$\mathrm{Diff}(\mathbb{R}^n)$ as a Milnor-Lie group

By Helge Glöckner of Darmstadt

(Received March 7, 2002; revised version March 4, 2005)

Abstract. We describe a construction of the Lie group structure on the diffeomorphism group $\mathrm{Diff}(\mathbb{R}^n)$, modelled on the space $\mathcal{D}(\mathbb{R}^n, \mathbb{R}^n)$ of $\mathbb{R}^n$-valued test functions on $\mathbb{R}^n$, in John Milnor's setting of infinite-dimensional Lie groups. New tools are introduced to simplify this task.

1. Introduction

It is well-known that the diffeomorphism group $\mathrm{Diff}(K)$ of a compact smooth manifold K can be made a Fréchet-Lie group, modelled on the Fréchet space of smooth vector fields on K. Since all popular basic notions of "smooth" mappings coincide for mappings between open subsets of Fréchet spaces (cf. [9] and [10, Thm. 4.11 (a), Thm. 12.8]), it does not matter much which framework of differential calculus and corresponding concept of Lie groups is used here; discussions based on smooth maps in the sense of Michal-Bastiani (also known as Keller's C^∞_c-maps) can be found in [8, p. 92] and [14]; a discussion in the "convenient setting" of analysis by Frölicher, Kriegl and Michor is given in [1, Thm. 4.7.5] and [10, §43].[1)]

The situation changes dramatically if one considers the diffeomorphism group $\mathrm{Diff}(M)$ of a non-compact, finite-dimensional smooth manifold M, which one would like to model on the LF-space $\mathcal{D}(M, TM)$ of compactly supported, smooth vector fields on M. In this case, Michal-Bastiani smoothness of mappings on $\mathcal{D}(M, TM)$ (which implies continuity) is a much stronger condition than being smooth in the convenient sense, already on $\mathcal{D}(\mathbb{R}, T\mathbb{R}) \cong \mathcal{D}(\mathbb{R})$: The self-map $\mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$, $\gamma \mapsto \gamma \circ \gamma - \gamma(0)$ of the space of real-valued test-functions on the line is smooth in the convenient sense, but discontinuous [6].

1991 *Mathematics Subject Classification.* Primary 58D05; Secondary 22E65, 46F05, 46T20.

Keywords and phrases. Diffeomorphism groups, non-compact manifolds, infinite-dimensional Lie groups, almost local mappings, spaces of test functions.

[1)] Omori's interpretation of $\mathrm{Diff}(K)$ as an ILB-Lie group [16] and Hamilton's interpretation of $\mathrm{Diff}(K)$ as a "tame" Lie group ([8, Thm. 2.3.5]) refer to stronger, specialized notions of smoothness specific to Fréchet spaces. Cf. also [11], where $\mathrm{Diff}(K)$ was first studied.

In the setting of Keller's C_c^∞-theory, Diff(M) was made a Lie group by Michor [12]. But the construction was rather complicated and soon after, Michor abandoned Keller's C_c^∞-theory in favour of the convenient setting of analysis. In this setting, Diff(M) was made a Lie group in [10], using a simpler construction (see [10], comments on p. 455). However, one had to pay a price: Instead of the quite natural topology on Diff(M) used in Keller's C_c^∞-theory (corresponding to the locally convex topology on $\mathcal{D}(M,TM)$), which makes Diff(M) a topological group, the convenient approach equips Diff(M) with a properly finer topology which does not make Diff(M) a topological group: the group multiplication is discontinuous (cf. [17]).

In this article, we introduce a certain class of mappings

$$f\colon\ \mathcal{D}(M,E)\ \to \mathcal{D}(N,F)$$

between spaces of vector-valued test functions (the "almost local mappings;" see Definition 3.1). Being almost local is a mild regularity property, which is satisfied (at least locally) by all mappings encountered in the construction of diffeomorphism groups. Now the gist is that an almost local map f is smooth (in the Michal-Bastiani sense) if and only if its restriction to each of the steps $C_K^\infty(M,E)$ of the directed system is smooth (Theorem 3.2). If E is finite-dimensional, then $C_K^\infty(M,E)$ is a Fréchet space and hence smoothness of mappings on this space coincides with smoothness in the convenient sense, which is (frequently) easily verified. In this way, we can profit from the advantages of both settings of analysis: One the one hand, we can work with the natural topologies and ensure smoothness in the stronger sense of Keller's C_c^∞-theory (where smooth maps are, in particular, continuous). On the other hand, once we have verified that a mapping of interest is almost local, we can use the powerful tools of convenient calculus to check its smoothness.

To illustrate the effectiveness of this idea, we describe in Sections 4–7 a new construction of the Lie group structure on the diffeomorphism group of $\mathbb{R}^n$. We remark that the concept of an almost local map can be adapted to mappings between spaces of sections in vector bundles (see [5], [7] and [4, Defn. F.29 & Thm. F.30], where in fact a slightly more general definition of almost local maps is given).[2)] In [7], almost local maps (and related novel tools, "patched maps") are used to construct the Lie group structure on the diffeomorphism group Diff(M) for σ-compact M, and also to verify that Diff(M) is a regular Lie group in Milnor's sense.[3)] The author believes that the novel arguments and simplifications become particularly clear in the easiest possible case of $\mathbb{R}^n$ treated here, unveiled by the additional technical machinery needed for the manifold case.

Following the pattern of Sections 4–7, it is also possible to create Lie group structures on other versions of diffeomorphism groups. Novel examples are the Fréchet-Lie group $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ of diffeomorphisms differing from $\mathrm{id}_{\mathbb{R}^n}$ by an $\mathbb{R}^n$-valued rapidly decreasing map, or the Fréchet-Lie group $\mathrm{Diff}_b(\mathbb{R}^n)$ of diffeomorphisms differing from $\mathrm{id}_{\mathbb{R}^n}$ by a

2) While the cited papers use abstract functional analytic methods to discuss almost local maps, our present approach is quite explicit.

3) For $\mathrm{Diff}_c(M)$, this was asserted (without proof) in [13], using different terminology. For compact M, the proof is given in [14]. Regularity in the convenient sense was proved in [10].

bounded smooth map with bounded partial derivatives of all orders (details will be given in [18]; it is even possible to replace $\mathbb{R}^n$ with a Banach space here). In [4], a variant of the present approach is used to turn diffeomorphism groups of finite-dimensional smooth manifolds over local fields into Lie groups.

2. Preliminaries

We recall one possible definition of smooth maps in the sense of Michal-Bastiani (see [2], [8], [12], [14] for more information): Let E and F be locally convex spaces, and U be an open subset of E. A map $f: U \to F$ is *smooth* if the two-sided directional derivatives $d^1f(x,v) := df(x,v) := \lim_{t\to 0} t^{-1}(f(x+tv) - f(x)) \in F$ exist for all $(x,v) \in U \times E$, the higher iterated differentials $d^kf := d(d^{k-1}f): U \times E^{2^k-1} \to F$ exist for all $2 \leq k \in \mathbb{N}$, and all of the mappings f, d^1f, d^2f, ... are continuous.

If M is a σ-compact finite-dimensional smooth manifold and E a locally convex space, then $C^\infty(M,E)$, the space of E-valued smooth mappings on M, is a locally convex space in a natural way; given a compact subset $K \subseteq M$, the closed vector subspace $C^\infty_K(M,E) := \{\gamma \in C^\infty(M,E) : \gamma|_{M\setminus K} = 0\} \subseteq C^\infty(M,E)$ is given the induced topology. The space of E-valued test functions is $\mathcal{D}(M,E) := \bigcup_K C^\infty_K(M,E) = \varinjlim_K C^\infty_K(M,E)$ (with K running through the compact subsets of M), equipped with the locally convex direct limit topology. It induces the given topology on each subspace $C^\infty_K(M,E)$ (see [3] for all this). We abbreviate $\mathcal{D}(M) := \mathcal{D}(M,\mathbb{R})$.

3. Almost local mappings between spaces of test functions

Suppose that $f: \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ is a mapping whose restriction to $C^\infty_K(\mathbb{R})$ is smooth, for each compact subset K of $\mathbb{R}$. Then f need not be smooth, and in fact not even continuous, as the example $f(\gamma) := \gamma \circ \gamma - \gamma(0)$ shows (see [6]). Roughly speaking, the pathology in this example is caused by the extreme nonlocality of f: For each $\varepsilon > 0$ and $x \in \mathbb{R}$, there are $\gamma, \eta \in \mathcal{D}(\mathbb{R})$ which coincide off $[-\varepsilon, \varepsilon]$, but such that $f(\gamma)(x) \neq f(\eta)(x)$. In contrast, no problems arise when the values $f(\gamma)(x)$ only depend on $\gamma(y)$ for y close to x (and in slightly more general situations), in a sense to be made precise presently. In order to be useful elsewhere, we formulate our result for mappings between open subsets of spaces of test functions on (σ-compact) finite-dimensional manifolds, with values in locally convex spaces. For our discussion of Diff($\mathbb{R}^n$), it would be sufficient to consider the special case where $M = N = \mathbb{R}^n$ and both E and F are finite-dimensional real vector spaces.

Definition 3.1. Let M and N be finite-dimensional smooth manifolds, E and F locally convex spaces, and P be an open subset of $\mathcal{D}(M,E)$. A map $f: P \to \mathcal{D}(N,F)$ is called *almost local*[4)] if there exist sequences $(U_n)_{n\in\mathbb{N}}$ and $(\widetilde{U}_n)_{n\in\mathbb{N}}$ of relatively compact open subsets $U_n \subseteq M$ and open neighbourhoods $\widetilde{U}_n \subseteq M$ of the closures

[4)] More precisely, we should call such mappings "special almost local maps," because in the meantime a more general notion of almost local maps has been developed (see [5], [7], [4, Defn. F.29]).

$\overline{U_n}$, as well as sequences $(V_n)_{n\in\mathbb{N}}$ and $(\widetilde{V}_n)_{n\in\mathbb{N}}$ of open, relatively compact subsets $V_n \subseteq N$ and open neighbourhoods $\widetilde{V}_n \subseteq N$ of the closures $\overline{V_n}$, such that the following conditions are satisfied:

(a) $(U_n)_{n\in\mathbb{N}}$ is an open cover of M, and $(\widetilde{U}_n)_{n\in\mathbb{N}}$ is locally finite.

(b) $(V_n)_{n\in\mathbb{N}}$ is an open cover of N, and $(\widetilde{V}_n)_{n\in\mathbb{N}}$ is locally finite.

(c) For all $n \in \mathbb{N}$ and $\gamma, \eta \in P$ such that $\gamma|_{U_n} = \eta|_{U_n}$, we have $f(\gamma)|_{V_n} = f(\eta)|_{V_n}$.

(d) $\widetilde{U}_n$ and $\widetilde{V}_n$ are coordinate neighbourhoods for each $n \in \mathbb{N}$, *i.e.*, there are C^∞-diffeomorphisms $\phi_n : \widetilde{U}_n \to A_n$ and $\psi_n : \widetilde{V_n} \to B_n$ onto open subsets A_n and B_n of $\mathbb{R}^a$ and $\mathbb{R}^b$, resp., where $a := \dim(M)$, $b := \dim(N)$.

The following result is the technical backbone of our discussion of $\mathrm{Diff}(\mathbb{R}^n)$.

Theorem 3.2. (*Smoothness Theorem*). *Let M and N be finite-dimensional smooth manifolds, E and F be locally convex spaces, P be an open subset of $\mathcal{D}(M,E)$, and $f : P \to \mathcal{D}(N,F)$ be a mapping. If $f_K := f|_{P\cap C^\infty_K(M,E)}$ is smooth for every compact subset $K \subseteq M$ and f is almost local, then f is smooth.*

Proof. The proof will be given in various steps. For convenience of notation, we abbreviate $P_j := P \times \mathcal{D}(M,E)^{2^j-1} \subseteq \mathcal{D}(M,E)^{2^j}$ for each $j \in \mathbb{N}$, and identify $\mathcal{D}(M,E)^{2^j}$ with $\mathcal{D}(M,E^{2^j})$ in the natural way.

3.3. We claim: *$d^j f : P_j \to \mathcal{D}(N,F)$ exists, for each $j \in \mathbb{N}$. Furthermore, $d^j f|_{P_j\cap C^\infty_K(M,E^{2^j})}$ is smooth, for each compact subset K of M, and $d^j f(\gamma)|_{V_n} = d^j f(\gamma_1)|_{V_n}$ for all $\gamma, \gamma_1 \in P_j$ such that $\gamma|_{U_n} = \gamma_1|_{U_n}$.*

Case $j = 1$: Given $\gamma \in P$ and $\eta \in \mathcal{D}(M,E)$, there is $\varepsilon > 0$ such that $\gamma + \,]-\varepsilon,\varepsilon[\,\eta \subseteq P$. Set $K := \mathrm{supp}(\gamma) \cup \mathrm{supp}(\eta)$; then $\gamma + t\eta \in P \cap C^\infty_K(M,E)$ for all $t \in \mathbb{R}^\times$, $|t| < \varepsilon$. Since f_K is smooth, the limit $df(\gamma,\eta) = \lim_{t\to 0} t^{-1}(f(\gamma+t\eta) - f(\gamma))$ exists, and is given by $df_K(\gamma,\eta)$. Accordingly, for each compact subset K of M,

$$df|_{P_1\cap C^\infty_K(M,E^2)} = d(f_K)\,,$$

identifying $\mathcal{D}(M,E)^2$ with $\mathcal{D}(M,E^2)$ and $C^\infty_K(M,E)^2$ with $C^\infty_K(M,E^2)$ in the obvious way. Here $(df)_K = d(f_K) : P_1 \cap C^\infty_K(M,E^2) \to \mathcal{D}(N,F)$ is smooth, and $df(\gamma,\eta)|_{V_n} = df(\gamma_1,\eta_1)|_{V_n}$ for all $(\gamma,\eta), (\gamma_1,\eta_1) \in P_1$ which coincide on U_n, since $(\gamma + t\eta)|_{U_n} = (\gamma_1 + t\eta_1)|_{U_n}$ and thus $t^{-1}(f(\gamma+t\eta) - f(\gamma))|_{V_n} = t^{-1}(f(\gamma_1+t\eta_1) - f(\gamma_1))|_{V_n}$ in the calculation of the directional derivatives.

Induction step: If the claim holds for all functions f satisfying the hypotheses of the theorem and all $j \in \{1,\ldots,r\}$ (where $r \in \mathbb{N}$), then $d^r f$ may play the role of f, and thus $d^{r+1} f = d(d^r f)$ has the required properties by the case $j = 1$. Thus **3.3** holds.

3.4. It remains to prove that $d^j f$ is continuous for each $j \in \mathbb{N}_0$. As, by **3.3**, the hypotheses of the proposition are satisfied when f is replaced with $d^j f$, it suffices to show that f is continuous. Now, given $\gamma_0 \in P$, apparently $g := f(\gamma_0 + \bullet) - f(\gamma_0) :$

$P - \gamma_0 \to \mathcal{D}(N,F)$ is almost local and $g|_{(P-\gamma_0)\cap C^\infty_K(M,E)}$ is smooth, for each compact subset $K \subseteq M$. The mapping f is continuous at γ_0 if and only if g is continuous at 0. It therefore suffices to consider the case where P is an open zero-neighbourhood and $f(0) = 0$ (which we assume now), and show that f is continuous at 0.

3.5. Since f is almost local, we find sequences $(U_n)_{n\in\mathbb{N}}$, $(\widetilde{U}_n)_{n\in\mathbb{N}}$, $(V_n)_{n\in\mathbb{N}}$, $(\widetilde{V}_n)_{n\in\mathbb{N}}$, $(\phi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ as described in Definition 3.1. For each $n \in \mathbb{N}$, we choose $h_n \in \mathcal{D}(M)$ such that $K_n := \operatorname{supp}(h_n) \subseteq \widetilde{U}_n$ and h_n is identically 1 on U_n.

3.6. Let Γ_1 be a set of continuous seminorms on E defining its locally convex topology, and which is directed in the sense that for all $p_1, p_2 \in \Gamma_1$, there is $p \in \Gamma_1$ such that $p \geq p_i$ pointwise for $i \in \{1,2\}$. Given $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $p \in \Gamma_1$, the mapping

$$\|.\|_{n,k,p} \colon \mathcal{D}(M,E) \to [0,\infty[\,, \quad \|\gamma\|_{n,k,p} := \sup_{|\alpha|\leq k} \sup_{x\in\phi_n(U_n)} p(\partial^\alpha(\gamma\circ\phi_n^{-1})(x))$$

is a continuous seminorm on $\mathcal{D}(M,E)$ (using standard notation for multi-indices $\alpha \in (\mathbb{N}_0)^a$ and partial derivatives). The sets

$$\mathcal{U}(k,p,\varepsilon) := \{\gamma \in \mathcal{D}(M,E) \colon (\forall n \in \mathbb{N})\ \|\gamma\|_{n,k_n,p_n} < \varepsilon_n\}\,,$$

where $k = (k_n) \in (\mathbb{N}_0)^{\mathbb{N}}$, $p = (p_n) \in (\Gamma_1)^{\mathbb{N}}$, and $\varepsilon = (\varepsilon_n) \in (\mathbb{R}^+)^{\mathbb{N}}$, form a basis of open zero-neighbourhoods for $\mathcal{D}(M,E)$ (see, *e.g.*, [3, Prop. 4.8]).

Let Γ_2 be a directed set of continuous seminorms defining the locally convex topology on $\mathcal{D}(N,F)$; proceeding as above, we use the seminorms $q_n \in \Gamma_2$ and take suprema over $x \in \psi_n(V_n)$ to define seminorms $\|.\|_{n,k_n,q_n}$ on $\mathcal{D}(N,F)$, as well as a basis of open zero-neighbourhoods $\mathcal{V}(k,q,\varepsilon)$ for $\mathcal{D}(N,F)$, where $k \in (\mathbb{N}_0)^{\mathbb{N}}$, $q \in (\Gamma_2)^{\mathbb{N}}$, $\varepsilon \in (\mathbb{R}^+)^{\mathbb{N}}$.

3.7. To prove the continuity of f at 0, let arbitrary sequences $k = (k_n) \in (\mathbb{N}_0)^{\mathbb{N}}$, $\varepsilon = (\varepsilon_n) \in (\mathbb{R}^+)^{\mathbb{N}}$, and $q = (q_n) \in (\Gamma_2)^{\mathbb{N}}$ be given. Set $F_n := \{m \in \mathbb{N} \colon U_m \cap K_n \neq \emptyset\}$ for $n \in \mathbb{N}$. The covering $(U_m)_{m\in\mathbb{N}}$ being locally finite, F_n is a finite set. Furthermore, $N_m := \{n \in \mathbb{N} : m \in F_n\} = \{n \in \mathbb{N} : U_m \cap K_n \neq \emptyset\}$ is finite for each m, as U_m is relatively compact and $(\widetilde{U}_n)_{n\in\mathbb{N}}$ is locally finite.

Next, P being an open 0-neighbourhood, by **3.6** we find $c = (c_n) \in (\mathbb{N}_0)^{\mathbb{N}}$, $\pi = (\pi_n) \in (\Gamma_1)^{\mathbb{N}}$, and $\rho = (\rho_n) \in (\mathbb{R}^+)^{\mathbb{N}}$ such that $\mathcal{U}(c,\pi,\rho) \subseteq P$.

Since $f|_{P\cap C^\infty_{K_n}(M,E)}$ is continuous at 0 for $n \in \mathbb{N}$ and $f(0) = 0$, we find $\ell_n \in \mathbb{N}_0$, $p_n \in \Gamma_1$, and $r_n > 0$ such that $\ell_n \geq c_m$, $p_n \geq \pi_m$, and $r_n < \rho_m$ for all $m \in F_n$, and such that $\|f(\gamma)\|_{n,k_n,q_n} < \varepsilon_n$ for all $\gamma \in C^\infty_{K_n}(M,E)$ such that $\|\gamma\|_{m,\ell_n,p_n} < r_n$ for all $m \in F_n$ (note that the latter condition ensures $\gamma \in P$). As a consequence of the Leibniz Rule for the differentiation of products, there is $s_n \in\,]0, r_n]$ such that, for all $m \in F_n$, we have $\|h_n \cdot \gamma\|_{m,\ell_n,p_n} < r_n$, for all $\gamma \in \mathcal{D}(M,E)$ satisfying $\|\gamma\|_{m,\ell_n,p_n} < s_n$ (cf. [3, proof of Prop. 4.8]). Given $m \in \mathbb{N}$, choose $t_m > 0$ such that $t_m \leq s_n$ for all $n \in N_m$, set $\kappa_m := \sup\{\ell_n : n \in N_m\} \in \mathbb{N}_0$, and pick $u_m \in \Gamma_1$ such that $u_m \geq p_n$ for all $n \in N_m$. Set $t := (t_m)$, $\kappa := (\kappa_m)$, $u := (u_m)$.

Let $\gamma \in \mathcal{U}(\kappa,u,t) \subseteq \mathcal{U}(c,\pi,\rho) \subseteq P$. For each $n \in \mathbb{N}$, we have $h_n \cdot \gamma \in C^\infty_{K_n}(M,E)$ and $\|h_n \cdot \gamma\|_{m,\ell_n,p_n} < r_n$ for all $m \in F_n$. Hence $\|f(\gamma)\|_{n,k_n,q_n} = \|f(h_n\cdot\gamma)\|_{n,k_n,q_n} < \varepsilon_n$, noting that $f(\gamma)|_{V_n} = f(h_n \cdot \gamma)|_{V_n}$ since $h_n|_{U_n} \equiv 1$. Thus $f(\mathcal{U}(\kappa,u,t)) \subseteq \mathcal{V}(k,q,\varepsilon)$, and thus f is continuous at 0, as required. □

4. Smoothness of composition on $\mathrm{End}_c(\mathbb{R}^n)$

We study a monoid of smooth self-maps of $\mathbb{R}^n$ closely related to $\mathrm{Diff}(\mathbb{R}^n)$.

Definition 4.1. Let $\mathrm{End}_c(\mathbb{R}^n)$ be the set of all smooth mappings $\gamma\colon \mathbb{R}^n \to \mathbb{R}^n$ which coincide with $\mathrm{id}_{\mathbb{R}^n}$ outside some compact set. Thus $\mathrm{End}_c(\mathbb{R}^n) = \mathrm{id}_{\mathbb{R}^n} + \mathcal{D}(\mathbb{R}^n,\mathbb{R}^n) \subseteq C^\infty(\mathbb{R}^n,\mathbb{R}^n)$. Clearly $\mathrm{End}_c(\mathbb{R}^n)$ is a monoid under composition, with identity element $\mathrm{id}_{\mathbb{R}^n}$. We give $\mathrm{End}_c(\mathbb{R}^n)$ the smooth manifold structure making the bijection

$$\beta\colon \mathcal{D}(\mathbb{R}^n,\mathbb{R}^n) \to \mathrm{End}_c(\mathbb{R}^n), \quad \gamma \mapsto \mathrm{id}_{\mathbb{R}^n} + \gamma$$

a diffeomorphism.

It is easily verified that $\beta^{-1}(\beta(\gamma)\circ\beta(\eta)) = \eta + \gamma\circ(\mathrm{id}_{\mathbb{R}^n}+\eta)$. Thus, to establish smoothness of the composition map $\mathrm{End}_c(\mathbb{R}^n)\times\mathrm{End}_c(\mathbb{R}^n)\to\mathrm{End}_c(\mathbb{R}^n)$, we only need to show that $g\colon \mathcal{D}(\mathbb{R}^n,\mathbb{R}^n)\times\mathcal{D}(\mathbb{R}^n,\mathbb{R}^n)\to\mathcal{D}(\mathbb{R}^n,\mathbb{R}^n)$, $g(\gamma,\eta) := \gamma\circ(\mathrm{id}_{\mathbb{R}^n}+\eta)$ is a smooth map. The following fact will be used, which follows from [10, Cor. 3.13] (see [6, Appendix] for an elementary proof; [4, Prop. 11.3] for generalizations):

Lemma 4.2. *The composition map*

$$\Gamma\colon C^\infty(\mathbb{R}^n,\mathbb{R}^m)\times C^\infty(\mathbb{R}^d,\mathbb{R}^n)\to C^\infty(\mathbb{R}^d,\mathbb{R}^m), \quad \Gamma(\gamma,\eta) := \gamma\circ\eta$$

is smooth, for all $d,m,n\in\mathbb{N}_0$. Given $\gamma,\gamma_1\in C^\infty(\mathbb{R}^n,\mathbb{R}^m)$ and $\eta,\eta_1\in C^\infty(\mathbb{R}^d,\mathbb{R}^n)$, we have $d\Gamma(\gamma,\eta;\gamma_1,\eta_1) = d\gamma\circ(\eta,\eta_1)+\gamma_1\circ\eta$. □

Lemma 4.3. *The mapping $g:\mathcal{D}(\mathbb{R}^n,\mathbb{R}^n)\times\mathcal{D}(\mathbb{R}^n,\mathbb{R}^n)\to\mathcal{D}(\mathbb{R}^n,\mathbb{R}^n)$, $g(\gamma,\eta) := \gamma\circ(\mathrm{id}_{\mathbb{R}^n}+\eta)$ is smooth, with differential given by*

$$dg(\gamma,\eta;\gamma_1,\eta_1) = d\gamma\circ(\mathrm{id}_{\mathbb{R}^n}+\eta,\eta_1)+\gamma_1\circ(\mathrm{id}_{\mathbb{R}^n}+\eta) \quad \text{for } \gamma,\gamma_1,\eta,\eta_1\in\mathcal{D}(\mathbb{R}^n,\mathbb{R}^n).$$

Proof. Given $\gamma_0,\eta_0\in\mathcal{D}(\mathbb{R}^n,\mathbb{R}^n)$, we have to show that g is smooth on some open neighbourhood of (γ_0,η_0). Set $P := \{(\gamma,\eta)\in\mathcal{D}(\mathbb{R}^n,\mathbb{R}^n)^2 : \|\eta\|_{\sup} < \|\eta_0\|_{\sup}+1\}$, where $\|\eta\|_{\sup} := \max\{\|\eta(x)\|_\infty : x\in\mathbb{R}^n\}$, using the $\|.\|_\infty$-norm on $\mathbb{R}^n$. Then P is an open neighbourhood of (γ_0,η_0) in $\mathcal{D}(\mathbb{R}^n,\mathbb{R}^n)^2\cong\mathcal{D}(\mathbb{R}^n,\mathbb{R}^{2n})$. To see that $f := g|_P$ is smooth, we verify the hypotheses of the Smoothness Theorem.

f is almost local. Indeed, pick a bijection $j\colon\mathbb{N}\to\mathbb{Z}^n$, $k\mapsto j_k$ and set $V_k := B_2(j_k)$, $\widetilde{V_k} := B_3(j_k)$ for $k\in\mathbb{N}$, where $B_2(j_k)$ denotes the open ball of radius 2 about j_k in $\mathbb{R}^n$, with respect to the $\|.\|_\infty$-norm. Set $r := 3+\|\eta_0\|_{\sup}$. Then, for any $x\in V_k$ and $(\gamma,\eta)\in P$, we have $f(\gamma,\eta)(x) = \gamma(x+\eta(x))$, where $\|j_k-(x+\eta(x))\|_\infty < r$. Accordingly, $f(\gamma,\eta)|_{V_k}$ only depends on $(\gamma,\eta)|_{U_k}$, where $U_k := B_r(j_k)\supseteq V_k$. We set $\widetilde{U_k} := B_{2r}(j_k)$. Then (U_k), $(\widetilde{U}_k)$, (V_k), and $(\widetilde{V}_k)$ are sequences as described in Definition 3.1. Thus f is almost local.

It remains to show that, given any compact subset $K\subseteq\mathbb{R}^n$, the mapping $f_K := f|_{P\cap C^\infty_K(\mathbb{R}^n,\mathbb{R}^{2n})}$ is smooth. To see this, given K, we observe that if $(\gamma,\eta)\in P\cap C^\infty_K(\mathbb{R}^n,\mathbb{R}^n)^2$ and $f(\gamma,\eta)(x)\neq 0$ for some $x\in\mathbb{R}^n$, then $\gamma(x+\eta(x))\neq 0$ and thus $x+\eta(x)\in\mathrm{supp}(\gamma)\subseteq K$, entailing that $x\in K-\eta(x)\subseteq K+\overline{B_r(0)} =: R$. Hence f_K takes its values in the vector subspace $C^\infty_R(\mathbb{R}^n,\mathbb{R}^n)$ of $\mathcal{D}(\mathbb{R}^n,\mathbb{R}^n)$, which also is a closed vector subspace of $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ (with the same induced topology). It therefore suffices to show that f_K is smooth as a map into $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, and has the desired differential. Let $\Gamma : C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n) \to C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ be the composition map, which is smooth by Lemma 4.2. Considering f_K as a map into $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, we have $f_K(\gamma, \eta) = \Gamma(\gamma, \mathrm{id}_{\mathbb{R}^n} + \eta)$ and thus $f_K = \Gamma \circ h$, where

$$h\colon P \cap C^\infty_K(\mathbb{R}^n, \mathbb{R}^{2n}) \to C^\infty(\mathbb{R}^n, \mathbb{R}^{2n}), \quad h(\gamma, \eta) := (\gamma, \mathrm{id}_{\mathbb{R}^n} + \eta)$$

is a restriction of a continuous affine linear map and hence smooth, with $dh(\gamma, \eta; \gamma_1, \eta_1) = (\gamma_1, \eta_1)$ for all $(\gamma, \eta) \in P \cap C^\infty_K(\mathbb{R}^n, \mathbb{R}^{2n})$ and $(\gamma_1, \eta_1) \in C^\infty_K(\mathbb{R}^n, \mathbb{R}^{2n})$. By the Chain Rule, $f_K = \Gamma \circ h$ is smooth. Furthermore, for each K containing the support $\mathrm{supp}(\gamma_0, \eta_0)$, the Chain Rule, Lemma 4.2 and the above formula for dh yield

$$\begin{aligned} dg(\gamma_0, \eta_0; \gamma_1, \eta_1) &= d(f_K)(\gamma_0, \eta_0; \gamma_1, \eta_1) = d\Gamma\big(h(\gamma_0, \eta_0), dh(\gamma_0, \eta_0; \gamma_1, \eta_1)\big) \\ &= d\gamma_0 \circ (\mathrm{id}_{\mathbb{R}^n} + \eta_0, \eta_1) + \gamma_1 \circ (\mathrm{id}_{\mathbb{R}^n} + \eta_0) \end{aligned} \tag{4.1}$$

for all $(\gamma_1, \eta_1) \in C^\infty_K(\mathbb{R}^n, \mathbb{R}^n)^2$. Now Theorem 3.2 shows that f is smooth, entailing that g is smooth on an open neighbourhood of (γ_0, η_0). As $\mathcal{D}(\mathbb{R}^n, \mathbb{R}^{2n}) = \bigcup_K C^\infty_K(\mathbb{R}^n, \mathbb{R}^{2n})$, Equation (4.1) shows that dg has the asserted form. □

5. Global coordinates for $\mathrm{Diff}_c(\mathbb{R}^n)$

In this section, we show that the unit group $\mathrm{Diff}_c(\mathbb{R}^n) := \mathrm{End}_c(\mathbb{R}^n)^\times$ of $\mathrm{End}_c(\mathbb{R}^n)$ is open in $\mathrm{End}_c(\mathbb{R}^n)$. The latter being a topological monoid, we only need to show that $\mathrm{Diff}_c(\mathbb{R}^n)$ is a neighbourhood of $\mathrm{id}_{\mathbb{R}^n}$ in $\mathrm{End}_c(\mathbb{R}^n)$.

Lemma 5.4. *Let $U := \{\gamma \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n) : \max_{x \in \mathbb{R}^n} \|d\gamma(x, \bullet)\|_{op} < 1\}$, where $\|.\|_{op}$ is the operator norm on $\mathcal{L}(\mathbb{R}^n)$ with respect to the maximum-norm on $\mathbb{R}^n$. Then U is an open zero-neighbourhood in $\mathcal{D}(\mathbb{R}^n, \mathbb{R}^n)$, and $\beta(U) \subseteq \mathrm{Diff}_c(\mathbb{R}^n)$.*

Proof. Clearly U is open. Given $\gamma \in U$, set $r := \max_{x \in \mathbb{R}^n} \|d\gamma(x, \bullet)\|_{op} < 1$.

Step 1. For each $x \in \mathbb{R}^n$, we have $d\beta(\gamma)(x, \bullet) = \mathrm{id}_{\mathbb{R}^n} + d\gamma(x, \bullet) \in \mathcal{L}(\mathbb{R}^n)^\times$, as $\|d\gamma(x, \bullet)\|_{op} < 1$. By the Inverse Function Theorem, *$\beta(\gamma)$ is a local diffeomorphism.*

Step 2: *$\beta(\gamma)$ is injective.* In fact, suppose that $x = (x_i)$ and $y = (y_i)$ are distinct elements of $\mathbb{R}^n$. Then there is $j \in \{1, \ldots, n\}$ such that $|y_j - x_j| = \|y - x\|_\infty \neq 0$. The jth coordinate of $\beta(\gamma)(y) - \beta(\gamma)(x)$ is given by

$$\beta(\gamma)(y)_j - \beta(\gamma)(x)_j = y_j - x_j + \gamma(y)_j - \gamma(x)_j\,. \tag{5.2}$$

Since

$$|\gamma(y)_j - \gamma(x)_j| = \left|\int_0^1 d\gamma(x + t(y-x), y-x)_j\, dt\right| \leq r \cdot \|y-x\|_\infty = r|y_j - x_j| < |y_j - x_j|\,,$$

Equation (5.2) shows that $\beta(\gamma)(y)_j - \gamma(\beta)(x)_j \neq 0$ and thus $\beta(\gamma)(x) \neq \beta(\gamma)(y)$.

Step 3: *$\beta(\gamma)$ is surjective.* To see this, choose a connected, compact subset $L \neq \emptyset$ of $\mathbb{R}^n$ containing $K := \mathrm{supp}(\gamma)$ in its interior. Since $\beta(\gamma)(x) = x$ for all $x \in \mathbb{R}^n \setminus K$ and $\beta(\gamma)$ is injective, we deduce that $\beta(\gamma)(K) \subseteq K$, and thus $\beta(\gamma)(L) \subseteq L$. Set

$f := \beta(\gamma)|_L^L : L \to L$. Since $\beta(\gamma)$ is a local diffeomorphism and thus an open map, the mapping $f|_{L^0}$ is open (where L^0 denotes the interior of L in $\mathbb{R}^n$). Furthermore, $f|_{L\setminus K} = \mathrm{id}_L|_{L\setminus K}$ is an open map on the open subset $L \setminus K$ of L. Since $L = L^0 \cup (L \setminus K)$, we deduce that f is an open map. Thus $f(L)$ is a non-empty, open, compact subset of the connected topological space L, and therefore $\beta(\gamma)(L) = f(L) = L$. Since $\beta(\gamma)(\mathbb{R}^n \setminus L) = \mathbb{R}^n \setminus L$ (as $\beta(\gamma)|_{\mathbb{R}^n\setminus L} = \mathrm{id}_{\mathbb{R}^n}|_{\mathbb{R}^n\setminus L}$), we deduce that $\beta(\gamma)(\mathbb{R}^n) = \mathbb{R}^n$. Thus $\beta(\gamma) \in \mathrm{Diff}_c(\mathbb{R}^n)$. □

By the preceding, $\Omega := \beta^{-1}(\mathrm{Diff}_c(\mathbb{R}^n))$ is an open subset of $\mathcal{D}(\mathbb{R}^n, \mathbb{R}^n)$.[5)] We retain the current meaning of the symbols U and Ω for the remainder of the article.

6. Smoothness of inversion

In this section, we show that inversion is smooth on $\mathrm{Diff}_c(\mathbb{R}^n)$.

Given $\gamma \in \Omega$, define $\gamma^* := \beta^{-1}(\beta(\gamma)^{-1})$. Thus

$$\gamma^* + \gamma \circ (\mathrm{id}_{\mathbb{R}^n} + \gamma^*) = 0 \quad \text{and} \quad \gamma + \gamma^* \circ (\mathrm{id}_{\mathbb{R}^n} + \gamma) = 0\,. \tag{6.1}$$

The group multiplication on $\mathrm{Diff}_c(\mathbb{R}^n)$ being smooth, it suffices to show that $*$ is smooth on U. Note that, for each compact subset $K \subseteq \mathbb{R}^n$ and $\gamma \in C^\infty_K(\mathbb{R}^n, \mathbb{R}^n) \cap U$, we have $\gamma^* \in C^\infty_K(\mathbb{R}^n, \mathbb{R}^n)$ (see proof of Lemma 5.4).

Lemma 6.1. *For each compact subset K of $\mathbb{R}^n$, the mapping*

$$f : U \cap C^\infty_K(\mathbb{R}^n, \mathbb{R}^n) \to C^\infty_K(\mathbb{R}^n, \mathbb{R}^n), \qquad \gamma \mapsto \gamma^*$$

is smooth.

Proof. Since $C^\infty_K(\mathbb{R}^n, \mathbb{R}^n)$ is a Fréchet space and $U \cap C^\infty_K(\mathbb{R}^n, \mathbb{R}^n)$ an open subset, we only need to show that f is a c^∞-map in the sense of convenient differential calculus, viz. f is smooth along smooth curves. To verify this property, we proceed along the lines of [10, p. 455]. Let $c : \mathbb{R} \to C^\infty_K(\mathbb{R}^n, \mathbb{R}^n) \cap U$, $t \mapsto c_t$ be a smooth curve; we have to show that $f \circ c$ is a smooth curve. Define $c^\wedge : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $c^\wedge(t, x) := c_t(x)$, and $(f \circ c)^\wedge : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $(f \circ c)^\wedge(t, x) := (f(c_t))(x) = (c_t)^*(x)$. Then $c^\wedge$ is smooth by [10, Thm. 3.12], and $(f \circ c)^\wedge$ satisfies the equation

$$(f \circ c)^\wedge(t, x) + c^\wedge(t, x + (f \circ c)^\wedge(t, x)) = 0$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, by (6.1). Thus $H(t, x, (f \circ c)^\wedge(t, x)) = 0$ where

$$H : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad H(t, x, y) := y + c^\wedge(t, x + y) = y + c_t(x + y)\,.$$

The partial differential of H with respect to the y-variable is given by $d_3H(t, x, y, \bullet) = \mathrm{id}_{\mathbb{R}^n} + dc_t(x + y, \bullet) = d(\beta(c_t))(x + y, \bullet) \in \mathrm{GL}(\mathbb{R}^n)$, for all t, x, y. Note that, for fixed t and x, the equation $0 = H(t, x, y) = y + c_t(x + y)$ has a unique solution y. In fact, $y := (f \circ c)^\wedge(t, x)$ is one solution; if there was a second solution $y_1 \neq y$, we would have

$$y - y_1 = c_t(x + y_1) - c_t(x + y) = \int_0^1 dc_t(x + y + s \cdot (y_1 - y), y_1 - y)\, ds\,.$$

[5)] If $n = 1$, it easy to see that $\Omega = \{\gamma \in \mathcal{D}(\mathbb{R}) : (\forall x \in \mathbb{R})\ \gamma'(x) > -1\}$.

As $c_t \in U$, in this equation the right hand side has $\|.\|_\infty$-norm properly smaller than the left hand side (cf. proof of Lemma 5.4), which is absurd; thus a second solution $y_1 \neq y$ cannot exist. Due to global uniqueness of solutions, we deduce from the standard Implicit Function Theorem that $(f \circ c)^\wedge$ is a smooth map. Therefore $f \circ c$ is smooth ([10, Thm. 3.12] or [15, Thm. III.4]). $\square$

Given $R > 0$, set $U_R := \{\gamma \in U : \|\gamma\|_{\sup} < R\}$. Then U_R is an open subset of U, and $U = \bigcup_{R>0} U_R$.

Lemma 6.2. *We have $B_r(x) \subseteq (\mathrm{id}_{\mathbb{R}^n} + \gamma)(B_{r+R}(x))$, for all $r > 0$, $x \in \mathbb{R}^n$, and $\gamma \in U_R$.*

Proof. Since $\mathrm{id}_{\mathbb{R}^n} + \gamma = \beta(\gamma) \in \mathrm{Diff}(\mathbb{R}^n)$ is a bijection, for every $y \in B_r(x)$ there is a uniquely determined element $z \in \mathbb{R}^n$ such that $z + \gamma(z) = \beta(\gamma)(z) = y$. Then $\|z - x\|_\infty = \|y - \gamma(z) - x\|_\infty \leq \|y - x\|_\infty + \|\gamma(z)\|_\infty < r + R$, whence $z \in B_{r+R}(x)$. $\square$

Lemma 6.3. *If $R, r > 0$ and $x \in \mathbb{R}^n$, then for all $\gamma, \eta \in U_R$ such that $\gamma|_{B_{r+R}(x)} = \eta|_{B_{r+R}(x)}$, we have $\gamma^*|_{B_r(x)} = \eta^*|_{B_r(x)}$.*

Proof. Let $y \in B_r(x)$. Lemma 6.2 gives $z \in B_{r+R}(x)$ such that $z + \gamma(z) = y$. Then

$$\gamma^*(y) = \gamma^*(z + \gamma(z)) = -\gamma(z) = -\eta(z) = \eta^*(z + \eta(z)) = \eta^*(z + \gamma(z)) = \eta^*(y)\,,$$

using Equation (6.1) to obtain the second and forth equality. $\square$

Lemma 6.4. *The mapping $f\colon U \to \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n)$, $\gamma \mapsto \gamma^*$ is smooth.*

Proof. In view of the Smoothness Theorem 3.2 and Lemma 6.1, it suffices to prove that, for each $R > 0$, the mapping $f_R := f|_{U_R}$ is almost local. Define $V_k := B_2(j_k)$, $\widetilde{V}_k := B_3(j_k)$, $U_k := B_{2+R}(j_k)$, $\widetilde{U}_k := B_{3+R}(j_k)$ for $k \in \mathbb{N}$, where $j_\bullet$ denotes a bijection $\mathbb{N} \to \mathbb{Z}^n$. In view of Lemma 6.3, f_R, together with the sequences $(V_k)_{k\in\mathbb{N}}$ $(\widetilde{U}_k)_{k\in\mathbb{N}}$, $(V_k)_{k\in\mathbb{N}}$, and $(\widetilde{V}_k)_{k\in\mathbb{N}}$, satisfies the conditions formulated in Definition 3.1. $\square$

Summing up:

Theorem 6.5. $\mathrm{Diff}_c(\mathbb{R}^n)$*, equipped with the smooth manifold structure making the bijection $\alpha\colon \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n) \supseteq \Omega \to \mathrm{Diff}_c(\mathbb{R}^n)$, $\gamma \mapsto \mathrm{id}_{\mathbb{R}^n} + \gamma$ a diffeomorphism, is a smooth Lie group in the sense of Milnor* [14]. $\square$

7. The Lie group structure on Diff($\mathbb{R}^n$)

We show that the group $\mathrm{Diff}(\mathbb{R}^n)$ of all diffeomorphisms of $\mathbb{R}^n$ can be made a smooth Lie group in the sense of Milnor, containing $\mathrm{Diff}_c(\mathbb{R}^n)$ as an open submanifold.

Note that $\mathrm{Diff}_c(\mathbb{R}^n)$ is a normal subgroup of $\mathrm{Diff}(\mathbb{R}^n)$: Given $\gamma \in \mathrm{Diff}(\mathbb{R}^n)$ and $\eta = \mathrm{id}_{\mathbb{R}^n} + \sigma \in \mathrm{Diff}_c(\mathbb{R}^n)$, the diffeomorphism $I_\gamma(\eta) := \gamma \circ \eta \circ \gamma^{-1} \in \mathrm{Diff}(\mathbb{R}^n)$ satisfies $I_\gamma(\eta)(x) = x$ for all $x \in \mathbb{R}^n$ outside the compact set $\gamma(\mathrm{supp}(\sigma))$; thus $I_\gamma(\eta) \in \mathrm{Diff}_c(\mathbb{R}^n)$.

Theorem 7.1. *There is a uniquely determined smooth manifold structure on the group* $\mathrm{Diff}(\mathbb{R}^n)$ *turning it into a Lie group and making* $\mathrm{Diff}_c(\mathbb{R}^n)$*, equipped with the Lie group structure described in Theorem* 6.5*, an open submanifold.*

Proof. As $\mathrm{Diff}_c(\mathbb{R}^n) \subseteq \mathrm{Diff}(\mathbb{R}^n)$ already is a smooth Lie group, in view of the "local characterization of Lie groups" stated in [3, Prop. 1.13], we only need to show that, for each $\gamma \in \mathrm{Diff}(\mathbb{R}^n)$, the automorphism $J_\gamma \colon \mathrm{Diff}_c(\mathbb{R}^n) \to \mathrm{Diff}_c(\mathbb{R}^n)$, $\eta \mapsto \gamma \circ \eta \circ \gamma^{-1}$ of the normal subgroup $\mathrm{Diff}_c(\mathbb{R}^n) \subseteq \mathrm{Diff}(\mathbb{R}^n)$ is smooth. Thus, in terms of the global chart $\alpha \colon \Omega \to \mathrm{Diff}(\mathbb{R}^n)$, $\eta \mapsto \mathrm{id}_{\mathbb{R}^n} + \eta$, we have to show that

$$f := \alpha^{-1} \circ J_\gamma \circ \alpha \colon \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n), \quad \eta \mapsto \gamma \circ (\mathrm{id}_{\mathbb{R}^n} + \eta) \circ \gamma^{-1} - \mathrm{id}_{\mathbb{R}^n}$$

is smooth.

Step 1: *f is almost local.* Indeed, given any bijection $j_\bullet : \mathbb{N} \to \mathbb{Z}^n$, we define $V_k := \gamma(B_2(j_k))$, $\widetilde{V}_k := \gamma(B_3(j_k))$, $U_k := B_2(j_k)$, $\widetilde{U}_k := B_3(j_k)$ for $k \in \mathbb{N}$. Then f, together with the sequences of open sets $(U_k)_{k\in\mathbb{N}}$, $(\widetilde{U}_k)_{k\in\mathbb{N}}$, $(V_k)_{k\in\mathbb{N}}$, and $(\widetilde{V}_k)_{k\in\mathbb{N}}$, apparently satisfies the conditions of almost locality described in Definition 3.1.

Step 2: *$f|_{\Omega \cap C^\infty_K(\mathbb{R}^n,\mathbb{R}^n)}$ is smooth, for each compact subset $K \subseteq \mathbb{R}^n$.* To see this, note first that, for each $\eta \in \Omega \cap C^\infty_K(\mathbb{R}^n, \mathbb{R}^n)$, the map $f(\eta) = J_\gamma(\mathrm{id}_{\mathbb{R}^n} + \eta) - \mathrm{id}_{\mathbb{R}^n}$ vanishes outside $\gamma(\mathrm{supp}(\eta)) \subseteq \gamma(K)$. Thus $f|_{\Omega \cap C^\infty_K(\mathbb{R}^n,\mathbb{R}^n)}$ is a map into the closed vector subspace $C^\infty_{\gamma(K)}(\mathbb{R}^n, \mathbb{R}^n)$ of $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, and we only need to show that $f|_{\Omega \cap C^\infty_K(\mathbb{R}^n,\mathbb{R}^n)}$ is smooth as a map into $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. But this is clear, because

$$f(\eta) \;=\; \Gamma\big(\gamma, \Gamma(\mathrm{id}_{\mathbb{R}^n} + \eta, \gamma^{-1})\big) - \mathrm{id}_{\mathbb{R}^n} \;=\; \Gamma\big(\gamma, \Gamma(h(\eta), \gamma^{-1})\big) - \mathrm{id}_{\mathbb{R}^n}$$

for each $\eta \in \Omega \cap C^\infty_K(\mathbb{R}^n, \mathbb{R}^n)$, where $\Gamma \colon C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n) \to C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ is the composition map (whose smoothness we recalled in Lemma 4.2), and where $h : \Omega \cap C^\infty_K(\mathbb{R}^n, \mathbb{R}^n) \to C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, $h(\eta) := \mathrm{id}_{\mathbb{R}^n} + \eta$ is a restriction of a continuous affine linear map and hence smooth as well. Using the Chain Rule, we deduce that indeed $f|_{\Omega \cap C^\infty_K(\mathbb{R}^n,\mathbb{R}^n)}$ is smooth as a map into $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. The hypotheses of Theorem 3.2 having been verified, f is smooth. □

Acknowledgements

The research was supported by Deutsche Forschungsgemeinschaft (FOR 363/1-1).

References

[1] Frölicher, A. and A. Kriegl: Linear Spaces and Differentiation Theory; John Wiley, 1988

[2] Glöckner, H.: Infinite-dimensional Lie groups without completeness restrictions; in: Strasburger, A. et al. (Eds.): Geometry and Analysis on Finite- and Infinite-Dimensional Lie Groups; Banach Center Publications **55**, Warsaw, 2002

[3] ______: Lie group structures on quotient groups and universal complexifications for infinite-dimensional Lie groups; J. Funct. Analysis **194** (2002), 347–409

[4] ______: Lie groups over non-discrete topological fields; arXiv:math.GR/0408008

[5] ______: Differentiable mappings between spaces of sections; submitted

[6] ______: Discontinuous non-linear mappings on locally convex direct limits; submitted

[7] ______: Patched locally convex spaces, almost local mappings, and diffeomorphism groups of non-compact manifolds; in preparation

[8] Hamilton, R.: The inverse function theorem of Nash and Moser; Bull. Amer. Math. Soc. **7** (1982), 65–222

[9] Keller, H. H.: Differential Calculus in Locally Convex Spaces; Springer-Verlag, 1974

[10] Kriegl, A. and P. W. Michor: The Convenient Setting of Global Analysis; Amer. Math. Soc., 1997

[11] Leslie, J.: On a differential structure for the group of diffeomorphism; Topology **6** (1967), 263–271

[12] Michor, P. W.: Manifolds of Differentiable Mappings; Shiva, 1980

[13] Milnor, J.: On infinite-dimensional Lie groups; preprint, Institute for Advanced Studies, Princeton, 1982

[14] ______: Remarks on infinite-dimensional Lie groups; pp. 1008–1057 in: DeWitt, B., and R. Stora (Eds.): Relativity, Groups and Topology II; North-Holland, 1983

[15] Neeb, K.-H.: Infinite-dimensional groups and their representations; in: Huckleberry, A. and T. Wurzbacher (Eds.): Infinite-Dimensional Kähler Manifolds; Birkhäuser, 2001

[16] Omori, H.: Infinite-Dimensional Lie Groups; Amer. Math. Soc., 1997

[17] Tatsuuma, N., H. Shimomura and T. Hirai: On group topologies and unitary representations of inductive limits of topological groups and the case of the group of diffeomorphisms; J. Math. Kyoto Univ. **38** (1998), 551–578

[18] Walter, B.: Fréchet-Liegruppen von Diffeomorphismen von $\mathbb{R}^n$; Diplomarbeit, TU Darmstadt, in preparation (supervision: H. Glöckner)

TU Darmstadt
FB Mathematik AG 5
Schlossgartenstr. 7
64289 Darmstadt, Germany
E-mail:
gloeckner@mathematik.tu-darmstadt.de