The Fredholm property of pseudodifferential operators with non-smooth symbols on modulation spaces

Vladimir S. Rabinovich, Steffen Roch *

Abstract

The aim of the paper is to study the Fredholm property of pseudodifferential operators in the Sjöstrand class OPS_w where we consider these operators as acting on the modulation spaces $M^{2,p}(\mathbb{R}^N)$. These spaces are introduced by means of a time-frequency partition of unity. The symbol class S_w does not involve any assumptions on the smoothness of its elements.

In terms of their limit operators, we will derive necessary and sufficient conditions for operators in OPS_w to be Fredholm. In particular, it will be shown that the Fredholm property and, thus, the essential spectra of operators in this class are independent of the modulation space parameter $p \in (1, \infty)$.

1 Introduction

This paper is devoted to the study of the Fredholm property of pseudodifferential operators in the Sjöstrand class OPS_w. The class S_w of Sjöstrand symbols and the corresponding class OPS_w of pseudodifferential operators were introduced in [8, 9]. This class contains the Hörmander class $OPS^0_{0,0}$ and other interesting classes of pseudodifferential operators. One feature of the class S_w is that no assumptions on the smoothness of its elements are made.

Sjöstrand [8, 9] considers operators in OPS_w as acting on $L^2(\mathbb{R}^N)$. He proves the boundedness of these operators and shows that OPS_w is an inverse closed Banach subalgebra of the algebra $L(L^2(\mathbb{R}^N))$ of all bounded linear operators on $L^2(\mathbb{R}^N)$.

Applications in time-frequency analysis had lead to an increasing interest in pseudodifferential operators in classes similar to OPS_w but acting on several kinds of modulation spaces (see, for instance [1, 3, 2, 11]). These spaces are

*Supported by CONACYT project 43432.

defined by means of a so-called time-frequency partition of unity (i.e., a partition of unity on the phase space).

Whereas main emphasize in [1, 3, 2, 11] is on boundedness results, we are going to examine the Fredholm property of pseudodifferential operators in OPS_w on modulation spaces which seems to have not been considered earlier. Our approach is based on the limit operators method. An introduction into this method as well as several applications of limit operators to other quite general operator classes can be found in the monograph [6] (see also the references therein). For several of these operator classes (including OPS_w and the Hörmander class $OPS^0_{0,0}$), the limit operators approach seems to be the only available method to treat the Fredholm property.

The present paper is organized as follows. In Section 2 we recall some auxiliary material from [5] and [6]. In particular, we introduce the Wiener algebra $\mathcal{W}(\mathbb{Z}^N, X)$ of band-dominated operators with operator-valued coefficients acting on the spaces $l^p(\mathbb{Z}^N, X)$ where X is a Banach space. For operators belonging to the so-called rich subalgebra $\mathcal{W}^\$(\mathbb{Z}^N, X)$ of $\mathcal{W}(\mathbb{Z}^N, X)$ we formulate necessary and sufficient conditions for their Fredholmness. It will turn out that the Fredholm property and, thus, the essential spectrum of an operator $A \in \mathcal{W}^\$(\mathbb{Z}^N, X)$ are independent of $p \in (1, \infty)$.

Section 3 is devoted to modulation spaces and their discretizations. Given a time-frequency partition of unity by pseudodifferential operators

$$\sum_{\alpha \in \mathbb{Z}^{2N}} \Phi_\alpha^* \Phi_\alpha = I,$$

the modulation space $M^{2,p}(\mathbb{R}^N)$ is defined as the space of all distributions $u \in S'(\mathbb{R}^N)$ with

$$\|u\|_{M^{2,p}(\mathbb{R}^N)} := \left(\sum_{\alpha \in \mathbb{Z}^{2N}} \|\Phi_\alpha u\|^p_{L^2(\mathbb{R}^N)} \right)^{1/p} < \infty$$

if $p \in [1, \infty)$ and with

$$\|u\|_{M^{2,\infty}(\mathbb{R}^N)} := \sup_{\alpha \in \mathbb{Z}^{2N}} \|\Phi_\alpha u\|_{L^2(\mathbb{R}^N)} < \infty$$

in case $p = \infty$. In Section 4, we introduce the continuous analogue $\mathcal{W}(\mathbb{R}^N)$ of the discrete Wiener algebra $\mathcal{W}(\mathbb{Z}^N, X)$ by imposing conditions on the decay of the operators $\Phi_\alpha A \Phi^*_{\alpha-\gamma}$. More precisely, an operator A belongs to $\mathcal{W}(\mathbb{R}^N)$ if

$$\|A\|_{\mathcal{W}(\mathbb{R}^N)} := \sum_{\gamma \in \mathbb{Z}^{2N}} \sup_{\alpha \in \mathbb{Z}^{2N}} \|\Phi_\alpha A \Phi^*_{\alpha-\gamma}\|_{L(L^2(\mathbb{R}^N))} < \infty.$$

We prove that the operators in $\mathcal{W}(\mathbb{R}^N)$ act boundedly on $M^{2,p}(\mathbb{R}^N)$ for every $p \in [1, \infty]$ and that $\mathcal{W}(\mathbb{R}^N)$ is an inverse closed subalgebra of $L(M^{2,p}(\mathbb{R}^N))$.

Via discretization, the results recalled in Section 2 apply to yield necessary and sufficient conditions for the Fredholmness on $M^{2,p}(\mathbb{R}^N)$ of operators in the so-called rich subalgebra $\mathcal{W}^\$(\mathbb{R}^N)$ of $\mathcal{W}(\mathbb{R}^N)$. Moreover, the essential spectrum of $A \in W^\$(\mathbb{R}^N)$ proves to be independent of $p \in (1, \infty)$.

In the concluding fifth Section, we apply the description of operators in OPS_w derived in [1] to conclude that $OPS_w \subset W^\$(\mathbb{R}^N)$. Thus, the results of the previous sections specify to give Fredholm criteria for pseudodifferential operators in OPS_w acting on modulation spaces $M^{2,p}(\mathbb{R}^N)$ in terms of limit operators. One consequence is the independence of the essential spectrum of an operator $A \in OPS_w$ of the modulation space parameter p.

Notice that a criterion for the Fredholmness of pseudodifferential operators in $OPS^0_{0,0}$ acting on $L^2(\mathbb{R}^N)$ was obtained in [5] by similar techniques (see also Chapter 4 in [6]).

2 Operators in the discrete Wiener algebra

2.1 Band-dominated operators and $\mathcal{P}$-Fredholmness

Given a complex Banach space X, let $L(X)$ and $K(X)$ stand for the Banach algebra of all bounded linear operators on X and for its closed ideal of all compact operators, respectively. For each positive integer N, each real number $p \geq 1$, and each complex Banach space X, let $l^p(\mathbb{Z}, X)$ denote the Banach space of all functions $f : \mathbb{Z}^N \to X$ with

$$\|f\|_{l^p(\mathbb{Z}^N, X)} := \left(\sum_{x \in \mathbb{Z}^N} \|f(x)\|_X^p \right)^{1/p} < \infty.$$

Further, let $l^\infty(\mathbb{Z}, X)$ refer to the Banach space of all bounded functions $f : \mathbb{Z}^N \to X$ with norm

$$\|f\|_{l^\infty(\mathbb{Z}^N, X)} := \sup_{x \in \mathbb{Z}^N} \|f(x)\|_X,$$

and write $c_0(\mathbb{Z}^N, X)$ for the closed subspace of $l^\infty(\mathbb{Z}, X)$ which consists of all functions f with

$$\lim_{x \to \infty} \|f(x)\|_X = 0.$$

For $1 \leq p < \infty$, the Banach dual space of $l^p(\mathbb{Z}^N, X)$ can be identified in a standard way with $l^q(\mathbb{Z}^N, X^*)$ where $1/p + 1/q = 1$, and the dual of $c_0(\mathbb{Z}^N, X)$ is isomorphic to $l^1(\mathbb{Z}^N, X^*)$. Moreover, if X is a reflexive Banach space, then the spaces $l^p(\mathbb{Z}^N, X)$ are reflexive for $1 < p < \infty$. In case $X = H$ is a Hilbert space with inner product $\langle ., . \rangle_H$, then $l^2(\mathbb{Z}^N, H)$ becomes a Hilbert space on defining an inner product by

$$\langle f, g \rangle := \sum_{x \in \mathbb{Z}^N} \langle f(x), g(x) \rangle_H.$$

In what follows, we agree upon using the notation $E(X)$ to refer to one of the spaces $l^p(\mathbb{Z}^N, X)$ with $1 < p < \infty$ or $c_0(\mathbb{Z}^N, X)$, whereas we will write $E^\infty(X)$ if one of the spaces $E(X)$, $l^1(\mathbb{Z}^N, X)$ or $l^\infty(\mathbb{Z}^N, X)$ is taken into consideration.

For $n \in \mathbb{N}$, we denote the operator of multiplication by the characteristic function of the discrete cube $I_n := \{x \in \mathbb{Z}^N : |x|_\infty := \max_{1\le j\le N} |x_j| \le n\}$ by P_n. This operator acts boundedly on each of the spaces $E^\infty(X)$. We let $\mathcal{P}$ refer to the set of all operators P_n with $n \in \mathbb{N}$ and set $Q_n := I - P_n$. Following the terminology introduced in [6], an operator $K \in L(E^\infty(X))$ is called *$\mathcal{P}$-compact* if

$$\lim_{n\to\infty} \|KQ_n\|_{E^\infty(X)} = \lim_{n\to\infty} \|Q_n K\|_{E^\infty(X)} = 0.$$

We denote the set of all $\mathcal{P}$-compact operators by $K(E^\infty(X)), \mathcal{P})$ and write $L(E^\infty(X)), \mathcal{P})$ for the set of all operators $A \in L(E^\infty(X))$ for which both AK and KA are $\mathcal{P}$-compact whenever K is $\mathcal{P}$-compact. Then $L(E^\infty(X)), \mathcal{P})$ is a closed subalgebra of $L(E^\infty(X))$ which contains $K(E^\infty(X)), \mathcal{P})$ as a closed ideal.

Definition 2.1 *An operator $A \in L(E^\infty(X), \mathcal{P})$ is called a $\mathcal{P}$-*Fredholm operator *if the coset $A + K(E^\infty(X), \mathcal{P})$ is invertible in the quotient algebra*

$$L(E^\infty(X), \mathcal{P})/K(E^\infty(X), \mathcal{P}),$$

i.e., if there exist operators $B, C \in L(E^\infty(X), \mathcal{P})$ and $K, L \in K(E^\infty(X), \mathcal{P})$ such that $BA = I + K$ and $AC = I + L$.

This definition is equivalent to the following one.

Definition 2.2 *An operator $A \in L(E^\infty(X), \mathcal{P})$ is $\mathcal{P}$-Fredholm if and only if there exist an $m \in \mathbb{N}$ and operators $L_m, R_m \in L(E^\infty(X), \mathcal{P})$ such that*

$$L_m A Q_m = Q_m A R_m = Q_m.$$

$\mathcal{P}$-Fredholmness is often referred to as *local invertibility at infinity.* If X has finite dimension, then these definitions become equivalent to the usual definition of Fredholmness, which says that an operator is Fredholm if both its kernel and its cokernel have finite dimension.

For $k \in \mathbb{Z}^N$, let $\hat{V}_k$ stand for the operator of shift by k,

$$(\hat{V}_k u)(x) = f(x - k), \quad x \in \mathbb{Z}^N.$$

Clearly, $\hat{V}_k \in L(E^\infty(X))$ and $\|\hat{V}_k\|_{L(E^\infty(X))} = 1$.

Definition 2.3 *A* band operator *on $E^\infty(X)$ is a finite sum of the form $\sum_\alpha a_\alpha \hat{V}_\alpha$ where $\alpha \in \mathbb{Z}^N$ and $a_\alpha \in l^\infty(\mathbb{Z}^N, L(X))$. An operator is* band-dominated *if it is the uniform limit in $L(E^\infty(X))$ of a sequence of band operators.*

In case $X = \mathbb{C}$ and $N = 1$, and with respect to the standard basis of $E^\infty(X)$, band operators are given by matrices with finite band width, which justifies this notion. Observe also that the class of band operators is independent of the concrete space $E^\infty(X)$ whereas the class of band-dominated operators depends on $E^\infty(X)$ heavily. We denote this class by $\mathcal{A}(E^\infty(X))$. It is easy to see that $\mathcal{A}(E^\infty(X))$ is a closed subalgebra both of $L(E^\infty(X))$ and of $L(E^\infty(X), \mathcal{P})$.

Definition 2.4 *Let $A \in L(E^\infty(X))$, and let $h : \mathbb{N} \to \mathbb{Z}^N$ be a sequence which tends to infinity. An operator $A_h \in L(E^\infty(X))$ is called a* limit operator *of A with respect to the sequence h if*

$$\begin{aligned}\lim_{n\to\infty} &\|P_k(\hat{V}_{-h(n)}A\hat{V}_{h(n)} - A_h)\|_{L(E^\infty(X))} \\ &= \lim_{n\to\infty} \|(\hat{V}_{-h(n)}A\hat{V}_{h(n)} - A_h)P_k\|_{L(E^\infty(X))} = 0 \end{aligned} \tag{1}$$

for every $k \in \mathbb{N}$. The set of all limit operators of A will be denoted by $\sigma_{op}(A)$ and is called the operator spectrum *of A. Let further $\mathcal{H}$ stand for the set of all sequences $h : \mathbb{N} \to \mathbb{Z}^N$ which tend to infinity, and let $\mathcal{A}^\$(E^\infty(X))$ refer to the set of all operators $A \in \mathcal{A}(E^\infty(X))$ enjoying the following property: Every sequence $h \in \mathcal{H}$ possesses a subsequence g for which the limit operator A_g exists. We refer to the operators in $\mathcal{A}^\$(E^\infty(X))$ as* rich *band-dominated operators.*

Obviously, richness is a compactness condition with respect to the convergence defined by (1).

The following is our main result on $\mathcal{P}$-Fredholmness of rich band-dominated operators. For its proof see [6], Theorem 2.2.1.

Theorem 2.5 *An operator $A \in \mathcal{A}^\$(E^\infty(X))$ is $\mathcal{P}$-Fredholm if and only if each of its limit operators is invertible and if the norms of their inverses are uniformly bounded, i.e.,*

$$\sup\left\{\|(A_h)^{-1}\|_{L(E^\infty(X))} : A_h \in \sigma_{op}(A)\right\} < \infty.$$

2.2 The discrete Wiener algebra

The statement of Theorem 2.5 gets a more satisfactory form for band-dominated operators which belongs to the discrete Wiener algebra, in which case the uniform boundedness of the inverses of the limit operators follows from their invertibility.

Let $\mathcal{W}(\mathbb{Z}^N, X)$ denote the set of all band-dominated operators of the form

$$A = \sum_{\alpha\in\mathbb{Z}^N} a_\alpha \hat{V}_\alpha$$

where the coefficients $a_\alpha \in l^\infty(\mathbb{Z}^N, L(X))$ are subject to the condition

$$\|A\|_{\mathcal{W}(\mathbb{Z}^N, X)} := \sum_{\alpha\in\mathbb{Z}^N} \|a_\alpha\|_{l^\infty(\mathbb{Z}^N, L(X))} < \infty. \tag{2}$$

Provided with usual operations and with the norm (2), the set $\mathcal{W}(\mathbb{Z}^N, X)$ becomes a Banach algebra, the so-called *discrete Wiener algebra.* The estimate

$$\|A\|_{L(E^\infty(X))} \leq \|A\|_{\mathcal{W}(\mathbb{Z}^N, X)}$$

shows that $\mathcal{W}(\mathbb{Z}^N, X)$ is a non-closed subalgebra of $\mathcal{A}(E^\infty(X))$.

One of the remarkable properties of the discrete Wiener algebra is its inverse closedness.

Proposition 2.6 *The Wiener algebra $\mathcal{W}(\mathbb{Z}^N, X)$ is inverse closed in every algebra $L(E^\infty(X))$.*

Otherwise stated: If an operator $A \in \mathcal{W}(\mathbb{Z}^N, X)$ acts on $E^\infty(X)$ and is invertible there, then $A^{-1} \in \mathcal{W}(\mathbb{Z}^N, X)$ again. A proof is in [6], Theorem 2.5.2. An immediate consequence of the inverse closedness is the independence of the spectrum of an operator $A \in \mathcal{W}(\mathbb{Z}^N, X)$, thought of as acting on one of the spaces $E^\infty(X)$, of the concrete choice of that space.

Set $\mathcal{W}^\$(\mathbb{Z}^N, X) := \mathcal{W}(\mathbb{Z}^N, X) \cap \mathcal{A}^\$(E^\infty(X))$, and let $A \in \mathcal{W}^\$(\mathbb{Z}^N, X)$. We consider this operator on one of the spaces $E^\infty(X)$ and determine its limit operators with respect to this space. It turns out that the operator spectrum of A does not depend on the choice of that space and that all limit operators of A belong to the Wiener algebra $\mathcal{W}(\mathbb{Z}^N, X)$ again. The following is Theorem 2.5.7 in [6].

Theorem 2.7 *Let X be a reflexive Banach space. The following assertions are equivalent for an operator $A \in \mathcal{W}^\$(\mathbb{Z}^N, X)$:*

(a) *there is a space $E(X)$ on which A is $\mathcal{P}$-Fredholm;*

(b) *there is a space $E(X)$ such that all limit operators of A are invertible on that space;*

(c) *all limit operators of A are invertible on $l^\infty(\mathbb{Z}^N, X)$;*

(d) *all limit operators of A are invertible on $l^\infty(\mathbb{Z}^N, X)$ and the norms of their inverses are uniformly bounded;*

(e) *all limit operators of A are invertible on all spaces $E^\infty(X)$ and the $L(E^\infty(X))$-norms of their inverses are uniformly bounded;*

(f) *A is $\mathcal{P}$-Fredholm operator on each of the spaces $E(X)$.*

Let $A \in L(E^\infty(X), \mathcal{P}))$. We say that the complex number λ belongs to the $\mathcal{P}$-spectrum of A if the operator $A - \lambda I$ is not $\mathcal{P}$-Fredholm on $E^\infty(X)$. We denote the $\mathcal{P}$-spectrum of A by $\sigma_{\mathcal{P}}(A|E^\infty(X))$ or shortly by $\sigma_{\mathcal{P}}(A)$. The common spectrum of A will be denoted by $\sigma(A|E^\infty(X))$ or simply by $\sigma(A)$.

Theorem 2.8 *Let X be a reflexive Banach space and $A \in \mathcal{W}^\$(\mathbb{Z}^N, X)$. Then the $\mathcal{P}$-spectrum of A, considered as an operator on $E(X)$, is equal to*

$$\sigma_{\mathcal{P}}(A|E(X)) = \bigcup_{A_h \in \sigma_{op}(A)} \sigma(A_h|E(X)). \tag{3}$$

Moreover, neither the operator spectrum of A, nor the $\mathcal{P}$-spectrum of A, nor the spectra of the limit operators of A on the right-hand side of (3) *depend on the choice of $E(X)$.*

If the space X has a finite dimension, then the $\mathcal{P}$-spectrum of A is the common essential spectrum of that operator, that is, the spectrum of the coset $A + K(E^\infty(X))$ in the Calkin algebra $L(E^\infty(X))/K(E^\infty(X))$. In this setting, the rich Wiener algebra coincides with the full Wiener algebra. Hence, Theorem 2.8 has the following corollary.

Theorem 2.9 *Let X be a finite dimensional space. Then the essential spectrum of $A \in \mathcal{W}(\mathbb{Z}^N, X)$ does not depend on the choice of $E(X)$, and it is given by* (3).

3 Operators on modulation spaces

In the following two sections we define the modulation spaces and consider the continuous counterparts of the band-dominated operators and the Wiener algebra. The discrete and the continuous world are linked by a certain discretization operation which we are going to introduce first.

3.1 Time-frequency discretization

Recall that a function $a \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ belongs to the Hörmander class $S^0_{0,0}$ if, for all $r, t \in \mathbb{N}$,

$$|a|_{r,t} := \sum_{|\alpha| \le r, |\beta| \le t} \sup_{(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N} |\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| < \infty. \tag{4}$$

Let $a \in S^0_{0,0}$. The associated pseudodifferential operator $Op(a)$ (also written as $a(x, D)$) is defined at $u \in S(\mathbb{R}^N)$ by

$$(Op(a)u)(x) := (2\pi)^{-N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} a(x, \xi) e^{i\langle x-y, \xi \rangle} u(y) \, dy \, d\xi. \tag{5}$$

The function a is called the symbol of $Op(a)$, and the class of all pseudodifferential operators with symbols in $S^0_{0,0}$ is denoted by $OPS^0_{0,0}$. Standard references on pseudodifferential operators are [12, 7, 10], to mention only a few.

It is well-known that $OPS^0_{0,0}$ forms an algebra with respect to the usual sum and composition of operators. Further, the operators $Op(a) \in OPS^0_{0,0}$ are bounded both on the Schwartz space $S(\mathbb{R}^N)$ and on the Lebesgue space $L^2(\mathbb{R}^N)$, and

$$\|Op(a)\|_{L(L^2(\mathbb{R}^N))} \le C|a|_{2k, 2l} \quad \text{if } 2k > N, \, 2l > N. \tag{6}$$

The latter fact is known as the Calderon-Vaillancourt theorem.

Let $A : S(\mathbb{R}^N) \to S(\mathbb{R}^N)$ be a bounded linear operator. An operator A^t is called the *formal adjoint* of A if

$$\langle Au, v\rangle = \langle u, A^t v\rangle \quad \text{for all } u, v \in S(\mathbb{R}^N). \tag{7}$$

If $A \in OPS^0_{0,0}$, then its formal adjoint A^t is a pseudodifferential operator in $OPS^0_{0,0}$ again. Furthermore, if $A \in OPS^0_{0,0}$ acts on $L^2(\mathbb{R}^N)$, then its Hilbert space adjoint A^* also belongs to $OPS^0_{0,0}$. Hence, (7) can be used to define the action of $A \in OPS^0_{0,0}$ on the space of tempered distributions $S'(\mathbb{R}^N)$.

Our next goal is to introduce the time-frequency discretization (which is called bi-discretization in [6]). For $\gamma = (\alpha, \beta) \in \mathbb{Z}^N \times \mathbb{Z}^N$, set $U_\gamma := V_\beta E_\alpha$, where

$$(V_\beta u)(x) := u(x - \beta) \quad \text{and} \quad (E_\alpha u)(x) := e^{i\langle \alpha, x\rangle} u(x).$$

The operators U_γ are unitary on $L^2(\mathbb{R}^N)$, and $U^*_\gamma = E_{-\alpha} V_{-\beta} = U^{-1}_\gamma$. Note that these operators, together with the scalar unitary operator $e^{ir} I$ with $r \in \mathbb{Z}$ form a noncommutative group, the so-called discrete Heisenberg group. In particular, one has

$$U^*_\alpha = e^{i\langle \alpha_1, \alpha_2\rangle} U_{-\alpha}, \qquad U_\alpha U_\beta = e^{i\langle \alpha_2, \beta_1\rangle} U_{\alpha+\beta} \tag{8}$$

and

$$U^*_\alpha U_\beta = e^{i\langle \alpha_2, \alpha_1 - \beta_1\rangle} U_{\beta-\alpha} = e^{i\langle \beta_2, \alpha_1 - \beta_1\rangle} U^*_{\alpha-\beta}$$

where $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^N \times \mathbb{Z}^N$.

Let $f \in C^\infty_0(\mathbb{R}^N)$ be a non-negative function such that $f(x) = f(-x)$ for all x and such that $f(x) = 1$ if $|x_i| \le 2/3$ for all $i = 1, \ldots, N$ and $f(x) = 0$ if $|x_i| \ge 3/4$ for at least one i. Define a nonnegative function φ on $\mathbb{R}^N$ by

$$\varphi^2(x) := \frac{f(x)}{\sum_{\beta\in\mathbb{Z}^N} f(x-\beta)}$$

and set $\varphi_\alpha(x) := \varphi(x - \alpha)$ for $\alpha \in \mathbb{Z}^N$. The family $(\varphi_\alpha)_{\alpha\in\mathbb{Z}^N}$ forms a partition of unit on $\mathbb{R}^N$ in sense that

$$\sum_{\alpha\in\mathbb{Z}^N} \varphi^2_\alpha(x) = 1 \qquad \text{for each } x \in \mathbb{R}^N.$$

For $\gamma = (\alpha, \beta) \in \mathbb{Z}^N \times \mathbb{Z}^N$, define ϕ_γ on $\mathbb{R}^N \times \mathbb{R}^N$ by

$$\phi_\gamma(x, \xi) := \varphi_\alpha(x)\varphi_\beta(\xi),$$

and write Φ_γ for the pseudodifferential operator $Op(\phi_\gamma)$. It is evident that

$$\Phi_\gamma u = \varphi_\alpha \varphi_\beta(D) u = \varphi_\alpha Op(\varphi_\beta) u$$

at $u \in S'(\mathbb{R}^N)$, and the formal adjoint of the operator Φ_γ acts as

$$\Phi^*_\gamma u = \varphi_\beta(D)\varphi_\alpha u = Op(\varphi_\beta)\varphi_\alpha u$$

at $u \in S'(\mathbb{R}^N)$.

The operators Φ_γ induce a partition of unity on the phase space $\mathbb{R}^N \times \mathbb{R}^N$ in the sense that

$$\sum_{\gamma \in \mathbb{Z}^{2N}} \Phi_\gamma^* \Phi_\gamma u = \sum_{\gamma \in \mathbb{Z}^{2N}} \Phi_\gamma \Phi_\gamma^* u = u \qquad \text{for each } u \in S'(\mathbb{R}^N) \tag{9}$$

where the series converge in $S'(\mathbb{R}^N)$. With these notations, we define the operator G of *time-frequency discretization* by

$$(Gu)_\gamma := \Phi_0 U_\gamma^* u \qquad \text{where } \gamma \in \mathbb{Z}^{2N} \text{ and } u \in S'(\mathbb{R}^N),$$

that is, we consider Gu as a vector-valued function on $\mathbb{Z}^{2N}$ with values in $S'(\mathbb{R}^N)$.

Now we are in a position to define the announced modulation spaces $M^{2,p}(\mathbb{R}^N)$ which will provide the frame for a localization of functions in the time-frequency domain. The modulation spaces under consideration were introduced in [4] where they are used to study the Fredholm property of pseudodifferential operators in $OPS^0_{0,0}$. Similar (but different) modulation spaces are considered in [3] (see also Chapter 11 of [2]).

Definition 3.1 *For $p \in [1, \infty)$, let $M^{2,p}(\mathbb{R}^N)$ denote the space of all distributions $u \in S'(\mathbb{R}^N)$ such that $(Gu)_\gamma \in L^2(\mathbb{R}^N)$ for every $\gamma \in \mathbb{Z}^{2N}$ and*

$$\|u\|_{M^{2,p}(\mathbb{R}^N)} := \left(\sum_{\gamma \in \mathbb{Z}^{2N}} \|(Gu)_\gamma\|^p_{L^2(\mathbb{R}^N)} \right)^{1/p} < \infty, \tag{10}$$

and let $L^{2,\infty}(\mathbb{R}^N)$ stand for the space of all distributions $u \in S'(\mathbb{R}^N)$ with $(Gu)_\gamma \in L^2(\mathbb{R}^N)$ for every $\gamma \in \mathbb{Z}^{2N}$ and

$$\|u\|_{M^{2,\infty}(\mathbb{R}^N)} := \sup_{\gamma \in \mathbb{Z}^{2N}} \|(Gu)_\gamma\|^p_{L^2(\mathbb{R}^N)} < \infty. \tag{11}$$

Since U_γ is a unitary operator on $L^2(\mathbb{R}^N)$, one can replace $(Gu)_\gamma = \Phi_0 U_\gamma^* u$ by $\Phi_\gamma u = U_\gamma \Phi_0 U_\gamma^* u$ in the definitions (10) and (11) of the norms.

The following proposition is taken from [4]. It summarizes basic properties of modulation spaces.

Proposition 3.2 *(a) $M^{2,p}(\mathbb{R}^N)$ is a Banach space for each $p \in [1,\infty]$, and $M^{2,2}(\mathbb{R}^N) = L^2(\mathbb{R}^N)$.*
(b) For $p \in [1,\infty)$, every linear continuous functional on $M^{2,p}(\mathbb{R}^N)$ is of the form

$$v \mapsto \int_{\mathbb{R}^N} u(x)\overline{v(x)}\, dx, \tag{12}$$

with some distribution $u \in M^{2,q}(\mathbb{R}^N)$ where $1/p + 1/q = 1$. Hence, the Banach dual $M^{2,p}(\mathbb{R}^N)^*$ can be identified with $M^{2,q}(\mathbb{R}^N)$, and $M^{2,p}(\mathbb{R}^N)$ is reflexive for

$p \in (1, \infty)$.
(c) The Schwartz space $S(\mathbb{R}^N)$ *is contained in* $M^{2,p}(\mathbb{R}^N)$ *for each* $p \in [1, \infty]$, *and it lies dense in* $M^{2,p}(\mathbb{R}^N)$ *for each* $p \in [1, \infty)$.
(d) $M^{2,p}(\mathbb{R}^N)$ *is contained in* $S'(\mathbb{R}^N)$ *in the sense that* $u \in M^{2,p}(\mathbb{R}^N)$ *defines a linear functional on* $S(\mathbb{R}^N)$ *acting at* φ *via*

$$u(\varphi) := \int_{\mathbb{R}^N} u(x)\varphi(x)\, dx.$$

Moreover, if $u_n \to 0$ *in* $M^{2,q}(\mathbb{R}^N)$, *then* $u_n(\varphi) \to 0$ *for each function* $\varphi \in S(\mathbb{R}^N)$.

Notice that the operators $U_\gamma = V_\beta E_\alpha$ are bijective isometries on each of the spaces $M^{2,p}(\mathbb{R}^N)$ with $p \in [1, \infty]$ and that $U_\gamma^{-1} = E_{-\alpha} V_{-\beta}$.

Proposition 3.3 *The operator* $G : M^{2,p}(\mathbb{R}^N) \to l^p(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$ *is an isometry, and the operator* G_l^{-1} *defined at* $f \in l^p(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$ *by*

$$G_l^{-1} f := \sum_{\gamma \in \mathbb{Z}^{2N}} U_\gamma \Phi_0^* f(\gamma) \tag{13}$$

is a left inverse for G.

Proof. The isometry of G is evident, and the equality $G_l^{-1} G = I$ follows from

$$G_l^{-1} G u = \sum_{\gamma \in \mathbb{Z}^{2N}} U_\gamma \Phi_0^* \Phi_0 U_\gamma^* u = \sum_{\gamma \in \mathbb{Z}^{2N}} \Phi_\gamma^* \Phi_\gamma u = u,$$

which holds for every $u \in M^{2,p}(\mathbb{R}^N)$ due to (9) and Proposition 3.2 (d). ∎

Thus, the operator $Q := G G_l^{-1} : l^p(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N)) \to l^p(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$ is a projection for all $p \in [1, \infty]$. We denote its range by $\mathcal{R}_p(Q)$. Then

$$G : M^{2,p}(\mathbb{R}^N) \to \mathcal{R}_p(Q)$$

becomes an isometric bijection, and each operator $A \in L(M^{2,p}(\mathbb{R}^N))$ becomes similar to the operator

$$A_G := G A G_l^{-1}|_{\mathcal{R}_p(Q)} : \quad \mathcal{R}_p(Q) \to \mathcal{R}_p(Q).$$

We extend A_G to an operator $\Gamma(A)$ acting on all of $l^p(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$ by setting

$$\Gamma(A) := A_G Q + I - Q = G A G_l^{-1} + I - Q$$

and call $\Gamma(A)$ the *time-frequency discretization* of A. Clearly,

$$G_l^{-1} \Gamma(A) G = G_l^{-1} (G A G_l^{-1} + I - G G_l^{-1}) G = A.$$

Proposition 3.4 $Q \in \mathcal{W}(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$.

Proof. The definitions of G and G_l^{-1} imply that Q acts at $f \in l^p(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$ by

$$(Qf)(\delta) = \sum_{\gamma \in \mathbb{Z}^{2N}} \Phi_0 U_\delta^* U_{\delta-\gamma} \Phi_0^* f(\delta - \gamma) = \sum_{\gamma \in \mathbb{Z}^{2N}} R_\gamma(\delta)(\hat{V}_\gamma f)(\delta)$$

where $R_\gamma(\delta) := \Phi_0 U_\delta^* U_{\delta-\gamma} \Phi_0^*$ and where $\hat{V}_\gamma$ denotes again the discrete shift operator $(\hat{V}_\gamma f)(\delta) := f(\delta - \gamma)$ on $l^p(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$. Choose $2k > N$. In [6], Proposition 4.3.2, it is shown that then

$$\begin{aligned} \|R_\gamma(\delta)\|_{L(L^2(\mathbb{R}^N))} &= \|\Phi_0 U_\delta^* U_{\delta-\gamma} \Phi_0^*\|_{L(L^2(\mathbb{R}^N))} \\ &= \|U_\delta \Phi_0 U_\delta^* U_{\delta-\gamma} \Phi_0^* U_{\delta-\gamma}^*\|_{L(L^2(\mathbb{R}^N))} \\ &= \|\Phi_\delta \Phi_{\delta-\gamma}^*\|_{L(L^2(\mathbb{R}^N))} \\ &\leq C(1+|\alpha|)^{-2k}(1+|\beta|)^{-2k} \end{aligned} \tag{14}$$

with a constant C independent of $\gamma = (\alpha, \beta)$. Consequently,

$$\sum_{\gamma \in \mathbb{Z}^{2N}} \|R_\gamma(\delta)\|_{L(L^2(\mathbb{R}^N))} \leq C \sum_{(\alpha,\beta) \in \mathbb{Z}^N \times \mathbb{Z}^N} (1+|\alpha|)^{-2k}(1+|\beta|)^{-2k} < \infty$$

showing that $\|Q\|_{\mathcal{W}(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))} < \infty$. ■

3.2 Fredholmness and time-frequency discretization

Our next goal is to point out the relation between the Fredholmness of an operators acting on a modulation space $M^{2,p}(\mathbb{R}^N)$ and the $\mathcal{P}$-Fredholmness of its time-frequency discretization. Beginning with this subsection, we assume $p \in (1, \infty)$ unless otherwise stated.

Proposition 3.5 *(a) For every $n \in \mathbb{N}$, the operators $P_n Q$ and QP_n are compact on $l^p(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$.*

(b) The projection Q belongs to $L(l^p(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N)), \mathcal{P})$.

(c) For every operator $A \in L(M^{2,p}(\mathbb{R}^N))$, its discretization $\Gamma(A)$ belongs to $L(l^p(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N)), \mathcal{P})$.

(d) If $K \in L(l^p(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N)))$ is a $\mathcal{P}$-compact operator of the form $K = QKQ$, then $G_l^{-1}KG$ is compact on $M^{2,p}(\mathbb{R}^N)$.

(e) The operator $A \in L(M^{2,p}(\mathbb{R}^N))$ is invertible if and only if the operator $\Gamma(A) \in L(l^p(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N)))$ is invertible.

(f) The operator $A \in L(M^{2,p}(\mathbb{R}^N))$ is Fredholm if and only if the operator $\Gamma(A) \in L(l^p(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N)))$ is $\mathcal{P}$-Fredholm.

This proposition is proved in [5] for $p = 2$, see also Proposition 4.2.2 in [6]. The proof for general $p \in (1, \infty)$ runs similarly.

Definition 3.6 *Let $A \in L(M^{2,p}(\mathbb{R}^N))$, and let $h : \mathbb{N} \to \mathbb{Z}^{2N}$ be a sequence tending to infinity. We say that the operator $A_h \in L(M^{2,p}(\mathbb{R}^N))$ is a* limit operator *of A with respect to the sequence h if*

$$U_{h(m)}^{-1} A U_{h(m)} \to A_h \qquad \textit{and} \qquad U_{h(m)}^{-1} A^* U_{h(m)} \to A_h^*$$

strongly as $m \to \infty$. The set $\sigma_{op}(A)$ of all limit operators of A is called the operator spectrum *of A.*

The following proposition describes the relation between the time-frequency discretization of the limit operators of A and the limit operators of the time-frequency discretization of A. Its proof for $p = 2$ is in [5] and Proposition 4.2.5 in [6]. The case of general $p \in (1, \infty)$ can be treated analogously.

Proposition 3.7 *Let $A \in L(M^{2,p}(\mathbb{R}^N))$, and let $h : \mathbb{N} \to \mathbb{Z}^{2N}$ be a sequence tending to infinity such that the limit operator A_h of A with respect to h exists. Then there is a subsequence g of h such that the limit operator $\Gamma(A)_g$ of $\Gamma(A)$ with respect to g exists, and there is an isometric isomorphism T_g mapping $l^p(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$ onto itself such that*

$$\Gamma(A)_g = T_g^{-1} \Gamma(A_h) T_g.$$

We still need the counterparts of the notions of band and band-dominated operators for operators on modulation spaces.

Definition 3.8 *An operator $A \in L(S'(\mathbb{R}^N))$ is called a* band operator *if there exists an $R > 0$ such that $\Phi_\alpha A \Phi_\beta^* = 0$ for all subscripts $\alpha, \beta \in \mathbb{Z}^{2N}$ with*

$$|\alpha - \beta| := \max_{1 \le i \le 2N} |\alpha_i - \beta_i| > R.$$

An operator $A \in L(M^{2,p}(\mathbb{R}^N))$ is called band-dominated *if it is the limit of a sequence of band operators converging to A in the norm of $L(M^{2,p}(\mathbb{R}^N))$.*

It is easy to check that the class of all band-dominated operators on $M^{2,p}(\mathbb{R}^N)$ is a closed subalgebra of $L(M^{2,p}(\mathbb{R}^N))$. We denote this algebra by $\mathcal{A}(M^{2,p}(\mathbb{R}^N))$. Further we call $A \in \mathcal{A}(M^{2,p}(\mathbb{R}^N))$ a *rich operator* if every sequence $h : \mathbb{N} \to \mathbb{Z}^{2N}$ which tends to infinity possesses a subsequence g for which the limit operator A_g exists. The set of all rich operators forms a closed subalgebra of $\mathcal{A}(M^{2,p}(\mathbb{R}^N))$ which we denote by $\mathcal{A}^\$(M^{2,p}(\mathbb{R}^N))$.

Proposition 3.9 *(a) If $A \in \mathcal{A}(M^{2,p}(\mathbb{R}^N))$, then $\Gamma(A) \in \mathcal{A}(l^p(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$.*
(b) If $A \in \mathcal{A}^\$(M^{2,p}(\mathbb{R}^N))$, then $\Gamma(A) \in \mathcal{A}^\$(l^p(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$.

Proof. We prove assertion (a) only. The second statement follows from (a) and Proposition 3.7. First let A be a band operator on $M^{2,p}(\mathbb{R}^N)$. Then, for $u \in l^p(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$,

$$\begin{aligned}(A_G u)(\delta) &= \sum_{\theta \in \mathbb{Z}^{2N}} \Phi_0 U_\delta^* A U_\theta \Phi_0^* u(\theta) = \sum_{\gamma \in \mathbb{Z}^{2N}} \Phi_0 U_\delta^* A U_{\delta-\gamma} \Phi_0^* u(\delta - \gamma) \\ &= \sum_{\gamma \in \mathbb{Z}^{2N}} A_\gamma(\delta)(\hat{V}_\gamma u)(\delta) \qquad (15)\end{aligned}$$

where $A_\gamma(\delta) := \Phi_0 U_\delta^* A U_{\delta-\gamma} \Phi_0^*$. Since A is a band operator, all series in (15) have a finite number of non-vanishing items only. Indeed,

$$\|A_\gamma(\delta)\|_{L(L^2(\mathbb{R}^N))} = \|\Phi_\delta A \Phi_{\delta-\gamma}^*\|_{L(L^2(\mathbb{R}^N))} = 0$$

if $|\gamma| > R$ with $R > 0$ being large enough. Hence, A_G is a band operator. That the operator A_G is band-dominated whenever A is so follows by an evident approximation argument (take into account that $G : M^{2,p}(\mathbb{R}^N) \to \mathcal{R}_p(Q)$ and $G_l^{-1} : \mathcal{R}_p(Q) \to M^{2,p}(\mathbb{R}^N)$ are isometries). Finally, since the projection Q belongs to the discrete Wiener algebra due to Proposition 3.4 (and is, thus, band-dominated), the operator $\Gamma(A) = A_G Q + (I - Q)$ is band-dominated for each band-dominated operator A. ∎

Combining Propositions 3.5, 3.7 and Theorem 2.5 we arrive at the following Fredholm criterion for rich band-dominated operators on modulation spaces.

Theorem 3.10 *An operator $A \in \mathcal{A}^{\$}(M^{2,p}(\mathbb{R}^N))$ is Fredholm if and only if all limit operators A_h of A are invertible and if the norms of their inverses are uniformly bounded, i.e.,*

$$\sup_{A_h \in \sigma_{op}(A)} \|A_h^{-1}\|_{L(M^{2,p}(\mathbb{R}^N))} < \infty.$$

4 The Wiener algebra on $\mathbb{R}^N$

We define the continuous analogue of the discrete Wiener algebra by imposing conditions on the decay of the norms $\|\Phi_\alpha A \Phi_{\alpha-\gamma}^*\|_{L(L^2(\mathbb{R}^N))}$ as γ tends to infinity.

Definition 4.1 *A linear operator $A : S'(\mathbb{R}^N) \to S'(\mathbb{R}^N)$ belongs to the* Wiener algebra $\mathcal{W}(\mathbb{R}^N)$ *if*

$$\|A\|_{\mathcal{W}(\mathbb{R}^N)} := \sum_{\gamma \in \mathbb{Z}^{2N}} \sup_{\alpha \in \mathbb{Z}^{2N}} \|\Phi_\alpha A \Phi_{\alpha-\gamma}^*\|_{L(L^2(\mathbb{R}^N))} < \infty. \qquad (16)$$

The Wiener algebra $\mathcal{W}(\mathbb{R}^N)$ contains sufficiently many interesting operators. So we will see in the next section that $\mathcal{W}(\mathbb{R}^N)$ contains the pseudodifferential operators with non-smooth symbols in the Sjöstrand class OPS_w and, thus, also the Hörmander class $OPS^0_{0,0}$. Here are some basic properties of $\mathcal{W}(\mathbb{R}^N)$.

Proposition 4.2 *(a)* $\mathcal{W}(\mathbb{R}^N) \subset L(M^{2,p}(\mathbb{R}^N))$, *and*

$$\|A\|_{L(M^{2,p}(\mathbb{R}^N))} \leq \|A\|_{\mathcal{W}(\mathbb{R}^N)}$$

for each $p \in [1, \infty]$ *and* $A \in \mathcal{W}(\mathbb{R}^N)$.
(b) Provided with the norm (16)*, the set* $\mathcal{W}(\mathbb{R}^N)$ *becomes a unital Banach algebra.*
(c) The Banach dual operator A^* *of an operator* $A \in \mathcal{W}(\mathbb{R}^N)$ *considered as acting on* $M^{2,p}(\mathbb{R}^N)$ *belongs* $\mathcal{W}(\mathbb{R}^N)$*, too.*

Proof. (a) First let $p \in [1, \infty)$. Then

$$\begin{aligned}
\|Au\|^p_{M^{2,p}(\mathbb{R}^N)} &= \sum_{\gamma\in\mathbb{Z}^{2N}} \|\Phi_\gamma A u\|^p_{L^2(\mathbb{R}^N)} \\
&= \sum_{\gamma\in\mathbb{Z}^{2N}} \left\| \Phi_\gamma A \sum_{\delta\in\mathbb{Z}^{2N}} \Phi^*_\delta \Phi_\delta u \right\|^p_{L^2(\mathbb{R}^N)} \\
&\leq \left(\sum_{\gamma\in\mathbb{Z}^{2N}} \|\Phi_\gamma A \Phi^*_{\gamma-\alpha}\|_{L(L^2(\mathbb{R}^N))} \left\| \sum_{\delta\in\mathbb{Z}^{2N}} \Phi_{\gamma-\alpha} u \right\|_{L^2(\mathbb{R}^N)} \right)^p \\
&\leq \sum_{\gamma\in\mathbb{Z}^{2N}} \left(\sum_{\alpha\in\mathbb{Z}^{2N}} k_A(\gamma-\alpha) \|\Phi_\alpha u\|_{L^2(\mathbb{R}^N)} \right)^p
\end{aligned}$$

where $k_A(\alpha) := \sup_{\gamma\in\mathbb{Z}^N} \|\Phi_\gamma A \Phi^*_{\gamma-\alpha}\|_{L(L^2(\mathbb{R}^N))}$. Since k_A is a sequence in $l^1(\mathbb{Z}^{2N})$,

$$\|Au\|_{M^{2,p}(\mathbb{R}^N)} \leq \sum_{\gamma\in\mathbb{Z}^{2N}} k_A(\gamma) \left(\sum_{\alpha\in\mathbb{Z}^{2N}} \|\Phi_\alpha u\|^p_{L^2(\mathbb{R}^N)} \right)^{1/p} = \|A\|_{\mathcal{W}(\mathbb{R}^N)} \|u\|_{M^{2,p}(\mathbb{R}^N)}.$$

In the same way, one gets the estimate

$$\|Au\|_{M^{2,\infty}(\mathbb{R}^N)} \leq \sum_{\gamma\in\mathbb{Z}^{2N}} k_A(\gamma) \sup_{\alpha\in\mathbb{Z}^{2N}} \|\Phi_\alpha u\|_{L^2(\mathbb{R}^N)} = \|A\|_{\mathcal{W}(\mathbb{R}^N)} \|u\|_{M^{2,\infty}(\mathbb{R}^N)}.$$

(b) It is easy to verify that

$$\|AB\|_{\mathcal{W}(\mathbb{R}^N)} \leq \|A\|_{\mathcal{W}(\mathbb{R}^N)} \|B\|_{\mathcal{W}(\mathbb{R}^N)},$$

and estimate (14) shows that the identity operator I belongs to $\mathcal{W}(\mathbb{R}^N)$. Hence, $\mathcal{W}(\mathbb{R}^N)$ is a unital algebra, and its completeness with respect to the norm (16) follows straightforwardly.

(c) Let A^* be the Banach adjoint operator of A acting on $M^{2,p}(\mathbb{R}^N)$, that is

$$\int_{\mathbb{R}^N} Au\overline{v}\, dx = \int_{\mathbb{R}^N} u\overline{A^*v}\, dx, \tag{17}$$

where $u \in M^{2,p}(\mathbb{R}^N)$ and $v \in M^{2,q}(\mathbb{R}^N)$ with $1/p + 1/q = 1$. The operator A is bounded on $L^2(\mathbb{R}^N)$ since $A \in \mathcal{W}(\mathbb{R}^N)$ (Proposition 4.3.4 in [6]). Since (17) holds for arbitrary $u, v \in S(\mathbb{R}^N)$, this identity states that A^* is the adjoint operator to A considered as acting on $L^2(\mathbb{R}^N)$. Hence,

$$\|\Phi_\alpha A^* \Phi^*_{\alpha-\gamma}\|_{L(L^2(\mathbb{R}^N))} = \|\Phi_{\alpha-\gamma} A^* \Phi^*_\alpha\|_{L(L^2(\mathbb{R}^N))},$$

which implies that

$$\begin{aligned} \sum_{\gamma \in \mathbb{Z}^{2N}} \sup_{\alpha \in \mathbb{Z}^{2N}} \|\Phi_\alpha A^* \Phi^*_{\alpha-\gamma}\|_{L(L^2(\mathbb{R}^N))} &= \sum_{\gamma \in \mathbb{Z}^{2N}} \sup_{\alpha \in \mathbb{Z}^{2N}} \|\Phi_{\alpha-\gamma} A^* \Phi^*_\alpha\|_{L(L^2(\mathbb{R}^N))} \\ &= \|\Phi_\alpha A^* \Phi^*_{\alpha-\gamma}\|_{L(L^2(\mathbb{R}^N))} < \infty, \end{aligned}$$

whence finally $A^* \in \mathcal{W}(\mathbb{R}^N)$. ■

Proposition 4.3 *(a) If $A \in \mathcal{W}(\mathbb{R}^N)$, then the operators GAG_l^{-1} and $\Gamma(A)$ belong to the discrete Wiener algebra $\mathcal{W}(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$.*
(b) Conversely, if $B \in \mathcal{W}(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$, then $G_l^{-1}AG$ lies in $\mathcal{W}(\mathbb{R}^N)$.

The proof runs as that of Proposition 3.9; compare also [5] and Proposition 4.3.5 in [6].

Proposition 4.4 *The algebra $\mathcal{W}(\mathbb{R}^N)$ is inverse closed on each of the spaces $M^{2,p}(\mathbb{R}^N)$ with $p \in [1, \infty]$, i.e., if $A \in \mathcal{W}(\mathbb{R}^N)$ is invertible in $L(M^{2,p}(\mathbb{R}^N))$, then $A^{-1} \in \mathcal{W}(\mathbb{R}^N)$.*

Proof. Let $A \in \mathcal{W}(\mathbb{R}^N)$ be invertible on $M^{2,p}(\mathbb{R}^N)$. Then $\Gamma(A)$ belongs to $\mathcal{W}(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$ by Proposition 4.3, and $\Gamma(A)$ is invertible on $l^p(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$ by Proposition 3.5 (e). From Proposition 2.6 we infer that $\Gamma(A)^{-1}$ lies in the discrete Wiener algebra $\mathcal{W}(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$, and since

$$G_l^{-1}\Gamma(A)^{-1}GA = G_l^{-1}\Gamma(A)^{-1}GAG_l^{-1}G = G_l^{-1}\Gamma(A)^{-1}\Gamma(A)QG = I,$$

one has $G_l^{-1}\Gamma(A)^{-1}G = A^{-1} \in \mathcal{W}(\mathbb{R}^N)$ due to Proposition 4.3 (b). ■

We fix a $p \in [1, \infty]$ and define the *rich Wiener algebra* by

$$\mathcal{W}^\$(\mathbb{R}^N) := \mathcal{W}(\mathbb{R}^N) \cap \mathcal{A}^\$(M^{2,p}(\mathbb{R}^N)).$$

Thus, an operator A belongs to $\mathcal{W}^\$(\mathbb{R}^N)$ if every sequence $h : \mathbb{N} \to \mathbb{Z}^{2N}$ possesses a subsequence g for which the limit operator A_g of A with respect to strong convergence on $M^{2,p}(\mathbb{R}^N)$ exists. It is easy to see that the limit operators A_g belong to $\mathcal{W}(\mathbb{R}^N)$ again. Thus, the definition of $\mathcal{W}^\$(\mathbb{R}^N)$ does not depend on the concrete choice of the parameter $p \in [1, \infty]$.

Theorem 4.5 *The following conditions are equivalent for $A \in \mathcal{W}^\$(\mathbb{R}^N)$:*
(a) A is a Fredholm operator on $M^{2,p}(\mathbb{R}^N)$ for a certain $p \in (1, \infty)$;
(b) A is a Fredholm operator on $M^{2,p}(\mathbb{R}^N)$ for each $p \in (1, \infty)$;
(c) there is a $p \in [1, \infty]$ such that all limit operators of A are invertible on $M^{2,p}(\mathbb{R}^N)$;
(d) all limit operators of A are invertible on every space $M^{2,p}(\mathbb{R}^N)$ with $p \in [1, \infty]$;
(e) all limit operators of A are uniformly invertible on each of the spaces $M^{2,p}(\mathbb{R}^N)$ with $p \in [1, \infty]$.

This is an immediate consequence of Theorem 2.7 and Proposition 3.5 (e) and (f). The preceding theorem has the following corollary for the essential spectrum of an operator A in the rich Wiener algebra when considered on $M^{2,p}(\mathbb{R}^N)$, i.e., for the spectrum of the coset $A + K(M^{2,p}(\mathbb{R}^N))$ in the corresponding Calkin algebra.

Theorem 4.6 *Let $A \in \mathcal{W}^\$(\mathbb{R}^N)$. Then the essential spectrum $\sigma_{ess}A$ of A considered on $M^{2,p}(\mathbb{R}^N)$ is equal to*

$$\sigma_{ess}(A|M^{2,p}(\mathbb{R}^N)) = \bigcup_{A_h \in \sigma_{op}(A)} \sigma(A_h|M^{2,p}(\mathbb{R}^N)).$$

Both the operator spectrum of A, the essential spectrum of A, and the common spectra of the limit operators of A are independent of $p \in (1, \infty)$.

5 Fredholm properties of pseudodifferential operators in the Sjöstrand class

We start with recalling the definition of the class of symbols of pseudodifferential operators introduced by Sjöstrand [8] in 1994; see also [9]. We introduce this class for $\mathbb{R}^n$ with arbitrary $n \in \mathbb{N}$. Later, we let $n = 2N$.

Let $\chi \in S(\mathbb{R}^n)$ be a function with $\int_{\mathbb{R}^n} \chi(x)\, dx = 1$. A function $a : \mathbb{R}^n \to \mathbb{C}$ belongs to the Sjöstrand class $S_w(\mathbb{R}^n)$ if

$$\|a\|_{S_w(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \sup_{k \in \mathbb{Z}^n} \left| \int_{\mathbb{R}^n} e^{i\langle x,\, \xi\rangle} a(x)\chi(x-k)\, dx \right| d\xi < \infty. \tag{18}$$

Provided with the norm (18), $S_w(\mathbb{R}^n)$ becomes a Banach space. Notice that a change of the function χ gives rise to an equivalent norm on $S_w(\mathbb{R}^n)$ and leads, thus, to the same class of symbols.

We have to mention another descriptions of the Sjöstrand class $S_w(\mathbb{R}^n)$. In 1997, Boulkhemair [1] introduced the class $\mathcal{B}(\mathbb{R}^n)$ of all functions $a : \mathbb{R}^n \to \mathbb{C}$

which own the property

$$\|a\|_{\mathcal{B}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \sup_{x\in\mathbb{Z}^n} \left| \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \hat{a}(\xi)\chi(\xi-\eta)d\xi \right| d\eta < \infty \tag{19}$$

where $\hat{a}$ refers to the Fourier transform of a in the sense of distributions. The norm (19) can be also written as

$$\|a\|_{\mathcal{B}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \|\chi(D-\eta)u\|_{L^\infty(\mathbb{R}^n)} d\eta$$

and is further equivalent to the norm

$$\|a\|_{\mathcal{B}(\mathbb{R}^n)} := \sum_{l\in\mathbb{Z}^n} \|\chi(D-l)u\|_{L^\infty(\mathbb{R}^n)}. \tag{20}$$

Moreover, Boulkhemair proved that the Sjöstrand class $S_w(\mathbb{R}^n)$ and his class $\mathcal{B}(\mathbb{R}^n)$ coincide. As a consequence of this fact, he derived the following very convenient constructive characterization of $S_w(\mathbb{R}^n)$.

Proposition 5.1 ([1]) *A distribution $a \in S'(\mathbb{R}^n)$ belongs to $S_w(\mathbb{R}^n)$ if and only if there exist a compact subset Q of $\mathbb{R}^n$ and a sequence of functions $(a_k)_{k\in\mathbb{Z}^n}$ in $L^\infty(\mathbb{R}^n)$ with supp $(\hat{a}_k) \subseteq Q$ and*

$$\sum_{k\in\mathbb{Z}^n} \|a_k\|_{L^\infty(\mathbb{R}^n)} < \infty,$$

such that

$$a(x) = \sum_{k\in\mathbb{Z}^n} e^{i\langle x,\,k\rangle} a_k(x)$$

almost everywhere.

Let now $n = 2N$ and $a \in S_w(\mathbb{R}^{2N})$. As usual, we write the independent variable in $\mathbb{R}^{2N}$ as $(x,\,\xi) \in \mathbb{R}^N \times \mathbb{R}^N$. Then the pseudodifferential operator with symbol a is defined by

$$(Op(a)u)(x) := (2\pi)^{-N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^{i\langle x-y,\,\xi\rangle} a(x,\,\xi)u(y)\,dy\,d\xi$$

where $u \in S(\mathbb{R}^N)$. Let $OPS_w = OPS_w(\mathbb{R}^{2N})$ stand for the class of all pseudodifferential operators with symbols in $S_w(\mathbb{R}^{2N})$. It has been shown in [8] that the operators in OPS_w are bounded on $L^2(\mathbb{R}^N)$ and that OPS_w is an inverse closed subalgebra of $L(L^2(\mathbb{R}^N))$, i.e., if $A \in OPS_w$ is invertible on $L^2(\mathbb{R}^N)$, then $A^{-1} \in OPS_w$ again.

The Sjöstrand class OPS_w contains several interesting classes of pseudodifferential operators. For instance, the Hörmander class $OPS^0_{0,0}$ is contained in $OPS_w(\mathbb{R}^n)$ which can be checked as follows. Let a be in $C_b^\infty(\mathbb{R}^n)$, i.e., let

$$|a|_m := \sum_{|\alpha|\le m} \sup_{r\in\mathbb{R}^n} |\partial^\alpha a(x)| < \infty$$

for all $m \in \mathbb{N}$ (note that $S^0_{0,0} = C_b^\infty(\mathbb{R}^N \times \mathbb{R}^N)$). Then $\chi(D)a = k_0 * a$ where $k_0 \in S(\mathbb{R}^n)$ is given by

$$k_0(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \chi(\xi)\, d\xi.$$

Consequently, for $m \in \mathbb{N}$ and all multi-indices l,

$$\begin{aligned}(\chi(D-l)a)(x) &= \int_{\mathbb{R}^n} e^{-i\langle l,x-y\rangle} k_0(x-y)a(y)\,dy \\ &= \langle l\rangle^{-2m} \int_{\mathbb{R}^n} e^{-i\langle l,x-y\rangle} \langle D_y\rangle^{2m}(k_0(x-y)a(y))\,dy\end{aligned}$$

with the standard notations

$$\langle l\rangle := (1+|l|_2^2)^{1/2} \quad \text{and} \quad \langle D_y\rangle^2 := I - \Delta_y.$$

The latter estimate implies

$$\|\chi(D-l)a\|_{L^\infty(\mathbb{R}^n)} \le C_m \langle l\rangle^{-2m} |a|_{2m}$$

since $\partial_x^\alpha k_0 \in S(\mathbb{R}^n)$ for all multi-indices α. ■

Similar classes of pseudodifferential operators have been considered in [2], see also [6].

To prove the inclusion of OPS_w into the Wiener algebra in Proposition 5.3 below we need the following estimates.

Proposition 5.2 *Let Q be a compact subset of $\mathbb{R}^n$, and let $f \in S'(\mathbb{R}^n)$ be a distribution with supp $\hat{f} \subset Q$. Then $f \in C^\infty$, and for every multi-index α,*

$$\|\partial^\alpha f\|_{L^\infty(\mathbb{R}^n)} \le C_\alpha \|f\|_{L^\infty(\mathbb{R}^n)}$$

where the constant C_α depends on α only.

Proof. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be such that $\hat{f}\phi = \hat{f}$. Since $\hat{f} \in \mathcal{E}'(\mathbb{R}^n)$, the compactly supported distributions, one has

$$f(x) = (2\pi)^{-n} \hat{f}(\phi e_{-x})$$

where $e_{-x}(\xi) := e^{-i\langle x,\xi\rangle}$. Consequently,

$$(\partial^\alpha f)(x) = (2\pi)^{-n}\hat{f}(\psi_{\alpha,x}) = (2\pi)^{-n}\int_{\mathbb{R}^n} f(y)e_y(\psi_{\alpha,x})\,dy$$

where $\psi_{\alpha,x} \in C_0^\infty(\mathbb{R}^n)$ is given by

$$\psi_{\alpha,x}(\xi) = (-i\xi)^\alpha\phi(\xi)e^{-i\langle x,\xi\rangle}.$$

The linear functional e_y is continuous on $C_0^\infty(\mathbb{R}^n)$. Hence,

$$(2\pi)^{-n}e_y(\psi_{\alpha,x}) = (2\pi)^{-n}\int_{\mathbb{R}^n}(-i\xi)^\alpha\phi(\xi)e^{-i\langle x-y,\xi\rangle}d\xi =: h_\alpha(x-y).$$

Integrating by parts one finds $h_\alpha \in L^1(\mathbb{R}^n)$. Thus,

$$(\partial^\alpha f)(x) = \int_{\mathbb{R}^n} h_\alpha(x-y)f(y)\,dy,$$

whence

$$\|\partial^\alpha f\|_{L^\infty(\mathbb{R}^n)} \le \|h_\alpha\|_{L^1(\mathbb{R}^n)}\|f\|_{L^\infty(\mathbb{R}^n)}$$

for every multi-index α. ■

Proposition 5.3 $OPS_w(\mathbb{R}^{2N}) \subset \mathcal{W}(\mathbb{R}^N)$.

Proof. Let $a \in S_w(\mathbb{R}^N \times \mathbb{R}^N)$. By Proposition 5.1, a can be represented as

$$a(x,\xi) = \sum_{(\alpha,\beta)\in\mathbb{Z}^N\times\mathbb{Z}^N} e^{i\langle x,\alpha\rangle + i\langle\xi,\beta\rangle}a_{\alpha\beta}(x,\xi) \tag{21}$$

where $\operatorname{supp}\hat{a}_{\alpha\beta}$ is contained in a compact subset Q of $\mathbb{R}^{2N}$ and

$$\sum_{(\alpha,\beta)\in\mathbb{Z}^N\times\mathbb{Z}^N}\|a_{\alpha\beta}\|_{L^\infty(\mathbb{R}^{2N})} < \infty.$$

Then

$$Op(a) = \sum_{(\alpha,\beta)\in\mathbb{Z}^N\times\mathbb{Z}^N} E_\alpha Op(a_{\alpha\beta})V_\beta \tag{22}$$

and

$$\begin{aligned}
&\|\Phi_{(\gamma_1,\gamma_2)}Op(a)\Phi^*_{(\delta_1,\delta_2)}\| \\
&\quad = \|\Phi_0 U^*_{(\gamma_1,\gamma_2)}Op(a)U_{(\delta_1,\delta_2)}\Phi_0^*\| \\
&\quad \le \left\|\sum_{(\alpha,\beta)\in\mathbb{Z}^N\times\mathbb{Z}^N} e^{i\langle\alpha,\gamma_2\rangle}\Phi_0 E_{\alpha-\gamma_1}V_{-\gamma_2}Op(a_{\alpha\beta})V_{\beta+\delta_2}E_{\delta_1}\Phi_0^*\right\| \\
&\quad \le \sum_{(\alpha,\beta)\in\mathbb{Z}^N\times\mathbb{Z}^N}\|\Phi_0 E_{\alpha-\gamma_1}V_{-\gamma_2}Op(a_{\alpha\beta})V_{\beta+\delta_2}E_{\delta_1}\Phi_0^*\|.
\end{aligned}$$

By Proposition 5.2,

$$\|\partial_x^\gamma \partial_\xi^\delta a_{\alpha\beta}\|_{L^\infty(\mathbb{R}^{2N})} \leq C_{\gamma\delta} \|a_{\alpha\beta}\|_{L^\infty(\mathbb{R}^{2N})}. \tag{23}$$

Hence (see, for instance, [6], Proposition 4.1.16),

$$\begin{aligned} &\|\Phi_0 E_{\alpha-\gamma_1} V_{-\gamma_2} Op(a_{\alpha\beta}) V_{\beta+\delta_2} E_{\delta_1} \Phi_0^*\| \\ &\quad \leq \; C\, |a_{\alpha\beta}|_{2k_1,\, 2k_2} (1 + |\alpha - \gamma_1 + \delta_1|)^{-2k_1} (1 + |\beta + \delta_2 - \gamma_2|)^{-2k_2}, \end{aligned}$$

where $2k_1 > N$ and $2k_2 > N$, and where the constant C is independent of $a_{\alpha\beta}$. From (23) one concludes that

$$|a_{\alpha\beta}|_{2k_1,\, 2k_2} \leq C\, \|a_{\alpha\beta}\|_{L^\infty(\mathbb{R}^N)}$$

with a constant C independent of $a_{\alpha\beta}$ again. So one finally has

$$\begin{aligned} &\|\Phi_{(\gamma_1,\, \gamma_2)} Op(a) \Phi^*_{(\delta_1,\, \delta_2)}\| \\ &\quad \leq \; C \sum_{(\alpha,\, \beta) \in \mathbb{Z}^N \times \mathbb{Z}^N} \|a_{\alpha\beta}\|_{L^\infty(\mathbb{R}^N)} (1 + |\alpha - \gamma_1 - \delta_1|)^{-2k_1} (1 + |\beta + \delta_2 - \gamma_2|)^{-2k_2} \\ &\quad =: \; h(\gamma_1 - \delta_1,\, \gamma_2 - \delta_2) \end{aligned}$$

with a sequence $h \in l^1(\mathbb{Z}^N \times \mathbb{Z}^N)$. Consequently, $Op(a) \in \mathcal{W}(\mathbb{R}^N)$. ∎

The following corollary follows immediately from the preceding proposition in combination with Proposition 4.2 (a).

Corollary 5.4 *Let $a \in S_w(\mathbb{R}^{2N})$ be represented as in (21), and let $p \in [1, \infty]$. Then*

$$\|Op(a)\|_{L(M^{2,\, p}(\mathbb{R}^N))} \leq C \sum_{(\alpha,\, \beta) \in \mathbb{Z}^N \times \mathbb{Z}^N} \|a_{\alpha\beta}\|_{L^\infty(\mathbb{R}^N)}$$

with a constant C independent of $a_{\alpha\beta}$.

We say that the symbol a belongs to the class $\mathcal{R}(\mathbb{R}^{2N})$ if there are integers $k_1,\, k_2$ with $2k_1 > N$ and $2k_2 > N$ such that a can be represented as

$$a(y) = \sum_{\gamma \in \mathbb{Z}^{2N}} e^{i\langle \gamma,\, y \rangle} a_\gamma(y)$$

where $y = (x,\, \xi) \in \mathbb{R}^N \times \mathbb{R}^N$, and where the functions $a_\gamma \in S^0_{0,\, 0}$ satisfy

$$\sum_{\gamma \in \mathbb{Z}^{2N}} |a_\gamma|_{2k_1,\, 2k_2} < \infty. \tag{24}$$

Proposition 5.5 *The classes $\mathcal{R}(\mathbb{R}^{2N})$ and $S_w(\mathbb{R}^{2N})$ coincide.*

Proof. Let $a \in \mathcal{R}(\mathbb{R}^{2N})$. Then

$$\begin{aligned}
&\sum_{l \in \mathbb{Z}^{2N}} \|\chi(D-l)a\|_{L^\infty(\mathbb{R}^{2N})} \\
&\quad \leq \sum_{l \in \mathbb{Z}^{2N}} \sum_{\gamma \in \mathbb{Z}^{2N}} \|\chi(D-l-\gamma)a_\gamma\|_{L^\infty(\mathbb{R}^{2N})} \\
&\quad \leq C \sum_{\gamma \in \mathbb{Z}^{2N}} |a_\gamma|_{2k_1,\,2k_2} \sum_{l \in \mathbb{Z}^{2N}} (1+|l_1|)^{-2k_1}(1+|l_2|)^{-2k_2} < \infty,
\end{aligned}$$

whence the inclusion $\mathcal{R}(\mathbb{R}^{2N}) \subset S_w(\mathbb{R}^{2N})$. The reverse inclusion follows from Proposition 5.1. ■

The following observation will be needed to prove the richness of the operators in $OPS_w(\mathbb{R}^{2N})$.

Lemma 5.6 *Let $(A_j)_{j\in\mathbb{N}}$ be a sequence of bounded linear operators on a Hilbert space H with*

$$\sum_{j\in\mathbb{N}} \|A_j\| < \infty, \tag{25}$$

and let $A := \sum_{j\in\mathbb{N}} A_j$. Furthermore, let $(U_m)_{m\in\mathbb{N}}$ be a sequence of unitary operators on H such that the sequences $(U_m^* A_j U_m)_{m\in\mathbb{N}}$ converge strongly as $m \to \infty$ to certain operators $\tilde{A}_j$ for every j. Then the sequence $(U_m^* A U_m)_{m\in\mathbb{N}}$ converges strongly to $\tilde{A} := \sum_{j\in\mathbb{N}} \tilde{A}_j$.

Proof. Let $u \in H$ and $\varepsilon > 0$. By condition (25), there is an $n_0 \in \mathbb{N}$ such that

$$\sum_{j>n_0} \|A_j u\| < \frac{\varepsilon}{3}, \tag{26}$$

and due to strong convergence, there is an $m_0 \in \mathbb{N}$ such that, for $m > m_0$,

$$\max_{1\leq j\leq n_0} \|(\tilde{A}_j - U_m^* A_j U_m)u\| < \frac{\varepsilon}{3n_0}.$$

Hence, given arbitrary $u \in H$ and $\varepsilon > 0$, one finds an $m_0 \in \mathbb{N}$ such that

$$\|(\tilde{A} - U_m^* A U_m)u\| \leq \sum_{j=1}^{n_0} \|(\tilde{A}_j - U_m^* A_j U_m)u\| + 2\sum_{j>n_0} \|A_j u\| < \varepsilon$$

for $m \geq m_0$. ■

Proposition 5.7 $OPS_w(\mathbb{R}^{2N}) \subset \mathcal{W}^\$(\mathbb{R}^N)$.

Proof. Let $A := Op(a) \in OPS_w(\mathbb{R}^N \times \mathbb{R}^N)$. By Proposition 5.5, the operator A can be written as

$$A = \sum_{(\alpha,\,\beta)\in\mathbb{Z}^N\times\mathbb{Z}^N} E_\alpha Op(a_{\alpha\beta}) V_\beta$$

where

$$\sum_{(\alpha,\,\beta)\in\mathbb{Z}^N\times\mathbb{Z}^N} \|Op(a_{\alpha\beta})\| < \infty.$$

Let $h : m \mapsto h_m := (h'_m,\, h''_m) \in \mathbb{Z}^N \times \mathbb{Z}^N$ be a sequence which tends to infinity. Then, evidently,

$$U^*_{h_m} A U_{h_m} = \sum_{(\alpha,\,\beta)\in\mathbb{Z}^N\times\mathbb{Z}^N} (U^*_{h_m} E_\alpha U_{h_m})\,(U^*_{h_m} Op(a_{\alpha\beta}) U_{h_m})\,(U^*_{h_m} V_\beta U_{h_m}).$$

Since $U_\gamma = V_\beta E_\alpha$ and $U^*_\gamma = E_{-\alpha} V_{-\beta}$, one has

$$U^*_{h_m} E_\alpha U_{h_m} = e^{-i\langle\alpha,\, h''_m\rangle} E_\alpha \quad \text{and} \quad U^*_{h_m} V_\beta U_{h_m} = e^{i\langle\beta,\, h'_m\rangle} V_\beta.$$

In [6], Lemma 4.2.4, it is verified that there is a subsequence g of h such that the functions

$$\varphi_m : \alpha \mapsto e^{-i\langle\alpha,\, g''_m\rangle} \quad \text{and} \quad \gamma_m : \beta \mapsto e^{-i\langle\beta,\, g'_m\rangle}$$

converge uniformly with respect to $\alpha,\, \beta \in \mathbb{Z}^N$ to certain limit functions φ and γ as $m \to \infty$. Clearly, $|\varphi(\alpha)| = |\gamma(\beta)| = 1$ for each $\alpha,\, \beta \in \mathbb{Z}^N$. It is also easy to see that

$$U^*_{g_m} Op(a_{\alpha\beta}) U_{g_m} = Op(a^{g_m}_{\alpha\beta})$$

where

$$a^{g_m}_{\alpha\beta}(x,\, \xi) := a_{\alpha\beta}(x + g'_m,\, \xi + g'_m).$$

According to the Arzéla-Ascoli Theorem, one further finds a subsequence k of g such that the functions $a^{k_m}_{\alpha\beta}$ converge to a limit function $a^k_{\alpha\beta}$ in the topology of $C^\infty(\mathbb{R}^{2N})$. This implies (compare [6], Theorem 4.3.15) that $a^k_{\alpha\beta} \in S^0_{0,\,0}$ and that

$$U^*_{k_m} Op(a_{\alpha\beta}) U_{k_m} \to Op(a^K_{\alpha\beta}) \quad \text{strongly as } m \to \infty.$$

Applying the standard Cantor diagonal process, we finally obtain that every sequence h has a subsequence l such that

$$U^*_{l_m} (E_\alpha Op(a_{\alpha\beta}) V_\beta) U_{l_m} \to \varphi(\alpha)\gamma(\beta) E_\alpha Op(a^l_{\alpha\beta}) V_\beta$$

strongly as $m \to \infty$. Hence, the strong convergence

$$U^*_{l_m} A U_{l_m} \to A_l := \sum_{(\alpha,\,\beta)\in\mathbb{Z}^N\times\mathbb{Z}^N} \varphi(\alpha)\gamma(\beta) E_\alpha Op(a^l_{\alpha\beta}) V_\beta \tag{27}$$

as $m \to \infty$ follows from Lemma 5.6, and the strong convergence of the adjoint sequences

$$U^*_{l_m} A^* U_{l_m} \to A^*_l := \sum_{(\alpha,\,\beta)\in\mathbb{Z}^N\times\mathbb{Z}^N} \bar\varphi(\alpha)\bar\gamma(\beta) V_{-\beta} \left[Op(a^l_{\alpha\beta})\right]^* E_{-\alpha}$$

can be checked similarly in the same way. ■

Now Theorem 17 implies the following final results on the Fredholmness of pseudodifferential operators in the Sjöstrand class acting on modulation spaces.

Theorem 5.8 *The following conditions are equivalent for $A \in OPS_w$:*
(a) A is a Fredholm operator on $M^{2,p}(\mathbb{R}^N)$ for a certain $p \in (1, \infty)$;
(b) A is a Fredholm operator on $M^{2,p}(\mathbb{R}^N)$ for each $p \in (1, \infty)$;
(c) there is a $p \in [1, \infty]$ such that all limit operators of A are invertible on $M^{2,p}(\mathbb{R}^N)$;
(d) all limit operators of A are invertible on every space $M^{2,p}(\mathbb{R}^N)$ with $p \in [1, \infty]$;
(e) all limit operators are uniformly invertible on each of the spaces $M^{2,p}(\mathbb{R}^N)$ with $p \in [1, \infty]$.

Corollary 5.9 *Let $A \in OPS_w$. Then the essential spectrum $\sigma_{ess}(A)$ of A considered as an operator on $M^{2,p}(\mathbb{R}^N)$ does not depend on $p \in (1, \infty)$, and*

$$\sigma_{ess}(A|M^{2,p}(\mathbb{R}^N)) = \bigcup_{A_h \in \sigma_{op}(A)} \sigma(A_h|L^2(\mathbb{R}^N)).$$

References

[1] A. Boulkhemair, Remark on a Wiener type pseudodifferential algebra and Fourier integral operators. – Math. Res. Letter **4**(1997), 53 – 67.

[2] K. Gröchenig, Foundation of Time-Frequency Analysis. – Birkhäuser Verlag, Boston, Basel, Berlin 2000.

[3] K. Gröchenig, C. Heil, Modulation spaces and pseudodifferential operators. – Integral Equations Oper. Theory **34**(1999), 4, 439 – 457.

[4] V. S. Rabinovich, On the Fredholmness of pseudodifferential operators on $\mathbb{R}^n$ in the scale of $L_{2,p}$-spaces. – Sib. Mat. Zh. **29**(1988), 4, 149 – 161 (Russian, Engl. transl. Sib. Math. J. **29**(1988), 4, 635 – 646).

[5] V. S. Rabinovich, S. Roch, Wiener algebra of operators and applications to pseudodifferential operators. – J. Anal. Appl. **23**(2004), 3, 437 – 482.

[6] V. S. Rabinovich, S. Roch, B. Silbermann, Limit Operators and their Applications in Operator Theory. – Operator Theory: Advances and Applications **150**, Birkhäuser Verlag, Basel, Boston, Berlin 2004.

[7] M. Shubin, Pseudodifferential operators and Spectral Theory. - Springer-Verlag, Berlin, Heidelberg, New York, Tokio 2001.

[8] J. Sjöstrand, An algebra of pseudodifferential operators. – Math. Res. Lett. **1**(1994), 185 – 192.

[9] J. Sjöstrand, Wiener type algebras of pseudodifferential operators. – Sém. Eq. aux Dér. Part., Ecole Polytechnique, 1994/1995, Exposé n. IV.

[10] E. M. Stein, Harmonic Analysis, Real-Variable Methods, Orthogonality, and Oscillatory Integrals. – Princeton Univ. Press, Princeton, New Jersey, 1993.

[11] K. Tachizawa, The boundedness of pseudodifferential operators on modulation spaces. – Math. Nachr. **168**(1994), 263 – 277.

[12] M. E. Taylor, Pseudodifferential Operators. - Princeton Univ. Press, Princeton, New Jersey, 1981.