Assertion Games
to Justify Classical Reasoning

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Abstract. To establish and justify general methods of linguistic reasoning, we introduce a use of sentences by means of 'assertion rules' which partially have a narrow relationship to the proceeding in [7: 1., 2.] and help us to demonstrate that one can rely on certain logical inference rules (§1, §2). As assertion rules we choose only such rules which can be formulated with words which are as unambiguous as possible. Therefore, we at first introduce the particles $\wedge, \vee, \exists$, and $\neg$ only. Let the resulting use of sentences be said to be the 'primary' use. However, we cannot define a subjunction (material implication) which indicates that, according to this use, one may conclude the succedent from the antecedent.

Therefore, we subsequently liberalize the primary use of sentences in §3. This liberalization establishes a 'classical' use which permits to apply classical logic, and can be justified by the following (and some other) facts: An elementary sentence may be asserted classically iff (i.e. if and only if) it may be asserted primarily. A sentence of the form $\forall x\,[A(x) \rightarrow B(x)]$ (which is defined suitably) expresses that, for any value r of the variable x, if $A(r)$ may be asserted classically, then $B(r)$ may *at once* (and generally later on) be asserted so. For this 'inferential purpose', the classical use of sentences of that form is also not unnecessarily restricted. If we replace the primary use by the classical use, only dispensable means of speech get lost. (Details will be discussed in §6.)

In §4 we especially deal with the concept of infinity on the example of the set $\mathbb{N}$ of natural numbers. The infinity of $\mathbb{N}$ is considered as a 'deontic' one. This means that we shall never be *obliged* to terminate the construction of natural numbers. So we avoid the ontological assumption concerning the infinity of $\mathbb{N}$.

In §5 we investigate a use of sentences which include indicators (as "this ant", e.g.) or objectual variables. This use depends on situations. To eliminate this dependency we introduce objectual quantification.

In §7 - §10 we deal with a ramified type theory in a cumulative version: In §7 and §8 we introduce an extension of a union of higher order languages by means of variables $x, y, \ldots$ for constants of arbitrary order, and variables for tuples $(c_1, \ldots, c_j)$ of arbitrary order and arbitrary length $j \in \mathbb{N}^+$. So we may simply identify types with orders. In that language, 'type-free' equations $x = y$ are definable. We extend the primary and the classical use to that language, show that their sentences are non-circular, and that their formulas are invariant under $(=)$.

In §9 we deal with singular description terms.

In §10 we even introduce higher order languages which also contain formulas with indicators and objectual variables, and enable a quantification which combines both substitutional and objectual quantification. We show that we may commute any consecutive existential quantifiers that occur in formulas of those expanded languages.

Contents.

§0. Introduction

By an **assertion** we understand an act: One asserts a proposition in general simply by pronouncing it as a complete sentence to a listener or writing it for a reader. Statements we also include among assertions. However, there exist linguistic acts in the shape of assertions which are not to be understood as assertions or are not meant seriously. (Examples are fictitious, fictional or jocular expressions.) This can be said additionally or can result from the context. Nevertheless, we can in general decide whether one has - or ourselves have - asserted a particular sentence because assertions which are unjustified (as lies, e.g.) or have not been accepted or believed are assertions as well (or are said to be assertions here).

Let us raise the question how it is possible to understand assertions. To this we consider the sentence: "Paul had a temperature of 39.2° C yesterday evening." A listener can understand this in so far as he is used to the rule by which this sentence should be asserted only after measuring temperature and getting the corresponding result. Such examples lead us to the

Thesis: Sentences become understandable because it is usual or agreed to restrict assertions in a regular way, and so to omit asserting particular sentences finally or temporarily. (This explanation requires, of course, some completion.)

The underlying standards of assertion are generally only tacitly valid. To obtain means of linguistic reasoning (especially inference rules) which can be shown to be serviceable and reliable we shall contrive a use of sentences by fixing explicit 'assertion rules' in addition to certain conventional standards. However, we do *not* intend to describe or explain how fluent speakers argue *factually*. Instead, we intend to establish and justify an argumentation technique which is efficient, uncomplicated, clearly arranged, and easy to use.

As assertion rules we shall choose only such rules that can be formulated with words which are as unambiguous as possible - so that we can check in as many cases as possible whether we have broken such a rule. However, it would not be sufficient for the mentioned purpose to take only 'formal' rules like such of a calculus since it follows from a result of GÖDEL that there does not exist a calculus, K, such that any first order arithmetic sentence is assertible iff it is deducible by the rules of K.

Recourse to prohibition rules: Which kind of rules are particularly appropriate to stipulate a use of assertions? General positive commands of the form "Whenever a happens, then do b !" have the disadvantage that one mostly has no opportunity to act upon them. An example for this fact is given by the following rule: "Whenever two sentences A, B have been asserted, then assert $A \wedge B$ too." Since we would have to speak without end by this rule, it suggests itself to replace it by an analogous permission. That an act is permitted means that it is not forbidden or that it must not be forbidden. To forbid an act means to ask or demand, not to do it. Note that the meaning of prohibitions can be demonstrated by means of punishment or blame, for instance. Therefore and with regard to the above mentioned thesis we shall take prohibition rules as assertion rules.

A calculus, e.g., is a system K of rules that *allow* to successively perform certain schematic operations on strings of symbols. This means, however, that in the context of K such an operation is *forbidden* unless it is explicitly permitted by the rules of K.

Material inference rules and means for their formulations: Requisite for reasoning are inference rules by which, for particular formulas $A_1(x)$, $A_2(x)$, and $B(x)$, e.g., we may conclude $B(r)$ from $A_1(r)$ and $A_2(r)$, for any value r of the variable x. To indicate that one may conclude so we intend to write

$$\forall x\, [A_1(x) \wedge A_2(x) \rightarrow B(x)]$$

(cf. [12, p.104]). To this end the assertion of this sentence should be restricted to the condition that, for all values r of x, if $A_1(r)$ and $A_2(r)$ may be asserted, then $B(r)$ may be asserted at once. But our understanding of this condition is particularly problematic, because it contains the words "for all" and "if - then". So we shall define sentences of the forms $A \rightarrow B$ and $\forall x\, A(x)$ by means of $\wedge$ (and), $\neg$ (not), and $\exists$ (for some). However, those sentences will serve their 'inferential purposes' only after a subsequent liberalization (§3) of the initially introduced use of that language.

Remarks on some well-known approaches to logic.

(1) An intuitionistic approach starts from a concept of proof which can inductively be defined (cf. [1], [4]). We formulate that for subjunction only:

> A proof of $A \rightarrow B$ is a 'construction' c that converts each proof of A into a proof of B.

That c is a proof of $A \to B$ means in detail that, for all pertinent p, if p is a proof of A, then $c(p)$ is a proof of B. Especially for $0 = 1$ in place of B, this means that there does not exist a proof of A. In these explanations, however, we have already used the words "for all", "if - then", and "not" of everyday speech. Besides, (for compound A) it is in general not decidable whether a pertinent construction p is a proof of A. For this reason, in [5] c as above has been replaced by a pair (c, d) where d is a 'demonstration' which shows that c converts each proof of A into a proof of B. However, this concept of proof - which is critically discussed in [1, p. 232] and [14] - is rather intricate and yet somewhat problematic. - The following approach avoids definitions which are circular or seriously involved in an infinite regressus:

(2) An approach to logic by dialogues (see [9, pp. 60ff.]). Let us consider material dialogue games as in [9, pp. 75 - 83]. To show their suitability we need especially the following theorem by which the 'modus ponens' may be applied: If there exist strategies to win dialogues for A as well as for $A \to B$, then there also exists a strategy to win dialogues for B. This theorem is a composite proposition of a metalanguage. Even the attempt to dialogically interprete it meets with difficulties. Moreover, to formulate a proof of that theorem - or of a more general so called *Cut Theorem* - we need sentences of the metalanguage in which the connective "if - then" and the quantifier "for all" occur iteratively. (Note also that the mentioned strategies are meant to be strategies to win dialogues against *any* opponents.) In the proof of the Cut Theorem, the comprehension of those particles succeed by linguistic habits and on the context. However, some ways of reasoning applied in that proof are just to be justified for the object language by means of the Cut Theorem.

In [6], K. LORENZ has taken other frame rules for dialogues as a basis, and has used constructive ordinal numbers. For the pertinent investigations one must have seen before that those alleged ordinals can in fact be used as ordinals, i.e. enable transfinite induction. - In **Proof Theory** (cf. [13] or [16], e.g.) occur analogous problems.

The problem of beginning

As we have partially seen, the considered approaches to logic are involved in different kinds of circularity or regressus. However, there is a more general problem: How can one be initially entitled to any reasoning at all (cf. [2], [15])? For a justification of rules for arguing, or for a proof that they are reliable we already need reliable argumentations.

To reduce this problem, we do not only stipulate certain rules to restrict assertions; we also agree that those assertions must not be restricted by further rules. Then we can see that the stipulated rules for compound sentences may be 'inverted' (cf. the 'Inversionsprinzip' in [7, §4]). So the latter rules and their inverses may also be used as inference rules. - Since our colloquial language can partially be understood to be ruled in that way, we may also perform particular colloquial reasonings by applying the mentioned rules. (Later we shall see that also other rules may be applied.)

If we start any metatheoretical reasoning with sentences which are said to be *assumed*, *presupposed*, or the like, we shall treat them like asserted sentences - so that the conclusions which we draw from them may be asserted as soon as those assumptions have rightly been asserted. In this way we understand colloquial conditionals.

We shall not presuppose that there ('actually' or 'potentially') exist infinitely many objects like (mental or other) constructions, proofs, or numbers, for instance.

§1. An assertion game

Elementary sentences and their use

We presuppose that we already dispose of concepts of certain '**elementary sentences**', of '**internal rules**' (of assertion concerning elementary sentences), and of '**external rules**'. (Here we include usage among rules.) For our purposes it suffices to formulate some claims on these concepts and some explanations.

Let elementary (or 'atomic') sentences be not as usual composed of other formulas (i.e. sentences or sentence forms) by means of connectives, quantifiers, or set theoretical particles. - In the following we write $E, E_1, \ldots$ for elementary sentences.

For numerous elementary sentences, E, we have learned in practice that E may be asserted only after we have made a particular perception or observation or have got a special result of an act as a measuring, for instance. (Details are beyond the scope of our topic). Let at least such rules of assertion (which are valid in our community) be **external**. We say that E has been **anchored** to mean that the present assertion of E would not violate an external rule.

Several elementary sentences can also depend on each other by rules concerning assertions of those sentences. Examples are the following rules by which one may assert "(All) beetles are insects" and "Beetles are no flies": If the sentence "This is a beetle" may be asserted, then also the sentence "This is an insect" may be asserted, but "This is a fly" must not be asserted. (See also the 'Prädikatorenregeln' in [9, p. 182].) Thus sentences as "Beetles are insects" and "Beetles are no flies" are based on our common linguistic usage (see [3]). [Here we need not deal with the question whether they can also be justified by (fictitious) definitions as "beetle $\rightleftharpoons$ insect and winged and with wing-cases and ...".]

Rules which are valid in our community and by which elementary sentences depend on each other as in the above examples can be considered as particular **internal**

rules. As an internal rule we also take the prohibition to assert a particular elementary sentence, $\bot$. Let us say that E has been **rejected** to mean that the assertion of E would (generally together with already accomplished assertions of other elementary sentences) violate an internal rule. - As internal rules for elementary sentences we at first admit - apart from the mentioned prohibition to assert $\bot$ - only rules with individual cases by which, for certain elementary sentences $E_1, \ldots, E_n$, and E, it is forbidden to do both, to assert all of the sentences $E_1, \ldots, E_n$ and to reject E. For such an individual internal rule we simply write

(int) $$E_1, \ldots, E_n \ \Rightarrow \ E.$$

Commentary: From (int) the following results:
If $E_1, \ldots, E_n$ have been asserted, it is forbidden to reject E.
But if E has been rejected, it is forbidden to assert all of the sentences $E_1, \ldots, E_n$.
If, moreover, $E_2, \ldots, E_n$ have been asserted, then E_1 is rejected.

In §5 we shall also admit other similar internal rules for elementary sentences.

Example 1: Let a and b range over signs for length (as '25 cm', e.g.). Let $L(s, a)$ mean that a stick s has the length a, and suppose that we have the internal rules

$$\begin{aligned} a = b,\ a \neq b \ &\Rightarrow \ \bot \\ L(s,a),\ L(s,b) \ &\Rightarrow \ a = b. \end{aligned}$$

$\bot$ is agreed to be rejected. So if we assert $a \neq b$, then $a = b$ becomes rejected. So if we also assert $L(s, a)$, then $L(s, b)$ becomes rejected.

By the latter rule, we may assert only one result of a measurement of s. This is not suitable for a branch s which can grow or for a stick which can change its length otherwise. In this case, a formulation as "s has the length a at t" (where t denotes a moment) may be more adequate. Nevertheless, in certain cases only *experience* can show (in general without giving final certainty, however) whether or how far a linguistic rule can be useful.

Now we consider the internal rules of the form (int) as a calculus which operates on elementary sentences where '$\Rightarrow$' indicates the permitted deduction steps. Let us agree that if E is deducible by those rules from other elementary sentences which have already been anchored and asserted, then also E passes for anchored. (Let this agreement be external.)

Notes: 1. An internal rule $\Rightarrow E$ without premises can be inconsistent with other internal rules. In this case it should not be accepted.
2. It is not totally impossible that an elementary sentence becomes both, anchored and rejected. (Therefore we do not use the word "verified" for "anchored".)
3. There is the danger that forbidden assertions will be be performed, since different persons can assert elementary sentences without knowing which sentences are asserted by the other persons. So it may especially occur that the assertions of every single of those persons do not violate an internal rule, if we disregard the assertions of

the other persons, but the assertions of all of those persons violate together internal rules. If such a violation becomes public, we should partially cancel our previous assertions and come to a corresponding agreement.

A material first order language $\mathcal{L}$

In the following we only deal with sentences which do not depend by rules of assertion (i.e. exclusive of rules of politeness or regard, e.g) on particular linguistic contexts or situations (or on the involved persons, e.g.) so that those sentences may be asserted repeatedly if they may be asserted at some time. In §5, however, we shall also deal with other sentences as "*This* is a beetle" which may only be asserted in particular situations (as while showing a particular animal).

At first we only consider formulas that are elementary formulas (of a certain class) or are composed of them by means of $\wedge$ (and), $\vee$ (or), $\neg$ (not), and $\exists$ (for some) as usual. Let the class of those formulas be denoted by $\mathcal{L}$. We assume here that the concepts of variables (of $\mathcal{L}$) and of values of variables have already been introduced. Those values are supposed to be constants, i.e. certain strings of symbols in which no variables occur. For short we say

"formula"	for	"formula of $\mathcal{L}$",
"sentence"	for	"sentence of $\mathcal{L}$".

(Sentences are formulas in which all occurrences of variables are bound.)

In more detail, we presuppose the following: For any string x of symbols we know how to decide whether x is a variable. For any variable x and any string c of symbols we know how to decide whether c is a value of x. Any two occurrences of variables in any string of symbols do not overlap. - Elementary formulas are not as usual composed of other formulas by means of logical or set theoretical particles. Elementary formulas include outer brackets (which, however, we generally omit). Further brackets occur only pairwise as usual in elementary formulas. From an elementary formula there results an elementary sentence if we replace the occurring variables by arbitrary values of them.

If F is a formula and $x_1, \ldots, x_n$ $(n \geq 1)$ are distinct variables, then let also

$$\exists\, x_1, \ldots, x_n\, F$$

be a formula of $\mathcal{L}$ (in which $x_1, \ldots, x_n$ occur bound). - As 'metavariables' we use

F, G, H	for	formulas
A, B, C	for	sentences
x, y, z	for	variables
$\underline{x}$	for	lists $x_1, \ldots, x_n$ of distinct variables
$\underline{r}$	for	lists $r_1, \ldots, r_m$ of constants
$A\underline{x}$ or $A(\underline{x})$	for	formulas in which at most the variables $\underline{x}$ occur free.

Definition: $\underline{r}$ is a **value** of $\underline{x}$ iff $\underline{r}$ has as many members as $\underline{x}$ (i.e. $m = n$) and r_i is a value of x_i for $i = 1, \ldots, n$. In this case, by $A\underline{r}$ we denote the sentence which is obtained from a given formula $A\underline{x}$ by substituting $\underline{r}$ for $\underline{x}$, i.e. r_i for each free occurrence of x_i $(i = 1, \ldots, n)$. - We write

$$\natural A \text{ for: to assert } A.$$

Now, (int) can also more detailed be written as: $\natural E_1, \ldots, \natural E_n \;\Rightarrow\; \natural E$.

Though we have partially formalized the considered language $\mathcal{L}$ by using symbols as logical particles, $\mathcal{L}$ is a material (assertoric) language, not merely a formal language. Since we shall introduce a use of sentences such that some but not all sentences may be asserted, we need not additionally give interpretations which assign meanings to sentences (in a realistic, mentalistic, or other manner). Nevertheless, some relation to 'reality' is given by external rules. - In §4, §5, §7, and §8 we shall also consider some expansions of $\mathcal{L}$.

The primary game

Leading idea: We should generally assert a sentence only if we know arguments for it, which we could, therefore, also assert to ourselves in our mind before asserting that sentence. Accordingly, we choose rules of assertion by which an assertion is forbidden until a certain condition is satisfied. Thus, that a sentence *may* be asserted means that its assertion would no longer violate an appertaining rule.

Given certain external rules as well as internal rules for elementary sentences (as above), the **primary game** is defined to be the 'assertion game' with those rules and the below quoted rules for compound sentences. Also these rules, which are conditional, possibly temporary prohibitions of assertions, are said to be **internal**. The first of them is to be read as: Assert $(A \wedge B)$ only after A has been asserted and B has been asserted. (Accordingly, let "$:\Rightarrow$" be short for "only after".)

P(∧)	$\natural (A \wedge B)$	$:\Rightarrow$	$\natural A$ and $\natural B$
P(∨)	$\natural (A \vee B)$	$:\Rightarrow$	$\natural A$ or $\natural B$ (or both)
P(∃)	$\natural \exists \underline{x}\, A\underline{x}$	$:\Rightarrow$	for some value $\underline{r}$ of $\underline{x}$: $\natural A\underline{r}$
P(¬)	$\natural \neg A$	$:\Rightarrow$	A has been rejected,

where the latter condition means that, by the internal rules, A must not (or no longer) be asserted. (This does in general require that certain elementary sentences have been asserted. Note also that if we assert an elementary sentence which has not yet been rejected, then it should also not be rejected later on. - A more detailed explanation of "rejected" will be given below.)

Let this game not contain other rules of assertion. The rules of the primary game are said to be **primary rules**. - The rule P(∧) can also be substituted by two rules.

The internal rules are formulated by means of our colloquial language, which we have learned exemplarily. This is not problematic for the rules P($\wedge$), P($\vee$), and P($\exists$) since we know how to decide at any time, and for any sentence of the form $A \wedge B$, $A \vee B$, or $\exists \underline{x}\, A\underline{x}$, whether we know that the present assertion of it would not violate the pertinent primary rule. (This is a reason for which we have chosen those rules.) - To explain P($\neg$) we at first give some examples:

Example 2: For sentences A and B which have been asserted according to the primary rules, and for any sentence C we may successively also assert $A \wedge B$, and $(A \wedge B) \vee C$ since this would not violate the corresponding primary rules. Therefore, $\neg\, [(A \wedge B) \vee C]$ must not be asserted.

Example 3: $\exists x\, (Ax \wedge \neg Ax)$ is rejected, since before asserting this sentence we should have asserted $Ar \wedge \neg Ar$ for some value r of x, and hence also Ar as well as $\neg Ar$, for which, however, Ar should also have been rejected. Accordingly, $\neg \exists x\, (Ax \wedge \neg Ax)$ may be asserted.

Example 4: Let s denote a stick with a length of *approximately* 25 cm, and let E now be short for "s is 24.8 cm in length", which can only be anchored by a measurement of s with the result 24.8 cm. Suppose that E can be rejected only by asserting another result of a measurement of s. Let us assume, however, that s has been burnt before measuring. Then E can neither be anchored nor rejected. So we must not assert E, neither $\neg E$.

This example shows that if E is an elementary sentence, the assertion of $\neg E$ does in general not become permitted by a mere hindrance to anchor E.

By the rule P($\neg$), the assertability of $\neg A$ is restricted to the condition that A is rejected, which means, in more detail, that performing any series of assertions which ends with that of A and satisfies P($\wedge$), P($\vee$), and P($\exists$) would (generally together with already accomplished assertions of further elementary sentences) violate an internal rule.

If we speak so about *any* (all) assertion series of such a kind we do not only mean assertion series which will *really* individually be performed, imagined, or considered. We ignore want of time and opportunity to perform, imagine, or consider assertions. - However, the condition that a sentence A is rejected is not in any case decidable. This fact corresponds to a theorem of GÖDEL by which there does not exist an effective procedure by which one can, for any first order arithmetical sentence A, decide whether A may be asserted. (This theorem is also relevant to other approaches to logic.)

As in the above Examples 2 and 3, the primary rules for compound sentences can be inverted for the following reasons: The primary game does not contain other rules to restrict assertions of those sentences, and the use of every compound sentence is determined non-circularly since it only depends on the use of its predecessors in the following sense.

Definition: C is said to be a **predecessor** of D iff C can be deduced *from* D by at least one application of the following rules (where '$\Rightarrow$' indicates the deduction steps):

$$\begin{array}{rclrcl} A \wedge B & \Rightarrow & A; & A \wedge B & \Rightarrow & B; \\ A \vee B & \Rightarrow & A; & A \vee B & \Rightarrow & B; \\ \neg A & \Rightarrow & A; & \exists \underline{x}\, A\underline{x} & \Rightarrow & A\underline{r} \quad \text{(for values } \underline{r} \text{ of } x). \end{array}$$

(Thus, A and B are the 'immediate predecessors' of $A \wedge B$ and of $A \vee B$, etc.)

The mentioned non-circularity means that no sentence is a predecessor of itself. Though this proposition belongs to a metalanguage, it can be understood as in the primary game. Sentences of $\mathcal{L}$ or of a similar metalanguage which have a sufficiently small complexity are obviously non-circular. Moreover, as generally known, the non-circularity of all sentences of $\mathcal{L}$ can be proved by induction on their complexity. However, we have not yet established that method of proof. So we stipulate for the present that by a sentence is to be understood a non-circular sentence only. - Now we consider the following '**inverses**' of internal rules:

$$\begin{array}{lrcl} \mathrm{I}(\wedge) & A, B & \Rightarrow & A \wedge B \\ \mathrm{I}(\vee) & A & \Rightarrow & A \vee B \\ & B & \Rightarrow & A \vee B \\ \mathrm{I}(\exists) & A\underline{r} & \Rightarrow & \exists \underline{x}\, A\underline{x} \quad \text{(for values } \underline{r} \text{ of } \underline{x}). \end{array}$$

These rules have the following property: After asserting the premises (on the left) of an individual case of an inverse rule the assertion of its conclusion (on the right) would not violate a primary rule. According to this property, in certain cases we may successively assert several sentences in a proper order (see Example 2 above).

This shows a narrow relationship of our introduction of compound sentences to that given in [7] (see specially [7: §4, §7]), which starts with rules as I($\wedge$), I($\vee$), and I($\exists$) as permission rules. The latter approach seems to have the advantage that colloquial phrases as "or" and "for some" do not occur in its rules. However, it must be supplied by the agreement that one may assert a sentence (which is neither elementary nor a negation) only if *there exists* a deduction of it from asserted elementary sentences or negations by the indicated rules.

Also the rule P($\neg$) can be inverted: If A has been rejected, then $\neg A$ may be asserted (cf. Example 3). However, the following example shows that a rule to restrict assertions can generally be inverted for *non-circular* sentences only: Suppose that an extension of the language $\mathcal{L}$ contains a particular sentence A_0 (as $\{x : x \notin x\} \in \{x : x \notin x\}$, e.g.) whose assertion is restricted by the rule: $\natural\, A_0 \;:\Rightarrow\; \natural\, \neg A_0$. Then this rule cannot be inverted. Note that A_0 is a predecessor of itself.

A corresponding assertion rule for universal sentences would be the following:

$$\natural\, \forall \underline{x}\, A\underline{x} \;\; :\Rightarrow \;\; \text{for all values } \underline{r} \text{ of } \underline{x} : \natural\, A\underline{r}.$$

This would not be useful if $\underline{x}$ has infinitely many values. So we define instead

$$\forall \underline{x}\, F \;\; \rightleftharpoons \;\; \neg \exists \underline{x}\, \neg F.$$

Moreover, we define a 'subjunction' by

$$F \to G \;\; \rightleftharpoons \;\; \neg\,(F \wedge \neg G).$$

However, a sentence $\forall x\,(Ax \to Bx)$ defined so does in general not yet express that, for any value r of x, one may conclude Br from Ar (cf. §0). To this end we shall liberalize the primary game in §3. We shall also return to this point in §6.

In the primary game, *definitions* can be employed by the following additional rules of assertion:

$$\natural\, E(\tilde{r}_1, \ldots, \tilde{r}_n) \;\; :\Rightarrow \;\; \natural\, E(r_1, \ldots, r_n)$$

if $\tilde{r}_1 \rightleftharpoons r_1, \;\ldots,\; \tilde{r}_n \rightleftharpoons r_n$ are definitions of 'new' constants $\tilde{r}_1, \ldots, \tilde{r}_n$, and if no other constants introduced by definitions occur in $E(\tilde{r}_1, \ldots, \tilde{r}_n)$.

$$\natural\, \tilde{A} \;\; :\Rightarrow \;\; \natural\, A$$

if $\tilde{A} \rightleftharpoons A$ is a definition of a 'new' sentence $\tilde{A}$.
However, we should only use such definitions for which these rules may also be inverted. - Rules belonging to iterated definitions can be applied successively.

§2. Admissibility of inference rules

Now we deal with further rules which may be applied. For the present, we restrict our investigations to inference rules of the form

$$\mathcal{R}: \qquad \mathcal{A}_1, \ldots, \mathcal{A}_n \;\; \Rightarrow \;\; \neg \mathcal{B}$$

$(n = 0, 1, 2)$ with sentence schemes $\mathcal{A}_i$ and $\mathcal{B}$ in which metavariables (for sentences, formulas, variables, constants, or proper terms) may occur - and from which sentences are obtained by replacing those metavariables with arbitrary values of them. We say that an **individual case** of $\mathcal{R}$ results from $\mathcal{R}$ by such a substitution.

Here we consider only rules of the special form $\mathcal{R}$ with a *negation* on the right for the following two reasons: 1. For such rules the following concept of admissibility is definable compactly. 2. A rule such as $A, B \Rightarrow (A \wedge B) \vee C$ must not immediately be 'applied', since if we want to assert its conclusion $(A \wedge B) \vee C$ we should previously assert not only its premises A, B but also $A \wedge B$. - However, in §3 we shall introduce a concept of 'classical admissibility' for rules of the general form $\mathcal{A}_1, \ldots, \mathcal{A}_n \Rightarrow \mathcal{B}$.

That an inference rule may be applied means that for every individual case of it whose premises have been asserted without violating a primary rule, also its conclusion may be asserted at once (and later). The following admissibility is a somewhat stronger condition (cf. the Lemma 1 below):

Definition: $\mathcal{R}$ (as above) is said to be **admissible** iff, by the internal rules, it is forbidden for any individual case $A_1, \ldots, A_n \Rightarrow \neg B$ of $\mathcal{R}$ to assert all of the sentences $A_1, \ldots, A_n$, and B.

This condition belongs to a metalanguage and can be formalized by

$$\neg\exists\, ..\, (\mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_n \wedge \mathcal{B})$$

where '..' indicates a list of all metavariables occurring behind. Accordingly, the formalization of the admissibility of an individual rule $A_1, \ldots, A_n \Rightarrow \neg B$ is simply

$$\neg\,(A_1 \wedge \ldots \wedge A_n \wedge B).$$

This shows that the just mentioned metalanguage is an expansion of the object language $\mathcal{L}$. - We have just used abbreviations such as

$$A_1 \wedge A_2 \wedge B \;\rightleftharpoons\; (A_1 \wedge A_2) \wedge B.$$

To show that all admissible rules of the above form $\mathcal{R}$ may be applied we have to analyse the concept of admissibility and the more basic concept of rejection. To this we employ the particles $\underline{\wedge}, \underline{\vee}, \underline{\rightarrow}, \underline{\leftrightarrow}, \underline{\exists}$, and $\underline{\forall}$ in a metalanguage. The particles $\underline{\wedge}, \underline{\vee}$, and $\underline{\exists}$ are to be used in the same way as $\wedge, \vee$, and $\exists$, respectively. However, for $\underline{\forall}$ and $\underline{\rightarrow}$ we fix the following rules (where t, u range over moments):

P($\underline{\forall}$): Assert $\underline{\forall} X\, \mathcal{P}(X)$ at t only if, for all values C of X, we may successively assert certain sentences inclusive of $\mathcal{P}(C)$ (or $\mathcal{P}(C)$ alone) immediately after t.

P($\underline{\rightarrow}$): Assert $\mathcal{P} \underline{\rightarrow} \mathcal{Q}$ at t only if, in case that $\mathcal{P}$ may be asserted at u, then immediately after t and u we may successively assert certain sentences inclusive of $\mathcal{Q}$ (or $\mathcal{Q}$ alone).

These rules serve the 'inferential purpose' mentioned in the abstract and in §0. The occurring condition that $\mathcal{P}$ *may* be asserted means that it is not forbidden by the pertinent rules to assert $\mathcal{P}$. This condition is, of course, generally not decidable. However, we apply these rules only to sentences of 'small complexity'. By this means we try to reduce the problem of beginning partially.

Definitions: Let α, β range over (finite or empty) lists of sentences of $\mathcal{L}$.
Let α_e denote a list of all members of α that are elementary sentences.
Let $\alpha_\neg$ denote a list of all members of α that are negations.

Let $\alpha\colon A_1,\ldots,A_n$ mean that α ends with $A_1,\ldots,A_n$ and the sequence of assertions of all members of α (in the succession of α) would satisfy P($\wedge$), P($\vee$), and P($\exists$). But generally we need not distinguish lists from finite sets. So we also use the set theoretical particles $\in,\subseteq$ and $=$, and we write $\alpha\beta$ for the union of α and β.
Let $!\sim\natural\,\alpha_e\alpha_\neg$ mean that, by internal rules, we must not assert all members of $\alpha_e\alpha_\neg$, i.e. the assertions of those members would altogether (with or without further assertions) violate some internal rules.
Let Γ,Δ range over lists of *elementary* sentences of $\mathcal{L}$.
If Γ is a list of all elementary sentences that have been asserted to a moment t, then, by the following definitions, $\mathrm{Rej}_\Gamma A$ means that A is rejected at t (and later), and $\Gamma\models\alpha_\neg$ means that all members of $\alpha_\neg$ may be asserted at t (and later):

$$\begin{aligned}\mathrm{Rej}_\Gamma A &\;\rightleftharpoons\; \underline{\forall}\,\alpha\colon A.\,!\sim\natural\,\Gamma\alpha_e\alpha_\neg;\\ \Gamma\models\alpha_\neg &\;\rightleftharpoons\; \underline{\forall}\,(\neg C)\in\alpha_\neg.\,\mathrm{Rej}_\Gamma C.\end{aligned}$$

For elementary sentences we especially have: $\mathrm{Rej}_\Gamma E \;\underline{\leftrightarrow}\; !\sim\natural\,\Gamma E$.
By **(a) - (d)** we denote several remarks, lemmata, or the like.

(a)

$$\begin{aligned}\mathrm{Rej}_\Gamma A \;\underline{\wedge}\; \Gamma\subseteq\Delta &\;\underline{\rightarrow}\; \mathrm{Rej}_\Delta A;\\ \Gamma\models\alpha_\neg \;\underline{\wedge}\; \Gamma\subseteq\Delta &\;\underline{\rightarrow}\; \Delta\models\alpha_\neg.\end{aligned}$$

Proof: Let $\mathrm{Rej}_\Gamma A$ and $\Gamma\subseteq\Delta$. If, moreover, $\alpha : A$, then $!\sim\natural\,\Gamma\alpha_e\alpha_\neg$ and hence $!\sim\natural\,\Delta\alpha_e\alpha_\neg$. So $\mathrm{Rej}_\Delta A$. - Let $\Gamma\models\alpha_\neg$ and $\Gamma\subseteq\Delta$. If, moreover, $(\neg C)\in\alpha_\neg$, then $\mathrm{Rej}_\Gamma C$ and hence $\mathrm{Rej}_\Delta C$. So $\Delta\models\alpha_\neg$. □

(b) $\quad !\sim\natural\,\Gamma\alpha_\neg\beta_\neg \;\underline{\wedge}\; \Gamma\models\alpha_\neg \;\underline{\rightarrow}\; !\sim\natural\,\Gamma\beta_\neg \quad$ (where $\beta_\neg$ may also be empty).

Demonstration: Let $!\sim\natural\,\Gamma\alpha_\neg\beta_\neg$ and $\Gamma\models\alpha_\neg$. Suppose that we assert (all members of) $\Gamma\beta_\neg$, and that we also assert $\alpha_\neg$ after Γ. Then these assertions violate together certain internal rules, but the particular rules for the members of $\alpha_\neg$ are not violated. So those violated rules are violated by the assertions of $\Gamma\beta_\neg$ alone. But this result does not depend on the additional supposition that we assert $\alpha_\neg$ after Γ. So we must not assert $\Gamma\beta_\neg$. □

(c) Every individual case of an admissible inference rule is also admissible.

Proof: Let Γ be all elementary sentences asserted to t. Let $A_1,A_2\Rightarrow\neg B$, e.g., be an individual case of a rule, $\mathcal{A}_1,\mathcal{A}_2\Rightarrow\neg\mathcal{B}$, that is admissible at t, i.e. $\mathrm{Rej}_\Gamma\exists..(\mathcal{A}_1\wedge\mathcal{A}_2\wedge\mathcal{B})$. Then for all $\alpha\colon A_1,A_2$ and all $\beta\colon B$, the assertions of all the sentences α,β, $(A_1\wedge A_2),(A_1\wedge A_2\wedge B),\exists..(\mathcal{A}_1\wedge\mathcal{A}_2\wedge\mathcal{B})$ would not violate P($\wedge$), P($\vee$) or P($\exists$). So we have $!\sim\natural\,\Gamma\alpha_e\beta_e\alpha_\neg\beta_\neg$. So $A_1,A_2\Rightarrow\neg B$ is also admissible at t. □

Definitions: That a sentence has been asserted **according to the internal rules** is to mean that it has been asserted in a deduction-like process which begins only with assertions of elementary sentences that have not yet been rejected altogether or with assertions of negations of sentences that have been rejected (or with assertions of sentences of both sorts), and then only applies the inverse rules I(∧), I(∨), I(∃) (cf. Example 2). - We write $\natural^{+}_{\Delta}A$ to mean that A has been asserted according to the internal rules, and that Δ is a list of all elementary sentences asserted to the same time. - Obviously

$$\natural^{+}_{\Delta}A \ \underline{\rightarrow}\ \exists\alpha{:}A.\,(\alpha_e \subseteq \Delta\ \underline{\wedge}\ \Delta \models \alpha_\neg).$$

(d) $\quad \mathrm{Rej}_\Gamma A\ \underline{\wedge}\ \natural^{+}_{\Delta}A\ \underline{\rightarrow}\ !{\sim}\natural\,\Gamma\Delta.$

So as long as we do not violate the internal rules by asserting elementary sentences, we cannot do both, reject A and assert A according to the internal rules.

Proof: Let $\mathrm{Rej}_\Gamma A$ and $\natural^{+}_{\Delta}A$. So for some $\alpha : A$ we have $\alpha_e \subseteq \Delta$, $\Delta \models \alpha_\neg$, and hence, by (a), $\Gamma\Delta \models \alpha_\neg$. But because of $\mathrm{Rej}_\Gamma A$ we have $!{\sim}\natural\,\Gamma\alpha_e\alpha_\neg$, so $!{\sim}\natural\,\Gamma\Delta\alpha_\neg$, and so, by (b), $!{\sim}\natural\,\Gamma\Delta$. □

Lemma 1: If $A_1, \ldots, A_n \Rightarrow \neg B$ is an individual case of an admissible inference rule, and if $A_1, \ldots, A_n$ have been asserted according to the internal rules, then B has been rejected (so that $\neg B$ may be asserted at once).

Proof: Let Γ be a list of all elementary sentences asserted to t. Assume that $A_1, A_2 \Rightarrow \neg B$, e.g., is admissible at t (cf. (c)), and that $\natural^{+}_{\Delta}A_1, A_2$. So $\mathrm{Rej}_\Gamma\,(A_1 \wedge A_2 \wedge B)$, and for some $\alpha : A_1, A_2$ we have $\alpha_e \subseteq \Delta$ and $\Delta \models \alpha_\neg$. For all $\beta : B$ we obtain $!{\sim}\natural\,\Gamma\alpha_e\beta_e\alpha_\neg\beta_\neg$, hence $!{\sim}\natural\,\Gamma\Delta\beta_e\alpha_\neg\beta_\neg$, and hence, by (b), $!{\sim}\natural\,\Gamma\Delta\beta_e\beta_\neg$. So $\mathrm{Rej}_{\Gamma\Delta}B$. □

Now we consider the case that forbidden assertions have been performed (by mistake, e.g.). If, especially, the already performed assertions of elementary sentences violate together internal rules, we should not accept all of those assertions. Accordingly, we shall define sets, Σ_k ($k \in \mathbb{N}$), of asserted elementary sentences that do not together violate internal rules if we ignore assertions of sentences that do not belong to those sets. To this end we at first define the sets T_k of those elementary sentences which have been anchored and asserted to the k^{th} moment of such assertions. (The use of natural numbers, k, and the principle of arithmetical induction will be justified in §4. This justification does not depend upon the following remarks.)

Let $T_0 \rightleftharpoons \emptyset$, and let T_{k+1} be the set of all elementary sentences E that have been 'stated' (which is here to mean 'anchored and asserted') so early that all elementary sentences that have been stated before accomplishing the first statement of E belong to T_k. This can be formalized in the shape

$$E \in T_{k+1} \ \rightleftharpoons\ \sharp\, E\ \underline{\wedge}\ \underline{\forall} D\,(\sharp_{<E}\, D\ \underline{\rightarrow}\ D \in T_k),$$

where $\sharp\, E$ means that E has been stated, and $\sharp_{<E}\, D$ means that D has been stated before accomplishing the first statement of E. For $k > 0$, let t_k denote the earliest moment to which all elements of T_k have been stated. - We obtain

$$\underline{\forall}\, k, m \in \mathbb{N}. \,(k < m \;\underline{\rightarrow}\; T_k \subseteq T_m).$$

The proof, by arithmetical induction, is straightforward, if we at first deal with the case $m = k + 1$. (Note that we possibly have $!\!\sim \natural\, T_k$ for some $k \in \mathbb{N}$.)

Using the abbreviation $\Delta_{k+1} \rightleftharpoons T_{k+1} \setminus T_k$, we recursively define sets Σ_k $(\subseteq T_k)$ by $\Sigma_0 \rightleftharpoons \emptyset$ and

$$E \in \Sigma_{k+1} \;\rightleftharpoons\; E \in \Sigma_k \Delta_{k+1} \;\underline{\wedge}\; (!\!\sim \natural\, \Sigma_k \Delta_{k+1} \;\underline{\rightarrow}\; E \in \Sigma_k).$$

So we have

$$\Sigma_{k+1} = \begin{cases} \Sigma_k, & \text{if } !\!\sim \natural\, \Sigma_k \Delta_{k+1}, \\ \Sigma_k \Delta_{k+1} & \text{otherwise.} \end{cases}$$

By induction we obtain, for all $k \in \mathbb{N}$, *not* $!\!\sim \natural\, \Sigma_k$ (i.e. we must not assert $!\!\sim \natural\, \Sigma_k$). So it would be appropriate to accept only assertions of those elements of T_k which belong to Σ_k and to modify the definition of 'rejected' thus: That A is rejected *at* t_k means that $\mathrm{Rej}_{\Sigma_k} A$.

To show that certain inference rules are admissible we shall only apply the rules P($\wedge$), P($\vee$), P($\exists$), their inverses, inference rules which have previously been shown to be admissible, and Lemma 1.

Definition: For formulas F in which the distinct variables $x_1, \ldots, x_n$ but no others occur free,

$$\begin{aligned} & \exists.F \;\rightleftharpoons\; \exists\, x_1, \ldots, x_n\, F, \\ \text{specially} \quad & \exists.A \;\rightleftharpoons\; A, \quad \text{for sentences } A \;\; (\text{i.e. } n = 0). \end{aligned}$$

Proposition: The following inference rules are admissible:

R01	$A,\, \neg\,(A \wedge B) \;\Rightarrow\; \neg B$
	$B,\, \neg\,(A \wedge B) \;\Rightarrow\; \neg A$
R02	$\neg\,(A \vee B) \;\Rightarrow\; \neg A$
	$\neg\,(A \vee B) \;\Rightarrow\; \neg B$
R03	$\neg \exists \underline{x}\, A\underline{x} \;\Rightarrow\; \neg A\underline{r}$ (for values $\underline{r}$ of $\underline{x}$)
R1	$\Rightarrow\; \neg\exists.(F \wedge \neg F)$
R2	$\Rightarrow\; \neg\exists.[(F \wedge G) \wedge \neg\,(G \wedge F)]$
R3*a*	$\Rightarrow\; \neg\exists.\{[(F \wedge G) \wedge H] \wedge \neg\,[F \wedge (G \wedge H)]\}$
R3*b*	$\Rightarrow\; \neg\exists.\{[F \wedge (G \wedge H)] \wedge \neg\,[(F \wedge G) \wedge H]\}$
R4	$\neg\exists.G \;\Rightarrow\; \neg\exists.(F \wedge G)$
R5	$\neg\exists.(F \wedge G),\; \neg\exists.(F \wedge \neg G) \;\Rightarrow\; \neg\exists.F.$

These rules have the form $\mathcal{A}_1, \ldots, \mathcal{A}_n \Rightarrow \neg\mathcal{B}$. To show that such a rule is admissible, we at first note the sentence schemes

$$\mathcal{B}, \mathcal{A}_1, \ldots, \mathcal{A}_n$$

(with "a." for "assumptions" behind). Thereafter we treat them like sentences which have been asserted: We note schemes for sentences which ought to have been asserted before by the rules P(∧), P(∨), or P(∃) - or may then be asserted by their inverses, or schemes for negations of sentences which ought to have been rejected by rules which have already been proved to be admissible (see Lemma 1). We continue doing so until we obtain a 'contradiction' $\mathcal{C}, \neg\mathcal{C}$. - Substitutions of all variables occurring free in the considered formulas by values of them will sometimes be indicated by *.

Ad R01:

$$\begin{array}{ll} A,\ B,\ \neg\,(A \wedge B) & \text{a.} \\ A \wedge B & \text{I}(\wedge). \end{array}$$

Ad R03:

$$\begin{array}{ll} A\underline{r},\ \neg\exists\underline{x}A\underline{x} & \text{a.} \\ \exists\underline{x}A\underline{x} & \text{I}(\exists). \end{array}$$

Ad R02: Analogously. - Ad R1: See §1, Example 3.
Ad R2:

$$\begin{array}{lll} & \exists.[(F \wedge G) \wedge \neg\,(G \wedge F)] & \text{a.} \\ \text{For some *:} & (F^* \wedge G^*) \wedge \neg\,(G^* \wedge F^*) & \text{P}(\exists) \\ & F^* \wedge G^*,\quad \neg\,(G^* \wedge F^*) & \text{P}(\wedge) \\ & F^*,\ \ G^* & \text{P}(\wedge) \\ & G^* \wedge F^* & \text{I}(\wedge). \end{array}$$

Ad R3: Analogously by means of the partial scheme:

$$\begin{array}{c} (F^* \wedge G^*) \wedge H^* \\ F^* \wedge G^*,\ \ H^* \\ F^*,\ \ G^*, \\ G^* \wedge H^* \\ F^* \wedge (G^* \wedge H^*). \end{array}$$

Ad R4:

$$\begin{array}{lll} & \exists.(F \wedge G),\ \neg\exists.G & \text{a.} \\ \text{for some *:} & F^* \wedge G^*,\ \ G^*,\ \ \exists.G. & \end{array}$$

Ad R5:

$$\begin{array}{lllll} & \exists.F, & \neg\exists.(F \wedge G), & \neg\exists.(F \wedge \neg G) & \text{a.} \\ \text{for some * :} & F^*, & \neg(F^* \wedge G^*), & \neg(F^* \wedge \neg G^*) & \text{P}(\exists), \text{R03} \\ & & \neg G^* & \neg\neg G^* & \text{R01.} \end{array}$$

By a **term** t we understand a string of symbols in which variables may occur and from which a constant results by any substitution of all free occurring variables by values of them. - The literal equality of any two strings of symbols ϱ and σ will be indicated by '$\varrho \equiv \sigma$'.

Definition: For lists $\underline{x} \equiv x_1, \ldots, x_n$ $(n \geq 0)$ of distinct variables and lists $\underline{t} \equiv t_1, \ldots, t_n$ of terms, let

$$F^{\underline{x}}_{\underline{t}} \quad \text{result from} \quad F$$

by substituting t_i for each free occurrence of x_i $(i = 1, \ldots, n)$. - Moreover, we use the following sentences of a metalanguage:

$$\begin{aligned} \mathrm{N}(x, F) &\rightleftharpoons x \text{ does not occur free in } F. \\ \mathrm{Fr}(t, x, F) &\rightleftharpoons t \text{ is free for } x \text{ in } F, \text{ this means} \end{aligned}$$

1) every substitution instance of t is a value of x,
2) F^x_t is a formula of $\mathcal{L}$, and
3) each free occurrence of a variable in t is also free in F^x_t wherever t is substituted for x in F.

Example: y is *not* free for x in $\exists y\,(x < y)$ since y occurs free in y but the occurrence of y substituted for x is bound in $\exists y\,(y < y)$.

Remark: $\mathrm{Fr}(t, x, F)$ is an abbreviation of a composite formula of a metalanguage. We use it in the same way as a composite formula of our object language $\mathcal{L}$ in the primary game. [In that metalanguage we may also apply Proposition 3.4 (§3).]

Lemma 2: If $\mathrm{Fr}(t, x, F)$, if $x, \underline{y}$ is a list of all distinct variables which occur free in F or t, and if $r, \underline{s}$ is a value of $x, \underline{y}$, then

$$(*) \qquad (F^x_t)^{x,\underline{y}}_{r,\underline{s}} \equiv (F^{\underline{y}}_{\underline{s}})^x_{t^*} \quad \text{for} \quad t^* \equiv t^{x,\underline{y}}_{r,\underline{s}}.$$

Proof: Consider the following diagrams of partially simultaneous and partially successive substitutions of the free occurrences of the variables $x, \underline{y}$:

$$\begin{array}{cc} x, \underline{y} & x, \underline{y} \\ t, \underline{y} & x, \underline{s} \\ t^*, \underline{s} & t^*, \underline{s}. \end{array}$$

Proposition: The following rules are admissible:

$$\begin{array}{lrcl} \text{R6} & \mathrm{Fr}(t, x, F) & \Rightarrow & \neg\exists.(F^x_t \wedge \neg\exists x\, F). \\ \text{R7} & \mathrm{N}(x, F),\ \neg\exists.(F \wedge G) & \Rightarrow & \neg\exists.(F \wedge \exists x\, G). \end{array}$$

Proofs: Ad R6: Let $\mathrm{Fr}(t, x, F)$. By using the denotations from above we may argue as follows:

$$\begin{array}{rll} & \exists.(F^x_t \wedge \neg\exists x\, F) & \text{a.} \\ \text{for some } r, \underline{s}: & (F^x_t \wedge \neg\exists x\, F)^{x,\underline{y}}_{r,\underline{s}} & \mathrm{P}(\exists) \\ & (F^{\underline{y}}_{\underline{s}})^x_{t^*},\ \neg\exists x\, F^{\underline{y}}_{\underline{s}} & \mathrm{P}(\wedge), (*) \\ & \exists x\, F^{\underline{y}}_{\underline{s}} & \mathrm{I}(\exists). \end{array}$$

Ad R7: Let $\mathrm{N}(x, F)$, and let $\underline{y}$ be a list of all distinct variables occurring free in $F \wedge \exists x\, G$. Then we may argue as follows:

$$\begin{array}{rll} & \exists \underline{y}\,(F \wedge \exists x\, G),\ \ \neg\exists x, \underline{y}\,(F \wedge G) & \text{a.} \\ \text{for some } \underline{s}, r: & F^{\underline{y}}_{\underline{s}},\ \ G^{x,\underline{y}}_{r,\underline{s}} & \\ & (F \wedge G)^{x,\underline{y}}_{r,\underline{s}} & \\ & \exists x, \underline{y}\,(F \wedge G). & \end{array}$$

On occasion we write '$\Leftrightarrow$' to combine two inference rules, and we use the definition

$$\forall.F \ \rightleftharpoons\ \neg\exists.\neg F, \quad \text{specially} \quad \forall.A \ \rightleftharpoons \neg\neg A.$$

Proposition: Admissible are the following rules:

$$\begin{array}{lrcl} \mathrm{R8}a & \neg\exists.G,\ \neg\exists.(F \wedge \neg G) & \Rightarrow & \neg\exists.F \\ \mathrm{R8}b & \neg\exists.\neg G,\ \neg\exists.(F \wedge G) & \Rightarrow & \neg\exists.F \\ \mathrm{R9} & \neg\exists.(G \wedge F) & \Rightarrow & \neg\exists.(F \wedge G) \\ \mathrm{R10} & \neg\exists.[F \wedge (G \wedge H)] & \Leftrightarrow & \neg\exists.[(F \wedge G) \wedge H] \\ \mathrm{R11}a & \forall.\neg F & \Leftrightarrow & \neg\exists.F; \\ \mathrm{R11}b & \neg\neg\neg A & \Leftrightarrow & \neg A \\ \mathrm{R11}c & \forall.(F \to G) & \Leftrightarrow & \neg\exists.(F \wedge \neg G) \\ \mathrm{R11}d & \forall.(F \to \neg G) & \Leftrightarrow & \neg\exists.(F \wedge G) \end{array}$$

For the proof that a rule $\mathcal{A}_1, \ldots, \mathcal{A}_n \Rightarrow \neg\mathcal{B}$ is admissible, it also suffices to deduce $\neg\mathcal{B}$ from $\mathcal{A}_1, \ldots, \mathcal{A}_n$ by repeated application of admissible rules (since $\neg\mathcal{B}$ contradicts the omited assumption $\mathcal{B}$). We shall proceed so in the following.

Ad R8a:

$$\begin{array}{ll} \neg\exists.G,\ \neg\exists.(F \wedge \neg G) & \text{premises} \\ \neg\exists.(F \wedge G) & \text{by R4} \\ \neg\exists.F & \text{by R5} \end{array}$$

Ad R9:

$$\begin{array}{ll} \neg\exists.(G \wedge F) & \text{prem.} \\ \neg\exists.[(F \wedge G) \wedge \neg\,(G \wedge F)] & \text{R2} \\ \neg\exists.(F \wedge G) & \text{R8a.} \end{array}$$

Ad R11a($\Rightarrow$):

$$\begin{array}{ll} \neg\exists.\neg\neg F & \text{prem.} \\ \neg\exists.(F \wedge \neg F) & \text{R1} \\ \neg\exists.F & \text{R8b.} \end{array}$$

R11b and R11c are special cases of R11a.

Ad R11d($\Rightarrow$):

$\forall.(F \to \neg G)$	prem.
$\neg\exists.(F \wedge \neg\neg G)$	R11c
$\neg\exists.(\neg\neg G \wedge F)$	R9
$\neg\exists.(G \wedge \neg\neg G \wedge F)$	R4, 10
$\neg\exists.(F \wedge G \wedge \neg\neg G)$	R9, 10
$\neg\exists.(F \wedge G \wedge \neg G)$	R1, 4, 10
$\neg\exists.(F \wedge G)$	R5. $\square$

For the remaining rules and the rules quoted in §3 we can also prove their admissibility in this way 'deductively'. To this end we need not yet justify the corresponding *general* method of deduction. This will later be possible by means of induction on the number of deduction steps (cf. §4).

§3. An approach to classical logic

Unsolved problems as the (arithmetical) conjecture of GOLDBACH yield examples of sentences A for which neither A nor $\neg A$ may be asserted up to now so that $A \vee \neg A$ must also not yet be asserted in the primary game. Correspondingly, the *'tertium non datur'* was said to be 'onbetrouwbaar' by L.E.J. BROUWER (1908). Nevertheless, $\neg(A \vee \neg A)$ must not be asserted for any sentence A. This follows from the admissibility of the rules

$$\neg(A \vee \neg A) \;\Rightarrow\; \neg A$$
$$\neg(A \vee \neg A) \;\Rightarrow\; \neg\neg A$$

(see R02). Hence, for arbitrary sentences A, we have

$$\neg\neg(A \vee \neg A).$$

Accordingly, there exist sentences B such that $\neg\neg B$ may indeed be asserted but B must not yet be asserted. Hence, the rule

$$\neg\neg B \;\Rightarrow\; B$$

should not be applied merely thoughtlessly. Note, however, that the inverse rule

$$B \;\Rightarrow\; \neg\neg B$$

is admissible due to R1 and R01. - To make the previous rule admissible, too, and so to obtain the *tertium non datur* and even the whole classical logic we liberalize the primary game by using the complete *asserted* sentences as abbreviations of their double negations.

In §6 we shall justify this '**classical use**' of assertions by showing that it satisfies what has been stated in the second section of the abstract.

Definition: An inference rule $\mathcal{A}_1, \ldots, \mathcal{A}_n \Rightarrow \mathcal{B}$ is said to be **classically admissible** iff

$$\neg\neg\mathcal{A}_1, \ldots, \neg\neg\mathcal{A}_n \;\;\Rightarrow\;\; \neg\neg\mathcal{B}$$

is admissible.

3.1. Proposition: If an inference rule of the form

$$\mathcal{A}_1, \ldots, \mathcal{A}_n \;\Rightarrow\; \neg\mathcal{B}$$

$(n \geq 0)$ is admissible, then it is also classically admissible.

Proof for $n = 2$: Let $\mathcal{A}_1, \mathcal{A}_2 \;\;\Rightarrow\;\; \neg\mathcal{B}$ be admissible, i.e., let

$$\neg\,\exists..(\mathcal{A}_1 \wedge \mathcal{A}_2 \wedge \mathcal{B}).$$

Then the following rules are also admissible:

$$\mathcal{A}_2, \mathcal{B} \;\;\Rightarrow\;\; \neg\mathcal{A}_1 \;\;\Rightarrow\;\; \neg\neg\neg\mathcal{A}_1$$

and hence likewise

$$\neg\neg\mathcal{A}_1, \mathcal{A}_2 \;\;\Rightarrow\;\; \neg\mathcal{B}$$
$$\neg\neg\mathcal{A}_1, \neg\neg\mathcal{A}_2 \;\;\Rightarrow\;\; \neg\mathcal{B} \;\;\Rightarrow\;\; \neg\neg\neg\mathcal{B}. \;\Box$$

Due to our definition $\forall\underline{x}\, F \rightleftharpoons \neg\,\exists\underline{x}\,\neg F$, R11b, and R03,

$$\forall\underline{x}\, A\underline{x} \;\;\Rightarrow\;\; A\underline{r} \quad \text{(for values } \underline{r} \text{ of } \underline{x}\text{)}$$

is classically admissible. Therefore, $\forall.F$ may be read as "F is universally true". Accordingly, a rule of the form

$$\forall.\mathcal{F}_1, \ldots, \forall.\mathcal{F}_n \;\;\Rightarrow\;\; \forall.\mathcal{G}$$

will be abbreviated by

$$\mathcal{F}_1, \ldots, \mathcal{F}_n \;\;\Rightarrow\;\; \mathcal{G}.$$

Moreover, in the below rules R22 - R25 we shall (as usual) remove the 'syntactical' premises $\mathrm{Fr}(t, x, F)$ and $\mathrm{N}(x, H)$ to the rear.

3.2. Proposition: The following rules are admissible (by the latter abbreviation):

R12 $\quad \Rightarrow \;\; F \to F$

R13 $\quad F,\; F \to G \;\;\Rightarrow\;\; G$ (modus ponens)

R14 $\quad F \to G,\; G \to H \;\;\Rightarrow\;\; F \to H$

R15a $\quad \Rightarrow \;\; F \wedge G \to F$

R15b $\Rightarrow \;\; F \wedge G \rightarrow G$

R16 $H \rightarrow F,\; H \rightarrow G \;\; \Rightarrow \;\; H \rightarrow F \wedge G$

R17 $F \wedge G \rightarrow H \;\; \Leftrightarrow \;\; F \rightarrow (G \rightarrow H)$

R18 $F \;\; \Rightarrow \;\; H \rightarrow F$

R19 $F,\; G \;\; \Leftrightarrow \;\; F \wedge G$ (three rules)

Definition: $F \leftrightarrow G \; \rightleftharpoons \; (F \rightarrow G) \wedge (G \rightarrow F)$.

R20 $\Rightarrow \;\; F \leftrightarrow \neg\neg F$

R21 $F \rightarrow G \;\; \Leftrightarrow \;\; \neg G \rightarrow \neg F$

R22 $\Rightarrow \;\; F^x_t \rightarrow \exists x\, F,$ if $\mathrm{Fr}(t, x, F)$

R23 $F \rightarrow H \;\; \Rightarrow \;\; \exists x\, F \rightarrow H,$ if $\mathrm{N}(x, H)$

R24 $\Rightarrow \;\; \forall x\, F \rightarrow F^x_t,$ if $\mathrm{Fr}(t, x, F)$

R25 $H \rightarrow F \;\; \Rightarrow \;\; H \rightarrow \forall x\, F,$ if $\mathrm{N}(x, H)$.

Here we only prove the admissibility of R16. To this end we give a deduction of the following rule at first:

R16* $H \rightarrow F \;\; \Rightarrow \;\; H \rightarrow H \wedge F.$

$\forall.\,(H \rightarrow F)$	premise
$\neg\,\exists.\,(H \wedge \neg F)$	R11
$\neg\,\exists.\,(\neg\,(H \wedge F) \wedge H \wedge \neg F)$	R4, 10
$\neg\,\exists.\,(\neg\,(H \wedge F) \wedge H \wedge F)$	R1, 9, 10
$\neg\,\exists.\,(\neg\,(H \wedge F) \wedge H)$	R5
$\forall.\,(H \rightarrow H \wedge F)$	R9, 11.

Ad R16 (sketch):

$H \rightarrow F, \quad H \rightarrow G$	premises
$H \wedge F \rightarrow G$	R15, 14
$H \rightarrow H \wedge F \rightarrow H \wedge F \wedge G \rightarrow F \wedge G$	R16* etc. □

3.3. Proposition: The following rules concerning $(\vee)$ are admissible:

R26a $\Rightarrow \;\; \neg\,\exists.[F \wedge \neg\,(F \vee G)]$

R26b $\Rightarrow \;\; \neg\,\exists.[G \wedge \neg\,(F \vee G)]$

R27 $\Rightarrow \;\; \neg\,\exists.[\neg F \wedge \neg G \wedge (F \vee G)].$

R28a $\Rightarrow \;\; F \rightarrow F \vee G$

R28b $\Rightarrow\ G \to F \vee G$

R29 $F \to H,\ G \to H\ \Rightarrow\ F \vee G \to H.$

The admissibility of R26 - R27 can immediately be checked. R28 - R29 are due to the preceding rules. - It is well-known that also other inference rules and other methods of classical reasoning are due to the rules R1 - R29 and $\neg\neg A \Rightarrow A$. We shall make use of this fact within the framework of $\mathcal{L}$ and other languages.

On the classical use of elementary sentences

To justify this use we need the following result (concerning the primary game):

3.4. Proposition: For elementary sentences E (as considered so far), we ought to assert $\neg\neg E$ only after E has been anchored (see §1).

To obtain this proposition, we assign to every elementary sentence E a new 'auxiliary sentence', $-E$. As an *external* rule we lay down this: Assert $-E$ only after E has been rejected due to the internal rules *except* the following one. As an *internal* rule we lay down:

$$E, -E\ \Rightarrow\ \bot.$$

Let the primary game not contain other rules concerning $-E$. Note that E may be rejected due to the latter rule (i.e. $-E$ may be asserted) only if E has been rejected without regard to this rule.

For the proof of 3.4 we need some further preliminaries:

Definition: An n-tuple $(E_1, \ldots, E_n)$ of elementary sentences is said to be *absolutely rejected* iff it is forbidden by the internal rules to assert all of its members $E_1, \ldots, E_n$. (Here "absolutely" means: "independent of which elementary sentences have been asserted".)

3.5. Lemma: If $(E_1, \ldots, E_n)$ is absolutely rejected, then from $E_1, \ldots, E_n$ there is deducible $\bot$ by the internal rules (see §1).

Proof: All absolutely rejected tuples can be derived by the following rules:
1. $\Rightarrow (\bot)$;
2. $(D_1, \ldots, D_m, E) \Rightarrow (D_1, \ldots, D_m, E_1, \ldots, E_n)$
   if $E_1, \ldots, E_n \Rightarrow E$ is an internal rule $(m \geq 0)$;
3. $(D_1, \ldots, D_m) \Rightarrow (E_1, \ldots, E_n)$ if $\{D_1, \ldots, D_m\} \subseteq \{E_1, \ldots, E_n\}$.

So we easily obtain 3.5 by induction on the number of applications of these rules (cf. §4, arithmetical induction). □

Proof of 3.4: Suppose that we have successively asserted $-E$ and $\neg E$. Then (by the rule $E, -E \Rightarrow \bot$) E has been rejected so that $\natural\neg E$ does not violate a rule. So we can argue as follows: We ought to assert $\neg\neg E$ only if the following is the case: $\neg E$ has been rejected, so that (by the previous argument) $-E$ has been rejected. Thus, by 3.5, there exist elementary sentences $E_1, \ldots, E_n$ which have already been asserted so that from them and $-E$ there is deducible $\bot$ by the internal rules. Since $E_1, \ldots, E_n$ ought not to be rejected and the rule $E, -E \Rightarrow \bot$ is the only internal rule concerning $-E$, it follows that E is deducible from $E_1, \ldots, E_n$ by the internal rules. Since $E_1, \ldots, E_n$ ought to be anchored, also E passes for anchored. $\square$

The auxiliary sentences $-E$ will not be considered otherwise in this paper.

§4. An approach to arithmetic

The following is due to [7, §13] or [9, II.1]. As natural numbers we can simply use the **numerals** which are defined to be the figures $0, 0', 0'', 0''', \ldots$ constructible in the calculus K($\mathbb{N}$) with the two rules

$$\begin{array}{rcll} & \Rightarrow & 0 & \text{(start with 0)} \\ k & \Rightarrow & k' & \text{(from } k \text{ infer } k'). \end{array}$$

Here, k may at any time be replaced by a figure already constructed in K($\mathbb{N}$). For 'original' constants r we read $r \mathrel{\varepsilon} \mathbb{N}$ as "r is constructible in K($\mathbb{N}$)". (In the metalanguage used in §7 we shall write '$\in$' for 'ε'.) - However, also certain '*new*' signs may be used as abbreviations or singular descriptions for elements of $\mathbb{N}$ (e.g. 3 for $0'''$, and $357 \rightleftharpoons 3 \times 10^2 + 5 \times 10 + 7$ after introducing addition etc.). - In the following, k, m, n stand for arbitrary numerals.

Let the **equality** on $\mathbb{N}$ be the literal equality. Accordingly, $k_0 = m_0$ is to mean that this equation is deducible in the calculus K($=$) with the two rules

$$\Rightarrow 0 = 0; \quad k = m \;\Rightarrow\; k' = m'.$$

The 'deontic infinity' of $\mathbb{N}$: Because of lack of time and material we can *really* construct only finitely many numerals. However, we shall never be *obliged* to terminate the constructions in K($\mathbb{N}$). - Moreover, by the rules of K($\mathbb{N}$) we successively obtain only *diverse* numerals $0, 0', 0'', \ldots$. These facts suggest to say that $\mathbb{N}$ is *infinite*. (We shall return to this point.)

For 'external reasons' it is impossible to apply the rules of K($\mathbb{N}$) infinitely many times. So we cannot construct infinitely long 'numerals', which end with $\ldots'''$. However, with respect to arithmetical induction, such figures should also be excluded from $\mathbb{N}$ *by internal rules*. Therefore, we replace the rules of K($\mathbb{N}$) by the following Γ-rules and, similarly, the rules of K($=$) by the following Δ-rules.

Given any first order language $\mathcal{L}$ as considered so far, we extend it by introducing new sentences of the forms $\Gamma|\varrho|$, $r \varepsilon \mathbb{N}$, $\Delta|\varrho = \sigma|$, and $r = s$ where $\Gamma, |, \varepsilon, \mathbb{N}, \Delta$, and $=$ (which is short for $=_{\mathbb{N}}$, e.g.) are particular *new* symbols, r and s range over arbitrary values of variables of $\mathcal{L}$, and ϱ and σ range over strings of (atomic) symbols occurring in those values. To introduce the use of sentences of those forms we include just the following 'Γ-' and 'Δ-rules' among the **internal rules** of the language expanded so:

$$
\begin{array}{rcl}
\Gamma|0| & \Rightarrow & \bot \\
\Gamma|\varrho'| & \Rightarrow & \Gamma|\varrho| \\
r \varepsilon \mathbb{N} & :\Rightarrow & \neg\,\Gamma|r|
\end{array}
\qquad
\begin{array}{rcl}
\Delta|0 = 0| & \Rightarrow & \bot \\
\Delta|\varrho' = \sigma'| & \Rightarrow & \Delta|\varrho = \sigma| \\
r = s & :\Rightarrow & \neg\,\Delta|r = s|.
\end{array}
$$

We have omitted the sign $\natural$ in these rules. We include the sentences of the forms $\Gamma|\varrho|$ and $\Delta|\varrho = \sigma|$ among the elementary sentences and understand, for convenience, sentences of the forms $r \varepsilon \mathbb{N}$ and $r = s$ to be abbreviations of $\neg\,\Gamma|r|$ and $\neg\,\Delta|r = s|$, respectively.

Commentary: By the Γ-rules it is just forbidden to assert $\Gamma|0|$, $\Gamma|0'|$, $\Gamma|0''|$, ... so that we may assert $0 \varepsilon \mathbb{N}$, $0' \varepsilon \mathbb{N}$, $0'' \varepsilon \mathbb{N}$, ... but no other sentences of the form $r \varepsilon \mathbb{N}$. [An infinitely long 'numeral' Ω ending with $\ldots'''$ (as mentioned above) cannot be distinguished from Ω' so that $\Gamma|\Omega|$ is not rejected. We shall, however, not make use of these informal remarks.] - Similarly, by the Δ-rules it is just forbidden to assert $\Delta|0 = 0|$, $\Delta|0' = 0'|$, $\Delta|0'' = 0''|$, ... So we may assert $0 = 0$, $0' = 0'$, $0'' = 0''$, ... but no other equations between numerals. - In this §4 we write x, y for variables which range over numerals, and z for variables which range over numerals *at least.*

Propositions:

(a) $\forall z\,(z \varepsilon \mathbb{N} \leftrightarrow z' \varepsilon \mathbb{N})$ (cf. the infinity of $\mathbb{N}$)

(b) $\forall x, y\,(x = y \leftrightarrow x' = y')$

(c) $\forall x\,\neg\,(x' = 0)$, $\forall y\,\neg\,(0 = y')$.

Proofs: (a)(←) We ought to assert $\exists z\,\big(\neg\,(z \varepsilon \mathbb{N}) \wedge z' \varepsilon \mathbb{N}\big)$ only if, for some r, $r \varepsilon \mathbb{N}$ is rejected and $r' \varepsilon \mathbb{N}$ has been asserted. To this, $\Gamma|r'|$ should be rejected by the rule $\Gamma|r'| \Rightarrow \Gamma|r|$, so that also $\Gamma|r|$ must be rejected (inversion). But then $r \varepsilon \mathbb{N}$ may be asserted ('contradiction'). By these arguments, $\exists z\,\big(\neg\,(z \varepsilon \mathbb{N}) \wedge z' \varepsilon \mathbb{N}\big)$ is rejected. So we may assert its negation and so, by R11, $\forall z\,(z \varepsilon \mathbb{N} \leftarrow z' \varepsilon \mathbb{N})$. - (a)(→), (b), and (c) can be proved similarly. □

Principle of arithmetical induction: Admissible is the rule

$$A(0),\ \forall x\,[A(x) \to A(x')] \ \Rightarrow\ \forall x\,A(x).$$

Demonstration: Given a formula $A(x)$. Since the figures denoted by ϱ are generally no constants, we use new sentences $\Lambda|\varrho|$ instead of $\varrho \varepsilon \mathbb{N} \wedge A(\varrho)$, the assertions of which we restrict just by the internal rules $\Lambda|\varrho| :\Rightarrow \neg\,\Gamma|\varrho|$ and $\Lambda|k| :\Rightarrow A(k)$, for numerals k. (So we have $\Lambda|k| \leftrightarrow A(k)$. Recall that x ranges over $\mathbb{N}$ only.) - Assume now that we have $A(0)$, $\forall x\ [A(x) \rightarrow A(x')]$, and $n \,\varepsilon\, \mathbb{N}$. So it is forbidden by the rules $\Gamma|0| \Rightarrow \perp$ and $\Gamma|\varrho'| \Rightarrow \Gamma|\varrho|$ to assert $\Gamma|n|$. In the same way, it is forbidden by the rules $\neg\,\Lambda|0| \Rightarrow \perp$ and $\neg\,\Lambda|\varrho'| \Rightarrow \neg\,\Lambda|\varrho|$ (which are classically admissible) to assert $\neg\,\Lambda|n|$, i.e. $\neg A(n)$. So we have $\neg\,\exists x\,\neg A(x)$. $\square$

The argument beginning with "in the same way" has already the form of an induction principle on a metalevel. However, this argument is justified by the fact that we have *no other* rules than the Γ-rules to restrict assertions of the form $\natural\,\Gamma|\varrho|$.

We shall also apply other induction principles that can be explained by arithmetical induction.

Similarly but by considering the Δ-rules we obtain the **'Induction principle for equations'** which says that the following rule is admissible:

$$A(0,0),\ \forall x,y\,[A(x,y) \rightarrow A(x',y')] \ \Rightarrow\ \forall x,y\,[x=y \rightarrow A(x,y)].$$

By this principle we can conclude: $r = s \rightarrow r \,\varepsilon\, \mathbb{N} \wedge s \,\varepsilon\, \mathbb{N}$. - By (b) and arithmetical induction, we easily obtain $n = n$. Moreover, we also have

(d) $\qquad k = m \ \wedge\ A(k) \ \rightarrow\ A(m).$

Proof: Let e be a variable for the 'empty figure' as well as for figures which can be constructed from it by applying the calculus rule: $q \Rightarrow {}'q$. Then we have

$$\forall e\,[A(0e) \rightarrow A(0e)] \text{ and } \forall x,y\,\{\forall e\,[A(xe) \rightarrow A(ye)] \rightarrow \forall e\,[A(x'e) \rightarrow A(y'e)]\}.$$

Now we obtain (d) by the induction principle for equations. $\square$

From (d) follows the comparativity, $k = m \wedge k = n \rightarrow m = n$, and hence the symmetry and transitivity of the equality on $\mathbb{N}$. This relation is, therefore, an equivalence relation under which all formulas considered are invariant.

Recursion as a way of generating relations: Addition in $\mathbb{N}$, e.g., can be introduced by fixing the following assertion rules for new sentences:

$$\begin{aligned} \mathrm{Add}(k,0,n) &\ :\Rightarrow\ n = k \\ \mathrm{Add}(k,m',n) &\ :\Rightarrow\ \exists x[\mathrm{Add}(k,m,x) \wedge n = x']. \end{aligned}$$

These rules can be considered as special cases of

$$\begin{aligned} \underline{r} \,\varepsilon\, S_0 &\ :\Rightarrow\ A(\underline{r}) \\ \underline{r} \,\varepsilon\, S_{m'} &\ :\Rightarrow\ B(\underline{r}, m, S_m) \end{aligned}$$

where $S \rightleftharpoons \mathrm{I}\underline{x}yZ(A(\underline{x}), B(\underline{x},y,Z))$. $A(\ldots)$ and $B(\ldots)$ are permitted to be formulas of an *extended* object language $\mathcal{L}^+$ which is the least language containing certain elementary formulas as well as formulas of the shape $\underline{t} \varepsilon Z$, and is closed under $\wedge, \vee, \neg, \exists$, and ε I (with the '*induction operator*' 'I'). Z is assumed to be a variable for sets or relations (as S_m) which are definable in $\mathcal{L}^+$. Such variables may be bound by 'I' but must not be bound by '∃' in formulas of $\mathcal{L}^+$. - The latter rules can also be inverted since the assertions on their left are not to be subjected to additional restrictions, and even the language $\mathcal{L}^+$ is non-circular as can be shown by induction on the outlined construction of formulas of $\mathcal{L}^+$ (see [19, pp. 426 ff., 452] or 7.3). The predicator 'Add' represents a function, '+'. An introduction of singular description terms (as terms of the form $s+t$, e.g.) will be sketched in §9. - As easily seen, all recursive functions are definable in $\mathcal{L}^+$.

For constructive or predicative **analysis** in the sense of [8] inclusive of measure theory and functional analysis (as in [18], e.g) there suffice real numbers which are given by rational Cauchy sequences definable in $\mathcal{L}^+$. (See also the end of §7.)

We have unproblematically obtained the above result (a), by which $\mathbb{N}$ is infinite. This result, however, has substantial consequences. To give an example, we consider the power (Pow) of natural numbers. The proposition

$$9^{9^9} \varepsilon \mathbb{N}$$

(in which an iterated singular description occurs) can be considered as an abbreviation of the composite sentence

$$\exists x, y \, [\mathrm{Pow}(9,9,x) \ \wedge \ \mathrm{Pow}(9,x,y) \ \wedge \ y \, \varepsilon \, \mathbb{N}],$$

a generalization of which can inductively be proved by a well known procedure. It is, however, not possible *really* to construct a figure n by the rules of K($\mathbb{N}$) which satisfies $9^{9^9} = n$. The existential sentence $9^{9^9} \varepsilon \mathbb{N}$ must, therefore, not actually be asserted in the primary game for the whole history of mankind. Nevertheless, it would *not* violate a rule successively to perform proper assertions and ultimately to assert $9^{9^9} \varepsilon \mathbb{N}$. Accordingly, we may assert the double negation of this sentence, which, therefore, can be understood classically.

A more general problem concerns sentences of the form $\forall x \, \varepsilon \, K. \, \exists y \, A(x,y)$, i.e. $\forall x \, (x \, \varepsilon \, K \rightarrow \exists y \, A(x,y))$. At best we can proof such a sentence *directly* by describing an effective procedure, p, and showing that

$$(*) \qquad \forall x \, \varepsilon \, K. \, \big\{ \exists y \, (p : x \mapsto y) \ \wedge \ \forall y \, [(p : x \mapsto y) \rightarrow A(x,y)] \big\}.$$

Here, $p : x \mapsto y$ is to mean that p with the input x prescribes to produce the output y finally.

The assertion of (*) shows for any $k \, \varepsilon \, K$ how one can 'on principle' find an m satisfying $\neg\neg A(k,m)$. In many cases, however, p with a 'large' input k will not really yield an output m in available time. How can we understand the existence of such an m? To this, we consider p as a system of rules by which certain successions of action steps are permitted (or even required) and the others are forbidden. Every permitted step is assumed to be uniquely determined by the input and the preceding steps. We include the rules of p among the internal rules of the primary game.

If an input, k, is given, the 'classical existence' of a corresponding output (i.e. $\neg\neg\exists y\,(p : k \to y)$) means that it is permitted (i.e. not forbidden) by the rules of p to perform certain steps which finally yield an output.

Since for every application of p we can really perform a limited number T of steps at most, it suffices for the investigation of our prospect of success to consider only such procedures that stop by a command after $\leq T$ steps. If such a procedure p is applied to a given input k, then after $\leq T$ steps we obtain either an output or the result that there does not exist an output. This result, however, would contradict (*). So (*) indicates for any $k \varepsilon K$ how we can find an m with $\neg\neg A(k, m)$ within the time T. (However, because of the limitation on the computing time available for p, (*) generally holds only for *smaller* sets K than without this limitation.)

§5. Objectual quantification

Indicators and denotations: Sentences as "All ants are mortal" or "Some apples are red" have the form "All P are Q" or "Some P are Q", respectively, or - in a 'modern' manner of writing - $\forall u\,(Pu \to Qu)$ or $\exists u\,(Pu \wedge Qu)$, respectively. However, the use of such sentences cannot adequately be reconstructed as in §1 since we have not enough proper names for ants or apples, e.g., as values of the variable u at our disposal. So we also consider sentences as "This is an ant" or "This ant has only five legs" with '**indicators**' as "this" or "this ant", which can *temporarily* be used like proper names for objects (as solids or events, e.g.).

Under a **denotation** of an object by an indicator or of an indicator by an object we understand a naming which, however, is in general only valid in a special situation (or context). Such a denotation can result, for instance, from pointing at that object and pronouncing that indicator at the same time.

In many cases, an object in question cannot be shown to a listener so that the corresponding denotation is restricted to the speaker. However, if he has said to *himself* "This ant has only five legs", e.g., then he may say to any listener that there *exists* an ant with five legs only (cf. P($\exists$ den) below). Accordingly, we need *not presuppose* here that the denoted objects do not depend on the concerned persons. (Astronomical constellations, e.g., do so). - We avoid the definition of denotations as *functions* that map sets of indicators into sets of objects. (This definition is somewhat problematic for the present.) - We shall not deal with several other problems concerning denotations as, for instance, the danger of misunderstandings which can result from not sufficiently clear bounds of situations.

As indicators we shall use 'objectual variables' (for certain sorts of objects). They are to be distinguished from the previously considered 'substitutional variables' whose free occurrences may be substituted by constants (or especially proper names) (cf. [10, §26]). - Since indicators can temporarily be used as names of objects, we even admit that they occur free in sentences of $\mathcal{L}$. This means, sentences are formulas in which perhaps indicators but not substitutional variables occur free. However, we stipulate for the present that indicators do not occur in constants. (In §10 we shall omit this restriction.) - As metavariables will be used u, v, w for indicators (objectual variables), and $\underline{u}, \underline{v}$ for lists of distinct indicators. A denotation of all members of such a list $\underline{u}$ is briefly said to be a denotation of $\underline{u}$. (If occasion arises, $\underline{u}$ is also permitted to be empty.)

A denotation of a single indicator is said to be **simple**. We distinguish any two simple denotations that are created by different acts of naming. If $u_1, \ldots, u_k$ are different indicators, and γ_i is a simple denotation of $u_i (i = 1, \ldots, k)$, then we say that $(\gamma_1, \ldots, \gamma_k)$ is a denotation of $u_1, \ldots, u_k$ or of any permutation of these indicators. Let all denotations considered in the following be composed in this way from simple denotations of different indicators.

If γ is a denotation of $\underline{u} \equiv u_1, \ldots, u_k$, and $\delta \rightleftharpoons (\delta_1, \ldots, \delta_m)$ is a further denotation, then let $\gamma\delta$ result from γ by adding those members δ_i of δ that are not denotations of members of $\underline{u}$. (That is, let $\gamma\delta$ coincide with γ on $\underline{u}$ and otherwise with δ.)

We compactly write, for instance, $\natural\, A|\delta$ for the assertion of A in a situation in which the denotation δ is valid. Then δ is assumed to be a denotation of at least all indicators occurring free in A. Now we extend the primary game by transferring its rules (see §1) to such assertions as follows:

P($\wedge$)	$\natural\,(A \wedge B)\vert\delta$	$:\Rightarrow$	$\natural\, A\vert\delta$ and $\natural\, B\vert\delta$
P($\vee$)	$\natural\,(A \vee B)\vert\delta$	$:\Rightarrow$	$\natural\, A\vert\delta$ or $\natural\, B\vert\delta$
P($\exists$)	$\natural\, \exists\underline{x}\, A\underline{x}\vert\delta$	$:\Rightarrow$	for some value $\underline{r}$ of $\underline{x} : \natural\, A\underline{r}\vert\delta$
P($\exists$ den)	$\natural\, \exists\underline{u}\, A\vert\delta$	$:\Rightarrow$	for some denotation γ of $\underline{u} : \natural\, A\vert\gamma\delta$
P($\neg$)	$\natural\, \neg A\vert\delta$	$:\Rightarrow$	A has been rejected in δ.

(Here "in δ" means "in a situation in which δ is valid".) Let the primary game also contain certain corresponding external rules and internal rules for elementary sentences (cf. §1), but let it not contain further rules for compound sentences. Accordingly, the above primary rules for compound sentences may be inverted.

P($\exists$ den) is to introduce a kind of 'objectual' (or 'denotational') quantification (cf. [10, §26]). - Note that if A is a sentence in which only the indicators $\underline{u}$ occur free, it does not depend on the situation whether $\exists\underline{u}\, A$ may be asserted.

In the following, under a **situation** we understand especially a situation in which a denotation of all indicators occurring free in the considered sentences is valid. Sometimes, however, we shall omit referring to such denotations. If we speak so about *several* assertions or rejections, we assume that they all depend on the same denotation.

Rules analogous to R1 - R29 are also admissible for sentences with indicators. Examples are the following rules which are analogous to R6 and R7, respectively:

$$\Rightarrow \quad \neg\exists.\, \exists \underline{u}, v\, (F \wedge \neg\exists v\, F);$$
$$\neg\exists.\, \exists \underline{u}, v\, (F \wedge G) \quad \Rightarrow \quad \neg\exists.\, \exists \underline{u}\, (F \wedge \exists v\, G), \text{ if } v \text{ does not occur free in } F.$$

(In every individual case of these rules, the list $\underline{u}, v$ can also be replaced by any permutation of it.) Therefore, in R22 - R25, both x and t may be replaced by the same metavariable for indicators. - Since constants (especially values of substitutional variables) are assumed not to contain indicators, it is also allowed to commute consecutive existential quantifiers, e.g., when one is substitutional and the other objectual.

5.1. Proposition: Let $E_1(\underline{x}), \ldots, E_n(\underline{x})$, and $E(\underline{x})$ be elementary formulas in which at most the substitutional variables $\underline{x}$ and perhaps several indicators, $\underline{u}$, occur. Let $E_1(\underline{x}), \ldots, E_n(\underline{x}) \Rightarrow E(\underline{x})$ be an internal rule by which one must not, for any value $\underline{r}$ of $\underline{x}$, assert all of the sentences $E_1(\underline{r}), \ldots, E_n(\underline{r})$ and reject $E(\underline{r})$ in the same situation. (This rule can also be formulated by use of metavariables in place of $\underline{x}$.) Then we have $\forall \underline{x}\, \forall \underline{u}\, \big[E_1(\underline{x}) \wedge \ldots \wedge E_n(\underline{x}) \rightarrow E(\underline{x})\big]$.

The proof is straightforward.

In the following, let an *equation* $u = v$ between indicators (or proper names) be used to express that u and v denote the same object in the present situation. (To correctly decide whether this is the case we must in general *know how* to delimit and recognize objects. The following investigations, however, do not depend upon details.) - To induce that $u = v \wedge Eu \rightarrow Ev$ 'holds' for elementary sentences Eu it is sufficient to take

$$u = v,\ Eu \ \Rightarrow\ Ev.$$

as an internal rule. Here Eu may especially be an equation $u = w$. Let, moreover, $u = u$ pass for anchored. Accordingly an *equivalence relation* is given by those equations. (This may help to adjust our way to identify empirical objects.) - By induction on the complexity of sentences Au (of $\mathcal{L}$) we even obtain $u = v \wedge Au \rightarrow Av$.

If δ is a simple denotation of u, and $v \not\equiv u$, then let $\delta[u/v]$ be the simple denotation of v which results from δ by replacing u with v. Accordingly, we fix the internal rule

$$\Rightarrow \quad \natural\, u = v | (\delta, \delta[u/v]) \quad (\text{if } u \not\equiv v).$$

('Realistically' spoken, u and v are to denote the same object in $(\delta, \delta[u/v])$.) As a further internal rule we take

$$\natural\, E|\gamma \ \Rightarrow\ \natural\, E|\delta$$

if γ and δ contain the same simple denotation of every indicator occurring in E.

5.2. Proposition: Let $\underline{u} \equiv u_1, \ldots, u_k$ and $\underline{v} \equiv v_1, \ldots, v_k$ be disjoint lists of distinct indicators. Then each denotation of $\underline{u}$ can be extended to the members of $\underline{v}$ in such a way that we may assert $u_i = v_i$ for $i = 1, \ldots, k$ in this extension. For every sentence A in which the members of $\underline{v}$ do not occur we may assert

$$\exists \underline{u}\, A \;\leftrightarrow\; \exists \underline{v}\, A^{\underline{u}}_{\underline{v}} \quad \text{and} \quad \forall \underline{u}\, A \;\leftrightarrow\; \forall \underline{v}\, A^{\underline{u}}_{\underline{v}} \quad \text{in each situation.}$$

Proof: Let $\delta \rightleftharpoons (\delta_1, \ldots, \delta_k)$ where δ_i is a simple denotation of u_i $(i = 1, \ldots, k)$. Let $\delta' \rightleftharpoons (\delta_1, \ldots, \delta_k, \delta_1[u_1/v_1], \ldots, \delta_k[u_k/v_k])$. Then we have $u_i = v_i$ in δ' and so $A \leftrightarrow A^{\underline{u}}_{\underline{v}}$ in every situation in which δ' is valid. Now we can apply R22-R25, and R14. (For, in the primary game it is not forbidden to realize the simple components of δ'.) $\Box$

The following proposition will serve to justify the classical use of sentences in §6. To this end we take, for any sentence B considered so far, the new sentences $+B$ and $-+B$. Moreover, we fix the rules

P(+) $\qquad \natural + B \;:\Rightarrow\; \natural B,$
P(+, −) $\qquad \natural + B, \natural - +B \;\Rightarrow\; \natural \bot$

as internal rules, and

$\mathrm{P}_{\mathrm{ext}}(-)$ $\qquad \natural - +B \;:\Rightarrow\;$ B has been rejected

as an external rule. The latter condition on the right is equivalent to the fact that $+B$ has been rejected according to the internal rules with the exception of P(+, −).

Remark: In the primary game we may employ the rule $B \Rightarrow +B$ (i.e. the inversion of P(+)). This, however, is only due to the external rule $\mathrm{P}_{\mathrm{ext}}(-)$. Accordingly, the rule $B \Rightarrow +B$ is not classically admissible (which can be seen for $B \equiv A \vee \neg A$, e.g.), and we should generally not assert $B \rightarrow +B$.

5.3. Proposition: Let $E_1, \ldots, E_n$ be elementary sentences which include the indicators $\underline{u}$ at most $(n \geq 1)$. Then, in the primary game, we ought to assert $\neg\neg + \exists \underline{u}\, (E_1 \wedge \ldots \wedge E_n)$ in δ only if, for some denotation γ of $\underline{u}$, the sentences $E_1, \ldots, E_n$ have been anchored in $\gamma\delta$.

5.3 can similarly (but somewhat simpler) be proved as 3.4. To this consider $+\exists \underline{u}\, (E_1 \wedge \ldots \wedge E_n)$ in place of E.

Literal equality

In §1 we have pointed out that the primary rules for compound sentences may be inverted. A necessary reason for this fact is that the components of those sentences are uniquely determined by them. Further syntactical properties of formulas concerning substitutions of variables by values of them have tacitly been used in §2 to obtain contradictions of the form $\mathcal{F}^*, \neg\mathcal{F}^*$ in the proofs that the rules R1 - R3, e.g., are admissible. One of those properties means that, for any substitution $*$ of variables, if $F \equiv G$ then $F^* \equiv G^*$. Here the letters F, G are used as *objectual* variables for *occurrences* of formulas at arbitrary places, also outside of such equations.

To enable proofs of such syntactical properties of formulas of $\mathcal{L}$ we have particularly to investigate the literal equality of strings of symbols. This equality can similarly be introduced as the equality on $\mathbb{N}$ (see §4) or as in [7, §5, §9], e.g. However, in this introduction and the subsequent investigations we have to use objectual variables for occurrences of strings of symbols. But here we do not prove the mentioned properties of formulas. We only give an introduction of literal equality.

By a 'letter' or a 'word', respectively, we here understand an *occurrence* of a (connected) atomic symbol or of a string of symbols, resp., which is written down at a particular place. We include letters among words. We say that a letter a is a copy of another letter b to mean that a literally equals b. Let $o_1, \ldots, o_k$ be copies of certain distinct 'original letters'. As metavariables we use

a, b	for	indicators for copies of original letters
t, u, v, w	for	indicators for words composed of copies of original letters, and indicators especially for copies of original letters.

Let $v \equiv^\circ o_i$ $(i = 1, \ldots, k)$ and $v \doteq ta$ be elementary sentences, where

$v \equiv^\circ o_i$	means:	v literally equals the original letter with the copy o_i.
$v \doteq ta$	means:	v is composed of t and a in this order of succession.

It is clear how to anchor sentences of the forms $v \equiv^\circ o_i$, $v \doteq ta, v = w$, and $a \neq b$ (where '=' again designates the identity of denoted objects, and '≠' their diversity).

Now we can introduce a sequence of relations $(\equiv_m)$ by fixing the following internal rules (in which we omit both the sign ♮ and referring to denotations):

$$
\begin{array}{rcl}
v = w,\ Ev & \Rightarrow & Ew \quad \text{(as so far)} \\
a = b,\ a \neq b & \Rightarrow & \bot \\
v \equiv^\circ o_i,\ v \equiv^\circ o_j & \Rightarrow & \bot \quad \text{if} \quad 1 \leq i < j \leq k \\
v \equiv^\circ o_i,\ v \doteq ta & \Rightarrow & \bot \\
b \doteq ta & \Rightarrow & \bot \\
v \doteq ta,\ v \doteq ub & \Rightarrow & t = u,\ a = b \quad \text{(two rules)} \\
v \doteq ta,\ w \doteq tb & \Rightarrow & v = w
\end{array}
$$

$$
\begin{array}{rcl}
v \equiv_0 w & :\Rightarrow & (v \equiv^\circ o_1 \ \wedge\ w \equiv^\circ o_1) \ \vee\ \ldots\ \vee\ (v \equiv^\circ o_k \ \wedge\ w \equiv^\circ o_k) \\
v \equiv_{n'} w & :\Rightarrow & \exists t, a, u, b\,(v \doteq ta \ \wedge\ w \doteq ub \ \wedge\ t \equiv_n u \ \wedge\ a \equiv_0 b) \\
v \equiv w & :\Rightarrow & \exists \kappa \in \mathbb{N}.\ v \equiv_\kappa w.
\end{array}
$$

The latter three rules may also be inverted since we do not take further rules for the sentences introduced here. By a similar argument, we ought to assert $\neg\,(a \equiv^\circ o_1)$ only after we have asserted $a \equiv^{\circ} o_i$ for some $i = 2, \ldots, k$. So we may assert $\neg\neg\,(a \equiv_0 a)$. So by induction on the construction of words as considered here we obtain: $\neg\neg\,(v \equiv v)$ may be asserted in any situation in which v denotes such a word.

If one of the symbols (except 'o_1',...'o_k') used in the latter rules equals an original letter, it must be replaced by another one. - The rules concerning '$\doteq$' should be justified. (We should not take both | and ||, e.g., as original letters.)

The above mentioned syntactical properties of formulas especially concern the literal equality of components of our object language. For the investigations of those properties we require only finitely many sentences of a metalanguage. So the length and complexity of those sentences are limited. Therefore, we do not require a preceding general theory of the syntactical properties of the applied metalanguage.

§6. Purposes of assertions in the classical game

Let the **classical game** be that assertion game in which a sentence A of $\mathcal{L}$ may be asserted iff $\neg\neg A$ may be asserted in the primary game.

The rules R1 - R29 (see §2 and §3) inclusive of $\neg\neg A \Rightarrow A$ and analogous rules for sentences with indicators are admissible in the classical game. This means that in this game we may apply classical logic. - In the following we show which purposes assertions of different kinds of sentences can serve in the classical game, and that this game preserves all means of speech which are indispensable for those purposes.

Due to 3.4 (which also holds for elementary sentences containing indicators), we should assert an elementary sentence, E, in the classical game only if E has become anchored. Accordingly, for the listener or reader the 'classical' assertion of E can substitute a first hand knowledge of an anchoring of E, in particular a perception or observation, or the result of an investigation of objects.

Due to R01 and R03, in the classical game the rules

$$\begin{array}{rcl} A,\ A \rightarrow B & \Rightarrow & B \\ \forall x\, Ax & \Rightarrow & Ar \quad \text{(for values } r \text{ of } x\text{)} \end{array}$$

have the property that their premises may be asserted only if the pertinent conclusion may *already* be asserted. So we have in the classical game:

$\natural\,(A \rightarrow B)$ can serve the listener or reader as the advice to assert B (perhaps to himself only) as soon as A may be asserted.

$\natural\,\forall x\, Ax$ can serve as a substitute for $\natural\, Ar$, for any value r of x.

Similarly, $\natural\,\forall u\, A|\delta$ can serve as a substitute for $\natural\, A|\gamma\delta$ when $\gamma\delta$ is valid for any additional denotation γ of u.

By means of conjunction we can - more clearly arranged - write $A_1 \wedge A_2 \wedge A_3 \rightarrow B$ for $A_1 \rightarrow [A_2 \rightarrow (A_3 \rightarrow B)]$.

The following holds in the primary game. An assertion of the form $\natural\, A \vee B$ can without loss of information be replaced by the shorter assertion $\natural\, A$ or $\natural\, B$. In the same way, the assertion of an existential sentence, $\exists x\, Ax$, is dispensable since it can be replaced by the assertion of Ar, for some value r of x.

However, what we have just stated does not hold for existential sentences $\exists u\, A$ with indicators (objectual variables) u in place of x. Note that sentences A with free occurring indicators may in general be asserted only in particular situations. On the other hand, many empirically obtained facts can - in any situations - be summarized to sentences of the form $\exists \underline{u}\, (E_1 \wedge \ldots \wedge E_n)$ with elementary components $E_1, \ldots, E_n$. Accordingly, we have quoted Proposition 5.3 by which even from the classical assertion of $+\exists \underline{u}\, (E_1 \wedge \ldots \wedge E_n)$ we can conclude that $E_1, \ldots, E_n$ have been anchored for some denotation of $\underline{u}$. This shows how far the means of speech of the classical game are sufficient to inform about empirical datas.

Sentences of the form $\forall x\, \varepsilon\, K.\, \exists y\, A(x, y)$ have been investigated at the end of §4. A generalization of those investigations should still be worked out.

Since we dispose of certain admissible inference rules, composite formulas can be used as marks for something of data processing. As is well known, all inference rules of 'constructive' or 'intuitionistic' logic are also admissible in classical logic. Hence, (especially mathematical) composite formulas are in the classical game at least as useful as processing marks as in a language in which only intuitionistic logic is available. - For purposes, however, which have not been regarded here, a more restrictive use of assertions may be more suitable than the classical use.

Hypothetical assertions

In everyday speech and in empirical sciences one necessarily proceeds more liberally than in our classical game. So one does not only assert established facts but also uses universal hypotheses or conjectures, which often do not even get cited. If H is the conjunction of all current hypotheses, we could use (assert) certain sentences B as short for $H \rightarrow B$. Indeed, for any admissible inference rule $\mathcal{A}_1, \ldots, \mathcal{A}_n \;\Rightarrow\; \mathcal{B}$ the rule $H \rightarrow \mathcal{A}_1, \ldots, H \rightarrow \mathcal{A}_n \;\Rightarrow\; H \rightarrow \mathcal{B}$ is also admissible. (Here, the $\mathcal{A}_i$'s and $\mathcal{B}$ represent sentence schemes in which metavariables for sentences, formulas, constants, variables, indicators, or terms may occur.) But as soon as H becomes rejected, it becomes obviously unserviceable to assert sentences of the form $H \rightarrow B$ (or abbreviations of them). Accordingly, if H contains (probably) untrue hypotheses (such as simplifications of conjectures) we can instead of $H \rightarrow B$ better use the statement that B has been *deduced* from H and already (justly) asserted sentences of a given class, K, by the rules of classical logic (e.g.). This statement reminds of *necessity*, say "B is *necessary* with respect to (H, K)" (cf. [9, p.111], e.g.). (The set K should be chosen considering particular purposes. It might be a set of physical or medical sentences, e.g., that can possibly be verified.) Then the sets S_i of all sentences that are deducible at successive times t_i $(i = 0, 1, 2, \ldots)$ form a monotonic increasing sequence $S_0, S_1, S_2, \ldots$.

We can include sentences of of the form "B is deducible from H by the rules of Σ" (where Σ denotes a system of inference rules) in the object language. In order to see this we consider, for any $n \in \mathbb{N}$, the set of all sentences that are deducible from H by the rules of Σ in $\leq n$ steps of deduction. These sets can be defined by recursion on n. So their sequence can be obtained by means explained in §4, and so their union can be defined. (Note that a statement of deducibility refers to sentences. To indicate this fact, we can enclose them in quotation marks.)

Sometimes we are convinced that if we perform a certain action a, then - after an additional time δ - we shall obviously have attained a purpose e_1 *or* another purpose e_2, for instance. We explain the intended effect of the advice then to act as if the according hypothesis $\forall\tau\,(A^{\tau-\delta} \rightarrow +(E_1^\tau \vee E_2^\tau))$ (with τ for moments) holds in the classical game: By this advice, we should act as if the following holds: If $A^{\tau-\delta}$ may be asserted in the classical game, then $+(E_1^\tau \vee E_2^\tau)$ may also be asserted in that game - and hence E_1^τ or E_2^τ will have been anchored (due to an analogue to 5.3). (This anchoring will be anticipated, if we assert $A^{\tau-\delta}$ before τ.

§7. Preliminaries on higher order languages

In the following we speak 'about' so-called abstract objects like sets and relations. But we do not presuppose that they exist independently of the signs by which they are given ('designated', 'denoted', or the like). What we shall say of sets, e.g., can be accounted for as a mere manner of speaking. If we say that a sign S is (or designates) a set we only mean the following: 1. For all relevant constants (or, especially, names or indicators) c, a notation such as $c \in S$ (or $c \,\varepsilon\, S$, or "c is an element of S") serves as a sentence of the pertinent language. 2. Within that language, S is to be used 'abstractively' so that any occurrence of S in any asserted sentence of that language may be replaced by any other set sign that is said to be equal to S. (However, we do not simply identify sets that are extensionally equal. So one may prefer the word "attribute" or "property" in place of "set".) - We shall deal with a ramified type theory (cf. [7], [11], [13], [17], e.g.) in a cumulative version.

Given a set $\mathcal{E}$ of elementary formulas in which certain constants may occur. Those constants are said to be of order 0. We shall introduce sets of order 1, whose elements are constants of order 0 (or objects denoted by them), sets of order 2, whose elements are constants of order 0 or sets of order 1, etc. So a set of order n contains only elements that have orders $< n$. However, a set of order n will also be said to have any order larger than n.

To this end, we shall construct a set $\mathcal{A}$ of (first or) higher order sentences and introduce an assertion game, which contains certain '*primary rules*' to restrict assertions of sentences belonging to $\mathcal{A}$. Since this '*primary game*' does not contain further rules of assertion, and since all sentences of $\mathcal{A}$ can be shown to be non-circular, the primary rules for sentences of $\mathcal{A} \setminus \mathcal{E}$ can be *inverted* so that both, those rules and their inverses, can also be used as inference rules. By this means, even all usual inference rules of classical logic can (as in §2, §3) be shown to be admissible in the '*classical game*' which is given by the agreement that a sentence may be asserted in this game iff its double negation may be asserted in the primary game (cf. §3).

So our first *main task* will be to show that all sentences of $\mathcal{A}$ are non-circular.

Now we incompletely sketch the higher order languages that will be introduced in §8. Assume that we already dispose of certain **elementary formulas** and terms, which are said to be **original terms**. All variables that occur in those formulas or terms are said to be of order 0. Let

$\mathcal{V}_0$ = set of all variables of order 0
$\mathcal{T}_{\mathrm{Or}}$ = set of all original terms, $\mathcal{V}_0 \subset \mathcal{T}_{\mathrm{Or}}$
$\mathcal{E}$ = set of all elementary formulas (to be considered).

$\mathcal{V}_0$ is permitted to contain variables of several sorts. (Of course, $\mathcal{V}_0$ is supposed to contain denumerably many variables of every of those sorts.) Let **constants** / **sentences** be terms / formulas, respectively, without free occurring variables.

We shall introduce the following sets of higher order terms and formulas:

$\mathcal{T}_n$ = set of all (simple) terms of order n,
$\mathcal{F}_n$ = set of all formulas of order n.

Here and in the following, m, n range over (signs of) ordinal numbers belonging to a given set Ω with $\mathbb{N} \subseteq \Omega$. We define

$\mathcal{C}_n \rightleftharpoons$ set of all constants belonging to $\mathcal{T}_n$,
$\overline{\mathcal{C}}_n \rightleftharpoons \bigcup_{j \in \mathbb{N}^+} C_n^j$,

which is the set of all j-tuples $(c_1, \ldots, c_j)$ of constants $c_i \in \mathcal{C}_n$ with arbitrary length $j \in \mathbb{N}^+ \rightleftharpoons \mathbb{N} \setminus \{0\}$. Let also be given two disjunct denumerable sets $\mathcal{V}$ and $\overline{\mathcal{V}}$ of 'new' variables, which do not occur in elements of $\mathcal{T}_{\mathrm{Or}} \cup \mathcal{E}$. We shall use the elements of $\mathcal{V}$ as variables for elements of $\bigcup_{n \in \Omega} \mathcal{C}_n$, i.e. for constants of arbitrary order, and the elements of $\overline{\mathcal{V}}$ as variables for elements of $\bigcup_{n \in \Omega} \overline{\mathcal{C}}_n$, i.e. for arbitrary tuples of constants. - Moreover, let

$\overline{\mathcal{T}}_n \rightleftharpoons \bigcup_{j \in \mathbb{N}^+} \mathcal{T}_n^j \cup \overline{\mathcal{V}}$.

So $\overline{\mathcal{C}}_n$ is the set of all constants belonging to $\overline{\mathcal{T}}_n$.

As signs of the object language for $\mathcal{C}_n, \overline{\mathcal{C}}_n$, and $\in$ we shall use $C_n, \overline{C}_n$, and ε, respectively. In this introduction, $x, x_1, x_2, \ldots$ range over variables of $\mathcal{V}_0 \cup \mathcal{V}$, and $\overline{x}, \overline{y}$ over variables of $\overline{\mathcal{V}}$.

All elements of $\mathcal{C}_n \setminus \mathcal{C}_0$ will be introduced as subsets of $\bigcup_{m<n} \overline{\mathcal{C}}_m$. A constant of the form $\{\overline{x} \mathrel{\varepsilon} \overline{C}_m \colon A(\overline{x})\}$ will denote the set of all elements $c \in \overline{\mathcal{C}}_m$ satisfying $A(c)$. A sentence of the form $\exists x \mathrel{\varepsilon} C_m.\, A(x)$ is to mean that there exists a constant c of order m satisfying $A(c)$. By this means, j-ary relations ($j \in \mathbb{N}^+$) can be described in the form

$$\{(x_1, \ldots, x_j) \mathrel{\varepsilon} C_m^j \colon A(x_1, \ldots, x_j)\} \rightleftharpoons$$
$$\{\overline{x} \mathrel{\varepsilon} \overline{C}_m \colon \exists x_1 \mathrel{\varepsilon} C_m. \ldots \exists x_j \mathrel{\varepsilon} C_m. (\overline{x} =_m (x_1, \ldots, x_j) \wedge A(x_1, \ldots, x_j))\}.$$

(To this end, the sign '$=_m$' must previously be introduced suitably.) - So we at first demand that

$$\begin{array}{rll} s \in \mathcal{T}_n & \text{if} & s \in \mathcal{T}_{\text{Or}} \cup \mathcal{V}, \\ \{\overline{x} \mathrel{\varepsilon} \overline{C}_m \colon F\} \in \mathcal{T}_n & \text{if} & F \in \mathcal{F}_n,\ m < n, \\ E \in \mathcal{F}_n & \text{if} & E \in \mathcal{E}, \\ (F \wedge G), (F \vee G) \in \mathcal{F}_n & \text{if} & F, G \in \mathcal{F}_n, \\ (\neg F) \in \mathcal{F}_n & \text{if} & F \in \mathcal{F}_n, \\ (\exists x \mathrel{\varepsilon} C_m.\, F) \in \mathcal{F}_n & \text{if} & F \in \mathcal{F}_n,\ m < n, \\ (s \mathrel{\varepsilon} t) \in \mathcal{F}_n & \text{if} & s \in \overline{\mathcal{T}}_n,\ t \in \mathcal{T}_n. \end{array}$$

We shall replace these and certain further demands by corresponding rules of construction. - Note that we need not deal with complicated types that include information about 'arities' of relations. So we may simply identify types with orders.

For mathematical purposes we want also to dispose of sequences R of relations $R(0), R(1), R(2), \ldots \in \mathcal{C}_n$ satisfying

$$(\underline{c}, k) \mathrel{\varepsilon} R(l) \leftrightarrow (\underline{c}) \mathrel{\varepsilon} \overline{C}_m \ \wedge\ k < l \ \wedge\ A((\underline{c}), k, R(k))$$

for all tuples $(\underline{c}) \equiv (c_1, \ldots, c_j)$ of constants and all $k, l \in \Omega$, if any formula $A(\overline{x}, \mu, z) \in \mathcal{F}_n$ and any ordinal $m < n$ are given. By this 'recursive characterization', $R(l)$ depends upon the relations $R(k)$ with numbers $k < l$ only. - We designate R by $(\mathrm{J}\overline{x} \mathrel{\varepsilon} \overline{C}_m, \mu, z \colon A(\overline{x}, \mu, z))$. Accordingly, we demand:

$$(\mathrm{J}\overline{x} \mathrel{\varepsilon} \overline{C}_m, \mu, z \colon F)(q) \in \mathcal{T}_n \quad \text{if} \quad F \in \mathcal{F}_n,\ q \in \mathcal{T}(\Omega),\ m < n,\ \mu \in \mathcal{V}(\Omega), z \in \mathcal{V}$$

where $\mathcal{T}(\Omega)$ ($\subseteq \mathcal{T}_{\text{Or}}$) is a given set of terms whose substitution instances are elements of Ω, and $\mathcal{V}(\Omega) = \mathcal{V}_0 \cap \mathcal{T}(\Omega)$ is a set of variables for elements of Ω. ('J' is an 'induction operator'; cf. §4) - Then it can be shown that there also exists a sequence S of relations $S(0), S(1), S(2), \ldots \in \mathcal{C}_n$ satisfying

$$\begin{array}{rcl} c \mathrel{\varepsilon} S(0) & \leftrightarrow & c \mathrel{\varepsilon} \overline{C}_m \ \wedge\ A(c) \\ c \mathrel{\varepsilon} S(k+1) & \leftrightarrow & c \mathrel{\varepsilon} \overline{C}_m \ \wedge\ B(c, k, S(k)) \end{array}$$

for all $c \in \bigcup_{n \in \Omega} \overline{\mathcal{C}}_n$ and all $k \in \mathbb{N}$, if the formulas $A(\overline{x}), B(\overline{x}, \mu, z) \in \mathcal{F}_n$ and the order $m < n$ are given. (For purposes of classical reasoning, the particles $\rightarrow$, $\leftrightarrow$, and $\forall$ can be defined as in §1 and §3.)

We want to introduce equations $x = y$ such that all formulas considered are invariant under $(=)$, i.e. satisfy $c = d \wedge A(c) \to A(d)$ for all constants c, d and all formulas $A(x)$ of arbitrary orders. To this end, equal constants must especially have the same order, and equal sets must contain the same elements:

$$\begin{aligned} c = d \quad &\to \quad \forall \mu \, \varepsilon \, C_0 . \, (c \, \varepsilon \, C_\mu \; \leftrightarrow \; d \, \varepsilon \, C_\mu) \\ c = d \, \wedge \, \neg \, (c \, \varepsilon \, C_0) \quad &\to \quad c \subseteq d \, \wedge \, d \subseteq c \end{aligned}$$

where $\mu \in \mathcal{V}(\Omega)$ (again), and $c \subseteq d$ means that c is a subset of d (see below). Since the formulas $c \, \varepsilon \, C_\mu$ and $c \subseteq d$ should belong to the object language to be introduced, we demand and define the following (where $\exists \overline{x} \, \varepsilon \, t . \, F$ is to be read as "For some $\overline{x}$, $\overline{x} \, \varepsilon \, t$ and F"):

$$\begin{aligned} (t \, \varepsilon \, C_q) \in \mathcal{F}_n \quad &\text{if} \quad t \in \mathcal{T}_n, \; q \in \mathcal{T}(\Omega) \\ (\exists \overline{x} \, \varepsilon \, t . \, F) \in \mathcal{F}_n \quad &\text{if} \quad t \in \mathcal{T}_n, \; F \in \mathcal{F}_n \\ \forall \overline{x} \, \varepsilon \, s . \, F \quad &\rightleftharpoons \quad \neg \, \exists \overline{x} \, \varepsilon \, s . \, \neg F \\ s \subseteq t \quad &\rightleftharpoons \quad \forall \overline{x} \, \varepsilon \, s . \, \overline{x} \, \varepsilon \, t \, \wedge \, \neg \, (s \, \varepsilon \, C_0) \, \wedge \, \neg \, (t \, \varepsilon \, C_0) . \end{aligned}$$

Notice, however, that if q (is or) contains a variable, we do not rank C_q with the terms of $\bigcup_{n \in \Omega} \mathcal{T}_n$.

Now we presuppose: Let $(=_0)$ be an equivalence relation on $\mathcal{C}_0$ (which has already been introduced and is suitable for certain purposes). Assume that all terms of $\mathcal{T}_{\mathrm{Or}}$ and all formulas of $\mathcal{E}$ are invariant under $(=_0)$. For terms s, t of any order we define

$$\begin{aligned} s \sim t \quad &\rightleftharpoons \quad \forall \mu \, \varepsilon \, C_0 . \, (s \, \varepsilon \, C_\mu \; \leftrightarrow \; t \, \varepsilon \, C_\mu) \\ s = t \quad &\rightleftharpoons \quad s =_0 t \, \vee \, (s \subseteq t \, \wedge \, t \subseteq s \, \wedge \, s \sim t) . \end{aligned}$$

Of course, we demand that

$$(s =_0 t) \in \mathcal{F}_n \quad \text{if} \quad s, t \in \mathcal{T}_n .$$

Then it can be shown that all formulas of $\bigcup_{n \in \Omega} \mathcal{F}_n$ are invariant under $(=)$. This is our *second main task*.

The 'type-free' relations $(\subseteq)$, $(\sim)$, and $(=)$ are definable in our object language but they are neither elements of $\mathcal{C}$ nor elements of elements of $\mathcal{C}$.

Given a formula $A(x)$, a tuple $c \equiv (c_1, \ldots, c_j) \in \overline{\mathcal{C}}_m$ of constants, and some $i = 1, \ldots, j$. Then $A(c_i)$ means that the i^{th} component of c satisfies $A(x)$. Since our object language also contains variables $\overline{y}$ for such tuples c of constants, we postulate, in addition, that the object language contains a formula expressing that the i^{th} component of any given value of $\overline{y}$ belongs to $\mathcal{C}_m$ and satisfies $A(x)$. For that formula we take $\exists x \, \varepsilon \, \pi_m(\overline{y}, i) . \, A(x)$ (with π for "projection"). Generalizing we demand

$$(\exists x \, \varepsilon \, \pi_m(s, p) . \, F) \in \mathcal{F}_n \;\text{ if }\; m < n, \; s \, \varepsilon \, \overline{\mathcal{T}}_n, \; p \in \mathcal{T}(\mathbb{N}^+), \; F \in \mathcal{F}_n$$

where $\mathcal{T}(\mathbb{N}^+)$ ($\subseteq \mathcal{T}_{\rm Or}$) is a given set of terms (inclusive of variables) whose substitution instances are elements of $\mathbb{N}^+$. Of course, we want to obtain that

$$\exists x \,\varepsilon\, \pi_m((c_1, \ldots, c_j), i).\, A(x) \;\leftrightarrow\; c_i \,\varepsilon\, C_m \wedge A(c_i)$$

$(i = 1, \ldots, j)$ holds in the object language.

For constructive or predicative *analysis* in the sense of [8] inclusive of measure theory and functional analysis (as in [18], e.g.) there suffice real numbers that are given by first order Cauchy sequences of rational numbers. In the domain of those real numbers there converges every real Cauchy sequence that is *given by a corresponding first order double sequence of rational numbers.* Suitable for predicative analysis are functions $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^j$, such that if $\alpha_1, \ldots, \alpha_j$ are sequences *of the mentioned sort* which satisfy $(\alpha_1, \ldots, \alpha_j) : \mathbb{N} \to A$ then $f \circ (\alpha_1, \ldots, \alpha_j)$ is a sequence *of that sort.* So we can pursue predicative analysis in languages of low orders (as stressed in [17], see also [8, p.3]). - Nevertheless, to designate orders or types we also admit transfinite ordinal numbers.

§8. Higher order languages

In this §8 we consider only the case that all elements of $\mathcal{V}_0$ are 'substitutional' variables that range over certain constants (or especially proper names). In §10 we shall also consider formulas containing objectual variables.

Now we go into details of constructing the object language. Let $\mathcal{C}_0$ be the set of all constants of order 0 ($\mathcal{C}_0 \subseteq \mathcal{T}_{\rm Or}$). For all $w \in \mathcal{V}_0$ let $\mathcal{C}(w) \subseteq \mathcal{C}_0$ be the set of all values of w. Two variables $w, x \in \mathcal{V}_0$ are assumed to be of the **same sort** iff $\mathcal{C}(w) = \mathcal{C}(x)$. - The following sets (which we have already mentioned in §7) are supposed to be decidable: $\mathcal{E}, \mathcal{T}_{\rm Or}, \mathcal{V}_0, \mathcal{V}, \overline{\mathcal{V}}$, and $\mathcal{C}(w)$ for every $w \in \mathcal{V}_0$. - In a metalanguage we use, for instance, the particles $\in, \subset, \subseteq$ as well as $\underline{\wedge}$, $\underline{\vee}$, $\underline{\rightarrow}$, $\underline{\leftrightarrow}$, $\underline{\forall}$, and $\underline{\exists}$. In place of $\underline{\wedge}$ we sometimes write the comma.

So, for the metalanguage we presuppose that the mentioned (and some other) particles have already been introduced. However, the sentences of the metalanguage which we actually use in the following investigations are of small complexity. Therefore, those sentences are not connected with problems (as that of non-circularity) which we want to solve for the object language in the following.

For any element Φ of $\mathcal{T}_{\rm Or} \cup \mathcal{E}$ we define:

$$\begin{array}{rcl} \mathcal{V}_0(\Phi) & \rightleftharpoons & \text{set of all variables occurring (free) in } \Phi, \\ * \in \mathcal{S}_0(\Phi) & \rightleftharpoons & * \text{ is a substitution of all variables } w \in \mathcal{V}_0(\Phi) \text{ by} \\ & & \text{values } w^* \text{ of them (so that } \underline{\forall}\, w \in \mathcal{V}_0(\Phi).\, w^* \in \mathcal{C}(w)), \\ \mathcal{T}(w) & \rightleftharpoons & \{r \in \mathcal{T}_{\rm Or} : \underline{\forall} * \in \mathcal{S}_0(r).\, r^* \in \mathcal{C}(w)\} \quad \text{for} \quad w \in \mathcal{V}_0. \end{array}$$

$\mathcal{T}(w)$ is the set of all original terms whose substitution instances are elements of $\mathcal{C}(w)$. We have $\mathcal{T}(w) \cap \mathcal{C}_0 = \mathcal{C}(w)$ for all $w \in \mathcal{V}_0$.

Let an ordered set $(\Omega, <)$, $\Omega \subseteq \mathcal{C}_0$, be introduced which includes $\mathbb{N}$ (in the usual succession) as an initial segment and permits all applications of (transfinite) induction that will be performed in the following. The equality $(=)$ in Ω is assumed to be the literal equality $(\equiv)$. The elements of Ω are said to be **ordinals** (ordinal numbers). We suppose the following: If $k \in \Omega$ then $k' \in \Omega$ where k' is the (immediate) successor of k; for any two ordinals k, l it is decidable whether $k < l$. - Examples for Ω are $\mathbb{N}$ and the set of all ordinals of the form $\omega^k \cdot n_k + \omega^{k-1} \cdot n_{k-1} + \ldots + \omega \cdot n_1 + n_0$ where $k, n_0, \ldots, n_k \in \mathbb{N}$, and $n_k > 0$ if $k > 0$.

As variables for elements of Ω / $\mathbb{N}^+$, respectively, we use elements of a denumerable and decidable set $\mathcal{V}(\Omega)$ / $\mathcal{V}(\mathbb{N}^+) \subseteq \mathcal{V}_0$. As metavariables we take: i, j for elements of $\mathbb{N}^+$; k, l, m, n for elements of Ω; and λ, μ, ν for elements of $\mathcal{V}(\Omega)$. Let $\mathcal{T}(\Omega) = \mathcal{T}(\lambda) = \Omega \cup \bigcup_{x \in \mathcal{V}(\Omega) \cup \mathcal{V}(\mathbb{N}^+)} \{x, x', x'', \ldots\}$. The equations $(q = r)$ and inequations $(q < r)$ with $q, r \in \mathcal{T}(\Omega)$ are assumed to be elements of $\mathcal{E}$. - Let $\mathcal{T}(\mathbb{N}^+) = \mathcal{T}(\kappa) = \mathbb{N}^+ \cup \bigcup_{x \in \mathcal{V}(\mathbb{N}^+)} \{x, x', x'', \ldots\}$ where $\kappa \in \mathcal{V}(\mathbb{N}^+)$.

We admit that $\mathcal{E}$ contains formulas of the shape $((s_1, \ldots, s_j) \,\varepsilon\, P)$ with $s_1, \ldots, s_j \in \mathcal{T}_{\mathrm{or}}$ but $P \notin \mathcal{T}_{\mathrm{or}}$. Here, P may especially be $\mathbb{N}$ or Ω (if $j = 1$). So let $\mathbb{N}, \Omega \notin \mathcal{C}_0$.

For original terms and elementary formulas we presuppose (where s^w_r is defined as in §2):

P1: $s \in \mathcal{T}_{\mathrm{or}},\ w \in \mathcal{V}_0,\ r \in \mathcal{T}(w) \ \underline{\rightarrow}\ s^w_r \in \mathcal{T}_{\mathrm{or}}$.
P2: $E \in \mathcal{E},\ w \in \mathcal{V}_0,\ r \in \mathcal{T}(w) \ \underline{\rightarrow}\ E^w_r \in \mathcal{E}$.

We shall write $\underline{s}$ for lists $s_1, \ldots, s_j$; $(\underline{s})$ for tuples $(s_1, \ldots, s_j)$ of simple terms s_i; and '$s_1, \ldots, s_j \in \mathcal{T}_n$' for '$s_1 \in \mathcal{T}_n, \ldots, s_j \in \mathcal{T}_n$', e.g. Let

$$\mathcal{W} \rightleftharpoons \mathcal{V}_0 \cup \mathcal{V},\ \ \overline{\mathcal{W}} \rightleftharpoons \mathcal{W} \cup \overline{\mathcal{V}}.$$

In the following $w, x, y, z, x_1, x_2, \ldots$ range over arbitrary variables (belongig to $\overline{\mathcal{W}}$), and $\overline{x}, \overline{y}$ over elements of $\overline{\mathcal{V}}$. Distinctly denoted variables are assumed to be distinct. Accordingly, a list $\underline{x}$ of variables is assumed to be a list of distinct variables. ($\underline{x}$ is to be distinguished from $\overline{x}$.)

Induktive definitions of $\mathcal{T}_n$, $\overline{\mathcal{T}}_n$ and $\mathcal{F}_n$ $(n \in \Omega)$:
Sentences (belonging to a metalanguage) of the forms $(s \in \mathcal{T}_n)$, $(s \in \overline{\mathcal{T}}_n)$ and $(F \in \mathcal{F}_n)$ are to be verified by their deductions by the following '$\mathcal{T}, \mathcal{F}$-rules'. (In these rules, $\Rightarrow$ indicates the permitted deduction steps; the pertinent conditions for applications of these rules are quoted behind the word "if".)

$$
\begin{array}{rcll}
 & \Rightarrow & s \in \mathcal{T}_n & (\text{if } s \in \mathcal{T}_{\text{Or}}) \\
 & \Rightarrow & z \in \mathcal{T}_n & (\text{if } z \in \mathcal{V}) \\
F \in \mathcal{F}_n & \Rightarrow & \{\overline{x}\ \varepsilon\ \overline{C}_m : F\} \in \mathcal{T}_n & (\text{if } m < n) \\
F \in \mathcal{F}_n & \Rightarrow & (\mathrm{J}\overline{x}\ \varepsilon\ \overline{C}_m, \mu, z : F)(q) \in \mathcal{T}_n & (\text{if } m < n,\ z \in \mathcal{V},\ q \in \mathcal{T}(\Omega)\,) \\
 & \Rightarrow & \overline{x} \in \overline{\mathcal{T}}_n & \\
s_1, \ldots, s_j \in \mathcal{T}_n & \Rightarrow & (s_1, \ldots, s_j) \in \overline{\mathcal{T}}_n & \\
 & \Rightarrow & E \in \mathcal{F}_n & (\text{if } E \in \mathcal{E}) \\
F, G \in \mathcal{F}_n & \Rightarrow & (F \wedge G), (F \vee G) \in \mathcal{F}_n & (\text{two rules}) \\
F \in \mathcal{F}_n & \Rightarrow & (\neg F) \in \mathcal{F}_n & \\
F \in \mathcal{F}_n & \Rightarrow & (\exists x\ \varepsilon\ C_m.\, F) \in \mathcal{F}_n & (\text{if } m < n,\ x \in \mathcal{W}) \\
s \in \overline{\mathcal{T}}_n,\ t \in \mathcal{T}_n & \Rightarrow & (s\ \varepsilon\ t) \in \mathcal{F}_n & \\
t \in \mathcal{T}_n,\ F \in \mathcal{F}_n & \Rightarrow & (\exists \overline{x}\ \varepsilon\ t.\, F) \in \mathcal{F}_n & \\
s \in \overline{\mathcal{T}}_n,\ F \in \mathcal{F}_n & \Rightarrow & (\exists x\ \varepsilon\ \pi_m(s,p).\, F) \in \mathcal{F}_n & (\text{if } m < n,\ x \in \mathcal{W},\ p \in \mathcal{T}(\mathbb{N}^+)) \\
s, t \in \mathcal{T}_n & \Rightarrow & (s =_0 t) \in \mathcal{F}_n & \\
s \in \overline{\mathcal{T}}_n & \Rightarrow & (s\ \varepsilon\ C^p_q) \in \mathcal{F}_n & (\text{if } p \in \mathcal{T}(\mathbb{N}^+),\ q \in \mathcal{T}(\Omega)).
\end{array}
$$

Thus, $\mathcal{T}_0 = \mathcal{T}_{\text{Or}} \cup \mathcal{V}$, and $\mathcal{E} \subset \mathcal{F}_0$. - The following occurrences of variables in terms or formulas are said to be *bound*: x in $(\exists x\ \varepsilon\ C_m.\, F)$ and in $(\exists x\ \varepsilon\ \pi_m(s,p).\, F)$, $\overline{x}$ in $\{\overline{x}\ \varepsilon\ \overline{C}_m : F\}$ and in $(\exists \overline{x}\ \varepsilon\ t.\, F)$, and $\overline{x}, \mu, z$ in $(J\overline{x}\ \varepsilon\ \mathcal{C}_m, \mu, z : F)$. All other occurrences of variables in terms or formulas are said to be *free*, i.e. not bound. - We presuppose, of course, that if $(\Phi \in \mathcal{T}_n), (\Phi \in \overline{\mathcal{T}}_n)$, or $(\Phi \in \mathcal{F}_n)$ is a conclusion of one of the latter rules except the first or seventh, then Φ does not belong to $\mathcal{T}_{\text{Or}} \cup \mathcal{E}$. - Sometimes we shall as usual omit brackets from formulas. - Definitions:

$$
\begin{array}{l}
\mathcal{A}_n \rightleftharpoons \text{ set of all sentences of order } n\ (\subset \mathcal{F}_n); \\
\mathcal{T} \rightleftharpoons \bigcup_{n\in\Omega} \mathcal{T}_n;\ \overline{\mathcal{T}} \rightleftharpoons \bigcup_{n\in\Omega} \overline{\mathcal{T}}_n;\ \mathcal{F} \rightleftharpoons \bigcup_{n\in\Omega} \mathcal{F}_n; \\
\mathcal{C} \rightleftharpoons \bigcup_{n\in\Omega} \mathcal{C}_n;\ \overline{\mathcal{C}} \rightleftharpoons \bigcup_{n\in\Omega} \overline{\mathcal{C}}_n;\quad \text{and } \mathcal{A} \rightleftharpoons \bigcup_{n\in\Omega} \mathcal{A}_n.
\end{array}
$$

Notice that $C^j_m, C_m, \overline{C}_m \notin \mathcal{C}$. Similarly, $C^p_q, \pi_m(s,p) \notin \mathcal{T}$. - As metavariables we shall use: p for elements of $\mathcal{T}(\mathbb{N}^+)$; q for elements of $\mathcal{T}(\Omega)$; r, s, t for terms (i.e. elements of $\mathcal{T} \cup \overline{\mathcal{T}}$); F, G, H for formulas (i.e. elements of $\mathcal{F}$); a, b, c, d for constants (i.e. elements of $\mathcal{C} \cup \overline{\mathcal{C}}$); A, B for sentences (i.e. elements of $\mathcal{A}$); and, for instance, $A(x_1, \ldots, x_j)$ for formulas, in which at most the variables $x_1, \ldots, x_j$ occur free.

To formulate assertion rules for sentences of $\mathcal{A}$ we use some **definitions**:

$$
\begin{array}{rcl}
\mathcal{C}_m(x) & \rightleftharpoons & \left\{ \begin{array}{ll} \mathcal{C}(x) & \text{for } x \in \mathcal{V}_0 \\ \mathcal{C}_m & \text{for } x \in \mathcal{V} \\ \overline{\mathcal{C}}_m & \text{for } x \in \overline{\mathcal{V}} \end{array} \right. \\
(c_1, \ldots, c_j) \in \mathcal{C}_m(x_1, \ldots, x_k) & \rightleftharpoons & j = k\ \underline{\wedge}\ \underline{\forall} i \le j.\ c_i \in \mathcal{C}_m(x_i).
\end{array}
$$

For $\Phi \in \mathcal{T} \cup \overline{\mathcal{T}} \cup \mathcal{F}$ and $\mathcal{U} \subseteq \overline{\mathcal{W}}$ let

$$\mathcal{U}(\Phi) \ \rightleftharpoons\ \text{set of all variables} \in \mathcal{U} \text{ that occur free in } \Phi.$$

A term s / a formula F is said to be *invariant* under an equivalence relation $(\approx)$ on $\mathcal{C}_n$ iff for all $(\underline{c}), (\underline{d}) \in \mathcal{C}_n(\underline{w})$ with $\{\underline{w}\} = \{w_1, \ldots, w_j\} = \overline{\mathcal{W}}(\underline{s})$ / $\overline{\mathcal{W}}(F)$, respectively, the following holds:

$$\begin{array}{lcl} \underline{c} \approx \underline{d} & \rightarrow & s\frac{\underline{w}}{\underline{c}} \approx s\frac{\underline{w}}{\underline{d}}, \\ \underline{c} \approx \underline{d} & \rightarrow & (F\frac{\underline{w}}{\underline{c}} \leftrightarrow F\frac{\underline{w}}{\underline{d}}), \text{ respectively,} \end{array}$$

where

$$(\underline{c}) \approx (\underline{d}) \;\rightleftharpoons\; \underline{c} \approx \underline{d} \;\rightleftharpoons\; c_1 \approx d_1 \wedge \ldots \wedge c_j \approx d_j$$

and the substitutions $\frac{\underline{w}}{\underline{c}}$ and $\frac{\underline{w}}{\underline{d}}$ are defined as in §2. (This has been formulated somewhat beforehand.)

Presupposition: Let $(\approx_0)$ be an equivalence relation on $\mathcal{C}_0$ (which has already been introduced) under which all terms of $\mathcal{T}_{\mathrm{Or}}$ and all formulas of $\mathcal{E}$ are invariant. For all $w \in \mathcal{V}_0$, let the scope $\mathcal{C}(w)$ of values of w be invariant, i.e., for all c, d with $c \approx_0 d$ and $c \in \mathcal{C}(w)$ let also be $d \in \mathcal{C}(w)$. - For all $k, l \in \Omega$ let: $k \approx_0 l \;\leftrightarrow\; k = l$.

Let the '**primary game**' contain the following '**primary rules**' of assertion: Assume that we have already agreed upon certain primary rules (or usage) for elementary sentences ($\in \mathcal{E}$). Let each of those rules be 'external' or 'internal' (see §1). For the sentences of $\mathcal{A} \setminus \mathcal{E}$ we now stipulate the following primary rules, which we include among the internal rules:

$$\begin{array}{rcl} \natural\,(A \wedge B) & :\Rightarrow & \natural\,A \text{ and } \natural\,B \\ \natural\,(A \vee B) & :\Rightarrow & \natural\,A \text{ or } \natural\,B \\ \natural\,\neg A & :\Rightarrow & A \text{ rejected (see §1)} \\ \natural\,\exists x\,\varepsilon\,C_m.\,A(x) & :\Rightarrow & \text{for some } c:\ \natural\,c \in \mathcal{C}_m(x),\ \natural\,A(c) \\ \natural\,c\,\varepsilon\,\{\overline{x}\,\varepsilon\,\overline{C}_m\colon A(\overline{x})\} & :\Rightarrow & \natural\,c \in \overline{\mathcal{C}}_m,\ \natural\,A(c), \end{array}$$

for $R(\nu) \rightleftharpoons (\mathrm{J}\overline{x}\,\varepsilon\,\overline{C}_m, \mu, z\colon A(\overline{x}, \mu, z))(\nu) \in \mathcal{T}$:

$$\begin{array}{rcl} \natural\,(\underline{c}, k)\,\varepsilon\,R(l) & :\Rightarrow & \natural\,(\underline{c}) \in \overline{\mathcal{C}}_m,\ \natural\,k < l,\ \natural\,A((\underline{c}), k, R(k)) \\ \natural\,\exists\overline{x}\,\varepsilon\,b(\overline{x}).\,A(\overline{x}) & :\Rightarrow & \text{for some } c:\ \natural\,c\,\varepsilon\,b(c),\ \natural\,A(c) \\ \natural\,\exists x\,\varepsilon\,\pi_m((c_1, \ldots, c_j), i).\,A(x) & :\Rightarrow & \natural\,c_i \in \mathcal{C}_m(x),\ \natural\,A(c_i) \quad \text{(if } i \leq j) \\ \natural\,\exists x\,\varepsilon\,\pi_m((c_1, \ldots, c_j), i).\,A(x) & :\Rightarrow & \natural\,\bot \quad \text{(if } i > j) \\ \natural\,\exists x\,\varepsilon\,\pi_m(s, p).\,A(x) & :\Rightarrow & \natural\,\bot \quad \text{(if } \overline{\mathcal{W}}(s, p) = \{x\}\text{)} \\ \natural\,c =_0 d & :\Rightarrow & \natural\,c, d \in \mathcal{C}_0,\ \natural\,c \approx_0 d \\ \natural\,c\,\varepsilon\,C_m^j & :\Rightarrow & \natural\,c \in \mathcal{C}_m^j. \end{array}$$

For $a \notin \bigcup_{j \in \mathbb{N}^+} \big(\mathcal{C}^j \times \Omega\big)$ let $(a\,\varepsilon\,R(l))$ pass for rejected. For $d \in \mathcal{C}_0$ let $(c\,\varepsilon\,d)$ pass for rejected, too. - Assertions of 'auxiliary sentences' of the forms $(c \in \mathcal{C}_m(x))$, $(c \in \overline{\mathcal{C}}_m)$, $(k < l)$, and $(c \in \mathcal{C}_m^j)$ ought, of course, to be justified additionally. - *Assertions of sentences of $\mathcal{A}$ are not to be restricted besides.*

To see that the primary rules for sentences of $\mathcal{A} \setminus \mathcal{E}$ can be inverted we shall prove that all sentences of $\mathcal{A}$ are non-circular in the following sense, and we presuppose that the assertability of sentences of $\mathcal{E} \cup \{(c \approx_0 d) \colon c, d \in \mathcal{C}_0\}$ is fixed without referring to assertions of sentences of $\mathcal{A} \setminus \mathcal{E}$.

Definitions: A sentence C is said to be a **predecessor** of D iff C is deducible from D by at least one application of the following rules (where $\Rightarrow$ again indicates the deduction steps):

$$\begin{array}{rcll} A \wedge B & \Rightarrow & A,\ B & \text{(two rules)} \\ A \vee B & \Rightarrow & A,\ B & \text{(two rules)} \\ \neg A & \Rightarrow & A & \\ \exists x \,\varepsilon\, C_m .\, A(x) & \Rightarrow & A(c), & \text{if } c \in \mathcal{C}_m(x) \\ c \,\varepsilon\, \{\overline{x} \,\varepsilon\, \overline{C}_m : A(\overline{x})\} & \Rightarrow & A(c), & \text{if } c \in \overline{\mathcal{C}}_m, \end{array}$$

for $R(\nu) \rightleftharpoons (\mathrm{J}\overline{x} \,\varepsilon\, \overline{C}_m, \mu, z \colon A(\overline{x}, \mu, z))(\nu) \in \mathcal{T}$:

$$(\underline{c}, k) \,\varepsilon\, R(l) \quad\Rightarrow\quad A((\underline{c}), k, R(k)), \qquad \text{if } (\underline{c}) \in \overline{\mathcal{C}}_m \text{ and } k < l,$$

for terms $b(\overline{x})$ of the form $\{\overline{y} \,\varepsilon\, \overline{C}_m \colon G\}$ or $(\mathrm{J}\overline{y} \,\varepsilon\, \overline{C}_m, \lambda, w \colon G)(l)$:

$$\begin{array}{rcll} \exists \overline{x} \,\varepsilon\, b(\overline{x}) .\, A(\overline{x}) & \Rightarrow & c \,\varepsilon\, b(c),\ A(c), & \text{if } c \in \overline{\mathcal{C}}_m \\ \exists x \,\varepsilon\, \pi_m((c_1, \ldots, c_j), i) .\, A(x) & \Rightarrow & A(c_i), & \text{if } i \le j,\ c_i \in \mathcal{C}_m(x) \\ c =_0 d & \Rightarrow & c \approx_0 d, & \text{if } c, d \in \mathcal{C}_0 . \end{array}$$

In every individual case of any of these rules the conclusion is said to be an **immediate predecessor** of the premise, iff the conditions quoted behind the word “if” are satisfied. However, we do not include those conditions (which are formulated as auxiliary sentences) among the predecessors. Sentences that do not occur as premises of the just mentioned rules have no predecessors.

A sentence is said to be **non-circular** iff it is not a predecessor of itself.

For the announced proof that all sentences are non-circular we need some preliminaries. At first we define

$$\mathcal{T}_n(w) \quad\rightleftharpoons\quad \begin{cases} \mathcal{T}(w) & \text{for } \ w \in \mathcal{V}_0 \\ \mathcal{T}_n & \text{for } \ w \in \mathcal{V} \\ \overline{\mathcal{T}}_n & \text{for } \ w \in \overline{\mathcal{V}}. \end{cases}$$

8.1. Lemma: Let $w, y \in \overline{\mathcal{W}}$. Then we have:

$$\begin{array}{rcl} s \in \mathcal{T}_n(y),\ r \in \mathcal{T}_n(w) & \underline{\rightarrow} & s^w_r \in \mathcal{T}_n(y); \\ F \in \mathcal{F}_n,\ r \in \mathcal{T}_n(w) & \underline{\rightarrow} & F^w_r \in \mathcal{F}_n . \end{array}$$

Regard that $\mathcal{T}(\Omega), \mathcal{T}(\mathrm{I\!N}^+) \in \{\mathcal{T}(y) \colon y \in \mathcal{V}_0\}$.

Proof: At first let $y \in \mathcal{V}_0$, and $s \in \mathcal{T}(y)$. This means that if $\underline{x}$ is a list of all distinct elements of $\mathcal{V}(s)$, and $\underline{a} \in \mathcal{C}(\underline{x})$, then $s^{\underline{x}}_{\underline{a}} \in \mathcal{C}(y)$. Let $w \in \mathcal{V}_0(s)$ (otherwise $s^w_r \equiv s$) and $r \in \mathcal{T}(w)$. Let $\underline{z}$ be a list of all distinct elements of $\mathcal{V}_0(s^w_r)$, and let $\underline{y}$ be $\underline{z}$ without w. Then for all $\underline{c} \in \mathcal{C}(\underline{z})$, by setting $r^\circ \rightleftharpoons r^{\underline{z}}_{\underline{c}}$ and $\underline{b} \rightleftharpoons \underline{y}^{\underline{z}}_{\underline{c}}$, we have $r^\circ \in \mathcal{C}(w)$ and hence $(s^w_r)^{\underline{z}}_{\underline{c}} \equiv s^{w,\,\underline{y}}_{r^\circ,\underline{b}} \in \mathcal{C}(y)$. So $s^w_r \in \mathcal{T}(y)$. -
Now let $y \in \mathcal{V} \cup \overline{\mathcal{V}}$. We prove 7.1 in this case by **induction on the $\mathcal{T}, \mathcal{F}$-rules** (i.e. on the number of steps of construction by the $\mathcal{T}, \mathcal{F}$-rules): Let $r \in \mathcal{T}_n(w)$, and let $*$ denote the substitution of r for w. We write "I.H." for "induction hypothesis". - For arbitrary elements Φ of $\mathcal{T}_n \cup \overline{\mathcal{T}}_n \cup \mathcal{F}_n$ we conclude $\Phi^* \in \mathcal{T}_n \cup \overline{\mathcal{T}}_n \cup \mathcal{F}_n$ from the I.H.: $s^* \in \mathcal{T}_n / \ s^* \in \overline{\mathcal{T}}_n / \ F^* \in \mathcal{F}_n$, respectively, holds for all terms s and formulas F for which a previous deduction of $(s \in \mathcal{T}_n) / \ (s \in \overline{\mathcal{T}}_n) / \ (F \in \mathcal{F}_n)$ by the $\mathcal{T}, \mathcal{F}$-rules is required for a deduction of $(\Phi \in \mathcal{T}_n)$, $(\Phi \in \overline{\mathcal{T}}_n)$ or $(\Phi \in \mathcal{F}_n)$.

▷ Let $\Phi \in \mathcal{T}_{\text{or}} \subset \mathcal{T}_n$. If $w \in \mathcal{V}_0$ then $r \in \mathcal{T}(w)$, so that (by **P1**) $\Phi^* \in \mathcal{T}_{\text{or}}$.
If $w \notin \mathcal{V}_0$ then w does not occur in Φ; therefore, $\Phi^* \equiv \Phi \in \mathcal{T}_n$.

▷ Let $\Phi \equiv z \in \mathcal{V} \subset \mathcal{T}_n$. If $w \equiv z$ then $z^* \equiv r \in \mathcal{T}_n(w) = \mathcal{T}_n$.
If $w \not\equiv z$ then $z^* \equiv z \in \mathcal{T}_n$.

▷ Let $\Phi \equiv (\mathrm{J}\overline{x} \,\varepsilon\, \overline{C}_m, \mu, z \colon F)(q) \in \mathcal{T}_n$. So $F \in \mathcal{F}_n, m < n$, and so (by I.H.) $F^* \in \mathcal{F}_n$.
If $w \notin \{\overline{x}, \mu, z\}$ then $\Phi^* \equiv (\mathrm{J}\overline{x} \,\varepsilon\, \overline{C}_m, \mu, z \colon F^*)(q^*) \in \mathcal{T}_n$ (since $q^* \in \mathcal{T}(\Omega)$).
If $w \in \{\overline{x}, \mu, z\}$ then $\Phi^* \equiv (\mathrm{J}\overline{x} \,\varepsilon\, \overline{C}_m, \mu, z \colon F)(q^*)$, thus again $\Phi^* \in \mathcal{T}_n$.

▷ Let $\Phi \equiv (s_1, \ldots, s_j) \in \overline{\mathcal{T}}_n$. Then $s_1, \ldots, s_j \in \mathcal{T}_n$. So (by I.H.) $s_1^*, \ldots, s_j^* \in \mathcal{T}_n$, and so $\Phi^* \equiv (s_1^*, \ldots, s_j^*) \in \overline{\mathcal{T}}_n$.

The remaining steps of induction can be performed analogously. □

Definitions: Let $\Phi \in \mathcal{T} \cup \overline{\mathcal{T}} \cup \mathcal{F}$.

$$\begin{array}{rcl}
\mathcal{A}^{\text{nc}} & \rightleftharpoons & \text{set of all non-circular sentences.} \\
* \in \mathcal{S}_n(\Phi) & \rightleftharpoons & * \text{ is a substitution of all variables } w \in \overline{\mathcal{W}}(\Phi) \\
 & & \text{by constants } w^* \in \mathcal{C}_n(w) \text{ and satisfies} \\
 & & \underline{\forall}\, w \in \mathcal{V}(\Phi).\ \underline{\forall}\ a \in \overline{\mathcal{C}}_n.\ (a \,\varepsilon\, w^*) \in \mathcal{A}^{\text{nc}} \\
F \in \mathcal{F}_n^{\text{nc}} & \rightleftharpoons & \underline{\forall} * \in \mathcal{S}_n(F).\ F^* \in \mathcal{A}^{\text{nc}}. \\
* \in \mathcal{S}_n & \rightleftharpoons & \underline{\exists}\, \Phi \in \mathcal{T} \cup \overline{\mathcal{T}} \cup \mathcal{F}.\ * \in \mathcal{S}_n(\Phi).
\end{array}$$

Remark: By 8.1 we have:

$$\begin{array}{rcl}
s \in \mathcal{T}_n,\ * \in \mathcal{S}_n(s) & \underline{\rightarrow} & s^* \in \mathcal{C}_n. \\
s \in \overline{\mathcal{T}}_n,\ * \in \mathcal{S}_n(s) & \underline{\rightarrow} & s^* \in \overline{\mathcal{C}}_n. \\
F \in \mathcal{F}_n,\ * \in \mathcal{S}_n(F) & \underline{\rightarrow} & F^* \in \mathcal{A}_n.
\end{array}$$

Definition: Let $t\binom{x}{c}^*$ be the term which results from t if we at first replace all free occurrences of x by c, and then apply the substitution $*$. Let $\binom{x}{c}^*$ be the corresponding compound substitution.

8.2. Lemma: $\quad * \in \mathcal{S}_n,\ c \in \mathcal{C}_m(x),\ \mathcal{F}_m \subseteq \mathcal{F}_m^{\text{nc}},\ m < n \ \underline{\rightarrow}\ \binom{x}{c}^* \in \mathcal{S}_n.$

Proof: Let $* \in \mathcal{S}_n$, $c \in \mathcal{C}_m(x)$, $\mathcal{F}_m \subseteq \mathcal{F}_m^{\text{nc}}$, $m < n$, as well as $w \in \mathcal{V}$ with $w^* \in \mathcal{C}_n$, and $a \in \overline{\mathcal{C}}_n$. We only need show that $(a \;\varepsilon\; w(^x_c)^*) \in \mathcal{A}^{\text{nc}}$ (cf. the definition of $\mathcal{S}_n(\Phi)$). If $w \not\equiv x$ then $(a \;\varepsilon\; w(^x_c)^*) \equiv (a \;\varepsilon\; w^*) \in \mathcal{A}^{\text{nc}}$. Now let $w \equiv x$. So $(a \;\varepsilon\; w(^x_c)^*) \equiv (a \;\varepsilon\; c)$. If $a \in \overline{\mathcal{C}}_n \backslash \overline{\mathcal{C}}_m$ then $(a \;\varepsilon\; c)$ has no predecessors (since $c \in \mathcal{C}_m$) so that $(a \;\varepsilon\; c)$ is non-circular. If $a \in \overline{\mathcal{C}}_m$ then $(a \;\varepsilon\; c) \in \mathcal{A}_m \subseteq \mathcal{A}^{\text{nc}}$. $\square$

8.3. Theorem: All sentences of $\mathcal{A}$ are non-circular.

Proof: By 'composite induction' we show that $\mathcal{F}_n \subseteq \mathcal{F}_n^{\text{nc}}$ for all $n \in \Omega$: We start from the induction hypothesis

I.H.1: $\mathcal{F}_m \subseteq \mathcal{F}_m^{\text{nc}}$ for all $m < n$.

From this we conclude $\mathcal{F}_n \subseteq \mathcal{F}_n^{\text{nc}}$ by induction on the $\mathcal{T}, \mathcal{F}$-rules. To this end, for arbitrary formulas $H \in \mathcal{F}_n$ we infer $H \in \mathcal{F}_n^{\text{nc}}$ from I.H.1 and the further hypothesis

I.H.2: $F \in \mathcal{F}_n^{\text{nc}}$ for all formulas F such that the deduction of $(H \in \mathcal{F}_n)$ by the $\mathcal{T}, \mathcal{F}$-rules requires a previous deduction of $(F \in \mathcal{F}_n)$ by those rules.

Let $H \in \mathcal{F}_n$ and $* \in \mathcal{S}_n(H)$. We have to show that H^* is non-circular. To this it suffices to show that all immediate predecessors of H^* are non-circular.

▷ If $H \in \mathcal{E}$ then $H^* \in \mathcal{E} \cap \mathcal{A} \subset \mathcal{A}^{\text{nc}}$.

▷ Let $H \in \{(F \wedge G), (F \vee G)\}$ with $F, G \in \mathcal{F}_n$. By I.H.2 we have $F, G \in \mathcal{F}_n^{\text{nc}}$, thus $F^*, G^* \in \mathcal{A}^{\text{nc}}$ and so $H^* \in \{(F^* \wedge G^*), (F^* \vee G^*)\} \subset \mathcal{A}^{\text{nc}}$.

▷ Let $H \equiv (\neg F)$ with $F \in \mathcal{F}_n$. By I.H.2, $F \in \mathcal{F}_n^{\text{nc}}$. So $H^* \equiv (\neg F^*) \in \mathcal{A}^{\text{nc}}$.

▷ Let $H \equiv (\exists x \;\varepsilon\; C_m. F)$ with $m < n$ and $F \in \mathcal{F}_n$. By I.H.2, $F \in \mathcal{F}_n^{\text{nc}}$. Every immediate predecessor of H^* has the form $F(^x_c)^*$ with $c \in \mathcal{C}_m(x)$ and is, therefore, non-circular (since $(^x_c)^* \in \mathcal{S}_n$ holds by 8.2 and I.H.1). Thus, H^* is non-circular, too.

▷ Let $H \equiv (s \;\varepsilon\; t)$ with $s \in \overline{\mathcal{T}}_n$, $t \in \mathcal{T}_n$. By 8.1, $s^* \in \overline{\mathcal{C}}_n$.
Case 1: Let $t \in \mathcal{T}_{\text{or}}$. Then $t^* \in \mathcal{C}_0$. So $H^* \equiv (s^* \;\varepsilon\; t^*)$ has no predecessors.
Case 2: Let $t \in \mathcal{V}$. Because of $* \in \mathcal{S}_n(H)$ and $s^* \in \overline{\mathcal{C}}_n$ we have $H^* \equiv (s^* \;\varepsilon\; t^*) \in \mathcal{A}^{\text{nc}}$.
Case 3: $t \equiv (\mathrm{J}\overline{x} \;\varepsilon\; \overline{C}_m, \mu, z \colon F)(q)$ with $F \in \mathcal{F}_n$ and $m < n$. By I.H.2, $F \in \mathcal{F}_n^{\text{nc}}$. Let $t^* \equiv R(q^*) \rightleftharpoons (\mathrm{J}\overline{x} \;\varepsilon\; \overline{C}_m, \mu, z : A(\overline{x}, \mu, z))(q^*)$. Now we suppose that $((\underline{c}, h) \;\varepsilon\; R(k))$ is non-circular for all $(\underline{c}) \in \overline{\mathcal{C}}_m$ and all $h < k$. Then for all $(\underline{c}) \in \overline{\mathcal{C}}_m$ we obtain $(^{\overline{x},\, \mu,\, z}_{(\underline{c}), k, R(k)})^* \in \mathcal{S}_n$ (since $\overline{x}, \mu \notin \mathcal{V}$). Because of $F \in \mathcal{F}_n^{\text{nc}}$ it follows that, for all $(\underline{c}) \in \overline{\mathcal{C}}_m$, the following sentences are non-circular: $F(^{\overline{x},\, \mu,\, z}_{(\underline{c}), k, R(k)})^*$, i.e. $A((\underline{c}), k, R(k))$, and hence $((\underline{c}, k) \;\varepsilon\; R(l))$. Since we may conclude so for all $k, l \in \Omega$, it follows by induction on Ω especially that $H^* \equiv (s^* \;\varepsilon\; R(q^*))$ is non-circular.
The residual case 4: $t \equiv \{\overline{x} \;\varepsilon\; \overline{C}_m \colon F\}$ can even be treated simpler.

▷ Let $H \equiv (\exists \overline{x} \;\varepsilon\; t.\, F)$ with $t \in \mathcal{T}_n$ and $F \in \mathcal{F}_n$. By I.H.2, $F \in \mathcal{F}_n^{\text{nc}}$. As in the case "$H \equiv (s \;\varepsilon\; t)$" we also obtain $(\overline{x} \;\varepsilon\; t) \in \mathcal{F}_n^{\text{nc}}$. Every immediate predecessor of H^* has the form $(\overline{x} \;\varepsilon\; t)(^{\overline{x}}_c)^* \equiv (c \;\varepsilon\; t(^{\overline{x}}_c)^*)$ or $F(^{\overline{x}}_c)^*$ with $c \in \overline{\mathcal{C}}_m$ where $t(^{\overline{x}}_c)^*$ has the form $\{\overline{y} \;\varepsilon\; \overline{C}_m \ldots\}$ or $(\mathrm{J}\overline{y} \;\varepsilon\; \overline{C}_m \ldots)(l)$. (Otherwise, H^* has no predecessor.) If $\overline{x} \notin \overline{\mathcal{V}}(t)$ then $t(^{\overline{x}}_c)^* \equiv t^* \in \mathcal{C}_n$, so that $m < n$. If $\overline{x} \in \overline{\mathcal{V}}(t)$ then $t \notin \mathcal{T}_0$, so that also $t\,(\in \mathcal{T}_n)$

has the indicated form. So $m < n$, again. So, in every case, $c \in \overline{\mathcal{C}}_n$, hence $\binom{\overline{x}}{c}^* \in \mathcal{S}_n$ (since $\overline{x} \notin \mathcal{V}$). Therefore, the considered (i.e. all) immetiate predecessors of H^* are non-circular.

▷ For $H \equiv (\exists x \,\varepsilon\, \pi_m(s,p).\, F)$ we can argue as in the above case "$H \equiv (\exists x \,\varepsilon\, C_m.\, F)$".

▷ Let $H \equiv (s =_0 t)$ with $s, t \in T_n$. If H^* has an immediate predecessor, then this predecessor is $(s^* \approx_0 t^*)$ with $s^*, t^* \in \mathcal{C}_0$, which has no predecessor. So $H^* \in \mathcal{A}^{\mathrm{nc}}$.

▷ Let $H \equiv (s \,\varepsilon\, C^p_q)$ with $s \in \mathcal{T}_n$. Then $H^* \equiv (s^* \,\varepsilon\, C^{p^*}_{q^*})$ has no predecessors.

We have shown that $\mathcal{F}_n \subseteq \mathcal{F}^{\mathrm{nc}}_n$ for all $n \in \Omega$. It follows especially that all sentences are non-circular. □ - By 8.3 and the results of §2 and §3 we obtain:

8.3* Corollary: All primary rules for sentences of $\mathcal{A} \setminus \mathcal{E}$ can be inverted so that we may argue classically with sentences of $\mathcal{A}$ in the classical game.

In the proof of 8.3 we have only employed the property of $\mathcal{A}^{\mathrm{nc}}$ that this set is progressive in the following sense: A subset $\mathcal{X} \subseteq \mathcal{A}$ is said to be **progressive** iff it satisfies the following condition: For all $B \in \mathcal{A}$, if $A \in \mathcal{X}$ for all immediate predecessors A of B, then $B \in \mathcal{X}$. - So we have

8.4. Theorem: Every progressive subset of $\mathcal{A}$ includes $\mathcal{A}$.

Here, by a subset of $\mathcal{A}$ we mean a set which is given by a notation $\mathcal{X}$ such that, for any sentence A ($\in \mathcal{A}$), the notation $(A \in \mathcal{X})$ is introduced as a non-circular sentence of a metalanguage. We need not define the scope of these subsets in more detail since we shall apply 8.4 only in a particular case (in the proof of 10.4).

Now we repeat some former definitions, define an equivalence relation $(=)$ on $\mathcal{C}$, and show that all terms and formulas are invariant under that relation.

Definitions: For $x \in \mathcal{W}$ and $s, t \in \mathcal{T}$ we define:

$$
\begin{array}{rcll}
\forall x \,\varepsilon\, C_n.\, F & \rightleftharpoons & \neg\, \exists x \,\varepsilon\, C_n.\, \neg F & \\
\exists\, w\, F & \rightleftharpoons & \exists\, w \,\varepsilon\, C_0.\, F & \text{(if } w \in \mathcal{V}_0) \\
\forall\, w\, F & \rightleftharpoons & \forall\, w \,\varepsilon\, C_0.\, F & \text{(if } w \in \mathcal{V}_0) \\
\forall \overline{x} \,\varepsilon\, s.\, F & \rightleftharpoons & \neg\, \exists \overline{x} \,\varepsilon\, s.\, \neg F & \\
s \,\varepsilon\, C_q & \rightleftharpoons & (s) \,\varepsilon\, C^1_q & \\
s \subseteq t & \rightleftharpoons & \forall \overline{x} \,\varepsilon\, s.\, \overline{x} \,\varepsilon\, t \;\wedge\; \neg\,(s \,\varepsilon\, C_0) \;\wedge\; \neg\,(t \,\varepsilon\, C_0) & \\
s \sim t & \rightleftharpoons & \forall \mu\, (s \,\varepsilon\, C_\mu \;\leftrightarrow\; t \,\varepsilon\, C_\mu) & \\
s = t & \rightleftharpoons & s =_0 t \;\vee\; (s \subseteq t \;\wedge\; t \subseteq s \;\wedge\; s \sim t) & \\
s =_n t & \rightleftharpoons & s = t \;\wedge\; s \,\varepsilon\, C_n \;\wedge\; t \,\varepsilon\, C_n & \text{(for } n > 0) \\
s \,\varepsilon\, t & \rightleftharpoons & (s) \,\varepsilon\, t. &
\end{array}
$$

In the definitions of $(s \subseteq t)$ and $(s \sim t)$ let the variables $\overline{x}$ and μ not occur in s or t. - We easily obtain:

8.5. Lemma: $\quad c \,\varepsilon\, C_m \;\wedge\; c =_n d \;\rightarrow\; c =_m d.$

Definition: For $s, t \in \overline{\mathcal{T}}$ we define by means of variables $x, y, z \in \mathcal{V} \setminus \mathcal{V}(s,t)$ and κ for elements of $\mathbb{N}^+$:

$$\begin{aligned} s \,\varepsilon\, \overline{C}_m \;&\rightleftharpoons\; s \,\varepsilon\, \{\overline{x} \,\varepsilon\, \overline{C}_m : 0 = 0\} \\ s =_m t \;&\rightleftharpoons\; s \,\varepsilon\, \overline{C}_m \;\wedge\; t \,\varepsilon\, \overline{C}_m \;\wedge \\ &\quad \wedge\, \forall \kappa\, \forall x \,\varepsilon\, C_m.\, [\,\exists y \,\varepsilon\, \pi_m(s,\kappa).\, x = y \;\leftrightarrow\; \exists z \,\varepsilon\, \pi_m(t,\kappa).\, x = z\,]. \end{aligned}$$

8.6. Lemma: If $a \equiv (a_1, \ldots, a_j),\; b \equiv (b_1, \ldots, b_k)$ then

$$a =_m b \;\underline{\leftrightarrow}\; j = k \;\underline{\wedge}\; a_1 =_m b_1 \;\underline{\wedge}\; \ldots \;\underline{\wedge}\; a_j =_m b_j.$$

Proof: We now write A as short for $(a \,\varepsilon\, \overline{C}_m \wedge b \,\varepsilon\, \overline{C}_m)$ und use i as a metavariable for elements of $\mathbb{N}^+$. Then we have

$$\begin{aligned} a =_m b \quad &\underline{\leftrightarrow} \quad A \;\underline{\wedge}\; \underline{\forall} i\, \forall x \,\varepsilon\, C_m.\, [\,\exists y \,\varepsilon\, \pi_m(a,i).\, x = y \;\leftrightarrow\; \exists z \,\varepsilon\, \pi_m(b,i).\, x = z\,] \\ &\leftrightarrow \quad A \;\underline{\wedge}\; \underline{\forall} i\, \forall x \,\varepsilon\, C_m.\, [\,i \le j \;\wedge\; x = a_i \;\leftrightarrow\; i \le k \;\wedge\; x = b_i\,] \\ &\underline{\leftrightarrow} \quad j = k \;\underline{\wedge}\; \underline{\forall} i \le j.\, a_i =_m b_i. \quad \square \end{aligned}$$

Definitions: For $\underline{x} \equiv x_1, \ldots, x_j \in \mathcal{W}$:

$$\begin{aligned} \exists \underline{x} \,\varepsilon\, C_m.\, F \;&\rightleftharpoons\; \exists x_1 \,\varepsilon\, C_m \ldots \exists x_j \,\varepsilon\, C_m.\, F \\ C_m(\underline{x}) \;&\rightleftharpoons\; \{\overline{x} \,\varepsilon\, C_m : \exists \underline{x} \,\varepsilon\, C_m.\, \overline{x} =_m (\underline{x})\}. \end{aligned}$$

Remark: As easily seen, for all $c \in \overline{\mathcal{C}}$: $\;c \,\varepsilon\, C_m(\underline{x}) \;\underline{\leftrightarrow}\; c \in \mathcal{C}_m(\underline{x})$.

8.7. Lemma: $\quad c =_n d \;\rightarrow\; (c \,\varepsilon\, C_m(x) \leftrightarrow d \,\varepsilon\, C_m(x)).$

Proof: Suppose that $c =_n d$ and $c \,\varepsilon\, C_m(x)$, that is $c \in \mathcal{C}_m(x)$. For $x \in \mathcal{V}_0$ we have $\mathcal{C}_m(x) = \mathcal{C}(x) \subseteq \mathcal{C}_0$, hence $c \in \mathcal{C}_0$ and hence $c =_0 d$ (by 8.5), i.e. $c \approx_0 d$. Since $\mathcal{C}(x)$ is invariant under $(\approx_0)$, it follows that $d \in \mathcal{C}(x)$, i.e. $d \,\varepsilon\, C_m(x)$. - For $x \in \mathcal{V}$, $\mathcal{C}_m(x) = \mathcal{C}_m$. From this and $c \sim d$ we successively obtain $c \in \mathcal{C}_m,\; c \,\varepsilon\, C_m,\; d \,\varepsilon\, C_m$, $d \in \mathcal{C}_m$, and so $d \,\varepsilon\, C_m(x)$. $\square$

Remark: By 8.6, it follows that analogues to 8.5 and 8.7 also hold for elements of $\overline{\mathcal{C}}$ (in place of $\mathcal{C}$).

Definitions: For $\Phi \in \mathcal{T} \cup \overline{\mathcal{T}} \cup \mathcal{F}$, let $|\Phi|$ be the least $n \in \Omega$ with $\Phi \in \mathcal{T}_n \cup \overline{\mathcal{T}}_n \cup \mathcal{F}_n$. Moreover, for $\underline{w} \in \overline{\mathcal{W}}$ we define: $\;\mathcal{C}(\underline{w}) \rightleftharpoons \bigcup_{n \in \Omega} \mathcal{C}_n(\underline{w})$.

8.8. Lemma: If $\Phi \in \mathcal{T}\cup\overline{\mathcal{T}}\cup\mathcal{F}$, $w \in \overline{\mathcal{W}}(\Phi)$, and $c \in \mathcal{C}(w)$, then $|\Phi^w_c| = \max\{|\Phi|, |c|\}$.

Proof by induction on the construction of Φ by the $\mathcal{T}, \mathcal{F}$-rules: To this we give only some induction steps as examples. Let $\Phi \in \mathcal{T}\cup\overline{\mathcal{T}}\cup\mathcal{F}$, $w \in \overline{\mathcal{W}}(\Phi)$, and $c \in \mathcal{C}(w)$. We write $m{+}1$ for m'; $*$ for the substitution ^w_c, and "I.H." for "induction hypothesis".

▷ Let $\Phi \equiv (s \,\varepsilon\, C^p_q)$. Then $|\Phi| = |s|$. By 8.1, $p^* \in \mathcal{T}(\mathbb{N}^+)$, $q^* \in \mathcal{T}(\Omega)$, and so $\Phi^* \in \mathcal{F}$. If $w \in \overline{\mathcal{W}}(s)$ then (by I.H.): $|s^*| = \max\{|s|, |c|\}$, and so $|\Phi^*| = |(s^* \,\varepsilon\, C^{p^*}_{q^*})| = |s^*| = \max\{|s|, |c|\} = \max\{|\Phi|, |c|\}$. If $w \notin \overline{\mathcal{W}}(s)$ then $w \in \mathcal{V}_0(p) \cup \mathcal{V}_0(q)$, so $c \in \mathcal{C}(w) \subset \mathcal{C}_0$, $|c| = 0$, and so $|\Phi^*| = |(s \,\varepsilon\, C^{p^*}_{q^*})| = |s| = |\Phi| = \max\{|\Phi|, |c|\}$.

▷ Let $\Phi \equiv (\mathrm{J}\overline{x} \,\varepsilon\, \overline{C}_m, \mu, z \colon F)(q) \in \mathcal{T}$. Then $|\Phi| = \max\{m+1, |F|\}$. If $w \in \{\overline{x}, \mu, z\}$ then $w \in \mathcal{V}_0(q)$, so again $|c| = 0$, moreover $\Phi^* \equiv (\mathrm{J}\overline{x} \,\varepsilon\, \overline{C}_m, \mu, z : F)(q^*)$, and so $|\Phi^*| = \max\{m+1, |F|\} = |\Phi| = \max\{|\Phi|, |c|\}$. In case $w \notin \overline{\mathcal{W}}(F)$ we may conclude in the same way. Now let $w \notin \{\overline{x}, \mu, z\}$ and $w \in \overline{\mathcal{W}}(F)$. Then $\Phi^* \equiv (\mathrm{J}\overline{x} \,\varepsilon\, \overline{C}_m, \mu, z \colon F^*)(q^*)$, and by I.H.: $|F^*| = \max\{|F|, |c|\}$. So we obtain
$|\Phi^*| = \max\{m+1, |F^*|\} = \max\{m+1, |F|, |c|\} = \max\{|\Phi|, |c|\}$.

▷ Let $\Phi \equiv (\exists x \,\varepsilon\, \pi_m(s,p).\, F)$ with $s \in \overline{\mathcal{T}}$, $p \in \mathcal{T}(\mathbb{N}^+)$ und $F \in \mathcal{F}$. Then $|\Phi| = \max\{m+1, |s|, |F|\}$. Because of $w \in \overline{\mathcal{W}}(\Phi)$ we have $w \not\equiv x$. Again we have $p^* \in \mathcal{T}(\mathbb{N}^+)$ and so $\Phi^* \equiv (\exists x \,\varepsilon\, \pi_m(s^*, p^*).\, F^*) \in \mathcal{F}$. If $w \in \overline{\mathcal{W}}(s) \cup \overline{\mathcal{W}}(F)$ then
$|\Phi^*| = \max\{m+1, |s^*|, |F^*|\} =_{\text{I.H.}} \max\{m+1, |s|, |F|, |c|\} = \max\{|\Phi|, |c|\}$.
If $w \notin \overline{\mathcal{W}}(s) \cup \overline{\mathcal{W}}(F)$ then $w \in \mathcal{V}_0(p)$, so again $|c| = 0$ and thus
$|\Phi^*| = |\exists x \,\varepsilon\, \pi_m(s, p^*).\, F| = \max\{m+1, |s|, |F|\} = |\Phi| = \max\{|\Phi|, |c|\}$. -
The remaining steps of induction can be performed analogously. $\Box$ -
From 8.8 we obtain:

8.9. Corollary: If $s \in \mathcal{T}\cup\overline{\mathcal{T}}$, $\overline{\mathcal{W}}(s) = \{\underline{w}\}$, and $\underline{c}, \underline{d} \in \mathcal{C}(\underline{w})$ then: $\underline{c} \sim \underline{d} \;\rightarrow\; s^{\underline{w}}_{\underline{c}} \sim s^{\underline{w}}_{\underline{d}}$.

Definitions:

$$\begin{aligned} \exists\overline{x} \,\varepsilon\, \overline{C}_m.\, F \;&\rightleftharpoons\; \exists\overline{x} \,\varepsilon\, \{\overline{x} \,\varepsilon\, \overline{C}_m \colon 0 = 0\}.\, F \\ \forall\overline{x} \,\varepsilon\, \overline{C}_m.\, F \;&\rightleftharpoons\; \neg\,\exists\overline{x} \,\varepsilon\, \overline{C}_m.\, \neg F. \end{aligned}$$

Let I_n denote the set of all elements a of $\mathcal{C}_n$ for which the formula $(\overline{x} \,\varepsilon\, a)$ is invariant under $(=_n)$. Accordingly, for $a, b \in \mathcal{C}$ we define:

$$\begin{aligned} a \,\varepsilon\, I_n \;&\rightleftharpoons\; a \,\varepsilon\, C_n \;\wedge\; \forall\overline{x} \,\varepsilon\, \overline{C}_n.\, \forall\overline{y} \,\varepsilon\, \overline{C}_n.\, [\,\overline{x} =_n \overline{y} \;\rightarrow\; (\overline{x} \,\varepsilon\, a \;\leftrightarrow\; \overline{y} \,\varepsilon\, a)\,] \\ a \mathrel{\ddot{=}_n} b \;&\rightleftharpoons\; a =_n b \;\wedge\; a \,\varepsilon\, I_n \;\wedge\; b \,\varepsilon\, I_n. \end{aligned}$$

So we especially have: $a \,\varepsilon\, I_0 \;\leftrightarrow\; a \,\varepsilon\, C_0$, and $a \mathrel{\ddot{=}_0} b \;\leftrightarrow\; a =_0 b$. - For $a \equiv (a_1, \ldots, a_j)$ and $b \equiv (b_1, \ldots, b_j)$ define

$$a \mathrel{\ddot{=}_n} b \;\rightleftharpoons\; a_1 \mathrel{\ddot{=}_n} b_1 \;\wedge\; \ldots \;\wedge\; a_j \mathrel{\ddot{=}_n} b_j.$$

8.10. Lemma: If all elements of $\mathcal{F}_m$ are invariant under $(=_m)$, then for all $a, b \in \mathcal{C}_m \cup \overline{\mathcal{C}}_m$: $\quad a =_n b \; \rightarrow \; a \doteqdot_n b$.

Proof: Let $a \in \mathcal{C}_m$. Then the formula $(\overline{x} \; \varepsilon \; a)$ is a member of $\mathcal{F}_m$ and therefore, by hypothesis, invariant under $(=_m)$. So $a \; \varepsilon \; I_m$. Moreover, we have $\forall \overline{x} \; \varepsilon \; \overline{C}_n \, (\overline{x} \; \varepsilon \; a \; \rightarrow \; \overline{x} \; \varepsilon \; \overline{C}_m)$. By 8.5 (with $\overline{x}, \overline{y}$ in place of c, d) it follows that

$$\forall \overline{x} \; \varepsilon \; \overline{C}_n . \, \forall \overline{y} \; \varepsilon \; \overline{C}_n . \, [\, \overline{x} =_n \overline{y} \;\; \rightarrow \;\; (\overline{x} \; \varepsilon \; a \; \leftrightarrow \; \overline{x} \; \varepsilon \; a \; \wedge \; \overline{x} =_m \overline{y} \; \leftrightarrow \\ \leftrightarrow \; \overline{y} \; \varepsilon \; a \; \wedge \; \overline{x} =_m \overline{y} \; \leftrightarrow \; \overline{y} \; \varepsilon \; a) \,].$$

So $a \; \varepsilon \; I_n$. From this we obtain 8.10 for $a, b \in \mathcal{C}_m$ and so, by 8.6, also for $a, b \in \overline{\mathcal{C}}_m$. □

8.11. Theorem: All elements of $\mathcal{T}_n \cup \overline{\mathcal{T}}_n \cup \mathcal{F}_n$ are invariant under $(=_n)$.

Proof by compound induction: We start from the hypothesis

I.H.1: For all $m < n$, all elements of $\mathcal{T}_m \cup \overline{\mathcal{T}}_m \cup \mathcal{F}_m$ are invariant under $(=_m)$.

From this we conclude *at first*: All elements of $\mathcal{T}_n \cup \overline{\mathcal{T}}_n \cup \mathcal{F}_n$ are invariant under $(\doteqdot_n)$. To this end, we consider any element $\Phi \in \mathcal{T}_n \cup \overline{\mathcal{T}}_n \cup \mathcal{F}_n$ and infer from I.H.1 and the following hypothesis I.H.2 that Φ is invariant under $(\doteqdot_n)$:

I.H.2: Invariant under $(\doteqdot_n)$ are all terms and formulas that must have been shown to be elements of $\mathcal{T}_n \cup \overline{\mathcal{T}}_n \cup \mathcal{F}_n$ in order to show (by the $\mathcal{T}, \mathcal{F}$-rules) that Φ is an element of $\mathcal{T}_n \cup \overline{\mathcal{T}}_n \cup \mathcal{F}_n$.

So we suppose that $\Phi \in \mathcal{T}_n \cup \overline{\mathcal{T}}_n \cup \mathcal{F}_n$ and $\overline{\mathcal{W}}(\Phi) = \{\underline{w}\}$ with $\underline{w} \rightleftharpoons w_1, \ldots, w_j$. Moreover, let $\underline{c}, \underline{d} \in \mathcal{C}(\underline{w})$ and $\underline{c} \doteqdot_n \underline{d}$. We consider the substitutions $* \rightleftharpoons \frac{\underline{w}}{\underline{c}}$ und $\dagger \rightleftharpoons \frac{\underline{w}}{\underline{d}}$, and write "invariant" as short for "invariant under $(\doteqdot_n)$".

▷ Let $\Phi \equiv s \in \mathcal{T}_{\mathrm{or}}$. In s occur only variables w_i of $\mathcal{V}_0$. So we have $c_i, d_i \in \mathcal{C}_0$. Because of $c_i =_n d_i$ it follows (by 8.5) that $c_i =_0 d_i$, i.e. $c_i \approx_0 d_i$, so (by the hypothesis on $(\approx_0)$): $s^* =_0 s^\dagger$, $s^*, s^\dagger \in \mathcal{C}_0$, and so $s^*, s^\dagger \; \varepsilon \; I_n$, too.

▷ Let $\Phi \in \mathcal{V} \cup \overline{\mathcal{V}}$. Then $\Phi \equiv w_1$. So $\Phi^* \equiv c_1$ and $\Phi^\dagger \equiv d_1$. Since $c_1 \doteqdot_n d_1$ we have $\Phi^* \doteqdot_n \Phi^\dagger$.

▷ Let $\Phi \equiv R(q) \equiv (\mathrm{J}\overline{x} \; \varepsilon \; \overline{C}_m, \mu, z : F)(q) \in \mathcal{T}_n$ with $m < n$ and $F \in \mathcal{F}_n$. F is invariant by I.H.2. Let $F \equiv A(\overline{x}, \mu, z, \underline{w})$. Suppose that for all $k < l$ we have $R^*(k) \doteqdot_n R^\dagger(k)$. Then for all $\underline{a}, \underline{b}$ with $\underline{a} =_n \underline{b}$ it follows (by I.H.1, 8.7, and 8.10) that: $(\underline{a}) \; \varepsilon \; \overline{C}_m \; \rightarrow \; \underline{a} \doteqdot_n \underline{b}$. So (by 8.7) for all $k \in \Omega$:

$$(\underline{a}, k) \; \varepsilon \; R^*(l) \;\; \leftrightarrow \;\; (\underline{a}) \; \varepsilon \; \overline{C}_m \; \wedge \; k < l \; \wedge \; A((\underline{a}), k, R^*(k), \underline{c}) \\ \leftrightarrow \;\; (\underline{b}) \; \varepsilon \; \overline{C}_m \; \wedge \; k < l \; \wedge \; A((\underline{b}), k, R^\dagger(k), \underline{d}) \;\; \leftrightarrow \;\; (\underline{b}, k) \; \varepsilon \; R^\dagger(l).$$

From this it follows, by 8.9, that $R^*(l) \doteqdot_n R^\dagger(l)$. By induction on Ω we obtain this result for all $l \in \Omega$. So especially $\Phi^* \doteqdot_n \Phi^\dagger$.

▷ The case $\Phi \equiv \{\overline{x} \,\varepsilon\, \overline{C}_m \colon F\}$ can even be treated simpler.

▷ Let $\Phi \equiv (s_1, \ldots, s_k) \in \overline{\mathcal{T}}_n$. By I.H.2 we have $s_i^* \ddot{=}_n s_i^\dagger$. So $\Phi^* \ddot{=}_n \Phi^\dagger$.

▷ Let $\Phi \equiv E \in \mathcal{E}$. As in the case "$\Phi \equiv s \in \mathcal{T}_{\mathrm{Or}}$" we obtain $c_i =_0 d_i$, and so: $E^* \leftrightarrow E^\dagger$.

▷ Let $\Phi \in \big\{(F \wedge G), (F \vee G), (\neg F), (\exists x \,\varepsilon\, C_m.\, F)\big\}$ with invariant formulas $F, G \in \mathcal{F}_n$, and $m < n$. Then it easily follows that also Φ is invariant. Concerning the formula $(\exists x \,\varepsilon\, C_m.\, F)$ regard that, by I.H.1 and 8.10, we have: $a \,\varepsilon\, C_m \;\rightarrow\; a \ddot{=}_n a$.

▷ Let $\Phi \equiv (s \,\varepsilon\, t)$ where $s \in \overline{\mathcal{T}}_n$ and $t \in \mathcal{T}_n$ are invariant. So we have $s^* =_n s^\dagger$, $t^* \,\varepsilon\, I_n$ and $t^* =_n t^\dagger$. It follows that: $s^* \,\varepsilon\, t^* \;\leftrightarrow\; s^\dagger \,\varepsilon\, t^* \;\leftrightarrow\; s^\dagger \,\varepsilon\, t^\dagger$.

▷ Let $\Phi \equiv (\exists \overline{x} \,\varepsilon\, t.\, F)$ where $t \in \mathcal{T}_n$, $F \in \mathcal{F}_n$ are invariant. First let $\overline{x} \in \overline{\mathcal{V}}(t^*)$. For all $a \in \overline{\mathcal{C}}$ with $a \,\varepsilon\, t\binom{\overline{x}}{a}^*$ we have $|a| < |t\binom{\overline{x}}{a}^*| = \max\{|t^*|, |a|\}$ (by 8.8), so $|a| < |t^*| \le n$. This also holds for $\overline{x} \notin \overline{\mathcal{V}}(t^*)$. So in any case, by I.H.1 and 8.10: $a \,\varepsilon\, t\binom{\overline{x}}{a}^* \;\rightarrow\; a \ddot{=}_n a \;\rightarrow\; t\binom{\overline{x}}{a}^* \ddot{=}_n t\binom{\overline{x}}{a}^\dagger$. So: $a \,\varepsilon\, t\binom{\overline{x}}{a}^* \;\wedge\; F\binom{\overline{x}}{a}^* \;\rightarrow\; a \,\varepsilon\, t\binom{\overline{x}}{a}^\dagger \;\wedge\; F\binom{\overline{x}}{a}^\dagger$. So we obtain: $\Phi^* \;\rightarrow\; \Phi^\dagger$ and, in the same way: $\Phi^\dagger \;\rightarrow\; \Phi^*$.

▷ Let $\Phi \equiv (\exists x \,\varepsilon\, \pi_m(s,p).\, F)$ where $s \in \overline{\mathcal{T}}_n$, $p \in \mathcal{T}(\mathbb{N}^+)$, and $F \in \mathcal{F}_n$, which are invariant. At first let $x \in \mathcal{W} \setminus \mathcal{W}(s,p)$. Then $s^*, s^\dagger$ are constants, $p^*, p^\dagger \in \mathbb{N}^+$, $s^* \ddot{=}_n s^\dagger$ and $p^* \equiv p^\dagger$. Let $s^* \equiv (a_1, \ldots, a_j)$, $s^\dagger \equiv (b_1, \ldots, b_j)$, and $i \rightleftharpoons p^*$. In case $i \le j$ we have $a_i \ddot{=}_n b_i$, so that: $\Phi^* \;\leftrightarrow\; F\binom{x}{a_i}^* \leftrightarrow F\binom{x}{b_i}^\dagger \;\leftrightarrow\; \Phi^\dagger$. In case $i > j$ we have: $\Phi^* \;\leftrightarrow\; \bot \;\leftrightarrow\; \Phi^\dagger$. Finally, let $x \in \mathcal{W}(s,p)$. Then $x \in \mathcal{W}(s^*, p^*) \cap \mathcal{W}(s^\dagger, p^\dagger)$, so that again: $\Phi^* \;\leftrightarrow\; \bot \;\leftrightarrow\; \Phi^\dagger$.

▷ Let $\Phi \equiv (s \,\varepsilon\, C_q^p)$ with an invariant term $s \in \overline{\mathcal{T}}_n$. Since $s^* \sim s^\dagger$, $p^* \equiv p^\dagger$ and $q^* \equiv q^\dagger$ we have: $s^* \,\varepsilon\, C_{q^*}^{p^*} \;\leftrightarrow\; s^\dagger \,\varepsilon\, C_{q^*}^{p^*} \;\leftrightarrow\; s^\dagger \,\varepsilon\, C_{q^\dagger}^{p^\dagger}$.

▷ Let $\Phi \equiv (s =_0 t)$ with invariant $s, t \in \mathcal{T}_n$. Then $\Phi^* \;\rightarrow\; s^* \,\varepsilon\, C_0 \;\wedge\; t^* \,\varepsilon\, C_0$, so, by 8.5: $\Phi^* \;\rightarrow\; s^\dagger =_0 s^* =_0 t^* =_0 t^\dagger \;\rightarrow\; \Phi^\dagger$, and likewise: $\Phi^\dagger \;\rightarrow\; \Phi^*$.

On the assumption I.H.1 we have shown that all elements of $\mathcal{T}_n \cup \mathcal{F}_n$ are invariant under $(\ddot{=}_n)$. Now we consider any element $c \in \mathcal{C}_n$. All 'elements' a, b of c (if there are any) have a lower order than n, so that (by I.H.1 and 8.10): $a =_n b \;\rightarrow\; a \ddot{=}_n b$. Since the formula $(\overline{x} \,\varepsilon\, c)$ belongs to $\mathcal{F}_n$, it is invariant under $(\ddot{=}_n)$ and so under $(=_n)$. Therefore, $c \,\varepsilon\, I_n$. For all $c, d \in \mathcal{C}_n$ we obtain: $c =_n d \;\leftrightarrow\; c \ddot{=}_n d$. So all elements of $\mathcal{T}_n \cup \mathcal{F}_n$ are invariant under $(=_n)$. □

Definition:

$$
\begin{array}{lcll}
(s_1, \ldots, s_j) = (t_1, \ldots, t_j) & \rightleftharpoons & s_1 = t_1 \wedge \ldots \wedge s_j = t_j & \\
(s_1, \ldots, s_j) = (t_1, \ldots, t_k) & \rightleftharpoons & \bot & \text{if } j \neq k.
\end{array}
$$

It seems not to be possible to adequately define $(\overline{x} = \overline{y}), (\overline{x} = (t_1, \ldots, t_j))$, and $((s_1, \ldots, s_j) = \overline{y})$ as formulas of $\mathcal{F}$. Nevertheless, we obtain:

8.12. Corollary: All elements of $\mathcal{T} \cup \overline{\mathcal{T}} \cup \mathcal{F}$ are invariant under $(=)$.

Proof, for formulas, e.g.: Let $F \in \mathcal{F}$, $\{\underline{w}\} = \overline{\mathcal{W}}(F)$, $(\underline{c}), (\underline{d}) \in \mathcal{C}(\underline{w})$ and $\underline{c} = \underline{d}$. Then there exist $k, m \in \Omega$ such that $F \in \mathcal{F}_k$ and $\underline{c} =_m \underline{d}$. Let $n \rightleftharpoons \max\{k, m\}$. Then $F \in \mathcal{F}_n$, $\underline{c} =_n \underline{d}$, and so $(F^{\underline{w}}_{\underline{c}} \leftrightarrow F^{\underline{w}}_{\underline{d}})$ (by 8.11). $\square$

By 8.3 and 8.12 we have solved our main tasks mentioned in §7.

Definitions of j-ary relations (already mentioned in §7): For $\underline{x} \equiv x_1, \ldots, x_j$,

$$\begin{aligned} \{\underline{x} \,\varepsilon\, C_m \colon F\} &\rightleftharpoons \{\overline{x} \,\varepsilon\, \overline{C}_m \colon \exists \underline{x} \,\varepsilon\, C_m.\, (\overline{x} =_m (\underline{x}) \ \wedge\ F)\} \\ (\mathrm{J}\underline{x} \,\varepsilon\, C_m, \mu, z \colon F) &\rightleftharpoons (\mathrm{J}\overline{x} \,\varepsilon\, \overline{C}_m, \mu, z \colon \exists \underline{x} \,\varepsilon\, C_m.\, (\overline{x} =_m (\underline{x}) \ \wedge\ F)\}. \end{aligned}$$

So we have

$$(\underline{c}) \,\varepsilon\, \{\underline{x} \,\varepsilon\, C_m \colon A(\underline{x})\} \ \leftrightarrow\ (\underline{c}) \,\varepsilon\, C_m(\underline{x}) \ \wedge\ A(\underline{c}),$$

(If $x_i \in \mathcal{V}_0$ but $c_i \notin \mathcal{C}(x_i)$ for some member c_i of $\underline{c}$, then in general $A(\underline{c}) \notin \mathcal{A}$. In this case let $((\underline{c}) \,\varepsilon\, C_m(\underline{x}) \wedge A(\underline{c}))$ pass for rejected.) -
Similarly, for $R \rightleftharpoons (\mathrm{J}\underline{x} \,\varepsilon\, C_m, \mu, z : A(\underline{x}, \mu, z))$ we have

$$(\underline{c}, k) \,\varepsilon\, R(l) \ \leftrightarrow\ (\underline{c}) \,\varepsilon\, C_m(\underline{x}) \ \wedge\ k < l \ \wedge\ A(\underline{c}, k, R(k)).$$

Now we can prove the following Corollary by which one can 'define' sequences of relations $S(k)$, $k \in \mathbb{N}$, by ordinary *recursion* on $\mathbb{N}$ (cf. §4: Addition in $\mathbb{N}$).

8.13. Corollary: For any two formulas $A(\underline{x}), B(\underline{x}, \mu, z) \in \mathcal{F}_n$ with $\underline{x} \in \mathcal{W}$, $z \in \mathcal{V}$, and any $m < n$ there exists a term $S(\nu) \in \mathcal{T}_n$ such that for all $(\underline{c}) \in \overline{\mathcal{C}}$ and all $k \in \mathbb{N}$,

$$\begin{aligned} (\underline{c}) \,\varepsilon\, S(0) &\leftrightarrow (\underline{c}) \,\varepsilon\, C_m(\underline{x}) \ \wedge\ A(\underline{c}) \\ (\underline{c}) \,\varepsilon\, S(k') &\leftrightarrow (\underline{c}) \,\varepsilon\, C_m(\underline{x}) \ \wedge\ B(\underline{c}, k, S(k)). \end{aligned}$$

Proof: Let

$$\begin{aligned} D(\underline{x}, \mu, z) &\rightleftharpoons \big[\mu = 0 \ \wedge\ A(\underline{x})\big] \vee \\ &\quad\ \vee \exists\lambda \big[\mu = \lambda' \ \wedge\ B(\underline{x}, \lambda, \{\underline{x} \,\varepsilon\, C_m \colon (\underline{x}, \lambda) \,\varepsilon\, z\})\big] \\ R(k) &\rightleftharpoons (\mathrm{J}\underline{x} \,\varepsilon\, C_m, \mu, z \colon D(\underline{x}, \mu, z))(k) \\ S(k) &\rightleftharpoons \{\underline{x} \,\varepsilon\, C_m \colon (\underline{x}, k) \,\varepsilon\, R(k')\}. \end{aligned}$$

Then we have

$$\begin{aligned} (\underline{c}) \,\varepsilon\, S(0) &\leftrightarrow (\underline{c}, 0) \,\varepsilon\, R(0') \leftrightarrow (\underline{c}) \,\varepsilon\, C_m(\underline{x}) \ \wedge\ A(\underline{c}) \\ (\underline{c}) \,\varepsilon\, S(k') &\leftrightarrow (\underline{c}, k') \,\varepsilon\, R(k'') \leftrightarrow (\underline{c}) \,\varepsilon\, C_m(\underline{x}) \ \wedge\ B(\underline{c}, k, S(k)). \quad \square \end{aligned}$$

§9. ι-terms (singular description terms)

Now we introduce 'ι-terms', i.e. terms of the form $(\iota\, x\ \varepsilon\ C_m.\, F)$ ["the (unique) element x of C_m which satisfies F"]. Then, for instance, the notation of function application can be defined by $f(a) \rightleftharpoons (\iota y\ \varepsilon\ C_m.\, (a,y)\ \varepsilon\ f)$. - In $(\iota x\ \varepsilon\ C_m.\, F)$ all occurrences of x are said to be bound. A ι-term without free occurring variables is said to be a *ι-constant* (or a singular description).

To obtain a language containing formulas in which ι-terms may occur, we construct the sets $\mathcal{E}^\iota_n$, $\mathcal{T}^\iota_{\mathrm{Or},n}$, $\mathcal{T}(x)^\iota_n$ (with $x \in \mathcal{V}_0$), $\mathcal{T}^\iota_n$, $\overline{\mathcal{T}}^\iota_n$, and $\mathcal{F}^\iota_n$ by the following rules, in which we let $\mathcal{P}$ range over $\{\mathcal{E}, \mathcal{T}_{\mathrm{Or}}\} \cup \{\mathcal{T}(x) \colon x \in \mathcal{V}_0\}$:

$$
\begin{array}{rcll}
 & \Rightarrow & \Phi \in \mathcal{P}^\iota_n & (\text{if} \quad \Phi \in \mathcal{P}), \\
\Phi \in \mathcal{P}^\iota_n, F \in \mathcal{F}^\iota_n & \Rightarrow & \Phi^y_\tau \in \mathcal{P}^\iota_n & (\text{if} \quad \tau \equiv (\iota\, x\, \varepsilon\, C_0.\, F),\ n > 0, \\
 & & & \ x, y \in \mathcal{V}_0,\ \mathcal{C}(x) = \mathcal{C}(y)), \\
F \in \mathcal{F}^\iota_n & \Rightarrow & (\iota\, x\, \varepsilon\, C_m.\, F) \in \mathcal{T}^\iota_n & (\text{if} \quad m < n,\ x \in \mathcal{W}),
\end{array}
$$

and the rules which result from the $\mathcal{T}, \mathcal{F}$-rules by replacing the symbols $\mathcal{T}_n, \overline{\mathcal{T}}_n, \mathcal{F}_n$, and $\mathcal{P}$ (as above) with $\mathcal{T}^\iota_n, \overline{\mathcal{T}}^\iota_n, \mathcal{F}^\iota_n$, and $\mathcal{P}^\iota_n$, respectively. (Regard that $\mathcal{T}(\Omega), \mathcal{T}(\mathbb{N}^+) \in \{\mathcal{T}(x) \colon x \in \mathcal{V}_0\}$.)

In the second of the adduced rules already occurs "Φ^y_τ" ("the Ψ which results from Φ by substituting τ for y"), which is a description term of a metalanguage. This term can be eliminated in that rule by replacing it by "Ψ", e.g., and adding the condition "if Ψ results from Φ by substituting τ for y".

Definitions: $\mathcal{P}^\iota \rightleftharpoons \bigcup_{n \in \Omega} \mathcal{P}^\iota_n$, for each $\mathcal{P} \in \{\mathcal{T}(x) \colon x \in \mathcal{V}_0\} \cup \{\mathcal{T}_{\mathrm{Or}}, \mathcal{E}, \mathcal{T}, \overline{\mathcal{T}}, \mathcal{F}\}$;

$$
\begin{array}{rcl}
\mathcal{H} & \rightleftharpoons & \mathcal{E}^\iota \,\cup\, \{(s =_0 t) \colon s, t \in \mathcal{T}^\iota\} \,\cup \\
 & & \cup\, \{(s\ \varepsilon\ t) \colon s\ \varepsilon\ \overline{\mathcal{T}}^\iota,\ t\ \varepsilon\ \mathcal{T}^\iota\} \,\cup \\
 & & \cup\, \{(s\ \varepsilon\ C^p_q) \colon s\ \varepsilon\ \overline{\mathcal{T}}^\iota,\ p \in \mathcal{T}(\mathbb{N}^+)^\iota,\ q \in \mathcal{T}(\Omega)^\iota\} \,\cup \\
 & & \cup\, \{(\exists x \in \pi_m(s,p).\, F) \colon x \in \mathcal{W},\ m \in \Omega,\ s \in \overline{\mathcal{T}}^\iota,\ p \in \mathcal{T}(\mathbb{N}^+)^\iota,\ F \in \mathcal{F}^\iota\}.
\end{array}
$$

Let $\mathcal{C}^\iota_m /\ \overline{\mathcal{C}}^\iota_m$ be the set of all *constants* belonging to $\mathcal{T}^\iota_m /\ \overline{\mathcal{T}}^\iota_m$, respectively. Let $\mathcal{A}^\iota$ be the set of all *sentences* belonging to $\mathcal{F}^\iota$.

Now we stipulate the use of sentences containing ι-terms: By using the abbreviations

$$
\begin{array}{rcl}
\alpha & \rightleftharpoons & \iota\, x\, \varepsilon\, C_m.\, A(x) \\
A(=y) & \rightleftharpoons & \forall x\, \varepsilon\, C_m.\, [A(x)\ \leftrightarrow\ x = y]
\end{array}
$$

we lay down the 'primary rule':

R(ι) $\qquad \natural\, B(\alpha) \ :\Rightarrow\ \natural\, \exists y\, \varepsilon\, C_m.\, [A(=y)\ \wedge\ B(y)]$, which, however, we restrict to the following conditions: $B(y) \in \mathcal{H}$; $y \in \mathcal{W}$; there is a unique free occurrence of y in $B(y)$; the occurrence of α substituted for y in $B(y)$ begins on the left of all other occurrences of ι-constants in $B(y)$; moreover, y does not occur in $A(x)$ and has the same values as x [i.e. $\mathcal{C}(x) = \mathcal{C}(y)$]. If $B(\alpha)$ is given, y is the 'first' variable satisfying these conditions. Here "the first" means "the first with respect to a certain lexicographical ordering of $\mathcal{W}$". (So α, y, and the sentence on the right are uniquely determined by $B(\alpha)$.)

For sentences of $\mathcal{A}^\iota$ that do not belong to $\mathcal{H}$ or do not contain ι-constants, we take the same 'primary rules' of assertion as for sentences of $\mathcal{A}$ but with $\mathcal{C}^\iota_m / \overline{\mathcal{C}}^\iota_m$ in place of $\mathcal{C}_m / \overline{\mathcal{C}}_m$, respectively, in the rules for $(c \;\varepsilon\; \{\overline{x} \;\varepsilon\; \overline{C}_m : A(\overline{x})\})$, $((\underline{c}, k) \;\varepsilon\; R(l))$, $(\exists x \;\varepsilon\; \pi_m((\underline{c}), i).A(x))$, and $(c \;\varepsilon\; C^j_m)$. (Notice that $\mathcal{C}^\iota_0 = \mathcal{C}_0$.)

It can be shown (as sketched below) that all sentences of $\mathcal{A}^\iota$ are non-circular and, therefore, $R(\iota)$ and all other primary rules for sentences of $\mathcal{A}^\iota \backslash \mathcal{E}$ are invertible. Thus, in the classical game we may apply all inference rules of classical logic to sentences of $\mathcal{A}^\iota$. (Certain restrictions will be indicated below.)

Remarks: 1. $\exists y \;\varepsilon\; C_m . A(= y)$ means that there exists exactly one $c \in \mathcal{C}_m$ such that $A(c)$ holds. On the conditions of R(ι) we have:

$$\begin{aligned} \exists y \;\varepsilon\; C_m . A(= y) \;\rightarrow\; \big\{ B(\alpha) \;&\leftrightarrow\; \exists y \;\varepsilon\; C_m . [\, A(y) \wedge B(y) \,] \\ &\leftrightarrow\; \forall y \;\varepsilon\; C_m . [\, A(y) \rightarrow B(y) \,] \big\}. \end{aligned}$$

This statement can in a well-known way even be extended to *arbitrary* sentences $B(\alpha)$ with α at arbitrary places (see, e.g., [7, §9] or [9, pp. 170f.]). By R(ι), its inverse, and the latter remark we especially have

$$\begin{gathered} \alpha \;\varepsilon\; C_m \;\leftrightarrow\; \exists y \;\varepsilon\; C_m . A(= y) \;\rightarrow\; A(\alpha); \\ \exists y \;\varepsilon\; C_m . A(= y) \;\leftrightarrow\; A(\alpha), \quad \text{if } A(\alpha) \in \mathcal{H}. \end{gathered}$$

However, to obtain the mentioned results, we have at first to show that Lemma 8.1 also holds for terms and formulas containing ι-terms. Then in the definitions of $\mathcal{S}_n(\Phi)$ and $\mathcal{F}^{\text{nc}}_n$, and in Lemma 8.2 we have to replace $\mathcal{C}_n(w) \,/\, \overline{\mathcal{C}}_n \,/\, \mathcal{C}_m(x)$, respectively, by the set of all elements of $\mathcal{C}(w)^\iota_n \,/\, \overline{\mathcal{C}}^\iota_n \,/\, \mathcal{C}(x)^\iota_m$ in which no ι-constants occur. (But open ι-terms may occur in those elements.) Of course, we have also to supply the symbols $\mathcal{T}, \overline{\mathcal{T}}, \mathcal{F}, \mathcal{A}, \mathcal{S}_n, \mathcal{F}_n$, and $\mathcal{F}_m$ with the symbol ι.

In the following we say that a single occurrence, $\dot{\tau}$, of a ι-term in H is **free** in H to mean that all free occurrences of variables in $\dot{\tau}$ are also free in H. - It can be shown that if we replace a single free occurrence of a ι-term, $(\iota\, x \;\varepsilon\; C_m . F)$, in a formula of $\mathcal{F}^\iota_n$ by a variable y with $\mathcal{C}(y) = \mathcal{C}(x)$, then there again results a formula of $\mathcal{F}^\iota_n$. (However, we do not go into details here.)

To prove that all sentences of $\mathcal{F}^\iota$ are *non-circular* (with respect to an adequate predecessor relation), we can proceed similarly as in the proof of 8.3, but we have to add the following step of induction and to replace the induction hypothesis I.H.2 by:

I.H.2': $F \in (\mathcal{F}_n^\iota)^{\mathrm{nc}}$ for all formulas F for which $(F \in \mathcal{F}_n^\iota)$ has a shorter deduction by the $\mathcal{T}^\iota, \mathcal{F}^\iota$-rules than $(H \in \mathcal{F}_n^\iota)$.

The announced induction step is: ▷ Let $H \in \mathcal{F}_n^\iota \cap \mathcal{H}$ be a formula in which a ι-term occurs free. Then there is a *unique* free occurrence in H of a ι-term $\tau \rightleftharpoons (\iota\, x \,\varepsilon\, C_m.\, F)$, say, that begins on the left of all other free occurrences of ι-terms in H. Let G result from H by replacing that occurrence of τ with a variable $y \in \mathcal{W}$ with $\mathcal{C}(y) = \mathcal{C}(x)$ that does not occur in H. So $H \equiv G^y_\tau$, $F, G \in \mathcal{F}_n^\iota$, and $m < n$. The deductions of $(F \in \mathcal{F}_n^\iota)$ and $(G \in \mathcal{F}_n^\iota)$ by the $\mathcal{T}^\iota, \mathcal{F}^\iota$-rules are shorter than the deduction of $(H \in \mathcal{F}_n^\iota)$. So, by I.H.2', $F, G \in (\mathcal{F}_n^\iota)^{\mathrm{nc}}$. Similarly as the proof of 8.3 we obtain $(\exists y \,\varepsilon\, C_m.(F(=y) \wedge G)) \in (\mathcal{F}_n^\iota)^{\mathrm{nc}}$. But for any $* \in \mathcal{S}_n^\iota(H)$, $H^* \equiv (G^*)^y_{\tau^*}$ has (apart from the variable y) the unique predecessor $\exists y \,\varepsilon\, C_m.\,(F(=y)^* \wedge G^*)$ (cf. R(ι)) which is non-circular by the previous argument. So $H \in (\mathcal{F}_n^\iota)^{\mathrm{nc}}$.

To complete the proof we have previously to prove the following lemma and some others: Let $\mathcal{F}_n^{\iota-} \,/\, \mathcal{A}_n^{\iota-}$ denote the set of all elements of $\mathcal{F}_n^\iota \,/\, \mathcal{A}_n^\iota$, respectively, in which no ι-terms / ι-constants occur free. If $H \in \mathcal{F}_n^{\iota-}$ and $* \in \mathcal{S}_n^\iota(H)$, then $H^* \in \mathcal{A}_n^{\iota-}$.

To formulate further results we need the

Definitions: If $x \in \mathcal{V} \,/\, \overline{\mathcal{V}}$, let $\mathcal{T}(x) \rightleftharpoons \mathcal{T} \,/\, \overline{\mathcal{T}}$, respectively. (For $x \in \mathcal{V}_0$, $\mathcal{T}(x)$ has already been defined.)

For $x \in \overline{\mathcal{W}}$ and $\Phi \in \mathcal{T}^\iota \cup \overline{\mathcal{T}}^\iota \cup \mathcal{F}^\iota$, a term t is said to be **free for** x in Φ iff $t \in \mathcal{T}(x)^\iota$ and every free occurrence of a variable in t is also free in Φ^x_t wherever t is substituted for x in Φ.

For $x \in \overline{\mathcal{W}}$, let $\mathcal{T}(x)^{\circ\iota}$ be the set of all terms $t \in \mathcal{T}(x)^\iota$ such that $t \in \mathcal{T}(x)$ or there exist distinct variables $w_1, \ldots, w_j \in \mathcal{W}$, ι-terms $\tau_1, \ldots, \tau_j$, and a term $s \in \mathcal{T}(x)$ such that τ_i is free for w_i in s $(i = 1, \ldots, j)$ and t results from s by the simultaneous substitution of $\tau_1, \ldots, \tau_j$ for $w_1, \ldots, w_j$.

For terms $s, t, t_1, \ldots, t_j \in \mathcal{T}(x)^{\circ\iota}$ which are free for x in $G(x)$ $(x \in \mathcal{W})$, and terms $r \in \overline{\mathcal{T}}^{\circ\iota}$ which are free for $\overline{x}$ in $H(\overline{x})$, we can successively prove

$$\forall \underline{y} \,\varepsilon\, C_n.\,[\, t \,\varepsilon\, C_m \;\wedge\; G(t) \;\rightarrow\; \exists x \,\varepsilon\, C_m.\, G(x)\,]$$
$$\forall \underline{y} \,\varepsilon\, C_n.\,[\, \forall x \,\varepsilon\, C_m.\, G(x) \;\wedge\; t \,\varepsilon\, C_m \;\rightarrow\; G(t)\,]$$
$$\forall \underline{y} \,\varepsilon\, C_n.\,[\, s = t \;\wedge\; G(s) \;\rightarrow\; G(t)\,]$$
$$\forall \underline{y} \,\varepsilon\, C_n.\,[\, r \,\varepsilon\, \{\overline{x} \,\varepsilon\, \overline{C}_m \colon H(\overline{x})\} \;\leftrightarrow\; r \,\varepsilon\, \overline{C}_m \;\wedge\; H(r)\,],$$
$$\forall \underline{y} \,\varepsilon\, C_n.\,[\, \exists x \,\varepsilon\, \pi_m((t_1, \ldots, t_j), i).\, G(x) \;\leftrightarrow\; t_i \,\varepsilon\, C_m \wedge G(t_i)\,],$$

if $\underline{y}$ is a list of all distinct variables occurring free in $[\ldots]$, and if, in the latter line, $i \leq j$ and $x \,\varepsilon\, \mathcal{W} \setminus \mathcal{W}(t_1, \ldots, t_j)$. - An analogous statement also holds for $(\mathrm{J}\overline{x} \,\varepsilon\, C_m, \mu, z \colon H)(q)$ in place of $\{\overline{x} \,\varepsilon\, C_m \colon H(\overline{x})\}$.

For the proof of the first of these statements, we can at first prove it especially for ι-constants t (where $\underline{y}$ may be assumed to be empty), and then for ι-terms t that are free for x in $G(x)$.

Summary (informally): In the scope of $\mathcal{T}^{\iota} \cup \overline{\mathcal{T}}^{\iota} \cup \mathcal{F}^{\iota}$ we may argue classically. ι-terms and even elements of $\mathcal{T}^{\circ\iota}$ may be used like 'ordinary' terms.

§10. Objectual variables in higher order languages

Now we admit that **objectual variables** (cf. §5) occur in terms of $\mathcal{T}_{\mathrm{Or}}$ and in formulas of $\mathcal{E}$, and thus also in higher order terms and formulas.

To simplify our investigations we use two sorts of objectual variables, i.e. *boundable* ones and *unboundable* ones. (This is compatible with the procedure in §5 since, by 5.2, objectual variables which are bound by quantifiers may be replaced by 'new' ones.) However, we drop the word "boundable". The unboundable objectual variables are concisely said to be **indicators.** They must not be bounded. We include the objectual variables among $\mathcal{V}_0$. However, we do not include any indicator among $\overline{\mathcal{W}}\,(\rightleftharpoons \mathcal{V}_0 \cup \mathcal{V} \cup \overline{\mathcal{V}})$. Besides, we shall use all other former assumptions concerning $\mathcal{T}_{\mathrm{Or}}, \mathcal{E}, \mathcal{V}_0, \mathcal{V}$, and $\overline{\mathcal{V}}$ as well as all former notations and the construction rules for higher order terms and formulas. Especially, let $\mathcal{C}_m$ or $\mathcal{A}$, respectively, be the set of all elements of $\mathcal{T}_m$ or $\mathcal{F}$ in which no elements of $\overline{\mathcal{W}}$ occur free. So (in contrast to §5) indicators are constants and can occur in other constants. - For every objectual variable ξ let $\mathcal{C}(\xi)$ contain denumerably many indicators but no other constants.

As metavariables we use: ξ, η for objectual variables, $\underline{\xi}$ for lists of distinct objectual variables, u, v, w for indicators, and $\underline{u}, \underline{v}, \underline{w}$ for lists of *distinct* indicators. Moreover, we use β, γ, δ to denote arbitrary denotations (of indicators, cf. §5). -
$\gamma\,\Delta\,\{\underline{u}\}$ is to mean that γ is a denotation of $\underline{u}$ but not of further indicators. (If $\underline{u}$ is empty, then $\gamma\,\Delta\,\{\underline{u}\}$ means that γ is 'empty'. In this case, let $\gamma\delta$ coincide with δ.)
$\gamma\,\Delta\,\underline{u}$ is to mean that γ is a denotation of $\underline{u}$ and perhaps of further indicators.
Let, for instance, $\mathrm{ind}(c - A(x))$ designate the set of all indicators occurring in c but not in $A(x)$.

Now we fix the following 'primary rules' for sentences of the language expanded so. In these rules (which we include among the *internal rules* of the *primary game*) we compactly write $\natural\,A|\delta$, e.g., for the assertion of A in a situation in which δ is valid. Then δ is assumed to be a denotation of at least all indicators occurring in A. We also write "under δ" as short for "in a situation in which δ is valid."

$$\begin{array}{rcl}
\natural\,(A \wedge B)|\delta & :\Rightarrow & \natural\,A|\delta \text{ and } \natural\,B|\delta \\
\natural\,(A \vee B)|\delta & :\Rightarrow & \natural\,A|\delta \text{ or } \natural\,B|\delta \\
\natural\,\neg A|\delta & :\Rightarrow & A \text{ rejected under } \delta \\
\natural\,\exists x\,\varepsilon\,C_m.\,A(x)|\delta & :\Rightarrow & \text{for some } c \in \mathcal{C}_m(x) : \\
 & & \text{for some } \gamma\,\Delta\,\mathrm{ind}(c - A(x)) : \;\; \natural\,A(c)|\gamma\delta \\
 & & \text{(cf. 10.0 below)} \\
\natural\,c\,\varepsilon\,\{\overline{x}\,\varepsilon\,\overline{C}_m \colon A(\overline{x})\}|\delta & :\Rightarrow & \natural\,c \in \overline{\mathcal{C}}_m,\; \natural\,A(c)|\delta,
\end{array}$$

for $R \rightleftharpoons (\mathrm{J}\overline{x}\ \varepsilon\ \overline{C}_m, \mu, z\colon A(\overline{x}, \mu, z))$:

$$
\begin{array}{rcl}
\natural\, (\underline{c}, k)\ \varepsilon\ R(l)|\delta & :\Rightarrow & \natural\, (\underline{c}) \in \overline{\mathcal{C}}_m,\ \natural\, k < l,\ \natural\, A((\underline{c}), k, R(k))|\delta \\
\natural\, \exists\overline{x}\ \varepsilon\ b(\overline{x}).\ A(\overline{x})|\delta & :\Rightarrow & \text{for some } \ c \in \overline{\mathcal{C}}\colon \\
 & & \text{for some } \ \gamma\,\Delta\,\mathrm{ind}(c-(b(\overline{x}), A(\overline{x})))\ : \\
 & & \natural\, c\ \varepsilon\ b(c)|\gamma\delta,\ \natural\, A(c)|\gamma\delta \\
\natural\, \exists x\ \varepsilon\ \pi_m((c_1, \ldots, c_j), i).\ A(x)|\delta & :\Rightarrow & \natural\, c_i \in \mathcal{C}_m(x),\ \natural\, A(c_i)|\delta \quad (\text{if } \ i \leq j) \\
\natural\, \exists x\ \varepsilon\ \pi_m((c_1, \ldots, c_j), i).\ A(x)|\delta & :\Rightarrow & \natural\, \bot \qquad (\text{if } \ i > j) \\
\natural\, \exists x\ \varepsilon\ \pi_m(s, p).\ A(x)|\delta & :\Rightarrow & \natural\, \bot \qquad (\text{if } \ \overline{\mathcal{W}}(s, p) = \{x\}) \\
\natural\, c =_0 d|\delta & :\Rightarrow & \natural\, c, d \in \mathcal{C}_0,\ \natural\, c \approx_0 d|\delta \\
\natural\, c\ \varepsilon\ C_m^j|\delta & :\Rightarrow & \natural\, c \in \mathcal{C}_m^j.
\end{array}
$$

For any $d \in \mathcal{C}_0$ let $(c\ \varepsilon\ d)$ pass for rejected under every denotation. For R as above and $a \notin \bigcup_{j \in \mathbb{N}^+} \left(\mathcal{C}^j \times \Omega\right)$, let also $(a\ \varepsilon\ R(l))$ pass for rejected under every denotation. - Let the primary game also contain corresponding external rules and internal rules for elementary sentences, but let it not contain further rules for sentences of $\mathcal{A} \setminus \mathcal{E}$.

We presuppose that $(\approx_0)$ (occurring in the last but one rule) is a given equivalence relation on $\mathcal{C}_0$ under which all formulas of $\mathcal{E}$ and all terms of $\mathcal{T}_{\mathrm{Or}}$ are invariant - so that, for all that $c, d \in \mathcal{C}(x)$, $x \in \mathcal{V}_0$, $a(x) \in \mathcal{T}_{\mathrm{Or}}$, and all $E(x) \in \mathcal{E}$ we may assert $c \approx_0 d \to a(c) \approx_0 a(d)$ and $c \approx_0 d \wedge E(c) \to E(d)$ *under all denotations* of the occurring indicators. - For all $x \in \mathcal{V}_0$ let $\mathcal{C}(x)$ be invariant under $(\approx_0)$.

(The rule for $\exists x\ \varepsilon\ C_m.\ A(x)$ comprises the both rules P($\exists$) and P($\exists$ den) from §5. This is compatible with our agreement that indicators may occur in constants.)

The *predecessor relation* can adequately be defined as in §8. So, all sentences considered here are non-circular, so that the adduced primary rules may be inverted.

10.0. Remarks: The existential quantifiers introduced here combine both substitutional and (in general) objectual quantification. We shall especially show that we may commute consecutive existential quantifiers that have been introduced. (This result does not contradict the well-known fact that in the context of Quines discussion (see [10, §28]) it is in general not allowed to commute consecutive existential quantifieres when one is objectual and the other substitutional.) - Regard also that our distinction of objectual variables from indicators and, therefore, the requirement of substitutions of indicators for objectual variables are not semantically essential.

Example: In the context of Quines discussion it is not true that every non-empty set has a subset containing a unique element. Nevertheless, we can argue as follows. Let a set be given by a constant $d \rightleftharpoons \{\xi\ \varepsilon\ C_0\colon D(\xi)\} \in \mathcal{C}_1$ together with a denotation $\delta\,\Delta\,\mathrm{ind}(d)$. Suppose that this set contains at least one element:

$$\exists\xi\ (\xi\ \varepsilon\ d) \quad \text{holds under} \quad \delta.$$

Then there exist an indicator $u \in \mathcal{C}(\xi)$ and a denotation γ of $\operatorname{ind}(u-d)$ such that

$$u \,\varepsilon\, d \quad \text{under} \quad \gamma\delta.$$

From this we conclude the following (where $\{u\}_0$ is short for $\{\xi \,\varepsilon\, C_0\colon \xi = u\}$):

$$\begin{array}{c} u \,\varepsilon\, d \;\wedge\; \{u\}_0 = \{u\}_0 \quad \text{under} \quad \gamma\delta \\ \exists\xi\,(\xi \,\varepsilon\, d \;\wedge\; \{u\}_0 = \{\xi\}_0) \quad \text{under} \quad \gamma\delta \\ \exists y \,\varepsilon\, C_1.\, \exists\xi\,(\xi \,\varepsilon\, d \;\wedge\; y = \{\xi\}_0) \quad \text{under} \quad \delta, \end{array}$$

which means that d has a subset, y, containing a unique element. So under δ we may assert: $\exists\xi\,(\xi \,\varepsilon\, d) \;\rightarrow\; \exists y \,\varepsilon\, C_1.\, \exists\xi\,(\xi \,\varepsilon\, d \;\wedge\; y = \{\xi\}_0)$ and, therefore,

$$\exists\xi\, \exists y \,\varepsilon\, C_1.\, (\xi \,\varepsilon\, d \;\wedge\; y = \{\xi\}_0) \;\rightarrow\; \exists y \,\varepsilon\, C_1.\, \exists\xi\,(\xi \,\varepsilon\, d \;\wedge\; y = \{\xi\}_0),$$

which, however, generally becomes untrue if we use a purely substitutional variable instead of y. Notice that the above unit set is given by the 'pair' $\{u\}_0|\gamma\delta$.

The following serves to prove that (in the classical game) we may argue classically within $\mathcal{A}$, and that all formulas of $\mathcal{F}$ are invariant under $(=)$. (This will be stated as Corollary 10.5.)

Assumptions: For every indicator u, there exist denumerably many ξ such that $u \in \mathcal{C}(\xi)$. If $\mathcal{C}(\xi) \cap \mathcal{C}(\eta) \neq \emptyset$, then $\mathcal{C}(\xi) = \mathcal{C}(\eta)$. - For all $x \in \mathcal{V}_0$ and all $u \in \mathcal{C}(\xi)$, assume that

$$\begin{array}{rcl} t \in \mathcal{T}(x) & \rightharpoondown & t^u_\xi \in \mathcal{T}(x) \\ t \in \mathcal{T}_{\mathrm{or}} & \rightharpoondown & t^u_\xi \in \mathcal{T}_{\mathrm{or}} \\ E \in \mathcal{E} & \rightharpoondown & E^u_\xi \in \mathcal{E}. \end{array}$$

Remark: By **P1, P2** of §8 and 8.1, the same also holds after interchanging ξ with u.

10.1. Lemma: If $(\underline{u}), (\underline{v}) \in \mathcal{C}(\underline{\xi})$, $x \in \mathcal{W}$, then

$$t \in \mathcal{T}_n(x) \;\rightharpoondown\; t^{\underline{u}}_{\underline{\xi}}, t^{\underline{u}}_{\underline{v}} \in \mathcal{T}_n(x).$$

Proof: Let $u, v \in \mathcal{C}(\xi)$. Similarly as in the proof of 8.1 we obtain:

$$t \in \mathcal{T}_n(x) \;\rightharpoondown\; t^u_\xi \in \mathcal{T}_n(x) \;\rightharpoondown\; t^u_v \equiv (t^u_\xi)^\xi_v \in \mathcal{T}_n(x).$$

So we obtain 10.1 by induction on the length of the list $\underline{\xi}$. $\square$

To enable further simplified proofs we shall simulate the above introduced quantification by substitutional quantification. To this we introduce notations (designations) of simple denotations, which can also be considered as proper names of objects denoted by those denotations. (A notation of a simple denotation created by a certain act of naming can be given by the pertinent indicator, the name of the actor, and the date, e.g.) *However, to obtain the desired results (Corollary 10.5) we need not presuppose that such notations will really be used.*

Remark: Denotations of *several* indicators generally result from acts of naming objects at different places at different times. However, several simple denotations can be designated by notations, which can be united to linguistic contexts (in place of situations) anywhere at any time. In such a context several denotations can *together* become valid.

Let every simple denotation of an indicator be represented by a notation (of a certain sort). These notations are said to be **new constants**. Generally we do not distinguish them from the simple denotations designated by them. - Let also the compound denotations considered here be represented by notations $(\beta_1, \ldots, \beta_j)$ composed of distinct new constants $\beta_1, \ldots, \beta_j$ which designate simple denotations of distinct indicators.

Definition: Let $\mathcal{C}(u) \rightleftharpoons \mathcal{C}(\xi)$ if $u \in \mathcal{C}(\xi)$.

Now we construct a further language in which the new constants occur as additional constants of order 0. Let $\hat{\mathcal{T}}_{\mathrm{Or}}$ be the set of all 'terms' that result from elements of $\mathcal{T}_{\mathrm{Or}}$ by replacing every occurring indicator u by a new constant γ which satisfies $\gamma \,\Delta\, \{v\}$ for some $v \in \mathcal{C}(u)$. In the same way, let $\hat{\mathcal{E}}$ result from $\mathcal{E}$, and $\hat{\mathcal{C}}(x)$ from $\mathcal{C}(x)$ (if $x \in \mathcal{V}_0$). Let the terms and formulas belonging to $\hat{\mathcal{T}}_n \cup \hat{\overline{\mathcal{T}}}_n \cup \hat{\mathcal{F}}_n$ be constructed from the elements of $\hat{\mathcal{T}}_{\mathrm{Or}} \cup \hat{\mathcal{E}}$ in the same way as the elements of $\mathcal{T}_n \cup \overline{\mathcal{T}}_n \cup \mathcal{F}_n$ from those of $\mathcal{T}_{\mathrm{Or}} \cup \mathcal{E}$. That is, any Ψ is an element of $\hat{\mathcal{T}}_n \,/\, \hat{\overline{\mathcal{T}}}_n \,/\, \hat{\mathcal{F}}_n$, respectively, iff the sentence (of a metalanguage) $(\Psi \in \hat{\mathcal{T}}_n) \,/\, (\Psi \in \hat{\overline{\mathcal{T}}}_n) \,/\, (\Psi \in \hat{\mathcal{F}}_n)$ is deducible by the corresponding '$\hat{\mathcal{T}}, \hat{\mathcal{F}}$-rules'. Let $\hat{\mathcal{C}}_n/\, \hat{\mathcal{A}}_n$, respectively, be the set of all elements of $\hat{\mathcal{T}}_n/\, \hat{\mathcal{F}}_n$ without free occurring elements of $\overline{\mathcal{W}}$. Define $\hat{\mathcal{A}} \rightleftharpoons \bigcup_{n\in\Omega} \hat{\mathcal{A}}_n$ etc.

For the sentences of $\hat{\mathcal{A}} \setminus \hat{\mathcal{E}}$ we stipulate the primary rules from §8. Moreover, we shall transfer the primary rules for sentences of $\mathcal{E}$ also to sentences of $\hat{\mathcal{E}}$. As a further internal rule for sentences of $\mathcal{E}$ we need:

$$\natural\, E|\gamma \;\Rightarrow\; \natural\, E|\delta,$$

if γ and δ contain the same simple denotations of all indicators occurring in $\mathcal{E}$ (cf. §5). - In the following we apply the definition of $\delta[u/v]$ (see §5), which we complete by $\delta[u/u] \rightleftharpoons \delta$.

Now let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be a lexicographical ordering of all new constants. Let the indicators t_i ($i \in \mathbb{N}^+$) be determined by $\alpha_i \,\Delta\, \{t_i\}$. For any $j \in \mathbb{N}^+$ let j^* be the least $k \in \mathbb{N}^+$ such that $t_k \in \mathcal{C}(t_j)$ and $t_k \not\equiv t_{i^*}$ for all $i < j$. Define $t_j^* \rightleftharpoons t_{j^*}$ and $\alpha_j^* \rightleftharpoons \alpha_j[t_j/t_j^*]$.

Thus in case $t_i^* \equiv t_j^*$ we have $i = j$ and so $\alpha_i^* \equiv \alpha_j^*$. Moreover, $\alpha_i^* \,\Delta\, \{t_i^*\}$ and $(t_i = t_i^*)|\alpha_i\alpha_i^*$.

To transfer the primary rules for sentences of $\mathcal{E} \cup \{(s \approx_0 t) : s, t \in \mathcal{T}_{\text{or}}\}$ to sentences of $\hat{\mathcal{E}} \cup \{(s \approx_0 t) : s, t \in \hat{\mathcal{T}_{\text{or}}}\}$, we fix the following internal rule:

$$\natural\, E(\beta_1, \ldots, \beta_j) \;\; :\Rightarrow \;\; \natural\, E(u_1^*, \ldots, u_j^*)|(\beta_1^*, \ldots, \beta_j^*),$$

if $E(\xi_1, \ldots, \xi_j) \in \mathcal{E} \cup \{(s \approx_0 t) : s, t \in \mathcal{T}_{\text{or}}\}$, $u_i \in \mathcal{C}(\xi_i)$ and $\beta_i \,\Delta\, \{u_i\}$ $(i = 1, \ldots, j)$, and if the variables $\xi_1, \ldots, \xi_j$ occurring in $E(\xi_1, \ldots, \xi_j)$ and the new constants $\beta_1, \ldots, \beta_j$ are distinct from each other. - This rule can also be *inverted* since we do not fix other assertion rules for sentences like $E(\beta_1, \ldots, \beta_j)$.

In the following we use a metalanguage which extends the object language and in which sentences of the form $(A|\delta)$ ("A holds under δ") are introduced by means of the only assertion rule: $\natural(A|\delta) :\Rightarrow \;\natural A|\delta$ (which may be inverted). In this language we argue classically.

10.2. Lemma: All formulas of $\hat{\mathcal{F}}$ are invariant under $(=)$.

Proof: Due to 8.12 we need only show that all formulas of $\hat{\mathcal{E}}$ and all terms of $\hat{\mathcal{T}_{\text{or}}}$ are invariant under $(\approx_0)$. First let $E(x, \underline{w}) \in \mathcal{E}$, $b(\underline{w}), c(\underline{w}) \in \mathcal{C}(x) \subset \mathcal{T}_{\text{or}}$, and let $E(x, \underline{w}), b(\underline{w}), c(\underline{w})$ only contain the indicators $\underline{w} \rightleftharpoons w_1, \ldots, w_j$. Moreover, let $\delta_i \,\Delta\, \{w_i\}$ $(i = 1, \ldots, j)$, $\underline{\delta} \rightleftharpoons \delta_1, \ldots, \delta_j$ and $\delta \rightleftharpoons (\delta_1, \ldots, \delta_j)$. Then we have

$$\begin{aligned} b(\underline{\delta}) \approx_0 c(\underline{\delta}),\; E(b(\underline{\delta}), \underline{\delta}) \;&\underline{\rightarrow}\; (b(\underline{w}^*) \approx_0 c(\underline{w}^*)|\delta^*),\; (E(b(\underline{w}^*), \underline{w}^*)|\delta^*) \\ &\underline{\rightarrow}\; (E(c(\underline{w}^*), \underline{w}^*)|\delta^*) \;\underline{\rightarrow}\; E(c(\underline{\delta}), \underline{\delta}). \end{aligned}$$

Similarly we obtain that all terms of $\hat{\mathcal{T}_{\text{or}}}$ are invariant under $(\approx_0)$. $\square$

Definition: If $\Phi \in \mathcal{T} \cup \overline{\mathcal{T}} \cup \mathcal{F}$ and $\beta \equiv (\beta_1, \ldots, \beta_j)$ where $\beta_i \,\Delta\, \{u_i\}$ $(i = 1, \ldots, j)$ and $u_1, \ldots, u_j$ are distinct indicators, then let Φ^β result from Φ by substituting β_i for u_i $(i = 1, \ldots, j)$.

10.3. Lemma: For any $\hat{c} \in \hat{\mathcal{C}}_m(x)$ with $x \in \mathcal{W}$ there exist $c \in \mathcal{C}_m(x)$ and a denotation β such that $\beta \,\Delta\, \text{ind}(c)$ and $\hat{c} = c^\beta$.

Proof: By induction on the $\hat{\mathcal{T}}, \hat{\mathcal{F}}$-rules we can show that for every element Ψ of $\hat{\mathcal{T}}_n$, $\hat{\overline{\mathcal{T}}}_n$, or $\hat{\mathcal{F}}_n$, respectively, there is an element Φ of $\mathcal{T}_n$, $\overline{\mathcal{T}}_n$, or $\mathcal{F}_n$ such that Ψ results from Φ if we replace every indicator u occurring in Φ by a new constant γ with $\gamma \,\Delta\, \{v\}$ for some $v \in \mathcal{C}(u)$. -
Now let $\hat{c} \in \hat{\mathcal{C}}_m(x)$. Then $\hat{c}$ results from a constant $c \equiv c(u_1, \ldots, u_j) \in \mathcal{C}_m(x)$ containing the distinct indicators $u_1, \ldots, u_j$ but no further ones, if we replace them by new constants $\gamma_1, \ldots, \gamma_j$ such that there exist $v_i \in \mathcal{C}(u_i)$ with $\gamma_i \,\Delta\, \{v_i\}$ $(i = 1, \ldots, j)$.

So $\hat{c} \equiv c(\gamma_1, \ldots, \gamma_j)$. Let $\beta_i \rightleftharpoons \gamma_i[v_i/u_i]$. For every $i = 1, \ldots, j$ we have $v_i = u_i|\gamma_i\beta_i$, so $v_i^* = u_i^*|\gamma_i^*\beta_i^*$, and so $\gamma_i = \beta_i$. - But there exist distinct $\xi_1, \ldots, \xi_j$ such that $u_i \in \mathcal{C}(\xi_i)$ for $i = 1, \ldots, j$. By 10.1, $c(\xi_1, \ldots, \xi_j) \in \hat{\mathcal{T}}$. So, by 10.2, $c(\xi_1, \ldots, \xi_j)$ is invariant under (=). So we obtain $\hat{c} = c(\gamma_1, \ldots, \gamma_j) = c(\beta_1, \ldots, \beta_j) = c^\beta$. □

10.4. Proposition: For all $A \in \mathcal{A}$ and all $\delta \,\Delta\, \mathrm{ind}(A)$ we have: $A^\delta \;\underline{\leftrightarrow}\; (A|\delta)$.

Proof: Let $\mathcal{X}$ be the class of all sentences A satisfying $A^\delta \;\underline{\leftrightarrow}\; (A|\delta)$ for all $\delta \,\Delta\, \mathrm{ind}(A)$. By 8.4 it suffices to show that $\mathcal{X}$ is progressive. Only the following three statements are not obvious: 1. $\mathcal{E} \cap \mathcal{A} \subseteq \mathcal{X}$. 2. $(\exists x \;\varepsilon\; C_m.\, A(x)) \in \mathcal{X}$ if ('hypothesis') $A(c) \in \mathcal{X}$ for all $c \in \mathcal{C}_m(x)$. 3. The corresponding statement concerning $(\exists \overline{x} \;\varepsilon\; b(\overline{x}).\, A(\overline{x}))$. - Here we only prove 1. and 2. Ad 1.: Let $A \equiv E(\underline{w}) \in \mathcal{E}$ where $\underline{w} \equiv w_1, \ldots, w_h$ is a list of all indicators occurring in A. Let $\delta \rightleftharpoons (\delta_1, \ldots, \delta_h)$ with $\delta_i \,\Delta\, \{w_i\}$ $(i = 1, \ldots, h)$, and $\underline{\delta} \rightleftharpoons \delta_1, \ldots, \delta_h$. Then $w_i = w_i^*|\delta_i\delta_i^*$ and so

$$(E(\underline{w})|\delta) \;\underline{\leftrightarrow}\; (E(\underline{w}^*)|\delta^*) \;\underline{\leftrightarrow}\; E(\underline{\delta}) \;\equiv\; A^\delta.$$

For the proof of 2. we write $A(x, \underline{w})$ for $A(x)$, where $\underline{w}$ is assumed to be a list of all occurring indicators. Then, by using primary rules and their inverses, from the adduced hypothesis we obtain for all δ as used in the proof of 1.:

$$\begin{array}{lll}
 & (\exists x \;\varepsilon\; C_m.\, A(x, \underline{w})|\delta) & \\
\underline{\leftrightarrow} & \underline{\exists} c \in \mathcal{C}_m(x).\, \underline{\exists}\gamma \,\Delta\, \mathrm{ind}(c - \underline{w}).\, (A(c, \underline{w})|\gamma\delta) & \\
\underline{\leftrightarrow} & \underline{\exists} c \in \mathcal{C}_m(x).\, \underline{\exists}\gamma \,\Delta\, \mathrm{ind}(c - \underline{w}).\, A(c, \underline{w})^{\gamma\delta} & \text{(by hypothesis)} \\
 & \big[\text{now notice that: } A(c, \underline{w})^{\gamma\delta} \equiv A(c^{\gamma\delta}, \underline{\delta})\,\big] & \\
\underline{\rightarrow} & \underline{\exists} c \in \mathcal{C}_m(x).\, \underline{\exists}\beta \,\Delta\, \mathrm{ind}(c).\, A(c^\beta, \underline{\delta}) & \\
\underline{\leftrightarrow} & \underline{\exists} \hat{c} \in \hat{\mathcal{C}}_m(x).\, A(\hat{c}, \underline{\delta}) & \text{(}(\underline{\leftarrow})\text{ by 10.3)} \\
\underline{\leftrightarrow} & \exists x \;\varepsilon\; C_m.\, A(x, \underline{w})^\delta. &
\end{array}$$

Now we prove the converse of '$\underline{\rightarrow}$' (above). Assume that we have $c \equiv c(\underline{u}) \in \mathcal{C}_m(x)$ and $\beta \,\Delta\, \mathrm{ind}(c) = \{\underline{u}\}$, $\underline{u} \equiv u_1, \ldots, u_j$. Then there are $\underline{v} \rightleftharpoons v_1, \ldots, v_j \notin \{\underline{u}, \underline{w}\}$ such that $(\underline{v}) \in \mathcal{C}(\underline{u})$. By 10.1 we obtain: $c(\underline{v}) \in \mathcal{C}_m(x)$. For $i = 1, \ldots, j$ let $\gamma_i \rightleftharpoons \beta_i[u_i/v_i]$. As in the proof of 10.3 we obtain $\beta_i = \gamma_i$, which we sum up to $\underline{\beta} = \underline{\gamma}$. Moreover, $\{\underline{v}\} = \mathrm{ind}(c(\underline{v})) = \mathrm{ind}(c(\underline{v}) - \underline{w})$. By these results and since the formulas of $\hat{\mathcal{F}}$ are invariant under (=), we obtain:

$$A(c^\beta, \underline{\delta}) \;\underline{\rightarrow}\; A(c(\underline{\beta}), \underline{\delta}) \;\underline{\rightarrow}\; A(c(\underline{\gamma}), \underline{\delta}) \;\underline{\rightarrow}\; A(c(\underline{v}), \underline{w})^{\gamma\delta},$$

and so the converse of '$\underline{\rightarrow}$'. The residual proof is left to the reader. □

10.5. Corollary: In the classical game we may apply all inference rules of classical logic even to sentences of $\mathcal{A}$ in which indicators or objectual variables may occur. All formulas of $\mathcal{F}$ are invariant under (=).

Proof: If $A_1, A_2 \Rightarrow B$ (e.g.) is either an individual case of an inference rule of classical logic or has the form: $\underline{c} = \underline{d}$, $A(\underline{c}) \Rightarrow A(\underline{d})$, then (by 10.4, 8.3*, and 8.12) we have: $(A_1|\delta) \ \underline{\wedge} \ (A_2|\delta) \ \underline{\rightarrow} \ A_1^\delta \ \underline{\wedge} \ A_2^\delta \ \underline{\rightarrow} \ B^\delta \ \underline{\rightarrow} \ (B|\delta)$. $\square$

By 10.5 we have solved our 'main tasks' mentioned in §7. It follows especially that we may commute any consecutive existential quantifiers that we have introduced (cf. the remark 10.0).

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