The Topology of Gauge Groups

- preprint -

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Abstract

In this paper we develop an effective way to access the topology of the gauge group for a smooth K-principal bundle $\mathcal{P} = (K, \pi, P, M)$ with possibly infinite-dimensional structure group K over a compact manifold with corners M. For this purpose we introduce the concept of a not necessarily finite-dimensional manifold with corners and show that $C^{\infty}(M, K)$ is a Lie group if M is a compact manifold with corners. This enables us in the second section to consider the gauge group $\mathrm{Gau}(\mathcal{P})$, with a natural topology on it, as an infinite-dimensional Lie group if M is compact and K is locally exponential. In the last section we discuss some applications. We show that the inclusion $\mathrm{Gau}(\mathcal{P}) \hookrightarrow \mathrm{Gau}_c(\mathcal{P})$ of smooth into continuous gauge transformations is a weak homotopy equivalence, apply this result to the calculation of $\pi_n(\mathrm{Gau}(\mathcal{P}))$.

Keywords: manifold with boundary; manifold with corners; infinite-dimensional manifold; infinite-dimensional Lie group; mapping group; gauge group; Kac-Moody group; topology of gauge groups; homotopy groups of gauge groups

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Introduction

Let $\mathcal{P} = (K, \pi, P, M)$ be a smooth K-principal bundle with possibly infinite-dimensional structure group K over a finite-dimensional manifold with corners M. Then the group of vertical bundle automorphisms, shortly called gauge group and denoted by $\mathrm{Gau}(\mathcal{P})$, can be identified with the space of smooth K-equivariant mappings $C^{\infty}(P, K)^K$, where K acts on itself by conjugation and we will identify $\mathrm{Gau}(\mathcal{P})$ with $C^{\infty}(P, K)^K$ throughout this paper. To avoid confusion we stress that the objects called gauge groups in the physical literature (the group K from the bundle $\mathcal{P}$) are called structure groups in our setting. If K is abelian or the bundle is trivial, then $C^{\infty}(P, K)^K$ is isomorphic to the mapping group $C^{\infty}(M, K)$, which carries a Lie group structure if M is compact [Glö02a]. In this paper we show that if M is compact and K is locally exponential (i.e. K has an exponential function restricting to a local diffeomorphism on some zero neighbourhood in $\mathfrak{k}$), then $C^{\infty}(P, K)^K$, equipped with a natural topology, can be turned into an infinite-dimensional Lie group. This applies in particular if K is finite-dimensional and M is compact. This statement (without the requirement on K being locally exponential) is frequently treated in the literature as a folklore statement, but to the author no rigorous proof is known. Unfortunately the proof given in [KM97, Theorem 42.21] has a serious gap since the topology constructed there is not the natural one, but a much finer topology which might even become discrete.

Gauge groups occur naturally as infinite-dimensional symmetry groups in so-called pure Yang-Mills theories. The objects of interest there are solutions of gauge-invariant equations of motion for connections on $\mathcal{P}$, hence elements of the moduli space $\mathrm{Conn}(\mathcal{P})/\mathrm{Gau}(\mathcal{P})$ (cf. [DV80]). Thus there is a natural interest in the analysis of the topology of these groups.

Another interesting fact on gauge groups is their relation to Kac-Moody groups. If $M = \mathbb{S}^1$, then $C^\infty(P,K)^K$ is isomorphic to the twisted loop group

$$C^\infty_\tau(R,K) = \{f \in C^\infty(\mathbb{R},K) : f(x+n) = \tau^n(f(x))\},$$

where $\tau : K \to K$ denotes a fixed automorphism of K. If τ is of finite order then these groups are special examples of Kac-Moody groups, which have been intensively studied in the mathematical and physical literature (cf. [PS86] and [Mic87]).

We now describe our results in more detail. The key to the Lie group structure on $C^\infty(P,K)^K$ is to combine the local triviality $\mathcal{P}$ with existing results on mapping groups (cf. [PS86, Section 3.2], [Nee01, Theorem II.1] and [Glö02a, Section 3.2]). For this approach we are forced to deal with a compact subset $\overline{V}$ of M, for which there exists a smooth section $\sigma : \overline{V} \to P$, as a manifold and we introduce the notion of a manifold with corners for this purpose. This notion uses differentiablitily on non-open domains $V \subseteq E$ with dense interior of a locally convex space E as in [Mic80], hence a map $f : V \to F$ is defined to be continuously differentiable if it is so on $\mathrm{int}(V)$ and the differential $\mathrm{int}(V) \times E \ni (x,v) \mapsto df(x).v \in F$ extends continuously to $V \times E$. This definition the appropriate one for a treatment of mapping spaces (cf. [Woc05]). However, the Whitney extension theorem [Whi34] and [KM97, Theorem 24.5] imply that our definition of smooth maps coincides with the usual one used e.g. in [Lee03] or [Lan99].

If M and N are manifolds with corners we define smooth maps, tangent bundles (which also turn out to be manifolds with corners) and to each smooth map $f : M \to N$ a tangent map $Tf : TM \to TN$. Since on this elementary level everything behaves as in the case of smooth manifolds, we can easily transfer the corresponding result on mapping groups $C^\infty(M,K)$ from the smooth case to the case where M is a compact manifold with corners.

If $\mathcal{P} = (K,\pi,P,M)$ is a smooth K-principal bundle and $U \subseteq M$ is open with a smooth section $\sigma : U \to P$, then the restriction of $f \in C^\infty(P,K)^K$ to $\pi^{-1}(U)$ corresponds to an element of $C^\infty(U,K)$. This correspondence is the key to the Lie group structure on $C^\infty(P,K)^K$. If M is compact, we can identify $C^\infty(P,K)^K$ with a closed subgroup of the Lie group $\prod_{i=1}^n C^\infty(\overline{V}_i,K)$, where $(\overline{V_i})_{i=1,\ldots,n}$ is a cover of M by appropriate compact manifolds with corners. Since closed subgroups of infinite-dimensional Lie groups may not be Lie groups [Bou89b, Exercise III.8.2], we are forced to incorporate the assumption that K is locally exponential to derive charts for the Lie group structure on $C^\infty(P,K)^K$.

Theorem (Lie group structure on Gau($\mathcal{P}$)). *If $\mathcal{P} = (K,\pi,P,M)$ is a smooth K-principal bundle with compact base M (possibly with corners) and locally exponential structure group K, then $\mathrm{Gau}(\mathcal{P}) \cong C^\infty(P,K)^K$ carries the structure of a smooth locally exponential Lie group.*

In the case where K is locally exponential and Fréchet, this theorem can be taken as a substitute for [KM97, Theorem 42.21], since for Fréchet-Lie groups the notion of differentiability used here and in [KM97] coincide. It also turns out that the topology on $C^\infty(P,K)^K$ obtained in this way coincides with the natural subgroup topology induced from the topological group $C^\infty(P,K)$ and that for different choices of the cover $(\overline{V}_i)_{i=1,\ldots,n}$ we obtain the same Lie group structure on $C^\infty(P,K)^K$.

In the last section we analyse the topology on $C^\infty(P,K)^K$ in more detail. First we establish an approximation result which allows us to access the topology on $C^\infty(P,K)^K$ from that on $C(P,K)^K$. This technical part of the paper was inspired by [Hir76, Section 2.2] and [Nee02, Section A.3]. Since our definition of the topology on spaces of smooth mappings differs from the one given in [Hir76], we derive the corresponding statements explicitly. Eventually we get the same results for $C^{\infty}(P,K)^{K}$ as in [Nee02] for $C^{\infty}(M,K)$ provided that K is locally exponential.

Theorem (Weak homotopy equivalence for Gau($\mathcal{P}$)). *If $\mathcal{P}=(K,\pi,P,M)$ is a smooth K-principal bundle with compact base M (possibly with corners) and locally exponential structure group K, then the natural inclusion $C^{\infty}(P,K)^{K}\hookrightarrow C(P,K)^{K}$ of smooth into continuous gauge transformations is a weak homotopy equivalence, i.e. the induced mappings $\pi_n(C^{\infty}(P,K)^{K})\to\pi_n\left(C(P,K)^{K}\right)$ are isomorphisms of groups for $k\in\mathbb{N}_0$.*

Using this result we turn to the calculation of homotopy groups of $C^{\infty}(P,K)^{K}$. Initially we treat a purely topological situation. First we consider the case where the base space M is a compact surface, possibly with boundary. If $\partial M\neq\emptyset$, then $\mathcal{P}$ turns out to be trivial and we thus get an explicit description of $\pi_n(C(P,K)^{K})$ and of $\pi_n(C^{\infty}(P,K)^{K})$ in terms of $\pi_{n+1}(K)$ and $\pi_n(K)$ similar to the non-boundary case for mapping groups (cf. [GN05]). In the case where $\partial M=\emptyset$ we use an explicit version of the classification of continuous K-principal bundles with connected structure group over compact surfaces to show that $C(P,K)^{K}$ can be described in terms of the genus g of M and a based loop $\gamma\in C_*(\mathbb{S}^1,K)$. We may thus identify $C(P,K)^{K}$ with a closed subgroup $G_{g,\gamma}$ of $C(B,K)$, where B is the closed unit disk. We next derive an explicit description of $\pi_n((G_{g,\gamma})_*)$ in terms of $\pi_{n+2}(K)$ and $\pi_{n+1}(K)$, where $(G_{g,\gamma})_*$ denotes the subgroup of pointed maps in $G_{g,\gamma}$. To this end the way is just as in the case of mapping groups, but while $C(M,K)$ is isomorphic to $C_*(M,K)\rtimes K$, a similar statement for $C(P,K)^{K}$ seems not to be true. But if K is locally contractible, then the evaluation map $\mathrm{ev}:G_{g,\gamma}\to K$ having $(G_{g,\gamma})_*$ as kernel admits locally continuous sections and we thus obtain a long exact homotopy sequence for $\pi_n(C(P,K)^{K})$ which decays into short exact sequences.

Similar considerations apply also to the case of bundles over spheres, i.e. the case $M=\mathbb{S}^r$. Due to this explicit formulae and the weak homotopy equivalence for Gau($\mathcal{P}$) we obtain the following theorem.

Theorem (Homotopy groups of Gau($\mathcal{P}$)). *Let $\mathcal{P}=(K,\pi,P,M)$ be a smooth K-principal bundle, M be a compact orientable surface of genus $g\geq 1$ and K be locally exponential and connected. If M has empty boundary, then we have for each $n\in\mathbb{N}_0$ exact sequences*

$$0\to\pi_{n+2}(K)\oplus\pi_{n+1}(K)^{2g}\to\pi_n(\mathrm{Gau}(\mathcal{P}))\to\pi_n(K)\to 0.$$

If ∂M in non-empty and has m components, then we have for each $n\in\mathbb{N}_0$ isomorphisms

$$\pi_n(\mathrm{Gau}(\mathcal{P}))\cong\pi_{n+1}(K)^{2g+m-1}\oplus\pi_n(K).$$

If $\mathcal{P}=(K,\pi,P,\mathbb{S}^r)$ is a smooth K-principal bundle over the r-sphere and K is locally exponential, then we have for each $n\in\mathbb{N}_0$ exact sequences

$$0\to\pi_{n+r}(K)\to\pi_n(\mathrm{Gau}(\mathcal{P}))\to\pi_n(K)\to 0.$$

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I Smooth Mappings and Manifolds

I.1 Notions of Differential Calculus

In this section we present the elementary notions of differential calculus on locally convex spaces and for not necessarily open domains. The notion for open subsets of locally convex spaces has been worked on and with during the last two decades and is due to [Ham82] and [Mil83]. The notion for sets with dense interior used here is due to [Mic80]. Many proofs for manifolds with corners (cf. Definition I.6) carry over from the case of smooth manifolds and are frequently omitted in this text.

Definition I.1. Let E and F be a locally convex spaces and $U \subseteq E$ be open. We say that $f : U \to F$ is *continuously differentiable* or *of class* C^1 if it is of class C^0 (i.e. continuous), for each $v \in E$ the differential quotient

$$df(x).v := \lim_{h \to 0} \frac{f(x+hv) - f(x)}{h}$$

exists and if the map $df : U \times E \to F$ is continuous. If $n > 1$ we inductively define f to be *of class* C^n if it is of class C^1 and df is of class C^{n-1}, saying that the map $d^n f$ inductively defined by $d^n f := d^{n-1}(df)$ is continuous. We say that f is of class C^∞ or *smooth* if it is of class C^n for all $n \in \mathbb{N}_0$. We denote the set of maps from U to F of class C^1, C^n and C^∞ respectively by $C^1(U, E)$, $C^n(U, E)$ and $C^\infty(U, E)$. This is the notion of differentiability used in [Ham82], [Mil83] and [Glö02b] and it will be the notion throughout this paper.

Remark I.2. (cf. [Nee02, Remark 3.2]) We briefly recall the basic definitions underlying the convenient calculus from [KM97]. Again let E and F be locally convex spaces. A curve $f : \mathbb{R} \to E$ is called smooth if it is smooth in the sense of Definition I.1. Then the c^∞-topology on E is the final topology induced from all smooth curves $f \in C^\infty(\mathbb{R}, E)$. If E is a Fréchet space, then the c^∞-topology is again a locally convex vector topology which coincides with the original topology [KM97, Theorem 4.11]. If $U \subseteq E$ is c^∞-open then $f : U \to F$ is said to be of class C^∞ or smooth if

$$f_* \left(C^\infty(\mathbb{R}, U)\right) \subseteq C^\infty(\mathbb{R}, F),$$

e.g. if f maps smooth curves to smooth curves. The chain rule [Glö02a, Proposition 1.15] implies that each smooth map in the sense of Definition I.1 is smooth in the convenient sense. On the other hand [KM97, Theorem 12.8] implies that on a Frèchet space a smooth map in the convenient sense is smooth in the sense of Definition I.1. Hence for Fréchet spaces the two notions coincide.

Definition I.3. Let E and F be a locally convex space, and let $U \subseteq E$ be a set with dense interior. We say that a map $f : U \to F$ is *continuously differentiable* or *of class* C^1 if it is of class C^0 (i.e. continuous), $f_{\text{int}} := f|_{\text{int}(U)}$ is of class C^1 (in the sense of Definition I.1) and the map

$$d\left(f_{\text{int}}\right) : \text{int}(U) \times E \to F, \ \ (x, v) \mapsto d\left(f_{\text{int}}\right)(x).v$$

extends to a continuous map on $U \times E$, which is called the *differential* df of f. If $n > 1$ we inductively define f to be *of class* C^n if if is of class C^1 and df is of class C^{n-1} for $n > 1$, saying that the maps inductively defined by $d^n f := d^{n-1}(df)$ are continuous. We say that f is of class C^∞ or *smooth* if f is of class C^n for all $n \in \mathbb{N}_0$.

Remark I.4. Since $\text{int}(U \times E^{2n-1}) = \text{int}(U) \times E^{2n-1}$ we have for $n = 1$ that $(df)_{\text{int}} = d\left(f_{\text{int}}\right)$ and we inductively obtain $(d^n f)_{\text{int}} = d^n\left(f_{\text{int}}\right)$. Hence the higher differentials $d^n f$ are defined to be the continuous extensions of the differentials $d^n(f_{\text{int}})$ and thus we have that a map $f : U \to F$ is smooth if and only if

$$d^n\left(f_{\text{int}}\right) : \text{int}(U) \times E^{2^n - 1} \to F$$

has a continuous extension $d^n f$ to $U \times E^{2^n-1}$ for all $n \in \mathbb{N}$.

Lemma I.5. *If E, E' and F are locally convex spaces, $U \subseteq E$, $U' \subseteq E'$ have dense interior $f : U \to U'$, $g : U' \to F$ with $f(\text{int}(U)) \subseteq \text{int}(U')$ are of class C^1, then $g \circ f : U \to F$ is of class C^1 and its differential is given by*

$$d(g \circ f)(x).v = dg(f(x)).df(x, v).$$

In particular it follows that $g \circ f$ is smooth if g and f are so.

Proof. This follows easily from the chain rule for locally convex spaces [Glö02a, Proposition 1.15] and $(g\circ f)_{\mathrm{int}} = g\circ f_{\mathrm{int}} = g_{\mathrm{int}}\circ f_{\mathrm{int}}$, where the last equality holds due to $f(\mathrm{int}(U))\subseteq \mathrm{int}(U')$. □

Definition I.6. Let E be a locally convex space, $\lambda_1,\ldots,\lambda_n$ be continuous functionals and denote $E^+ := \bigcap_{i=1}^n \lambda_i^{-1}(\mathbb{R}_0^+)$. If M is a Hausdorff space, then a collection $(U_i,\varphi_i)_{i\in I}$ of homeomorphisms $\varphi_i : U_i \to \varphi(U_i)$ onto open subsets $\varphi_i(U_i)$ of E^+ is called a *differential structure* on M (cf. [Lee03] for the finite-dimensional case) of co-dimension n if

i) for each $m\in M$ there exists an φ_i with $m\in U_i$, called a chart around m,

ii) for each pair of charts $\varphi_i : U_i\to E^+$ and $\varphi_j : U_j \to E^+$ with $U_i\cap U_j\neq\emptyset$ we have that the *coordinate change*

$$\varphi_i\left(U_i\cap U_j\right)\ni x\mapsto \varphi_j\left(\varphi_i^{-1}(x)\right)\in\varphi_j(U_i\cap U_j)$$

is smooth in the sense of Definition I.3.

Two differentiable structures $(U_i,\varphi_i)_{i\in I}$ and $(U_i,\varphi_i)_{i\in I'}$ are called *compatible* if their union $(U_i,\varphi_i)_{i\in I\cup I'}$ is again a differential structure and a maximal differential structure with respect to compatibility is called an *atlas*. Furthermore, M together with an atlas $(U_i,\varphi_i)_{i\in I}$ is called a *manifold with corners* of co-dimension n.

Remark I.7. Note that the previous definition of a manifold with corners coincides for $E=\mathbb{R}^n$ with the one given in [Lee03] and in the case of co-dimension 1 and a Banach space E with the definition of a manifold with boundary in [Lan99], but our notion of smoothness differs. In both cases a map f defined on a non-open subset $U\subseteq E$ is said to be smooth if for each point $x\in U$ there exist an open neighbourhood $V_x\subseteq E$ of x and a smooth map f_x defined on V_x with $f=f_x$ on $U\cap V_x$.

The notion of smoothness used here is due to [Mic80, Definition 2.1] and is the appropriate one for a treatment of mapping spaces. In the finite-dimensional case, the Whitney extension theorem [Whi34] yields that our notion of smoothness coincides with the one used e.g. in [Lee03] and in the case of Banach spaces and co-dimension 1, [KM97, Theorem 24.5] also implies that the notions coincides with the one from [Lan99].

Lemma I.8. *If M is a manifold with corners of modelled on the locally convex space E and $\varphi : U\to E^+$ and $\psi : V\to E^+$ are two charts with $U\cap V\neq\emptyset$, then*

$$\psi\circ\varphi^{-1}(\mathrm{int}(\varphi(U\cap V)))\subseteq \mathrm{int}(\psi(U\cap V)).$$

Proof. Denote by $\alpha : \varphi(U\cap V)\to\psi(U\cap V)$ and $\beta : \psi(U\cap V)\to\varphi(U\cup V)$ the corresponding coordinate changes. We claim that $d\alpha(x) : E\to E$ is an isomorphism if $x\in\mathrm{int}(\varphi(U\cap V))$. Since β maps a neighbourhood W_x of $\alpha(x)$ into $\mathrm{int}(\varphi(U\cap V))$ we have $d\alpha(\beta(x')).\big(d\beta(x').v\big) = v$ for $v\in E$ and $x'\in\mathrm{int}(W_x)$ (cf. Lemma I.5). Since $x',v\mapsto d\alpha(\beta(x')).\big(d\beta(x').v\big)$ is continuous and $\mathrm{int}(W_x)$ is dense in W_x, we thus have that $v\mapsto d\beta(\alpha(x)).v$ is a continuous inverse for $d\alpha(x)$.

Now suppose $x\in\mathrm{int}(\varphi(U\cap V))$ and $\alpha(x)\notin\mathrm{int}(\psi(U\cap V))$. Then $\lambda_i(\alpha(x))=0$ for some $i\in\{1,\ldots,n\}$ and thus there exists an $v\in E$ such that $\alpha(x)+tv\in\psi(U\cap V)$ for $t\in[0,1]$ and $\alpha(x)+tv\notin\psi(U\cap V)$ for $t\in[-1,0)$. But then $v\notin\mathrm{im}(d\alpha(x))$, contradicting the surjectivity of $d\alpha(x)$. □

Definition I.9. If M is a manifold with corners, then

$$\mathrm{int}(M) := \{m\in M : \varphi(m)\in\mathrm{int}(\varphi(U)) \text{ for each chart } \varphi \text{ around } m\}$$

is called the *interior of* M and $\partial M := M\backslash\mathrm{int}(M)$ is called the *boundary of* M.

Remark I.10. Note that Lemma I.8 implies that ∂M is the set of all points $m\in M$ which are mapped to $\partial E^+ = \bigcup_{i=1}^n \ker(\lambda_i)$ by an arbitrary chart $\varphi : U\to E^+$ around m.

Definition I.11. A map $f : M \to N$ between manifolds with corners is said to be *of class* C^n, respectively *smooth*, if $f\left(\operatorname{int}(M)\right) \subseteq \operatorname{int}(N)$ and the map

$$\varphi(U \cap f^{-1}(V)) \ni x \mapsto \psi\left(f\left(\varphi^{-1}(x)\right)\right) \in \psi(V)$$

is of class C^n respectively smooth for each pair $\varphi : U \to E^+$ and $\psi : V \to F^+$ of charts on M and N in the sense of Definition I.3.

Remark I.12. For a map f to be smooth it suffices to check that

$$\varphi(U \cap f^{-1}(V)) \ni x \mapsto \varphi'(f(\varphi^{-1}(x))) \in \psi(V)$$

maps $\operatorname{int}(\varphi(U \cap f^{-1}(V)))$ into $\operatorname{int}\psi(V)$ and is smooth in the sense of Definition I.3. for each $m \in M$ and an arbitrary pair of charts $\varphi : U \to E^+$ and $\psi : V \to F_+$ around m and $f(m)$ due to Lemma I.5 and Lemma I.8.

Definition I.13. If M is a manifold with corners with differentiable structure $(U_i, \varphi_i)_{i \in I}$, which is modelled on the locally convex space E, then the *tangent space* $T_m M$ in $m \in M$ is defined to be

$$T_m := (E \times I_m) \,/ \sim\,,$$

where $I_m := \{i \in I : m \in U_i\}$ and

$$(x, i) \sim \left(d\left(\varphi_j \circ \varphi_i^{-1}\right)(\varphi_i(m)).x, j\right).$$

The set $TM := \cup_{m \in M}\{m\} \times T_m M$ is called the *tangent bundle* of M.

Remark I.14. Note that the tangent spaces $T_m M$ are isomorphic for all $m \in M$, including the boundary points.

Proposition I.15. *The tangent bundle TM is a manifold with corners and the map $\pi : TM \to M$, $(m, [x, i]) \mapsto m$ is smooth.*

Proof. Fix a differentiable structure $(U_i, \varphi_i)_{i \in I}$ on M. Then each U_i is a manifold with corners with respect to the differential structure (U_i, φ_i) on U_i. We endow each TU_i with the topology induced from the mappings

$$\begin{aligned} \operatorname{pr}_1 &: TU_i \to M, \quad (m, v) \mapsto m \\ \operatorname{pr}_2 &: TU_i \to E, \quad (m, v) \mapsto v, \end{aligned}$$

and endow TM with the topology making each map $TU_i \hookrightarrow TM$, $(m, v) \mapsto (x, [v, i])$ a topological embedding. Then $\varphi_i \circ \operatorname{pr}_1 \times \operatorname{pr}_2 : TU_i \to \varphi(U_i) \times E$ defines a differential structure on TM and from the very definition it follows immediately that π is smooth. □

Corollary I.16. *If M and N are manifolds with corners, then a map $f : M \to N$ is of class C^1 if $f(\operatorname{int}(M)) \subseteq \operatorname{int}(N)$, $f_{\text{int}} := f|_{\operatorname{int}(M)}$ is of class C^1 and $Tf_{\text{int}} : T(\operatorname{int}(M)) \to T(\operatorname{int}(N)) \subseteq TN$ extends continuously to TM. If, in addition, f is of class C^n for $n \geq 2$, then the map*

$$Tf : TM \to TN, \quad (m, [x, i]) \mapsto \left(f(m), \left[d\left(\varphi_j \circ f \circ \varphi_i^{-1}\right)(\varphi_i(m)).x, j\right]\right)$$

is well-defined and of class C^{n-1}.

Definition I.17. If M is a manifold with corners, then for $n \in \mathbb{N}_0$ the *higher tangent bundles* $T^n M$ are the inductively defined manifolds with corners $T^0 M := M$ and $T^n := T\left(T^{n-1}M\right)$. If N is a manifold with corners and $f : M \to N$ is of class C^n, then the *higher tangent maps* $T^m f : T^m M \to T^m N$ are the inductively defined maps $T^0 f := f$ and $T^m f := T(T^{m-1}f)$ if $1 < m \leq n$.

Corollary I.18. *If M, N and O are manifolds with corners and $f : M \to N$ and $g : N \to O$ with $f(\mathrm{int}(M)) \subseteq \mathrm{int}(N)$ and $g(\mathrm{int}(N)) \subseteq \mathrm{int}(O)$ are of class C^n, then $f \circ g : M \to O$ is of class C^n and we have $T^m(g \circ f) = T^m f \circ T^m g$ for all $m \leq n$.*

Proposition I.19. *If M is a finite-dimensional paracompact manifold with corners and $(U_i)_{i\in I}$ is an open cover of M, then there exists a smooth partition of unity $(f_i)_{i\in I}$ subordinated to this open cover.*

Proof. The construction in [Hir76, Theorem 2.1] actually yields smooth functions $f_i : U_i \to \mathbb{R}$ also in the sense of Definition I.11. □

II The Gauge Group

II.1 The Gauge Group as infinite-dimensional Lie Group

We now turn to the investigation of the smooth Lie group structure on $\mathrm{Gau}(\mathcal{P})$. Having fixed in Definition I.1 the notion of smoothness for locally convex spaces it is in particular clear what the notion of a smooth Lie group is in this context.

Proposition II.1. *Let G be a group with a smooth manifold structure on $U \subseteq G$ modelled on the locally convex space E. Furthermore assume that there exists $V \subseteq U$ open such that $e \in V$, $VV \subseteq U$, $V = V^{-1}$ and*

i) $V \times V \to U$, $(g, h) \mapsto gh$ is smooth,

ii) $V \to V$, $g \mapsto g^{-1}$ is smooth,

iii) for all $g \in G$ there exists an open unit neighbourhood $W \subseteq U$ such that $g^{-1}Wg \subseteq U$ and the map $W \to U$, $h \mapsto g^{-1}hg$ is smooth.

Then there exists a unique manifold structure on G such that V is an open submanifold of G which turns G into a Lie group.

Remark II.2. If M is a (not necessarily finite-dimensional) manifold with corners and E is a locally convex space, each $f \in C^\infty(M, E)$ defines a continuous map $T^n f : T^n M \to T^n E$ (cf. Corollary I.16). Since $T^n E \cong E^{2^n}$ is a trivial bundle, all relevant information about $T^n f$ is already contained in its last component $d^n f := \mathrm{pr}_{2^n} \circ T^n f$. Hence we endow $C^\infty(M, E)$ with the topology for which the canonical map

$$C^\infty(M, E) \hookrightarrow \prod_{n\in\mathbb{N}_0} C(T^n M, E)_c \ \ f \mapsto (d^n f)_{n\in\mathbb{N}_0}$$

is a topological embedding (cf. [Glö02b, Definition 3.1]). Since each $C(T^n M, E)$ is a topological vector space so is $C^\infty(M, E)$ and if E is, moreover, locally convex, so is $C^\infty(M, E)$. If $\mathfrak{k}$ is a locally convex topological Lie algebra, then an easy compactness argument shows that $C^\infty(M, \mathfrak{k})$ also is a locally convex topological Lie algebra with respect to the pointwise Lie bracket (cf. [GN05]).

Proof. The proof of [Bou89b, Proposition III.1.9.18] carries over without changes. □

Proposition II.3. *If M is a compact manifold with corners and K is a smooth Lie group modelled on the locally convex space $\mathfrak{k} = L(G)$, then $C^\infty(M, K)$ is a Lie group modelled on the locally convex space $C^\infty(M, \mathfrak{k})$ w.r.t. pointwise multiplication and the topology induced from the chart*

$$\varphi_* : C^\infty(M, W) \to C^\infty(M, \mathfrak{k}), \ \ \gamma \mapsto \varphi \circ \gamma,$$

where $\varphi : W \to \varphi(W) \subseteq \mathfrak{k}$ is a chart of K around e with $\varphi(e) = 0$.

Proof. The proof of the smooth case in [Glö02b, Section 3.2] carries over. The results on the mapping spaces $C^\infty(U,E)$ used in the proof of [Glö02b, Section 3.2] only depend on the existence of tangent maps and their properties as continuous maps and carry over to the case of a manifold with corners in exactly the same way. A more detailed description of the proof can be found in [Woc05]. $\square$

Remark II.4. Let $\mathcal{P}=(K,\pi,P,M)$ be a continuous K-principal bundle [Ste51, Section I.8]. Then $\mathcal{P}$ can be described by an open cover $(U_i)_{i\in I}$ and continuous maps $k_{ij}:U_i\cap U_j\to K$ satisfying $k_{ij}(x)\cdot k_{jl}(x)=k_{il}(x)$ for $x\in U_i\cap U_j\cap U_l$. Then the quotient

$$P=\bigcup_{i\in I}\{i\}\times U_i\times K/\sim$$

with $(i,x,k)\sim(j,x',k')\Leftrightarrow x=x'$ and $k_{ij}(x)\cdot k=k'$ is homeomorphic to P (cf. [Ste51, Section I.3]). Then the bundle projection is given by $\pi([(i,x,k)])=x$ and K acts on P by $([i,x,k])\cdot k'=([i,x,k\cdot k'])$. Then $\sigma_i:U_i\to P,\ x\mapsto[(i,x,e)]$ is a continuous section and

$$\Omega_i:\pi^{-1}(U_i)=\{[(i,x,k)]\in P:k\in U_i\}\to U_i\times K\ \ [(i,x,k)]\mapsto(x,k)$$

is a local trivialisation.

Definition II.5. We say that a continuous K-principal bundle $\mathcal{P}=(K,\pi,P,M)$ is a *smooth K-principal bundle* if K is a Lie group, M is a manifold with corners and each k_{ij} is smooth (in in the sense of Definition I.3). We then introduce a *differential structure on P* by defining each Ω_i from Remark II.4 to be a diffeomorphism, i.e. if N is a manifold with corners, a map $f:P\to N$ is smooth if $f\circ\Omega_i:U_i\times K\to N$ is smooth for each $i\in I$ and a map $g:N\to P$ is smooth if $\Omega_i\circ g$ is smooth for each $i\in I$. In addition, we denote by

$$\mathrm{Gau}(\mathcal{P}):=\{f\in\mathrm{Diff}(P):(\forall p\in P)(\forall k\in K)f(p\cdot k)=f(p)\cdot k\text{ and }\pi\circ f=\pi\}$$

the group of vertical bundle automorphisms or shortly the *gauge group* of $\mathcal{P}$.

Definition II.6. If $\mathcal{P}=(K,\pi,P,M)$ is a continuous (respectively smooth) K-principal bundle, then a subset $V\subseteq M$ is said to be *trivial*, if there exists $U\subseteq M$ open with $V\subseteq U$ and a continuous (respectively smooth) map $\sigma:U\to P$ satisfying $\pi\circ\sigma=\mathrm{id}_U$.

Remark II.7. Note that in our definition of a smooth principal bundle we permit the base M to be a *manifold with corners*. If we denote by

$$C^\infty(P,K)^K:=\{\gamma\in C^\infty(P,K):(\forall p\in P)(\forall k\in K)\gamma(p\cdot k)=k^{-1}\gamma(p)\cdot k\}$$

the group of K-equivariant smooth maps from P to K, then the map

$$C^\infty(P,K)^K\ni f\mapsto\big(p\mapsto p\cdot f(p)\big)\in\mathrm{Gau}(\mathcal{P})$$

is an isomorphism and we will throughout this paper identify $\mathrm{Gau}(\mathcal{P})$ with $C^\infty(P,K)^K$ via this map. The corresponding algebraic counterpart is the gauge algebra

$$\mathfrak{gau}(\mathcal{P}):=C^\infty(P,\mathfrak{k})^K:=\{\xi\in C^\infty(P,\mathfrak{k}):(\forall p\in P)(\forall k\in K)\xi(p\cdot k)=\mathrm{Ad}(k^{-1}).\xi(p)\}.$$

Since for each $p\in P$ the evaluation map is continuous, it follows that $C^\infty(P,K)^K$ is a closed subgroup of the topological group $C^\infty(P,K)$ and that $\mathfrak{gau}(\mathcal{P})$ is a closed subspace of $C^\infty(P,\mathfrak{k})$.

Definition II.8. a) If M is a manifold with corners, K is a Lie group and $f\in C^\infty(M,K)$, then the *left logarithmic derivative* $\delta^l(f)$ of f is the $\mathfrak{k}$-valued 1-form $\delta^l(f).X=d\lambda_{f^{-1}}.df.X$.

b) If K is a Lie group, E is a locally convex space and $\tau:K\times E\to E$ is a smooth representation, then $\dot\tau:\mathfrak{k}\times E\to E$, $(x,y)\mapsto d\tau(e,y)(x,0)$ is called the *derived representation.* In the special case of the adjoint representation, we get $\dot\tau(x,y)=\mathrm{ad}(x,y)=[x,y]$.

Lemma II.9. *Let M be a manifold with corners, K be a Lie group and $\tau : K \times E \to E$ be a smooth representation on the locally convex space E. If $k : M \to K$ and $f : M \to E$ are smooth, then we have*

$$d\left(\tau(k^{-1}).f\right).X = \tau(k^{-1}).df.X - \dot{\tau}\left(\delta^l(k).X\right).\left(\tau(k^{-1}).f\right)$$

with $\tau(k^{-1}).f : M \to E$, $m \mapsto \tau\left(k(m)^{-1}\right).f(m)$. If $\tau = \mathrm{Ad}$ is the adjoint representation of K on $\mathfrak{k}$, then we have

$$d\left(\mathrm{Ad}(k^{-1}).f\right).X = \mathrm{Ad}(k^{-1}).df.X - \left[\delta^l(k).X, \mathrm{Ad}(k^{-1}).f\right]$$

Proof. (cf. [Nee04, Lemma II.7]) We write $\tau(k^{-1}, f)$ instead of $\tau(k^{-1}).f$, interpret it as a function of two variables and calculate

$$\begin{aligned} d\left(\tau(k^{-1}, f)\right)(X, X) &= d\left(\tau(k^{-1}, f)\right)\left((0, X) + (X, 0)\right) = d_2\left(\tau(k^{-1}, f)\right).X + d_1\left(\tau(k^{-1}).f\right).X \\ &= \tau(k^{-1}, df.X) + d\tau(\cdot, f).T(\iota \circ k).X = \tau(k^{-1}).(df.X) + d\tau(\cdot, f).T(\iota \circ \lambda_k \circ \lambda_{k^{-1}} \circ k).X \\ &= \tau(k^{-1}).(df.X) - d\left(\tau(\cdot, f) \circ \rho_{k^{-1}}\right).\delta^l(k)(X) = \tau(k^{-1}).(df.X) - \dot{\tau}\left(\delta^l(k)(X), \tau(k^{-1}, f)\right), \end{aligned}$$

where $d_1\tau$ (respectively d_2) denotes the differential of τ with respect to the first (respectively second) variable, keeping constant the second (respectively first) variable. □

Lemma II.10. *If $\mathcal{P} = (K, \pi, P, M)$ is a smooth K-principal bundle with finite-dimensional base M, then there exists an open cover $(V_i)_{i\in I}$ such that each $\overline{V_i}$ is trivial and a manifold with corners. If, moreover, M is compact then we may assume the cover to be finite.*

Proof. For each $m \in M$ there exists an open neighbourhood U and a chart $\varphi : U \to (\mathbb{R}^n)^+$ such that U is trivial. Then there exists an $\varepsilon > 0$ such that $\varphi(U) \cap (\varphi(m) + [-\varepsilon, \varepsilon]^n) \subseteq \mathbb{R}^n$ is a manifold with corners and we set $\overline{V}_m := \varphi^{-1}(\varphi(U) \cap (\varphi(m) + (-\varepsilon, \varepsilon)^n))$. Then $(V_m)_{m\in M}$ has the desired properties and if M is compact it has a finite subcover. □

Proposition II.11. *If $\mathcal{P} = (K, \pi, P, M)$ is a smooth K-principal bundle with finite-dimensional base M (possibly with corners) and $(V_i)_{i\in I}$ is an open cover of M such that each $\overline{V_i}$ is a trivial subset and a manifold with corners, then $\mathfrak{gau}(\mathcal{P})$ is isomorphic to the Lie algebra*

$$\mathfrak{g}(\mathcal{P}) := \left\{(\xi_i)_{i\in I} \in \bigoplus_{i\in I} C^\infty(\overline{V_i}, \mathfrak{k}) : (\forall x \in V_i \cap V_j)\ \xi_i(x) = \mathrm{Ad}(k_{ij}(x)).\xi_j(x)\right\}$$

with the subspace-topology induced from $\bigoplus_{i\in I} C^\infty(\overline{V_i}, \mathfrak{k})$.

Proof. First we note that $\mathfrak{g}(\mathcal{P})$ is a topological Lie algebra with respect to the pointwise Lie bracket (cf. Remark II.2). Since each $\overline{V_i}$ is trivial and a manifold with corners, there exist smooth sections $\sigma_i : U_i \to P$. For each $\xi \in \mathfrak{gau}(\mathcal{P})$ the element $(\eta_i)_{i\in I}$ with $\eta_i = \xi \circ \sigma_i|_{\overline{V_i}}$ clearly defines an element of $\mathfrak{g}(\mathcal{P})$. Note that $\xi \circ \sigma_i$ is smooth since $\mathrm{int}(\overline{V_i}) = V_i$ and $\sigma_i(V_i) = \Omega_i^{-1}(V_i \times \{e\})$ (cf. Corollary I.16). Since σ_i is continuous, the map $\xi \mapsto (\eta_i)_{i\in I}$ is continuous since each η_i is a pullback. On the other hand, given $(\eta_i)_{i\in I} \in \mathfrak{g}(\mathcal{P})$, the map

$$\xi : P \to \mathfrak{k},\ \ p \mapsto \mathrm{Ad}\left(k^{-1}\right).\eta_i\left(\pi(p)\right)\ \text{ if } p \in \pi^{-1}(\overline{V_i}) \text{ and } p = \sigma_i(k) \cdot k$$

is well-defined, smooth and K-equivariant. Since the map $\varphi : \mathfrak{g}(\mathcal{P}) \to C^\infty(P, \mathfrak{k})^K$, $(\eta_i)_{i\in I} \mapsto \xi$ is obviously an isomorphism of abstract Lie algebras and has a continuous inverse it remains to check that it is continuous, i.e. that

$$\mathfrak{g}(\mathcal{P}) \ni (\eta_i)_{i\in I} \mapsto d^n\xi \in C(T^nP, \mathfrak{k})$$

is continuous for all $n \in \mathbb{N}_0$. If $K \subseteq T^nP$ is compact, then $(T^n\pi)(K) \subseteq T^nM$ is compact and hence it is covered by finitely many $T^nV_{i_1}, \ldots, T^nV_{i_m}$ and thus $\left(T^n\left(\pi^{-1}(\overline{V_i})\right)\right)_{i=i_1,\ldots,i_m}$ is a finite closed cover of $K \subseteq T^nP$. Hence it suffices to show that the map

$$\mathfrak{g}(\mathcal{P}) \ni (\eta_i)_{i\in I} \mapsto d^n\left(\xi|_{\pi^{-1}(U_i)}\right) \in C(T^n\pi^{-1}(U_i), \mathfrak{k})$$

is continuous for $n \in \mathbb{N}_0$ and $i \in I$ and we may thus w.l.o.g. assume that $\mathcal{P}$ is trivial. In the trivial case we have $\xi = \mathrm{Ad}(k^{-1}).(\eta \circ \pi)$ if $p \mapsto (\pi(p), k(p))$ defines a global trivialisation. We shall make the case $n = 1$ explicit. The other cases can be treated similarly and since the formulae get quite long we skip them here.

Given any open zero neighbourhood, which we may assume to be $\lfloor K, V \rfloor$ with $K \subseteq TP$ compact and $0 \in V \subseteq \mathfrak{k}$ open, we have to construct an open zero neighbourhood O in $C^\infty(M, \mathfrak{k})$ such that $\varphi(O) \subseteq \lfloor K, V \rfloor$. For $\eta' \in C^\infty(M, \mathfrak{k})$ and $X_p \in K$ we get with Lemma II.9

$$d(\varphi(\eta'))(X_p) = \mathrm{Ad}(k^{-1}(p)).d\eta'(T\pi(X_p)) - [\delta^l(k)(X_p), \mathrm{Ad}(k^{-1}(p)).\eta'(\pi(p))].$$

Since $\delta^l(K) \subseteq \mathfrak{k}$ is compact, there exists an open zero neighbourhood $V' \subseteq \mathfrak{k}$ such that

$$\mathrm{Ad}(k^{-1}(p)).V' + [\delta^l(k)(X_p), \mathrm{Ad}(k^{-1}(p)).V'] \subseteq V$$

for each $X_p \in K$. Since $T\pi : TP \to TM$ is continuous, $T\pi(K)$ is compact and we may set $O = \lfloor T\pi(K), V' \rfloor$. □

Definition II.12. A Lie group K is said to have an *exponential function* if for each $X \in \mathfrak{k}$ the initial value problem

$$\gamma(0) = e, \quad \gamma'(t) = \gamma(t).X$$

has a solution $\gamma_X \in C^\infty(\mathbb{R}, K)$ and the exponential function

$$\exp_K : \mathfrak{k} \to K, \;\; X \mapsto \gamma_X(1)$$

is smooth. Furthermore, if there exists a zero neighbourhood $V \subseteq k$ such that $\exp_K|_V$ is a diffeomorphism onto some open unit neighbourhood, then K is said to be *locally exponential.*

Remark II.13. The Fundamental Theorem of Calculus for locally convex spaces (cf. [Glö02a, Theorem 1.5]) implies that a Lie group K can have at most one exponential function. If K is a Banach-Lie group (i.e. $\mathfrak{k}$ is a Banach space), then K is locally exponential due to the existence of solutions of differential equations, their smooth dependence on initial values [Lan99, Chapter IV] and the Inverse Mapping Theorem for Banach spaces [Lan99, Theorem I.5.2]. For a more detailed treatment of locally exponential Lie groups we refer to [GN05, Chapter 4]

Lemma II.14. *If K is a Lie group with exponential function and $X, Y \in \mathfrak{k}$ such that $[X, Y] = 0$, then $\exp_K(X)\exp_K(Y) = \exp_K(X + Y)$.*

Proof. This follows from $\delta^l(f \cdot g) = \delta^l(f) + \mathrm{Ad}(f).\delta^l(g)$. □

Lemma II.15. *If K and K' are Lie groups with exponential function, then for each Lie group homomorphism $\alpha : K \to K'$ and the induced Lie algebra homomorphism $d\alpha(e) : \mathfrak{k} \to \mathfrak{k}'$ the diagram*

$$\begin{array}{ccc} K & \xrightarrow{\alpha} & K' \\ \uparrow{\scriptstyle \exp_K} & & \uparrow{\scriptstyle \exp_{K'}} \\ \mathfrak{k} & \xrightarrow{d\alpha(e)} & \mathfrak{k}' \end{array}$$

commutes.

Proof. For $X \in \mathfrak{k}$ consider the curve

$$\tau : \mathbb{R} \to K, \;\; t \mapsto \exp_K(tX).$$

Then $\gamma := \alpha \circ \tau$ is a curve such that $\gamma(0) = e$ and $\gamma(1) = \alpha\big(\exp_K(X)\big)$ with left logarithmic derivate $\delta^l(\gamma) = d\alpha(e).X$. □

Definition II.16. If $\mathcal{P} = (K, \pi, P, M)$ is a smooth K-principal bundle with compact base M and $\overline{V}_1, \ldots, \overline{V}_n$ is an open cover such that each $\overline{V_i}$ is trivial and a manifold with corners, then we denote by $G(\mathcal{P})$ the group

$$G(\mathcal{P}) := \Big\{ (\gamma_i)_{i=1,\ldots,n} \in \prod_{i=1}^{n} C^\infty(\overline{V_i}, K) : (\forall x \in V_i \cap V_j)\ \gamma_i(x) = k_{ij}(x)\gamma_j k_{ji}(x) \Big\}$$

with respect to pointwise group operations.

Remark II.17. The group $G(\mathcal{P})$ is isomorphic to $C^\infty(P, K)^K$ via the map

$$(\gamma_i)_{i=1,\ldots,n} \mapsto \Big(p \mapsto k^{-1}\gamma_i(\pi(p))\, k \quad \text{if} \quad p = \sigma_i(\pi(p)) \cdot k \in \pi^{-1}(\overline{V_i}) \Big),$$

if $\sigma_i : V_i \to P$ are smooth sections. Note that the map on the right hand side is well-defined and smooth. Since for $x \in \overline{V_i}$ the evaluation map $\mathrm{ev}_x : C^\infty(\overline{V_i}, K) \to K$ is continuous, the group $G(\mathcal{P})$ is a closed subgroup of the Lie group $\prod_{i=1}^{n} C^\infty(\overline{V_i}, K)$. But since an infinite-dimensional Lie group may posses closed subgroups that are no Lie groups (cf. [Bou89b, Exercise III.8.2]), this does not automatically yield a Lie group structure on $G(\mathcal{P})$.

Lemma II.18. a) *Let $\mathcal{P} = (K, \pi, P, M)$ be a smooth K-principal bundle with compact base M (possibly with corners) and locally exponential structure group K. If $\exp_K$ restricts to a diffeomorphism from the open neighbourhood $W' \subseteq \mathfrak{k}$ to the open unit neighbourhood $W := \exp_K(W')$, then the map*

$$\varphi_* : U := G(\mathcal{P}) \cap \prod_{i=1}^{n} C^\infty(\overline{V_i}, W) \to \mathfrak{g}(\mathcal{P}), \quad (\exp_K \circ \xi_i)_{i=1,\ldots,n} \mapsto (\xi_i)_{i=1,\ldots,n},$$

induce a smooth manifold structure on U. Furthermore, the conditions $i) - iii)$ of Proposition II.1 are satisfied such that $G(\mathcal{P})$ can be turned into a smooth Lie group.

b) *In the setting of* a)*, the topology on $C^\infty(P, K)^K$, which is induced from the isomorphism of abstract groups $G(\mathcal{P}) \cong C^\infty(P, K)^K$, coincides with the subspace topology induced from the topological group $C^\infty(P, K)$.*

c) *In the setting of* a)*, we have $L(G(\mathcal{P})) \cong \mathfrak{g}(\mathcal{P})$.*

Proof. a) First we note that φ_* is well-defined since $\exp_K|_{W'}$ is bijective. If $(\gamma_i)_{i=1,\ldots,n} \in U$, then $\varphi_*((\gamma_i)_{i=1,\ldots,n})$ is contained in $\mathfrak{g}(\mathcal{P})$ since $\exp_K(\mathrm{Ad}(k).X) = k{\cdot}\exp_K(X){\cdot}k^{-1}$ holds for all $k \in K$ and $X \in W'$ (cf. Lemma II.15). Furthermore the image of φ_* is $U' := \mathfrak{g}(\mathcal{P}) \cap \prod_{i=1}^{n} C^\infty(\overline{V_i}, W')$ since $\exp_K|_{W'}$ is bijective. Since U' is open in $\mathfrak{g}(\mathcal{P})$ and φ_* is bijective, it induces a smooth manifold structure on U.

Let $W_0 \subseteq W$ be an open unit neighbourhood with $W_0 \cdot W_0 \subseteq W$ and $W_0^{-1} = W_0$. Then $U_0 := G(\mathcal{P}) \cap \prod_{i=1}^{n} C^\infty(\overline{V_i}, W_0)$ is an open unit neighbourhood in U with $U_0 \cdot U_0 \subseteq U$ and $U_0 = U_0^{-1}$. Since each $C^\infty(\overline{V_i}, K)$ is a topological group there exist for each $(\gamma_i)_{i=1,\ldots,n}$ open unit neighbourhoods $U_i \subseteq C^\infty(\overline{V_i}, K)$ with $\gamma_i \cdot U_i \cdot \gamma_i^{-1} \subseteq C^\infty(\overline{V_i}, W)$. Since $C^\infty(\overline{V_i}, W_0)$ is open in $C^\infty(\overline{V_i}, K)$, so is $U_i' := U_i \cap C^\infty(\overline{V_i}, W)$ Hence

$$(\gamma_i)_{i=1,\ldots,n} \cdot (G(\mathcal{P}) \cap (U_1' \times \cdots \times U_n')) \cdot (\gamma_i^{-1})_{i=1,\ldots,n} \subseteq U$$

and conditions $i) - iii)$ of Proposition II.1 are satisfied.

b) With the same argument as in the proof of Proposition II.11, we may assume that the bundle is trivial. We thus have to show that the map

$$\varphi : C^\infty(M, K) \to C^\infty(P, K)^K, \quad \varphi(\gamma)(p) = k(p)^{-1} \cdot \gamma(\pi(p)) \cdot k(p),$$

where $p \mapsto (\pi(p), k(p))$ defines a global trivialisation, is an isomorphism of topological groups with respect to the subspace topology on $C^\infty(P, K)^K$ induced from $C^\infty(P, K)$. The map $C^\infty(M, K) \ni f \mapsto f \circ \pi \in C^\infty(P, K)$ is continuous since

$$C^\infty(M, K) \ni f \mapsto T^k(f \circ \pi) = T^k f \circ T^k \pi = (T^k\pi)_*(T^k f) \in C(T^k P, T^k K)$$

is continuous (as a composition of a pullback an the map $f \mapsto T^k f$, which defines the topology on $C^\infty(M, K)$).Since conjugation in $C^\infty(P, K)$ is continuous, it follows that φ is continuous. Since the map $f \mapsto f \circ \sigma$ is also continuous (with the same argument), the assertion follows.

c) This follows immediately from $L(C^\infty(\overline{V_i}, K)) \cong C^\infty(\overline{V_i}, \mathfrak{k})$ (cf. [Glö02b, Section 3.2]). $\square$

Theorem II.19 (Lie group structure on Gau($\mathcal{P}$)). *If $\mathcal{P} = (K, \pi, P, M)$ is a smooth K-principal bundle with compact base M (possibly with corners) and locally exponential structure group K, then* $\mathrm{Gau}(\mathcal{P}) \cong C^\infty(P, K)^K$ *carries the structure of a smooth locally exponential Lie group.*

Proof. First we show that if M is a compact manifold with corners and K has an exponential function, then

$$(\exp_K)_* : C^\infty(M, \mathfrak{k}) \to C^\infty(M, K) \;\; \xi \mapsto \exp_K \circ \, \xi$$

is an exponential function for $C^\infty(M, K)$. For $X \in \mathfrak{k}$ let $\gamma_X \in C^\infty(\mathbb{R}, K)$ be the solution of the initial value problem $\gamma(0) = e$, $\gamma'(t) = \gamma(t).X$ (cf. Definition II.12). If $\xi \in C^\infty(M, \mathfrak{k})$, then $\Gamma(t, m) = \gamma_{\xi(m)}(t)$ depends smoothly on t and m and thus represents an element Γ_ξ of $C^\infty(\mathbb{R}, C^\infty(M, K))$. Then $\xi \mapsto \Gamma_\xi(1) = \exp_K \circ \gamma$ is an exponential function for $C^\infty(M, K)$. The proof of the preceding lemma yields immediately that the restriction of $(\xi_i)_{i=1,\ldots,n} \mapsto (\exp_K \circ \xi)_{i=1,\ldots,n}$ to $\mathfrak{g}(\mathcal{P}) \cap \prod_{i=1}^n C^\infty(\overline{V_i}, W')$ is a diffeomorphism and thus $\mathrm{Gau}(\mathcal{P}) \cong C^\infty(P, K)^K \cong G(\mathcal{P})$ is locally exponential. $\square$

Remark II.20. For two different covers $\overline{V}_1, \ldots, \overline{V}_n$ and $\overline{V}'_1, \ldots, \overline{V}'_m$ we have isomorphisms of groups $G(\mathcal{P}) \cong C^\infty(P, K)^K \cong G(\mathcal{P})'$. Since the corresponding map from $\mathfrak{g}(\mathcal{P})$ to $\mathfrak{g}(\mathcal{P})'$ is an isomorphism (Proposition II.11) and since $G(\mathcal{P})$ and $G(\mathcal{P})'$ are locally exponential, this implies that the isomorphism $G(\mathcal{P}) \cong G(\mathcal{P})'$ is also an isomorphism of smooth Lie groups (cf. [GN05, Chapter 4]). In this sense the topology on $C^\infty(P, K)^K$ is independent on the chosen cover $\overline{V}_1, \ldots, \overline{V}_n$.

III Applications

III.1 Approximations of Continuous Gauge Transformations

We now investigate the approximation of continuous gauge transformations by smooth ones. This was inspired by [Nee02, Section A.3] and [Hir76, Chapter 2]. It will be convenient to consider the space $C(X, G)$ of continuous maps from the Hausdorff space X into the topological group G either with the topology of compact convergence, denoted by $C(X, G)_c$, or with the compact-open topology, denoted by $C(X, G)_{c.o.}$. Since these two topologies coincide [Bou89a, Theorem X.3.4.2] we will use them interchangeably.

Definition III.1. If $\mathcal{P} = (K, \pi, P, M)$ is a continuous K-principal bundle (cf. [Ste51, Section I.8]), then we denote by

$$\mathrm{Gau_c}(\mathcal{P}) := \{f \in \mathrm{Homeo}(P) : (\forall p \in P)(\forall k \in K) f(p \cdot k) = f(p) \cdot k \text{ and } \pi \circ f = \pi\}$$

the group of *continuous gauge transformations* of P.

Remark III.2. The same mapping as in the smooth case (cf. Remark II.7 yields an isomorphism

$$\mathrm{Gau_c}(\mathcal{P}) \cong C(P, K)^K := \{\gamma \in C(P, K) : (\forall p \in P)(\forall k \in K)\; \gamma(p \cdot k) = k^{-1}\gamma(p)k\},$$

and $C(P, K)^K$ is a topological group as a closed subgroup of $C(P, K)_c$. We equip $\mathrm{Gau_c}(\mathcal{P})$ with the topology defined by this isomorphism. Denote by $(V_i)_{i \in I}$ an open cover of M such that there exist continuous sections $\sigma_i : V_i \to P$. Then $G := \prod_{i \in I} C(V_i, K)_c$ is a topological group with

$$G_c(\mathcal{P}) := \Big\{ (\gamma_i)_{i=1,\ldots,n} \in \prod_{i \in I} C(V_i, K) : (\forall x \in V_i \cap V_j)\; \gamma_i(x) = k_{ij}(x)\gamma_j(x)k_{ji}(x) \Big\}$$

as a closed subgroup. Then

$$G_c(\mathcal{P}) \ni (\gamma_i)_{i\in I} \mapsto \Big(p \mapsto k_i(p)^{-1} \cdot \gamma_i(\pi(p)) \cdot k_i(p) \text{ if } p \in \pi^{-1}(V_i)\Big) \in C(P,K)^K,$$

where $k_i \in C(\pi^{-1}(V_i), K)$ is defined by $p = \sigma_i(\pi(p)) \cdot k_i(p)$, defines an isomorphism of groups and a straightforward verification shows that this map also defines an isomorphism of topological groups.

If M is compact, then there exists a finite open cover $(V_i')_{i=1,\ldots,n}$ of Msuch that for each $\overline{V_i'}$ is contained in some V_j. Since each $C(\overline{V_i'}, K)$ is a Lie group [GN05], the same argumentation as in the proof of Lemma II.18 shows that $C(P,K)^K$, with the subspace-topology from $C(P,K)_c$, can be turned into a Lie group.

Remark III.3. We recall some facts from set-theoretic topology. If a topological space is second countable, then it is Lindelöf [Mun75, Theorem 4.1.3], i.e. every open cover of X has a countable subcover. If it is, in addition, locally compact, then it is σ-compact [Dug66, Theorem XI.7.2]. Furthermore, any σ-compact space is paracompact [Dug66, Theorem XI.7.3] and thus normal [Bre93, Theorem I.12.5].

Proposition III.4. *If M is a finite-dimensional σ-compact manifold with corners, then for each locally convex space E the space $C^\infty(M,E)$ is dense in $C(M,E)_c$. If $f \in C(M,E)$ has compact support and U is an open neighbourhood of* $\operatorname{supp}(f)$*, then each neighbourhood of f in $C(M,E)$ contains a smooth function whose support is contained in U.*

Proof. The proof of [Nee02, Theorem A.3.1] carries over without changes. □

Corollary III.5. *If M is a finite-dimensional σ-compact manifold with corners and V is an open subset of the locally convex space E, then $C^\infty(M,V)$ is dense in $C(M,V)_c$.*

Lemma III.6. *Let M be a finite-dimensional σ-compact manifold with corners, E be a locally convex space, $W \subseteq E$ be open and convex and let $f : M \to W$ be continuous. If $L \subseteq M$ is closed and $U \subseteq M$ is open such that f is smooth on a neighbourhood of $L\backslash U$, then each neighbourhood of f in $C(M,E)_c$ contains a continuous map $g : M \to W$, which is smooth on a neighbourhood of L and which equals f on $M\backslash U$.*

Proof. (cf. [Hir76, Theorem 2.5]) Let $A \subseteq M$ be an open set containing $L\backslash U$ such that $f\big|_A$ is smooth. Then $L\backslash A \subseteq U$ is closed in M so that there exists $V \subseteq U$ open with

$$L\backslash A \subseteq V \subseteq \overline{V} \subseteq U$$

(cf. Remark III.3). Then $\{U, M\backslash\overline{V}\}$ is an open cover of M, and there exists a smooth partition of unity $\{f_1, f_2\}$ subordinated to this cover. Then

$$G_f : C(M,W)_c \to C(M,E)_c, \;\; G_f(\gamma)(x) = f_1(x)\gamma(x) + f_2(x)f(x)$$

is continuous since $\gamma \mapsto f_1\gamma$ and $f_1\gamma \mapsto f_1\gamma + f_2 f$ are continuous.

If γ is smooth on $A \cup V$ then so is $G_f(\gamma)$, because f_1 and f_2 are smooth, f is smooth on A and $f_2\big|_V \equiv 0$. Note that $L \subseteq A \cup (L\backslash A) \subseteq A \cup V$, so that $A \cup V$ is an open neighbourhood of L. Furthermore we have $G_f(\gamma) = \gamma$ on V and $G_f(\gamma) = f$ on $M\backslash U$. Since $G_f(f) = f$, there is for each open neighbourhood O of f an open neighbourhood O' of f such that $G_f(O') \subseteq O$. By the preceding Corollary there is a smooth function $h \in O'$ such that $g := G_f(h)$ has the desired properties. □

Lemma III.7. *Let M be a finite-dimensional σ-compact manifold with corners, K be a Lie group, $W \subseteq K$ be diffeomorphic to an open convex subset of $\mathfrak{k}$ and $f : M \to W$ be continuous. If $L \subseteq M$ is closed and $U \subseteq M$ is open such that f is smooth on a neighbourhood of $L\backslash U$, then each neighbourhood of f in $C(M,W)_{c.o.}$ contains a map which is smooth on a neighbourhood of L and which equals f on $M\backslash U$.*

Proof. Let $\varphi : W \to \varphi(W) \subseteq \mathfrak{k}$ be the postulated diffeomorphism. If $\lfloor K_1, V_1 \rfloor \cap \ldots \cap \lfloor K_n, V_n \rfloor$ is an open neighbourhood of $f \in C(M,K)_{c.o.}$, where we may assume that $V_i \subseteq W$, then $\lfloor K_1, \varphi(V_1) \rfloor \cap \ldots \cap \lfloor K_n, \varphi(V_n) \rfloor$ is an open neighbourhood of $\varphi \circ f$ in $C(M, \varphi(W))_c$. We apply Lemma III.6 to this open neighbourhood to obtain a map h. Then $\varphi^{-1} \circ h$ has the desired properties. □

Proposition III.8. *Let M be a finite-dimensional second countable manifold with corners, K be a Lie group and $f \in C(M,K)$. If $L \subseteq M$ is closed and $U \subseteq M$ is open such that f is smooth on a neighbourhood of $L \backslash U$, then each open neighbourhood O of f in $C(M,K)_{c.o.}$ contains a map g, which is smooth on a neighbourhood of L and equals f on $M \backslash U$.*

Proof. We recall the properties of the topology on M from Remark III.3. If f is smooth on the open neighbourhood A of $L\backslash U$, then there exists an open set $A' \subseteq M$ such that $L \backslash U \subseteq A' \subseteq \overline{A'} \subseteq A$. We choose an open cover $(W_j)_{j \in J}$ of $f(M)$, where each W_j is an open subset of K diffeomorphic to an open zero neighbourhood of $\mathfrak{k}$ and set $V_j := f^{-1}(W_j)$. Since each $x \in M$ has an open neighbourhood $V_{x,j}$ with $\overline{V_{x,j}}$ compact and $\overline{V_{x,j}} \subseteq V_j$ for some $j \in J$, we may redefine the cover $(V_j)_{j \in J}$ such that $\overline{V_j}$ is compact and $f(\overline{V_j}) \subseteq W_j$ for all $j \in J$.

Since M is paracompact, we may assume that the cover $(V_j)_{j\in J}$ is locally finite, and since M is normal, there exists a cover $(V_i')_{i \in I}$ such that for each $i \in I$ there exists a $j \in J$ such that $\overline{V_i'} \subseteq V_j$. Since M is also Lindelöf, we may assume that the latter is countable, i.e. $I = \mathbb{N}^+$. Hence M is also covered by countably many of the V_j and we may thus assume $\overline{V_i'} \subseteq V_i$ and $f(\overline{V_i}) \subseteq W_i$ for each $i \in \mathbb{N}^+$ Furthermore we set $V_0 := \emptyset$ and $V_0' := \emptyset$. Observe that both covers are locally finite by their construction. Define

$$L_i := L \cap \overline{V_i'} \backslash (V_0' \cup \ldots \cup V_{i-1}')$$

which is closed and contained in V_i. Since $L \backslash A' \subseteq U$ we then have $L_i \backslash A' \subseteq V_i \cap U$ and there exist open subsets $U_i \subseteq V_i \cap U$ such that $L_i \backslash A' \subseteq U_i \subseteq \overline{U_i} \subseteq V_i \cap U$. We claim that there exist functions $g_i \in O$, $i \in \mathbb{N}_0$, satisfying

$$g_i = g_{i-1} \text{ on } M \backslash \overline{U_i} \text{ for } i > 0$$
$$g_i(\overline{V_j}) \subseteq W_j \text{ for all } i, j \in \mathbb{N}_0$$
$$g_i \text{ is smooth on a neighbourhood of } L_0 \cup \ldots \cup L_i \cup \overline{A'}.$$

For $i = 0$ we have nothing to show, hence we assume that the g_i are defined for $i < a$. We consider the set

$$Q := \{\gamma \in C(V_a, W_a) : \gamma = g_{a-1} \text{ on } V_a \backslash \overline{U_a}\},$$

which is a closed subspace of $C(V_a, W_a)_{c.o.}$. Then the map

$$F : Q \to C(M, W_a), \quad F(\gamma)(x) = \begin{cases} \gamma(x) & \text{if } x \in \overline{U_a} \\ g_{a-1}(x) & \text{if } x \in M \backslash \overline{U_a} \end{cases}$$

is continuous since $\overline{U_a}$ is closed. Note that, by induction, $g_{a-1}(V_a) \subseteq W_a$, whence $g_{a-1}|_{V_a} \in Q$. Since F is continuous and $F(g_{a-1}|_{V_a}) = g_{a-1}$, there exists an open set $O' \subseteq C(V_a, W_a)$ containing $g_{a-1}|_{V_a}$ such that $F(O' \cap Q) \subseteq O$.

Since $(V_j)_{j \in \mathbb{N}_0}$ is locally finite and $\overline{V_j}$ is compact, the set $\{j \in \mathbb{N}_0 : \overline{U_a} \cap \overline{V_j} \neq \emptyset\}$ is finite and hence

$$O'' = O' \cap \bigcap_{j \in \mathbb{N}_0} \lfloor \overline{U_a} \cap \overline{V_j}, W_j \rfloor$$

is an open neighbourhood of $g_{a-1}|_{V_a}$ in $C(V_a, W_a)_{c.o.}$ by induction. We now apply Lemma III.7 with to the manifold with corners V_a, the closed set $L_a' := (L \cap \overline{V_a'}) \cup (\overline{A'} \cap V_a) \subseteq V_a$, the open set $U_a \subseteq V_a$, $g_{a-1}|_{V_a} \in Q \subseteq C(V_a, W_a)$ and the open neighbourhood O'' of $g_{a-1}|_{V_a}$. Due to the construction we have $L_a \backslash U_a \subseteq A' \cap V_a$ and $L \cap \overline{V_a'} \subseteq L_0 \cup \ldots \cup L_a$. Hence we have

$$L_a' \backslash U_a \subseteq (L_0 \cup \ldots \cup L_{a-1} \cup (L_a \backslash U_a)) \cup (\overline{A'} \cap V_a \backslash U_a) \subseteq L_1 \cup \ldots \cup L_{a-1} \cup (\overline{A'} \cap V_a)$$

so that $g_{a-1}|_{V_a}$ is smooth on a neighbourhood of $L_a' \backslash U_a$. We thus obtain a map $h \in O''$ which is smooth on a neighbourhood of L_a' and which coincides with $g_{a-1}|_{V_a}$ on $V_a \backslash U_a \supseteq V_a \backslash \overline{U_a}$, hence is contained in $O'' \cap Q$, and we set $g_a := F(h)$. Since $h(\overline{U_a} \cap \overline{V_j}) \subseteq W_j$ and $g_{a-1}(\overline{V_j}) \subseteq W_j$, we have $F(h)(V_j) \subseteq W_j$. Furthermore $F(h)$ inherits the smoothness properties from g_{a-1} on $M \backslash \overline{U_a}$, from h on V_a and since $L_a \subseteq L \cap \overline{V_a'}$, it has the desired smoothness properties on M. This finishes the construction of the g_i.

We now construct g. First we set $m(x) := \max\{i : x \in \overline{V_i}\}$ and $n(x) := \max\{i : x \in V_i\}$. Then obviously $n(x) \leq m(x)$ and each $x \in M$ has a neighbourhood on which $g_{n(x)}, \ldots, g_{m(x)}$ coincide since $\overline{U_i} \subseteq V_i$ and $g_i = g_{i-1}$ on $M \backslash U_i$. Hence $g(x) := g_{n(x)}(x)$ defines a continuous function on M. If $x \in L$, then $x \in L_0 \cup \ldots \cup L_{n(x)}$ and thus g is smooth on a neighbourhood of x. If $x \in M \backslash U$, then $x \notin U_1 \cup \ldots \cup U_{n(x)}$ and thus $g(x) = f(x)$. □

Proposition III.9. *If $\mathcal{P} = (K, \pi, P, M)$ is a smooth K-principal bundle with locally exponential structure group K and M is a finite-dimensional second countable manifold with corners, then* $\mathrm{Gau}(\mathcal{P}) \cong C^\infty(P, K)^K$ *is dense in* $\mathrm{Gau_c}(\mathcal{P}) \cong C(P, K)^K$ *with the topology defined in Remark III.2.*

Proof. We recall the properties of the topology on M from Remark III.3. Let $(U_i')_{\in I}$ be a locally finite open cover of M such that for each $i \in I$ there exists a smooth section $\sigma_i : U_i' \to P$. Since M is locally compact, we may assume that $\overline{U_i'}$ is compact and since M is normal, there exist open covers $(V_j)_{j \in J}$, $(V_j')_{j \in J}$ and $(U_j)_{j \in J}$ such that for each $j \in J$ there exists some $i \in I$ with $\overline{V_j} \subseteq V_j' \subseteq \overline{V_j'} \subseteq U_j \subseteq \overline{U_j} \subseteq U_i'$. Hence we may assume that that $I = J$ and furthermore that $I = J = \mathbb{N}$ since M is Lindelöf. Since $(U_i')_{i \in \mathbb{N}}$ is locally finite, the same holds for the other covers.

Each element of $C(P, K)^K$ is represented by an element $(\gamma_i)_{i \in \mathbb{N}}$ of $\prod_{i \in \mathbb{N}} C(V_i, K)$ satisfying $\gamma_i(x) = k_{ij}(x)\gamma_j(x)k_{ji}(x)$ if $x \in V_i \cap V_j$. Then

$$O = \Big\{ (\gamma_i')_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} C(V_i, K) : \gamma_i'(x) = k_{ij}(x)\gamma_j'(x)k_{ji}(x) \text{ and } \gamma_i'(L_{i,l}) \subseteq W_{i,l} \Big\},$$

for finitely many $L_{i,l} \subseteq V_i$ compact, $W_{i,l} \subseteq K$ open (i.e. assume for $i, l \leq m$) is a basic open neighbourhood of $(\gamma_i)_{i \in \mathbb{N}}$. Since each $L_{i,l}$ is compact, there exists an open unit neighbourhood $W \subseteq K$ such that

$$W \cdot \gamma_i(L_{i,l}) \subseteq W_{i,l} \text{ for all } 1 \leq i, l \leq m \tag{1}$$

and that there exists a chart $\exp_K^{-1}|_W := \varphi : W \to \mathfrak{k}$ such that $\varphi(e) = 0$ and $\varphi(W)$ is an open convex zero neighbourhood in $\mathfrak{k}$. Furthermore, for each $i \in \mathbb{N}$ denote by W_i an open unit neighbourhood such that $W_i \subseteq W$, $\varphi(W_i) \subseteq \mathfrak{k}$ is convex and $(W_i)^i \subseteq W$.

Since each k_{ij} is defined on $U_i' \cap U_j'$, we may extend each γ_i to a continuous function on U_i' by defining $\gamma_i(x) = k_{ij}(x)\gamma_j(x)k_{ji}(x)$ if $x \in U_i' \cap V_j$. We now construct inductively $\widetilde{\gamma}_i \in C^\infty(U_i', K)$ such that

$$\widetilde{\gamma}_i(L_{i,l}) \subseteq W_{i,l} \text{ for all } i, l \in \{1, \ldots, m\} \tag{2}$$

$$\widetilde{\gamma}_i = k_{ij}\widetilde{\gamma}_j k_{ji} \text{ pointwise on a neighbourhood of } x \text{ for each } x \in \overline{V_i} \cap \overline{V_j} \tag{3}$$

$$k_{ji}(x)\widetilde{\gamma}_i(x)k_{ij}(x)\gamma_j(x)^{-1} \in W_j \text{ if } i < j \text{ and } x \in U_i \cap U_j. \tag{4}$$

Then $\widetilde{\gamma}_a|_{V_a}$ represents an element of $C^\infty(P, K)^K$ which is contained in O and hence establishes the assertion.

For $i=1$ denote $X_j := \overline{U_1} \cap \overline{U_j}$ and a compactness argument using the locally finiteness of $(U_i)_{i\in\mathbb{N}}$ shows that $\{j \in \mathbb{N} : X_j \neq \emptyset\}$ is finite. Furthermore $\{j \in \mathbb{N} : x \in X_j\}$ is also finite. If $X_j \neq \emptyset$, then for $x \in X_j$ there exists an open unit neighbourhood $W_{x,j}$ such that

$$k_{j1}(x) \cdot W_{x,j}^2 \cdot \gamma_1(x) \cdot k_{1j}(x) \cdot W_{x,j}^2 \cdot \gamma_j(x) \subseteq W_j \tag{5}$$

and we set $W_x := \bigcap_{\{j: x\in X_j\}} W_{x,j}$. Then the continuity of k_{j1}, k_{1j} and γ_j yields an open neighbourhood $U_{x,j} \subseteq X_j$ of x such that

$$\left.\begin{array}{rl} k_{j1}(y)^{-1} \cdot k_{j1}(y') & \in W_x \\ k_{1j}(y)^{-1} \cdot k_{1j}(y') & \in W_x \\ \gamma_j(y)^{-1} \cdot \gamma_j(y') & \in W_x \end{array}\right\} \text{ if } y, y' \in U_{x,j}. \tag{6}$$

Furthermore we may assume w.l.o.g. that $\gamma_1(\overline{U_{x,j}}) \subseteq W_x \cdot \gamma_1(x)$. Since X_j is compact, it is covered by finitely many $U_{x_1,j}, \ldots U_{x_n,j}$ and then

$$O_j := \lfloor \overline{U_{x_1,j}}, W_{x_1} \cdot \gamma_1(x_1) \rfloor \cap \ldots \cap \lfloor \overline{U_{x_n,j}}, W_{x_n} \cdot \gamma_1(x_n) \rfloor$$

is an open neighbourhood of γ_1 in $C(U_1', K)_{c.o.}$. To obtain $\widetilde{\gamma}_1$ we apply Proposition III.8 to the manifold with corners U_1', the closed set $\emptyset$, the open set U_1' and the open neighbourhood

$$\lfloor L_{1,1}, W_{1,1} \rfloor \cap \ldots \cap \lfloor L_{1,m}, W_{1,m} \rfloor \cap \bigcap_{\{j: X_j \neq \emptyset\}} O_j$$

of $f = \gamma_1$. Then (2) holds with $i = 1$ and (3) is trivially satisfied. To check (4), we first observe that each $x \in U_1 \cap U_j$ is contained in some $U_{x_r,j} \subseteq \overline{U_1} \cap \overline{U_j}$ for $r \in \{1, \ldots, n\}$ and hence $\widetilde{\gamma}_1(x) \in W_{x_r} \cdot \gamma_1(x_r)$. We thus have

$$\begin{aligned} k_{j1}(x)\widetilde{\gamma}_1(x)k_{1j}(x)\gamma_j(x)^{-1} &\subseteq k_{j1}(x)W_{x_r}\gamma_1(x_r)k_{1j}(x)\gamma_j(x)^{-1} \\ &= k_{j1}(x_r)k_{j1}(x_r)^{-1}k_{j1}(x)W_{x_r}\gamma_1(x_r)k_{1j}(x_r)k_{1j}(x_r)^{-1}k_{1j}(x)\gamma_j(x)^{-1}\gamma_j(x_r)\gamma_j(x_r)^{-1} \\ &\subseteq k_{j1}(x_r)W_{x_r}^2\gamma_1(x_r)k_{1j}(x_r)W_{x_r}^2\gamma_j(x_r)^{-1} \subseteq W_j \end{aligned}$$

This finishes the construction of γ_1.

Having defined the $\widetilde{\gamma}_i$ inductively for $i < a$ we now construct $\widetilde{\gamma}_a$. The reader might want to choose $a = 3$ with $V_1 \cap V_2 \cap V_3 \neq \emptyset$ to follow the construction. First we have to interpolate between the differences of $k_{ai}\widetilde{\gamma}_i k_{ia}$ for different $i < a$ on $U_a \backslash (V_1 \cup \ldots \cup V_{a-1})$. For this sake we first construct $\eta \in C(U_a, K)$ as follows:

$$U_a \cap U_1 \backslash \overline{V_1}, \ldots, U_a \cap U_{a-1} \backslash \overline{V_{a-1}}, U_a \cap (V_1' \cup \ldots \cup V_{a-1}'), U_a \backslash (\overline{V_1'} \cup \ldots \cup \overline{V_{a-1}'})$$

is an open cover of U_a and there exists a subordinated partition of unity $f_1, \ldots, f_{a-1}, g, h$. If

$$x \in U_a \cap (U_{j_1} \backslash \overline{V_{j_1}}) \cap \ldots \cap (U_{j_r} \backslash \overline{V_{j_r}}) \tag{7}$$

where $j_1 < \ldots < j_r$, and $\{j_1, \ldots, j_r\} \subseteq \{1, \ldots, a-1\}$ is maximal such that (7) holds, set

$$\begin{aligned} \widetilde{\eta}(x) := f_{j_1}(x) \star \big(k_{aj_1}(x) \cdot \widetilde{\gamma}_{j_1}(x) \cdot k_{j_1 a}(x) \cdot \gamma_a(x)^{-1}\big) \cdot \ldots \\ \ldots \cdot f_{j_r}(x) \star \big(k_{aj_r}(x) \cdot \widetilde{\gamma}_{j_r}(x) \cdot k_{j_r a}(x) \cdot \gamma_a(x)^{-1}\big), \end{aligned}$$

where $\lambda \star k := \exp_K(\lambda\varphi \cdot (k))$ for $\lambda \in [0,1]$ and $k \in W$. Then $\widetilde{\eta}$ is a well-defined and continuous map on

$$U_a \cap \big((U_1 \cup \ldots \cup U_{a-1}) \backslash (\overline{V_1} \cup \ldots \cup \overline{V_{a-1}})\big),$$

since each $x \in \partial\overline{V_l}$ or $x \in \partial U_l$ for $l < a$ has a neighbourhood on which f_l vanishes. For $x \in U_a \cap (U_1 \cup \ldots \cup U_{a-1})$ we now set

$$\overline{\eta}(x) := \begin{cases} \widetilde{\eta}(x) & \text{if } x \notin V_1' \cup \ldots \cup V_{a-1}' \\ \widetilde{\eta}(x) \cdot g(x) \star \big(k_{am(x)}(x) \cdot \widetilde{\gamma}_{m(x)}(x) \cdot k_{m(x)a}(x) \cdot \gamma_a(x)^{-1}\big) & \text{if } x \in V_1' \cup \ldots \cup V_{a-1}', \end{cases}$$

where $m(x) := \max\{i < a : x \in \overline{V_i}\}$. Then $\overline{\eta}$ is a continuous map on $U_a \cap (U_1 \cup \ldots \cup U_{a-1})$ since each $x \in \partial(U_a \cup (V_1' \cup \ldots \cup V_{a-1}'))$ has a neighbourhood on which g vanishes and each $x \in \overline{V_a} \cap \overline{V_i}$ has a neighbourhood on which

$$k_{am(x)}\gamma_{m(x)}k_{m(x)a} = k_{ai}\widetilde{\gamma}_i k_{ia}$$

holds pointwise due to (3). For $x \in U_a'$ we now set

$$\eta(x) := \begin{cases} \gamma_a(x) & \text{if } x \notin U_1 \cup \ldots \cup U_{a-1} \\ \overline{\eta}(x) \cdot \gamma_a(x) & \text{if } x \in U_i \cup \ldots \cup U_{a-1}. \end{cases}$$

This defines a continuous map on U_a' since $U_a' \supseteq U_a$ and each $x \in (U_a \cap (U_1 \cup \ldots \cup U_{a-1}))$ has a neighbourhood on which $f_1, \ldots, f_{a-1}$ and g vanish, whence $\overline{\eta} = 1$.

We now check the properties of η. For $x \in (\overline{V_1} \cup \ldots \cup \overline{V_{a-1}}) \cap \overline{V_a}$ there exists a neighbourhood V_x of x such that $h(x') = 0$ and

$$k_{am(x')}(x') \cdot \widetilde{\gamma}_{m(x')}(x') \cdot k_{m(x')a}(x') = k_{ai}(x')\widetilde{\gamma}_i(x') \cdot k_{ia}(x')$$

if $x' \in \overline{V_i} \cap V_x$, whence $\overline{\eta}(x') = k_{am(x')}(x') \cdot \widetilde{\gamma}_{m(x')}(x') \cdot k_{m(x')a} \cdot \gamma_a(x')^{-1}$ (cf. Lemma II.14). Then we have

$$\eta(x') = k_{am(x')}(x') \cdot \widetilde{\gamma}_{m(x')}(x') \cdot k_{m(x')a} = k_{ai}(x') \cdot \widetilde{\gamma}_i(x') \cdot k_{ia}(x')$$

and thus η is smooth on V_x. Furthermore we have $\eta(x) \in W_{a,l}$ if $x \in L_{a,l}$ due to (1), (4) and the construction of $\overline{\eta}$.

We now construct $\widetilde{\gamma}_a$ from η. If $\overline{U_a} \cap \overline{U_j} \neq \emptyset$, then a similar argument as in the construction of $\widetilde{\gamma}_1$, with γ_1 substituted by η, yields an open neighbourhood O_j of η such that

$$k_{ja}(x) \cdot \eta'(x) \cdot k_{ja}(x) \cdot \gamma_j(x)^{-1} \in W_j$$

if $x \in U_a \cap U_j$ and $\eta' \in O_j$. This will ensure (4). As we have seen before, η is smooth on a neighbourhood of $\overline{V_a} \cap (\overline{V_1} \cup \ldots \cup \overline{V_{a-1}})$ and to ensure (3), we choose a closed set $L \subseteq U_a'$ such that $\overline{V_a} \cap (\overline{V_1} \cup \ldots \cup \overline{V_{a-1}}) \subseteq \operatorname{int}(L)$ and η is smooth on a neighbourhood of L.

We now apply Proposition III.8 to the manifold with corners U_a', the closed set L, the open set $U_a' \backslash L$ and the open neighbourhood

$$\lfloor L_{a,1}, W_{a,1} \rfloor \cap \ldots \cap \lfloor L_{a,m}, W_{a,m} \rfloor \cap \bigcap_{\{j : \overline{U_a} \cap \overline{U_j} \neq \emptyset\}} O_j$$

of η to obtain $\widetilde{\gamma}_a \in C^\infty(U_a', K)$. □

Lemma III.10. *Let $\mathcal{P} = (K, \pi, P, M)$ be a smooth K-principal bundle, M be compact, K be locally exponential and let $W' \subseteq \mathfrak{k}$ be an open convex zero neighbourhood such that $\exp_K : W' \to \exp_K(W') =: W$ is a diffeomorphism onto an open unit neighbourhood of K. If $(\gamma_i)_{i=1,\ldots,n} \in G(\mathcal{P})$ represents an element of $C^\infty(P, K)^K$ (cf. Remark II.17), which is close to identity, in the sense that $\gamma_i(\overline{V_i}) \subseteq W$, then $(\gamma_i)_{i=1,\ldots,n}$ is homotopic to the constant map $(x \mapsto e)_{i=1,\ldots,n}$.*

Proof. Since the map

$$\varphi_* : U := G(\mathcal{P}) \cap \prod_{i=1}^{n} C^\infty(\overline{V_i}, W) \to \mathfrak{g}(\mathcal{P}), \quad (\exp_K \circ \xi_i)_{i=1,\ldots,n} \mapsto (\xi_i)_{i=1,\ldots,n},$$

is a chart of $G(\mathcal{P})$ (cf. Lemma II.18) and $\varphi_*(U) \subseteq \mathfrak{g}(\mathcal{P})$ is convex, the map

$$[0,1] \ni t \mapsto \varphi_*^{-1}\big(t \cdot \varphi_*((\gamma_i)_{i=1,\ldots,n})\big) \in G(\mathcal{P})$$

defines the desired homotopy. □

Theorem III.11 (Weak homotopy equivalence for $\mathrm{Gau}(\mathcal{P})$). *If $\mathcal{P} = (K,\pi,P,M)$ is a smooth K-principal bundle with compact base M (possibly with corners) and locally exponential structure group K, then the natural inclusion $C^{\infty}(P,K)^K \hookrightarrow C(P,K)^K$ of smooth into continuous gauge transformations is a weak homotopy equivalence, i.e. the induced mappings $\pi_n(C^{\infty}(P,K)^K) \to \pi_n\left(C(P,K)^K\right)$ are isomorphisms of groups for $k \in \mathbb{N}_0$.*

Proof. To check surjectivity, consider the continuous K-principal bundle $\mathrm{pr}^*(\mathcal{P})$ obtained form $\mathcal{P}$ by pulling it back along the projection $\mathrm{pr} : \mathbb{S}^k \times M \to M$. Then $\mathrm{pr}^*(\mathcal{P})$ is isomorphic to $(K, \mathrm{id}\times\pi, \mathbb{S}^k\times P, \mathbb{S}^k\times M)$, where K acts trivially on the first factor of $\mathbb{S}^k \times P$. We have with respect to this action $C(\mathrm{pr}^*(P),K)^K \cong C(\mathbb{S}^k\times P,K)^K$ and $C^{\infty}(\mathrm{pr}^*(P))^K \cong C^{\infty}(\mathbb{S}^k\times P,K)^K$. The isomorphism $C(\mathbb{S}^k,G_0) \cong C_*(\mathbb{S}^k,G_0) \rtimes G_0 = C_*(\mathbb{S}^k,G) \rtimes G_0$, where $C_*(\mathbb{S}^k,G)$ denotes the space of base-point-preserving maps from $\mathbb{S}^k$ to G, yields $\pi_n(G) = \pi_0(C_*(\mathbb{S}^k,G)) \cong \pi_0(C(\mathbb{S}^k,G_0))$ for any topological group G. We thus get a map

$$\pi_n(C^{\infty}(P,K)^K) = \pi_0(C_*(\mathbb{S}^k,C^{\infty}(P,K)^K)) \cong \pi_0(C(\mathbb{S}^k,C^{\infty}(P,K)^K_0)) \xrightarrow{\eta} \pi_0(C(\mathbb{S}^k,C(P,K)^K_0)),$$

where η is induced by the inclusion $C^{\infty}(P,K)^K \hookrightarrow C(P,K)^K$.

If $f \in C(\mathbb{S}^k\times P,K)$ represents an element $[F] \in \pi_0(C(\mathbb{S}^k,C(P,K)^K_0))$ (recall $C(P,K)^K \cong G_{\mathrm{c}}(\mathcal{P}) \subseteq \prod_{i=1}^n C(V_i,K)$ and $C(\mathbb{S}^k,C(V_i,K)) \cong C(\mathbb{S}^k\times V_i,K)$), then there exists $\widetilde{f} \in C^{\infty}(\mathbb{S}^k\times P,K)^K$ which is contained in the same connected component of $C(\mathbb{S}^k\times P,K)^K$ as f (cf. Proposition III.9). Since $\widetilde{f}$ is in particular smooth in the second argument, it follows that $\widetilde{f}$ represents an element $\widetilde{F} \in C(\mathbb{S}^k,C^{\infty}(P,K)^K)$. Since the connected components and the arc components of $C(\mathbb{S}^k\times P,K)^K$ coincide (since it is a Lie group, cf. Remark III.2), there exists a path

$$\tau : [0,1] \to C(\mathbb{S}^k\times P,K)^K_0$$

such that $t \mapsto \tau(t)\cdot f$ is a path connecting f and $\widetilde{f}$. Since $\mathbb{S}^k$ is connected it follows that $C(\mathbb{S}^k\times P,K)^K_0 \cong C(\mathbb{S}^k,C(P,K)^K)_0 \subseteq C(\mathbb{S}^k,C(P,K)^K_0)$. Thus τ represents a path in $C(\mathbb{S}^k,C(P,K)^K_0))$ connecting F and $\widetilde{F}$ whence $[F] = [\widetilde{F}] \in \pi_0(C(\mathbb{S}^k,C(P,K)^K_0))$. That $\pi_n(\mathrm{incl})$ is injective follows with Lemma III.10 as in [Nee02, Theorem A.3.7]. □

III.2 Calculating Homotopy Groups of the Gauge Group

In this section we will apply the results from the previous section to the calculation of the homotopy groups $\pi_n(\mathrm{Gau}(\mathcal{P}))$ for bundles over compact orientable surfaces with and without boundary. The technical results of the previous section and the first part of the following will enable us to obtain this result in a quite elegant way.

Throughout this section we will frequently use the following facts from general topology. If X/R is a quotient of the topological space X by an equivalence relation R, then the continuous functions on X/R are in on-to-one correspondence with the continuous functions on X, which are constant on the equivalence classes of R [Bou89a, §I.3.4]. A consequence of this is that if X is covered by the closed sets $(X_i)_{i\in I}$ and $f_i : X_i \to Y$ are continuous satisfying $f_i|_{X_i\cap X_j} = f_j|_{X_j\cap X_i}$, then $f : X \to Y$, $x \mapsto f_i(x)$ if $x \in X_i$, is continuous (cf. [Bou89a, §I.2.5]).

Throughout this section we will deal with orientable surfaces, but the results carry over with very little changes to non-orientable surfaces. Since the formulae will always vary a bit between these two cases we stick to the orientable case for convenience and shortness. The results carry over to the non-orientable case in the same manner (cf. Remark III.27)

Lemma III.12. *If $\mathcal{P} = (K,\pi,P,M)$ is a continuous K-principal bundle over $\mathbb{S}^1$ with connected structure group K, then $\mathcal{P}$ is trivial.*

Proof. This is [Ste51, Corollary 18.6]. □

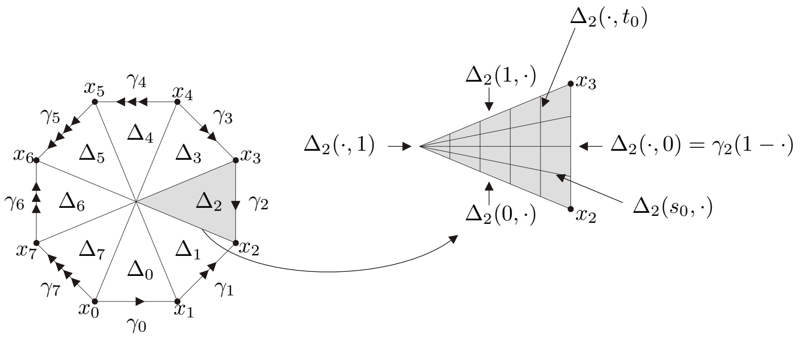

Figure 1: Coordinates on the regular 8-gon

Remark III.13. We introduce some notation: If M is a compact orientable surface (i.e. a compact connected orientable 2-manifold) of genus $g \geq 1$, then M is homeomorphic to a quotient of a regular polygon B (which we may identify with a subset of $\mathbb{C}$) with $4g$ vertices [Mas67, Theorem 5.1]. The quotient is constructed via affine maps $\gamma_i : [0,1] \to B$ such that

$$\begin{aligned} &\gamma_i(0) = x_i,\ \gamma_i(1) = x_{i+1}, \quad \text{if } [i] \in \{[0],[1]\} \text{ in } \mathbb{Z}_4, \\ &\gamma_i(0) = x_{i+1},\ \gamma_i(1) = x_i, \quad \text{if } [i] \in \{[2],[3]\} \text{ in } \mathbb{Z}_4 \end{aligned}$$

for $i \leq 0 < 4g-1$, where $x_0, \ldots, x_{4g-1}$ denote the ordered vertices of B (i.e. $\arg(x_i) < \arg(x_{i+1})$) and $x_{4g} = x_0$ (for convenience).

Let $\varphi = \frac{2\pi}{8g}$, denote $x_0 = (-\tan(\varphi), -1)$ and $x_1 = (\tan(\varphi), -1)$ and let $\Delta : [0,1]^2 \to \mathbb{C} \cong \mathbb{R}^2$, be the affine map of $[0,1]^2$ onto the simplex $(x_0, x_1, 0)$ with $\Delta(0,0) = x_0$, $\Delta(1,0) = x_1$ and $\Delta([0,1] \times \{1\}) = \{0\}$. Furthermore denote by Δ_i the composition of Δ and the rotation around 0 by $\frac{2\pi}{4g}$ (cf. Figure 1). Then $B = \bigcup_{i=0}^{4g-1} \operatorname{im}(\Delta_i)$ and we have $\Delta_i(0,0) = x_i$ and $\Delta_i(1,0) = x_{i+1}$, whence

$$\Delta_i(s,0) = \begin{cases} \gamma_i(s) & \text{if } [i] \in \{[0],[1]\} \text{ in } \mathbb{Z}_4 \\ \gamma_i(1-s) & \text{if } [i] \in \{[2],[3]\} \text{ in } \mathbb{Z}_4 \end{cases}$$

for $s \in [0,1]$. If we define an equivalence relation R by $\gamma_i(s) \sim \gamma_{i+2}(s)$ for $0 \leq i \leq 4g-1$, $[i] \in \{[0],[1]\}$ in $\mathbb{Z}_4$ and $s \in [0,1]$, then the classification in [Mas67, Theorem 5.1] implies that $M \cong B/R$.

Lemma III.14. *If $\mathcal{P} = (K, \pi, P, M)$ is a continuous K-principal bundle, M is a compact orientable surface of genus $g \geq 1$ identified with a quotient of B as in Remark III.13 and K is connected, then there exist $\gamma \in C_*(\mathbb{S}^1, K)$ and $\sigma \in C(B, P)$ such that*

$$\begin{aligned} &\sigma(\gamma_i(s)) = \sigma(\gamma_{i+2}(s)) && \text{if } i < 4g-3,\ [i] \in \{[0],[1]\} \text{ and} \\ &\sigma(\gamma_{4g-3}(s)) = \sigma(\gamma_{4g-1}(s)) \cdot \gamma(s) \end{aligned}$$

for $s \in [0,1]$, where we denote by $C_*(X,Y)$ the space of base-point preserving maps from X to Y and choose $\{0,1\} \in \mathbb{S}^1 \cong [0,1]/\{0,1\}$ as the base-point of $\mathbb{S}^1$.

Proof. We adopt the notation introduced in Remark III.13 and denote by $q : B \to M$ the corresponding quotient map. Furthermore we write Δ_i (respectively γ_i) for the map from $[0,1]^2$ (respectively $[0,1]$) to B as well as for its image in B.

Lemma III.12 yields that $P|_{q(\gamma_i)}$ is trivial, hence there exist continuous maps $\sigma_i : \gamma_i \to P$ satisfying $\sigma_i(\gamma_i(s)) = \sigma_{i+2}(\gamma_{i+2}(s))$ if $[i] \in \{[0],[1]\}$, $\sigma_i(\gamma_i(0)) = \sigma_i(\gamma_i(1))$ if $0 \leq i \leq 4g-3$ and $\pi \circ \sigma_i = q|_{\gamma_i}$. Then [Bre93, Theorem VII.6.4] and [Bre93, Corollary VII.6.12] imply that we may inductively construct a continuous map

$$\sigma' : \bigcup_{i=0}^{4g-2} \Delta_i \to P.$$

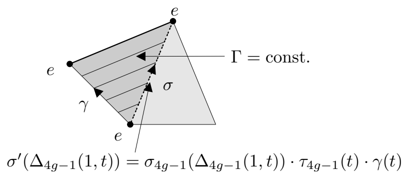

Figure 2: Construction of Γ

(cf. the following construction of σ''). For the extension of σ' to B we apply [Bre93, Theorem VII.6.4] to the diagram

$$\begin{array}{ccc} L & \xrightarrow{\ f\ } & P \\ \Big\downarrow \text{incl} & & \Big\downarrow \pi \\ \Delta_{4g-1} & \xrightarrow{\ q\ } & M \end{array}$$

where L denotes the subcomplex $\gamma_{4g-1} \cup (\Delta_{4g-1} \cap \Delta_{4g-2})$ and f is defined by $f(\gamma_{4g-1}(s)) = \sigma'(\gamma_{4g-3}(s))$ and $f(\Delta_{4g-1}(0,t)) = \sigma'(\Delta_{4g-2}(1,t))$. This yields a continuous map $\sigma'' : \Delta_{4g-1} \to P$ and we define γ by

$$\sigma'(\Delta_0(0,t)) = \sigma''(\Delta_{4g-1}(1,t)) \cdot \gamma(t). \tag{8}$$

This defines a continuous map γ on $[0,1]$ and since $q : B \to M$ maps all vertices to one single point in M, we have $\gamma(0) = e$. Due to $\Delta_i(s,1) = \Delta_i(s',1)$ for all $s, s' \in [0,1]$ we also have $\sigma''(\Delta_{4g-1}(s,1)) = \sigma'(\Delta_0(s',1))$ and thus $\gamma(1) = \gamma(0) = e$, whence $\gamma \in C_*(\mathbb{S}^1, K)$. Now define $\Gamma : \Delta_{4g-1} \to K$ by

$$\Gamma(\lambda\Delta_{4g-1}(1,t) + (1-\lambda)\Delta_{4g-1}(1-t,0)) = \gamma(t) \text{ for } \lambda \in [0,1],$$

which is continuous and satisfies $\Gamma(\Delta_{4g-1}(0,t)) = e$, $\Gamma(\Delta_{4g-1}(1,t)) = \gamma(t)$ and $\Gamma(\Delta_{4g-1}(s,0)) = \gamma(1-s)$ (cf. Figure 2). Then

$$\sigma : B \to P,\ \ \Delta_i(s,t) \mapsto \begin{cases} \sigma'(\Delta_i(s,t)) & \text{if } i < 4g-1 \\ \sigma''(\Delta_{4g-1}(s,t)) \cdot \Gamma(\Delta_{4g-3}(s,t)) & \text{if } i = 4g-1 \end{cases}$$

has the desired properties. □

Remark III.15. The preceding lemma provides an explicit mappings from the classification

$$\mathrm{Bun}(K,M) \cong [M, BK] \cong H^2(M, \pi_1(K)) \cong \mathrm{Hom}(H_2(M), \pi_1(K)) \cong \pi_1(K),$$

of K-principal bundles over M with connected K (where BK is the classifying space of K). The first isomorphism is [Hus66, Theorem 4.13.1], the second is a consequence of [Bre93, Corollary VII.13.16], the third is [Bre93, Theorem V.7.2] and the fact that $H_1(M) = \mathbb{Z} \oplus \mathbb{Z}$ is projective, and the last isomorphism is $H_2(M) = \mathbb{Z}$. In this sense $[\gamma] \in \pi_0(C_*(\mathbb{S}^1, K)) \cong \pi_1(K)$ can be seen as obstruction for the existence of a global section. Since the bundle is determined by g and $\gamma \in C_*(\mathbb{S}^1, K)$, we will denote it shortly by $\mathcal{P}_{g,\gamma}$.

Lemma III.16. *If $\mathcal{P}_{g,\gamma}$ denotes a continuous K-principal bundle as in Remark III.15, then the continuous gauge group $\mathrm{Gau_c}(\mathcal{P}_{g,\gamma}) \cong C(P,K)^K$ is isomorphic to*

$$G_{g,\gamma} := \{f \in C(B,K) : f(\gamma_i(s)) = f(\gamma_{i+2}(s)) \text{ if } i < 4g-3, [i] \in \{[0],[1]\} \text{ in } \mathbb{Z}_4 \text{ and }$$
$$f(\gamma_{4g-3}(s)) = \gamma(s)^{-1} f(\gamma_{4g-1}(s))\gamma(s) \text{ for } s \in [0,1]\}.$$

Proof. The pull-back $\sigma^* : C(P,K)^K \to C(B,K)$ provides the desired isomorphism. □

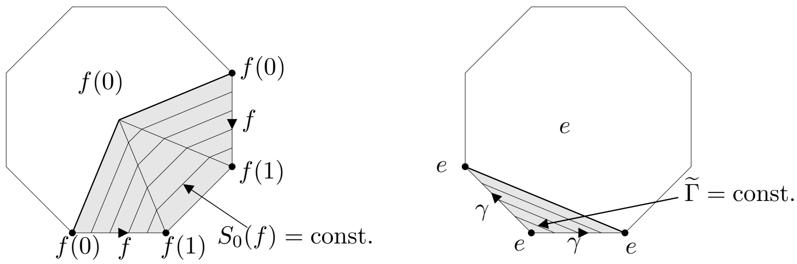

Figure 3: Construction of $S_1(f)$ and $\widetilde{\Gamma}$

Proposition III.17. *If $\mathcal{P}_{g,\gamma}$ denotes a continuous K-principal bundle as in Remark III.15, then the normal subgroup*

$$(G_{g,\gamma})_* := \{f \in G_{g,\gamma} : f(\Delta_0(0,0)) = e\}$$

is homeomorphic to the direct product

$$C_*(\mathbb{S}^2, K) \times C_*(\mathbb{S}^1, K)^{2g}.$$

Proof. First we remark that the vanishing of $f \in G_{g,\gamma}$ in $\Delta_0(0,0)$ implies that f vanishes on each $x_0, \ldots, x_{4g-1}$. We identify $C_*(\mathbb{S}^2, K)$ with the normal subgroup $N := \{f \in G_{g,\gamma} : f|_{\partial B} \equiv e\}$. Then N is the kernel of the restriction map

$$\text{res} : (G_{g,\gamma})_* \to C_*(\mathbb{S}^1, K)^{2g}, \quad f \mapsto (f|_{\gamma_{4i}}, f|_{\gamma_{4i+1}})_{i=0,\ldots,g-1}.$$

We now construct a continuous splitting of this map. For $f \in C([0,1], K)$ and $0 \leq i \leq 4g-3$ we define by

$$S_i(f)(\Delta_j(s,t)) = \begin{cases} f(s) & \text{if } \Delta_j(s,t) = \lambda\Delta_i(s,0) + (1-\lambda)\Delta_i(1, 1-s) \text{ for } \lambda \in [0,1] \\ f(1-t) & \text{if } j = i+1 \\ f(s) & \text{if } \Delta_j(s,t) = \lambda\Delta_{i+2}(0, 1-s) + (1-\lambda)\Delta_{i+2}(1-s, 0) \text{ for } \lambda \in [0,1] \\ f(0) & \text{else} \end{cases}$$

a continuous map on B (c.f. Figure 3). We now set

$$\widetilde{\Gamma}\big(\lambda\Delta_{4g-1}(1-s, 0) + (1-\lambda)\Delta_0(s,0)\big) = \gamma(s) \text{ for } s, \lambda \in [0,1] \tag{9}$$

and since $\gamma(0) = e$ we may extend $\widetilde{\Gamma}$ to a continuous function on B by setting if to e if it is not defined in (9). Since $S_i(f)$ depends continuously on f (consider the topology of compact convergence), the map

$$\sigma : C_*(\mathbb{S}^1, K)^{2g} \to G_{g,\gamma}, \quad (f_i)_{i=0,\ldots,2g-1} \mapsto S_0(f_0) \cdot S_1(f_1) \cdot S_4(f_2) \cdot S_5(f_3) \cdot S_8(f_4) \cdot \ldots$$
$$\ldots \cdot S_{4g-7}(f_{2g-3}) \cdot S_{4g-4}(f_{2g-2}) \cdot \widetilde{\Gamma}^{-1} \cdot S_{4g-3}(f_{2g-1}) \cdot \widetilde{\Gamma}$$

defines a continuous section of the restriction map. □

Proposition III.18. *If $\mathcal{P}_{g,\gamma}$ denotes a continuous K-principal bundle as in Remark III.15, then we have*

$$\pi_n((G_{g,\gamma})_*) \cong \pi_{n+2}(K) \oplus \pi_{n+1}(K)^{2g}.$$

Proof. Since for any Hausdorff space X

$$\pi_n(C_*(\mathbb{S}^n, X)) = \pi_0(C_*(\mathbb{S}^k, C_*(\mathbb{S}^n, X)) \cong \pi_0(C_*(\mathbb{S}^k \wedge \mathbb{S}^n, X)) \cong \pi_0(C_*(\mathbb{S}^{k+n}, X)) = \pi_{n+n}(X),$$

this is an immediate consequence of Proposition III.17 □

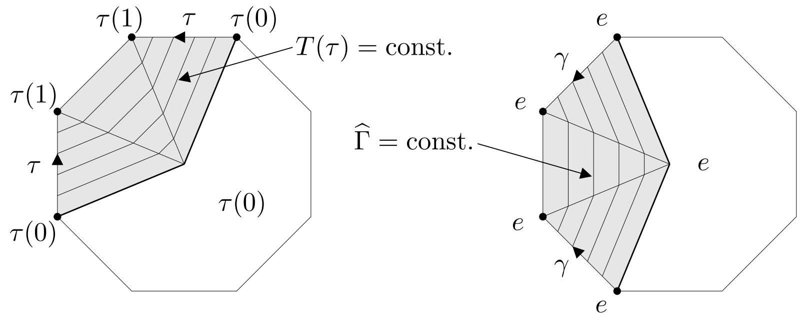

Figure 4: Construction of $T(\tau)$ and $\widehat{\Gamma}$

Lemma III.19. *If K is a locally contractible topological group, then these exist an open unit neighbourhood $V \subseteq K$ and a continuous map*

$$V \ni k \mapsto \tau_k \in C([0,1],K)_c$$

such that $\tau_k(0) = e$, $\tau_k(1) = k$ and $\tau_e \equiv e$.

Proof. Since K is locally contractible, there exist open unit neighbourhoods U, V and a continuous map $F : [0,1] \times V \to U$ such that $F(0,k) = e$, $F(1,k) = k$ for all $k \in V$ and $F(t,e) = e$ for all $t \in [0,1]$. For $k \in V$ we set $\tau_k := F(\cdot, k)$, which is a continuous path and satisfies $\tau_k(0) = e$, $\tau_k(1) = k$ and $\tau_e(t) = e$ for $t \in [0,1]$.

Let $U' \subseteq C([0,1],K)_c$ be an open unit neighbourhood, which we may assume to be $C([0,1],W)$ for some open unit neighbourhood W. Since F is continuous, there exists a unit neighbourhood V' such that $F([0,1] \times V') \subseteq W$, whence $\tau_k \in U'$ for all $k \in V \cap V'$. Thus the map $V \ni k \mapsto \tau_k \in C([0,1],K)_c$ is continuous. $\square$

Lemma III.20. *If $\mathcal{P}_{g,\gamma}$ denotes a continuous K-principal bundle as in Remark III.15 and if K is connected and locally contractible, then the sequence*

$$0 \to (G_{g,\gamma})_* \stackrel{\text{incl}}{\longrightarrow} G_{g,\gamma} \stackrel{\text{ev}_0}{\longrightarrow} K \to 0,$$

where ev_0 its the evolution map in $\Delta_0(0,0)$, is exact and admits local continuous sections, i.e. is an extension of topological groups.

Proof. Since ev_0 is a homomorphism of topological groups, it suffices to construct an continuous section on a identity neighbourhood. Let $V \ni k \mapsto \tau_k \in C([0,1]K)$ be the map from Lemma III.19. If for $\tau \in C([0,1],K)$ we denote by τ^- the path $s \mapsto \tau(1-s)$ and if we set $T(\tau) := (S_{4g-4})(\tau)$ and $\widehat{\Gamma} = S_{4g-3}(\gamma)$ (with S_{4g-4} and S_{4g-3} defined as in the proof of Proposition III.17, cf. Figure 4), we obtain the map

$$V \ni x \mapsto T(\tau_x) \cdot \widehat{\Gamma} \cdot T({\tau_x}^-) \cdot \widehat{\Gamma}^{-1} \in G_{g,\gamma}$$

as a local continuous section of ev_0. $\square$

Lemma III.21. *If $\mathcal{P} = (K, \pi, P, \mathbb{S}^r)$ is a continuous K-principal bundle over the r-sphere, K is connected and locally contractible and $C_*(P,K)^K$ denotes the group of continuous pointed gauge transformations (i.e those gauge transformations that vanish on $p_0 \cdot K$, where $p_0 \in P$ is the base-point), then the sequence*

$$0 \to C_*(P,K)^K \to C(P,K)^K \stackrel{\text{ev}_{p_0}}{\longrightarrow} K \to 0$$

defines an extension of topological groups. Furthermore we have $C_*(P,K)^K \cong C_*(\mathbb{S}^r, K)$.

Proof. Denote by $q : \mathbb{D}^n \to \mathbb{S}^n$ the usual quotient map, i.e. q maps $\partial\mathbb{D}^n$ to the base-point $x_0 = \pi(p_0)$ in $\mathbb{S}^n$ and denote by $x \mapsto \bar{x}$ the antipode map on $\mathbb{S}^r$. Then $\mathbb{S}^r - \bar{x}_0$ is contractible and thus there exists a continuous section $\sigma : \mathbb{S}^r - \bar{x}_0 \to P$ and a continuous map $k : P\backslash\pi^{-1}(\bar{x}_0) \to K$ satisfying $p = \sigma(\pi(p))\cdot k(p)$. Denoting by $V \ni k \mapsto \tau_k \in C([0,1],K)$ the map from Lemma III.19, the map

$$V \ni k \mapsto \Big(T_k : \mathbb{S}^r \to K,\ \ x \mapsto \tau_k(\|x\|)\Big) \in C(\mathbb{D}^r, K)_c$$

defines a continuous map on some unit neighbourhood $V \subseteq K$ with $T_k(k)|_{\partial\mathbb{D}^r} \equiv k$ and $T_k(0) = e$. Then

$$\widetilde{T}_k : P\backslash\pi^{-1}(\bar{x}_0),\ \ p \mapsto k(p)^{-1} \cdot T_k(\pi(p)) \cdot k(p)$$

defines a continuous map satisfying $\widetilde{T}_k(p\cdot k) = k^{-1}\widetilde{T}_k(p)\cdot k$ wherever defined. But since $P\backslash\pi^{-1}(\bar{x}_0)$ is dense in P, $\bar{x}_0 = 0$ and $T_k(0) = e$ we may extend $\widetilde{T}_k$, by setting it to e on $\pi^{-1}(\bar{x}_0)$, to an element of $C(P,K)^K$ which we also denote by $\widetilde{T}_k$. Since $k \mapsto T_k$ is continuous we also have that

$$V \ni k \mapsto \widetilde{T}_k \in C(P,K)^K$$

is continuous and it clearly defines a section of ev_{p_0}. Since K is connected, the same construction also shows surjectivity ov ev_{p_0} and hence that the sequence is an extension of topological groups.

The previous extension-argument based only on the fact that $T_k(\bar{x}_0) = e$ and thus the same argument with $\bar{x}_0$ substituted by x_0 shows that $C_*(\mathbb{S}^r, K) \cong C_*(P,K)^K$. □

Proposition III.22. *If $\mathcal{P}_{g,\gamma}$ denotes a continuous K-principal bundle as in Remark III.15 and K is locally contractible, then we have for each $n \in \mathbb{N}_0$ exact sequences*

$$0 \to \pi_{n+2}(K) \oplus \pi_{n+1}(K)^{2g} \to \pi_n(G_{g,\gamma}) \to \pi_n(K) \to 0.$$

If $\mathcal{P} = (K, \pi, P, \mathbb{S}^r)$ is a continuous K-principal bundle over the r-sphere, K is connected and locally contractible, then we have for each $n \in \mathbb{N}_0$ exact sequences

$$0 \to \pi_{n+r}(K) \to \pi_n(C(P,K)^K) \to \pi_n(K) \to 0.$$

Proof. First we recall

$$\pi_n(C_*(\mathbb{S}^r,K)) \cong \pi_0(C_*(\mathbb{S}^n, C_*(\mathbb{S}^r,K))) \cong \pi_0(C_*(\mathbb{S}^n \wedge \mathbb{S}^r, K)) \cong \pi_0(C_*(\mathbb{S}^{n+r},K)) \cong \pi_{n+r}(K).$$

Since the exact sequences from Lemma III.20 and III.21 are fibrations [Bre93, Corollary VII.6.12], the exact homotopy sequence from [Bre93, Theorem VII.6.7] and Proposition III.18 yield long exact sequences

$$\ldots \to \pi_{n+1}(K) \xrightarrow{\delta_{n+1}} \pi_n((G_{g,\gamma})_*) \to \pi_n(G_{g,\gamma}) \to \pi_n(K) \xrightarrow{\delta_n} \pi_{n-1}((G_{g,\gamma})_*) \to \ldots$$
$$\ldots \to \pi_{n+1}(K) \xrightarrow{\delta_{n+1}} \pi_{n+r}(K) \to \pi_n(C(P,K)^K) \to \pi_n(K) \xrightarrow{\delta_n} \pi_{n+r-1}(K) \to \ldots$$

We show that the connecting homomorphisms δ_n vanish for $n \in \mathbb{N}^+$. Since $\pi_n((G_{g,\gamma})_*) \cong \pi_{n+2}(K)\oplus\pi_{n+1}(K)^{2g}$ this will prove the assertion. The only fact that we will use is that the local sections from Lemma III.20 and Lemma III.21 both yield the constant map e when evaluated at $e \in K$. Hence we assume from now on that we are given an extension of topological groups $G_* \hookrightarrow G \overset{\mathrm{ev}}{\twoheadrightarrow} K$, where G consists of K-valued continuous mappings. The connecting homomorphism is constructed as follows (cf. [Bre93, Theorem VII.6.7]). For an arbitrary $[f] \in \pi_n(K)$, i.e. $f \in C_*(\mathbb{S}^n, K)$, and the usual quotient map $q : \mathbb{D}^n \to \mathbb{S}^n$, i.e. q maps $\partial\mathbb{D}^n$ to the base-point in $\mathbb{S}^n$, we have $f' := f \circ q \in C(\mathbb{D}^n, K)$ with $f'|_{\partial\mathbb{D}^n} \equiv e$. Since $G_* \hookrightarrow G \overset{\mathrm{ev}}{\twoheadrightarrow} K$ is a fibration we can lift f' to a map $F \in C(\mathbb{D}^n, G)$ with $\mathrm{ev}_0 \circ F = f'$, whence $F|_{\partial\mathbb{D}^n}$ has values in $\ker(\mathrm{ev}) = G_*$. With $\partial\mathbb{D}^n \cong \mathbb{S}^{n-1}$ we then have $\delta_n([f]) = [F|_{\partial\mathbb{D}^n}] \in \pi_{n-1}(G_*)$.

We construct a lift F of f' as follows (cf. [Bre93, Theorem VII.6.11]). Since $\mathrm{ev} : G \to K$ is locally trivial K has an open cover of trivial sets and we pull this cover with f^{-1} back to an

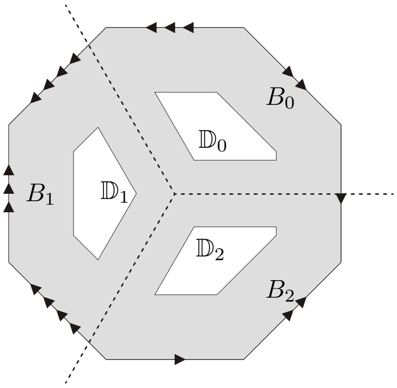

Figure 5: A surface with boundary

open cover of $\mathbb{D}^n$. Then compactness yields a closed cover of $\mathbb{D}^n = [0,1]^n$ consisting of closed cubes $W_{(i_1,\dots,i_n)}$ with $(i_1,\dots,i_n) \in \{1,\dots,m\}^{\{1,\dots,n\}}$ such that each each $W_{i_1,\dots,i_n}$ is a translate of $[0,\frac{1}{m}]^n$, $W_{i,\dots,1} = [0,\frac{1}{m}]^n$, $W_{i_1,\dots,i_l+1,\dots,i_n} = (0,\dots,\frac{1}{m},\dots,0) + W_{i_1,\dots,i_l,\dots,i_n}$ and that each $f'(W_{i_1,\dots,i_l,\dots,i_n})$ is contained in a trivial subset of K. Order the $(i_1,\dots,i_n)$ lexicographically, i.e. $(i_1,\dots,i_n) < (j_1,\dots,j_n)$ if $i_k = j_k$ for $k \leq l+1$ and $i_l < j_j$. This results in a total order on the $(i_1,\dots,i_n)$ and we will refer to these tuples by a single letter i or j and to their order by $i < j$.

Since $f(W_i)$ is contained in a trivial subset of K there exist sections $\sigma_i : f(W_i) \to G$ and since $f'|_{\partial\mathbb{D}^n} \equiv e$ we may assume that σ_i is the section defined on the unit neighbourhood V from the proof of Lemma III.20 or Lemma III.21 if $i_k = 0$ or $i_k = m$ for some $k \leq n$. We thus may assume that $\sigma_i(f'(x_1,\dots,x_n))$ is the constant map e if $x_k = 0$ for some $k \leq n$ since then $f'(x_1,\dots,x_n) = e$.

We define the lift F inductively. Set $F|_{W_1} = \sigma_1 \circ f|_{W_1}$ and assume in the following that F is defined on $W' := \cup_{j<i} W_j$. Then F and $F_i := \sigma_i \circ f'|_{W_i}$ may differ on $W' \cap W_i$, which is a union of sets of the form

$$U_l := [\frac{i_1}{m}, \frac{i_1+1}{m}] \times \ldots \times \{i_l\} \times \ldots \times [\frac{i_n}{m}, \frac{i_n+1}{m}]$$

Let $\tau : U_l \to G_*$ correspond to the difference between F and F_i on U_l, i.e. $F_i \cdot \tau = F$ pointwise on U_i and extend τ to W_i by defining it to be constant on lines orthogonal to U_l. Then $F' = F_i \cdot \tau$ is a continuous map on W_i that coincides with F on U_l. Note that F' and F_i coincide (i.e. $\tau = e$) wherever F and F_i coincide. Iterating this procedure yields a map $\widetilde{F} : W_i \to G$ that lifts $f'|_{W_i}$ (since we modified F_i only by multiplying it with elements of the fibre G_*) and that coincides with F on $W' \cap W_i$ and thus defines a lift of f on $W' \cup W_i$.

Due to the assumption that $\sigma_i \circ f_i(x_1,\dots,x_n) \equiv e$ if $x_k = 0$ for some $k \leq n$, the maps $\sigma_i \circ f|_{W_i}$ coincide on $\partial\mathbb{D}^n$ and thus the constructed lift F yields only the constant map e on $\partial\mathbb{D}^n$. Hence $\delta_n([f]) = [F|_{\partial\mathbb{D}^n}]$ is trivial. □

Lemma III.23. *If $\mathcal{P} = (K,\pi,P,M)$ is a continuous K-principal bundle, M is a compact connected orientable surface with non-empty boundary and K is connected, then $\mathcal{P}$ is trivial.*

Proof. We may assume that M is obtained by cutting m arbitrary open disjoint disks $\mathbb{D}_0,\dots,\mathbb{D}_{m-1}$ out of a compact connected orientable surface [Mas67, Theorem 10.1], cf. Figure 5. If the genus $g \geq 1$ we adopt the notation from Remark III.13. Denote by $B_i \subseteq B$ the closure of $\{x \in B : \frac{2\pi}{m} i < \arg(x) < \frac{2\pi}{m}(i+1)\}$. We may arrange the disks as subsets of B that such that $\mathbb{D}_i \subseteq \operatorname{int}(B_i)$. Since $\mathcal{P}|_{\partial B_i}$ is trivial (cf. Lemma III.12), there exist continuous sections $\sigma_i : \partial B_i \to P$, which we may assume to coincide on $B_i \cap B_{i+1}$ if $i < m-1$ and on $B_0 \cap B_{m-1}$. Then [Bre93, Theorem VII.6.4] implies that we may extend each σ_i to a section $S_i : B_i \to P$ and thus

$$B \backslash (\mathbb{D}_1 \cup \dots \cup \mathbb{D}_m) \ni x \mapsto S_i(x) \in P$$

if $x \in B_i$ is a well-defined continuous section. Since the construction does not depend on how the points on the boundary are identified, the same construction carries over to the case where ∂B is identified to one point in the quotient, i.e. $g = 0$ □

Remark III.24. Since $H^2(M) = 0$ one can also obtain the result of the preceding lemma with the argumentation from Remark III.15.

Proposition III.25. *If $\mathcal{P} = (K, \pi, P, M)$ is a continuous K-principal bundle, M is a compact connected orientable surface with non-empty boundary obtained by cutting m open disjoint disks out of a surface of genus $g \geq 1$ and K is connected, then we have*

$$\pi_n(C(P,K)^K) \cong \pi_{n+1}(K)^{2g+m-1} \oplus \pi_n(K).$$

Proof. Since $\mathcal{P}$ is trivial we have $C(P,K)^K \cong C(M,K) \cong C_*(M,K) \rtimes K$ whence it suffices to show $\pi_n(C_*(M,K)) \cong \pi_{n+1}(K)^{2g+m-1}$. Note that $m \geq 1$ since m is the number of connected components of ∂M (cf. [Mas67, Theorem 10.1]). If M is obtained from the surface M', then we may construct a continuous section of the restriction map

$$\text{res} : C_*(M,K) \to C_*(\mathbb{S}^1,K)^{2g}, \ \ f \mapsto (f|_{\gamma_{4i}}, f|_{\gamma_{4i+1}})_{i=0,\ldots,g-1}$$

with the continuous section σ from the proof of Lemma III.17 (for the trivial K-principal bundle over M', i.e. $\gamma \equiv e$) and restricting $\sigma(f_0,\ldots,f_{g-1}) \in C_*(M',K)$ to M. The kernel of the restriction map consists of those pointed continuous functions on M which vanish on ∂B and hence it is isomorphic to $C_*(D_{m-1},K)$, where D_{m-1} denotes the closed unit disk $\overline{\mathbb{D}}$ in $\mathbb{R}^2$ with $m-1$ disjoint open disks $\mathbb{D}_1,\ldots,\mathbb{D}_{m-1}$ cut out.

We may assume w.l.o.g. that the base-point of D_{m-1} is in $\partial\mathbb{D}_{m-1}$. Since $\partial\mathbb{D}_{m-1} \subseteq \overline{\mathbb{D}}\backslash\mathbb{D}_{m-1}$ is a retract of $\overline{\mathbb{D}}\backslash\mathbb{D}_{m-1}$ we may embed $C_*(\mathbb{S}^1,K)$ into $C_*(D_{m-1},K)$ such that the restriction map

$$C_*(D_{m-1},K) \to C_*(\partial\mathbb{D}_{m-1},K) \cong C_*(\mathbb{S}^1,K)$$

has a continuous global section. The kernel of this map are those pointed continuous functions on D_{m-1} which vanish on $\partial\mathbb{D}_{m-1}$ and hence it is isomorphic to $C_*(D_{m-2},K)$. Thus we get inductively

$$C_*(M,K) \cong C_*(\mathbb{S}^1,K)^{2g} \times C_*(\mathbb{S}^1,K)^{m-1} \times C_*(\overline{\mathbb{D}},K).$$

Since $\overline{\mathbb{D}} \cong [0,1]^2$, $\mathbb{S}^k = \mathbb{S}^{k-1} \wedge \mathbb{S}$, $[0,1]^2 \wedge [0,1]^k = [0,1]^{k+2}/A \cong [0,1]^{k+2}$, where A is some contractible space in the boundary of $[0,1]^{k+2}$ and $\mathbb{S}^1 \wedge [0,1] \cong [0,1]^2$, we inductively get $\mathbb{S}^k \wedge \overline{\mathbb{D}} \cong [0,1]^{k+2}$. Since $[0,1]^{k+2}$ is contractible this implies

$$\pi_n(C_*(\overline{\mathbb{D}},K)) = \pi_0(C_*(\mathbb{S}^k, C_*(\overline{\mathbb{D}},K))) \cong \pi_0(C_*(\mathbb{S}^k \wedge \overline{\mathbb{D}},K)) = \{e\}$$

and thus

$$\pi_n(C_*(\mathbb{S}^1,K)) \cong \pi_0(C_*(\mathbb{S}^k \wedge \mathbb{S}^1,K)) = \pi_0(C_*(\mathbb{S}^{k+1},K)) = \pi_{n+1}(K)$$

yields the assertion. □

Theorem III.26 (Homotopy groups of Gau($\mathcal{P}$)). *Let $\mathcal{P} = (K,\pi,P,M)$ be a smooth K-principal bundle, M be a compact orientable surface of genus $g \geq 1$ and K be locally exponential and connected. If M has empty boundary, then we have for each $n \in \mathbb{N}_0$ exact sequences*

$$0 \to \pi_{n+2}(K) \oplus \pi_{n+1}(K)^{2g} \to \pi_n(\text{Gau}(\mathcal{P})) \to \pi_n(K) \to 0.$$

If ∂M in non-empty and has m components, then we have for each $n \in \mathbb{N}_0$ isomorphisms

$$\pi_n(\text{Gau}(\mathcal{P})) \cong \pi_{n+1}(K)^{2g+m-1} \oplus \pi_n(K).$$

If $\mathcal{P} = (K,\pi,P,\mathbb{S}^r)$ is a smooth K-principal bundle over the r-sphere and K is connected and locally exponential, then we have for each $n \in \mathbb{N}_0$ exact sequences

$$0 \to \pi_{n+r}(K) \to \pi_n(\text{Gau}(\mathcal{P})) \to \pi_n(K) \to 0.$$

Proof. Since a Lie group is locally contractible, this is Lemma III.16, Proposition III.22, Proposition III.25 and Theorem III.11. □

Remark III.27. If M is a non-orientable surface of genus $g \geq 1$, K is a connected topological group and $\mathcal{P} = (K, \pi, P, M)$ is a continuous K-principal bundle, then it is homeomorphic to a quotient of a regular Polygon B with $2g$ vertices. The quotient B/R is constructed similarly as in Remark III.13 by the equivalence relation R defined by $\gamma_i(s) \sim \gamma_{i+1}(s)$ if $i = 0$ in $\mathbb{Z}_2$. We thus obtain a map $\sigma : B \to P$ satisfying $\sigma(\gamma_{2i}(s)) = \sigma(\gamma_{2i+1}(s))$ if $i < g-1$, $i = 0$ in $\mathbb{Z}_2$ and $\sigma(\gamma_{2g-2}(s)) = \sigma(\gamma_{2-1}(s)) \cdot \gamma(s)$ with $\gamma \in C_*(\mathbb{S}^1, K)$.

The remainder constructions of this section carry over in the same way and we thus obtain the same results (i.e. Proposition III.22, Proposition III.25 and Theorem III.26) for the non-orientable case.

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