Finite sections of band-dominated operators with almost periodic coefficients

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Abstract

We consider the sequence of the finite sections R_nAR_n of a band-dominated operator A on $l^2(\mathbb{Z})$ with almost periodic coefficients. Our main result says that if the compressions of A onto $\mathbb{Z}^+$ and $\mathbb{Z}^-$ are invertible, then there is a distinguished subsequence of (R_nAR_n) which is stable. Moreover, this subsequence proves to be fractal, which allows us to establish the convergence in the Hausdorff metric of the singular values and pseudoeigenvalues of the finite section matrices.

1 Introduction

Given a non-empty subset $\mathbb{I}$ of the set $\mathbb{Z}$ of the integers, let $l^2(\mathbb{I})$ stand for the Hilbert space of all sequences $(x_n)_{n\in\mathbb{I}}$ of complex numbers with $\sum_{n\in\mathbb{I}} |x_n|^2 < \infty$. We identify $l^2(\mathbb{I})$ with a closed subspace of $l^2(\mathbb{Z})$ in the natural way, and we write $P_\mathbb{I}$ for the orthogonal projection from $l^2(\mathbb{Z})$ onto $l^2(\mathbb{I})$.

The set of the non-negative integers will be denoted by $\mathbb{Z}^+$, and we write P in place of $P_{\mathbb{Z}^+}$ and Q in place of the complementary projection $I - P$. Thus, $Q = P_{\mathbb{Z}^-}$ where $\mathbb{Z}^-$ refers to the set of all negative integers. For $k \in \mathbb{Z}$, define the *shift operator*

$$U_k : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}), \quad (x_n) \mapsto (y_n) \text{ with } y_n = x_{n-k}.$$

Further, each function $a \in l^\infty(\mathbb{Z})$ induces a *multiplication operator*

$$a : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}), \quad (x_n) \mapsto (a_n x_n).$$

Notice that the shifted multiplication operator $U_{-k}aU_k$ is a multiplication operator again:

$$(U_{-k}aU_kx)_n = (aU_kx)_{n+k} = a_{n+k}x_n.$$

Definition 1.1 *A function $a \in l^\infty(\mathbb{Z})$ is called* almost periodic *if the set of all multiplication operators $U_{-k}aU_k$ with $k \in \mathbb{Z}$ is relatively compact in the norm topology of $L(l^2(\mathbb{Z}))$ or, equivalently, in the norm topology of $l^\infty(\mathbb{Z})$. We denote* *the set of all almost periodic functions on* $\mathbb{Z}$ *by* $AP(\mathbb{Z})$, *and we write* $\mathcal{A}_{AP}(\mathbb{Z})$ *for the norm closure in* $L(l^2(\mathbb{Z}))$ *of the set of all operators*

$$A = \sum_{k=-K}^{K} a_k U_k \quad \textit{with } a_k \in AP(\mathbb{Z}).$$

The operators in $\mathcal{A}_{AP}(\mathbb{Z})$ *are called* band-dominated operators with almost periodic coefficients.

Is is easy to see that $AP(\mathbb{Z})$ and $\mathcal{A}_{AP}(\mathbb{Z})$ are C^*-subalgebras of $l^\infty(\mathbb{Z})$ and $\mathcal{A}_{AP}(\mathbb{Z})$, respectively.

For each positive integer n, set

$$P_n := P_{\{0,\,1,\,\ldots,\,n-1\}} \quad \text{and} \quad R_n := P_{\{-n,\,-n+1,\,\ldots,\,n-1\}}.$$

The projections R_n converge *-strongly to the identity operator on $l^2(\mathbb{Z})$, and the projections P_n converge *-strongly to the identity operator on $l^2(\mathbb{Z}^+)$ when considered as acting on $l^2(\mathbb{Z}^+)$ and to the projection P when considered as acting on $l^2(\mathbb{Z})$. For each operator $A \in \mathcal{A}_{AP}(\mathbb{Z})$, we consider the sequences (R_nAR_n) and (P_nPAPP_n) of its finite sections. These sequences converge *-strongly to A and PAP, respectively. Hence, they can be viewed as approximation methods for these operators. The finite sections sequences (R_nAR_n) resp. (P_nPAPP_n) are said to be *stable* if the operators $R_nAR_n : \operatorname{im} R_n \to \operatorname{im} R_n$ resp. $P_nPAPP_n : \operatorname{im} P_n \to \operatorname{im} P_n$ are invertible for sufficiently large n and if the norms of their inverses are uniformly bounded.

The stability of the finite section method for band-dominated operators with *arbitrary* l^∞-coefficients has been studied in [11, 12]. The crucial observation employed there is that the stability of the sequence (R_nAR_n) is equivalent to the Fredholmness of an associated band-dominated operator which can be treated by means of the limit operators method. The resulting criterion says that the sequence (R_nAR_n) is stable if and only if the operator PAP is invertible and if a whole family of so-called limit operators associated with that sequence is uniformly invertible. Similarly, the stability of (P_nPAPP_n) is equivalent to the invertibility of PAP plus the uniform invertibility of an associated limit operator family. The precise statements can be found in [11, 12, 14].

In the present paper we will show if $A \in \mathcal{A}_{AP}(\mathbb{Z})$ and if the operators PAP and QAQ are invertible then one can always find a *subsequence* of (R_nAR_n) resp. of (P_nPAPP_n) which is stable. Moreover, this subsequence can be effectively determined in many situations. Thus, the uniform invertibility of the (in general, infinite) family of limit operators is replaced by the invertibility of the single operator QAQ.

The motivation to consider suitable subsequences of (R_nAR_n) comes from a special class of band-dominated operators with almost periodic coefficients: the *block Laurent operators with continuous generating function*. These are the operators on $l^2(\mathbb{Z})$ with matrix representation $(a_{i-j})_{i,j\in\mathbb{Z}}$ with respect to the standard basis of $l^2(\mathbb{Z})$ where a_j is the jth Fourier coefficient of a continuous function $a : \mathbb{T} \to \mathbb{C}^{l\times l}$,

$$a_j := \frac{1}{2\pi}\int_0^{2\pi} a(e^{it})e^{-ijt}\,dt.$$

The block Laurent operator with generating function a will be denoted by $L(a)$. Since every continuous function on $\mathbb{T}$ can be uniformly approximated by a polynomial, block Laurent operators with continuous generating function are band-dominated operators with l-periodic (hence, almost periodic) coefficients. If $L(a)$ is a block Laurent operator, then the operator

$$T(a) := PL(a)P : l^2(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+)$$

is called the associated *block Toeplitz operator with generating function a.*

Let, for simplicity, $l = 2$ and write the jth Fourier coefficient a_j of the continuous function $a : \mathbb{T} \to \mathbb{C}^{2\times 2}$ as

$$a_j = \begin{pmatrix} a^j_{00} & a^j_{01} \\ a^j_{10} & a^j_{11} \end{pmatrix}.$$

Then the standard finite sections sequence (P_nPAPP_n) for the block Toeplitz operator $A = T(a)$ starts with

$$(\,a^0_{00}\,), \quad \begin{pmatrix} a^0_{00} & a^0_{01} \\ a^0_{10} & a^0_{11} \end{pmatrix}, \quad \begin{pmatrix} a^0_{00} & a^0_{01} & a^{-1}_{00} \\ a^0_{10} & a^0_{11} & a^{-1}_{10} \\ a^1_{00} & a^1_{01} & a^0_{00} \end{pmatrix}, \quad \begin{pmatrix} a^0_{00} & a^0_{01} & a^{-1}_{00} & a^{-1}_{01} \\ a^0_{10} & a^0_{11} & a^{-1}_{10} & a^{-1}_{11} \\ a^1_{00} & a^1_{01} & a^0_{00} & a^0_{01} \\ a^1_{10} & a^1_{11} & a^0_{10} & a^0_{11} \end{pmatrix}, \ \ldots$$

These finite sections do not completely reflect the 2×2-block structure of the operator $T(a)$. It is thus much more natural to consider the *subsequence* $(P_{2n}PAPP_{2n})$ of (P_nPAPP_n) which starts with

$$\begin{pmatrix} a^0_{00} & a^0_{01} \\ a^0_{10} & a^0_{11} \end{pmatrix}, \quad \begin{pmatrix} a^0_{00} & a^0_{01} & a^{-1}_{00} & a^{-1}_{01} \\ a^0_{10} & a^0_{11} & a^{-1}_{10} & a^{-1}_{11} \\ a^1_{00} & a^1_{01} & a^0_{00} & a^0_{01} \\ a^1_{10} & a^1_{11} & a^0_{10} & a^0_{11} \end{pmatrix}, \ \ldots$$

where each finite section is a 2×2-block Toeplitz matrix, too. In fact, it is the sequence $(P_{ln}PT(a)PP_{ln})$ which is usually referred to as *the* finite sections sequence for the $l \times l$-block Toeplitz operator $T(a)$ rather than the sequence $(P_nPT(a)PP_n)$ itself. The stability of the sequence $(P_{ln}PT(a)PP_{ln})$ for a block Toeplitz operator $T(a)$ with continuous generating function is well understood (see [7, 4, 5], for instance). It is stable if and only if the operators $PL(a)P$ and $QL(a)Q$ are invertible. The same results holds for the stability of the finite sections sequence $(R_{ln}L(a)R_{ln})$, simply because the operators $R_{ln}L(a)R_{ln}$

and $P_{2ln}PT(a)PP_{2ln}$ possess the same matrix representation with respect to the standard basis of $l^2(\mathbb{Z})$.

The paper is organized as follows. We start with some simple observations concerning band-dominated operators with almost periodic coefficients and their limit operators. Thereby we will learn how to choose a distinguished subsequence of the sequences (R_nAR_n) and (P_nPAPP_n) such that the above mentioned results hold. Then we will prove the stability results. We will *not* derive them from the stability theorem for the finite sections method for general band-dominated operators from [11, 12, 14]. Rather we prefer to show that these results follow in a completely elementary way from basic properties of band-dominated operators with almost periodic coefficients, in the very same manner as the stability of the finite sections method for Toeplitz operators with (scalar-valued) continuous generating functions has been proved in [9], Theorem 4.45 (see also [2] and Section 1.3.3 in [8]).

We will have occasion to observe that many properties of band-dominated operators with almost periodic coefficients are unexpected close to those of block Laurent operators with continuous generating function (= band-dominated operators with periodic coefficients). Thus, for readers which are familiar with Toeplitz and Hankel operators, it might be helpful to introduce the following notations for every band-dominated operator A:

$$T(A) := PAP, \quad \widetilde{A} := JAJ, \quad \text{and} \quad H(A) := PAQJ$$

where J stands for the flip operator

$$J : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}), \quad (x_n) \mapsto (y_n) \quad \text{with} \quad y_n := x_{-n-1}.$$

Then one has

$$T(\widetilde{A}) = PJAJP = JQAQJ \quad \text{and} \quad H(\widetilde{A}) = PJAJQJ = JQAP,$$

and equalities like $PABP = PAPBP + PAQBP$ can be written as

$$T(AB) = T(A)T(B) + H(A)H(\widetilde{B})$$

which reminds of a basic identity relating Toeplitz and Hankel operators.

Finally we would like to mention that the results of this paper can be transferred to l^p-spaces over $\mathbb{Z}$ and $\mathbb{Z}^+$ with $1 < p < \infty$ without great effort. For spectral and pseudospectral approximation on such spaces see [3] and [13], whereas the splitting property of the singular values is treated in [15].

2 Limit operators of band-dominated operators with almost periodic coefficients

We start with recalling the definition of a limit operator of a given operator. Let $\mathcal{H}$ refer to the set of all sequences $h : \mathbb{Z}^+ \to \mathbb{Z}$ which tend to infinity.

Definition 2.1 *An operator $A_h \in L(l^2(\mathbb{Z}))$ is called a* strong limit operator *of the operator $A \in L(l^2(\mathbb{Z}))$* with respect to the sequence $h \in \mathcal{H}$ *if*

$$U_{-h(k)} A U_{h(k)} \to A_h \qquad \text{as } k \to \infty \tag{1}$$

*-strongly. The sets of all strong limit operators of a given operator A will be denoted by $\sigma_{op,\,s}(A)$, and we refer to this set as the strong operator spectrum of A. *Further, let $\mathcal{H}_{A,\,s}$ stand for the set of all sequences $h \in \mathcal{H}$ such that* (1) *holds with respect to the *-strong topology. Analogously, we call A_h a* norm limit operator of A *if* (1) *holds with respect to norm convergence, and we introduce the related* norm operator spectrum $\sigma_{op,\,n}(A)$ *of A and the corresponding class $\mathcal{H}_{A,\,n}$.*

In [10, 11, 12, 14] we have exclusively worked with limit operators in the *-strong sense (simply because the norm operator spectrum proved to be to small to be of any use in general). But for band-dominated operators with almost periodic coefficients, one can work in the norm topology as well.

Lemma 2.2 *For $A \in \mathcal{A}_{AP}(\mathbb{Z})$, one has $\sigma_{op,\,s}(A) = \sigma_{op,\,n}(A)$.*

Proof. The inclusion $\supseteq$ is obvious. The reverse inclusion holds for operators of multiplication by an almost periodic function due to the definition of the class $AP(\mathbb{Z})$. Then it holds also for band operators with almost periodic coefficients. For the proof in the general case, approximate the operator A in the norm topology by a sequence (A_n) of band operators with almost periodic coefficients. Let $g_0 := h \in \mathcal{H}_{A,\,s}$. Then there is a subsequence g_1 of g_0 which belongs to $\mathcal{H}_{A_1,\,n}$. Further, there is a subsequence g_2 of g_1 with $g_2 \in \mathcal{H}_{A_2,\,n}$. We proceed in this way and find, for every positive integer k, a subsequence g_k of g_{k-1} with $g_k \in \mathcal{H}_{A_k,\,n}$. The sequence g defined by $g(k) := g_k(k)$ is a subsequence of each sequence g_k. Thus, all limit operators $(A_k)_g$ exist with respect to norm convergence. Then also the limit operator A_g exists with respect to norm convergence, whence $A_h \in \sigma_{op,\,n}(A)$. ■

It follows in particular that $\mathcal{H}_{A,\,n}$ is not empty if $A \in \mathcal{A}_{AP}(\mathbb{Z})$.

Lemma 2.3 *Let $A \in \mathcal{A}_{AP}(\mathbb{Z})$ and $h \in \mathcal{H}_{A,\,n}$. Then $(A_h)_{-h} = A$.*

This follows immediately from

$$\|U_{h(n)} A_h U_{-h(n)} - A\| = \|A_h - U_{-h(n)} A U_{h(n)}\| \to 0.$$

Lemma 2.4 *If $A \in \mathcal{A}_{AP}(\mathbb{Z})$, then $A \in \sigma_{op,\,n}(A)$.*

Proof. Let h be any sequence in $\mathcal{H}_{A,\,n}$. We define a sequence $(n_k)_{k\ge1}$ as follows. Let $n_1 = 0$. If n_k is already defined for some $k \ge 1$, then we choose $n_{k+1} > n_k$ such that

$$|h(n_{k+1}) - h(n_k)| \ge k + 1 \tag{2}$$

which is possible since $h \in \mathcal{H}$. Set $g(k) := h(n_k) - h(n_{k+1})$. Then

$$\begin{aligned}
&\|U_{-g(k)}AU_{g(k)} - A\| \\
&\quad = \|U_{h(n_{k+1})}U_{-h(n_k)}AU_{h(n_k)}U_{-h(n_{k+1})} - A\| \\
&\quad \leq \|U_{h(n_{k+1})}(U_{-h(n_k)}AU_{h(n_k)} - A_h)U_{-h(n_{k+1})}\| + \|U_{h(n_{k+1})}A_hU_{-h(n_{k+1})} - A\| \\
&\quad \leq \|U_{-h(n_k)}AU_{h(n_k)} - A_h\| + \|U_{h(n_{k+1})}A_hU_{-h(n_{k+1})} - A\| \to 0
\end{aligned}$$

as $k \to \infty$. Thus, $\lim U_{-g(k)}AU_{g(k)} = A$ in the norm. Since condition (2) ensures that $g \in \mathcal{H}$, we have $g \in \mathcal{H}_{A,n}$ and $A_g = A$. ■

In case of $l \times l$-block Laurent operators (= band-dominated operators with l-periodic coefficients) this result is obvious: the sequence $g(k) := lk$ belongs to $\mathcal{H}_{L(a),n}$ and $L(a)_g = L(a)$.

3 Band-dominated operators with almost periodic coefficients on $l^2(\mathbb{Z}^+)$

Here we consider compressions of band-dominated operators with almost periodic coefficients onto $l^2(\mathbb{Z}^+)$. Notice that the compression of an operator of multiplication by an almost periodic function a to $l^2(\mathbb{Z}^+)$ (considered as a subspace of $l^2(\mathbb{Z})$) is no longer almost periodic unless the trivial case $a = 0$.

Definition 3.1 *Let $A \in \mathcal{A}_{AP}(\mathbb{Z})$. The we call PAP a* band-dominated operator with AP coefficients on $l^2(\mathbb{Z}^+)$. *The smallest closed subalgebra of $L(l^2(\mathbb{Z}^+))$ which contains all band-dominated operators with AP coefficients on $l^2(\mathbb{Z}^+)$ will be denoted by $\mathcal{A}_{AP}(\mathbb{Z}^+)$.*

Evidently, $\mathcal{A}_{AP}(\mathbb{Z}^+)$ is a C^*-subalgebra of $L(l^2(\mathbb{Z}^+))$.

Lemma 3.2 *For $A \in \mathcal{A}_{AP}(\mathbb{Z})$, one has $\|A\| = \|PAP\|$.*

In case of periodic coefficients, this simply says that $\|L(a)\| = \|T(a)\|$.

Proof. Choose a sequence $h \in \mathcal{H}_{A,n}$ which converges to $+\infty$ and for which $A_h = A$. (Starting with a suitable sequence h in the proof of Lemma 2.4 one easily gets a sequence with these properties.) Then $h \in \mathcal{H}_{P,s}$ and $P_h = I$. Hence, $h \in \mathcal{H}_{PAP,s}$ and $(PAP)_h = A_h = A$. This implies the assertion since

$$\|A\| = \|A_h\| = \|(PAP)_h\| \leq \|PAP\| \leq \|A\|$$

where we have used the elementary estimate $\|B_h\| \leq \|B\|$ for limit operators (Proposition 1.2.2 in [12]). ■

Corollary 3.3 *Let B, $C \in \mathcal{A}_{AP}(\mathbb{Z})$. If $PBP = PCP$, then $B = C$.*

This follows from Lemma 3.2 with $A := B - C$. One can consider the statement of the preceding corollary as a rigidity property of band-dominated operators with AP coefficients: The restriction of an operator $A \in \mathcal{A}_{AP}(\mathbb{Z})$ onto $l^2(\mathbb{Z}^+)$ can be extended to an operator in $\mathcal{A}_{AP}(\mathbb{Z})$ in exactly one manner. The extension of a Toeplitz operator $T(a)$ is just the Laurent operator $L(a)$.

Lemma 3.4 *Let* $A \in \mathcal{A}_{AP}(\mathbb{Z})$. *Then*
(a) $\|A\| \le \|A + K\|$ *for each compact operator* $K \in L(l^2(\mathbb{Z}))$;
(b) $\|PAP\| \le \|PAP + K\|$ *for each compact operator* $K \in L(l^2(\mathbb{Z}^+))$.

Proof. Let h be as in the proof of Lemma 3.2, and let K be compact. Then, in both cases, $h \in \mathcal{H}_{K,n}$ and $K_h = 0$. Thus,

$$\|A\| = \|A_h\| = \|(A + K)_h\| \le \|A + K\|$$

and, by Lemma 3.2,

$$\|PAP\| = \|A\| = \|A_h\| = \|(PAP + K)_h\| \le \|PAP + K\|$$

which implies assertions (a) and (b), respectively. ∎

Lemma 3.5 *One has*

$$\mathcal{A}_{AP}(\mathbb{Z}^+) = \{PAP + K : A \in \mathcal{A}_{AP}(\mathbb{Z}),\ K \in L(l^2(\mathbb{Z}^+)) \text{ compact}\}, \tag{3}$$

and each operator $B \in \mathcal{A}_{AP}(\mathbb{Z}^+)$ *can be written as* $PAP + K$ *with* $A \in \mathcal{A}_{AP}(\mathbb{Z})$ *and* K *compact in a unique way.*

The well known analogue of (3) for Toeplitz operators ([8], Theorem 1.51) is

$$\mathcal{A}_{\mathbb{C}}(\mathbb{Z}^+) = \{T(a) + K : a \in C(\mathbb{T}),\ K \text{ compact}\}$$

where $\mathcal{A}_{\mathbb{C}}(\mathbb{Z}^+)$ stands for the smallest closed subalgebra of $L(l^2(\mathbb{Z}^+))$ which contains all Toeplitz operators with continuous generating function (= all restrictions of band-dominated operators with constant coefficients to $l^2(\mathbb{Z}^+)$).

Proof. Denote the right-hand side of (3) by $\mathcal{A}'$ for a moment. The inclusion $\mathcal{A}' \subseteq \mathcal{A}_{AP}(\mathbb{Z}^+)$ holds since $PAP \in \mathcal{A}_{AP}(\mathbb{Z}^+)$ by definition and since $K \in \mathcal{A}_{\mathbb{C}}(\mathbb{Z}^+)$ as mentioned above. For the reverse inclusion notice that the operator

$$PAPBP - PABP = -PAQBP$$

is compact for each pair of band-dominated operators A, B (for the operator PAQ is of finite rank if A is a band operator). Hence, all finite sums of products $\sum_i \prod_j PA_{ij}P$ with band-dominated operators A_{ij} belong to $\mathcal{A}'$, and the implication $\mathcal{A}_{AP}(\mathbb{Z}^+) \subseteq \mathcal{A}'$ will follow once we have shown that $\mathcal{A}'$ is closed.

Let $(PA_nP + K_n)$ be a Cauchy sequence in $\mathcal{A}'$. By Lemma 3.2 and Lemma 3.4 (b),

$$\|A_n - A_m\| = \|P(A_n - A_m)P\| \leq \|(PA_nP + K_n) - (PA_mP + K_m)\|.$$

Thus, (A_n) is a Cauchy sequence in $\mathcal{A}_{AP}(\mathbb{Z})$. Let $A \in \mathcal{A}_{AP}(\mathbb{Z})$ denote its limit. Then PA_nP converges to PAP in the norm, which implies that (K_n) is a Cauchy sequence, too. Its limit K is compact. So we finally get that PA_nP+K_n converges in the norm to $PAP + K$ which obviously is in $\mathcal{A}'$. ∎

Lemma 3.6 *Let $A \in \mathcal{A}_{AP}(\mathbb{Z})$. Then A is invertible if and only if PAP is a Fredholm operator on $l^2(\mathbb{Z}^+)$.*

In particular, the block Laurent operator $L(a)$ with continuous generating function a is invertible if and only if the Toeplitz operator $T(a)$ is Fredholm.

Proof. If PAP is a Fredholm operator, then every strong limit operator $(PAP)_h$ of PAP is invertible (Proposition 1.2.9 in [12]). Choosing a sequence h such that $(PAP)_h = A$ gives the invertibility of A. The reverse implication holds for arbitrary band-dominated operators A since PAQ and QAP are compact. ∎

4 Distinguished finite sections methods

Definition 4.1 *Let $A \in \mathcal{A}_{AP}(\mathbb{Z})$. By a* distinguished sequence *for A we mean a monotonically increasing sequence $h : \mathbb{Z}^+ \to \mathbb{Z}^+$ which belongs to $\mathcal{H}_{A,n}$ and for which $A_h = A$. If h is a distinguished sequence for A, then the sequences $(P_{h(n)}PAPP_{h(n)})$ and $(R_{h(n)}AR_{h(n)})$ are called the associated* distinguished finite sections methods *for PAP and A, respectively.*

Theorem 4.2 *Let $A \in \mathcal{A}_{AP}(\mathbb{Z})$ and let h be a distinguished sequence for A. Let further L be a compact operator on $l^2(\mathbb{Z}^+)$. Then the sequence $(P_{h(n)}(PAP + L)P_{h(n)})$ is stable if and only if the operators $PAP + L$ and QAQ are invertible.*

Of course, this result implies the well known criterion for the stability of the finite sections method $(P_{ln}T(a)P_{ln})$ for the block Toeplitz operator $T(a)$ with continuous function $a : \mathbb{T} \to \mathbb{C}^{l\times l}$: This method is stable if and only if the Toeplitz operator $T(a) = PL(a)P$ itself and the associated Toeplitz operator $T(\tilde{a}) = JQL(a)QJ$ with $\tilde{a}(t) := a(1/t)$ is invertible.

In what follows we will several times make use of the following elementary lemma.

Lemma 4.3 (Kozak) *Let X be a linear space, P a projection, $Q := I - P$ and A an invertible linear operator on X. Then the operator $PAP|_{\mathrm{im}\,P}$ is invertible if and only if the operator $QA^{-1}Q|_{\mathrm{im}\,Q}$ is invertible, and*

$$(PAP)^{-1}P = PA^{-1}P - PA^{-1}Q(QA^{-1}Q)^{-1}QA^{-1}P. \tag{4}$$

Proof of Theorem 4.2. First we show that if $PAP+L$ and QAQ are invertible, then the distinguished finite sections sequence $(P_{h(n)}(PAP+L)P_{h(n)})$ is stable.

The invertibility of $PAP+L$ implies those of A by Lemma 3.6, and the invertibility of QAQ implies those of $PA^{-1}P$ by Kozak's lemma. Thus one has

$$P = PAA^{-1}P = PAPA^{-1}P + PAQA^{-1}P$$

and

$$PAP + L = (PA^{-1}P)^{-1} - PAQA^{-1}P(PA^{-1}P)^{-1} =: (PA^{-1}P)^{-1} + L - K \quad (5)$$

where $K := PAQA^{-1}P(PA^{-1}P)^{-1}$ is compact due to the compactness of PAQ.

We claim that the finite sections method $(P_{h(n)}(PA^{-1}P)^{-1}P_{h(n)})$ for the operator $(PA^{-1}P)^{-1}$ is stable if the operator QAQ is invertible. By Kozak's lemma again, the sequence $(P_{h(n)}(PA^{-1}P)^{-1}P_{h(n)})$ is stable if and only if the sequence $(Q_{h(n)}PA^{-1}PQ_{h(n)})$ with $Q_n := I - P_n : l^2(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+)$ is stable, i.e., if the operators

$$Q_{h(n)}PA^{-1}PQ_{h(n)}|_{\operatorname{im} Q_{h(n)}}$$

are invertible for sufficiently large n and if the norms of their inverses are uniformly bounded. This happens if and only if the operators

$$\begin{aligned}
&U_{-h(n)}Q_{h(n)}PA^{-1}PQ_{h(n)}U_{h(n)}|_{\operatorname{im}(U_{-h(n)}Q_{h(n)}U_{h(n)})} \\
&\quad = U_{-h(n)}Q_{h(n)}U_{h(n)}\; U_{-h(n)}A^{-1}U_{h(n)}\; U_{-h(n)}Q_{h(n)}U_{h(n)}|_{\operatorname{im} P} \\
&\quad = PU_{-h(n)}A^{-1}U_{h(n)}P|_{\operatorname{im} P} \qquad (6)
\end{aligned}$$

are invertible for sufficiently large n and if the norms of their inverses are uniformly bounded. Since h is a distinguished sequence for A, one has

$$\|U_{-h(n)}AU_{h(n)} - A\| \to 0$$

which implies

$$\|U_{-h(n)}A^{-1}U_{h(n)} - A^{-1}\| \to 0.$$

Hence, (6) converges in the norm to $PA^{-1}P$. Since this operator is invertible as mentioned above, the operators in (6) are invertible for sufficiently large n, and their inverses are uniformly bounded. This proves the claim.

Now (5) gives

$$P_{h(n)}(PAP + L)P_{h(n)} = P_{h(n)}(PA^{-1}P)^{-1}P_{h(n)} + P_{h(n)}(L - K)P_{h(n)},$$

i.e., the sequence $(P_{h(n)}(PAP+L)P_{h(n)})$ we are interested in is a compact perturbation of the stable sequence $(P_{h(n)}(PA^{-1}P)^{-1}P_{h(n)})$. Since $(PA^{-1}P)^{-1}+L-K = PAP + L$ is an invertible operator by hypothesis, the perturbation theorem for approximation methods (Corollary 1.22 in [8]) implies the stability of the finite sections method $(P_{h(n)}(PAP + L)P_{h(n)})$.

Conversely, we have to show that the stability of that sequence implies the invertibility of the operators $PAP+L$ and QAQ. This follows in a standard way from

$$P_{h(n)}(PAP+L)P_{h(n)} \to PAP+L \qquad {}^*\text{-strongly}$$

and

$$U_{-h(n)}P_{h(n)}(PAP+L)P_{h(n)}U_{h(n)} \to QAQ \qquad {}^*\text{-strongly}$$

which holds for every distinguished sequence h. ■

Next we consider the finite section method for operators in $\mathcal{A}_{AP}(\mathbb{Z})$. We will need one more simple lemma.

Lemma 4.4 *Let $A \in \mathcal{A}_{AP}(\mathbb{Z})$, and let h be a sequence in $\mathcal{H}_{A,n}$ with $A_h = A$. Then $2h$ and $-h$ are sequences in $\mathcal{H}_{A,n}$ with $A_{2h} = A$ and $A_{-h} = A$.*

This follows easily from

$$\begin{aligned}
&\|U_{-2h(n)}AU_{2h(n)} - A\| \\
&\quad \le \|U_{-2h(n)}AU_{2h(n)} - U_{-h(n)}AU_{h(n)}\| + \|U_{-h(n)}AU_{h(n)} - A\| \\
&\quad \le 2\,\|U_{-h(n)}AU_{h(n)} - A\| \to 0
\end{aligned}$$

and

$$\begin{aligned}
\|U_{h(n)}AU_{-h(n)} - A\| &= \|U_{h(n)}(A - U_{-h(n)}AU_{h(n)})U_{-h(n)}\| \\
&\le \|A - U_{-h(n)}AU_{h(n)}\| \to 0.
\end{aligned}$$

Theorem 4.5 *Let $A \in \mathcal{A}_{AP}(\mathbb{Z})$, and let h be a distinguished sequence for A. Furthermore, let L be a compact operator on $l^2(\mathbb{Z})$. Then the sequence $(R_{h(n)}(A+L)R_{h(n)})$ is stable if and only if the operators $A+L$, PAP and QAQ are invertible.*

In case $L=0$, the invertibility of $A+L=A$ follows from the invertibility of PAP due to Lemma 3.6. Hence, in this case, the stability of the finite section method is equivalent to the invertibility of PAP and QAQ.

Proof. The crucial observation is that

$$\begin{aligned}
&\|U_{h(n)}R_{h(n)}AR_{h(n)}U_{-h(n)} - P_{2h(n)}PAPP_{2h(n)}\| \\
&\quad = \|P_{2h(n)}U_{h(n)}AU_{-h(n)}P_{2h(n)} - P_{2h(n)}PAPP_{2h(n)}\| \\
&\quad \le \|U_{h(n)}AU_{-h(n)} - A\| \to 0
\end{aligned}$$

by the preceding lemma. The same lemma states furthermore that $2h$ is a distinguished sequence for A. Thus, if PAP and QAQ are invertible, then $(P_{2h(n)}PAPP_{2h(n)})$ is a stable sequence by Theorem 4.2. Since

$$(P_{2h(n)}PAPP_{2h(n)}) \qquad \text{and} \qquad (U_{h(n)}R_{h(n)}AR_{h(n)}U_{-h(n)})$$

differ by a sequence which tends to zero in the norm, the latter sequence is stable, too. But then, clearly, the sequence $(R_{h(n)}AR_{h(n)})$ is stable. Since $A+L$ is invertible by hypothesis, the stability of the compactly perturbed sequence $(R_{h(n)}(A+L)R_{h(n)})$ follows via the perturbation theorem (Corollary 1.22 in [8]) again. The reverse implication in Theorem 4.5 follows as in the proof of Theorem 4.2. ■

In the following examples we are going to make the previous constructions more explicit.

Example A: Multiplication operators. For each real number $\alpha \in [0,\, 1)$, the function

$$a : \mathbb{Z} \to \mathbb{C}, \quad n \mapsto e^{2\pi i\alpha n} \tag{7}$$

is almost periodic. Indeed, for every integer k, $U_{-k}aU_k$ is the operator of multiplication by the function a_k with $a_k(n) = a(n+k) = e^{2\pi i\alpha k}a(n)$, i.e.,

$$U_{-k}aU_k = e^{2\pi i\alpha k}a. \tag{8}$$

Let $(U_{-k(n)}aU_{k(n)})$ by any sequence in $\{U_{-k}aU_k : k \in \mathbb{Z}\}$. Due to the compactness of $\mathbb{T}$, there are a subsequence $(e^{2\pi i\alpha k(n(r))})_{r\geq 1}$ of $(e^{2\pi i\alpha k(n)})_{n\geq 1}$ and a real number β such that

$$e^{2\pi i\alpha k(n(r))} \to e^{2\pi i\beta} \qquad \text{as } r \to \infty.$$

Thus, the functions $a_{k(n(r))} = e^{2\pi i\alpha k(n(r))}a$ converge uniformly to $e^{2\pi i\beta}a$, whence the almost periodicity of a. For the operator spectrum of the operator aI one finds

$$\sigma_{op,\,s}(aI) = \sigma_{op,\,n}(aI) = \begin{cases} \{e^{2\pi il/q}\, a : l = 1,\, 2,\, \dots,\, q\} & \text{if} \quad \alpha = 2p/q \in \mathbb{Q}, \\ \{e^{it}\, a : t \in \mathbb{R}\} & \text{if} \quad \alpha \notin \mathbb{Q}, \end{cases}$$

Here, p and q are relatively prime integers with $q > 0$. Indeed, the inclusion $\subseteq$ follows immediately from (8). The reverse inclusion is evident in case $\alpha \in \mathbb{Q}$. If $\alpha \notin \mathbb{Q}$, then it follows from a theorem by Kronecker which states that the set of all numbers $e^{2\pi i\alpha k}$ with integer k lies dense in the unit circle $\mathbb{T}$.

In case $\alpha = p/q \in \mathbb{Q}$, the sequence a is q-periodic, and $h(n) = qn$ is a distingiushed sequence for the multiplication operator aI. To get a distinguished sequence h for aI in case $\alpha \notin \mathbb{Q}$, too, one has to ensure that

$$\lim_{n\to\infty} e^{2\pi i\alpha h(n)} = 1$$

(cp. (8)). For develop $\alpha \in (0,\, 1)$ into a continued fraction

$$\alpha = \lim_{n\to\infty} \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots \, b_{n-1} + \cfrac{1}{b_n}}}}$$

with uniquely determined positive integers b_i. Write this continued fraction as p_n/q_n with positive and relatively prime integers p_n, q_n. These integers satisfy the recursions

$$p_n = a_n p_{n-1} + p_{n-2}, \qquad q_n = a_n q_{n-1} + q_{n-2} \tag{9}$$

with $p_0 = 0$, $p_1 = 1$, $q_0 = 1$ and $q_1 = a_1$, and one has for all $n \geq 1$

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}. \tag{10}$$

These facts can be found in any book on continued fractions. From (10) we conclude that

$$|\alpha q_n - p_n| \leq q_n \left| \alpha - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n} \to 0,$$

whence

$$e^{2\pi i \alpha q_n} = e^{2\pi i (\alpha q_n - p_n)} \to 1.$$

Since moreover $q_1 < q_2 < \ldots$ due to the recursion (9), this shows that the sequence $h(n) := q_n$ belongs to $\mathcal{H}_{A,n}$ and that $A_h = A$, i.e. h is a distinguished sequence for the operator aI with a as in (7). ■

Example B: Almost Mathieu operators. These are the operators $H_{\alpha,\lambda,\theta} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ given by

$$(H_{\alpha,\lambda,\theta} x)_n := x_{n+1} + x_{n-1} + \lambda x_n \cos 2\pi(n\alpha + \theta)$$

with real parameters α, λ and θ. Thus, $H_{\alpha,\lambda,\theta}$ is a band operator with almost periodic coefficients, and

$$H_{\alpha,\lambda,\theta} = U_{-1} + U_1 + aI \qquad \text{with} \qquad a(n) = \lambda \cos 2\pi(n\alpha + \theta).$$

For a treatment of the spectral theory of Almost Mathieu operators see [1]. As in Example A one gets

$$U_{-k} H_{\alpha,\lambda,\theta} U_k = U_{-1} + U_1 + a_k I$$

with

$$\begin{aligned} a_k(n) &= a(n+k) = \lambda \cos 2\pi((n+k)\alpha + \theta) \\ &= \lambda(\cos 2\pi(n\alpha + \theta) \cos 2\pi k\alpha - \sin 2\pi(n\alpha + \theta) \sin 2\pi k\alpha). \end{aligned} \tag{11}$$

We will only consider the non-periodic case, i.e., we let $\alpha \in (0, 1)$ be irrational. As in the previous example, we write α as a continued fraction with nth approximant p_n/q_n such that (10) holds. Then

$$\cos 2\pi\alpha q_n = \cos 2\pi(\alpha q_n - p_n) = \cos 2\pi q_n(\alpha - p_n/q_n) \to \cos 0 = 1$$

and, similarly, $\sin 2\pi\alpha q_n \to 0$. Further we infer from (11) that

$$|(a_{q_n}) - a)(n)| \le |\lambda|\, |1 - \cos 2\pi\alpha q_n| + |\lambda|\, |\sin \pi\alpha q_n|.$$

Hence, $a_{q_n} \to a$ uniformly. Thus, $h(n) := q_n$ defines a distinguished sequence for the Almost Mathieu operator $H_{\alpha,\lambda,\theta}$. Notice that this sequence depends on the parameter α only. Theorems 4.2 and 4.5 imply the following.

Corollary 4.6 *Let $A := H_{\alpha,\lambda,\theta}$ be an Almost Mathieu operator and h a distinguished sequence for A. Then the following conditions are equivalent:*
(a) the distinguished finite sections method $(P_{h(n)}PAPP_{h(n)})$ for PAP is stable;
(b) the distinguished finite sections method $(R_{h(n)}AR_{h(n)})$ for A is stable;
(c) the operators PAP and QAQ are invertible.

If $\theta = 0$, then the Almost Mathieu operator $A = H_{\alpha,\lambda,0}$ is flip invariant, i.e., $JAJ = A$. So we observe in this case that the third condition in Corollary 4.6 is equivalent to the invertibility of PAP alone.

For a different approach to the numerical treatment of Almost Mathieu and other operators in irrational rotation algebras see [6].

5 The algebra of the finite sections method

In what follows we fix a strongly monotonically increasing sequence $h : \mathbb{Z}^+ \to \mathbb{Z}^+$. Define

$$\mathcal{A}_{AP,h}(\mathbb{Z}) := \{A \in \mathcal{A}_{AP}(\mathbb{Z}) : h \in \mathcal{H}_{A,n} \text{ and } A_h = A\}.$$

Thus, an operator $A \in \mathcal{A}_{AP}(\mathbb{Z})$ belongs to $\mathcal{A}_{AP,h}(\mathbb{Z})$ if and only if h is a distinguished sequence for PAP. By (a slightly improved version of) Lemma 2.4, every operator $A \in \mathcal{A}_{AP}(\mathbb{Z})$ belongs to one of the sets $\mathcal{A}_{AP,h}(\mathbb{Z})$ with a suitably chosen sequence h.

It is easy to check that $\mathcal{A}_{AP,h}(\mathbb{Z})$ is a C^*-subalgebra of $L(l^2(\mathbb{Z}))$ which is moreover shift invariant, i.e., $U_{-k}AU_k$ belongs to this algebra for each $k \in \mathbb{Z}$ whenever A does. It is also clear that all Laurent operators with continuous and complex-valued generating function belong to each of the algebras $\mathcal{A}_{AP,h}(\mathbb{Z})$.

Let $\mathcal{A}_{AP,h}(\mathbb{Z}^+)$ refer to the smallest closed subalgebra of $L(l^2(\mathbb{Z}^+))$ which contains all operators PAP with $A \in \mathcal{A}_{AP,h}(\mathbb{Z})$. For instance, all Toeplitz operators with continuous and complex-valued generating function lie in this algebra. Hence, $\mathcal{A}_{AP,h}(\mathbb{Z}^+)$ also contains all compact operators, and one can show as in Lemma 3.5 that

$$\mathcal{A}_{AP,h}(\mathbb{Z}^+) = \{PAP + K : A \in \mathcal{A}_{AP,h}(\mathbb{Z}),\ K \in L(l^2(\mathbb{Z}^+)) \text{ compact}\}. \quad (12)$$

Let $\mathcal{F}_h$ stand for the set of all bounded sequences (A_n) of matrices $A_n \in \mathbb{C}^{h(n)\times h(n)}$. Provided with pointwise defined operations and the supremum norm, $\mathcal{F}_h$ becomes a C^*-algebra. As earlier, we will identify the matrices A_n with operators acting on $\operatorname{im} P_{h(n)}$. Finally, we let $\mathcal{S}_{AP,\,h}(\mathbb{Z}^+)$ denote the smallest closed subalgebra of $\mathcal{F}_h$ which contains all sequences $(P_{h(n)}PAPP_{h(n)})$ with operators $A \in \mathcal{A}_{AP,\,h}(\mathbb{Z})$. The following result describes this algebra completely. For, introduce

$$W_n : l^2(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+), \quad (x_n)_{n\geq 0} \mapsto (x_{n-1},\, x_{n-2},\, \ldots.\, x_0,\, 0,\, 0,\, \ldots).$$

Theorem 5.1 *The algebra $\mathcal{S}_{AP,\,h}(\mathbb{Z}^+)$ consists exactly of all sequences of the form*

$$(P_{h(n)}PAPP_{h(n)} + P_{h(n)}KP_{h(n)} + W_{h(n)}LW_{h(n)} + C_{h(n)}) \tag{13}$$

with $A \in \mathcal{A}_{AP,\,h}(\mathbb{Z})$, K, $L \in L(l^2(\mathbb{Z}^+))$ compact and $\|C_{h(n)}\| \to 0$ as $n \to \infty$, and each sequence in $\mathcal{S}_{AP,\,h}(\mathbb{Z}^+)$ can be written in the form (13) *in a unique way.*

The Toeplitz analogue of Theorem 5.1 is well known (Theorem 1.53 in [8], for instance): the smallest closed subalgebra $\mathcal{S}_{\mathbb{C}}(\mathbb{Z}^+)$ of $\mathcal{F}_{\mathrm{id}}$ which contains all sequences $(P_nT(a)P_n)$ with a continuous function $a : \mathbb{T} \to \mathbb{C}$ consists exactly of all sequences of the form

$$(P_nT(a)P_n + P_nKP_n + W_nLW_n + C_n)$$

where a is continuous, K and L are compact, and (C_n) is a sequence tending to zero in the norm.

Proof of Theorem 5.1. First let A and B be arbitrary band-dominated operators and n a positive integer. Then

$$\begin{aligned} &P_nPAPP_n\; P_nPBPP_n \\ &\quad = P_nPAPBPP_n - P_nPAPQ_nPBPP_n \\ &\quad = P_nPABPP_n - P_nPAQBPP_n - P_nPAPQ_nPBPP_n. \end{aligned} \tag{14}$$

Since

$$PQ_nP = U_nPU_{-n}, \qquad PW_nPJ = PU_nQ, \qquad JPW_nP = QU_{-n}P \tag{15}$$

we obtain

$$\begin{aligned} P_nPAPQ_nPBPP_n &= W_nJ\; JPW_nP\; AU_nPU_{-n}B\; PW_nPJ\; JW_n \\ &= W_nJQU_{-n}P\; AU_nPU_{-n}B\; PU_nQJW_n. \end{aligned} \tag{16}$$

Further we conclude from

$$W_nJQQ_{-n}Q = 0 \qquad \text{and} \qquad QU_nQJW_n = 0$$

and from (16) that

$$P_nPAPQ_nPBPP_n = W_nJQU_{-n}\; AU_nPU_{-n}B\; U_nQJW_n.$$

Together with (14) this gives

$$\begin{aligned} & P_nPAPP_n\, P_nPBPP_n \\ & \quad = P_nPABPP_n - P_nPAQBPP_n - W_nJQU_{-n}\, AU_nPU_{-n}B\, U_nQJW_n \\ & \quad = P_nPABPP_n + P_nKP_n - W_nJQ\, U_{-n}AU_nPU_{-n}BU_n\, QJW_n \end{aligned} \tag{17}$$

with a compact operator $K = -PAQBP$. Now let especially $A, B \in \mathcal{A}_{AP,h}(\mathbb{Z})$ and replace n in (17) by $h(n)$. Since

$$\|U_{-h(n)}AU_{h(n)}PU_{-h(n)}BU_{h(n)} - APB\| \to 0,$$

we obtain from (17) the identity

$$\begin{aligned} & P_{h(n)}PAPP_{h(n)}\, P_{h(n)}PBPP_{h(n)} \\ & \quad = P_{h(n)}PABPP_{h(n)} + P_{h(n)}KP_{h(n)} + W_{h(n)}LW_{h(n)} + C_{h(n)} \end{aligned}$$

with compact operators K and $L := -JQAPBQJ$ and with

$$\|C_{h(n)}\| = \|W_{h(n)}JQ(U_{-h(n)}AU_{h(n)}PU_{-h(n)}BU_{h(n)} - APB)QJW_{h(n)}\| \to 0.$$

Thus, the (non-closed) dense subalgebra of $\mathcal{S}_{AP,h}(\mathbb{Z}^+)$ which is generated by all sequences of the form $(P_{h(n)}PAPP_{h(n)})$ with $A \in \mathcal{A}_{AP,h}(\mathbb{Z})$ is contained in the set $\mathcal{S}'$ of all sequences of the form (13). The inclusion $\mathcal{S}_{AP,h}(\mathbb{Z}^+) \subseteq \mathcal{S}'$ will follow once we have shown that $\mathcal{S}'$ is closed.

For this goal, notice that for each sequence $\mathbf{A} = (A_n) \in \mathcal{S}'$ with

$$A_n := P_{h(n)}PAPP_{h(n)} + P_{h(n)}KP_{h(n)} + W_{h(n)}LW_{h(n)} + C_{h(n)}$$

the sequences $(A_nP_{h(n)})$ and $(W_{h(n)}A_nW_{h(n)})$ converge *-strongly to $W(\mathbf{A}) := PAP + K$ and $\widetilde{W}(\mathbf{A}) := JQAQJ + L = PJAJP + L$, respectively. The first of these assertions is evident. The second one follows since, by (15),

$$W_{h(n)}PAPW_{h(n)} = JJW_{h(n)}PAPW_{h(n)}JJ = JQU_{-h(n)}PAPU_{h(n)}QJ \to JQAQJ$$

*-strongly. By the Banach-Steinhaus theorem, the linear mappings W and $\widetilde{W}$ are continuous. Thus, if $(\mathbf{A}_k)$ is a Cauchy sequence in $\mathcal{S}'$, then $(W(\mathbf{A}_k)) = (PA_kP + K_k)$ is a Cauchy sequence in $\mathcal{A}_{AP,h}(\mathbb{Z}^+)$. As in the proof of Lemma 3.5 one concludes that this sequence converges to an operator $PAP + K$ with $A \in \mathcal{A}_{AP,h}(\mathbb{Z})$ and with a compact operator K. Further, $(\widetilde{W}(\mathbf{A}_k)) = (PJA_kJP + L_k)$ is a Cauchy sequence, too. Since $\|PJA_kJP - PJAJP\| \to 0$ as we have just seen, (L_k) is a Cauchy sequence which converges to a compact operator L. Moreover, standard arguments show that the set of all sequences in $\mathcal{F}_h$ which tend to zero in the norm is closed in $\mathcal{F}_h$. This finally shows that the sequence $(\mathbf{A}_k)$ converges in the norm of $\mathcal{F}_h$ to a sequence of the form

$$\mathbf{A} := (P_{h(n)}PAPP_{h(n)} + P_{h(n)}KP_{h(n)} + W_{h(n)}LW_{h(n)} + C_{h(n)})$$

with $\|C_{h(n)}\| \to 0$ which clearly belongs to $\mathcal{S}'$. Thus, $\mathcal{S}'$ is closed.

For the reverse implication $\mathcal{S}' \subseteq \mathcal{S}_{AP,h}(\mathbb{Z}^+)$ we have to show that

$$(P_{h(n)}KP_{h(n)} + W_{h(n)}LW_{h(n)} + C_{h(n)}) \in \mathcal{S}_{AP,h}(\mathbb{Z}^+)$$

for arbitrary compact operators K and L and arbitrary zero sequences $(C_{h(n)})$. But this is clear since all finite sections sequences for Toeplitz operators with continuous and complex-valued generating function belong to $\mathcal{S}_{AP,h}(\mathbb{Z}^+)$, hence, $\mathcal{S}_{\mathbb{C}}(\mathbb{Z}^+) \subseteq \mathcal{S}_{AP,h}(\mathbb{Z}^+)$, and since all sequences of the form $(P_nKP_n+W_nLW_n+C_n)$ with compact operators K, L and with a zero sequence (C_n) belong to $\mathcal{S}_{\mathbb{C}}(\mathbb{Z}^+)$ as mentioned above. ∎

In the preceding proof, we have defined linear mappings W and $\widetilde{W}$ on $\mathcal{S}'$. Due to the coincidence of $\mathcal{S}'$ with $\mathcal{S}_{AP,h}(\mathbb{Z}^+)$ these mappings are defined on the algebra $\mathcal{S}_{AP,h}(\mathbb{Z}^+)$, and it is easy to see that they act as *-homomorphisms from this algebra into $\mathcal{A}_{AP,h}(\mathbb{Z}^+)$.

As in proof of Theorem 1.54 in [8], a twice application of the perturbation theorem gives the following stability result for sequences in $\mathcal{S}_{AP,h}(\mathbb{Z}^+)$.

Theorem 5.2 *A sequence $\mathbf{A} = (A_n) \in \mathcal{S}_{AP,h}(\mathbb{Z}^+)$ is stable if and only if the two operators $W(\mathbf{A})$ and $\widetilde{W}(\mathbf{A})$ are invertible.*

Corollary 5.3 *The algebra $\mathcal{S}_{AP,h}(\mathbb{Z}^+)/\mathcal{G}$ is *-isomorphic to the C^*-subalgebra of $L(l^2(\mathbb{Z}^+)) \times L(l^2(\mathbb{Z}^+))$ which consists of all pairs $(W(\mathbf{A}), \widetilde{W}(\mathbf{A}))$ with $\mathbf{A} \in \mathcal{S}_{AP,h}(\mathbb{Z}^+)$.*

Indeed, since $W(\mathcal{G}) = 0$ for each sequence $\mathbf{G} \in \mathcal{G}$, the mapping

$$\mathcal{S}_{AP,h}(\mathbb{Z}^+)/\mathcal{G} \to L(l^2(\mathbb{Z}^+)) \times L(l^2(\mathbb{Z}^+)), \quad \mathbf{A} + \mathcal{G} \mapsto (W(\mathbf{A}), \widetilde{W}(\mathbf{A}))$$

is correctly defined. It turns out that this mapping is a *-homomorphism which, by Theorem 5.2, preserves spectra. Elementary C^*-arguments show that then this mapping is an isomorphism.

6 Spectral approximation

Another corollary to Theorem 5.2 states that the algebra $\mathcal{S}_{AP,h}(\mathbb{Z}^+)$ is fractal in the following sense. Let again $\mathcal{F}$ stand for the algebra of all matrix sequences with dimension function δ. For each strongly monotonically increasing sequence $\eta : \mathbb{Z}^+ \to \mathbb{Z}^+$, let $\mathcal{F}_\eta$ refer to the algebra of all matrix sequences with dimension function $\delta \circ \eta$. There is a natural *-homomorphism $R_\eta : \mathcal{F} \to \mathcal{F}_\eta$ given by

$$R_\eta : (A_n) \mapsto (A_{\eta(n)});$$

thus, $A_{\eta(n)}$ is a $\delta(\eta(n)) \times \delta(\eta(n))$-matrix.

Definition 6.1 *A C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$ with $\mathcal{G} \subseteq \mathcal{A}$ is called* fractal *if, for every strongly monotonically increasing sequence $\eta : \mathbb{Z}^+ \to \mathbb{Z}^+$, there is a mapping $\pi_\eta : R_\eta \mathcal{A} \to \mathcal{F}/\mathcal{G}$ such that*

$$\pi_\eta(R_\eta \mathbf{A}) = \mathbf{A} + \mathcal{G} \qquad \textit{for each sequence } \mathbf{A} \in \mathcal{A}.$$

Thus, the coset $\mathbf{A} + \mathcal{G} \in \mathcal{A}/\mathcal{G}$ can be reconstructed from each subsequence of $\mathbf{A}$.

Theorem 6.2 *The subalgebra $\mathcal{S}_{AP,h}(\mathbb{Z}^+)$ of $\mathcal{F}$ is fractal.*

This follows immediately from Corollary 5.3 in combination with Theorem 1.69 in [8].

Fractal subalgebras of $\mathcal{F}$ are distinguished by the excellent convergence properties of their elements. For a general account on this topic, see the third chapter of [8]. Here we will mention only a few facts which arise immediately from Corollary 5.3 and the general results provided in [8].

For each element A on a unital C^*-algebra, let $\sigma(A)$ refer to the spectrum of A and $\sigma_{sing}(A)$ to the set of all square roots of the points in $\sigma(A^*A)$. Thus, for an $n \times n$-matrix A, $\sigma_{sing}(A)$ is just the set of the singular values of that matrix.

Corollary 6.3 *Let $\mathbf{A} := (A_n) \in \mathcal{S}_{AP,h}(\mathbb{Z}^+)$ be a self-adjoint sequence. Then the spectra $\sigma(A_n)$ converge in the Hausdorff metric to the spectrum of the coset $\mathbf{A} + \mathcal{G}$ in $\mathcal{S}_{AP,h}(\mathbb{Z}^+)/\mathcal{G}$ which, on its hand, coincides with $\sigma(W(\mathbf{A})) \cup \sigma(\widetilde{W}(\mathbf{A}))$.*

Corollary 6.4 *Let $\mathbf{A} := (A_n) \in \mathcal{S}_{AP,h}(\mathbb{Z}^+)$. Then the sets of the singular values $\sigma_{sing}(A_n)$ converge in the Hausdorff metric to $\sigma_{sing}(\mathbf{A} + \mathcal{G})$ in $\mathcal{S}_{AP,h}(\mathbb{Z}^+)/\mathcal{G}$ which is equal to $\sigma_{sing}(W(\mathbf{A})) \cup \sigma_{sing}(\widetilde{W}(\mathbf{A}))$.*

Let $\varepsilon > 0$. The *ε-pseudospectrum* $\sigma^{(\varepsilon)}(A)$ of an element A of a C^*-algebra with identity element I is the set of all $\lambda \in \mathbb{C}$ for which either $A - \lambda I$ is not invertible or $\|(A - \lambda I)^{-1}\| \geq 1/\varepsilon$.

Corollary 6.5 *Let $\varepsilon > 0$ and $\mathbf{A} := (A_n) \in \mathcal{S}_{AP,h}(\mathbb{Z}^+)$. Then the ε-pseudospectra $\sigma^{(\varepsilon)}(A_n)$ converge in the Hausdorff metric to $\sigma^{(\varepsilon)}(\mathbf{A} + \mathcal{G})$ in $\mathcal{S}_{AP,h}(\mathbb{Z}^+)/\mathcal{G}$ which coincides with $\sigma^{(\varepsilon)}(W(\mathbf{A})) \cup \sigma^{(\varepsilon)}(\widetilde{W}(\mathbf{A}))$.*

Another consequence of Corollary 5.3 is related with Fredholm sequences and the splitting phenomenon of their singular values. Given an $n \times n$-matrix A, let $0 \leq \sigma_1(A) \leq \sigma_2(A) \leq \ldots \leq \sigma_n(A) = \|A\|$ refer to the singular values of A, counted with respect to their multiplicity. A sequence $\mathbf{A} = (A_n) \in \mathcal{F}$ is a *Fredholm sequence* if there is a non-negative integer k such that

$$\liminf_{n \to \infty} \sigma_{k+1}(A_n) > 0,$$

and the smallest number k with this property is the *α-number* of $\mathbf{A}$. We denote it by $\alpha(\mathbf{A})$.

Corollary 6.6 *A sequence* $\mathbf{A} := (A_n) \in \mathcal{S}_{AP,h}(\mathbb{Z}^+)$ *is Fredholm if and only if its strong limit* $W(\mathbf{A})$ *is a Fredholm operator. In this case,* $\widetilde{W}(\mathbf{A})$ *is a Fredholm operator, too,*

$$\alpha(\mathbf{A}) = \dim\ker W(\mathbf{A}) + \dim\ker \widetilde{W}(\mathbf{A}), \tag{18}$$

and, moreover, $\lim_{n\to\infty} \sigma_{\alpha(\mathbf{A})}(A_n) = 0$.

The first part of the assertion holds for general band-dominated operators; see Theorem 5.7 (*b*) in [14]. The identity (18) and the final assertion follow from Theorem 6.12 in [8].

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