \left(\frac{\xi}{2}\right)^2\frac{1}{2} + \ldots\right]\\
&= \xi + \frac{\xi^3}{4} + \ldots - \xi - \frac{\xi^3}{8} + \ldots \quad \sim \frac{\xi^3}{8}\,,\\
\Rightarrow \quad \zeta(\xi) &\sim \xi^2\,\frac{\frac{\xi}{2}}{\frac{\xi^3}{8}} \to 4 \quad \text{as} \quad \xi \to 0\,.
\end{aligned}
\tag{4.18}
$$

This implies

$$
\begin{aligned}
\mathbf{\Omega}_{\mathrm{b}} &= 1 + \frac{8\,N^2}{\nu+1}\left(\frac{1-\left(\frac{\beta}{\gamma}\right)^2}{(\delta\,a)^2} + \frac{\left(\left(\frac{\beta}{\gamma}\right)+\nu\right)^2}{\zeta(\delta\,a) + 8\,N^2\,(1-\nu)}\right)\\
&= 1 + \frac{8\,N^2}{\nu+1}\left(\frac{1-\left(\frac{\beta}{\gamma}\right)^2}{N^2}\left(\frac{\ell_b}{a}\right)^2 + \frac{\left(\left(\frac{\beta}{\gamma}\right)+\nu\right)^2}{\zeta(\delta\,a) + 8\,N^2\,(1-\nu)}\right)\,, \text{ letting } \quad \delta\,a \to 0 \Rightarrow\\
&\sim 1 + \frac{8}{\nu+1}\left[1-\left(\frac{\beta}{\gamma}\right)^2\right]\left(\frac{\ell_b}{a}\right)^2 + \frac{8N^2}{(\nu+1)}\,\frac{\left(\left(\frac{\beta}{\gamma}\right)+\nu\right)^2}{4 + 8\,N^2\,(1-\nu)}\\
&= 1 + \frac{8}{(\nu+1)\,4\mu}\left[\frac{(\gamma-\beta)(\gamma+\beta)}{\gamma\,a^2}\right] + \frac{8N^2}{(\nu+1)}\,\frac{\left(\left(\frac{\beta}{\gamma}\right)+\nu\right)^2}{4 + 8\,N^2\,(1-\nu)}\,.
\end{aligned}
\tag{4.19}
$$

Hence, for $\mathbf{\Omega}_{\mathrm{b}}$ to remain bounded as $a \to 0$ (and $\gamma > 0$) one must have

$$
(\gamma+\beta)\,(\gamma-\beta) \le 0\,. \tag{4.20}
$$

[11] This experiment does not provide information on α and information on β is only obtained if $\gamma > 0$.

Since both factors must be positive anyway (3.5) it follows that either $(\gamma+\beta) = 0$ or $(\gamma-\beta) = 0$. In this case, the Cosserat model still shows size effects in bending (provided $\gamma > 0$) and bounded stiffness.

For (3.7) one has $(\gamma - \beta) = 0$. In this case then to leading order for $a \to 0$

$$\boldsymbol{\Omega}_{\mathrm{b}} = 1 + 8N^2 \frac{\nu + 1}{4 + 8\,N^2\,(1-\nu)} = 1 + \frac{2N^2}{1 + 2N^2\,(1-\nu)}\,(\nu+1)\,. \tag{4.21}$$

Adopting, however, (3.9) one has $(\gamma + \beta) = 0$, in which case

$$\boldsymbol{\Omega}_{\mathrm{b}} = 1 + \frac{8N^2}{\nu+1}\frac{(\nu-1)^2}{4 + 8\,N^2\,(1-\nu)} = 1 + \frac{2N^2}{1 + 2N^2\,(1-\nu)}\frac{(\nu-1)^2}{\nu+1}\,, \tag{4.22}$$

as leading order behaviour for $a \to 0$.

In size-experiments, assuming that $\mu_c > 0$ these last two formulas for $\boldsymbol{\Omega}_{\mathrm{b}}$ would allow us first to decide whether $\gamma + \beta$ or $\gamma - \beta$ are zero for $\nu \neq 0$ and second to determine μ_c viz N^2 independent of length scale parameters since ν and μ are already determined from classical tension experiments.

4.2.3 The response under pure bending for $\mu_c \to 0$

Similarly, one observes that for $\mu_c \to 0$ and $a > 0$ it holds that

$$\boldsymbol{\Omega}_{\mathrm{b}} = 1 + \frac{8}{\nu+1}\left[1 - \left(\frac{\beta}{\gamma}\right)^2\right]\left(\frac{\ell_b}{a}\right)^2\,, \tag{4.23}$$

and one recovers classical elasticity with $\boldsymbol{\Omega}_{\mathrm{b}} \equiv 1$ either by assuming $\beta^2 = \gamma^2$ i.e $(\gamma+\beta)\,(\gamma-\beta) = 0$ or letting simultaneously $\ell_b \to 0$.

4.2.4 The response under pure bending for $\mu_c > 0$, $a > 0$ but $\alpha, \beta, \gamma \to 0$

We assume again a fixed rate for $\alpha, \beta, \gamma \to 0$, hence $\frac{\beta}{\gamma}$ remains constant, while $\delta \to \infty$ and $\ell_b \to 0$. Analyzing (4.19) implies with $\zeta(\xi) \to \infty$ as $\xi \to \infty$ that to leading order

$$\boldsymbol{\Omega}_{\mathrm{b}} \sim 1 + \frac{8}{\nu+1}\left(1 - \left(\frac{\beta}{\gamma}\right)^2\right)\left(\frac{\ell_b}{a}\right)^2\,. \tag{4.24}$$

For $\ell_b \to 0$ we have in the limit $\boldsymbol{\Omega}_{\mathrm{b}} \equiv 1$ without any restrictions.

4.3 Bending of a semicircular ring

In [25] the problem of a semicircular ring with rectangular cross-section of edge length $h > 0$ bent by transverse radial shear resultants P has been treated and compared with the classical response. For the derivation of the three-dimensional solution in polar coordinates the plane-stress distribution is assumed which is consistent with the classical result. While analytical solution formulas are obtained, these are to long to be recorded here in their entity. In any case, they involve the modified Bessel functions of the first and second kind. The authors [25] define two characteristic length for this problem given as

$$\ell_1^2 = \frac{2\gamma}{E}\,, \quad \ell_2^2 = \frac{\gamma}{4\,N^2\,\mu} = \frac{\ell_b^2}{N^2}\,, \quad E > 0 \quad \text{classical Young's modulus}\,. \tag{4.25}$$

Let a be the inner ring diameter. The classical result for the radial displacement u_r at the edges is given by

$$u_r(\Theta, r)_{|\Theta=0,\pi} = \frac{\pi\,P}{E}\frac{1+\rho^2}{1-\rho^2+(1+\rho^2)\,\ln\rho}\,, \qquad \rho^2 = \left(1 + \frac{h}{a}\right)^2\,. \tag{4.26}$$

Considering $\frac{h}{a} \to 0$ and using the ln-expansion, the leading order classical result[12] is seen to be

$$u_r(\Theta, r)_{|\Theta=0,\pi} \sim \frac{3\,\pi\,P}{E}\frac{a^3}{h^3}\,. \tag{4.27}$$

[12]The smaller h or the larger P, the larger the radial displacement u_r.

The corresponding leading order micropolar result for $\frac{h}{a} \to 0$ reads, however,

$$u_r(\Theta, r)_{|\Theta=0,\pi} \sim \frac{3\,\pi\,P}{E}\,\frac{a^3}{h^3}\,\frac{1}{1+6\,\left(\frac{\ell_1}{h}\right)^2}\,. \tag{4.28}$$

If one assumes that $\gamma > 0$ ($\ell_1 > 0$) then the leading order micropolar behaviour will differ by several orders of magnitude as $h \to 0$ from the classical response. This is not acceptable and can only be avoided by taking $\gamma = 0$.

The authors of the analytical solution have been aware of this feature. In their discussion of the result they write [25, p.503]: "The thin ring displacement ... exhibits the same characteristic stiffening by micropolar effects as that found for other structural elements. Macroscopic homogeneity considerations require that $\frac{\ell_1}{h} \ll 1$ since any lengths associated with micropolar effects should be small compared to physical dimensions of the ring. Increases of a few percent in stiffness over the classical value are all that can be anticipated (to occur in experiments)." Bracket my addition.

4.4 Bending of a rectangular plate by lateral edge moments

An analytical solution formula for the cylindrical bending of a thin micropolar rectangular plate by lateral edge moments has been obtained in [24, 23]. The classical formula connecting the uniform lateral edge moments M_x and the corresponding infinitesimal curvature $\frac{\partial^2 u_3(x,y,z)}{\partial x^2}$ for small displacements u_3 in thickness direction reads

$$\begin{aligned}
M_x &= -D\,\boldsymbol{\Omega}_{\mathrm{b}}\,\frac{\partial^2 u_3(x,y,z)}{\partial x^2}\,, \quad \boldsymbol{\Omega}_{\mathrm{b}} \equiv 1\,,\\
D &= \frac{E}{(1-\nu^2)}\,\frac{h^3}{12}\,, \quad \text{classical bending stiffness}\,,\\
\nu &= \frac{\lambda}{2\mu^* + 2\lambda + \kappa} = \frac{\lambda}{2\,(\mu+\lambda)}\,, \quad \text{classical Poisson ratio,}\\
E &> 0\,, \quad \text{classical Young's modulus,} \quad h > 0\,, \quad \text{thickness of the plate}\,.
\end{aligned} \tag{4.29}$$

Similarly as before, we focus on the relative stiffness factor compared to classical elasticity. For the linear elastic Cosserat solid, formula (4.29) holds also but with $\boldsymbol{\Omega}_{\mathrm{b}} = 1$ replaced by

$$\boldsymbol{\Omega}_{\mathrm{b}} = 1 + \frac{12\,\gamma\,(1-\nu^2)}{E\,h^2}\,. \tag{4.30}$$

We note again that under condition (3.7) and (3.9) the terms in the solution formula make sense, as $\gamma > 0$. It should be observed that (4.30) is independent of the Cosserat couple modulus μ_c which can be understood by the underlying assumption of pure bending. For a continuous solid material, it is possible to consider the thickness h being ever smaller. If we want to ensure a bounded relative stiffness $\boldsymbol{\Omega}_{\mathrm{b}}$ as the thickness $h \to 0$, the only way to obtain this is by setting $\gamma = 0$.

4.5 Stress concentration along a cylindrical hole

In [18, p.222] or [17, p.238] the analytical solution for the stress distribution around a cylindrical hole with radius $r > 0$ of an infinite plate is recalled. The stress concentration factor K_t which classically is $K_t = 3$ turns for the linear Cosserat model into

$$\begin{aligned}
&K_t = \frac{3+F_1}{1+F_1} \le 3\,, \quad F_1 = 8\,(1-\nu)\,N^2\,\frac{1}{4+\frac{r^2}{c^2}+2\frac{r}{c}\frac{K_0(\frac{r}{c})}{K_1(\frac{r}{c})}}\,, \quad c^2 = \frac{\gamma\,(\mu^*+\kappa)}{\kappa\,(2\mu^*+\kappa)} = \frac{\ell_b^2}{N^2}\,,\\
&K_0(\xi),\ K_1(\xi) \quad \text{modified } \textbf{Bessel functions} \text{ of the second kind}\,.
\end{aligned} \tag{4.31}$$

In the genuine micropolar case, the stress intensities are somewhat weakened: the Cosserat solid has the ability to distribute the stresses more smoothly. We note that the stress intensity factor is independent of $\beta + \gamma$. For $\ell_b \to 0$ or $N^2 \to 0$ it is easy to see that $F_1 \to 0$ and we have the classical limit, as expected.

In order to investigate what happens for arbitrary small holes $r \to 0$, we observe that for small $\xi \to 0$ the leading order behaviour of the Bessel-functions is given as

$$K_0(\xi) \sim \ln \frac{2}{c_1^+ \, \xi}\,, \quad K_1(\xi) \sim \frac{1}{2}\Gamma(1)\frac{2}{\xi}\,, \qquad c_1^+ \,:\, \text{Euler's constant}\,. \tag{4.32}$$

Hence, $F_1 \to 2\,(1-\nu)\,N^2$ as $r \to 0$ while $\ell_b > 0$. Therefore, arbitrary small holes give rise to a non-classical stress intensity factor if $N^2 > 0$. However, this is not in principle in conflict with our physical understanding for a continuous solid since the absolute dimensions of the hole radius for an infinitely extended medium cannot have a specific influence.

5 The remaining linear elastic Cosserat solid

Gathering the results implied so far by the stipulation of bounded stiffness for $\mu_c > 0$ and arbitrary thin samples we have

1. torsion of a cylinder: $\beta + \gamma = 0$ or $\Psi = \frac{\beta+\gamma}{\alpha+\beta+\gamma} = \frac{3}{2}$.
2. bending of a cylinder: $(\beta+\gamma)\,(\gamma-\beta) = 0$.
3. bending of a curved bar and bending of a thin plate: $\gamma = 0$.

The only consistent choice with these three conditions is $\beta = \gamma = 0$. Since I have not found an analytical solution restricting the value for α we can still assume $\alpha > 0$. Altogether, we are left with the variational problem

$$\begin{aligned}
I(u,\overline{A}) &= \int_\Omega W(\nabla u,\overline{A}) - \langle f,u\rangle - \langle \overline{M},\overline{A}\rangle \,\mathrm{dV} \\
&\quad - \int_{\Gamma_S} \langle f_S,u\rangle - \langle \mathrm{axl}(\mathrm{skew}\,\overline{M}_S),\vec{n}\rangle \cdot \langle \mathrm{axl}\,\overline{A},\vec{n}\rangle \,\mathrm{dS} \mapsto \min.\ \text{w.r.t.}\ (u,\overline{A}), \qquad (5.1)\\
\overline{\varepsilon} &= \nabla u - \overline{A}, \quad u_{|\Gamma} = u_{\mathrm{d}}\,, \\
\forall\,\zeta \in & H^{\frac{1}{2}}(\Gamma): \int_\Gamma \zeta(x)\ \left[\langle \mathrm{axl}\,\overline{A},\vec{n}\rangle - \langle \mathrm{axl}\,\overline{A}_{\mathrm{d}},\vec{n}\rangle\right]\,\mathrm{dS} = 0\,, \\
W(\nabla u,\overline{A}) &= \mu\,\|\,\mathrm{sym}\,\nabla u\|^2 + \frac{\mu_c}{2}\,\|\,\mathrm{curl}\,u - 2\,\mathrm{axl}\,\overline{A}\|^2_{\mathbb{R}^3} + \frac{\lambda}{2}\,(\mathrm{Div}\,u)^2 + \frac{\alpha}{2}\left(\mathrm{Div}\,\mathrm{axl}\,\overline{A}\right)^2 . \\
\sigma.\vec{n}_{|\Gamma_S} &= f_S\,, \quad m.\vec{n}_{|\Gamma_S} = \alpha\,[\mathrm{Div}\,\mathrm{axl}\,\overline{A}]\cdot\vec{n} = \langle \mathrm{axl}(\mathrm{skew}(\overline{M}_S)),\vec{n}\rangle\,\vec{n}\,, \\
\sigma.\vec{n}_{|\partial\Omega\setminus(\Gamma_S\cup\Gamma)} &= 0\,, \quad m.\vec{n}_{|\partial\Omega\setminus(\Gamma_S\cup\Gamma)} = \alpha\,[\mathrm{Div}\,\mathrm{axl}\,\overline{A}]\cdot\vec{n} = 0\,.
\end{aligned}$$

Interestingly enough, we are still able to prove existence: the functional is quadratic and therefore convex, it is also coercitive in the space $H^{1,2}(\Omega,\mathbb{R}^3) \times H(\mathrm{Div},\Omega,\mathfrak{so}(3,\mathbb{R}))$. Here,

$$H(\mathrm{Div},\Omega,\mathfrak{so}(3,\mathbb{R})) := \{\phi \in L^2(\Omega,\mathbb{R}^3)\,|\,\mathrm{Div}\,\phi \in L^2(\Omega)\}\,, \tag{5.2}$$

is a Hilbert-space [28, p.27]. This is enough to show existence in this space by weak lower semicontinuity arguments. In the space $H(\mathrm{Div},\Omega,\mathfrak{so}(3,\mathbb{R}))$ it is (only) possible to prescribe weakly the normal components of $\mathrm{axl}\,\overline{A}$, i.e. $\langle \mathrm{axl}\,\overline{A},\vec{n}\rangle$ on Γ_N in the appropriate sense as duality pairing where $\vec{n}$ is the unit outer normal on Γ_N.

The crucial question we have to answer is whether for smooth functions $u,\phi \in C^\infty(\Omega,\mathbb{R}^3)$ with $u_{|\Gamma} = 0$ (no additional boundary condition on the axial vector ϕ) there exists a positive constant c_K such that

$$\begin{aligned}
&\int_\Omega \|\,\mathrm{sym}\,\nabla u(x)\|^2_{\mathbb{M}^{3\times3}} + \|\,\mathrm{curl}\,u(x) - \phi(x)\|^2_{\mathbb{R}^3} + \mathrm{tr}\,[\nabla u(x)]^2 + |\,\mathrm{Div}\,\phi(x)|^2\,\mathrm{dV} \\
&\quad \geq c_K\left(\|u\|^2_{H^{1,2}(\Omega,\mathbb{R}^3)} + \|\phi\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\,\mathrm{Div}\,\phi\|^2_{L^2(\Omega)}\right)\,. \qquad (5.3)
\end{aligned}$$

It is easy to see that if the left hand side is zero, then the right hand side is also zero by using Korn's first inequality on u. Then the usual contradiction argument together with weak lower semicontinuity (see e.g. [47, Th.4.10]) shows that the inequality must be true. This shows the uniform convexity of the problem in the space $H^{1,2}(\Omega,\mathbb{R}^3)\times H(\mathrm{Div},\Omega,\mathfrak{so}(3,\mathbb{R}))$. It implies that the minimizer exists and is unique and provides also for continuous dependence in this space.

It is also possible to consider a different boundary condition[13] for $\operatorname{axl}\overline{A}$ at Γ instead of the Dirichlet condition for the normal component. This condition introduces a coupling between the two otherwise independent fields $(u, \overline{A})$, which we call **ultra weak consistent coupling condition**, namely in the appropriate weak sense as duality pairing

$$\forall\, \zeta \in H^{\frac{1}{2}}(\Gamma): \int_\Gamma \zeta(x)\ \left[\langle 2\operatorname{axl}\overline{A}(x), \vec{n}\rangle - \langle \operatorname{curl} u(x), \vec{n}\rangle\right]\ \mathrm{dS} = 0\,. \tag{5.4}$$

We note that $\langle \operatorname{curl} u, \vec{n}\rangle$ has a meaning on the boundary Γ, since trivially $\operatorname{curl} u \in H(\mathrm{Div})$ provided that $\operatorname{curl} u \in L^2(\Omega)$, which is guaranteed for bounded energy I in (5.1) by Korn's first inequality. In smooth classical situations one is tempted to assume the identification of infinitesimal continuum rotation and microrotation

$$\forall\, x \in \overline{\Omega}: \quad \operatorname{skew} \nabla u(x) = \overline{A}(x) \quad \Leftrightarrow \quad \operatorname{curl} u(x) - 2\operatorname{axl}\overline{A}(x) = 0\,. \tag{5.5}$$

In this sense, the ultra weak consistent coupling condition (5.4) requires classical behaviour on the Dirichlet boundary Γ in a weakened sense and turned otherwise: it is consistent with classical response and does not introduce non-classical effects through the boundary conditions for the micrototations. It represents one condition for the three entries of $\overline{A}$.[14]

Now consider the minimization problem (5.1) without body couples and surface couples, i.e. $\overline{M}, \overline{M}_S = 0$ and let either $\overline{A}$ be free everywhere on the boundary $\partial\Omega$ or let $\overline{A}$ satisfy the ultra weak consistent coupling condition on the Dirichlet boundary Γ. The minimizer $(u, \overline{A}) \in H^{1,2}(\Omega, \mathbb{R}^3) \times H(\mathrm{Div}, \Omega, \mathfrak{so}(3, \mathbb{R}))$ to this problem exists but $\overline{A}$ must comply with $\operatorname{axl}\overline{A} = \frac{1}{2}\operatorname{curl} u$ in Ω, since this will yield zero curvature energy ($\mathrm{Div}\,\frac{1}{2}\operatorname{curl} u = 0$) and zero coupling energy $\|\operatorname{curl} u - 2\operatorname{axl}\overline{A}\|^2 = 0$. This is true for arbitrary inhomogeneous response of u, e.g. implied by considering inhomogeneous Dirichlet boundary data for u or classical inhomogeneous body forces. Hence, non-classical micropolar effects are in general not related to inhomogeneous response whatsoever in this model. In order to activate micropolar behaviour, we need other boundary conditions for the microrotation field $\overline{A}$! Then the curvature parameter α will come into play, but it will be linked with the prescribed boundary condition and the parameter $\mu_c > 0$ acts as a weighting factor for the influence of the non-consistent boundary condition on the final solution. **This suggests that α is not a material parameter since it will be difficult to determine this parameter independent of the boundary condition.** The only possible remaining linear Cosserat problem (5.1) ensuring bounded stiffness is not a viable possibility for the description of a continuous solid showing size effects.

6 Conclusion

That linear elastic Cosserat models may show singular stiffening behaviour has already been observed previously. In [43, p.17] we read "For some combinations of elastic constants, the apparent modulus tends to infinity as the bar or plate size goes to zero. Large stiffening effects might be seen in composite materials consisting of very stiff fibers or laminae in a compliant matrix. However, infinite stiffening effects are unphysical. For very slender specimens, it is likely that a continuum theory more general than Cosserat elasticity; or use of a discrete structural model, would be required to deal with the observed phenomena".

A possible explanation for the type of unphysical singular response in torsion and bending for slender specimens under condition (3.4) (notably $\mu_c > 0$) consists of the following interpretation: Cosserat elasticity assumes a perfectly rigid substructure. For ever smaller samples, the rigid

[13] This is not a consequence of the previously studied boundary conditions.

[14] An extension to the geometrically exact Cosserat model is straight forward. We require for the deformation $\varphi: \Omega \mapsto \mathbb{R}^3$ and the microrotation $\overline{R}: \Omega \mapsto \mathrm{SO}(3, \mathbb{R})$, $\overline{R} = \exp(\overline{A})$, $\overline{A} \in \mathfrak{so}(3, \mathbb{R})$, the **ultra weak consistent coupling condition**

$$\forall\, \zeta \in H^{\frac{1}{2}}(\Gamma): \int_\Gamma \zeta(x)\ \left[\langle 2\operatorname{axl}(\log \overline{R}(x)), \vec{n}\rangle - \langle \operatorname{curl} \varphi(x), \vec{n}\rangle\right]\ \mathrm{dS} = 0\,. \tag{5.6}$$

Whether this is "the" correct finite-strain extension is not yet clear. Note that in a purely planar setting $\langle \operatorname{curl} \varphi(x), \vec{n}\rangle = 0$, such that the axis of rotations $\overline{R}$ at the boundary must be perpendicular to the normal on the boundary. This is automatically satisfied by taking the axis of rotations parallel to e_3.

"core" of the material is mainly responsible for the macroscopic response. In torsion and bending, this "core" does not have the possibility to remain rigid, hence the unbounded stiffness.

All of the depicted problematic responses of a linear elastic Cosserat model arise in essence in those cases, where geometrical length scales of a specimen are taken to be smaller than the assumed Cosserat length scales. Indeed, in [25, p.503] it has been observed that " ...thickness should at least be an order of magnitude greater then the smaller of the two characteristic micropolar lengths to assure that macroscopic homogeneity, assumed in the development of the theory, is maintained."

The experimentally never observed singular stiffening effects can always be matched in the linear Cosserat model by chosing $\mu_c > 0$ small enough. In fitting of Cosserat parameters for continuous solids therefore the smallness of μ_c is linked to the smallest experimentally investigated specimen. Is it then surprising that the actually determined values for μ_c are orders of magnitudes smaller than the shear modulus μ?

However, the experimentalist Bell [4, p.161] notes "To be a material constant of a given solid, the numerical value, of course, must be independent of the size and shape of the specimen." This statement certainly applies to $\mu_c \geq 0$ in the Cosserat model. But from the foregoing arguments for the continuous solid the response would always depend on the investigated geometrical size of the specimen, contradicting the latter statement of Bell.

This problematic response of the linear Cosserat solid under the usually adopted constitutive condition (3.4) with $\mu_c > 0$ should not be construed, however, as being an inconsistency of the general, geometrically exact Cosserat model. Indeed, in [48] it has been shown that it is possible and physically meaningful to consider a geometrically exact Cosserat model, in which the Cosserat couple modulus μ_c is set to zero. Such a choice leads to a bounded stiffness in torsion (in terms of the above interpretation: the "core" can remain rigid in torsion and does not lead to unbounded stiffness since the coupling with the displacement is one order weaker), while smaller samples are still more rigid than larger samples. A linearization of such a model leads to classical linear elasticity.

Therefore, our way to resolve the apparent difficulty with stiffening behaviour is simple but unusual: **the Cosserat couple modulus $\mu_c \geq 0$ is not a material parameter, it can be set to zero; a linear elastic Cosserat model does not apply to any continuous solid materials with arbitrary microstructure.**

We repeat that this statement does not concern man-made structure-grid frameworks (or foams and bones) modelled as a genuine Cosserat continuum: here the spacing of the grid provides a natural lower bound for a length scale below which nothing exists and for which the investigation of ever smaller samples does not make sense. If, on the other side, the aim of the Cosserat model is to furnish a regularization scheme, then one may take any computationally convenient $\mu_c > 0$ as a **penalty parameter**.

Otherwise, it remains to be seen whether by giving up the rigidity of the "core", i.e. considering a micromorphic solid, one can avoid the singular stiffening behaviour in an infinitesimal context while keeping with the uniform positivity assumption and $\mu_c > 0$.

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Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times 3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times 3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3\times 3}$. The identity tensor on $\mathbb{M}^{3\times 3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[X] = \langle X, 1\!\!1\rangle$. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$. For $X \in \mathbb{M}^{3\times 3}$ we set for the deviatoric part $\operatorname{dev} X = X - \frac{1}{3}\operatorname{tr}[X]\, 1\!\!1 \in \mathfrak{sl}(3)$ where $\mathfrak{sl}(3)$ is the Lie-algebra of traceless matrices. The Lie-algebra of $\mathrm{SO}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \mid X^T X = 1\!\!1,\ \det[X] = 1\}$ is given by the set $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times 3} \mid X^T = -X\}$ of all skew symmetric tensors. The canonical identification of $\mathfrak{so}(3,\mathbb{R})$ and $\mathbb{R}^3$ is denoted by $\operatorname{axl}(\overline{A}) \in \mathbb{R}^3$ for $\overline{A} \in \mathfrak{so}(3,\mathbb{R})$. Finally, w.r.t. abbreviates with respect to.