Large Existence, Uniqueness and Regularity Classes of Stationary Navier-Stokes Equations in Bounded Domains of $\mathbb{R}^n$

R. Farwig, G.P. Galdi, H. Sohr

Abstract

Using the notion of very weak solutions, introduced recently, see [2], [9], [10], [16], we obtain a new and very large uniqueness class for solutions of the inhomogeneous Navier-Stokes system $-\Delta u + u\cdot\nabla u + \nabla p = f$, $\operatorname{div} u = k$, $u|_{\partial\Omega} = g$ with $u \in L^q$, $q \geq n$, and very general data classes for f, k, g such that u may have no differentiability property. If the data are sufficiently smooth we get a large class of unique and regular solutions extending the well known classical solution classes and generalize a regularity result of Gerhardt [17].

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1 Introduction and Main Result

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$, let $N = N(x) = \big(N_1(x), \ldots, N_n(x)\big)$ denote the outer normal at $x = (x_1, \ldots, x_n) \in \partial\Omega$, and let $1 < q < \infty$, $q' = \frac{q}{q-1}$. In Ω we consider the Navier-Stokes system

$$-\Delta u + u \cdot \nabla u + \nabla p = f, \ \operatorname{div} u = k, \ u|_{\partial\Omega} = g \tag{1.1}$$

with data $f = \operatorname{div} F, k, g$ satisfying

$$\begin{aligned} &F = (F_{i,j})_{i,j=1}^n \in L^r(\Omega), \quad k \in L^r(\Omega), \quad g \in W^{-\frac{1}{q},q}(\partial\Omega), \\ &\int_\Omega k\,dx = \int_{\partial\Omega} N \cdot g\,dS \text{ where } n \leq q < \infty,\ q' < r \leq q,\ \frac{1}{n} + \frac{1}{q} \geq \frac{1}{r}. \end{aligned} \tag{1.2}$$

Here the surface integral is well defined in the generalized sense $\int_{\partial\Omega} N \cdot g\,dS = \langle g, N\rangle_{\partial\Omega} = \langle N \cdot g, 1\rangle_{\partial\Omega}$ of a boundary distribution.

The notion of very weak solutions, introduced in principle by Amann [2], [3] for the $3D$-nonstationary case with $k = 0$, and in [9], [10], [16] for the $2D$- and $3D$-stationary case with $k \neq 0$, rests on the use of test functions in the space

$$C^2_{0,\sigma}(\overline{\Omega}) := \{v = (v_1, \dots, v_n) \in C^2(\overline{\Omega}); \operatorname{div} v = 0, v_{|_{\partial\Omega}} = 0\}. \tag{1.3}$$

Applying such a test function formally to (1.1) we obtain the following relation which is well defined for $u \in L^q$, $q \geq n$, and data (1.2):

$$\begin{aligned} -\langle u, \Delta w\rangle_\Omega \quad &+ \quad \langle g, N \cdot \nabla w\rangle_{\partial\Omega} - \langle uu, \nabla w\rangle_\Omega - \langle ku, w\rangle_\Omega \\ &= \quad -\langle F, \nabla w\rangle_\Omega, \quad w \in C^2_{0,\sigma}(\overline{\Omega}). \end{aligned} \tag{1.4}$$

Here $\langle \cdot, \cdot\rangle_\Omega$ means the usual L^q-$L^{q'}$-pairing in Ω, $\langle g, N\cdot\nabla w\rangle_{\partial\Omega}$ denotes the value of the distribution $g = (g_1, \dots, g_n) \in W^{-\frac{1}{q},q}(\partial\Omega)$ at the normal derivative $N\cdot\nabla w_{|_{\partial\Omega}}$, and $uu = (u_i u_j)^n_{i,j=1}$. Further we use the relation $u\cdot\nabla u = (u\cdot\nabla)u = \operatorname{div}(uu) - ku$, and the notation $f = \operatorname{div} F := \big(\sum^n_{i=1} D_i F_{ij}\big)^n_{j=1}$, $D_i = \partial/\partial x_i$, $i = 1, \dots, n$.

The main result, see Theorem 1.3 below, states that system (1.1) has a unique very weak solution $u \in L^q$ in the sense of (1.4) if the data in (1.2) are sufficiently small. Furthermore, we obtain the existence of a pressure $p \in W^{-1,q}(\Omega)$, well defined up to a constant, by de Rham's argument [28]. Concerning regularity we prove that every very weak solution is a weak solution or even a strong solution provided the data in (1.2) are sufficiently regular, see Theorem 1.6 below. These results will generalize the regularity result of Gerhardt [17] on weak solutions in the four-dimensional case to higher dimensions and are related to the result on the existence of (locally) strong solutions by Frehse & Růžička [11], [12] for dimensions $n \geq 5$, see Remark 1.7 below.

To understand the precise meaning of all terms in (1.4) let $\tau = \tau(x) = \big(\tau_1(x), \dots, \tau_{n-1}(x)\big)$ be a system of unit tangential vectors at $x \in \partial\Omega$ such that $\big(\tau(x), N(x)\big) = \big(\tau_1(x), \dots, \tau_{n-1}(x), N(x)\big)$ defines a Cartesian basis at x. Then $g(x)$ has the form

$$g(x) = \mathcal{L}_g\big(\tau(x)\big) + (N \cdot g)N(x) \tag{1.5}$$

where $\mathcal{L}_g\big(\tau(x)\big) \in \mathbb{R}^n$ means a suitable linear combination of $\tau_1(x), \dots, \tau_{n-1}(x)$ contained in the tangential plane at x, and $N \cdot g = N_1 g_1 + \dots N_n g_n$ denotes the normal component of $g(x)$. An elementary calculation, using $\operatorname{div} w = 0$ and assuming without loss of generality that $\big(\tau(x), N(x)\big)$ is the standard basis of $\mathbb{R}^n$, shows that $N \cdot \nabla w_{|_{\partial\Omega}}$ is contained in the tangential plane. Thus we obtain that

$$\langle g, N \cdot \nabla w\rangle_{\partial\Omega} = \langle \mathcal{L}_g(\tau_1, \dots, \tau_{n-1}), N \cdot \nabla w\rangle_{\partial\Omega}\,; \tag{1.6}$$

hence (1.4) contains only the tangential components of g.

Concerning the normal component $N\cdot g$ of g we have to require the additional (well defined) conditions

$$\operatorname{div} u = k \text{ in } \Omega, \;\; N \cdot u = N \cdot g \text{ on } \partial\Omega. \tag{1.7}$$

Thus, if (1.4) is satisfied for some vector field $u \in L^q(\Omega)$, we say by definition that

$$\mathcal{L}_{u|_{\partial\Omega}}(\tau_1, \ldots, \tau_{n-1}) := \mathcal{L}_g(\tau_1, \ldots, \tau_{n-1}) \in W^{-\frac{1}{q},q}(\partial\Omega) \tag{1.8}$$

is the tangential trace of u at $\partial\Omega$ in the sense of boundary distributions. Since the trace $N \cdot u|_{\partial\Omega} \in W^{-\frac{1}{q},q}(\partial\Omega)$ is well defined in the usual sense we get a precise meaning of the boundary trace $u|_{\partial\Omega} = g$ in (1.1).

Definition 1.1 Given data f, k, g as in (1.2), a vector field $u \in L^q(\Omega)$ is called a *very weak solution of* (1.1) if and only if the relation (1.4) and the conditions (1.7) are satisfied.

For the linearized system

$$-\Delta u + \nabla p = f, \quad \operatorname{div} u = k, \quad u|_{\partial\Omega} = g \tag{1.9}$$

we may omit the condition $q' < r$ in (1.2), caused by the nonlinear term $u \cdot \nabla u$, and suppose that the data $f = \operatorname{div} F, k, g$ satisfy

$$\begin{aligned} &F \in L^r(\Omega), \quad k \in L^r(\Omega), \quad g \in W^{-\frac{1}{q},q}(\partial\Omega), \\ &\int_\Omega k\,dx = \int_{\partial\Omega} N \cdot g\,dS \quad \text{with} \quad n \le q < \infty, \quad 1 < r \le q, \quad \frac{1}{n} + \frac{1}{q} \ge \frac{1}{r}\,. \end{aligned} \tag{1.10}$$

Definition 1.2 Given data f, k, g as in (1.10), a vector field $u \in L^q(\Omega)$ is called a *very weak solution of* (1.9) if and only if the relation

$$-\langle u, \Delta w\rangle_\Omega + \langle g, N \cdot \nabla w\rangle_{\partial\Omega} = -\langle F, \nabla w\rangle_\Omega \quad \text{for all } w \in C^2_{0,\sigma}(\overline{\Omega}) \tag{1.11}$$

and the conditions $\operatorname{div} u = k$, $N \cdot u|_{\partial\Omega} = N \cdot g$ are satisfied.

Our main result reads as follows.

Theorem 1.3 *Suppose the data $f = \operatorname{div} F, k, g$ satisfy* (1.2)*. Then there exists a constant $K = K(\Omega, q, r) > 0$ such that in the case*

$$\|F\|_{L^r(\Omega)} + \|k\|_{L^r(\Omega)} + \|g\|_{W^{-\frac{1}{q},q}(\partial\Omega)} \le K \tag{1.12}$$

there is a unique very weak solution $u \in L^q(\Omega)$ of (1.1) *satisfying the estimate*

$$\|u\|_{L^q(\Omega)} \le C\big(\|F\|_{L^r(\Omega)} + \|k\|_{L^r(\Omega)} + \|g\|_{W^{-\frac{1}{q},q}(\partial\Omega)}\big) \tag{1.13}$$

with $C = C(\Omega, q, r) > 0$. Moreover, there exists a pressure $p \in W^{-1,q}(\Omega)$ such that $-\Delta u + u \cdot \nabla u + \nabla p = f$ is satisfied in the sense of distributions.

Remark 1.4 (i) In the proof of Theorem 1.3 we first solve the linearized system (1.9), only assuming (1.10) in the sense of Definition 1.2. The solution $u \in L^q(\Omega)$ of (1.9) always exists, is unique, satisfies (1.13) and possesses the explicit representation formula (3.4), see below.

(ii) A well known scaling argument shows that the data conditions in (1.2) are optimal for the solution class $u \in L^q(\Omega)$ and that the estimate (1.13) is sharp. Thus we also get a new class of unique solutions $u \in L^q(\Omega)$ without any differentiability property. However, if the data f, k, g are sufficiently smooth we obtain the corresponding regularity properties for the solution u solving (1.1) in the classical weak or strong sense. Therefore, even in the case of more regular data f, k, g, the uniqueness class determined by (1.12) leads to a new large class also for classical solutions. These uniqueness and regularity assertions are described in the following two theorems.

Theorem 1.5 *Suppose the data $f = \operatorname{div} F, k, g$ satisfy (1.2), and let $u \in L^q(\Omega)$ be a very weak solution of (1.1). Then there exists a constant $K = K(\Omega, q, r) > 0$ such that under the condition*

$$\|u\|_q + \|k\|_r \le K \tag{1.14}$$

there is no other very weak solution $v \in L^q(\Omega)$ of (1.1) for the same data f, k, g.

Theorem 1.6 *Let $u \in L^q(\Omega)$ be a very weak solution of the Navier-Stokes system (1.1) with data $f = \operatorname{div} F$ and k, g as in (1.2).*

(i) *Assume that the data f, k, g satisfy the additional conditions*

$$F \in L^q(\Omega), \quad k \in L^q(\Omega) \quad \text{and} \quad g \in W^{1-1/q,q}(\partial\Omega).$$

Then $u \in W^{1,q}(\Omega)$, the equation $-\Delta u + u \cdot \nabla u + \nabla p = f$ holds in the sense of distributions with some pressure function $p \in L^q(\Omega)$, and $u|_{\partial\Omega} = g$ holds in the sense of the usual trace theorem.

(ii) *Assume that the data $f = \operatorname{div} F, k, g$ satisfy the additional conditions*

$$f \in L^s(\Omega), \quad F \in L^q(\Omega), \quad k \in W^{1,q}(\Omega) \quad \text{and} \quad g \in W^{2-1/q,q}(\partial\Omega)$$

where $s \in [\frac{n}{2}, \infty)$. Then $u \in D(A_s) + W^{2,q}(\Omega)$, the equation $-\Delta u + u \cdot \nabla u + \nabla p = f$ holds strongly in $L^{\tilde q}(\Omega)$, $\tilde q = \min(q, s)$, with some pressure function $p \in W^{1,\tilde q}(\Omega)$ and $u|_{\partial\Omega} = g$ holds in the sense of traces.

Remark 1.7 (i) Consider $f \in L^s(\Omega)$ with $\frac{n}{2} \le s < \infty$, and the "classical" case $k = 0$, $g = 0$. Then we can choose $q \ge \max(n, s)$ such that $\frac{\alpha}{n} + \frac{1}{q} = \frac{1}{s}$ with some $\alpha \in [0, 1]$. This yields $\frac{\alpha}{n} + \frac{1}{s'} = \frac{1}{q'}$ and the embedding $W^{1,q'}(\Omega) \subset L^{s'}(\Omega)$ which shows that there exists some $F \in L^q(\Omega)$ satisfying $f = \operatorname{div} F$. Thus Theorem 1.6 (ii) implies that every (very) weak solution u of (1.1) satisfies

$u \in D(A_s) + W^{2,q}(\Omega) \subset W^{2,s}(\Omega)$. In the case $s = \frac{n}{2}$ this result generalizes the well known regularity theorem for weak solutions of the Navier-Stokes equations in the four-dimensional case when $k = 0$ and $g = 0$, see [17].

(ii) For every right-hand side $f \in W^{-1,2}(\Omega)$ the classical Navier-Stokes equations (with $k = 0$, $g = 0$) have a weak solution $u \in W_0^{1,2}(\Omega) \cap L^2_\sigma(\Omega)$, see [28], Theorem 1.2 in Chapter II, §1. If additionally $u \in L^n$ which is guaranteed for $n = 3, 4$, this weak solution u is also a very weak solution in the sense of Definition 1.1, since $\tilde{V} := \overline{C^\infty_{0,\sigma}(\Omega)}^{H_1 \cap L^n} = \overline{C^2_{0,\sigma}(\overline{\Omega})}^{H_1 \cap L^n}$ on p. 161 in [28]. Now assuming $f \in L^s(\Omega)$, $s \geq \frac{n}{2}$, part (i) implies that u is a strong solution in $W^{2,s}(\Omega)$. This result is related to existence and regularity theorems in a series of papers, see e.g. [11], [12], where the existence of at least one locally regular solution $u \in W^{2,r}_{\mathrm{loc}}(\Omega)$, $r > 1$, was proved. If the data are sufficiently small in the sense of Theorem 1.3 to guarantee the unique existence of a very weak solution $u \in L^q$, we get $u \in W^{2,s}(\Omega)$ provided the external force f lies in $L^s(\Omega)$, $s \geq \frac{n}{2}$.

(iii) If $q = s$ in Theorem 1.6, (ii), then obviously $u \in W^{2,q}(\Omega)$.

2 Some preliminaries

Let $1 < q < \infty$ and $q' = \frac{q}{q-1}$ such that $\frac{1}{q} + \frac{1}{q'} = 1$. We need the usual spaces $L^q(\Omega)$, $L^q(\partial\Omega)$, $W^{\alpha,q}(\Omega)$, $W_0^{\alpha,q}(\Omega)$, $W^{-\alpha,q}(\Omega) = \big(W_0^{\alpha,q'}(\Omega)\big)'$, $W^{\alpha,q}(\partial\Omega)$, and $W^{-\alpha,q}(\partial\Omega) = \big(W^{\alpha,q'}(\partial\Omega)\big)'$, $0 \leq \alpha \leq 2$. The corresponding pairings are denoted by $\langle\cdot,\cdot\rangle_\Omega$ or $\langle\cdot,\cdot\rangle_{\partial\Omega}$, resp., and the corresponding norms are denoted by $\|\cdot\|_q = \|\cdot\|_{q,\Omega}$, $\|\cdot\|_{\pm\alpha;q,\Omega} = \|\cdot\|_{\pm\alpha;q}$, $\|\cdot\|_{q,\partial\Omega}$, and $\|\cdot\|_{\pm\alpha;q,\partial\Omega}$, respectively.

The spaces of smooth functions on Ω are denoted as usual by $C^j(\Omega)$, $C^j_0(\Omega)$, $C^j(\overline{\Omega})$ for $j = 0, 1, 2, \ldots$ and $j = \infty$. We set

$$\begin{aligned} C^j_0(\overline{\Omega}) &:= \{v \in C^j(\overline{\Omega});\ v|_{\partial\Omega} = 0\}, \\ C^j_{0,\sigma}(\overline{\Omega}) &:= \{v = (v_1, \ldots, v_n) \in C^j_0(\overline{\Omega});\ \operatorname{div} v = 0\}, \end{aligned}$$

and $C^j_{0,\sigma}(\Omega) := \{v \in C^j_0(\Omega);\ \operatorname{div} v = 0\}$. The space of distributions $C^\infty_0(\Omega)'$ is the dual space of $C^\infty_0(\Omega)$ in the usual topology, the duality pairing of which is again denoted by $\langle\cdot,\cdot\rangle_\Omega$. Correspondingly, we use the test function space $C^j(\partial\Omega)$, $j = 1, 2$, on the boundary $\partial\Omega$, and the corresponding distribution spaces $C^j(\partial\Omega)'$ with pairing $\langle\cdot,\cdot\rangle_{\partial\Omega}$.

Let $L^q_\sigma(\Omega)$ be the closure of $C^\infty_{0,\sigma}(\Omega)$ in the norm $\|\cdot\|_{L^q(\Omega)}$. The dual space $L^q_\sigma(\Omega)'$ of $L^q_\sigma(\Omega)$ is identified with $L^{q'}_\sigma(\Omega)$ by the pairing $\langle f, v\rangle_\Omega = \int_\Omega f \cdot v\, dx$. By analogy, we identify $L^q(\partial\Omega)' = L^{q'}(\partial\Omega)$ with pairing $\langle f, v\rangle_{\partial\Omega} = \int_{\partial\Omega} f \cdot v\, dS$.

We need some trace and extension properties for $\alpha = 1, 2$. The trace map $f \mapsto f|_{\partial\Omega}$ is a well defined bounded linear operator from $W^{\alpha,q}(\Omega)$ onto $W^{\alpha-\frac{1}{q},q}(\partial\Omega)$. Conversely, there exists a bounded linear operator $E^1 : W^{1-1/q,q}(\partial\Omega) \to W^{1,q}(\Omega)$ with $E^1(h)|_{\partial\Omega} = h$, and a bounded linear operator

$E^2 : W^{2-1/q,q}(\partial\Omega) \times W^{1-1/q,q}(\partial\Omega) \to W^{2,q}(\Omega)$ satisfying $E^2(h_1,h_2)|_{\partial\Omega} = h_1$, $N \cdot \nabla E^2(h_1,h_2)|_{\partial\Omega} = h_2$; see [24], Theorem 5.8, [29], 5.4.4.

Let $1 < r \le q$, $\frac{1}{n} + \frac{1}{q} \ge \frac{1}{r}$, and let $f = (f_1, \dots, f_m) \in L^q(\Omega)$, $\operatorname{div} f \in L^r(\Omega)$. Then, using E^1 with q replaced by q', the embedding estimate

$$\|E^1(h)\|_{r',\Omega} \le C\big(\|E^1(h)\|_{q',\Omega} + \|\nabla E^1(h)\|_{q',\Omega}\big), \quad C = C(\Omega,q,r) > 0,$$

and Green's identity $\langle \operatorname{div} f, E^1(h)\rangle_\Omega = \langle N \cdot f, h\rangle_{\partial\Omega} - \langle f, \nabla E^1(h)\rangle_\Omega$ for $h \in W^{1/q,q'}(\partial\Omega)$, we get $N \cdot f|_{\partial\Omega} \in W^{-\frac{1}{q},q}(\partial\Omega)$ and the estimate

$$\|N \cdot f\|_{-\frac{1}{q};q,\partial\Omega} \le C\big(\|f\|_{q,\Omega} + \|\operatorname{div} f\|_{r,\Omega}\big) \tag{2.1}$$

with $C = C(\Omega,q,r) > 0$.

Conversely, there is a linear operator $\hat{E} : W^{-\frac{1}{q},q}(\partial\Omega) \to L^q(h)$ satisfying $\operatorname{div} \hat{E}(h) \in L^r(\Omega)$, $N \cdot \hat{E}(h)|_{\partial\Omega} = h$ and the estimate

$$\|\hat{E}(h)\|_{q,\Omega} + \|\operatorname{div} \hat{E}(h)\|_{r,\Omega} \le C\|h\|_{-\frac{1}{q};q,\partial\Omega}, \quad h \in W^{-\frac{1}{q},q}(\partial\Omega), \tag{2.2}$$

with $C = C(\Omega,q,r) > 0$; see [25], Corollary 4.6, (4.10).

As an application we consider some gradient $\nabla H = (D_1 H, \dots, D_n H) \in L^q(\Omega)$ with $\Delta H \in L^r(\Omega)$, and apply (2.1) to ∇H and to the vector fields $f^{i,j} = \big(f_1^{i,j}, \dots, f_n^{i,j}\big)$, $0 \le i < j \le n$, satisfying $f_i^{i,j} := D_j H$, $f_j^{i,j} := -D_i H$ but $f_k^{ij} = 0$ if $i \ne k \ne j$. Obviously $\operatorname{div} f^{i,j} = D_i D_j H - D_j D_i H = 0$ in the sense of distributions. Then $N \cdot \nabla H|_{\partial\Omega}$ and $N \cdot f^{i,j}|_{\partial\Omega} \in W^{-\frac{1}{q},q}(\partial\Omega)$ are well defined by (2.1), and a calculation shows that each $D_k H$, $k = 1, \dots, n$, at $\partial\Omega$ is a linear combination of $N \cdot \nabla H|_{\partial\Omega}$ and $N \cdot f^{i,j}|_{\partial\Omega}$ with $1 \le i < j \le n$. Therefore we conclude from (2.1) that $\nabla H|_{\partial\Omega} \in W^{-\frac{1}{q},q}(\partial\Omega)$ is well defined and satisfies the estimate

$$\|\nabla H\|_{-\frac{1}{q};q,\partial\Omega} \le C\big(\|\nabla H\|_{q,\Omega} + \|\Delta H\|_{r,\Omega}\big) \tag{2.3}$$

with $C = C(\Omega,q,r) > 0$.

Consider the data $f = \operatorname{div} F, k, g$ as in (1.10), and the weak Neumann problem

$$\Delta H = k, \quad N \cdot \nabla H|_{\partial\Omega} = N \cdot g \tag{2.4}$$

where $\nabla H \in L^q(\Omega)$ is considered as a solution. Then we use $\hat{E}(h)$ with $h = N \cdot g \in W^{-\frac{1}{q},q}(\partial\Omega)$, and choose a solution $b(h) \in W_0^{1,r}(\Omega)$ of the equation $\operatorname{div} b(h) = \operatorname{div} \hat{E}(h) - k \in L^r(\Omega)$. Since

$$\int_\Omega (\operatorname{div} \hat{E}(h) - k)dx = \int_{\partial\Omega} N \cdot g \, dS - \int_\Omega k \, dx = 0,$$

such a solution exists, see [14], Theorem III, 3.2, or [27], and satisfies

$$\|b(h)\|_{q,\Omega} \le C_1\|\nabla b(h)\|_{r,\Omega} \le C_2\big(\|\mathrm{div}\,\hat{E}(h)\|_{r,\Omega} + \|k\|_{r,\Omega}\big) \tag{2.5}$$

with $C_j = C_j(\Omega, q, r) > 0$, $j = 1, 2$. Writing (2.4) in the form

$$\Delta H = \mathrm{div}\,\big(\hat{E}(h) - b(h)\big), \quad N \cdot \big(\nabla H - \hat{E}(h) - b(h)\big)\big|_{\partial\Omega} = 0, \tag{2.6}$$

we find, see [13], [25], a unique solution $\nabla H \in L^q(\Omega)$ satisfying

$$\|\nabla H\|_{q,\Omega} \le C_1\big(\|\hat{E}(h)\|_{q,\Omega} + \|b(h)\|_{q,\Omega}\big) \le C_2\big(\|N \cdot g\|_{-\frac{1}{q};q,\partial\Omega} + \|k\|_{r,\Omega}\big), \tag{2.7}$$

and therefore

$$\|\nabla H\|_{-\frac{1}{q};q,\partial\Omega} \le C\big(\|N \cdot g\|_{-\frac{1}{q};q,\partial\Omega} + \|k\|_{r,\Omega}\big) \tag{2.8}$$

with $C = C(\Omega, q, r) > 0$, $C_j = C_j(\Omega, q, r) > 0$, $j = 1, 2$.

Now approximate the data k, g in (2.4) by smooth function k_j, g_j, $j \in \mathbb{N}$, such that $\lim_{j\to\infty} \|k - k_j\|_{r,\Omega} = 0$ and $\lim_{j\to\infty} \|g - g_j\|_{-\frac{1}{q};q,\partial\Omega} = 0$. Let $\nabla H_j \in L^q(\Omega)$ be the corresponding solution of (2.4). Using (2.7), (2.8) with $\nabla H, k, g$ replaced by $\nabla H - \nabla H_j, g - g_j, k - k_j$ we see that $\lim_{j\to\infty} \|\nabla H - \nabla H_j\|_{q,\Omega} = 0$ and $\lim_{j\to\infty} \|\nabla H - \nabla H_j\|_{-\frac{1}{q};q,\partial\Omega} = 0$. Then, using the Stokes operator $A_{q'}$ and its inverse $A_{q'}^{-1}$, see below, we get the important identity

$$\begin{aligned}
\langle \nabla H, \Delta A_{q'}^{-1} v\rangle_\Omega &= \lim_{j\to\infty} \langle \nabla H_j, \Delta A_{q'}^{-1} v\rangle_\Omega \\
&= \lim_{j\to\infty} \big(\langle \nabla H_j, N \cdot \nabla A_{q'}^{-1} v\rangle_{\partial\Omega} + \langle \nabla \Delta H_j, A_{q'}^{-1} v\rangle_\Omega\big) \\
&= \langle \nabla H, N \cdot \nabla A_{q'}^{-1} v\rangle_{\partial\Omega}
\end{aligned} \tag{2.9}$$

for all $v \in L^{q'}_\sigma(\Omega)$ since $\mathrm{div}\, A_{q'}^{-1} v = 0$ and $A_{q'}^{-1} v\big|_{\partial\Omega} = 0$.

Let $f = (f_1, \dots, f_n) \in L^q(\Omega)$. Then as in (2.6) the weak Neumann problem

$$\Delta H = \mathrm{div}\, f, \quad N \cdot (\nabla H - f)\big|_{\partial\Omega} = 0$$

has a unique solution $\nabla H \in L^q(\Omega)$, see [13], [25], satisfying

$$\|\nabla H\|_{q,\Omega} \le C\|f\|_{q,\Omega} \tag{2.10}$$

with $C = C(\Omega, q) > 0$. Setting $P_q f := f - \nabla H$ we get the Helmholtz projection as a bounded linear operator from $L^q(\Omega)$ onto $L^q_\sigma(\Omega)$ satisfying $P_q^2 = P_q$ and $P_q' = P_{q'}$ where P_q' means the dual operator.

The Stokes operator A_q with domain $D(A_q) = L^q_\sigma(\Omega) \cap W^{1,q}_0(\Omega) \cap W^{2,q}(\Omega)$ and range $R(A_q) = L^q_\sigma(\Omega)$ defined by $A_q u := -P_q \Delta u$, $u \in D(A_q)$, is a densely defined closed operator satisfying $\langle A_q u, v\rangle_\Omega = \langle u, A_{q'} v\rangle_\Omega$ for $u \in D(A_q)$, $v \in D(A_{q'})$, and $A_q u = A_\gamma u$ for $1 < q, \gamma < \infty$, $u \in D(A_q) \cap D(A_\gamma)$. The fractional power

$A_q^\beta : D(A_q^\beta) \to L^q_\sigma(\Omega)$, $0 \le \beta \le 1$, with $D(A_q) \subseteq D(A_q^\beta) \subseteq L^q_\sigma(\Omega)$, is well defined and bijective; its inverse $A_q^{-\beta} = (A_q^\beta)^{-1}$ is bounded from $L^q_\sigma(\Omega)$ onto $R(A_q^{-\beta}) = D(A_q^\beta)$. Moreover, it holds $(A_q^\beta)' = A_{q'}^\beta$. We note that the norms $\|u\|_{2;q,\Omega}$ and $\|A_q u\|_{q,\Omega}$ are equivalent for $u \in D(A_q)$, as well as that the norms $\|u\|_{1;q,\Omega}$ and $\|A_q^{1/2} u\|_{q,\Omega}$ are equivalent for $u \in D(A_q^{1/2})$. Further it holds the embedding estimate

$$\|u\|_{q,\Omega} \le C\|A_\gamma^\beta u\|_{\gamma,\Omega}, \quad u \in D(A_\gamma^\beta),\, 1 < \gamma \le q < \infty,\, 2\beta + \frac{n}{q} = \frac{n}{\gamma}, \tag{2.11}$$

with $C = C(\Omega, q, \gamma) > 0$. Using $A_q^{1/2}$ we define the Yosida operators $J_m = (I + \frac{1}{m}A_q^{1/2})^{-1}$ for $m \in \mathbb{N}$. It is well known that there exists $C = C(\Omega, q) > 0$ such that

$$\|J_m\| + \left\|\tfrac{1}{m} A_q^{\frac{1}{2}} J_m\right\| \le C, \quad m \in \mathbb{N}, \tag{2.12}$$

in the operator norm on $L^q_\sigma(\Omega)$ and that $J_m u \to u$ in $L^q_\sigma(\Omega)$ as $m \to \infty$. See [4], [18], [19], [20], [23], [27], [29], concerning the Stokes operator.

Using (2.11) we get for $f = \operatorname{div} F$ and arbitrary $v \in L^{q'}_\sigma(\Omega)$ the estimate

$$\begin{aligned} |\langle f, A_{q'}^{-1} v\rangle_\Omega| &= |\langle F, \nabla A_{q'}^{-1} v\rangle_\Omega| = |\langle F, \nabla A_{r'}^{-\frac{1}{2}} A_{r'}^{-\frac{1}{2}} v\rangle_\Omega| \\ &\le C_1 \|F\|_{r,\Omega}\, \|A_{r'}^{-\frac{1}{2}} v\|_{r',\Omega} \le C_2 \|F\|_{r,\Omega}\, \|v\|_{q',\Omega} \end{aligned} \tag{2.13}$$

with $C_j = C_j(\Omega, q, r) > 0$, $j = 1, 2$. This proves the existence of a unique $\hat f \in L^q_\sigma(\Omega)$ satisfying $\langle f, A_{q'}^{-1} v\rangle_\Omega = \langle \hat f, v\rangle_\Omega$ for all $v \in L^{q'}_\sigma(\Omega)$, and the estimate

$$\|\hat f\|_{q,\Omega} \le C\|F\|_{r,\Omega}\,, \quad C = C(\Omega, q, r) > 0. \tag{2.14}$$

Similarly as in the theory of distributions, we set, by definition, $\hat f = A_q^{-1} P_q f \in L^q_\sigma(\Omega)$ giving this expression a generalizing meaning. Then $A_q^{-1} P_q f$ is well defined by the relation

$$\langle A_q^{-1} P_q f, v\rangle_\Omega = \langle f, A_{q'}^{-1} v\rangle_\Omega\,, \quad v \in L^{q'}_\sigma(\Omega). \tag{2.15}$$

More generally, let $f \in C_0^\infty(\Omega)'$ be any distribution such that $\langle f, w\rangle_\Omega$ is well defined (by any continuous extension) for all test functions $w \in D(A_{q'}^\beta)$, $0 \le \beta \le 1$, and satisfies the estimate

$$|\langle f, A_{q'}^{-\beta} v\rangle_\Omega| \le C_f \|v\|_{q',\Omega}\,,\ v \in L^{q'}_\sigma(\Omega). \tag{2.16}$$

Then $A_q^{-\beta} P_q f \in L^q_\sigma(\Omega)$ is well defined by the relation

$$\langle A_q^{-\beta} P_q f, v\rangle_\Omega = \langle f, A_{q'}^{-\beta} v\rangle_\Omega\,,\ v \in L^{q'}_\sigma(\Omega), \tag{2.17}$$

giving $A_q^{-\beta} P_q f$ a generalized meaning, and it holds

$$\|A_q^{-\beta} P_q f\|_q \le C_f. \tag{2.18}$$

As an example we mention the estimate

$$\|A_q^{-\frac{1}{2}} P_q \operatorname{div} w\|_q \le C\|w\|_q, \quad w \in L^q(\Omega),\ 1<q<\infty, \tag{2.19}$$

with $C = C(\Omega, q) > 0$. See [27], III, 2.5, 2.6, for similar definitions.

Let $w \in C^2_{0,\sigma}(\overline{\Omega})$ and $v = A_{q'} w$. Then, using (2.11) and the trace estimates, we obtain that

$$\begin{aligned} |\langle g, N\cdot\nabla A_{q'}^{-1} v\rangle_{\partial\Omega}| &\le C_1\|g\|_{-\frac{1}{q};q,\partial\Omega}\,\|\nabla A_{q'}^{-1} v\|_{\frac{1}{q};q',\partial\Omega} \\ &\le C_2\|g\|_{-\frac{1}{q};q,\partial\Omega}\,\|\nabla A_{q'}^{-1} v\|_{1;q',\Omega} \\ &\le C_3\|g\|_{-\frac{1}{q};q,\partial\Omega}\,\|v\|_{q',\Omega} \end{aligned} \tag{2.20}$$

with $C_j = C_j(\Omega, q) > 0$, $j = 1, 2, 3$. Since $L^q_\sigma(\Omega) = \big(L^{q'}_\sigma(\Omega)\big)'$, there is a unique $G \in L^q_\sigma(\Omega)$ satisfying

$$\begin{aligned} \langle G, v\rangle_\Omega &= \langle g, N\cdot\nabla A_{q'}^{-1} v\rangle_{\partial\Omega} \quad \text{for all } v \in L^{q'}_\sigma(\Omega), \\ \|G\|_{q,\Omega} &\le C\|g\|_{-\frac{1}{q};q,\partial\Omega} \end{aligned} \tag{2.21}$$

with $C = C(\Omega, q) > 0$.

Finally we need the density property

$$\overline{A_q C^2_{0,\sigma}(\overline{\Omega})}^{\|\cdot\|_{q,\Omega}} = L^q_\sigma(\Omega). \tag{2.22}$$

Indeed, consider $f \in L^q_\sigma(\Omega)$, choose $f_j \in C^\infty_{0,\sigma}(\Omega)$, $j \in \mathbb{N}$, with $\lim_{j\to\infty}\|f - f_j\|_{q,\Omega} = 0$ and let $u_j = A_q^{-1} f_j$. The regularity property in [26], p. 518, (9.13) shows that $u_j \in C^2_{0,\sigma}(\overline{\Omega})$ for $j \in \mathbb{N}$, and we see that $A_q u_j = f_j \to f$ in $L^q_\sigma(\Omega)$ as $j \to \infty$. This proves (2.22). Moreover, this proof shows that $C^2_{0,\sigma}(\overline{\Omega}) \subseteq D(A_q)$ is a core of $D(A_q)$.

3 Proof of Theorems

First we consider the data $f = \operatorname{div} F, k, g$ as in (1.10) and prove a representation formula for the solution $u \in L^q(\Omega)$ of the linearized system (1.9).

Consider the solution $\nabla H \in L^q(\Omega)$ of the system (2.4). From (2.8) we know that $\hat{g} := \nabla H|_{\partial\Omega} \in W^{-\frac{1}{q},q}(\partial\Omega)$ is well defined, and from (2.9) we conclude that $-\langle \nabla H, \Delta w\rangle_\Omega + \langle \hat{g}, N\cdot\nabla w\rangle_{\partial\Omega} = 0$ for all $w \in C^2_{0,\sigma}(\overline{\Omega})$, $v = A_{q'} w$, $w = A_{q'}^{-1} v$. This shows, see (1.11), that $u_1 := \nabla H$ is a very weak solution of the linear system

$$-\Delta u_1 + \nabla p_1 = 0, \quad \operatorname{div} u_1 = k, \quad u_1|_{\partial\Omega} = \hat{g}. \tag{3.1}$$

Next set $\tilde{g} := g - \hat{g} \in W^{-\frac{1}{q},q}(\partial\Omega)$ and choose $\tilde{G} \in L^q_\sigma(\Omega)$, using (2.21) with g replaced by $\tilde{g}$, such that $\langle \tilde{g}, N\cdot\nabla A_{q'}^{-1} v\rangle_{\partial\Omega} = \langle \tilde{G}, v\rangle_\Omega$, $v \in L^{q'}_\sigma(\Omega)$. Setting $w = A_{q'}^{-1} v$ we get

$$\langle \tilde{G}, \Delta w\rangle_\Omega = -\langle \tilde{G}, -P_{q'}\Delta w\rangle_\Omega = -\langle \tilde{G}, v\rangle_\Omega = -\langle \tilde{g}, N\cdot\nabla w\rangle_{\partial\Omega}$$

which shows that $u_2 := \tilde{G}$ is a very weak solution of the linear system

$$-\Delta u_2 + \nabla p_2 = 0, \quad \operatorname{div} u_2 = 0, \quad u_2|_{\partial\Omega} = \tilde{g}. \tag{3.2}$$

Finally, we set $u_3 := A_q^{-1} P_q f$, see (2.15), and conclude that u_3 is a very weak solution of the linear system

$$-\Delta u_3 + \nabla p_3 = f, \quad \operatorname{div} u_3 = 0, \quad u_3|_{\partial\Omega} = 0. \tag{3.3}$$

Combining (3.1), (3.2), (3.3) and using $\operatorname{div}(u_1 + u_2 + u_3) = k$ and $N \cdot (u_1 + u_2 + u_3)|_{\partial\Omega} = N \cdot g$ we see that $u \in L^q(\Omega)$ defined by

$$u := u_1 + u_2 + u_3 = \nabla H + \tilde{G} + A_q^{-1} P_q f \tag{3.4}$$

is a very weak solution of the linearized system (1.9). Using (2.7), (2.14) and (2.21) with G, g replaced by $\tilde{G}, \tilde{g}$, we obtain the estimate

$$\|u\|_{q,\Omega} \le C\big(\|F\|_{r,\Omega} + \|k\|_{r,\Omega} + \|g\|_{-\frac{1}{q};q,\partial\Omega}\big) \tag{3.5}$$

with $C = C(\Omega, q, r) > 0$.

To prove the uniqueness let v be another solution of (1.9) for the same data (1.10). Then $u - v$ is a very solution of (1.9) with data $f = 0$, $k = 0$, $g = 0$. From (1.11) we obtain that $-\langle u - v, \Delta w\rangle_\Omega = \langle u - v, A_{q'} w\rangle_\Omega$ for all $w \in C^2_{0,\sigma}(\overline{\Omega})$, and using (2.22) we get that $u - v = 0$, $u = v$. Therefore, each very weak solution of (1.9) with data (1.10) has the representation (3.4).

Observe that in the proof of (3.4) we only used that $A_q^{-1} P_q f \in L^q_\sigma(\Omega)$ is well defined in the sense of (2.17) with $\beta = 1$. Thus instead of $f = \operatorname{div} F$ with $F \in L^r(\Omega)$ we only need to assume that f is a distribution such that $A_q^{-1} P_q f \in L^q_\sigma(\Omega)$ is well defined with (2.16) – (2.18). In this case we define a very weak solution u of (1.9) replacing the term $-\langle F, \nabla w\rangle_\Omega$ in (1.11) by $\langle f, w\rangle_\Omega$, and obtaining for u the formula (3.4) and the estimate

$$\|u\|_{q,\Omega} \le C\big(\|A_q^{-1} P_q f\|_{q,\Omega} + \|k\|_{r,\Omega} + \|g\|_{-\frac{1}{q};q,\partial\Omega}\big) \tag{3.6}$$

with $C = C(\Omega, q, r) > 0$.

Proof of Theorem 1.3 Considering the nonlinear case suppose that the data $f = \operatorname{div} F, k, g$ satisfy the conditions (1.2). First assume that $u \in L^q(\Omega)$ is a given very weak solution of (1.1). Setting $\hat{f} := f - \operatorname{div}(uu) + ku$ we obtain that $A_q^{-1} P_q \hat{f} \in L^q_\sigma(\Omega)$ is well defined in the general sense (2.17), see (3.9), (3.10) below.

Therefore, u is a very weak solution of the linear system

$$-\Delta u + \nabla p = \hat{f}, \quad \operatorname{div} u = k, \quad u|_{\partial\Omega} = g, \tag{3.7}$$

and possesses the representation

$$u = \mathcal{F}(u) := \nabla H + \tilde{G} + A_q^{-1} P_q f - A_q^{-1} P_q \operatorname{div}(uu) + A_q^{-1} P_q (ku). \tag{3.8}$$

Next we show that $u = \mathcal{F}(u)$ has a solution $u \in L^q(\Omega)$ using Banach's fixed point principle in a standard way.

Indeed, using (2.15) and (2.11) we obtain similarly as in (2.13) that

$$\begin{aligned} |\langle A_q^{-1} P_q \operatorname{div}(uu), v\rangle_\Omega| &= |\langle uu, \nabla A_{q'}^{-1} v\rangle_\Omega| \\ &\leq C_1 \|uu\|_{q/2,\Omega} \|\nabla A_{q'}^{-1} v\|_{(q/2)',\Omega} \\ &\leq C_2 \|u\|_q^2 \, \|A_{q'}^{-\frac{1}{2}} v\|_{(q/2)',\Omega} \\ &\leq C_3 \|u\|_{q,\Omega}^2 \|v\|_{q',\Omega} \end{aligned} \tag{3.9}$$

and that

$$\begin{aligned} |\langle A_q^{-1} P_q (ku), v\rangle_\Omega| &= |\langle ku, A_{q'}^{-1} v\rangle_\Omega| \\ &\leq C_1 \|ku\|_{(\frac{1}{r}+\frac{1}{q})^{-1},\Omega} \|A_{q'}^{-1} v\|_{(1-\frac{1}{r}-\frac{1}{q})^{-1},\Omega} \\ &\leq C_2 \|k\|_{r,\Omega} \, \|u\|_{q,\Omega} \, \|v\|_{q',\Omega} \end{aligned} \tag{3.10}$$

for $v \in L_\sigma^{q'}(\Omega)$ and with C_1, C_2, C_3 depending on Ω, q, r. Here we need that $q' < r \leq q$, $q \geq n$. This shows that $-A_q^{-1} P_q \operatorname{div}(uu) + A_q^{-1} P_q (ku) \in L_\sigma^q(\Omega)$ is well defined for $u \in L^q(\Omega)$; moreover, we get the estimate

$$\|\mathcal{F}(u)\|_{q,\Omega} \leq C\big(\|u\|_{q,\Omega}^2 + \|k\|_{r,\Omega} \, \|u\|_{q,\Omega} + \|F\|_{r,\Omega} + \|k\|_{r,\Omega} + \|g\|_{-\frac{1}{q};q,\partial\Omega}\big), \tag{3.11}$$

with $C = C(\Omega, q, r) > 0$, which is rewritten in the form

$$\|\mathcal{F}(u)\|_{q,\Omega} \leq a\|u\|_{q,\Omega}^2 + b\|u\|_{q,\Omega} + c$$

with $a := C$, $b := C\|k\|_{r,\Omega}$, $c := C\big(\|F\|_{r,\Omega} + \|k\|_{r,\Omega} + \|g\|_{-\frac{1}{q};q,\partial\Omega}\big)$. In the same way we obtain that

$$\|\mathcal{F}(u) - \mathcal{F}(v)\|_{q,\Omega} \leq \big(a(\|u\|_{q,\Omega} + \|v\|_{q,\Omega}) + b\big)\|u - v\|_{q,\Omega} \tag{3.12}$$

for $u, v \in L^q(\Omega)$.

Assume that

$$4ac + 2b < 1 \tag{3.13}$$

and consider the closed ball $\mathcal{B} := \{u \in L^q(\Omega);\ \|u\|_{q,\Omega} \leq y_1\}$ where $y_1 = 2c\big(1 - b + \sqrt{1 + b^2 - (4ac + 2b)}\big)^{-1} > 0$ is the smallest root of the equation $y = ay^2 + by + c$. Setting $K = K(\Omega, q, r) := (4C^2 + 3C)^{-1}$ with C from (3.11) we see that (1.12) is sufficient for (3.13) to be satisfied. If $u \in \mathcal{B}$, we obtain that $\|\mathcal{F}(u)\|_{q,\Omega} \leq ay_1^2 + by_1 + c = y_1 \leq 2c$ and that $\mathcal{F}(u) \in \mathcal{B}$. Thus Banach's fixed point principle yields a unique $u \in \mathcal{B}$ with $u = \mathcal{F}(u)$. This u is a very weak solution of (3.7) and therefore also of (1.1). Further we see that $\|u\|_{q,\Omega} \leq y_1 \leq 2c$ which proves (1.13).

This completes the existence proof. The uniqueness of the solution u is a consequence of Theorem 1.5 when we use the estimate (1.13). Note that the constant $K = (4C^2+3C)^{-1}$ with C from (3.11) is only sufficient for the existence; in general, the uniqueness requires another constant. The assertion concerning p easily follows by de Rham's argument. Now Theorem 1.3 is completely proved. ■

Proof of Theorem 1.5 Given very weak solutions $u, v \in L^q(\Omega)$ where u satisfies (1.14) a calculation shows that $w = u - v \in L^q_\sigma(\Omega)$ is a very weak solution of the linear system

$$-\Delta w + \nabla p = \hat{f}, \quad \operatorname{div} w = 0 \text{ in } \Omega, \quad w\big|_{\partial\Omega} = 0,$$

with $\hat{f} = -\operatorname{div}(vw + wu) + kw$. Then the representation formula (3.4) yields the well defined relation

$$w = -A_q^{-1} P_q \operatorname{div}(vw + wu) + A_q^{-1} P_q(kw). \tag{3.14}$$

This equation can be written - first of all formally - also in the form

$$A_q^{\frac{1}{2}} w = -A_q^{-\frac{1}{2}} P_q \operatorname{div}(vw + wu) + A_q^{-\frac{1}{2}} P_q(kw).$$

First let $q > n$. Then we conclude using well known embedding theorems that

$$-A_q^{-\frac{1}{2}} P_q \operatorname{div}(vw + wu) + A_q^{-\frac{1}{2}} P_q(kw) \in L^{q/2}(\Omega). \tag{3.15}$$

Looking at (3.14) a duality argument shows that $w \in D(A_{q/2}^{1/2})$, yielding $w \in L^{q_1}(\Omega)$ where $\frac{1}{n} + \frac{1}{q_1} = \frac{2}{q}$, see (2.11). Since $q > n$ and consequently $q_1 > q$, we may repeat this argument and obtain in a finite of steps that $w \in D(A_2^{1/2})$. Then take in (3.14) the scalar product with $A_2^{1/2} w$, write $vw = uw - ww$ and use that $\langle \operatorname{div}(ww), w \rangle = 0$. Now the smallness assumption (1.14) and an absorption argument show that $\|A_2^{1/2} w\|_2 \leq 0$ yielding $w = 0$ and $u = v$.

If $q = n$ we need an additional smoothing step using Yosida operators $J_m = (I + \frac{1}{m} A_q^{1/2})^{-1}$, $m \in \mathbb{N}$, see [27], p. 298, concerning a similar procedure. Furthermore, we choose C_0^∞-functions k_j, v_j and u_j, $j \in \mathbb{N}$, satisfying $\|k - k_j\|_r \to 0$, and $\|v - v_j\|_n + \|u - u_j\|_n \to 0$ as $j \to \infty$. Then (3.14) will be rewritten, using

$w = J_m w + \frac{1}{m} A_q^{1/2} J_m w$ on the right-hand side, in the form

$$\begin{aligned} A_q^{\frac{1}{2}} J_m w &= -J_m A_q^{-\frac{1}{2}} P_q \operatorname{div}\big((v-v_j)J_m w + (J_m w)(u-u_j)\big) \\ &\quad -\frac{1}{m} J_m A_q^{-\frac{1}{2}} P_q \operatorname{div}\big((v-v_j)A_q^{\frac{1}{2}} J_m w + (A_q^{\frac{1}{2}} J_m w)(u-u_j)\big) \\ &\quad -J_m A_q^{-\frac{1}{2}} P_q \operatorname{div}(v_j w + w u_j) \qquad (3.16) \\ &\quad +J_m A_q^{-\frac{1}{2}} P_q\big((k-k_j)J_m w\big) + \frac{1}{m} J_m A_q^{-\frac{1}{2}} P_q\big((k-k_j)A_q^{\frac{1}{2}} J_m w\big) \\ &\quad +J_m A_q^{-\frac{1}{2}} P_q(k_j w) \\ &=: h_1 + h_2 + h_3 + h_4 + h_5 + h_6; \end{aligned}$$

see [27], V.1.8, p. 298 concerning this smoothing procedure.

Next choose $q_1 > q = n$ and $\alpha \in [0,1]$ such that $\frac{2+\alpha}{n} + \frac{1}{q_1} < 1$ and $\frac{1+\alpha}{n} \geq \frac{1}{r}$. If $n > 3$, then $\alpha = 1$ is possible. In the case $q = n = 3$ and consequently $r > q' = \frac{3}{2}$ we find $\alpha \in [0,1)$ to fulfill both conditions. Given $q_1 > q$ let $\rho > 1$ be defined by $\frac{1}{n} + \frac{1}{q_1} = \frac{1}{\rho}$. Using (2.12), (2.13), and (2.19), h_1 in (3.16) is estimated by

$$\begin{aligned} \|h_1\|_\rho &\leq C_1 \|(v-v_j)J_m w + (J_m w)(u-u_j)\|_\rho \\ &\leq C_2\big(\|v-v_j\|_n + \|u-u_j\|_n\big)\|J_m w\|_{q_1} \\ &\leq C_3\big(\|v-v_j\|_n + \|u-u_j\|_n\big)\|A_\rho^{\frac{1}{2}} J_m w\|_\rho. \end{aligned}$$

Concerning h_2 let $\rho_1 \in (1,n)$ be defined by $\frac{1}{n} + \frac{1}{\rho} = \frac{1}{\rho_1}$. Then by (2.12), (2.13), (2.19),

$$\begin{aligned} \|h_2\|_\rho &\leq C_1 \|A_\rho^{\frac{1}{2}} h_2\|_{\rho_1} \leq C_2 \|(v-v_j)A_q^{\frac{1}{2}} J_m w + (A_q^{\frac{1}{2}} J_m w)(u-u_j)\|_{\rho_1} \\ &\leq C_2\big(\|v-v_j\|_n + \|u-u_j\|_n\big)\|A_\rho^{\frac{1}{2}} J_m w\|_\rho. \end{aligned}$$

Moreover,

$$\|h_3\|_\rho \leq C\|v_j w + w u_j\|_\rho \leq C\big(\|v_j\|_{q_1} + \|u_j\|_{q_1}\big)\|w\|_n.$$

Next, since $r \geq \frac{n}{2}$,

$$\begin{aligned} \|h_4\|_\rho &\leq C_1 \|(k-k_j)J_m w\|_{\rho_1} \leq C_1 \|k-k_j\|_{n/2} \|J_m w\|_{q_1} \\ &\leq C_2 \|k-k_j\|_r \|A_\rho^{\frac{1}{2}} J_m w\|_\rho. \end{aligned}$$

Looking at the estimate of h_2 and (2.13), we get for h_5 with $\rho_2 > 1$ defined by

$\frac{1}{\rho_2} = \frac{\alpha}{n} + \frac{1}{\rho_1}$, that

$$\begin{aligned}
\|h_5\|_\rho &\le C_1 \|A_q^{-\frac{1}{2}} P_q\big((k-k_j)A_q^{\frac{1}{2}} J_m w\big)\|_{\rho_1} \\
&\le C_2 \|A_q^{\frac{\alpha}{2}-\frac{1}{2}}\big(P_q(k-k_j)A_q^{\frac{1}{2}} J_m w\big)\|_{\rho_2} \\
&\le C_3 \|(k-k_j)A_q^{\frac{1}{2}} J_m w\|_{\rho_2} \\
&\le C_3 \|k-k_j\|_{\frac{n}{1+\alpha}} \|A_q^{\frac{1}{2}} J_m w\|_\rho \\
&\le C_4 \|k-k_j\|_r \|A_q^{\frac{1}{2}} J_m w\|_\rho .
\end{aligned}$$

Finally,

$$\|h_6\|_\rho \le C_1 \|k_j w\|_{\rho_1} \le C_1 \|k_j\|_\rho \, \|w\|_n \le C_2 \|k_j\|_{q_1} \, \|w\|_n .$$

Summarizing the L^ρ-estimates of h_j, $1 \le j \le 6$, we get from (3.16) the estimate

$$\begin{aligned}
\|A_\rho^{\frac{1}{2}} J_m w\|_\rho \le\; & C_5\big(\|v-v_j\|_n + \|u-u_j\|_n + \|k-k_j\|_r\big)\|A_\rho^{\frac{1}{2}} J_m w\|_\rho \\
& + C_6\big(\|v_j\|_{q_1} + \|u_j\|_{q_1} + \|k_j\|_{q_1}\big)\|w\|_n
\end{aligned} \tag{3.17}$$

with constants $C_5, C_6 > 0$ independent of $m \in \mathbb{N}$. Now choose $j \in \mathbb{N}$ sufficiently large such that $\|v-v_j\|_n + \|u-u_j\|_n + \|k-k_j\|_r \le 1/(2C_5)$. Hence, for this fixed j and for every $m \in \mathbb{N}$

$$\|A_\rho^{\frac{1}{2}} J_m w\|_\rho \le M := 2C_6\big(\|v_j\|_{q_1} + \|u_j\|_{q_1} + \|k_j\|_{q_1}\big)\|w\|_n .$$

Since the graph of $A_\rho^{1/2}$ is weakly closed and since $J_m w \to w$ in $L^\rho_\sigma(\Omega)$, we conclude that $w \in D(A_\rho^{1/2})$. Hence $w \in L^{q_1}_\sigma(\Omega)$ where $q_1 > n$. The condition $q_1 > n$ was the starting point in the first part of the proof. Thus we may proceed as before to prove that $w = 0$. ∎

Proof of Theorem 1.6 (i) We use the vector-valued version of $E^1(g) \in W^{1,q}(\Omega)$ satisfying $E^1(g)|_{\partial\Omega} = g$ and the solution $b(g) \in W_0^{1,q}(\Omega)$ of the equation $\operatorname{div} b(g) = \operatorname{div}\big(u - E^1(g)\big) = k - \operatorname{div} E^1(g)$, see §2; note that $\int_\Omega \big(k - \operatorname{div} E^1(g)\big)dx = 0$. Setting

$$\hat{u} = u - \hat{E}, \quad \hat{E} = E^1(g) + b(g),$$

we see that $\hat{u}$ is a very weak solution of the linear system

$$-\Delta \hat{u} + \nabla p = \hat{f}, \quad \operatorname{div} \hat{u} = 0 \quad \text{in } \Omega, \quad \hat{u}|_{\partial\Omega} = 0,$$

where $\hat{f} = f + \operatorname{div} \nabla \hat{E} - \operatorname{div}(uu) + ku$. The linear representation formula (3.4) yields

$$\hat{u} = A_q^{-1} P_q \operatorname{div}(F + \nabla \hat{E} - uu) + A_q^{-1} P_q(ku). \tag{3.18}$$

Writing, first of all formally, (3.18) in the form

$$A_q^{\frac{1}{2}}\hat{u} = A_q^{-\frac{1}{2}}P_q \operatorname{div}(F + \nabla\hat{E} - uu) + A_q^{-\frac{1}{2}}P_q(ku),$$

we argue as in the proof of Theorem 1.5. If $q > n$, we obtain in a finite number of steps that $\hat{u} \in D(A_q^{1/2}) \subset W^{1,q}(\Omega)$ and consequently also $u \in W^{1,q}(\Omega)$.

If $q = n$, we use the same smoothing procedure as in the proof of Theorem 1.5. First write (3.18) in the form

$$\hat{u} = A_q^{-1}P_q \operatorname{div}(F + \nabla\hat{E}) - A_q^{-1}P_q \operatorname{div}\big(u(\hat{u} + \hat{E})\big) + A_q^{-1}P_q\big(k(\hat{u} + \hat{E})\big) \quad (3.19)$$

and choose $u_j \in C_0^\infty(\Omega)$, $j \in \mathbb{N}$, satisfying $\|u - u_j\|_n \to \infty$ as $j \to \infty$. Then using the Yosida operators $J_m = (I + \frac{1}{m}A_q^{1/2})^{-1}$ we get from (3.19) that

$$\begin{aligned} A_q^{\frac{1}{2}}J_m\hat{u} &= -J_mA_q^{-\frac{1}{2}}P_q\operatorname{div}\big((u - u_j)J_m\hat{u}\big) \\ &\quad -\tfrac{1}{m}J_mA_q^{-\frac{1}{2}}P_q \operatorname{div}\big((u - u_j)A_q^{\frac{1}{2}}J_m\hat{u}\big) \\ &\quad -J_mA_q^{-\frac{1}{2}}P_q \operatorname{div}(u_j\hat{u}) \\ &\quad +J_mA_q^{-\frac{1}{2}}P_q \operatorname{div}(F + \nabla\hat{E}) - J_mA_q^{-\frac{1}{2}}P_q \operatorname{div}(u\hat{E}) \\ &\quad +J_mA_q^{-\frac{1}{2}}P_q\, k(\hat{u} + \hat{E}) \\ &= h_1 + h_2 + h_3 + h_4 + h_5 + h_6. \end{aligned} \quad (3.20)$$

Choose $q_1 > q = n$ and define $\rho \in (1, n)$ by $\frac{1}{\rho} = \frac{1}{n} + \frac{1}{q_1}$. The functions h_1, h_2 and h_3 are estimated similarly as h_1, h_2, h_3 in the proof of Theorem 1.5; we get that

$$\|h_i\|_\rho \le C_1\|u - u_j\|_n\, \|A_\rho^{\frac{1}{2}}J_m\hat{u}\|_\rho + C_2\|u_j\|_{q_1}\, \|\hat{u}\|_n, \quad i = 1, 2, 3.$$

The last three functions h_i are easily seen to satisfy the estimate

$$\|h_4\|_\rho + \|h_5\|_\rho + \|h_6\|_\rho \le C\big((\|\hat{u}\|_n + \|\hat{E}\|_n)\|k\|_n + \|u\|_n\, \|\hat{E}\|_{W^{1,n}} + \|F + \nabla\hat{E}\|_n\big).$$

Choosing $j \in \mathbb{N}$ sufficiently large, the absorption principle and (3.20) show that

$$\|A_\rho^{\frac{1}{2}}J_m\hat{u}\|_\rho \le M \quad \text{for all } m \in \mathbb{N},$$

where $M = M(\|u_j\|_{q_1}, \|\hat{u}\|_n, \|k\|_n, \|\hat{E}\|_{W^{1,n}}, \|F\|_n) > 0$ is independent of $m \in \mathbb{N}$. Hence $\hat{u} \in D(A_\rho^{1/2}) \subset L^{q_1}(\Omega)$ and also $u \in L^{q_1}(\Omega)$ where $q_1 > q = n$. Now we can use the same argument as for the case $q > n$ to conclude that $u \in W^{1,q}(\Omega)$.

(ii) By part (i) we first obtain that $u \in W^{1,q}(\Omega)$. Then we use the vector-valued version of the extension operator $E^2(g, h_2) \in W^{2,q}(\Omega)$ with a suitably chosen function $h_2 \in W^{1-1/q,q}(\partial\Omega)$ such that $\operatorname{div} E^2(g, h_2)|_{\partial\Omega} = -k|_{\partial\Omega}$. Since $\int_\Omega \big(k - \operatorname{div} E^2(g, h_2)\big)dx = 0$ and $\big(k - \operatorname{div} E^2(g, h_2)\big)|_{\partial\Omega} = 0$, we find a solution

$b \in W_0^{2,q}(\Omega)$ of the equation $\operatorname{div} b = \operatorname{div}\big(u - E^2(g,h_2)\big) = k - \operatorname{div} E^2(g,h_2)$, see §2. Setting $\hat{u} = u - E^2(g,h_2) - b$, we see that $\hat{u}$ is a very weak solution of the linear system

$$-\Delta\hat{u} + \nabla p = \tilde{f}, \quad \operatorname{div}\hat{u} = 0 \quad \text{in } \Omega, \quad \hat{u}\big|_{\partial\Omega} = 0,$$

where $\tilde{f} = f + \Delta E^2(g,h_2) + \Delta b - \operatorname{div}(uu) + ku$.

If $q > n$, standard estimates directly show that $\operatorname{div}(uu) - ku = u\cdot\nabla u \in L^q(\Omega)$. Hence the solution $\hat{u}$ has the representation

$$\hat{u} = A_s^{-1}P_s f + A_q^{-1}P_q\big(\Delta E^2(g,h_2) + \Delta b\big) - A_q^{-1}P_q\big(\operatorname{div}(uu) - ku\big) \tag{3.21}$$

yielding $\hat{u} \in D(A_s) + D(A_q)$ and consequently $u \in D(A_s) + W^{2,q}(\Omega)$. Next, if $q = n$ and $s > n/2$, we find some $F^* \in L^{q^*}(\Omega)$ with $f = \operatorname{div} F^*$, $q^* > n$, see Remark 1.7, (i); the exponent $q^* > n$ can be chosen such that $k \in L^{q^*}$, $g \in W^{1-1/q^*,q^*}(\Omega)$. By part (i) we get $u \in W^{1,q^*}(\Omega)$. Now we conclude that $u \cdot \nabla u \in L^q(\Omega)$ which leads to $\hat{u} \in D(A_s) + W^{2,q}(\Omega)$ as in the case $q > n$. Finally, in the limit case $q = n$ and $s = n/2$ yielding $u \cdot \nabla u \in L^{q_1}(\Omega)$ for every $1 < q_1 < n$, (3.21) holds with the last term replaced by $A_{q_1}^{-1}P_{q_1}\big(\operatorname{div}(uu) - ku\big)$. Choosing $s < q_1 < n$ we get that $\hat{u} \in D(A_s) + D(A_{q_1}) \subset D(A_s)$. This completes the proof. ■

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Reinhard Farwig
Fachbereich Mathematik
Technische Universität Darmstadt
D-64283 Darmstadt
farwig@mathematik.tu-darmstadt.de

Giovanni Paolo Galdi
Department of Mechanical Engineering
University of Pittsburgh
15261 Pittsburgh, USA
galdi@engrng.pitt.edu

Hermann Sohr
Fakultät für Elektrotechnik, Informatik und Mathematik
Universität Paderborn Paderborn
D-33098 Paderborn
hsohr@math.uni-paderborn.de