Orbits of triples in the Shilov boundary of a bounded symmetric domain

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Abstract. Let $\mathcal{D}$ be a bounded symmetric domain of tube type, S its Shilov boundary, and G its group of biholomorphic automorphisms. We classify the orbits of the identity component G of the group of biholomorphic maps of $\mathcal{D}$ in the set $S \times S \times S$.
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Introduction

Let $\mathcal{D}$ be a bounded symmetric domain in a (finite dimensional) complex vector space V, realized as a circular domain, let $G := \mathrm{Aut}(\mathcal{D})_0$ be the identity component of its group of biholomorphic transforms of $\mathcal{D}$ and let S be its Shilov boundary. The action of any element of G extends to a neigbourhood of $\overline{\mathcal{D}}$, and hence G acts on S. It is well known that this action is transitive. The main result of the present paper is a classification of the G-orbits in the set $S \times S \times S$ of triples in S, when $\mathcal{D}$ is of *tube type*.

The action of G on $S \times S$ can be easily studied as an application of Bruhat theory, and the description of the orbits is the same, whether $\mathcal{D}$ is of tube type or not. But for triples, there is a drastic difference between tube type domains and non tube type domains. In the first case, there is a finite number of orbits, whereas there are an infinite number of orbits for a non tube type domain.

Let r be the rank of $\mathcal{D}$. The notion of r*-polydiscs* (and their corresponding Shilov boundaries called r*-torus*) plays an important role in the analysis of the orbits. On one hand they are the "complexifications" of the *maximal flats* of $\mathcal{D}$ (in the sense of the geometry of Riemannian symmetric spaces). On the other hand, a r-polydisc in the usual sense is a set of the form

$$\Delta^r = \left\{ \sum_{j=1}^{r} \zeta_j x_j \colon |\zeta_j| < 1, 1 \leq j \leq r \right\} ,$$

where the x_j are linearly independent elements in V. The space V has a natural structure of a positive hermitian Jordan triple system, and in particular, it has a natural (Banach) norm, called the *spectral norm*, for which the domain $\mathcal{D}$ is realized as the open unit ball. One of the results we prove is that such a polydisc, constructed on vectors x_j of norm 1 lies in $\mathcal{D}$ if and only if the $(x_j)_{1\leq j\leq r}$ form a *Jordan frame* for V.

Fix an r-torus $T \subseteq S$ arising as the Shilov boundary of an r-polydisc associated to a Jordan frame. The main step towards the classification of the orbits of G in $S \times S \times S$ is the result that any triple in S can be sent by an element of G to a triple in T. This requires that $\mathcal{D}$ is of tube type, and this property really distinguishes tube type domains from non tube type domains. Once this result is obtained, the classification becomes easy, because the problem is

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reduced to the case of a polydisc, and further, using the product structure, to the case of the unit disc in $\mathbb{C}$, where the situation is easy to analyze. The generalized Maslov index (see [CØ01], [Cl04]) comes in as a subtle invariant for triples.

A special case of this theorem was known before. If $\mathcal{D}$ is the Siegel domain (the unit ball in the space of complex symmetric matrices $\mathrm{Sym}_r(\mathbb{C})$, then the group G is the projective symplectic group $\mathrm{PSp}_{2r}(\mathbb{R}) := \mathrm{Sp}_{2r}(\mathbb{R})/\{\pm\mathbf{1}\}$, and the Shilov boundary of $\mathcal{D}$ can be identified with the Lagrangian manifold (the set of Lagrangian subspaces of $\mathbb{R}^{2r}$). Then the orbits of triples of Lagrangians have been described (see [KS90, p.492]), using linear symplectic algebra techniques. Related results can be found in [FMS04], and in particular their Proposition 4.3 (which they deduce from [KS90]) is, for this specific example, equivalent to our Theorem III.1. The main point of [FMS04] is a description of the orbits of the action of the maximal compact subgroup group $\mathrm{U}_n(\mathbb{C})$ of $\mathrm{Sp}_{2n}(\mathbb{R})$ on triples of Lagrangians are classified, but this is a different problem.

Since S is in particular a generalized flag manifold of G, i.e., of the form G/P for some parabolic subgroup P, the natural question arises to which extent results similar to the ones obtained in this paper could be valid for other generalized flag manifolds. The natural background for this problem is the following. If $P_1, \ldots, P_k$ are parabolic subgroups of G, then the product manifold

$$M := G/P_1 \times \ldots \times G/P_k$$

is called a multiple flag manifold of finite type if the diagonal action of G on M has only finitely many orbits. For $k=1$ we always have only one orbit, and for $k=2$ the finiteness of the set of orbits follows from the Bruhat decomposition of G. For $G = \mathrm{GL}_n(\mathbb{K})$ or $G = \mathrm{Sp}_{2n}(\mathbb{K})$ and $\mathbb{K}$ is an algebraically closed field of characteristic zero, it has been shown in [MWZ99/00] that finite type implies $k \leq 3$, and for $k=3$ the triples of parabolics leading to multiple flag manifolds of finite type are described and the G-orbits in these manifolds classified. The main technique to achieve these classifications was the representation theory of quivers. In [Li94], Littelmann considers general simple algebraic groups over $\mathbb{K}$ and describes all multiple flag manifolds of finite type for $k=3$ under the assumption that P_1 is a Borel subgroup and P_2, P_3 are maximal parabolics. Actually Littelmann considers the condition that $B = P_1$ has a dense orbit in $G/P_2 \times G/P_3$, but the results in [Vi86] show that this implies the finiteness of the number of B-orbits and hence the finiteness of the number of G-orbits in $G/B \times G/P_2 \times G/P_3$. From Littelmann's classification one can easily reads off that for a maximal parabolic P in G the triple product $(G/P)^3$ is of finite type if and only if the unipotent radical U of P is abelian and in two exceptional situations. If U is abelian, then P is the maximal parabolic defined by a 3-grading of $\mathfrak{g} = \mathbf{L}(G)$, so that G/P is the conformal completion of a Jordan triple (cf. [BN05] for a discussion of such completions in an abstract setting). The first exceptional case, where U is not abelian, corresponds to $G = \mathrm{Sp}_{2n}(\mathbb{K})$, where $G/P = \mathbb{P}_{2n-1}(\mathbb{K})$ is the projective space of $\mathbb{K}^{2n}$, U is the $(2n-1)$-dimensional Heisenberg group and the Levi complement is $\mathrm{Sp}_{2n-2}(\mathbb{K}) \times \mathbb{K}^\times$. In the other exceptional case $G = \mathrm{SO}_{2n}(\mathbb{K})$ and G/P is the highest weight orbit in the 2^n-dimensional spin representation of the covering group $\widetilde{G} = \mathrm{Spin}_{2n}(\mathbb{K})$ of G. Here $U \cong \Lambda^2(\mathbb{K}^n) \oplus \mathbb{K}^n$ also is a 2-step nilpotent group and the Levi complement acts like $\mathrm{GL}_n(\mathbb{K})$ on this group. It seems that the positive finiteness results have a good chance to carry over to the split forms of groups over more general fields and in particular to $\mathbb{K} = \mathbb{R}$, but for real groups not much seems to be known about multiple flag manifolds of finite type.

If $M = (G/P)^3$ is a multiple flag manifold of finite type and $P = U \rtimes L$ is a Levi decomposition of P, then L is the simultaneous stabilizer of a pair in $(G/P)^2$ with an open orbit, and this implies that the conjugation action of L on U has only finitely many orbits. A closely related but different problem is the question when the conjugation action of P on U has finitely many orbits. According to a result of Richardson, there is always a dense orbit, but this does not imply finiteness. For more specific results on this question we refer to [PR97] and [HR99].

It is perhaps worthwhile to stress that the proofs we give are one more occurrence of the interaction between complex analysis of a bounded symmetric domains and the geometry of convex sets in the normed space V. The notions of extremal points or faces of a convex set do play an important role in our study.

The contents of the paper is as follows. In Section I we first recall several facts on bounded symmetric domains. Our main sources are Loos' lecture notes [Lo77] and Satake's book [Sa80]. For results concerning Euclidean Jordan algebras we use [FK94]. The main result of Section I is a classification of the G-orbits in the set of quasi-invertible (=transversal) pairs in $\overline{\mathcal{D}}$ (Theorem I.18). For this classification we do not need that $\mathcal{D}$ is of tube type. For the analysis of G-orbits in $S \times S \times S$ we only need the simpler case of pairs (x, y), where $x \in S$. For this case we give a more direct shorter proof, but we think that the general case might also be useful in other situations.

The main tool for the classification of G-orbits in $S \times S \times S$ is the characterization of the transversality relation on $\overline{\mathcal{D}}$ in terms of faces of the compact convex set $\overline{\mathcal{D}}$: Two elements $x, y \in \overline{\mathcal{D}}$ are transversal if and only if they are not contained in a proper face of $\overline{\mathcal{D}}$ (Theorem II.12). This characterization is also valid for non tube type domains. A key concept for the classification is the notion of the rank of a face F of $\overline{\mathcal{D}}$. For an irreducibel domain $\mathcal{D}$ of rank r it takes values in the set $\{0, 1, \ldots, r\}$ and classifies the G-orbits in the set of faces of $\mathcal{D}$. It is normalized in such a way that the rank of $\overline{\mathcal{D}}$ as a face is zero and that the extreme points, i.e., the elements in the Shilov boundary, are faces of rank r. If $\mathrm{Face}(x_1, \ldots, x_n)$ denotes the face generated by the subset $\{x_1, \ldots, x_n\}$ of $\overline{\mathcal{D}}$, then the function

$$\overline{\mathcal{D}}^n \to \{0, 1 \ldots, r\}, \quad (x_1, \ldots, x_n) \mapsto \operatorname{rank} \mathrm{Face}(x_1, \ldots, x_n)$$

is an invariant for the G-action on $\overline{\mathcal{D}}^n$.

In these terms, two elements $x, y \in \overline{\mathcal{D}}$ are transversal if and only if $\operatorname{rank} \mathrm{Face}(x, y) = 0$. In Section III we use this fact to show that for a domain $\mathcal{D}$ of tube type every triple in S is conjugate to a triple in the Shilov boundary T of a maximal polydisc Δ^r defined by a Jordan frame. This reduces the classification of G-orbits in $S \times S \times S$ to the description of intersections of these orbits with T^3. This is fully achieved in Section V by assigning a 5-tuple of integer invariants to each orbit and by showing that triples with the same invariant lie in the same orbit. The first four components of this 5-tuple are

$$(\operatorname{rank} \mathrm{Face}(x_1, x_2, x_3), \operatorname{rank} \mathrm{Face}(x_1, x_2), \operatorname{rank} \mathrm{Face}(x_2, x_3), \operatorname{rank} \mathrm{Face}(x_1, x_3)).$$

The fifth component is defined as the Maslov index $\iota(x_1, x_2, x_3)$ which is discussed in some detail in Section IV. Note that if (x_1, x_2, x_3) is transversal in the sense that all pairs (x_1, x_2), (x_2, x_3), (x_3, x_1) are transversal, then the first four components of the invariant vanish, which implies that the G-orbits in the set of transversal triples are classified by the Maslov index.

We conclude the paper with a brief discussion of how the classification of the G-orbits in $S \times S$ can be interpreted in terms of the Bruhat decomposition of G. We thank L. Kramer and H. Rubenthaler for comments and references concerning multiple flag manifolds of finite type.

I. Classification of orbits of transversal pairs in the boundary

Let $\mathcal{D}$ be an irreducible circular bounded symmetric domain, so that $\mathcal{D}$ is the open unit ball for a norm on a complex vector space V ([Lo77, Th.4.1]). In this section we describe the G-orbits in the set of quasi-invertible pairs of elements in the closure of $\mathcal{D}$ (cf. Theorem I.18 below). Here we do not have to assume that $\mathcal{D}$ is of tube type.

I.1. The associated Jordan triple. On V we consider the hermitian Jordan triple product $\{\cdot, \cdot, \cdot\}: V^3 \to V$ that is uniquely determined by the property that for each $v \in V$ the vector field given by the function

$$\xi_v : V \to V, \quad z \mapsto v - Q(z).v = v - \{z, v, z\}$$

generates a one-parameter group of automorphisms of $\mathcal{D}$ ([Lo77, Lemma 4.3]). Note that for each $v \in V$ the map $(z, w) \mapsto \{z, v, w\}$ is symmetric and complex bilinear, and that the maps $z \mapsto \{a, z, b\}$ are antilinear. For $x, y \in V$ we define $Q(x)$ and $x \Box y \in \mathrm{End}(V)$ by

$$Q(x).y := \{x, y, x\} \quad \text{and} \quad x \Box y.z := \{x, y, z\}.$$

The Jordan triple structure on V used by Loos is $\{x,y,z\}' = 2\{x,y,z\}$, so that his quadratic representation is given by $Q'(x,y) = 2\{x,y,z\}$, but since Loos defines $Q'(x)$ as $\frac{1}{2}Q'(x,x)$, we obtain the same operators $Q(x) = Q'(x)$.

I.2. Tripotents and Peirce decomposition. An element $e \in V$ is called a *tripotent* if $e = \{e,e,e\}$. For a tripotent $e \in V$ let $V_j := V_j(e)$ denote the j-eigenspace of the operator $2e\Box e$. Then we obtain the corresponding *Peirce decomposition of* V:

$$V = V_0 \oplus V_1 \oplus V_2$$

([Lo77, Th. 3.13]). Since $e\Box e$ is a Jordan triple derivation, we have the Peirce rules

$$\{V_i, V_j, V_k\} \subseteq V_{i-j+k}, \tag{1.1}$$

which imply in particular that each space V_j is a Jordan subtriple. In addition, we have

$$V_0 \Box V_2 = V_2 \Box V_0 = \{0\}. \tag{1.2}$$

The Jordan triple V also carries a Jordan algebra structure, denoted $V^{(e)}$, given by

$$ab := L(a).b := \{a,e,b\}.$$

Then e is an idempotent in $V^{(e)}$ because $ee = \{e,e,e\} = e$, and the Peirce decomposition of V with respect to the tripotent e coincides with the Peirce decomposition of the Jordan algebra $V^{(e)}$ with respect to the idempotent e.

The multiplication operators in $V^{(e)}$ are given by $L(a) = a\Box e$, so that $L(e)\,|_{V_2} = \mathrm{id}_{V_2}$ implies that (V_2, e) is a unital Jordan subalgebra of $V^{(e)}$. For the quadratic representation in $V^{(e)}$ we have

$$P(e) = 2L(e)^2 - L(e^2) = 2L(e)^2 - L(e) = (2L(e) - \mathbf{1})L(e),$$

so that $P(e) = Q(e)^2$ vanishes on $V_0 \oplus V_1$ and restricts to the identity on V_2. It follows in particular that $(V_2, e, Q(e))$ is an involutive Jordan algebra (cf. [Lo77, Th. 3.13]).

I.3. Orbits in $\overline{\mathcal{D}}$. Two tripotents $e, f \in V$ are said to be *orthogonal* if $f \in V_0(e)$. In view of the Peirce rules (1.2), this implies $\{f,f,e\} = \{e,f,f\} = (e\Box f).f = 0$, so that we also have $e \in V_0(f)$, i.e., orthogonality is a symmetric relation. If this is the case, then $e+f$ also is a tripotent because the relations $e\Box f = f\Box e = 0$ lead to

$$\{e+f, e+f, e+f\} = \{e,e,e+f\} + \{f,f,e+f\} = \{e,e,e\} + \{f,f,f\} = e+f.$$

We call a non-zero tripotent e *primitive* if it cannot be written as a sum of two non-zero orthogonal tripotents and e is said to be *maximal* if there is no non-zero tripotent orthogonal to e. A maximal tuple $(c_1,\ldots,c_r)$ of mutually orthogonal primitive tripotents is called a *Jordan frame in* V and $r = \operatorname{rank}\mathcal{D}$ is called the *rank of* $\mathcal{D}$. We fix a Jordan frame $(c_1,\ldots,c_r)$. For $k = 0,1,\ldots,r$ we then obtain tripotents

$$e_k := c_1 + \ldots + c_k,$$

where it is understood that $e_0 = 0$.

We recall that each bounded symmetric domain $\mathcal{D}$ can be decomposed in a unique fashion as a direct product of indecomposable, also called *irreducible*, bounded symmetric domains:

$$\mathcal{D} = \mathcal{D}_1 \times \ldots \times \mathcal{D}_m. \tag{1.3}$$

Then the connected group $G := \mathrm{Aut}(\mathcal{D})_0$ satisfies

$$G \cong G_1 \times \ldots \times G_m, \qquad \text{where} \qquad G_j := \mathrm{Aut}(\mathcal{D}_j)_0. \tag{1.4}$$

If $\mathcal{D}$ is irreducible, then G has exactly $r+1$ orbits in the closure $\overline{\mathcal{D}}$ of $\mathcal{D}$ in V and $e_0, \ldots, e_r$ form a set of representatives (cf. [Sa80, Th. III.8.7]). For $k=0$ we have $G.e_0 = \mathcal{D}$ and for $k=r$ we obtain the *Shilov boundary* $G.e_r = S$ ([Sa80, Th. III.8.14]). We define the *rank of* $x \in \overline{\mathcal{D}}$ by

$$\operatorname{rank} x = k \quad \text{for} \quad x \in G.e_k$$

and observe that the rank function is G-invariant and classifies the G-orbits in $\overline{\mathcal{D}}$.

If $\mathcal{D}$ is not irreducible, then (1.3/4) imply that the orbit of $x = (x_1, \ldots, x_m) \in \overline{\mathcal{D}} = \prod_{j=1}^m \overline{\mathcal{D}_j}$ is determined by the m-tuple

$$(\operatorname{rank} x_1, \ldots, \operatorname{rank} x_m) \in \mathbb{N}_0^m.$$

Here $(0, \ldots, 0)$ corresponds to elements in $\mathcal{D}$ and $(\operatorname{rk} \mathcal{D}_1, \ldots, \operatorname{rk} \mathcal{D}_m)$ to elements in the product set $S = S_1 \times \ldots \times S_m$.

I.4. Spectral decomposition and spectral norm. Let K be the stabilizer of $0 \in \mathcal{D}$ in G. Then K acts as a group of automorphism on the Jordan triple V and each element $z \in V$ is conjugate under K to an element in $\operatorname{span}_{\mathbb{R}}\{c_1, \ldots, c_r\}$. For $k.z = \sum_{j=1}^r \lambda_j c_j$ the number

$$|z| := \max\{|\lambda_1|, \ldots, |\lambda_r|\}$$

is called the *spectral norm of* z. Then the elements $\widetilde{c}_j := k^{-1}.c_j$ are orthogonal tripotents with

$$z = \sum_{j=1} \lambda_j \widetilde{c}_j,$$

which is the spectral decomposition of z. The spectral norm $|\cdot|$ is indeed a norm on V with

$$\mathcal{D} = \{z \in V : |z| < 1\}. \tag{1.5}$$

The following theorem relates the holomorphic arc-components in $\partial\mathcal{D}$ to the tripotents in V.

Theorem I.5. ([Lo77, Th. 6.3]) *For each holomorphic arc-component A of $\partial\mathcal{D}$ there exists a tripotent e in A such that*

$$A = A_e := e + \mathcal{D}_e, \quad \textit{where} \quad \mathcal{D}_e := \mathcal{D} \cap V_0(e)$$

is a bounded symmetric domain in $V_0(e)$. The map $e \mapsto A_e$ yields a bijection from the set of non-zero tripotents of V onto the set of holomorphic arc-components of $\partial\mathcal{D}$. The Shilov boundary S coincides with the set of maximal tripotents.

An element $x \in \overline{\mathcal{D}}$ is contained in A_e if and only if

$$e = \lim_{n\to\infty} Q(x)^n.x. \tag{1.6}$$

■

I.6. Conformal completion of V. Let $G_{\mathbb{C}}$ denote the universal complexification of the connected real Lie group G and τ the anti-holomorphic involution of $G_{\mathbb{C}}$ for which G is the identity component of the fixed point group $G_{\mathbb{C}}^{\tau}$. Then the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G_{\mathbb{C}}$ has a faithful realization by polynomial vector fields of degree ≤ 2 on V, which leads to a 3-grading

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-,$$

where $V \cong \mathfrak{g}_+$ is the space of constant vector fields, $\mathfrak{g}_0$ consists of linear vector fields, and $\mathfrak{g}_-$ is the set of quadratic vector fields corresponding to the maps $z \mapsto Q(z).v = \{z, v, z\}$ for $v \in V$. By construction of the triple product, the vector fields ξ_v correspond to elements of the real Lie algebra $\mathfrak{g} = \mathbf{L}(G)$, which implies that τ maps the constant vector field v to the quadratic vector field $z \mapsto -\{z, v, z\}$. Hence τ reverses the grading of $\mathfrak{g}_{\mathbb{C}}$, i.e., $\tau(\mathfrak{g}_j) = \mathfrak{g}_{-j}$ for $j \in \{+, -, 0\}$. The Jordan triple structure on $V \cong \mathfrak{g}_+$ then satisfies

$$\{x, y, z\} = \frac{1}{2}[[x, \tau.y], z]. \tag{1.7}$$

The subgroups

$$G^{\pm} := \exp \mathfrak{g}_{\pm} \quad \text{and} \quad G^0 := \{g \in G_{\mathbb{C}} \colon (\forall j)\ \operatorname{Ad}(g)\mathfrak{g}_j = \mathfrak{g}_j\}$$

satisfy

$$G^{\pm} \cap G^0 = \{\mathbf{1}\} \quad \text{and} \quad (G^{\pm} \rtimes G^0) \cap G^{\mp} = \{\mathbf{1}\}.$$

Therefore $P^{\pm} := G^{\pm}G^0 \cong G^{\pm} \rtimes G^0$ are subgroups of $G_{\mathbb{C}}$, and we obtain an embedding

$$V \hookrightarrow X := G_{\mathbb{C}}/P^-, \quad v \mapsto \exp v \cdot P^-,$$

called the *conformal completion of* V. The elements of G^+ act on $V \subseteq X$ by translations

$$t_v \colon x \mapsto x + v \tag{1.8}$$

because $\exp v \exp x P^- = \exp(v + x)P^-$. We further have $\tau(G^{\pm}) = G^{\mp}$ and $\tau(G^0) = G^0$.

For $w \in V$ we write $\widetilde{t}_w$ for the map $X \to X$ induced by the element $\exp(-\tau(w)) = (\tau(\exp w))^{-1}$. For $v \in V$ the condition $\widetilde{t}_w.v \in V$, where V is considered as a subset of X, is then equivalent to the invertibility of

$$\mathbf{1} + \operatorname{ad} v \operatorname{ad}(-\tau.w) + \frac{1}{4}(\operatorname{ad} v)^2(\operatorname{ad} \tau.w)^2 = \mathbf{1} - \operatorname{ad} v \operatorname{ad}(\tau.w) + \frac{1}{4}(\operatorname{ad} v)^2 \circ \tau \circ (\operatorname{ad} \tau)^2 \circ \tau \tag{1.9}$$

([BN05, Cor. 1.10]). In view of (1.7), this is precisely the Bergman operator

$$B(v, w) = \mathbf{1} - 2v\Box w + Q(v)Q(w).$$

We further have in V the relation

$$\widetilde{t}_w.v = B(v, w)^{-1}.(v - Q(v).w). \tag{1.10}$$

I.7. Quasi-invertibility and transversality. A pair $(x, y) \in V$ is called *quasi-invertible* if $B(x, y) \in \operatorname{End}(V)$ is invertible. We write $x \top y$ if (x, y) is quasi-invertible and say that x is *transversal* to y. We write $x^{\top} := \{y \in V \colon x \top y\}$ for the set of all elements in V transversal to x.

In the Jordan algebra $V^{(y)}$ with the product $ab := \{a, y, b\}$ we have $L(a) = a\Box y$ and $P(a) = Q(a)Q(y)$ ([NØ04, App. A]), so that

$$B(x, y) = \operatorname{id}_V - 2L(x) + P(x),$$

and in the unital Jordan algebra $V^{(y)} \times \mathbb{R}$ with the identity element $\mathbf{1} := (0, 1)$ we have

$$\mathbf{1} - 2L(x) + P(x) = P(\mathbf{1}, \mathbf{1}) - 2P(\mathbf{1}, x) + P(x, x) = P(\mathbf{1} - x),$$

i.e., the quasi-invertibility of (x, y) is equivalent to the quasi-invertibility of x in the Jordan algebra $V^{(y)}$.

I.8. The $\mathfrak{sl}_2$-triple associated to a tripotent. Let $e \in V$ be a tripotent, $f := \tau(e)$, $h := [e, f]$ and $\mathfrak{g}_e := \operatorname{span}_{\mathbb{R}}\{h, e, f\}$. Then

$$[h, e] = 2\{e, e, e\} = 2e \quad \text{and} \quad [h, f] = \tau[\tau h, e] = -\tau[h, e] = -2\tau e = -2f,$$

so that $\mathfrak{g}_e \cong \mathfrak{sl}_2(\mathbb{R})$ is a 3-dimensional subalgebra of $\mathfrak{g}$ with $\mathfrak{g}_e^{\tau} = \mathbb{R}(e + f)$.

(a) The operator $\mathrm{ad}_V h = 2e\Box e$ is diagonalizable with possible eigenvalues $0, 1, 2$. The corresponding eigenspace decomposition $V = V_0 \oplus V_1 \oplus V_2$ is the Peirce decomposition of the Jordan algebra $V^{(e)}$ with multiplication $ab := \{a, e, b\}$ with respect to the idempotent e, i.e., $2L(e).v_j = jv_j$ for $j = 0, 1, 2$.

(b) We observe that $P(e) = 2L(e)^2 - L(e^2) = (2L(e) - \mathbf{1})L(e)$. For $\lambda \in \mathbb{R}$ we therefore have for

$$\begin{aligned} &B(e, (1-\lambda)e) = B((1-\lambda)e, e) = \mathbf{1} - (1-\lambda)2e\Box e + (1-\lambda)^2 Q(e)^2 \\ &= \mathbf{1} - (1-\lambda)2L(e) + (1-\lambda)^2 P(e) = \mathbf{1} - (1-\lambda)2L(e) + (1-\lambda)^2(2L(e) - \mathbf{1})L(e) \end{aligned}$$

the relation

$$B(e, (1-\lambda)e)v_j = \lambda^j v_j, \quad j = 0, 1, 2.$$

(c) From $Q(e) = Q(Q(e)e) = Q(e)^3$ we conclude that the antilinear map $Q(e)$ is diagonalizable over $\mathbb{R}$ with eigenvalues in $\{1, 0, -1\}$, so that $Q(e)^2 = P(e) = (2L(e) - \mathbf{1})L(e)$ implies that

$$\ker Q(e) = \ker P(e) = V_0 \oplus V_1. \tag{1.11}$$

From $V_0 \Box V_2 = V_2 \Box V_0 = \{0\}$ we obtain for $x, y \in V_0$:

$$\begin{aligned} &B(e+x, e+y).e = e - 2(e+x)\Box(e+y).e + Q(e+x)Q(e+y)e \\ &= e - 2e - 2x\Box y.e + Q(e+x)(Q(e).e + Q(y).e + 2\{e, e, y\}) \\ &= -e - 2(e\Box y).x + Q(e+x).e = -e + (Q(e).e + Q(x).e + \{e, e, x\}) = 0. \end{aligned}$$

Theorem I.9. ([Lo77, Th. 8.11]) *Let $e \in V$ be a tripotent and $V^{(e)}$ the corresponding Jordan algebra with product $ab = \{a, e, b\}$. Identifying $e \in V$ with an element of $\mathfrak{g}_+$, the partial Cayley transform corresponding to e is defined by $C_e := \exp\left(\frac{\pi}{4}(e - \tau.e)\right) \in G_{\mathbb{C}}$, and in Jordan theoretic terms it is given as a partially defined map on V by*

$$C_e = t_e \cdot B(e, (1-\sqrt{2})e) \cdot \widetilde{t}_e.$$

In particular

$$C_e^{-1}(V) \cap V = \{v \in V : B(e, v) \in \mathrm{GL}(V)\} = e^{\top}.$$ ■

In [Lo77] Loos writes $B(e, -e)^{\frac{1}{2}}$ instead of $B(e, (1-\sqrt{2})e)$, which makes sense because

$$B(e, (1-\sqrt{2})e)^2 = B(e, (1-2)e) = B(e, -e)$$

is diagonalizable and the eigenvalues $1, \sqrt{2}$ and 2 of $B(e, (1-\sqrt{2})e)$ are positive (I.8).

I.10. The preceding theorem implies in particular that the condition for an element $x \in V$ to lie in the domain of the Cayley transform is precisely the transversality condition $e \top x$. If x_2 is the Peirce component of x in V_2, then [Lo77, Prop. 10.3] says that $e \top x$ is equivalent to the invertibility of $e - x_2$ in the unital Jordan algebra (V_2, e).

Definition I.11. A hermitian scalar product $\langle \cdot, \cdot \rangle$ on V is said to be *associative* if for $x, y, z, w \in V$ we have

$$\langle \{x, y, z\}, w \rangle = \langle x, \{y, z, w\} \rangle,$$

which is equivalent to

$$(z\Box y)^* = y\Box z \quad \text{for} \quad y, z \in V.$$

According to [Lo77, Cor. 3.16], a scalar product with this property is given by

$$\langle x, y \rangle := \mathrm{tr}(x\Box y),$$

and for $0 \neq x \in V$ the operator $x\Box x$ is non-zero and positive semidefinite. In this sense $(V, \{\cdot, \cdot, \cdot\})$ is a *positive hermitian Jordan triple.* ■

Lemma I.12. *Let $e \in V$ be a tripotent, $V_j := V_j(e)$ its Peirce spaces, and $z \in V_0$ with $|z| \leq 1$. Further let $f := \lim_{n\to\infty} Q(z)^n.z$ denote the unique tripotent contained in the holomorphic arc-component of z. Then $\varphi(z) := Q(z+e)\,|_{V_1}: V_1 \to V_1$ is an antilinear operator which is symmetric with respect to the real scalar product $(z,w) := \operatorname{Re}\operatorname{tr}(z\Box w)$, and for $z \in V_1$ we have $\varphi(z)v = 2\{z,v,e\}$.*

If $|z| < 1$, then $\varphi(z) + \mathbf{1}$ is injective ($\mathbf{1}$ stands for id_{V_1}), and for $|z| = 1$ its kernel is

$$\mathrm{Fix}(-Q(e+f)) \cap V_1(f) \cap V_1(e).$$

Proof. For $v \in V_1$ we have

$$\varphi(z)v = \{z+e, v, z+e\} = Q(z)v + Q(e)v + 2Q(z,e)v,$$

and $Q(e)v \in V_{4-1} = V_3 = \{0\}$ as well as $Q(z)v \in V_{0-1} = V_{-1} = \{0\}$ by the Peirce relations (1.1), so that $\varphi(z)v = 2\{z,v,e\}$.

According to [Lo77, Lemma 6.7], the operator $\varphi(z)$ on V_1 is symmetric with respect to the real scalar product $(\cdot,\cdot)$ on V_1, hence diagonalizable over $\mathbb{R}$ with real eigenvalues.

Let $v \in V_1$ be an eigenvector for $\varphi(z)$ corresponding to the eigenvalue $\lambda \in \mathbb{R}$, i.e., $Q(z+e).v = \lambda v$. Inductively we get

$$Q(Q(z+e)^n.(z+e)).v = \lambda^{2n+1} \cdot v$$

for all $n \in \mathbb{N}_0$ from

$$\begin{aligned} Q(Q(z+e)^n.(z+e)).v &= Q(Q(z+e)Q(z+e)^{n-1}.(z+e)).v \\ &= Q(z+e)Q(Q(z+e)^{n-1}.(z+e))Q(z+e).v \\ &= Q(z+e)Q(Q(z+e)^{n-1}.(z+e)).\lambda v = \lambda Q(z+e).(\lambda^{2n-1}.v) = \lambda^{2n+1}v. \end{aligned}$$

Since the inclusion $V_0 \hookrightarrow V$ is isometric with respect to the spectral norm ([Lo77, Th. 3.17]), we have

$$e + z \in e + \overline{\mathcal{D}_e} = \overline{A_e} \subseteq \overline{\mathcal{D}},$$

and the limit $f = \lim_{n\to\infty} Q(z)^n.z$ is a tripotent in $V_0(e)$ (Theorem I.5).

As a consequence of the Peirce relations (1.2), we obtain

$$Q(e+z).(e+z) = Q(e)e + Q(z)z = e + Q(z)z,$$

and inductively

$$Q(e+z)^n.(e+z) = e + Q(z)^n.z \to e + f.$$

Therefore

$$\lim_{n\to\infty} \lambda^{2n+1}v = \lim_{n\to\infty} Q(Q(z+e)^n.(z+e)).v = Q(e+f).v,$$

and the existence of the limit implies that $|\lambda| \leq 1$. If $|\lambda| < 1$, then $Q(e+f).v = 0$, and otherwise $Q(e+f).v = \lambda v$. It follows in particular that each eigenvector for $Q(e+z)$ on V_1 also is an eigenvector of $Q(e+f)$.

Suppose that $|\lambda| = 1$. As a consequence of the Peirce rules, the sum $e+f$ is a Jordan tripotent (I.3), and from $Q(e+f).v = \lambda v$ and $\ker Q(e+f) = V_0(e+f) \oplus V_1(e+f)$ (I.8), we derive $v \in V_2(e+f)$, so that $(e+f)\Box(e+f) = e\Box e + f\Box f$ implies that $v \in V_1(f)$.

On the other hand $Q(e+f)$ is an antilinear involution of $V_2(e+f) \supseteq V_1(e) \cap V_1(f)$. We conclude that

$$\ker(\varphi(z) + \mathbf{1}) = \ker(\varphi(f) + \mathbf{1}) = \mathrm{Fix}(-Q(e+f)) \cap V_1(f) \cap V_1(e). \qquad \blacksquare$$

To classify the G-orbits of transversal pairs in $\overline{\mathcal{D}}$, we need a more explicit description of the image

$$\mathcal{D}^C := C_e(\mathcal{D})$$

of $\mathcal{D}$ under the partial Cayley transform C_e in terms of the Peirce decomposition of V. To this end, we introduce the following notation:

Definition I.13. Let $e \in V$ be a tripotent.

(1) $(V_2, e, Q(e))$ is a unital involutive Jordan algebra. We write $v^* := Q(e)v$ for the involution on V_2 and observe that $V_2 = E \oplus iE$ for $E := \{v \in V : v^* = v\}$. In this sense

$$\operatorname{Re} v = \tfrac{1}{2}(v + v^*) = \tfrac{1}{2}(v + Q(e)v)$$

is the component of v in the real form E of V_2. The real Jordan algebra E is euclidean and we write $E_+ := \{a^2 : a \in E\}$ for its closed positive cone. For $v, w \in E$ we write $v > w$ for $v - w \in \operatorname{int}(E_+)$ and $v \geq w$ for $v - w \in E_+$.

(2) For $z \in V_0$ we define the antilinear map

$$\varphi(z) : V_1 \to V_1, \quad v \mapsto 2\{e, v, z\} = Q(e + z).v$$

(Due to the different normalization, the factor 2 not present in [Lo77]).

(3) We also define a hermitian map

$$F : V_1 \times V_1 \to V_2, \quad (z, w) \mapsto \{z, w, e\}$$

with

$$F(z, w)^* = F(w, z) \quad \text{and} \quad F(z, z) > 0 \quad \text{for} \quad 0 \neq z \in V_1.$$

For $u \in V_0$ with $|u| < 1$ we further define a real bilinear map

$$F_u(z, w) = F(z, (\mathbf{1} + \varphi(u))^{-1}.w),$$

where we recall from Lemma I.12 that $\mathbf{1} + \varphi(u)$ is invertible. ■

In the following proposition the missing factor $\frac{1}{2}$ in front of F, compared to [Lo77], is due to our different normalization of the triple product.

Proposition I.14. ([Lo77, Th. 10.8]) *We have*

$$\mathcal{D}^C = C_e(\mathcal{D}) = \{v = v_2 + v_1 + v_0 \in V_2 \oplus V_1 \oplus V_0 : |v_0| < 1, \operatorname{Re}(v_2 - F_{v_0}(v_1, v_1)) > 0\}. \quad ■$$

To determine the closure of $\mathcal{D}^C$, we need the following lemma, because there might be elements $x_0 \in \partial\mathcal{D} \cap V_0$ for which the operator $\varphi(x_0) + \mathbf{1}$ is not invertible.

Lemma I.15. *Let F be a finite-dimensional euclidean vector space, $(A_n)_{n\in\mathbb{N}}$ a sequence of positive definite operators on F converging to A and $(v_n)_{n\in\mathbb{N}}$ a sequence of elements of F converging to v. If the sequence $A_n^{-\frac{1}{2}} v_n$ is bounded, then $v \in \operatorname{im}(A)$.*

Proof. Since A is symmetric, we have $\operatorname{im}(A) = \ker(A)^\perp$. Let $w \in \ker(A)$. We have to show that $\langle v, w\rangle = 0$. Since the sequence $A_n^{-\frac{1}{2}} v_n$ is bounded, it contains a convergent subsequence, and we may thus assume that it converges to some $u \in F$. Then we get

$$\langle v, w\rangle = \lim_{n\to\infty} \langle v_n, w\rangle = \lim_{n\to\infty} \langle A_n^{\frac{1}{2}} A_n^{-\frac{1}{2}} v_n, w\rangle = \lim_{n\to\infty} \langle A_n^{-\frac{1}{2}} v_n, A_n^{\frac{1}{2}} w\rangle = \langle u, A^{\frac{1}{2}} w\rangle = \langle u, 0\rangle = 0.$$

This completes the proof. ■

Lemma I.16. *For each element $v = v_2 + v_1 + v_0 \in \overline{\mathcal{D}^C}$ we have $v_1 \in \operatorname{im}(\mathbf{1} + \varphi(v_0))$.*

Proof. Let $(v^n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}^C$ converging to v and write v_j^n, $j = 0, 1, 2,$ for its Peirce components.

We pick a linear functional $f \in E^*$ in the interior of the dual cone of E_+, so that $f(x) > 0$ holds for $0 \neq x \in E_+$, and observe that this implies that

$$(v, w) := f(\operatorname{Re} F(v, w))$$

defines a real scalar product on V_1. The argument in [Lo77, p.10.6] shows that for each $z \in V_0$ the operator $\varphi(z)$ is symmetric with respect to this scalar product. According to Lemma I.12, all its eigenvalues λ satisfy $|\lambda| \leq 1$ and even $|\lambda| < 1$ for $|z| < 1$, so that $\mathbf{1} + \varphi(z)$ is a positive semidefinite symmetric operator which is positive definite for $|z| < 1$.

From $v^n \in \mathcal{D}^C$ we get

$$|v_0^n| < 1 \quad \text{and} \quad \operatorname{Re} F_{v_0^n}(v_1^n, v_1^n) \leq \operatorname{Re} v_2^n,$$

which implies that

$$\begin{aligned} f(v_2^n) &\geq f(\operatorname{Re} F_{v_0^n}(v_1^n, v_1^n)) = f(\operatorname{Re} F(v_1^n, (\mathbf{1} + \varphi(v_0^n))^{-1} v_1^n)) \\ &= (v_1^n, (\mathbf{1} + \varphi(v_0^n))^{-1} v_1^n) = ((\mathbf{1} + \varphi(v_0^n))^{-\frac{1}{2}} v_1^n, (\mathbf{1} + \varphi(v_0^n))^{-\frac{1}{2}} v_1^n). \end{aligned}$$

Therefore the sequence $(\mathbf{1} + \varphi(v_0^n))^{-\frac{1}{2}} v_1^n$ in V_1 is bounded, and Lemma I.15 implies that

$$v_1 = \lim_{n \to \infty} v_1^n \in \operatorname{im}(\mathbf{1} + \varphi(v_0)). \qquad \blacksquare$$

I.17. From the preceding lemma one easily derives an explicit description of the closure of $\mathcal{D}^C$ because the operator $(\mathbf{1} + \varphi(v_0))^{-1}$ is well-defined on $\operatorname{im}(\mathbf{1} + \varphi(v_0))$. This leads to

$$\overline{\mathcal{D}^C} = \Big\{ v \in V \colon |v_0| \leq 1, v_1 \in \operatorname{im}(\varphi(v_0) + \mathbf{1}), \operatorname{Re}\big(v_2 - F(v_1, (\mathbf{1} + \varphi(x_0))^{-1} v_1)\big) \geq 0 \Big\}.$$

Since we do not need this description in the following, we leave the details of its verification to the reader.

Theorem I.18. (Orbits of transversal pairs) *Let $\mathcal{D}$ be an irreducible bounded symmetric domain, not necessarily of tube type. If $(x, y) \in \overline{\mathcal{D}}$ is a transversal pair with* $\operatorname{rk} x = k$, *then there exists a $g \in G$ with $g.(x, y) = (e_k, z)$ with*

$$e_k = c_1 + \ldots + c_k \quad \textit{and} \quad z = -(c_{j+1} + \ldots + c_k) + \sum_{l=k+1}^{r} \lambda_l c_l, \quad -1 \leq \lambda_{k+1} \leq \ldots \leq \lambda_r \leq 1.$$

Proof. Since $\mathcal{D}$ is irreducible, G acts transitively on the set of elements of rank k, so that we may w.l.o.g. assume that $x = e := e_k$. We then have to show that each G_e-orbits in $e^\top \cap \overline{\mathcal{D}}$ contains an element of the form

$$-(c_{j+1} + \ldots + c_k) + \sum_{l=k+1}^{r} \lambda_l c_l, \quad -1 \leq \lambda_{k+1} \leq \ldots \leq \lambda_r \leq 1.$$

We recall the notation from Definition I.13. For $y > 0$ in E we then find with (I.7)

$$B(e - y, e) = \mathrm{id}_V - 2L(e - y) + P(e - y) = P(e - (e - y)) = P(y). \tag{1.12}$$

Let $Q := G_{A_e}$ denote the stabilizer of the holomorphic arc-component A_e of e in $\partial\mathcal{D}$ (which is a maximal parabolic subgroup of G). Then the group $Q^C := C_e \circ Q \circ C_e^{-1}$ acts naturally on $\mathcal{D}^C = C_e(\mathcal{D})$ and we also put

$$Q_e^C := C_e \circ G_e \circ C_e^{-1} \subseteq Q^C,$$

where G_e is the stabilizer of e in G.

From [Lo77, Lemma 10.7] we now obtain

$$Q^C = \{ t_b \circ t_{v + F(v,v)} \exp(2e \Box v) P(y) \exp(\xi_w) \cdot k \colon b \in iE, v \in V_1, 0 < y \in E, w \in V_0, k \in K_e \},$$

where $K_e := \{g \in G : g.0 = 0, g.e = e\} \subseteq \mathrm{Aut}(V)_e$ is the set of all automorphisms of the Jordan triple V fixing e and $P(y)$ is the quadratic representation of the Jordan algebra $V^{(e)}$ (cf. I.7). From the proof of [Lo77, Thm. 9.15] and the description of the Lie algebra $\mathbf{L}(Q^C)$ in [Lo77, Prop. 10.6] it follows that for $b \in iE, v \in V_1, 0 < y \in E$ and $k \in K_e$ we have

$$t_b \circ t_{v+F(v,v)} \exp(2e\Box v)P(y)k \in Q_e^C.$$

Moreover, the explicit calculations in the proof of [Lo77, Th. 10.8] further imply that the map

$$V_0 \to A_e = e + (\mathcal{D} \cap V_0), \quad w \mapsto \exp(\xi_w).e$$

is bijective and that the Cayley transform fixes each ξ_w. This implies that

$$Q_e^C = \{t_b \circ t_{v+F(v,v)} \exp(2e\Box v)P(y) \cdot k \colon b \in iE, v \in V_1, 0 < y \in E, k \in K_e\}.$$

We observe that for $v \in V_1$ the Peirce rules imply that $e\Box v$ is a nilpotent operator on V mapping $V_j \to V_{j+1}$. For $x = x_2 + x_1 + x_0 \in \overline{\mathcal{D}^C}$ the V_1-component of

$$t_{v+F(v,v)} \exp(2e\Box v).x$$

is given by

$$x_1 + v + \varphi(x_0).v,$$

and since $-x_1 \in \mathrm{im}(\mathbf{1} + \varphi(x_0))$ by Lemma I.16, there is a unique $v \in \mathrm{im}(\mathbf{1} + \varphi(x_0))$ with

$$t_{v+F(v,v)} \exp(2e\Box v).x \in V_2 \oplus V_0.$$

From that we conclude that each Q_e^C-orbit in V through an element $y = y_2 + y_1 + y_0 \in \overline{\mathcal{D}^C}$ contains an element of the form

$$x_2 + x_0 \quad \text{with} \quad |x_0| \leq 1 \quad \text{and} \quad \mathrm{Re}\, x_2 \geq 0.$$

Applying elements of the form t_v, $v \in iE$, we may further assume that $x_2 \in E$, so that we have an element in $E_+ \times \mathcal{D}_e$. From the explicit description of Q_e^C we derive that the intersection of the orbit of $x_2 + x_0 \in E + V_0$ with $E + V_0$ contains the orbit of $x_2 + x_0$ under the group $Q'' := P(E_+)K_e$.

The orbits of Q'' on the set $E_+ \times \overline{\mathcal{D}}_e$ are products of orbits of the automorphism group $G(E_+)$ of the symmetric cone E_+ in E and orbits of the identity component of the group K_e on $\mathcal{D}_e$. Since the action of the group K_e preserves the Peirce decomposition, it acts on $\mathcal{D}_e \subseteq V_0$ as a subgroup of $\mathrm{Aut}(V_0)$. The identity component of the latter group is obtained by exponentiating elements of the Lie subalgebra $V_0 + \tau(V_0) + [V_0, \tau(V_0)] \subseteq \mathfrak{g}_{\mathbb{C}}$ (here we use that $\mathcal{D}_e = \mathcal{D} \cap V_0$ is an irreducible bounded symmetric domain; cf. Th. I.5), and all the elements of this subalgebra commute with the element $e \in V_2$ by the Peirce rules (I.2). Hence the image of K_e in $\mathrm{Aut}(V_0)$ contains the identity component of $\mathrm{Aut}(V_0)$.

For $e = e_k = c_1 + \ldots + c_k$, the orbits of $G(E_+)_0$, which coincide with the orbits of the full group $G(E_+)$, are represented by the elements

$$e_0 = 0, e_1 = c_1, \ldots, e_j = c_1 + \ldots + c_j, \ldots, e_k = e$$

([FK94, Prop. IV.3.2]). Since $(c_{k+1}, \ldots, c_r)$ is a Jordan frame in V_0, each orbit of $\mathrm{Aut}(V_0)_0$ in V_0 contains an element of the form

$$\sum_{l=k+1}^{r} \lambda_l c_l, \quad \lambda_{k+1} \leq \ldots \leq \lambda_r$$

(cf. [FK94, Prop. X.3.2]).

Next we transfer this information back to the bounded picture, i.e., to G_e-orbits in $\overline{\mathcal{D}}$. According to [Lo77, Prop. 10.3], we have

$$C_e(x_2 + x_0) = C_e(x_2) + x_0 = (e + x_2)(e - x_2)^{-1} + x_0 \quad \text{for} \quad x_2 \in V_2, x_0 \in V_0. \tag{1.13}$$

For $e_j = c_1 + \ldots + c_j$, $j \leq k$, the element $e + e_j$ is invertible in V_2, and we obtain for $\widetilde{e}_j := (e_j - e)(e_j + e)^{-1} = -C_e(-e_j) = C_e^{-1}(e_j)$ that $C_e(\widetilde{e}_j) = e_j$. An explicit calculation in the associative Jordan algebra generated by $c_1, \ldots, c_k$ quickly shows that

$$\widetilde{e}_j = -(e - e_j) = -e + e_j = -c_{j+1} - \ldots - c_k.$$

This completes the proof. ■

I.19. For the special case $k = r$, i.e., $e \in S$, we have $V_0 = \{0\}$, so that $\mathcal{D}^C$ is the Siegel domain

$$\mathcal{D}^C = \{v = v_2 + v_1 \in V_2 \oplus V_1 = V : \mathrm{Re}(v_2 - F(v_1, v_1)) > 0\}$$

of type II. In this case the orbits of Q_e'' are represented by elements of the form $-e + e_j$, $j = 0, \ldots, r$, so that we obtain only finitely many orbits. Observe that $\mathrm{rk}(-e + e_j) = r - j$, so that, even if Q'' is not connected, it cannot have less orbits in $e^\top$ than its identity component.

In Section III we shall only need the following special case of Theorem I.18, for which we provide the following more direct proof.

Lemma I.20. *Suppose that $\mathcal{D}$ is irreducible and of tube type, let $x \in S$ and $z \in \overline{\mathcal{D}}$, and assume that $x \top z$. There exists $g \in G$ and an integer $k, 0 \leq k \leq r$* (†) *such that*

$$g(x) = e_r \quad \text{and} \quad g(z) = -\sum_{j=k+1}^{r} c_j = e_k - e_r \,.$$

Proof. As G is transitive on S, there is no restriction in assuming that $x = e := e_r$. Now the transversality condition is equivalent to z belonging to the domain $V^\times + e$ of the Cayley transform $C(z) := C_e(z) := (e+z)(e-z)^{-1}$ (cf. (1.13)). Set $\zeta = C(z)$ (Theorem I.9). Then $\zeta \in E_+ + iE$. The point e is sent by the Cayley transform "to infinity", in such a way that the stabilizer of e in G corresponds via conjugation by the Cayley transform to a subgroup of the affine group of $E^{\mathbb{C}}$, denoted by Q_e^C, namely the semi-direct product of the translations by an element of iE and the group $G(E_+)$ (after complexification to $E^{\mathbb{C}}$ of its action on E). By using a translation, we see that in the Q_e^C-orbit of ζ, there is an element of the form $\eta \in E_+$. Since $\mathcal{D}$ is irreducible, the $G(E_+)$-orbits in E_+ are known to be exactly the $r+1$ orbits of the elements $e_k = \sum_{j=1}^k c_j$, with $k = 0, 1, \ldots r$ (see [FK94, Prop. IV.3.2]). But now the inverse Cayley transform of the element $\displaystyle\sum_{j=1}^{k} c_j$ is the element $\displaystyle e_k - e = -\sum_{j=k+1}^{r} c_j$. Hence the result. ■

II. Transversality and faces

In this section we keep the notation from Section I. In particular $\mathcal{D}$ is a circular irreducible bounded symmetric domain of rank r in V. The main result of this section is that transversality of two elements $x, y \in \overline{\mathcal{D}}$ is equivalent to the geometric property that x and y do not lie in a proper face of the compact convex set $\overline{\mathcal{D}}$ (Theorem II.12).

Definition II.1. (a) We call a non-empty convex subset F of a convex set C a *face* if for $0 < t < 1$ and $c, d \in C$ the relation $tc + (1-t)d \in F$ implies $c, d \in F$. We write $\mathcal{F}(C)$ for the set of non-empty faces of C. A face F is called *exposed* if there exists a linear functional $f : V \to \mathbb{R}$ with

$$F = f^{-1}(\max f(C)).$$

An *extreme point* $e \in C$ is a point for which $\{e\}$ is a face, i.e., $tc + (1-t)d = e$ for $c, d \in C$ and $0 < t < 1$ implies $c = d = e$. We write $\mathrm{Ext}(C)$ for the set of extreme points of C.

The set of all faces of C has a natural order structure given by set inclusion whose maximal element is C itself. All extreme points of C are minimal elements of this set, but C need not have any extreme points.

Obviously, the intersection of any family of faces is a face. We thus define for a subset $M \subseteq C$ the *face generated by M* by

$$\mathrm{Face}(M) := \bigcap \{F \subseteq C : F \in \mathcal{F}(C), M \subseteq F\}.$$

(†) *If $k = r$, use the convention that $\sum_{j=r+1}^{r} c_j = 0$.*

(b) For a convex set C in the vector space V we write

$$\mathrm{algint}(C) := \{x \in C \colon (\forall v \in C - C)(\exists \varepsilon > 0)\ x + [0,\varepsilon]v \subseteq C\}$$

for its *algebraic interior*. If V is finite-dimensional, then $\mathrm{algint}(C)$ is the interior of C in the affine subspace it generates. ∎

Remark II.2. (a) Suppose that C is a convex subset of a finite-dimensional vector space having non-empty interior. Then all proper faces of C are contained in the boundary ∂C and, conversely, the Hahn–Banach Separation Theorem implies that each boundary point is contained in a proper exposed face.

(b) For any non-empty convex subset of a finite-dimensional real vector space the algebraic interior is non-empty. Hence every face F is generated by any element in its algebraic interior.

(c) Since every face E of a face F of C is also a face of C, faces of exposed faces of C are faces of C. On the other hand, every proper face is contained in an exposed face (see (a)), so that we obtain inductively, that for each face F there exists a sequence of faces

$$F_0 = F \subseteq F_1 \subseteq \ldots \subseteq F_n = C$$

for which F_i is an exposed face of F_{i+1} for $i = 0, \ldots, n-1$. ∎

Proposition II.3. *The proper faces of the convex set $\overline{\mathcal{D}}$ are the closures of the holomorphic arc-components in $\partial\mathcal{D}$ and the Shilov boundary is the set of extreme points of $\overline{\mathcal{D}}$.*

In particular the group G acts on the set $\mathcal{F}(\overline{\mathcal{D}})$ of faces of $\overline{\mathcal{D}}$.

Proof. For the fact that S is the set of extreme points of $\overline{\mathcal{D}}$ we refer to [Lo77, Th. 6.5].

Next we use [Sa80, Lemma III.8.11, Th. III.8.13] to see that the proper exposed faces F of $\overline{\mathcal{D}}$ are the closures of the holomorphic arc-components in $\partial\mathcal{D}$. Since the action of the group G on $\overline{\mathcal{D}}$ permutes the holomorphic arc-components in $\partial\mathcal{D}$, it also permutes the exposed faces of $\overline{\mathcal{D}}$.

We now claim that each face of $\overline{\mathcal{D}}$ is exposed. Since every face F of $\overline{\mathcal{D}}$ is generated by a suitable element $x \in F$ (Remark II.2), it suffices to show that the face generated by any element $x \in \partial\mathcal{D}$ is exposed. Let A_x be the holomorphic arc-component of $\partial\mathcal{D}$ containing x. Then $\overline{A_x}$ is an exposed face of $\overline{\mathcal{D}}$ with $\mathrm{algint}(\overline{A_x}) = A_x$ (Theorem I.5). Therefore the face generated by x coincides with $\overline{A_x}$, showing that every face of $\overline{\mathcal{D}}$ is exposed. ∎

Remark II.4. From the preceding proposition we know that the map $F \mapsto \mathrm{algint}(F)$ is a G-equivariant bijection between the set $\mathcal{F}(\overline{\mathcal{D}})$ of faces of $\overline{\mathcal{D}}$ and the set of holomorphic arc-components in $\overline{\mathcal{D}}$.

If $\mathcal{D}$ is irreducible, we define the *rank of a face* by $\mathrm{rk}\, F := k$ if $\mathrm{algint}(F)$ consists of elements of rank k. Since two holomorphic arc-components are conjugate under G if and only if their elements have the same rank (cf. Theorem I.5), the rank function

$$\mathrm{rk} \colon \mathcal{F}(\overline{\mathcal{D}}) \to \{0, \ldots, r\}$$

classifies the G-orbits in $\mathcal{F}(\overline{\mathcal{D}})$. The stabilizer of a proper face, resp., a holomorphic arc-component in $\partial\mathcal{D}$, is a maximal parabolic subgroup of G ([Sa80, Cor. III.8.6]).

If $\mathcal{D} = \mathcal{D}_1 \times \ldots \times \mathcal{D}_m$ is a direct product of the irreducible domains $\mathcal{D}_j$, then each face F of $\overline{\mathcal{D}}$ is a product $F_1 \times \ldots \times F_m$ of faces $F_j \in \mathcal{F}(\overline{\mathcal{D}}_j)$, so that the G-orbits in

$$\mathcal{F}(\mathcal{D}) \cong \mathcal{F}(\mathcal{D}_1) \times \ldots \times \mathcal{F}(\mathcal{D}_m)$$

are classified by the m-tuple $(\mathrm{rk}\, F_1, \ldots, \mathrm{rk}\, F_m)$. ∎

In the following we shall prove that for two elements $x, y \in \overline{\mathcal{D}}$ transversality is equivalent to the *geometric transversality relation* $\mathrm{Face}(x, y) = \overline{\mathcal{D}}$. We start with the easy implication.

Proposition II.5. *If $x, y \in \overline{\mathcal{D}}$ are transversal, then they are not contained in a proper face, i.e.,* $\mathrm{Face}(x,y) = \overline{\mathcal{D}}$.

Proof. If x and y are not geometrically transversal, then $F := \mathrm{Face}(x,y)$ is a proper face of $\overline{\mathcal{D}}$, hence of the form

$$F = F_e = e + (\overline{\mathcal{D}} \cap V_0(e)) = (e + V_0(e)) \cap \overline{\mathcal{D}}$$

for some tripotent $e \in V$ (Theorem I.5, Prop. II.3 and [Sa80, Lemma III.8.10] for the second equality). Then $x, y \in F$ implies that $x, y \in e + V_0(e)$, so that I.8 leads to $B(x,y).e = 0$. Thus x and y are not transveral. This proves the assertion. ∎

Example II.6. We consider the r-dimensional polydisc

$$\mathcal{D} := \Delta^r := \{z \in \mathbb{C}^r \colon \max_j |z_j| < 1\} \subseteq V = \mathbb{C}^r.$$

Let $(c_1, \ldots, c_r)$ denote the canonical basis of $\mathbb{C}^r$. The corresponding Jordan triple structure is given by

$$\{x, y, z\} = (x_1 \overline{y_1} z_1, \ldots, x_r \overline{y_r} z_r).$$

An element $z \in \mathbb{C}^r$ is a tripotent if $|z_j|^2 z_j = z_j$ holds for each j, which means that either $z_j = 0$ or $|z_j| = 1$. We have

$$\mathrm{rk}\, z = |\{j \colon |z_j| = 1\}|,$$

and the tripotents of maximal rank form the n-dimensional torus $S = \mathbb{T}^n$, the Shilov boundary of Δ^r.

Since the faces of $\overline{\mathcal{D}} = \overline{\Delta}^r$ are cartesian products of faces of the closed unit disc

$$\overline{\Delta} = \{z \in \mathbb{C} \colon |z| \leq 1\},$$

each face $F \in \mathcal{F}(\overline{\Delta^r})$ is a product $F_1 \times \cdots \times F_r$ of closed unit discs and points in the boundary of Δ. For a subset $M \subseteq \overline{\Delta}^r$, it follows that the face generated by M is given by

$$\mathrm{Face}(M) = F_1 \times \cdots \times F_r, \quad F_j = \begin{cases} \{s\} & \text{if } m_j = s \in \partial\Delta \text{ for all } m \in M \\ \Delta & \text{otherwise.} \end{cases}$$

It follows in particular that $x, y \in \overline{\mathcal{D}}$ are contained in a proper face if and only if $x_j = y_j \in \partial\Delta$ holds for some j.

For $k \leq r$ we consider the tripotent $e_k := c_1 + \ldots + c_k$. Then

$$V_2 = \mathbb{C}^k \times \{0\}^{r-k} \quad \text{and} \quad V_0 = \{0\}^k \times \mathbb{C}^{r-k}.$$

An element $x \in \overline{\Delta^r}$ is transversal to e_k if and only if $e_k - (x_1, \ldots, x_k, 0, \ldots, 0)$ is invertible in the unital Jordan algebra (V_2, e_k), which means that the first k components of x are different from 1 (I.10). That this is not the case means that one component x_j, $j \leq k$, equals 1, and therefore $\mathrm{Face}(e_k, x) \neq \overline{\mathcal{D}}$. If, conversely, $\mathrm{Face}(e_k, x) \neq \overline{\mathcal{D}}$, then e_k, x are contained in a proper face of $\overline{\Delta^r}$ which implies that $x_j = 1$ for some $j \leq k$. ∎

Proposition II.7. *Let $e \in V$ be a tripotent, $V = \sum_{j=0}^2 V_j$ the corresponding Peirce decomposition and $p_j \colon V \to V_j$ the projection along the other Peirce components. Then each V_j is a positive hermitian Jordan triple and we have*

$$\mathcal{D}_j = V_j \cap \mathcal{D} = p_j(\mathcal{D}).$$

In particular, each map $p_j \colon V \to V_j$ is a contraction with respect to the spectral norms determined by the domains $\mathcal{D}$ and $\mathcal{D}_j$.

Proof. Let $\langle \cdot, \cdot \rangle$ be an associative hermitian scalar product on V (Definition I.11). Then the Peirce decomposition is orthogonal with respect to $\langle \cdot, \cdot \rangle$, so that it provides an orthogonal decomposition of V into 3 Jordan subtriples ([Lo77, Th. 3.13]).

Clearly the restriction of the scalar product to each V_j provides an associative scalar product on V_j and for each $v \in V_j$ the operator $v\Box v$ is positive semidefinite on V, which implies in particular that its restriction to V_j is positive semidefinite. Hence each V_j is a positive hermitian Jordan triple.

According to [Lo77, Th. 3.17], the inclusion maps $V_j \hookrightarrow V$ are isometric with respect to the spectral norm, which means that

$$\mathcal{D}_j = V_j \cap \mathcal{D} = \{z \in V_j : |z| < 1\}$$

holds for the corresponding bounded symmetric domains.

To see that the projections p_j are contractive with respect to the spectral norm, let $v \in V$ and $v_j = p_j(v)$ its component in V_j. For each unit vector $w \in V_j$ the orthogonality of the Peirce decomposition implies that

$$\langle v\Box v.w, w\rangle = \sum_{k,l=0}^{2} \langle v_k\Box v_l.w, w\rangle = \sum_{k=0}^{2}\langle v_k\Box v_k.w, w\rangle \geq \langle v_j\Box v_j.w, w\rangle,$$

which leads for the spectral norm $|v_j|$ to

$$\begin{aligned} |v_j|^2 = \|v_j\Box v_j\|_{V_j} &= \sup\{\langle v_j\Box v_j.w, w\rangle : w \in V_j, \langle w, w\rangle = 1\} \\ &\leq \sup\{\langle v\Box v.w, w\rangle : w \in V_j, \langle w, w\rangle = 1\} \leq \sup\{\langle v\Box v.w, w\rangle : w \in V, \langle w, w\rangle = 1\} = |v|^2. \end{aligned}$$

Since the inclusion $V_j \hookrightarrow V$ is isometric, p_j is a contraction with respect to the spectral norm, and therefore $\mathcal{D}_j \subseteq p_j(\mathcal{D}) \subseteq \mathcal{D}_j$ proves equality. ■

Corollary II.8. *If F is a proper face of $\overline{\mathcal{D}}_j$, then $p_j^{-1}(F)$ is a proper face of $\overline{\mathcal{D}}$.* ■

Definition II.9. Suppose that $e \in V$ is a tripotent with $V_2(e) = V$, so that $Q(e)$ is an antilinear involution on V turning $(V, e, Q(e))$ into an involutive unital Jordan algebra. As in Section I, we endow V with the spectral norm $|z|$ whose open unit ball is $\mathcal{D}$.

A *state of the unital involutive Jordan algebra* V is a linear functional $f: V \to \mathbb{C}$ with

$$1 = f(e) = \|f\| := \sup |f(\mathcal{D})|.$$ ■

Remark II.10. If f is a state on V and $y \in \overline{\mathcal{D}}$ with $f(y) = 1$, then e and y lie in the proper face $\{z \in \overline{\mathcal{D}} : \operatorname{Re} f(z) = 1\}$. ■

Proposition II.11. *If $y \in \overline{\mathcal{D}}$ and $e - y$ is not invertible in the unital Jordan algebra (V, e), there exists a state f of V with $f(y) = 1$.*

Proof. We endow V with the associative scalar product $\langle z, w\rangle := \operatorname{tr}(z\Box w)$ (cf. Def. I.11).

By assumption $e - y$ is not invertible, which implies that the left multiplication $L(e-y) = (e - y)\Box e$ is not invertible. Pick $v \in \ker L(e - y)$ with $\langle v, v\rangle = 1$. We consider the linear functional

$$f: V \to \mathbb{C}, \quad f(z) := \langle L(z).v, v\rangle$$

satisfying $f(e) = \langle v, v\rangle = 1$ and

$$f(y) = \langle L(y).v, v\rangle = \langle L(e).v, v\rangle = f(e) = 1.$$

It remains to show that f is a state. Let $E := \{z \in V : z^* = Q(e)z = z\}$ denote the euclidean Jordan algebra with $V \cong E \otimes_{\mathbb{R}} \mathbb{C}$ and unit element e. We write E_+ for the closed positive cone in E. This is the set of all those elements z for which there exists a system $c_1, \ldots, c_k$ of orthogonal idempotents with $e = c_1 + \ldots + c_k$ and non-negative real numbers λ_j with

$$z = \sum_{j=1}^{k} \lambda_j c_j.$$

For such elements $z \in E_+$ we then have

$$f(z) = \sum_{j=1}^{k} \lambda_j \langle L(c_j).v, v\rangle = \sum_{j=1}^{k} \lambda_j \langle c_j \Box c_j.v, v\rangle \geq 0$$

because $L(c_j) = c_j \Box e = c_j \Box c_j$ follows from $c_j \Box (e - c_j) = 0$ (I.2) and the operators $c_j \Box c_j$ are positive semidefinite on V ([Lo77, Cor. 3.16]). We conclude that $f(E) \subseteq \mathbb{R}$, so that $f(z^*) = \overline{f(z)}$ for all $z \in V$.

From $Q(e)^{-1} = Q(e)$ we derive $Q(Q(e).z) = Q(e)Q(z)Q(e) = Q(e)Q(z)Q(e)^{-1}$, so that $Q(e): z \mapsto z^*$ is a Jordan triple automorphism of V, hence an isometry for the spectral norm $|\cdot|$ on V. This implies that $Q(e)\mathcal{D} = \mathcal{D}$ and therefore that for $z = x + iy \in \mathcal{D}$, $x, y \in E$, we have

$$|x| = \tfrac{1}{2}|z + z^*| \leq \tfrac{1}{2}(|z| + |z^*|) = |z|.$$

For the map $\mathrm{Re}: V \to E, z \mapsto \frac{1}{2}(z + z^*)$ this means that $\mathcal{D}_E := \mathcal{D} \cap E = \mathrm{Re}(\mathcal{D})$.

For the functional f we thus obtain

$$\|f\| = \sup |f(\mathcal{D})| = \sup \mathrm{Re}\, f(\mathcal{D}) = \sup f(\mathrm{Re}\, \mathcal{D}) = \sup f(\mathcal{D}_E).$$

In view of the Spectral Theorem for euclidean Jordan algebras ([FK94]), we have

$$\mathcal{D}_E = (e - E_+) \cap (-e + E_+) \subseteq e - E_+,$$

so that $f(z) \geq 0$ for $z \in E_+$ leads to $\|f\| = \sup f(\mathcal{D}_E) = f(e) = 1$. This means that f is a state. ■

Theorem II.12. *Two elements $x, y \in \overline{\mathcal{D}}$ are transversal if and only if they are not contained in a proper face, i.e.,*

$$x \top y \iff \mathrm{Face}(x, y) = \overline{\mathcal{D}}.$$

Proof. In view of Theorem II.3, geometric transversality is also invariant under the action of the group G. On the other hand transversality is invariant under G ([CØ01]), so that it suffices to assume that $x = e$ is a Jordan tripotent. In view of Proposition II.5, it suffices to show that if e is not transversal to $y \in \overline{\mathcal{D}}$, then both e and y lie in a proper face of $\overline{\mathcal{D}}$.

For $e = 0$ we have $\mathrm{Face}(x, e) = \overline{\mathcal{D}}$ because $e \in \mathcal{D} = \mathrm{algint}(\overline{\mathcal{D}})$ and also $e \top x$ for all $x \in \overline{\mathcal{D}}$ because $B(x, e) = \mathrm{id}_V$.

We may therefore assume that $e \neq 0$. We have to show that if e and y are not transversal, then they are contained in a proper face of $\overline{\mathcal{D}}$. That y is not transversal to e is equivalent to the element $e - y_2$ being not invertible in the unital Jordan algebra $V_2(e)$ (I.10). In view of Proposition II.11, combined with Remark II.10, e and y_2 are contained in a proper face F of the convex set $\overline{\mathcal{D}}_2$. Hence e and y are contained in the proper face $p_2^{-1}(F)$ of $\overline{\mathcal{D}}$ (Corollary II.8). ■

Example II.13. Let $p, q \in \mathbb{N}$, $r := \min(p, q)$, and $\|\cdot\|$ denote the euclidean norm on $\mathbb{C}^p$, resp., $\mathbb{C}^q$. On the matrix space $V := M_{p,q}(\mathbb{C}) \cong \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^p)$ we write $|X|$ for the corresponding operator norm. Then

$$\mathcal{D} := \{X \in M_{p,q}(\mathbb{C}): |X| < 1\}$$

is a bounded symmetric domain. The pseudo-unitary group $\mathrm{U}_{p,q}(\mathbb{C})$ acts transitively on $\mathcal{D}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z := (az + b)(cz + d)^{-1},$$

the effectivity kernel of this action is $\mathbb{T}\mathbf{1}$, so that $G = \mathrm{Aut}(\mathcal{D})_0 \cong \mathrm{PU}_{p,q}(\mathbb{C})$. The 3-grading of $\mathfrak{g}_{\mathbb{C}}$ is induced by the 3-grading of $\mathfrak{gl}_{p+q}(\mathbb{C})$ given by

$$\mathfrak{gl}_{p+q}(\mathbb{C})_+ = \begin{pmatrix} 0 & M_{p,q}(\mathbb{C}) \\ 0 & 0 \end{pmatrix}, \quad \mathfrak{gl}_{p+q}(\mathbb{C})_0 = \begin{pmatrix} \mathfrak{gl}_p(\mathbb{C}) & 0 \\ 0 & \mathfrak{gl}_q(\mathbb{C}) \end{pmatrix}$$

and

$$\mathfrak{gl}_{p+q}(\mathbb{C})_- = \begin{pmatrix} 0 & 0 \\ M_{q,p}(\mathbb{C}) & 0 \end{pmatrix}.$$

We further have

$$\mathfrak{u}_{p,q}(\mathbb{C}) = \Big\{ \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} : a^* = -a, d^* = -d \Big\}.$$

The vector field associated to the one-parameter group given by $\exp\Big(t\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Big)$ is given by $z \mapsto b - az - zd - zcz$, so that the Jordan triple structure on $V = M_{p,q}(\mathbb{C})$ satisfies $Q(z)(w) = zw^*z$, which leads to

$$\{a, b, c\} = \tfrac{1}{2}(ab^*c + cb^*a).$$

In particular the Bergman operator satisfies

$$B(v,w)z = z - 2v\Box w.z + Q(v)Q(w)z = z - (vw^*z + zw^*v) + v(wz^*w)^*v = (\mathbf{1} - vw^*)z(\mathbf{1} - w^*v).$$

From that it follows that $v\top w$ is equivalent to the invertibility of $\mathbf{1} - w^*v$ in the algebra $M_q(\mathbb{C})$.

An element $e \in M_{p,q}(\mathbb{C})$ is a tripotent if and only if $ee^*e = e$, which implies that ee^* and e^*e are orthogonal projections, and that e defines a partial isometry $\mathbb{C}^q \to \mathbb{C}^p$. If $K := \ker(e)$ and $R := \mathrm{im}(e)$, then the face F_e of $\overline{\mathcal{D}}$ consists of all matrices $z \in \overline{\mathcal{D}}$ with $z.v = e.v$ for $v \in \ker(e)^\perp$. For $k = \mathrm{rank}(e)$ and an orthonormal basis $v_1, \ldots, v_k$ of $\ker(e)^\perp$ and $w_i := e.v_i$, we have

$$F_e = \{z \in \overline{\mathcal{D}} \colon (\forall i)\ \langle zv_i, w_i\rangle = 1\}.$$

From this description of the faces of $\overline{\mathcal{D}}$ it follows that an element $z \in \overline{\mathcal{D}}$ is contained in a proper face if and only if its restriction to some one-dimensional subspace of $\mathbb{C}^q$ is isometric, i.e., if and only if $|z| = 1$. Two elements z, w generate a proper face if and only if there exists a unit vector $v \in \mathbb{C}^q$ for which $z.v = w.v$ is a unit vector in $\mathbb{C}^p$.

A Jordan frame is given by the matrices $c_j := E_{jj}$, $j = 1, \ldots, r$, with a single non-zero entry 1 in position (j,j). The rank of $\mathcal{D}$ is r and $e_r := c_1 + \ldots + c_r$ is a maximal tripotent with

$$S = G.e_r = \begin{cases} \{z \in M_{p,q}(\mathbb{C}) \colon z^*z = \mathbf{1}\} & \text{if } q \leq p \\ \{z \in M_{p,q}(\mathbb{C}) \colon zz^* = \mathbf{1}\} & \text{if } p \leq q. \end{cases}$$

For $q \leq p$ this is the set of isometries $\mathbb{C}^q \hookrightarrow \mathbb{C}^p$ and for $p \leq q$ this is the set of all adjoints of isometries $\mathbb{C}^p \to \mathbb{C}^q$.

Let $e_k := c_1 + \ldots + c_k$ be the canonical tripotent of rank k. Writing an element $z \in M_{p,q}(\mathbb{C})$ as a block matrix

$$z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \quad \text{with} \quad z_{11} \in M_k(\mathbb{C}), z_{12} \in M_{k,q-k}(\mathbb{C}), z_{21} \in M_{p-k,k}(\mathbb{C}), z_{22} \in M_{p-k,q-k}(\mathbb{C}),$$

we have

$$2\{e, e, z\} = ee^*z + ze^*e = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} + \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2z_{11} & z_{12} \\ z_{21} & 0 \end{pmatrix}.$$

This shows that

$$V_2(e_k) \cong M_k(\mathbb{C}), \quad V_1(e_k) \cong M_{k,q-k}(\mathbb{C}) \oplus M_{p-k,k}(\mathbb{C}) \quad \text{and} \quad V_0(e_k) \cong M_{p-k,q-k}(\mathbb{C}),$$

and therefore

$$F_e = \Big\{ \begin{pmatrix} \mathbf{1} & 0 \\ 0 & z \end{pmatrix} : z \in M_{p-k,q-k}(\mathbb{C}), |z| \leq 1 \Big\}.$$

For $k = r$ we see in particular that $V_0(e_r) = 0$. ∎

III. Orbits of triples in the Shilov boundary

In this section we obtain the key result for our classification of triples in S. We show that if $(c_1, \ldots, c_r)$ is a Jordan frame in V, then each G-orbit in $S \times S \times S$ meets the Shilov boundary $T \cong \mathbb{T}^r$ of the corresponding polydisc. We further show that the polydiscs arising in this result can also be characterized directly as the intersections of $\mathcal{D}$ with r-dimensional subspaces of V, or, equivalently, as isometric images of polydiscs under affine maps $\mathbb{C}^r \to V$, mapping Δ^r isometrically into $\mathcal{D}$. In particular we show that any such affine map is linear.

Theorem III.1. *Suppose that $\mathcal{D} \subseteq V$ is of tube type, $(c_1, \ldots, c_r)$ is a Jordan frame in V, and*

$$T := S \cap \operatorname{span}\{c_1, \ldots, c_r\} = \Big\{ \sum_{j=1}^r \lambda_j c_j \colon (\forall j)\ |\lambda_j| = 1 \Big\}$$

is the corresponding r-torus in S. Then for each triple $(e, f, g) \in S$ there exists a $g \in G$ with $g.e, g.f, g.h \in T$.

Proof. Since Jordan frames and G decompose according to the decomposition of $\mathcal{D}$ into products of irreducible domains, it suffices to prove the assertion for irreducible domains. We prove the assertion by induction on the rank r of $\mathcal{D}$.

Case 1: If $\operatorname{Face}(e, f, h)$ is proper, then its algebraic interior is a bounded symmetric domain $\mathcal{D}'$ of smaller rank and (e, f, h) are contained in its Shilov boundary. In fact, according to Theorem I.5 and Proposition II.3, for each face F of $\overline{\mathcal{D}}$ corresponding to the holomorphic arc-component $A = \operatorname{algint}(F)$, the Shilov boundary of A is given by

$$S_A = \operatorname{Ext}(\overline{A}) = \operatorname{Ext}(F) = \operatorname{Ext}(\overline{\mathcal{D}}) \cap F = S \cap F.$$

Since every element of $\operatorname{Aut}(\mathcal{D}')_0$ is the restriction of an element of $\operatorname{Aut}(\mathcal{D})$ ([Sa80, Lemma III.8.1]), in this case the result follows from the induction hypothesis if $r > 1$. If $r = 1$, then each proper face of $\overline{\mathcal{D}}$ is an extreme point, so that the assumption that e, f, h lie in a proper face implies $e = f = h$. In this case we further have $c_1 \in S$, so that the assertion follows from the transitivity of the action of G on S.

Case 2: We assume that some pair (e, f), (f, h) or (e, h) is transversal. We may w.l.o.g. assume that (e, f) is transversal. Then $\operatorname{Face}(e, f, h) \supseteq F(e, f) = \overline{\mathcal{D}}$ by Theorem II.12, and $G.(e, f)$ contains $(e, -e)$ because $\operatorname{rk} f = \operatorname{rk} e = r$ (Lemma I.20). Therefore the orbit of (e, f, h) contains an element of the form $(e, -e, h)$. Since $\mathcal{D}$ is of tube type, we have $V_0(e) = V_1(e) = \{0\}$, so that $Q(e)$ is invertible (cf. (1.11)), and $(V, e, Q(e))$ is a unital involutive Jordan algebra. In this Jordan algebra, S is the set of unitary elements, so that $h^* = Q(e)h = h^{-1}$ (Jordan inverse). Now the assertion follows from the Spectral Theorem for unitary elements in $(V, e, Q(e))$ (cf. [FK94, Prop. X.2.3]).

Case 3: $\operatorname{Face}(e, f, h) = \overline{\mathcal{D}}$, but neither (e, f), nor (f, h) or (e, h) is transversal. Since G acts transitively on S, we may w.l.o.g. assume that $e = e_r = c_1 + \ldots + c_r$. Consider the proper face $F := \operatorname{Face}(f, h)$ of $\overline{\mathcal{D}}$. Then we have

$$\overline{\mathcal{D}} = \operatorname{Face}(e, f, h) = \operatorname{Face}(\{e\} \cup F),$$

and for any $x \in \operatorname{algint}(F)$ we obtain

$$\overline{\mathcal{D}} = \operatorname{Face}(\{e\} \cup F) = \operatorname{Face}(e, x),$$

which means that e and x are transversal (Theorem II.12).

Now we need the classification of G-orbits in the set of transversal pairs, which shows that the pair (e, x) is conjugate to an element of the form $(e, -e + e_j)$ (Lemma I.20). The face

$$F' = \operatorname{Face}(-e + e_j) = -\operatorname{Face}(e - e_j) = -(e - e_j) + (V_0(e - e_j) \cap \mathcal{D}) = (e_j - e) + (V_2(e_j) \cap \mathcal{D})$$

is a bounded symmetric domain or rank j, and (e,f,h) is conjugate to a triple of the form (e,f',h') where f',h' are two elements in the Shilov boundary of F', where they are transversal because they generate F' as a face (Theorem II.12). Next we observe that the Peirce rules imply that by exponentiating elements of the centralizer of $e-e_j$ in $\mathfrak{g}$ we generate the identity component G^0 of the group $\mathrm{Aut}(\mathcal{D}\cap V_0(e-e_j))$ and its elements g act on e_j-e+z by

$$g.(e_j-e+z)=(e_j-e)+g.z$$

because they commute with the translation t_{e_j-e}. Now we conclude the proof by applying the special case of transversal elements which has already been taken care of, to see that the G^0-orbit of (e,f',h') intersects T. ■

Remark III.2. If $\mathcal{D}$ is not of tube type, then the Cayley transform $C=C_e$ leads to a realization of $\mathcal{D}$ as a Siegel domain $\mathcal{D}^C$ of type II, and since $C_e(-e)=0$, the stabilizer $G_{e,-e}$ of $\pm e$ in G corresponds to the stabilizer $Q^C_{e,-e}:=C_e(G_{e,-e})$ of 0 in the affine group Q^C_e, and the identity component of this group is $G(E_+)_0K_e$ (see the proof of Theorem I.18). The Shilov boundary of $\mathcal{D}^C$ is the set

$$\{(v_2,v_1)\in V=V_2\oplus V_1\colon \mathrm{Re}\,v_2=F(v_1,v_1)\},$$

and from this description it is clear that no element v_2+v_1 with $v_1\neq 0$ is conjugate under $Q^C_{e,-e}$ to an element in $\mathrm{span}_{\mathbb{R}}\{c_1,\ldots,c_r\}\subseteq V_2$. Therefore the condition that $\mathcal{D}$ is of tube type is necessary for the conclusion of Theorem III.1. ■

Example III.3. The simplest example of a bounded symmetric domain not of tube type is the matrix ball $\mathcal{D}\subseteq\mathbb{C}^n$ for $n>1$. Its rank is $r=1$ and in this case $G\cong\mathrm{PSU}_{n,1}(\mathbb{C})$ (cf. Example II.13).

To $z\in\mathcal{D}$ we assign the one-dimensional subspace $L_z:=\mathbb{C}\begin{pmatrix}z\\1\end{pmatrix}\in\mathbb{C}^{n+1}$. Endowing $\mathbb{C}^{n+1}$ with the indefinite hermitian form h given by

$$h(z,w):=z_1\overline{w_1}+\ldots+z_n\overline{w_n}-z_{n+1}\overline{w_{n+1}},$$

we see that $\mathcal{D}$ corresponds to the set of lines on which h is negative definite, and its Shilov boundary, the sphere $S\cong\mathbb{S}^{2n-1}$, corresponds to the set of isotropic lines. In this picture the action of $\mathrm{SU}_{n,1}(\mathbb{C})$ on $\mathcal{D}$ comes from the natural action of this group on the one-dimensional subspaces of $\mathbb{C}^{n+1}$.

Fixing a unit vector $e\in S$, the pair $(e,-e)$ corresponds to two different isotropic lines L_e and L_{-e} in $\mathbb{C}^{n+1}$, and the stabilizer of this pair in $\mathrm{U}_{n,1}(\mathbb{C})$ fixes the non-degenerate subspace L_e+L_{-e}, and also its orthogonal complement of dimension $n-1$. We conclude that $\mathrm{U}_{n,1}(\mathbb{C})_{e,-e}\cong\mathbb{R}^\times\times\mathrm{U}_{n-1}(\mathbb{C})$, and that no line $L_z\not\subseteq L_e+L_{-e}$ can be moved by $\mathrm{U}_{n,1}(\mathbb{C})$ into the plane L_e+L_{-e}. On the other hand, the set of isotropic lines in the plane L_e+L_{-e} corresponds to the circle in S obtained by intersecting S with the boundary of a one-dimensional disc $\Delta\subseteq\mathcal{D}$ of size 1, which in particular is a polydisc of maximal rank. This shows quite directly that there are triples in S that cannot be moved into the one-dimensional space $\mathbb{C}e$, so that Theorem III.1 does not hold.

That Theorem III.1 fails in this context, can be expressed quantitatively by the observation that

$$F(\mathbb{C}v_1,\mathbb{C}v_2,\mathbb{C}v_3):=\frac{h(v_1,v_2)h(v_2,v_3)h(v_3,v_1)}{h(v_2,v_1)h(v_3,v_2)h(v_1,v_3)}$$

is a well-defined function on the set of triples of pairwise different isotropic lines in $\mathbb{C}^{n+1}$ which is invariant under the pseudo-unitary group $U_{n,1}(\mathbb{C})$. The function F is related to the *Cartan invariant* (for a presentation and a generalization of this invariant we refer to [Cl05]). ■

Example III.4. The matrix ball $\mathcal{D} \subseteq M_n(\mathbb{C})$ is a symmetric domain of tube type with Shilov boundary $S = U_n(\mathbb{C})$, the unitary group. The maximal polydiscs in $\mathcal{D}$ are obtained by intersecting $\mathcal{D}$ with the set of all matrices that are diagonal with respect to some fixed orthonormal basis of $\mathbb{C}^n$ with respect to the standard scalar product. A particular Jordan frame consists of the matrix units $c_j := E_{jj}$, $j = 1, \ldots, n$, whose span is the set of diagonal matrices. Therefore Theorem III.1 states that each triple (s_1, s_2, s_3) of unitary matrices can be diagonalized by an element $g \in U_{n,n}(\mathbb{C})$, acting on $U_n(\mathbb{C})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = (az+b)(cz+d)^{-1}.$$

The compact subgroup $U_n(\mathbb{C}) \times U_n(\mathbb{C})$ acts linearly by $(a,d).z = azd^{-1}$, and under this group each pair (s_1, s_2) is conjugate to a pair of the form $(\mathbf{1}, s_2')$, where the stabilizer of $\mathbf{1}$ is the diagonal subgroup, acting on the second component by $(a, a^{-1}).s_2 = as_2a^{-1}$, so that s_2' can be diagonalized by conjugating with a suitable element $a \in \mathrm{U}_n(\mathbb{C})$. This means that diagonalizability of pairs reduces to classical linear algebra, but diagonalizability of triples requires the non-linear action of $U_{n,n}(\mathbb{C})$ and Theorem III.1.

A classification of the conjugation orbits of $U_n(\mathbb{C})$ in $U_n(\mathbb{C})^2$ is given in [FMS04], but since $U_n(\mathbb{C})$ is much smaller than $U_{n,n}(\mathbb{C})$, this classification leads to infinitely many orbits. ■

Polydisc in bounded symmetric domains

Let $\mathcal{D} \subseteq V$ be a bounded symmetric domain of rank r and $\Delta^r \subseteq \mathbb{C}^r$ the r-dimensional unit polydisc. We endow $\mathbb{C}^r$ with the metric defined by the sup-norm

$$|z| := \max\{|z_1|, \ldots, |z_r|\}$$

and V by the metric defined by the spectral norm, also denotes $|z|$.

Theorem III.5. *Any affine isometric map $f\colon \mathbb{C}^r \to V$ mapping $\overline{\Delta}^r$ into $\overline{\mathcal{D}}$ is linear and preserves the rank, i.e., for each $x \in \overline{\Delta}^r$ we have*

$$\operatorname{rk} f(x) = \operatorname{rk} x.$$

Moreover, it is a morphism of Jordan triples and $f(e_1, \ldots, e_r)$ is a Jordan frame.

Proof. Let $x_0 := f(0)$. Then $\ell(x) := f(x) - x_0$ defines an isometric linear map $\ell\colon \overline{\Delta}^r \to V$. Since ℓ is linear and isometric, it maps the open unit ball Δ^r in $\mathbb{C}^r$ into the open unit ball $\mathcal{D}$ of $(V, |\cdot|)$, so that it also maps $\overline{\Delta}^r$ isometrically into $\overline{\mathcal{D}}$.

Let $f_1, \ldots, f_r$ denote the images of the canonical basis in $\mathbb{C}^r$ under ℓ. Then the coordinate projections

$$\chi_j\colon L := \operatorname{span}\{f_1, \ldots, f_r\} = \operatorname{im}(\ell) \to \mathbb{C}, \quad \sum_j \lambda_j f_j \mapsto \lambda_j$$

are linear maps with $\|\chi_j\| = 1$ because $\ell\colon \mathbb{C}^r \to L$ is an isometric inclusion. Using the Hahn–Banach Theorem, we find extensions $\chi_j\colon V \to \mathbb{C}$ with the same norm. Then the map

$$\chi := (\chi_1, \ldots, \chi_r)\colon V \to \mathbb{C}^r$$

satisfies $\|\chi\| = 1$ and $\chi \circ \ell = \mathrm{id}$. It follows in particular that $\chi(\mathcal{D}) \subseteq \Delta^r$.

Since χ maps $\overline{\mathcal{D}}$ into $\overline{\Delta}^r$, we have an order-preserving map

$$\chi^*\colon \mathcal{F}(\overline{\Delta}^r) \to \mathcal{F}(\overline{\mathcal{D}}), \quad F \mapsto \chi^{-1}(F)$$

and the corresponding map

$$\ell^*\colon \mathcal{F}(\overline{\mathcal{D}}) \to \mathcal{F}(\overline{\Delta}^r), \quad F \mapsto \ell^{-1}(F)$$

satisfies

$$\ell^* \circ \chi^* = (\chi \circ \ell)^* = \mathrm{id}\,.$$

We conclude that χ^* is an order preserving injection. This entails in particular, that for each strictly increasing chain

$$F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_r$$

of faces of $\overline{\Delta}^r$, the images under χ^* form a strictly increasing chain of faces of $\overline{\mathcal{D}}$. Since r is the rank of $\mathcal{D}$, the maximal chains in $\mathcal{F}(\overline{\mathcal{D}})$ are of length r, which implies that χ^* preserves the rank of faces. Since the rank of an element $x \in \overline{\mathcal{D}}$ coincides with the rank of the face it generates, we further see that for $z \in \overline{\Delta}^r$ we have

$$\operatorname{rk} \ell(z) = \operatorname{rk} \operatorname{Face}(\ell(z)) = \operatorname{rk} \ell^*(\operatorname{Face}(z)) = \operatorname{rk}(\operatorname{Face}(z)) = \operatorname{rk} z.$$

Therefore ℓ preserves the rank.

Moreover, ℓ maps the Shilov boundary $\mathbb{T}^r$, consisting of the elements of maximal rank, into the Shilov boundary S of $\mathcal{D}$. The relation

$$f(\overline{\Delta}^r) = x_0 + \ell(\overline{\Delta}^r) \subseteq \overline{\mathcal{D}}$$

implies

$$-x_0 + \ell(\overline{\Delta}^r) = -(x_0 + \ell(\overline{\Delta}^r)) \subseteq \overline{\mathcal{D}},$$

so that for each $z \in \mathbb{T}^r$ we have

$$\ell(z) = \tfrac{1}{2}((\ell(z) + x_0) + (\ell(z) - x_0)) \in S,$$

so that $S = \operatorname{Ext}(\overline{\mathcal{D}})$ implies $x_0 = 0$, and hence $f = \ell$ is linear.

For $i \in \{1, \ldots, r\}$ we consider the corresponding face

$$F := \{z \in \overline{\Delta}^r : z_i = 1\} \in \mathcal{F}(\overline{\Delta}^r).$$

Then F is the closure of an $(r-1)$-dimensional affine polydisc, and $f\,|_F : F \to \overline{\mathcal{D}}$ is an affine isometry into a face $F_c \in \mathcal{F}(\overline{\mathcal{D}})$, where c is a primitive tripotent (Theorem I.5, Prop. II.3). Applying the first part of the proof with $\mathcal{D}$ replaced by $\operatorname{algint}(F')$ to the corresponding map

$$\overline{\Delta}^{r-1} \to F_c - c, \quad z \mapsto f(z_1, \ldots, z_{i-1}, 1, z_i, \ldots, z_r) - c,$$

we see that this map is linear, hence maps 0 to 0, which leads to $f(e_i) = c$. For $i \neq j$ the element $e_i + e_j \in \overline{\Delta}^r$ is contained in the face generated by e_i, which implies that $f(e_i + e_j) = f(e_i) + f(e_j)$ is contained in the face generated by $f(e_i)$. From Theorem I.5 we now derive

$$f(e_j) = f(e_i + e_j) - f(e_i) \in V_0(f(e_i)),$$

so that the primitive tripotents $f(e_i)$, $i = 1, \ldots, r$, are mutually orthogonal. Hence the linear map $f : \mathbb{C}^r \to V$ is a morphism of Lie triples systems. ■

Corollary III.6. *Suppose that $\mathcal{D}_1 \subseteq V_1$ and $\mathcal{D}_2 \subseteq V_2$ are circular bounded symmetric domains of the same rank. Then any affine isometric map $f : V_1 \to V_2$ mapping $\overline{\mathcal{D}}_1$ into $\overline{\mathcal{D}}_2$ is linear and rank-preserving.*

Proof. Let $r := \operatorname{rk} \mathcal{D}_1 = \operatorname{rk} \mathcal{D}_2$ and fix a polycylinder $\mathcal{D}_0 := \Delta^r \subseteq \mathcal{D}_1$ defined by a Jordan frame $(c_1, \ldots, c_r)$. For $V_0 := \operatorname{span}\{c_1, \ldots, c_r\}$ we then obtain by restriction an isometric map $f_0 : V_0 \to V_2$ mapping $\overline{\mathcal{D}}_0 \to \overline{\mathcal{D}}_2$. In view of Theorem III.5, this map is linear, which implies $f(0) = f_0(0) = 0$, and thus f is linear.

Moreover, f_0 is rank-preserving by Theorem III.5, which implies that f is also rank-preserving. ■

Corollary III.7. *If $r = \operatorname{rank}\mathcal{D}$, then any isometric linear embedding $f\colon \Delta^r \hookrightarrow \mathcal{D}$ is equivariant in the sense that there exists a subgroup $G_1 \subseteq \operatorname{Aut}(\mathcal{D}_0)$ and a surjective homomorphism $G_1 \to \operatorname{Aut}(\Delta^r)_0 \cong \mathrm{PSU}_{1,1}(\mathbb{C})^r$ such that f is equivariant with respect to the action of G_1 on Δ^r and $\mathcal{D}$.*

Proof. If $(e_1, \ldots, e_r)$ is the canonical basis in $\mathbb{C}^r$, then $(c_1, \ldots, c_r) := (f(e_1), \ldots, f(e_r))$ is a Jordan frame, so that

$$\mathfrak{g}_1 := \sum_{j=1}^{r} \mathfrak{g}_{c_j} \subseteq \mathfrak{g}$$

is isomorphic to $\mathfrak{su}_{1,1}(\mathbb{C})^r \cong \mathfrak{sl}_2(\mathbb{R})^r$ (see (I.8)), the Lie algebra of the group $\operatorname{Aut}(\Delta^r)_0 \cong \mathrm{PSU}_{1,1}(\mathbb{C})$. We may now put $G_1 := \langle \exp \mathfrak{g}_1 \rangle \subseteq G$, and the assertion follows. ∎

IV. The Maslov index

To define the integers classifying the G-orbits in $S \times S \times S$, we need in particular the Maslov index, a certain G-invariant function $\iota\colon S \times S \times S \to \mathbb{Z}$. In this section we explain how the Maslov index can be defined for bounded symmetric domains of tube type which are not necessarily irreducible, hence extending the definition given in [CØ01], [CØ03], [Cl04b]. Using Theorem III.1, we further derive a list of properties of the Maslov index and show that it can be characterized in an axiomatic fashion by these properties. Actually this was our original motivation to prove Theorem III.1.

Let us first consider the case of the unit disc Δ. Then the group G is $\mathrm{PSU}_{1,1}(\mathbb{C})$ acting by homographies on Δ, and its Shilov boundary is the unit circle $\mathbb{T}$. The *Maslov index*

$$\iota = \iota_{\mathbb{T}} : \mathbb{T} \times \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{Z}$$

is defined by

- $\iota(x, y, z) = 0$ if two of the elements of the triplet coincide.
- $\iota(x, y, z) = \pm 1$ if (x, y, z) is conjugate under G to $(1, -1, \mp i)$.

If Δ^r denotes the r-polydisc, then the identity component of $\operatorname{Aut}(\Delta^r)$ is $G = \mathrm{PSU}_{1,1}(\mathbb{C})^r$ and the Shilov boundary of Δ^r is $\mathbb{T}^r$. The Maslov index $\iota = \iota_{\mathbb{T}^r} : \mathbb{T}^r \longrightarrow \mathbb{R}$ is defined by

$$\iota((x_1, x_2, \ldots, x_r), (y_1, y_2, \ldots, y_r), (z_1, z_2, \ldots, z_r)) := \iota(x_1, y_1, z_1) + \iota(x_2, y_2, z_2) + \ldots + \iota(x_r, y_r, z_r)\ .$$

Now consider an irreducible bounded symmetric domain $\mathcal{D}$ of tube type with Shilov boundary S. The Maslov index $\iota = \iota_S : S \times S \times S \longrightarrow \mathbb{Z}$ is defined in [CØ01], [CØ03], [Cl04b]. As the definition is involved, we won't repeat it here, but it has the following property, which, in the light of Theorem III.1 and because of the invariance of this index under G, is characteristic: For any Jordan frame $(c_1, c_2, \ldots, c_r)$, let

$$T = \Big\{ \sum_{j=1}^{r} t_j c_j \colon |t_j| = 1, 1 \leq j \leq r \Big\}$$

be the r-torus which is the Shilov boundary of the associated r-polydisc. Then for any three points x, y, z in T, one has

$$\iota_S(x, y, z) = \iota_T(x, y, z). \tag{4.2}$$

Last, we extend now the definition of the Maslov index to any bounded symmetric domain $\mathcal{D}$ in the following way. Assume that $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2 \times \ldots \times \mathcal{D}_m$ is the decomposition of $\mathcal{D}$ as a product of irreducible domains. Then the identity component of the group of biholomorphic automorphisms of $\mathcal{D}$ is the product

$$G = \operatorname{Aut}(\mathcal{D}_1)_0 \times \operatorname{Aut}(\mathcal{D}_2)_0 \times \ldots \times \operatorname{Aut}(\mathcal{D}_m)_0,$$

and the Shilov boundary S of $\mathcal{D}$ is the product $S = S_1 \times S_2 \times \ldots \times S_m$ of the corresponding Shilov boundaries. Then the Maslov index $\iota = \iota_S$ is defined by

$$\iota(x, y, z) := \iota_{S_1}(x_1, y_1, z_1) + \iota_{S_2}(x_2, y_2, z_2) + \ldots + \iota_{S_r}(x_l, y_l, z_l) \ .$$

Theorem IV.1. *The Maslov index has the following properties :*

(M1) *It is invariant under the group G.*

(M2) *It is an alternating function with respect to any permutation of the three arguments.*

(M3) *It satisfies the cocycle property $\iota(x, y, z) = \iota(x, y, w) - \iota(x, z, w) + \iota(y, z, w)$.*

(M4) *It is additive in the sense that if $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$, so that $S = S_1 \times S_2$, then*

$$\iota_S(x, y, z) = \iota_S((x_1, x_2), (y_1, y_2), (z_1, z_2)) = \iota_{S_1}(x_1, y_1, z_1) + \iota_{S_2}(x_2, y_2, z_2) \ .$$

(M5) *If $\Phi : \mathcal{D}_1 \longrightarrow \mathcal{D}_2$ is an equivariant holomorphic embedding of bounded symmetric domains of tube type of equal rank, then $\iota_{S_2} \circ \Phi = \iota_{S_1}$.*

(M6) *It is normalized by $\iota_{\mathbb{T}}(1, -1, -i) = 1$ for the Shilov boundary $\mathbb{T}$ of the unit disc Δ.*

Proof. Properties (M1)-(M3) are known for irreducible domains ([CØ01], [Cl04]), and the extension of these properties to products of irreducible domains is obvious. Property (M4) obviously holds by the way we have defined the Maslov index.

For Property (M5), let r be the common rank of the two domains. We may assume that $\mathcal{D}_1$ and $\mathcal{D}_2$ are given in a circular realization as unit balls in spaces V_1, resp., V_2. Then $\varphi(0) \in \mathcal{D}_2$, and there is some $g_2 \in G_2 := \mathrm{Aut}(\mathcal{D}_2)_0$ with $g_2.\varphi(0) = 0$. Then $\psi(z) := g_2.\varphi(z)$ defines an equivariant embedding $\mathcal{D}_1 \to \mathcal{D}_2$ which is linear because $\psi(0) = 0$.

Let $(x, y, z) \in S_1$ and pick $g_1 \in G_1 := \mathrm{Aut}(\mathcal{D}_1)_0$ such that $g_1.(x, y, z)$ is contained in the span of a Jordan frame $(c_1, \ldots, c_r)$ (Theorem III.1), hence in the Shilov boundary T_1 of the corresponding polydisc Δ^r in $\mathcal{D}_1$. From the equivariance of φ we derive the existence of some $\widetilde{g}_1 \in G_2$ with $\varphi \circ g_1 = \widetilde{g}_1 \circ \varphi$. Then $\psi(\Delta^r)$ is a maximal polydisc in $\mathcal{D}_2$ with Shilov boundary $T_2 := \psi(T_1)$, so that (4.2) implies that

$$\begin{aligned}
\iota_{S_1}(x, y, z) &= \iota_{S_1}(g_1.x, g_1.y, g_1.z) = \iota_{T_1}(g_1.x, g_1.y, g_1.z) \\
&= \iota_{T_2}(\psi(g_1.x), \psi(g_1.y), \psi(g_1.z)) = \iota_{S_2}(\psi(g_1.x), \psi(g_1.y), \psi(g_1.z)) \\
&= \iota_{S_2}(g_2\varphi(g_1.x), g_2\varphi(g_1.y), g_2\varphi(g_1.z)) = \iota_{S_2}(\varphi(g_1.x), \varphi(g_1.y), \varphi(g_1.z)) \\
&= \iota_{S_2}(\widetilde{g}_1\varphi(x), \widetilde{g}_1\varphi(y), \widetilde{g}_1\varphi(z)) = \iota_{S_2}(\varphi(x), \varphi(y), \varphi(z)).
\end{aligned}$$

Property (M6) is a consequence of the definition. ■

Remark IV.2. Note that (M2) and (M3) mean that ι_S is a $\mathbb{Z}$-valued Alexander–Spanier 2-cocycle on S. ■

Before we turn to the general case in the following section, we recall the classification of triples in the circle, the Shilov boundary of the unit disc:

Example IV.3. We consider the case $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. Then $G = \mathrm{PSU}_{1,1}(\mathbb{C})$ acts by

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right].z = (az + b)(cz + d)^{-1}.$$

The Shilov boundary is $S = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Identifying S with the projective line $\mathbb{P}_1(\mathbb{R})$ and G with $\mathrm{PSL}_2(\mathbb{R})$, we immediately see that there are exactly two G-orbits in $S \times S$, represented by

$$(1, 1) \quad \text{and} \quad (1, -1),$$

i.e., the diagonal in $S \times S$ and the set $(S \times S)_\top$ of transversal pairs. Since the action of G on S preserves the orientation of a triple, it follows that we have 6 orbits in $S \times S \times S$, represented by

$$(1, 1, 1), \quad (1, 1, -1), \quad (1, -1, 1), \quad (1, -1, -1), \quad (1, -1, -i) \quad \text{and} \quad (1, -1, i).$$ ■

Remark IV.4. As a function assigning to any triple in the Shilov boundary of any bounded symmetric domain $\mathcal{D}$ an integer, the Maslov index is uniquely determined by the properties (M1), (M2) and (M4)-(M6).

In view of Example IV.3, the Maslov index for $\mathcal{D} = \Delta$ is uniquely determined by (M1), (M2) and (M6). By (M4) it is also determined for polydiscs.

If $\mathcal{D}$ is any bounded symmetric domain of rank r and $(s_1, s_2, s_3) \in S \times S \times S$, then Theorem III.1 implies that it can be conjugate by some $g \in G$ to a triple in the Shilov boundary $T \cong \mathbb{T}^r$ of a maximal polydisc, so that Corollary III.7, (M1) and (M5) lead to

$$\iota_S(s_1, s_2, s_3) = \iota_S(g.s_1, g.s_2, g.s_3) = \iota_T(g.s_1, g.s_2, g.s_3).$$

We conclude that ι_S is determined uniquely by (M1), (M2), together with (M3)-(M6). ■

A classical case: the Lagrangian manifold

Let E be a real vector space of dimension $2r$ and ω be a symplectic form on E. The symplectic group $\mathrm{Sp}(E, \omega)$ is the group of linear automorphisms which preserve ω. A *Lagrangian* is a maximal totally isotropic subspace of E, hence of dimension r. The set Λ_r of all Lagrangians is a compact submanifold of the Grassmannian $\mathrm{Gr}_r(E)$ of r-dimensional subspaces of E. Then the group $G := \mathrm{PSp}(E, \omega) := \mathrm{Sp}(E, \omega)/\{\pm\mathbf{1}\}$ acts transitively and effectively on Λ_r. Choosing a symplectic basis in E, we may identify E with $\mathbb{R}^r \times \mathbb{R}^r$, the symplectic form being the standard one, namely

$$\omega((\xi, \eta), (\xi', \eta')) = \xi^\top \eta' - \eta^\top \xi'. \tag{4.1}$$

Let us consider the complex vector space $V = \mathrm{Sym}_r(\mathbb{C})$ of complex $r \times r$ symmetric matrices, and let $\mathcal{D}$ be the unit ball with respect to the operator norm. The space V is an involutive unital Jordan algebra with real form $\mathrm{Sym}_r(\mathbb{R})$, involution $z^* = \overline{z}$ and Jordan product $x*y := \frac{1}{2}(xy+yx)$. The spectral norm on V coincides with the operator norm, and the unit ball is then a bounded symmetric domain. To make connection with symplectic geometry, observe that the graph of a symmetric matrix is a complex isotropic subspace in $\mathbb{C}^r \times \mathbb{C}^r$ for the symplectic structure (4.1). Let moreover h be the Hermitian form on $\mathbb{C}^r \times \mathbb{C}^r$ given by

$$h((\xi, \eta), (\xi', \eta')) = \xi^\top \overline{\xi'} - \eta^\top \overline{\eta'} = (\xi')^* \xi - (\eta')^* \eta.$$

The Hermitian form h has signature (r, r). Now to any $x \in V$, associate its graph

$$\ell_x = \{(\xi, x.\xi) : \xi \in \mathbb{C}^r\}.$$

The condition that x is in the unit ball is equivalent to the fact that $\mathbf{1} - xx^*$ is positive definite, which in turn implies that the restriction of h to ℓ_x is positive definite. Conversely, any (complex) Lagrangian in $\mathbb{C}^r \times \mathbb{C}^r$ on which the restriction of h is positive definite is the graph of some complex symmetric matrix in the unit ball. The Shilov boundary of $\mathcal{D}$ is the manifold of unitary symmetric matrices, and the corresponding graphs are the (complex) Lagrangians on which the restriction of the form h is identically 0. Let C be the map from $\mathbb{R}^r \times \mathbb{R}^r$ to $\mathbb{C}^r \times \mathbb{C}^r$ given by

$$C(\xi, \eta) = \Big(\frac{\xi + i\eta}{\sqrt{2}}, \frac{\xi - i\eta}{\sqrt{2}}\Big).$$

Then an elementary computation shows that the complexification of the image under C of a (real) Lagrangian is a (complex) Lagrangian on which the restriction of h is identically 0, and vice versa. This gives a one-to-one correspondence between Λ_r and S. Moreover the natural action of G on Λ_r is transferred to an action on S and realizes an isomorphism of the real symplectic group and the group $\mathrm{Sp}_{2r}(\mathbb{C}) \cap \mathrm{U}_{r,r}(\mathbb{C})$, which generalizes the isomorphism of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SU}_{1,1}(\mathbb{C})$.

The matrices $E_{11},\ldots,E_{rr}$ form a Jordan frame in $\mathrm{Sym}_r(\mathbb{C})$. The corresponding r-torus is

$$T := \left\{ \begin{pmatrix} e^{i\theta_1} & 0 & \ldots & 0 \\ 0 & e^{i\theta_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{i\theta_r} \end{pmatrix} : \theta_j \in \mathbb{R}, 1 \leq j \leq r \right\}.$$

The graph of an element of T is the r-space generated by

$$(e_1, e^{i\theta_1}e_1), (e_2, e^{i\theta_2}e_2), \ldots, (e_r, e^{i\theta_r}e_r),$$

or equivalently by

$$(e^{-i\frac{\theta_1}{2}}e_1, e^{i\frac{\theta_1}{2}}e_1), (e^{-i\frac{\theta_2}{2}}e_2, e^{i\frac{\theta_2}{2}}e_2), \ldots, (e^{-i\frac{\theta_r}{2}}e_r, e^{i\frac{\theta_r}{2}}e_r).$$

Observe that $(e^{-i\frac{\theta_j}{2}}e_j, e^{i\frac{\theta_j}{2}}e_j) = C(\cos\frac{\theta_j}{2}e_j, \sin\frac{\theta_j}{2}e_j)$ to get that the corresponding Lagrangian $\ell(\theta_1,\theta_2,\ldots,\theta_r)$ in Λ_r is generated by

$$\Big(\cos\frac{\theta_1}{2}e_1, -\sin\frac{\theta_1}{2}e_1\Big), \Big(\cos\frac{\theta_2}{2}e_2, -\sin\frac{\theta_2}{2}e_2\Big), \ldots, \Big(\cos\frac{\theta_r}{2}e_r, -\sin\frac{\theta_r}{2}e_r\Big).$$

In this case, one can then reformulate Theorem III.1 as follows.

Theorem IV.5. *Let ℓ_1,ℓ_2,ℓ_3 be three arbitrary Lagrangians in a symplectic vector space E of dimension $2r$. Then there exists a symplectic basis $e_1,e_2,\ldots,e_r,f_1,f_2,\ldots,f_r$ such that each of the three Lagrangians is generated by*

$$\cos\theta_1 e_1 + \sin\theta_1 f_1, \cos\theta_2 e_2 + \sin\theta_2 f_2, \ldots, \cos\theta_r e_r + \sin\theta_r f_r$$

for appropriate choices of the $(\theta_j)_{1\leq j\leq r}$. ■

The classification result (Theorem V.4 below) for the case $S = \Lambda_r$ can also be found in [KS90, p.492].

V. The classification of triples

In this section we complete the classification of G-orbits in the set $S\times S\times S$ of triples in S by first assigning to each triples an increasing 5-tuple of integers $N = (n_1,n_2,n_3,n_4,n_5) \in \{0,\ldots,r\}^5$ depending only on its orbit. Then we exhibit for each such 5-tuple a standard triple with this invariant, and finally we show that two different standard triples belong to different orbits.

Definition V.1. To any triple (x_1,x_2,x_3) in $S\times S\times S$, we may associate five integers:
(1) the ranks of the three faces (cf. Remark II.4):

$$n_{12} = \mathrm{rank}\,\mathrm{Face}(x_1,x_2), \quad n_{2,3} = \mathrm{rank}\,\mathrm{Face}(x_2,x_3), \quad n_{3,1} = \mathrm{rank}\,\mathrm{Face}(x_3,x_1)$$

(2) the rank of the face generated by the triple

$$n_{1,2,3} = \mathrm{rank}\,\mathrm{Face}(x_1,x_2,x_3)$$

(3) the Maslov index $\iota(x_1,x_2,x_3)$.

Clearly the action of G preserves these integers. ■

When x_1,x_2,x_3 are contained in the boundary of a polydiscs (cf. Section III), then these integral invariants are easy to compute (cf. Example II.6).

Lemma V.2. *Let* $e = \sum_{j+1}^r c_j$ *be a Peirce decomposition of the unit, and, for* $\kappa = 1,2,3$, *let*

$$x_\kappa = \sum_{j=1}^r \xi_j^{(\kappa)} c_j, \quad \textit{where} \quad |\xi_j^{(\kappa)}| = 1 \quad \textit{for all} \quad j \in \{1,\ldots,r\}.$$

Then

$$n_{\kappa,\kappa'} = |\{j : \xi_j^{(\kappa)} = \xi_j^{(\kappa')}\}|, \quad n_{1,2,3} = |\{j : \xi_j^{(1)} = \xi_j^{(2)} = \xi_j^{(3)}\}|,$$

and

$$\iota(x_1, x_2, x_3) = \sum_{j=1}^r \iota(\xi_j^{(1)}, \xi_j^{(2)}, \xi_j^{(3)}).$$

■

Definition V.3. We now describe the *standard triples* associated to a (fixed) Jordan frame $(c_1,\ldots,c_r)$. Let $N = (n_1,n_2,n_3,n_4,n_5)$ be a 5-tuple of integers such that

$$0 \le n_1 \le n_2 \le n_3 \le n_4 \le n_5 \le r\ .$$

Then the *standard triple of type* N is the triple (x_1^N, x_2^N, x_3^N) defined by

$$x_1^N = e_r = c_1 + \ldots + c_r, \qquad x_2^N = c_1 + c_2 + \ldots + c_{n_2} - c_{n_2+1} - \ldots - c_r,$$

$$x_3^N = c_1 + \ldots + c_{n_1} - c_{n_1+1} - \ldots - c_{n_3} + c_{n_3+1} + \ldots + c_{n_4} - ic_{n_4+1} - \ldots - ic_{n_5} + ic_{n_5+1} + \ldots + ic_r\ .$$

For this triple, one has

$$n_{1,2,3} = n_1, \quad n_{1,2} = n_2, \quad n_{1,3} = n_1 + n_4 - n_3, \quad n_{2,3} = n_1 + n_3 - n_2,$$

and

$$\iota(x_1^N, x_2^N, x_3^N) = n_5 - n_4 - (r - n_5) = 2n_5 - n_4 - r.$$

■

Theorem V.4. *If* $\mathcal{D}$ *is an irreducible bounded symmetric domain of tube type, then any triple in* S *is conjugate to one and only one of the standard triples.*

Proof. For the standard triples we have

$$n_1 = n_{1,2,3}, \quad n_2 = n_{1,2}, \quad n_3 = n_{2,3} + n_2 - n_1 = n_{2,3} + n_{1,2} - n_{1,2,3}, \tag{5.1}$$

$$n_4 = n_{1,3} + n_3 - n_1 = n_{1,3} + n_{2,3} + n_{1,2} - 2n_{1,2,3}, \tag{5.2}$$

and

$$n_5 = \tfrac{1}{2}(\iota(x_1^N, x_2^N, x_3^N) + n_4 + r) = \tfrac{1}{2}(\iota(x_1^N, x_2^N, x_3^N) + r + n_{1,3} + n_{2,3} + n_{1,2} - 2n_{1,2,3}). \tag{5.3}$$

Since the numbers $n_{1,2,3}$, $n_{1,2}$, $n_{2,3}$, $n_{3,1}$ and the Maslov index are G-invariant, it follows that for different values of N, the corresponding standard triples are not conjugate under G.

To show, conversely, that each triple $(e,f,h) \in S \times S \times S$ is conjugate to a standard triple, we first use Theorem III.1 to see that we may w.l.o.g. assume that (e,f,h) is contained in the torus

$$T := \Big\{ \sum_{j=1}^r \lambda_j c_j : (\forall j)\ |\lambda_j| = 1 \Big\}$$

defined by the Jordan frame $(c_1,\ldots,c_r)$. It is the Shilov boundary of the polydisc

$$\Delta^r := \Big\{ \sum_{j=1}^r \lambda_j c_j : (\forall j)\ |\lambda_j| < 1 \Big\}.$$

We write

$$e = \sum_{j=1}^r \xi_j^e c_j, \quad f = \sum_{j=1}^r \xi_j^f c_j \quad \text{and} \quad h = \sum_{j=1}^r \xi_j^h c_j.$$

From I.8 it follows that every element of $\mathrm{Aut}(\Delta^r)_0 \cong \mathrm{PSU}_{1,1}(\mathbb{C})^r$ is the restriction of an element of $\mathrm{Aut}(\mathcal{D})_0$, because

$$\mathfrak{g}_{c_1} + \ldots + \mathfrak{g}_{c_r} \cong \mathfrak{su}_{1,1}(\mathbb{C})^r = \mathbf{L}(\mathrm{Aut}(\Delta^r))$$

is a subalgebra of $\mathfrak{g} = \mathbf{L}(G)$. We may therefore assume that $\xi_j^e = 1$ for each j. Let

$$n_2 := |\{j : \xi_j^e = \xi_j^f\}| = |\{j : \xi_j^f = 1\}|.$$

Since each permutation of the set $\{c_1, \ldots, c_r\}$ is induced by an element of K, which acts transitively on the set of Jordan frames, we may w.l.o.g. assume that

$$f = c_1 + c_2 + \ldots + c_{n_2} - c_{n_2+1} - \ldots - c_r$$

because the $\mathrm{Aut}(\Delta)_0$-orbits in $\mathbb{T} \times \mathbb{T}$ are represented by $(1,1)$ and $(1,-1)$ (Example IV.3).

Let $n_1 := |\{j : \xi_j^e = \xi_j^f = \xi_j^h\}|$ and write

$$n_4 := |\{j : \xi_j^e = \xi_j^f \text{ or } \xi_j^e = \xi_j^h \text{ or } \xi_j^f = \xi_j^h\}|$$

for the number of components in which at least two elements of $\{e, f, h\}$ have the same entries. Then h has precisely n_1 entries 1 among the first n_2, and we may w.l.o.g. assume that they arise in position $j = 1, \ldots, n_1$. We may likewise assume that the components of e, f and h are mutually different for $j > n_4$. Then the entries of h in positions $n_1 + 1, \ldots, n_2$ can be moved by elements of the group $\mathrm{Aut}(\Delta)_0^{n_2 - n_1}$ acting on these components to -1. For $j \in \{n_2 + 1, \ldots, n_4\}$ the j-th component of h equals either 1 or -1. Moving the 1-entries with some element of K_e permuting $\{c_1, \ldots, c_r\}$ to the rightmost positions, we get entries -1 for $j = n_1 + 1, \ldots, n_3$ for some n_3 satisfying $n_2 \leq n_3 \leq n_4$. For $j > n_4$ we then have $\mathrm{Im}\, \xi_j^h \neq 0$, and after permuting the Jordan frame, we may assume that for some $n_5 \geq n_4$ we have $\mathrm{Im}\, \xi_j^h < 0$ for $j = n_4 + 1, \ldots, n_5$ and $\mathrm{Im}\, \xi_j^h > 0$ for $j > n_5$. We finally use elements of $\mathrm{Aut}(\Delta)_0$ fixing 1 and -1 to move each entry with negative imaginary part to $-i$ and the others to i (cf. Example IV.3). This proves that each triple is conjugate to a standard triple. ■

Remark V.5. In Theorem V.4, we have classified the G-orbits in the space of triples in S by the set of all 5-tuples $N = (n_1, n_2, n_3, n_4, n_5) \in \{0, \ldots, r\}$ satisfying the monotonicity condition

$$n_1 \leq n_2 \leq n_3 \leq n_4 \leq n_5.$$

The description the standard triples shows that each such tuples arises via (5.1)-(5.3). We claim that for the 5-tuple

$$(r_0, r_1, r_2, r_3, d) := \big(n_{1,2,3}, n_{1,2}, n_{2,3}, n_{3,1}, \iota(x_1^N, x_2^N, x_3^N)\big)$$

of integers we then have

(P1) $0 \leq r_0 \leq r_1, r_2, r_3 \leq r$.

(P2) $r_1 + r_2 + r_3 \leq r + 2r_0$.

(P3) $|d| \leq r + 2r_0 - (r_1 + r_2 + r_3)$.

(P4) $d \equiv r + r_1 + r_2 + r_3 \bmod 2$.

In fact, (P1) is clear,

$$r_1 + r_2 + r_3 = n_4 + 2r_0 \leq r + 2r_0,$$

$$|d| = |n_5 - n_4 - (r - n_5)| \leq n_5 - n_4 + r - n_5 = r - n_4 = r + 2r_0 - r_1 - r_2 - r_3,$$

and

$$d = n_5 - n_4 - (r - n_5) \equiv n_4 + r \equiv r + r_1 + r_2 + r_3 \mod 2.$$

Suppose, conversely, that $(r_0, r_1, r_2, r_3, d) \in \mathbb{Z}^5$ satisfies (P1)-(P4). We then define

$$n_1 := r_0, \quad n_2 := r_1, \quad n_3 := r_2 + r_1 - r_0, \quad n_4 := r_3 + r_2 + r_1 - 2r_0$$

and

$$n_5 = \tfrac{1}{2}(d + r_3 + r_2 + r_1 + r) - r_0.$$

Then (P4) implies $n_5 \in \mathbb{Z}$. From (P1/2) we immediately get $0 \leq n_1 \leq n_2 \leq n_3 \leq n_4 \leq r$. Further (P3) leads to $|d| \leq r - n_4$, and $n_4 \leq n_5$ follows from

$$2n_5 = d + r_3 + r_2 + r_1 + r - 2r_0 = d + r + n_4 \geq r + n_4 - (r - n_4) = 2n_4.$$

This is turn implies $n_5 = \frac{1}{2}(r + d + n_4) \leq r$. ■

The conditions (P1)-(P4) are well known conditions describing the classification of triples of Lagrangian subspace of symplectic vector spaces ([KS90]).

VI. Classification of orbits in $S \times S$

In this section we describe how the classification of G-orbits in $S \times S$ can be derived from the Bruhat decomposition of G, resp., the description of the orbits of the maximal parabolic subgroup G_e in G with $G/G_e \cong S$.

Throughout this section we assume $\mathcal{D}$ to be irreducible. Let $(c_1, c_2, \ldots, c_r)$ be a Jordan frame and put

$$\varepsilon_k = c_1 + c_2 \ldots + c_k - c_{k+1} - \ldots - c_r \quad \text{for} \quad k = 0, \ldots, r.$$

Moreover let $e = c_1 + \ldots + c_r = \varepsilon_r$, and observe that $\varepsilon_0 = -e$. The vector space

$$\mathfrak{a} = \bigoplus_{j=1}^{r} \mathbb{R}c_j$$

is a maximal flat in V in the sense of Loos ([Lo77]) and can be thought of as a Cartan subspace in the tangent space of $\mathcal{D}$ at the origin. The corresponding vector fields form a Cartan subspace of $\mathfrak{p}$. Denoting by γ_j the j-th coordinate in $\mathfrak{a}$ with respect to the basis $(c_1, c_2, \ldots, c_r)$, it is known that the (restricted) roots of $(\mathfrak{g}, \mathfrak{a})$ are $\pm\gamma_j \pm \gamma_k, \pm 2\gamma_j, 1 \leq j \neq k \leq r$ and, in addition, $\pm\gamma_j, 1 \leq j \leq r$ in the non tube type case. We choose as positive Weyl chamber in $\mathfrak{a}$ the one defined by the inequalities

$$\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_r \geq 0,$$

so that the corresponding simple roots are

$$\gamma_1 - \gamma_2, \ \gamma_2 - \gamma_3, \ldots, \gamma_{r-1} - \gamma_r, \ \gamma_r \ .$$

The Weyl group W is isomorphic to the semi-direct product $\mathfrak{S}_r \ltimes \mathbb{Z}_2^r$, where $\mathfrak{S}_r$ acts by permutation of the coordinates γ_j, and the j-th factor $\mathbb{Z}_2$ acts by changing the sign of the j-th coordinate.

The stabilizer G_e of the point $e \in S$ is known to be a maximal parabolic subgroup (cf. Sect. I). It is the standard parabolic subgroup associated to the subset

$$\Theta = \{\gamma_1 - \gamma_2, \ \gamma_2 - \gamma_3, \ldots, \gamma_{r-1} - \gamma_r\}$$

of the set of simple roots. The subgroup W^Θ of W generated by the reflections associated to the roots in Θ is just $\mathfrak{S}_r$, and double cosets in $W^\Theta \backslash W / W^\Theta$ correspond to orbits of $\mathfrak{S}_r$ in $\mathbb{Z}_2^r$, which are characterized by their number of sign changes. In particular, this shows that the elements $\varepsilon_j, 0 \leq j \leq r$, form a set of representatives of the W^Θ-orbits in $W.e$.

Theorem VI.1. *There are $r+1$ orbits of G in $S\times S$. A set of representatives of these orbits is given by the pairs $(e,\varepsilon_k), 0\le k\le r$.*

Proof. As G acts transitively on S, any orbit of G in $S\times S$ meets the subset $\{e\}\times S$. So the statement amounts to show that a G_e-orbit in S contains ε_k for some $k, 0\le k\le r$. By Bruhat's theory, the orbits of the parabolic subgroup G_e of G are in one-to-one correspondence with the W^Θ-double cosets in W. In view of the preceding discussion, this shows the result. ∎

Remark VI.2. The open orbit in S under the G_e-action (the big Bruhat's cell) corresponds to the point $-e$ and is nothing but the set of all points in S transversal to e. ∎

Definition VI.3. For $(x,y)\in S\times S$ we define their *transversality index* $\mu(x,y)$ to be the unique number $k\in\{0,\ldots,r\}$ such that (x,y) belongs to the G orbit of (e,ε_k). Clearly, the transversality index is invariant by the action of G, and two pairs are conjugate if and only if they have the same transversality index. Moreover, a pair (x,y) is transversal if and only if its transversality index is 0. ∎

Theorem VI.4. *A pair $(x,y)\in S\times S$ has transversality index k if and only if the face $F(x,y)$ generated by x and y has rank k.*

Proof. For $0\le k\le r$ let $e_k=c_1+c_2+\ldots+c_k$. Then the face generated by e and ε_k is

$$\mathrm{Face}(e,\varepsilon_k)=(e_k+V_0(e_k))\cap\overline{\mathcal{D}},$$

which has rank k. As any pair in $S\times S$ is conjugate to one of the pairs (e,ε_k), the theorem follows immediately. ∎

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