Discontinuous Non-Linear Mappings on Locally Convex Direct Limits

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Abstract

Let M be a non-compact, finite-dimensional manifold of positive dimension, $E \neq \{0\}$ be a locally convex space, and $C^\infty_c(M,E)$ be the space of compactly supported smooth E-valued maps on M. We describe discontinuous mappings $C^\infty_c(M,E) \to C^\infty_c(M,\mathbb{R})$ whose restriction to $C^\infty_K(M,E) := \{\gamma \in C^\infty(M,E)\colon \gamma|_{M\setminus K} = 0\}$ is smooth, for each compact subset K of M.

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Introduction

Let $E_1 \subseteq E_2 \subseteq \cdots$ be an ascending sequence of locally convex spaces which does not become stationary, and such that E_{n+1} induces the given topology on E_n, for each n. It is a well-known phenomenon that the topology on $E := \bigcup_{n\in\mathbb{N}} E_n$ making E the direct limit of the spaces E_n in the category of locally convex spaces (and continuous linear maps) can be properly coarser than the topology making E the direct limit of its subspaces E_n in the category of topological spaces. For example, this phenomenon occurs whenever each E_n is an infinite-dimensional Fréchet space (cf. [13, Prop. 4.26 (ii)]). In particular, the locally convex direct limit topology on the space $C^\infty_c(\mathbb{R}) = \varinjlim C^\infty_{[-n,n]}(\mathbb{R})$ of test functions is properly coarser than the topology of direct limit topological space (cf. also [3, p. 506]).

So, for abstract reasons, discontinuous mappings on the space of test functions $C^\infty_c(\mathbb{R})$ are known to exist whose restriction to $C^\infty_{[-n,n]}(\mathbb{R})$ is continuous for each $n \in \mathbb{N}$. In this article, we describe such a mapping explicitly, whose restriction to $C^\infty_{[-n,n]}(\mathbb{R})$ is not only continuous but actually smooth (Proposition 2.2). More generally, for every σ-compact, non-compact, finite-dimensional smooth manifold M of positive dimension and locally convex space $E \neq \{0\}$, we construct a discontinuous map $f\colon C^\infty_c(M,E) \to C^\infty_c(M,\mathbb{R})$ whose restriction to $C^\infty_K(M,E)$ is smooth, for each compact subset K of M. An analogous result is obtained for the space $C^\infty_c(M,E)$ of compactly supported smooth sections in a bundle of locally convex spaces $E \to M$ over M, with non-trivial fibre (Theorem 3.2).

Further developments. The preceding result is useful for the investigation of direct limit properties of infinite-dimensional Lie groups. As shown in [11], it entails that there are discontinuous (and hence non-smooth) mappings on the Lie group $\mathrm{Diff}_c(M) = \bigcup_K \mathrm{Diff}_K(M)$ of compactly supported smooth diffeomorphisms of M (as in [14] or [10]), whose restriction to $\mathrm{Diff}_K(M) := \{\phi \in \mathrm{Diff}(M)\colon \phi|_{M\setminus K} = \mathrm{id}_{M\setminus K}\}$ is smooth, for each compact subset $K \subseteq M$. A similar pathology occurs for the Lie group $C^\infty_c(M,G) = \bigcup_K C^\infty_K(M,G)$ of compactly supported smooth maps with values in a non-discrete finite-dimensional Lie group (as in [5]). In this way, we obtain one half of the following table, which describes whether $\mathrm{Diff}_c(M) = \varinjlim \mathrm{Diff}_K(M)$ and $C^\infty_c(M,G) = \varinjlim C^\infty_K(M,G)$ holds in the categories shown:

category \ group	$C^\infty_c(M,G)$	$\mathrm{Diff}_c(M)$
Lie groups	yes	yes
topological groups	yes	yes
smooth manifolds	no	no
topological spaces	no	no

For the proof, see [11] (cf. also [18] for related results).

The present constructions of pathological mappings are complemented by investigations in [8]–[10] (cf. also [7]). In these articles, a mild additional property is introduced which ensures that a map $f : C^\infty_c(M,E) \to C^\infty_c(N,F)$ between spaces of test functions (or compactly supported sections) satisfying this property (an "almost local" map) is indeed smooth if and only if it is smooth on $C^\infty_K(M,E)$ for each K. In contrast to these mappings, the pathological examples presented here are extremely non-local.

In the final section, we describe examples of discontinuous bilinear mappings which are continuous (and hence analytic) on each step of a directed sequence of subspaces.

1 Preliminaries

In this article, we are working in the setting of infinite-dimensional differential calculus known as Keller's C^∞_c-theory, based on smooth maps in the sense of Michal-Bastiani (see [4], [12], [14], [16] for further information).

Definition 1.1 Let E, F be locally convex spaces and $f : U \to F$ be a mapping, defined on an open subset U of E. We say that f is *of class* C^0 if f is continuous. If f is a continuous map such that the two-sided directional derivatives

$$df(x,v) = \lim_{t\to 0} \tfrac{1}{t}\,(f(x+tv) - f(x))$$

exist for all $(x,v) \in U \times E$, and the map $df : U \times E \to F$ so defined is continuous, then f is said to be *of class* C^1. Recursively, given $k \in \mathbb{N}$ we call f a mapping of class C^{k+1} if it is of class C^1 and df is of class C^k on the open subset $U \times E$ of $E \times E$. We set $d^{k+1}f := d(d^k f) = d^k(df) : U \times E^{2^{k+1}-1} \to F$ in this case. The function f is called *smooth* (or of class C^∞) if it is of class C^k for each $k \in \mathbb{N}_0$.

Definition 1.2 Let M be a finite-dimensional, σ-compact smooth manifold and E be a locally convex topological vector space. We equip the vector space $C^\infty(M,E)$ of E-valued smooth mappings γ on M with the topology of uniform convergence of $\partial^\alpha(\gamma \circ \kappa^{-1})$ on compact subsets of V, for each chart $\kappa : M \supseteq U \to V \subseteq \mathbb{R}^d$ of M and multi-index $\alpha \in \mathbb{N}_0^d$ (where $d := \dim(M)$). Given a compact subset $K \subseteq M$, we equip the vector subspace $C^\infty_K(M,E) := \{\gamma \in C^\infty(M,E) : \gamma|_{M\setminus K} = 0\}$ of $C^\infty(M,E)$ with the induced topology.

We give $C_c^\infty(M,E) := \bigcup_K C_K^\infty(M,E) = \varinjlim C_K^\infty(M,E)$ the locally convex direct limit topology. We abbreviate $C_c^\infty(M) := C_c^\infty(M,\mathbb{R})$, $C^\infty(M) := C^\infty(M,\mathbb{R})$, and $C_K^\infty(M) := C_K^\infty(M,\mathbb{R})$. Further details can be found, e.g., in [5].

2 Example of a discontinuous mapping on $C_c^\infty(\mathbb{R})$

We show that the map $f\colon C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R})$, $\gamma \mapsto \gamma\circ\gamma - \gamma(0)$ is discontinuous, although its restriction to $C_{[-n,n]}^\infty(\mathbb{R})$ is smooth, for each $n \in \mathbb{N}$.

The following fact is essential for our constructions. It follows from [13, Cor. 3.13] and is also a special case of [8, Prop. 11.3]. For the convenience of the reader, we offer a direct, elementary proof as an appendix.

Lemma 2.1 *The composition map*

$$\Gamma\colon C^\infty(\mathbb{R}^n,\mathbb{R}^m)\times C^\infty(M,\mathbb{R}^n)\to C^\infty(M,\mathbb{R}^m)\,,\qquad \Gamma(\gamma,\eta) := \gamma\circ\eta$$

is smooth, for each finite-dimensional, σ-compact smooth manifold M and $m,n\in\mathbb{N}_0$. $\square$

For the following proof, recall that the sets

$$\mathcal{V}(k,e) := \big\{\gamma \in C_c^\infty(\mathbb{R})\colon\ (\forall n\in\mathbb{Z})\,(\forall j\in\{0,\ldots,k_n\})\,(\forall x\in[n-\tfrac{1}{2},n+\tfrac{1}{2}])\ |\gamma^{(j)}(x)|<\varepsilon_n\big\}$$

form a basis of open zero-neighbourhoods for the topology on $C_c^\infty(\mathbb{R})$, where $k=(k_n)\in(\mathbb{N}_0)^{\mathbb{Z}}$ and $e=(\varepsilon_n)\in(\mathbb{R}^+)^{\mathbb{Z}}$ (cf. [17, § II.1]; see [5, Prop. 4.8]).

Proposition 2.2 $f\colon C_c^\infty(\mathbb{R})\to C_c^\infty(\mathbb{R})$, $\gamma\mapsto\gamma\circ\gamma-\gamma(0)$ *has the following properties:*

(a) *The restriction of f to a map $C_{[-n,n]}^\infty(\mathbb{R})\to C_c^\infty(\mathbb{R})$ is smooth (and hence continuous), for each $n\in\mathbb{N}$.*

(b) *f is discontinuous at $\gamma=0$.*

Proof. (a) Fix $n\in\mathbb{N}$; we have to show that $f|_{C_{[-n,n]}^\infty(\mathbb{R})}\colon C_{[-n,n]}^\infty(\mathbb{R})\to C_c^\infty(\mathbb{R})$ is smooth. The image of this map being contained in the closed vector subspace $C_{[-n,n]}^\infty(\mathbb{R})$ of $C_c^\infty(\mathbb{R})$, which also is a closed vector subspace of $C^\infty(\mathbb{R})$ (with the same induced topology), it suffices to show that the map $C_{[-n,n]}^\infty(\mathbb{R})\to C^\infty(\mathbb{R})$, $\gamma\mapsto\gamma\circ\gamma-\gamma(0)$ is smooth (see [9, Prop. 1.9] or [1, La. 10.1]). Now $\gamma\mapsto\gamma(0)$ being a continuous linear (and thus smooth) map, it suffices to show that $C_{[-n,n]}^\infty(\mathbb{R})\to C^\infty(\mathbb{R})$, $\gamma\mapsto\gamma\circ\gamma$ is smooth. This readily follows from Lemma 2.1.

(b) Consider the zero-neighbourhood $V := \mathcal{V}((|n|)_{n\in\mathbb{Z}},(1)_{n\in\mathbb{Z}})$ in $C_c^\infty(\mathbb{R})$. Let $k=(k_n)\in(\mathbb{N}_0)^{\mathbb{Z}}$ and $e=(\varepsilon_n)\in(\mathbb{R}^+)^{\mathbb{Z}}$ be arbitrary. We show that $f(\mathcal{V}(k,e))\not\subseteq V$. Since $f(0)=0$, this entails that f is discontinuous at $\gamma=0$. It is easy to construct a function $h\in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp}(h)\subseteq\,]-\frac{1}{2},\frac{1}{2}[$ and $h(x)=x^{k_0+1}$ for all $x\in[-\frac{1}{4},\frac{1}{4}]$. Then $rh\in\mathcal{V}(k,e)$ for some $r>0$. For $m\in\mathbb{N}$, we define $h_m\in C_c^\infty(\mathbb{R})$ via

$$h_m(x) := \frac{r}{m^{k_0}}h(mx).$$

Then $\operatorname{supp}(h_m) \subseteq]-\frac{1}{2m}, \frac{1}{2m}[$ and thus $h_m \in \mathcal{V}(k,e)$ since, for all $j = 0, \ldots, k_0$ and $x \in [-\frac{1}{2}, \frac{1}{2}]$, we have $|h_m^{(j)}(x)| = \frac{rm^j}{m^{k_0}} |h^{(j)}(mx)| < \varepsilon_0$. We now choose $n \in \mathbb{N}$ such that $n \geq k_0+2$. It is easy to construct a function $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi(x) = x - n$ for x in some neighbourhood of n in $\mathbb{R}$, and $\operatorname{supp}(\psi) \subseteq]n - \frac{1}{2}, n + \frac{1}{2}[$. Then $\phi := s \cdot \psi \in \mathcal{V}(k,e)$ for suitable $s > 0$. Choosing s small enough, we may assume that $\operatorname{im}(\phi) \subseteq [-1,1]$. The supports of ϕ and h_m being disjoint, we easily deduce from $\phi, h_m \in \mathcal{V}(k,e)$ that also $\gamma_m := \phi + h_m \in \mathcal{V}(k,e)$. Then $\gamma_m(0) = 0$, and since $\operatorname{im}(\phi) \subseteq [-1,1]$, we have $f(\gamma_m)(x) = (h_m \circ \phi)(x)$ for all $x \in W :=]n - \frac{1}{2}, n + \frac{1}{2}[$. For $x \in W$ sufficiently close to n, we have $\phi(x) = s \cdot (x-n) \in [-\frac{1}{4m}, \frac{1}{4m}]$ and thus $f(\gamma_m)(x) = r \cdot m \cdot s^{k_0+1} \cdot (x-n)^{k_0+1}$, whence $f(\gamma_m)^{(k_0+1)}(n) = r \cdot m \cdot s^{k_0+1} \cdot (k_0+1)!$. Thus $f(\gamma_m) \notin V$ for all $m \in \mathbb{N}$ such that $r \cdot m \cdot s^{k_0+1} \cdot (k_0+1)! \geq 1$, and so $f(\mathcal{V}(k,e)) \not\subseteq V$. As k and e were arbitrary, (b) follows. $\square$

Note that $\operatorname{supp}(f(\gamma)) \subseteq \operatorname{supp}(\gamma)$ here, for all $\gamma \in C_c^\infty(\mathbb{R})$.

Remark 2.3 Although the map f from Proposition 2.2 is discontinuous and thus not smooth in the Michal-Bastiani sense, it is easily seen to be smooth in the sense of convenient differential calculus (as any map f on a "regular" countable strict direct limit $E = \varinjlim E_n$ of complete locally convex spaces, all of whose restrictions $f|_{E_n}$ are smooth).[1]

3 Discontinuous mappings on $C_c^\infty(M,E)$

In this section, we generalize our discussion of $C_c^\infty(\mathbb{R})$ from Section 2 to the spaces $C_c^\infty(M,E) = \varinjlim C_K^\infty(M,E)$ of compactly supported smooth mappings on a σ-compact finite-dimensional smooth manifold M with values in a locally convex space E. We show:

Proposition 3.1 *If $E \neq \{0\}$, the manifold M is non-compact, and $\dim(M) > 0$, then there exists a mapping $f\colon C_c^\infty(M,E) \to C_c^\infty(M,\mathbb{R})$ such that*

(a) *The restriction of f to $C_K^\infty(M,E)$ is smooth, for each compact subset K of M.*

(b) *f is discontinuous at 0.*

In particular, the locally convex direct limit topology on $C_c^\infty(M,E) = \varinjlim C_K^\infty(M,E)$ is properly coarser than the topology making $C_c^\infty(M,E)$ the direct limit of the spaces $C_K^\infty(M,E)$ in the category of topological spaces.

Instead of proving this proposition directly, we establish an analogous result for spaces of sections in bundles of locally convex spaces, which is no harder to prove. Noting that the function space $C_c^\infty(M,E)$ is topologically isomorphic to the space $C_c^\infty(M, M \times E)$ of compactly supported smooth sections in the trivial bundle $\mathrm{pr}_M : M \times E \to M$, clearly Proposition 3.1 is covered by the ensuing discussions for vector bundles. For background material concerning bundles of locally convex spaces and the associated spaces of sections, the reader is referred to [9] (or also [8, Appendix F]).

[1] Regularity means that every bounded subset of E is contained and bounded in some E_n.

For the present purposes, we recall: if $\pi : E \to M$ is a smooth bundle of locally convex spaces over the finite-dimensional, σ-compact smooth manifold M, with typical fibre the locally convex space F, then one considers on the space $C^\infty(M, E)$ of all smooth sections the initial topology with respect to the family of mappings

$$\theta_\psi : C^\infty(M, E) \to C^\infty(U, F), \quad \theta_\psi(\sigma) := \sigma_\psi := \mathrm{pr}_F \circ \psi \circ \sigma|_U^{\pi^{-1}(U)},$$

which take a smooth section σ to its local representation $\sigma_\psi : U \to F$ with respect to the local trivialization $\psi : \pi^{-1}(U) \to U \times F$ of E. Given a compact subset $K \subseteq M$, the subspace $C^\infty_K(M, E) \subseteq C^\infty(M, E)$ of sections vanishing off K is equipped with the induced topology, and $C^\infty_c(M, E) := \bigcup_K C^\infty_K(M, E) = \varinjlim C^\infty_K(M, E)$ is given the locally convex direct limit topology.

Theorem 3.2 *Let M be a σ-compact, non-compact, finite-dimensional smooth manifold of dimension* $\dim(M) > 0$*, and $\pi : E \to M$ be a smooth bundle of locally convex spaces over M, whose typical fibre is a locally convex topological vector space $F \neq \{0\}$. Then there exists a discontinuous mapping $f : C^\infty_c(M, E) \to C^\infty_c(M, \mathbb{R})$ whose restriction to $C^\infty_K(M, E)$ is smooth, for each compact subset K of M.*

Proof. Let $d := \dim(M)$. Since M is non-compact, there exists a sequence $(U_n)_{n\in\mathbb{N}_0}$ of mutually disjoint coordinate neighbourhoods $U_n \subseteq M$ diffeomorphic to $\mathbb{R}^d$ such that local trivializations $\psi_n : \pi^{-1}(U_n) \to U_n \times F$ of E exist, and such that every compact subset of M meets only finitely many of the sets U_n. We define

$$\theta_{\psi_n} : C^\infty_c(M, E) \to C^\infty(U_n, F), \quad \theta_{\psi_n}(\sigma) := \sigma_{\psi_n} := \mathrm{pr}_F \circ \psi_n \circ \sigma|_{U_n}^{\pi^{-1}(U_n)}.$$

By definition of the topology on $C^\infty_c(M, E)$, the linear maps θ_{ψ_n} are continuous. For each $n \in \mathbb{N}_0$, let $\kappa_n : U_n \to \mathbb{R}^d$ be a C^∞-diffeomorphism; define $x_n := \kappa_n^{-1}(0)$. We choose a function $h \in C^\infty_c(\mathbb{R}^d, \mathbb{R})$ such that $h|_{[-1,1]^d} = 1$; we define $h_n \in C^\infty_c(M, \mathbb{R})$ via $h_n(x) := h(\kappa_n(x))$ if $x \in U_n$, $h_n(x) := 0$ if $x \in M \setminus U_n$. Let $K_n := \mathrm{supp}(h_n) \subseteq U_n$. We choose a continuous linear functional $0 \neq \lambda \in F'$, and pick $v \in F$ such that $\lambda(v) = 1$. Note that $A := \bigcup_{n\in\mathbb{N}} K_n$ is closed in M, the sequence $(K_n)_{n\in\mathbb{N}}$ of compact sets being locally finite. Let $\mu : \mathbb{R} \times F \to F$ be the scalar multiplication. The eventual definition of the mapping f we are looking for will involve the map $\Phi : E \to M \times \mathbb{R}$, defined via

$$\Phi|_{\pi^{-1}(U_n)} := (\pi|_{\pi^{-1}(U_n)}, \lambda \circ \mu \circ ((h_n \circ \pi)|_{\pi^{-1}(U_n)}, \mathrm{pr}_F \circ \psi_n)) \tag{1}$$

for $n \in \mathbb{N}$, and $\Phi|_{E\setminus\pi^{-1}(A)} := (\pi|_{E\setminus\pi^{-1}(A)}, 0)$. Note that Φ is well-defined as the function in Equation (1) coincides with $(\pi, 0)$ on the set $\bigcup_{n\in\mathbb{N}} \pi^{-1}(U_n \setminus A)$. Also note that Φ is a fibre-preserving mapping from E into the trivial bundle $M \times \mathbb{R}$. Furthermore, it is readily verified that Φ is a smooth. By [9, Thm. 5.9] (or [8, Rem. F.25 (a)]), the pushforward

$$C^\infty_c(M, \Phi) : C^\infty_c(M, E) \to C^\infty_c(M, M \times \mathbb{R}), \quad \sigma \mapsto \Phi \circ \sigma$$

is smooth. For later use, we introduce the continuous linear map

$$\Lambda := \theta_{\mathrm{id}_{M\times\mathbb{R}}} : C^\infty_c(M, M \times \mathbb{R}) \to C^\infty(M, \mathbb{R}).$$

Let $\iota\colon \mathbb{R}\to\mathbb{R}^d$ denote the embedding $t\mapsto (t,0,\ldots,0)$. The mapping f to be constructed will also involve the map $\Psi\colon C^\infty_c(M,E)\to C^\infty(\mathbb{R},\mathbb{R})$ defined via

$$\Psi := C^\infty(\mathbb{R},\lambda)\circ C^\infty(\kappa_0^{-1}\circ\iota, F)\circ\theta_{\psi_0},$$

where the pullback $C^\infty(\kappa_0^{-1}\circ\iota,F): C^\infty(U_n,F)\to C^\infty(\mathbb{R},F)$, $\gamma\mapsto\gamma\circ\kappa_0^{-1}\circ\iota$ and the pushforward $C^\infty(\mathbb{R},\lambda)\colon C^\infty(\mathbb{R},F)\to C^\infty(\mathbb{R},\mathbb{R})$, $\gamma\mapsto\lambda\circ\gamma$ are continuous linear mappings and thus smooth, by [5, La. 3.3, La. 3.7]. Being a composition of smooth maps, Ψ is smooth. We now define the desired map $f\colon C^\infty_c(M,E)\to C^\infty_c(M,\mathbb{R})$ via

$$f \;:=\; \Gamma\circ(\Psi,\Lambda\circ C^\infty_c(M,\Phi)) \;-\; \lambda\circ\mathrm{ev}_{x_0}\circ\theta_{\psi_0}$$

(co-restricted from $C^\infty(M,\mathbb{R})$ to $C^\infty_c(M,\mathbb{R})$), where

$$\Gamma\colon C^\infty(\mathbb{R},\mathbb{R})\times C^\infty(M,\mathbb{R})\to C^\infty(M,\mathbb{R}),\qquad \Gamma(\gamma,\eta):=\gamma\circ\eta$$

denotes composition, and $\mathrm{ev}_{x_0}: C^\infty(U_0,F)\to F$ the evaluation map $\gamma\mapsto\gamma(x_0)$. Here $\lambda\circ\mathrm{ev}_{x_0}\circ\theta_{\psi_0}$ is a continuous linear map and thus smooth. Explicitly, for $\sigma\in C^\infty_c(M,E)$

$$\begin{aligned} f(\sigma)(x) &= \Big(\lambda\circ\sigma_{\psi_0}\circ\kappa_0^{-1}\circ\iota\Big)\big(\lambda\big(h_n(x)\,\sigma_{\psi_n}(x)\big)\big)\\ &= \lambda\Big(\sigma_{\psi_0}\big(\kappa_0^{-1}(h_n(x)\cdot\lambda(\sigma_{\psi_n}(x)),\;0)\big)\Big)\;-\;\lambda(\sigma_{\psi_0}(x_0)) \end{aligned}$$

if $x\in U_n$ ($n\in\mathbb{N}$), whereas $f(\sigma)(x)=0$ if $x\in M\setminus A$.

Claim: *The restriction of f to $C^\infty_K(M,E)$ is smooth, for each compact subset K of M.* To see this, note that $f(C^\infty_K(M,E)\subseteq C^\infty_K(M,\mathbb{R})$, where $C^\infty_K(M,\mathbb{R})$ is a closed vector subspace of $C^\infty(M,\mathbb{R})$ and $C^\infty_c(M,\mathbb{R})$. Thus, it suffices to show that $f|_{C^\infty_K(M,E)}$ is smooth as a map into $C^\infty(M,\mathbb{R})$ ([9, Prop. 1.9], or [1, La. 10.1]). But this follows from the Chain Rule, as Γ is smooth by Lemma 2.1 and also the other constituents of f are smooth.

Claim: *f is discontinuous at the zero-section $\sigma=0$.* To see this, consider the set V of all $\gamma\in C^\infty_c(M,\mathbb{R})$ such that, for all $n\in\mathbb{N}$ and multi-indices $\alpha\in\mathbb{N}_0^d$ of order $|\alpha|\le n$, we have $|\partial^\alpha(\gamma\circ\kappa_n^{-1})(0)|<1$. It is easily verified that V is a symmetric, convex zero-neighbourhood in $C^\infty_c(M,\mathbb{R})$. Let U be any convex zero-neighbourhood in $C^\infty_c(M,E)$; we claim that $f(U)\not\subseteq V$. To see this, set $L_n:=\kappa_n^{-1}([-1,1]^d)$ for $n\in\mathbb{N}_0$. Then

$$\rho_n\colon C^\infty_{L_n}(M,E)\to C^\infty_{[-1,1]^d}(\mathbb{R}^d,F),\qquad \sigma\mapsto\sigma_{\psi_n}\circ\kappa_n^{-1}$$

is a topological isomorphism (cf. [9, La. 3.9, La. 3.10] or [8, La. F.9, La. F.15]) whose inverse gives rise to a topological embedding $j_n: C^\infty_{[-1,1]^d}(\mathbb{R}^d,F)\to C^\infty_c(M,E)$. The linear mapping $\phi:\mathbb{R}\to F$, $t\mapsto tv$ gives rise to a continuous linear map $C^\infty_{[-1,1]^d}(\mathbb{R}^d,\phi): C^\infty_{[-1,1]^d}(\mathbb{R}^d,\mathbb{R})\to C^\infty_{[-1,1]^d}(\mathbb{R}^d,F)$, $\gamma\mapsto\phi\circ\gamma$. Then $W_n:=(j_n\circ C^\infty_{[-1,1]^d}(\mathbb{R}^d,\phi))^{-1}(\frac{1}{2}U)$ is a convex zero-neighbourhood in $C^\infty_{[-1,1]^d}(\mathbb{R}^d,\mathbb{R})$. Thus, there exists $k_n\in\mathbb{N}_0$ and $\varepsilon_n>0$ such that $W_{k_n,\varepsilon_n}\subseteq W_n$, where W_{k_n,ε_n} is the set of all $\gamma\in C^\infty_{[-1,1]^d}(\mathbb{R}^d,\mathbb{R})$ such that $\sup\{|\partial^\alpha\gamma(x)|:$

$x \in [-1,1]^d\} < \varepsilon_n$ for all $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| \leq k_n$. We let $g \in C^\infty_{[-1,1]^d}(\mathbb{R}^d,\mathbb{R})$ be a function such that $g(y_1,\ldots,y_d) = y_1^{k_0+1}$ for all $y = (y_1,\ldots,y_d) \in [-\frac{1}{2},\frac{1}{2}]^d$. Then $rg \in W_{k_0,\varepsilon_0}$ for some $r > 0$. It is clear from the definition of W_{k_0,ε_0} that then also $\gamma_m \in W_{k_0,\varepsilon_0}$ for all $m \in \mathbb{N}$, where

$$\gamma_m \colon \mathbb{R}^d \to \mathbb{R}, \quad \gamma_m(y_1,\ldots,y_d) := \frac{r}{m^{k_0}}\, g(my_1, y_2, \ldots, y_d)\, .$$

Thus $\tau_m := j_0(\phi \circ \gamma_m) \in \frac{1}{2}U$.

Let $\ell := k_0+1$; we easily find $\eta \in W_{k_\ell,\varepsilon_\ell}$ such that, for suitable $s > 0$, we have $\eta(y) = s{\cdot}y_1$ for $y = (y_1,\ldots,y_d)$ in some zero-neighbourhood in $\mathbb{R}^d$. We define $\tau := j_\ell(\phi \circ \eta) \in \frac{1}{2}U$. Then $\sigma_m := \tau_m + \tau \in U$ by convexity of U. Consider $g_m := f(\sigma_m) \circ \kappa_\ell^{-1} \colon \mathbb{R}^d \to \mathbb{R}$. For $y \in [-1,1]^d$ sufficiently close to 0, we have $\eta(y) = sy_1$ and $m|\eta(y)| \leq \frac{1}{2}$. Thus

$$g_m(y) = \gamma_m(\eta(y), 0, \ldots, 0) = r \cdot m \cdot s^{k_0+1} \cdot y_1^{k_0+1},$$

entailing that $\frac{\partial^{k_0+1} g_m}{\partial y_1^{k_0+1}}(0) = r \cdot m \cdot s^{k_0+1} \cdot (k_0+1)!\,$. Hence $f(\sigma_m) \notin V$ for each $m \in \mathbb{N}$ such that $r \cdot m \cdot s^{k_0+1} \cdot (k_0+1)! \geq 1$. We have shown that $f(U) \not\subseteq V$ for any 0-neighbourhood U in $C^\infty_c(M,E)$, although $f(0) = 0$. Thus f is discontinuous at $\sigma = 0$. $\square$

4 Further examples

We describe various pathological bilinear mappings.

Proposition 4.1 *Let $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. The pointwise multiplication map*

$$\mu \colon C^\infty(\mathbb{R},\mathbb{K}) \times C^\infty_c(\mathbb{R},\mathbb{K}) \to C^\infty_c(\mathbb{R},\mathbb{K}), \quad \mu(\gamma,\eta) := \gamma \cdot \eta$$

is a hypocontinuous bilinear (and thus sequentially continuous) mapping on the locally convex direct limit

$$C^\infty(\mathbb{R},\mathbb{K}) \times C^\infty_c(\mathbb{R},\mathbb{K}) = \lim_{\longrightarrow} \left(C^\infty(\mathbb{R},\mathbb{K}) \times C^\infty_{[-n,n]}(\mathbb{R},\mathbb{K})\right),$$

whose restriction to $C^\infty(\mathbb{R},\mathbb{K}) \times C^\infty_{[-n,n]}(\mathbb{R},\mathbb{K})$ is continuous bilinear and thus $\mathbb{K}$-analytic, for each $n \in \mathbb{N}$. However, μ is discontinuous.

Proof. Using the Leibniz Rule for the differentiation of products of functions, it is easily verified that μ is separately continuous.[2] The spaces $C^\infty(\mathbb{R},\mathbb{K})$ and $C^\infty_c(\mathbb{R},\mathbb{K})$ being barrelled, this entails that μ is hypocontinuous and thus sequentially continuous [19, Thm. 41.2]. The restriction of μ to $C^\infty(\mathbb{R},\mathbb{K}) \times C^\infty_{[-n,n]}(\mathbb{R},\mathbb{K})$ is a sequentially continuous

[2] Alternatively, we can obtain the assertion as a special case of [9, Cor. 2.7] or [8, La. 4.5 (a) and Prop. 4.19 (d)], combined with the locally convex direct limit property.

bilinear mapping on a product of metrizable spaces and therefore continuous. To see that μ is discontinuous, consider the zero-neighbourhood

$$W := \{\gamma \in C^\infty_c(\mathbb{R}, \mathbb{K})\colon\; (\forall x \in \mathbb{R})\;\; |\gamma(x)| < 1\}$$

in $C^\infty_c(\mathbb{R}, \mathbb{K})$. If U is any zero-neighbourhood in $C^\infty(\mathbb{R}, \mathbb{K})$ and V any zero-neighbourhood in $C^\infty_c(\mathbb{R}, \mathbb{K})$, then there exists a compact subset K of $\mathbb{R}$ such that

$$(\forall \gamma \in C^\infty(\mathbb{R}, \mathbb{K}))\quad \gamma|_K = 0 \;\Rightarrow\; \gamma \in U.$$

Pick any $x_0 \in \mathbb{R} \setminus K$. There is a function $\phi \in C^\infty_c(\mathbb{R}, \mathbb{K})$ such that $\phi(x_0) \neq 0$ and $\operatorname{supp}(\phi) \subseteq \mathbb{R} \setminus K$. Then $r\phi \in V$ for some $r > 0$, and $t\phi \in U$ for all $t \in \mathbb{R}$. Choosing $t \geq \frac{1}{r \cdot |\phi(x_0)|^2}$, we have $(t\phi, r\phi) \in U \times V$ but $|\mu(r\phi, t\phi)(x_0)| = rt|\phi(x_0)|^2 \geq 1$, entailing that $\mu(U \times V) \not\subseteq W$. Thus μ is discontinuous at $(0, 0)$. □

Another instructive example is the following (compare also the examples in [2]):

Example 4.2 Let $E_1 \subset E_2 \subset \cdots$ be a strictly ascending sequence of Banach spaces, such that E_{n+1} induces the given topology on E_n. Set $E := \varinjlim E_n$ and $F := E'_b$. For example, we can take $E_n := L^2[-n, n]$, in which case $E = L^2_{\text{comp}}(\mathbb{R})$ and $F = L^2_{\text{loc}}(\mathbb{R}) = \varprojlim L^2[-n, n]$. Then $A_n := F \times E_n \times \mathbb{K} \times \mathbb{K}$ is a Fréchet space (and reflexive in the example $E_n = L^2[-n, n]$). The evaluation map $E'_n \times E_n \to \mathbb{R}$ being continuous as E_n is a Banach space, it is easy to see that A_n becomes a unital associative topological algebra via

$$(\lambda_1, x_1, z_1, c_1) \cdot (\lambda_2, x_2, z_2, c_2) \;:=\; \big(c_1\lambda_2 + c_2\lambda_1,\; c_1x_2 + c_2x_1,\; c_1z_2 + \lambda_1(x_2) + z_1c_2,\; c_1c_2\big)\,. \quad (2)$$

The multiplication can be visualized by considering $(\lambda, x, z, c) \in A_n$ as the 3-by-3 matrix

$$\begin{pmatrix} c & \lambda & z \\ 0 & c & x \\ 0 & 0 & c \end{pmatrix}.$$

The topological algebras A_n are very well-behaved: they have open groups of units, and inversion is a $\mathbb{K}$-analytic map. We can also use Formula (2) to define a multiplication map $\mu\colon A \times A \to A$ turning the direct limit locally convex space $A := F \times E \times \mathbb{K} \times \mathbb{K} = \varinjlim A_n$ into a unital, associative algebra. However, although the restriction of μ to $A_n \times A_n$ is a continuous bilinear map for each $n \in \mathbb{N}$, $\mu\colon A \times A = \varinjlim (A_n \times A_n) \to A$ is discontinuous (since the evaluation map $E'_b \times E \to \mathbb{R}$ is discontinuous, the space E not being normable). We refer to [6, Section 10] for more details.

Appendix: Proof of Lemma 2.1

We give a proof which is as elementary as possible, by reducing the assertion to the case $M = \mathbb{R}^d$. First, let M be a finite-dimensional, σ-compact smooth manifold, of dimension d. We choose an open cover $(U_j)_{j \in J}$ of M and C^∞-diffeomorphisms $\kappa_j\colon U_j \to \mathbb{R}^d$. Then

$$\Phi\,:\, C^\infty(M, \mathbb{R}^m) \;\to\; \prod_{j \in J} C^\infty(\mathbb{R}^d, \mathbb{R}^m) \;=:\; P\,, \qquad \Phi(\gamma) \;:=\; (\gamma \circ \kappa_j^{-1})_{j \in J}$$

is a topological embedding onto a closed vector subspace of the cartesian product P (cf. [9, La. 3.7]). Therefore Γ is smooth if and only if $\Phi\circ\Gamma$ is smooth ([9, Prop. 1.9] or [1, La. 10.1]), if and only if each component $\mathrm{pr}_j\circ\Phi\circ\Gamma$ is smooth [1, La. 10.3], where $\mathrm{pr}_j\colon P\to C^\infty(\mathbb{R}^d,\mathbb{R}^m)$ is the projection onto the j-coordinate. But

$$\mathrm{pr}_j(\Phi(\Gamma))(\gamma,\eta) \;=\; \gamma\circ\eta\circ\kappa_j^{-1} \;=\; \tilde{\Gamma}\big(\gamma, C^\infty(\kappa_j^{-1},\mathbb{R}^n)(\eta)\big)$$

for all $\gamma\in C^\infty(\mathbb{R}^n,\mathbb{R}^m)$ and $\eta\in C^\infty(M,\mathbb{R}^n)$, where

$$\tilde{\Gamma}\colon C^\infty(\mathbb{R}^n,\mathbb{R}^m)\times C^\infty(\mathbb{R}^d,\mathbb{R}^n)\to C^\infty(\mathbb{R}^d,\mathbb{R}^m)$$

is the composition map and $C^\infty(\kappa_j^{-1},\mathbb{R}^n): C^\infty(M,\mathbb{R}^n)\to C^\infty(\mathbb{R}^d,\mathbb{R}^n)$, $\eta\mapsto\eta\circ\kappa_j^{-1}$ is continuous linear and thus smooth, by [5, La. 3.7]. Hence $\mathrm{pr}_j\circ\Phi\circ\Gamma$ (and thus Γ) will be smooth if so is $\tilde{\Gamma}$.

By the reduction step just performed, it only remains to prove Lemma 2.1 for $M=\mathbb{R}^d$, which we assume now. We show by induction on $k\in\mathbb{N}_0$ that Γ is C^k.

The case $k=0$. Let $\gamma\in C^\infty(\mathbb{R}^n,\mathbb{R}^m)$, $\eta\in C^\infty(\mathbb{R}^d,\mathbb{R}^n)$ and $(\gamma_i,\eta_i)_{i\in\mathbb{N}}$ be a sequence in $C^\infty(\mathbb{R}^n,\mathbb{R}^m)\times C^\infty(\mathbb{R}^d,\mathbb{R}^n)$ converging to (γ,η). We have to show that

$$\begin{aligned}\delta_i \;&:=\; \Gamma(\gamma_i,\eta_i)-\Gamma(\gamma,\eta) \;=\; \gamma_i\circ\eta_i-\gamma\circ\eta\\ &=\; (\gamma_i-\gamma)\circ\eta_i \;+\; (\gamma\circ\eta_i-\gamma\circ\eta)\end{aligned}\tag{3}$$

converges to 0 in $C^\infty(\mathbb{R}^d,\mathbb{R}^m)$. To see this, we first check convergence in $C^0(\mathbb{R}^d,\mathbb{R}^m)$ (equipped with the topology of uniform convergence on compact sets). Given a compact set $K\subseteq\mathbb{R}^d$, the set $\bigcup_{i\in\mathbb{N}}\eta_i(K)$ is bounded and hence has compact closure L in $\mathbb{R}^d$. Now the first term in (3) converges uniformly to 0 on K since $\gamma_i-\gamma\to 0$ uniformly on L as $i\to\infty$. The second term converges uniformly to 0 on K since $\gamma|_L$ is uniformly continuous and $\eta_i\to\eta$ uniformly on K. Using the Chain Rule, for each fixed multi-index $\alpha\in\mathbb{N}_0^d$ of order ≥ 1, we find polynomials $P_\beta\in\mathbb{R}[(X_\gamma)_{\gamma\leq\alpha}]$ in indeterminates X_γ, for multi-indices $\beta\in\mathbb{N}_0^n$ of order $|\beta|\leq|\alpha|$, such that

$$\begin{aligned}\partial^\alpha\delta_i \;&=\; \sum_{|\beta|\leq|\alpha|}((\partial^\beta\gamma_i-\partial^\beta\gamma)\circ\eta_i)\cdot P_\beta((\partial^\gamma\eta_i)_{\gamma\leq\alpha})\\ &\quad+\sum_{|\beta|\leq|\alpha|}(\partial^\beta\gamma\circ\eta_i)\cdot(P_\beta((\partial^\gamma\eta_i)_{\gamma\leq\alpha})-P_\beta((\partial^\gamma\eta)_{\gamma\leq\alpha}))\\ &\quad+\sum_{|\beta|\leq|\alpha|}(\partial^\beta\gamma\circ\eta_i-\partial^\beta\gamma\circ\eta)\cdot P_\beta((\partial^\gamma\eta)_{\gamma\leq\alpha}).\end{aligned}$$

We easily deduce from this formula that $\partial^\alpha\delta_i$ converges to 0 as $i\to\infty$, uniformly on compact sets. We have shown that $\delta_i\to 0$ in $C^\infty(\mathbb{R}^d,\mathbb{R}^m)$. Thus Γ is continuous.

Induction step. Suppose that Γ is of class C^k, where $k\in\mathbb{N}_0$. Given $\gamma,\gamma_1\in C^\infty(\mathbb{R}^n,\mathbb{R}^m)$, $\eta,\eta_1\in C^\infty(\mathbb{R}^d,\mathbb{R}^n)$, we have

$$\tfrac{1}{t}\,(\Gamma(\gamma+t\gamma_1,\eta+t\eta_1)-\Gamma(\gamma,\eta)) \;=\; \tfrac{1}{t}\,(\gamma\circ(\eta+t\eta_1)-\gamma\circ\eta) \;+\; \gamma_1\circ(\eta+t\eta_1)\tag{4}$$

for $0 \neq t \in \mathbb{R}$. Given $t \in \mathbb{R}$, define $F_t \colon \mathbb{R}^d \to \mathbb{R}^m$ via

$$F_t(x) \;:=\; \int_0^1 H(x, st)\, ds\,,$$

where $H \colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^m$, $H(x,r) := d\gamma(\eta(x) + r\eta_1(x); \eta_1(x))$. Clearly H is smooth. It is easy to see that $F_t(x) \to F_0(x)$ uniformly for x in a compact set, as $t \to 0$. Furthermore, differentiating under the integral sign we find that $\partial^\alpha F_t(x) = \int_0^1 \partial^{(\alpha,0)} H(x,st)\, ds$ for $\alpha \in \mathbb{N}_0^d$, which converges uniformly for x in a compact set to $\partial^\alpha F_0(x)$ as $t \to 0$. Since

$$F_t \;=\; \tfrac{1}{t}\, (\gamma \circ (\eta + t\eta_1) - \gamma \circ \eta)$$

for $t \neq 0$, by the Mean Value Theorem, we see that the first term on the right hand side of (4) converges to $F_0 = (d\gamma) \circ (\eta, \eta_1) = \widetilde{\Gamma}(d\gamma, (\eta, \eta_1))$ in $C^\infty(\mathbb{R}^d, \mathbb{R}^m)$ as $t \to 0$, where $\widetilde{\Gamma} \colon C^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^m) \times C^\infty(\mathbb{R}^d, \mathbb{R}^n \times \mathbb{R}^n) \to C^\infty(\mathbb{R}^d, \mathbb{R}^m)$ is the composition map.

To tackle the second term, define $G_t := \gamma_1 \circ (\eta + t\eta_1) = \Gamma(\gamma_1, \eta + t\eta_1)$ for $t \in \mathbb{R}$. Since Γ is continuous by the above, we have $G_t \to G_0 = \gamma_1 \circ \eta$ in $C^\infty(\mathbb{R}^d, \mathbb{R}^m)$ as $t \to 0$. Thus the second term in Equation (4) converges to $\gamma_1 \circ \eta$.

Summing up, we have shown that $d\Gamma(\gamma, \eta; \gamma_1, \eta_1)$ exists, and is given by

$$d\Gamma(\gamma, \eta; \gamma_1, \eta_1) \;=\; \widetilde{\Gamma}(d\gamma, (\eta, \eta_1)) + \Gamma(\gamma_1, \eta). \tag{5}$$

The map $C^\infty(\mathbb{R}^n, \mathbb{R}^m) \to C^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^m)$, $\gamma \mapsto d\gamma$ is continuous linear (cf. [5, La. 3.8]), and Γ, $\widetilde{\Gamma}$ are C^k, by induction. Hence Equation (5) shows that $d\Gamma$ is C^k. Thus Γ is C^{k+1}, as required.

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