OPTIMAL APPROXIMATION OF SDE'S WITH ADDITIVE FRACTIONAL NOISE

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Abstract. We study pathwise approximation of scalar stochastic differential equations with additive fractional Brownian noise of Hurst parameter $H > 1/2$, considering the mean square L^2-error criterion. By means of the Malliavin calculus we derive the exact rate of convergence of the Euler scheme, also for non-equidistant discretizations. Moreover, we establish a sharp lower error bound that holds for arbitrary methods, which use a fixed number of bounded linear functionals of the driving fractional Brownian motion. The Euler scheme based on a discretization, which reflects the local smoothness properties of the equation, matches this lower error bound up to the factor 1.39.

1. Introduction

Let $B^H(t), t \in [0,1]$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$, i.e., B^H is a continuous centered Gaussian process with covariance kernel

$$K(s,t) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \qquad s,t \in [0,1].$$

For $H = 1/2$ fractional Brownian motion is a Brownian motion, while for $H \neq 1/2$ it is neither a semimartingale nor a Markov process. In particular, non-overlapping increments are negatively correlated if $H < 1/2$ and positively if $H > 1/2$. Moreover, it holds

$$(\mathbb{E}|B^H(t) - B^H(s)|^2)^{1/2} = |t-s|^H, \qquad s,t \in [0,1],$$

and almost all sample paths of B^H are Hölder continuous of any order $\lambda < H$.

We consider pathwise approximations of the stochastic differential equation

$$\begin{aligned} dX(t) &= a(t, X(t))\,dt + \sigma(t)\,dB^H(t), \qquad t \in [0,1], \qquad (1)\\ X(0) &= x_0, \end{aligned}$$

with $H \in (1/2, 1)$ and deterministic initial value $x_0 \in \mathbb{R}$. Here a and σ satisfy standard smoothness assumptions and equation (1) is an integral equation with all integrals being pathwise Riemann-Stieltjes integrals. See, e.g., Lin (1995), Zähle and Klingenhöfer (1999) and Nualart and Răşcanu (2002), also for the case of non-additive diffusion coefficients.

Approximation of stochastic differential equations driven by fractional Brownian motion is studied only in few articles. In particular, no results on lower error bounds are available up to now. Mainly, analytic methods like the Picard iteration (Lin

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(1995)), Wong-Zakai-type approximations (Lin (1995), Nourdin (2004), Boufoussi and Tudor (2004)) and the Kramers-Smoluchowski approximation (Boufoussi and Tudor (2004)) are considered, and uniform convergence of the approximation sequence for almost all sample paths is proved. Lin (1995) also shows that the Euler approximation of equation (1) converges uniformly in probability. Nourdin (2004) studies the approximation of autonomous differential equations driven by Hölder continuous functions and determines upper error bounds for the order of convergence of the equidistant Euler scheme and an equidistant Milstein-type scheme.

In this paper the error $e(\widehat{X})$ of an approximation $\widehat{X}$ of equation (1) will be measured as follows. The pathwise distance between X and $\widehat{X}$ in the L^2-norm $\|\cdot\|_2$ is taken and then averaged over all trajectories, i.e.,

$$e(\widehat{X}) = (\mathbb{E}\|X - \widehat{X}\|_2^2)^{1/2}.$$

First, we study the Euler approximation of equation (1) and wish to determine the best discretization in a strong asymptotic sense. Specifically, we consider regular sequences of discretizations generated by a density function h, i.e., the knots of these discretizations are quantiles of the density h.

Applying the Malliavin calculus for fractional Brownian motion, see, e.g., Alòs and Nualart (2003), we derive the exact rate of convergence of these non-equidistant Euler schemes, see Theorem 1. It turns out that the optimal density h^* is proportional to $\sigma^{1/(H+1/2)}$. For the error of the corresponding Euler scheme $\widehat{X}^E_{h^*,n}$ we obtain

$$\lim_{n\to\infty} n^H \cdot e(\widehat{X}^E_{h^*,n}) = \beta_H \cdot \|\sigma\|_{1/(H+1/2)}$$

with

$$\beta_H^2 = \frac{1}{(2H+1)(H+1)} - \frac{1}{6}.$$

Here n denotes the number of subintervals of the discretization, i.e., the number of evaluations of B^H.

Moreover we address the following questions: Can we reduce the error by switching to arbitrary discretizations or different approximation schemes? Furthermore, to which extent can we decrease the error by approximation schemes that can use arbitrary bounded linear functionals of the driving fractional Brownian motion?

To this end, we consider arbitrary approximation methods $\widehat{X}_n$ of equation (1), which apply n bounded linear functionals to a sample path of B^H. The n functionals may be determined sequentially. This data about B^H may then be used in any way to produce a approximation $\widehat{X}_n$. The quantity

$$e(n) = \inf_{\widehat{X}_n} e(\widehat{X}_n)$$

is the minimal error that can be achieved by approximations $\widehat{X}_n$ of this type.

We show that the minimal errors satisfy

$$\lim_{n\to\infty} n^H \cdot e(n) = \gamma_H \cdot \|\sigma\|_{1/(H+1/2)}$$

with

$$\gamma_H^2 = \frac{\sin(\pi H)\Gamma(2H)}{\pi^{2H+1}},$$

see Theorem 2.

Thus, the Euler scheme based on the optimal density h^* matches the minimal errors up to a constant factor, which only depends on the Hurst parameter H. Hence other approximations schemes, which may use arbitrary bounded linear functionals, can only decrease the error slightly, asymptotically. Moreover, there are no approximation schemes $\widehat{X}_n$ of the above type, which can achieve a better approximation rate than n^{-H}.

The paper is organized as follows. In Section 2 we state our assumptions on the drift- and diffusion coefficient and we provide basic properties of the solution in the mean square sense. Section 3 contains the results for the error of non-equidistant Euler schemes. The minimal error is addressed in Section 4. Proofs are postponed to Section 5.

2. Stochastic Differential Equations with Additive Fractional Noise

In the sequel let $H > 1/2$. Furthermore, we will assume throughout this article that the drift- and diffusion coefficient satisfy:

(A) $a \in C^{0,2}([0,1] \times \mathbb{R})$ and there exist constants $K_1, K_2, K_3 > 0$ such that

$$|a_x(t,x)| \le K_1, \qquad |a_{xx}(t,x)| \le K_2,$$

and

$$|a(t,x) - a(s,x)| \le K_3 \cdot (1 + |x|) \cdot |t-s|$$

for all $s, t \in [0,1]$ and $x \in \mathbb{R}$,

(B) $\sigma \in C^1([0,1])$,

(C) $\sigma(t) > 0$ for all $t \in [0,1]$.

Under these assumptions equation (1) has a unique pathwise solution X, i.e., almost all sample paths of the process X satisfy the integral equation

$$X(t) = x_0 + \int_0^t a(\tau, X(\tau))\, d\tau + \int_0^t \sigma(\tau)\, dB^H(\tau), \qquad t \in [0,1],$$

with all integrals being Riemann-Stieltjes integrals, and if $\widetilde{X}$ is another solution of equation (1), then X and $\widetilde{X}$ are indistinguishable. Moreover, almost all sample paths of X are Hölder continuous of every order $\lambda < H$, and it holds

$$\mathbb{E}\,\|X\|_\infty^p < \infty \tag{2}$$

for all $p > 1$. See Lin (1995), Nualart and Rǎşcanu (2002).

The assumptions (A), (B) and (C) are required for the analysis of approximations of equation (1). For existence of a unique pathwise Riemann-Stieltjes solution much weaker assumptions are sufficient. Compare, e.g., Lin (1995) resp. Nualart and Ouknine (2002).

The following Proposition characterizes the smoothness of the solution in the mean square sense.

Proposition 1. *Let X be the solution of equation (1). It holds*

$$\lim_{s \to 0} \frac{1}{s^H} \cdot \left(\mathbb{E}|X(t+s) - X(t)|^2\right)^{1/2} = |\sigma(t)| \quad \textit{uniformly in } t \in [0,1].$$

Hence the solution X behaves in mean square sense locally like a weighted fractional Brownian motion, although X is not necessarily Gaussian. The mean square Hölder exponent is given by the Hurst parameter H of the driving fractional Brownian motion, and the local mean square Hölder constant is determined by the diffusion coefficient σ.

Remark 1. Stochastic differential equations with non-additive fractional noise are studied, e.g., in Lin (1995), Zähle (1998), Nualart and Răşcanu (2002) and Nourdin (2004). Ferrante and Rovira (2004) also consider stochastic delay differential equations driven by fractional Brownian motion.

3. Non-Equidistant Euler Scheme

For any discretization

$$0 = t_0 < t_1 < \ldots < t_n = 1$$

the corresponding Euler scheme $\widehat{X}^E$ for equation (1) is given by

$$\widehat{X}^E(0) = x_0$$

and

$$\widehat{X}^E(t_{j+1}) = \widehat{X}^E(t_j) + a(t_j, \widehat{X}^E(t_j)) \cdot (t_{j+1} - t_j) + \sigma(t_j) \cdot (B^H(t_{j+1}) - B^H(t_j))$$

for $j = 0, \ldots, n-1$. A global approximation $\widehat{X}^E$ on $[0,1]$ is obtained by piecewise linear interpolation, i.e.,

$$\widehat{X}^E(t) = \frac{t_{j+1} - t}{t_{j+1} - t_j} \cdot \widehat{X}^E(t_j) + \frac{t - t_j}{t_{j+1} - t_j} \cdot \widehat{X}^E(t_{j+1})$$

for $t \in [t_j, t_{j+1}]$.

To determine the exact rate of convergence of the Euler scheme, we will restrict to regular sequences of discretizations generated by a strictly positive probability density function $h \in C([0,1])$, i.e.,

$$0 = t_{0,n} < t_{1,n} < \ldots < t_{n,n} = 1 \quad \text{with} \quad \int_0^{t_{j,n}} h(s)\, ds = \frac{j}{n}, \quad j = 1, \ldots, n-1. \tag{3}$$

So by choosing such a density h one gets a sequence of discretizations. If, e.g., $h = \mathrm{id}$, we obtain a sequence of equidistant discretizations.

We will use the notation $\widehat{X}^E_{h,n}$ for the Euler scheme based on the discretization given by (3). Clearly, good choices of h have to be related to the local smoothness of the solution of equation (1), i.e., the local Hölder constant σ and the Hölder exponent H.

Theorem 1. *It holds*

$$\lim_{n\to\infty} n^H \cdot e(\widehat{X}^E_{h,n}) \;=\; \beta_H \cdot \|\sigma \cdot h^{-H}\|_2$$

with

$$\beta_H^2 = \frac{1}{(2H+1)(H+1)} - \frac{1}{6}.$$

Theorem 1 shows that the order of convergence of the Euler scheme only depends on the Hurst parameter of the driving fractional Brownian motion. The minimal asymptotic constant is obtained by choosing the density

$$h^*(t) = \frac{1}{\|\sigma^{1/(H+1/2)}\|_1} \cdot |\sigma(t)|^{1/(H+1/2)}, \qquad t \in [0,1].$$

Corollary 1. *(1) For the equidistant Euler scheme it holds*

$$\lim_{n\to\infty} n^H \cdot e(\widehat{X}^E_{\mathrm{id},n}) \;=\; \beta_H \cdot \|\sigma\|_2.$$

(2) For the optimal density h^* we have

$$\lim_{n\to\infty} n^H \cdot e(\widehat{X}^E_{h^*,n}) \;=\; \beta_H \cdot \|\sigma\|_{1/(H+1/2)}.$$

Consequently, equidistant discretization leads only to the best asymptotic constant, if the diffusion coefficient is a constant mapping. For non-constant diffusion coefficients the error can be reduced asymptotically by the factor $\|\sigma\|_{1/(H+1/2)}/\|\sigma\|_2$.

The following example provides evidence that even for a moderate number of knots the Euler scheme based on the optimal density h^* is superior to the equidistant Euler scheme.

Example 1. We study the equation

$$dX(t) = 6 \cdot (1.01 - t)\, dB^H(t), \quad X(0) = 0 \tag{4}$$

by means of exact error formulas. Figure 1 shows the quantities $n^H \cdot e(\widehat{X}^E_{h^*,n})$ and $n^H \cdot e(\widehat{X}^E_{\mathrm{id},n})$, marked by $+$ resp. $\times$ in dependence of the number n of knots for $H = 0.7$. The solid lines correspond to the asymptotic constant of the error of the schemes. So, for equation (4) the non-equidistant scheme performs uniformly

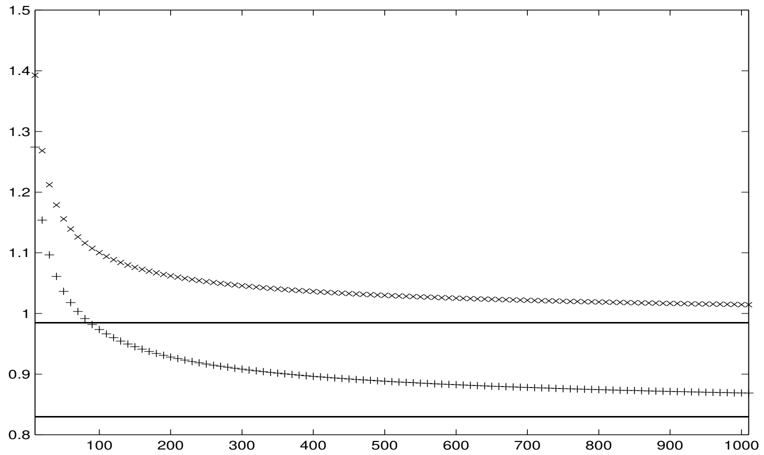

FIGURE 1. $n^H \cdot e(\widehat{X}^E_{\mathrm{id},n})$ and $n^H \cdot e(\widehat{X}^E_{h^*,n})$ for equation (4) vs. n for $H = 0.7$.

better than the equidistant one. Moreover, the asymptotic error formulas are in good accordance with the exact errors even for a small numbers of knots.

Remark 2. Regular sequences of discretizations are, e.g., widely studied and used for the approximation of stochastic processes and for the prediction of integrals of stochastic processes. See, e.g., Ritter (2000) for results and references. In the context of stochastic differential equations driven by Brownian motion regular sequences are studied, e.g., by Cambanis and Hu (1996).

Remark 3. Instead of regular sequences of discretizations we can use the following step size control for the Euler scheme, which is easier to implement. Let $h \in C([0,1])$ be a strictly positive probability density function and choose a basic step-size $\Delta > 0$. Set $t_0 = 0$ and

$$t_{k+1} = t_k + \Delta / h(t_k),$$

as long as the right hand side does not exceed one. Otherwise set $t_{k+1} = 1$. The total number of steps is $n(\Delta) = \min\{k \in \mathbb{N} : t_k = 1\}$. For the Euler scheme $\widehat{X}^E_{h,\Delta}$ based on this discretization the assertion of Theorem 1 holds with $n = n(\Delta)$.

4. Lower Bounds

The non-equidistant Euler scheme in the previous section uses a finite number of evaluations of B^H, i.e., a finite number of Dirac functionals is applied to the trajectories of the driving fractional Brownian motion. Now we determine sharp lower error bounds that hold for every approximation method, which applies n sequentially selected bounded linear functional to a sample path of B^H.

Let Λ^{lin} denote the class of all bounded linear functionals on $C([0,1])$ and assume that x_0 is known. Fix a and σ and consider the corresponding equation (1). Then an arbitrary approximation method $\widehat{X}_n$, based on x_0 and n sequentially selected bounded linear functionals, is defined by the measurable mappings

$$\psi_k : \mathbb{R}^k \to \Lambda^{\text{lin}}$$

for $k = 1, \dots, n$ and

$$\phi_n : \mathbb{R}^{n+1} \to L_2([0,1]).$$

The first functional, which will be applied to the trajectory of B^H, is

$$\Lambda_1 = \psi_1(x_0),$$

and the functionals Λ_k for $k = 2, \dots, n$ are given by

$$\Lambda_k = \psi_k(x_0, \Lambda_1(B^H), \dots, \Lambda_{k-1}(B^H)).$$

The data $x_0, \Lambda_1(B^H), \dots, \Lambda_n(B^H)$ is then used to compute a pathwise approximation

$$\widehat{X}_n = \phi_n(x_0, \Lambda_1(B^H), \dots, \Lambda_n(B^H)).$$

The quantity

$$e_2(n) = \inf_{\widehat{X}_n} e_2(\widehat{X}_n)$$

is the minimal error, which can be obtained by using such approximation methods. For fixed $\psi_1, \dots, \psi_n$ the best choice of ϕ_n is the conditional mean of X given the respective functionals applied to B^H. Hence the main difficulty in this theoretical minimization problem is the choice of the functionals, i.e., of the mappings $\psi_1, \dots, \psi_n$.

The number n can be considered as a coarse measure for the computational cost of the method $\widehat{X}_n$. Clearly, a more precise analysis of the computational cost should take at least the number of arithmetical operations performed by $\widehat{X}_n$ into account.

Theorem 2. *It holds*

$$\lim_{n\to\infty} n^H \cdot e(n) = \gamma_H \cdot \|\sigma\|_{1/(H+1/2)},$$

where

$$\gamma_H^2 = \frac{\sin(\pi H)\Gamma(2H)}{\pi^{1+2H}}.$$

Hence, the intrinsic difficulty of equation (1) is completely determined by the $L^{1/(H+1/2)}$-quasi-norm of the diffusion coefficient σ and the Hurst parameter H of the driving fractional Brownian motion. In particular, Theorem 2 implies that approximation schemes $\widehat{X}_n$ of the above type, which obtain a higher convergence rate than n^{-H}, do not exist.

Combining Theorem 1 and 2, we obtain that the non-equidistant Euler schemes obtain the optimal order of convergence. Moreover, by Corollary 1 we have that the Euler scheme based on the optimal density h^* is asymptotically optimal up to a constant factor, which only depends on H and not on the drift- or diffusion coefficient of the equation.

Corollary 2. *It holds*

$$\limsup_{n\to\infty} \frac{e(\widehat{X}^E_{h^*,n})}{e(n)} \leq \frac{\beta_H}{\gamma_H}.$$

The ratio of β_H/γ_H is a monotonically increasing function of H and we have

$$\frac{\pi}{\sqrt{6}} \leq \frac{\beta_H}{\gamma_H} \leq \frac{\sqrt{7}\pi}{6}.$$

Note that $\sqrt{7}\pi/6 \simeq 1.3853$. Thus, the considered arbitrary approximations methods can only be slightly better than the best Euler scheme, asymptotically.

Remark 4. Theorem 2 remains valid, if n sequentially selected bounded linear functionals of a trajectory of B^H on average are allowed. See Section 5.6.

Remark 5. Theorem 2 is also valid in the case $H = 1/2$, see Hofmann *et al.* (2002) for more general results. On the other hand, if one restricts in this case to methods that may use only point evaluations of the driving Brownian motion, then the corresponding minimal errors satisfy

$$\lim_{n\to\infty} n^{1/2} \cdot e(n) = \beta_{1/2} \cdot \|\sigma\|_1,$$

see Hofmann *et al.* (2000). The ratio $\beta_{1/2}/\gamma_{1/2} = \pi/\sqrt{6}$ is the well known gap between linear and standard information.

5. Proofs

Unspecified constants, depending only on K_1, K_2, K_3, x_0, $\|\sigma\|_\infty$ and $\|\sigma'\|_\infty$ will be denoted by c, regardless of their value. Note that the assumptions (A) on the drift coefficient a imply a linear growth condition and a global Lipschitz condition with respect to the state space variable, i.e.,

$$
\begin{aligned}
&(\tilde{\mathrm{A}}1) \quad \forall x \in \mathbb{R},\ \forall t \in [0,1]: \quad |a(t,x)| \leq c \cdot (1+|x|),\\
&(\tilde{\mathrm{A}}2) \quad \forall x,y \in \mathbb{R},\ \forall t \in [0,1]: \quad |a(t,y)-a(t,x)| \leq c \cdot |y-x|.
\end{aligned}
$$

5.1. **Proof of Proposition 1.** Let $0 \leq t \leq t+s \leq 1$. We have

$$
\begin{aligned}
X(t+s)-X(t) = \int_t^{t+s} a(\tau, X(\tau))\, d\tau + \int_t^{t+s} \sigma'(\tau)(B^H(t+s) - B^H(\tau))\, d\tau \\
+ \sigma(t)(B^H(t+s)-B^H(t)).
\end{aligned}
$$

We get by $(\tilde{\mathrm{A}}1)$

$$
\mathbb{E}\left| \int_t^{t+s} a(\tau, X(\tau))\, d\tau \right|^2 \leq c \cdot (1+\mathbb{E}||X||_\infty^2) \cdot s^2.
$$

Moreover, we have

$$
\mathbb{E}\left| \int_t^{t+s} \sigma'(\tau)(B^H(t+s)-B^H(\tau))\, d\tau \right|^2 \leq c \cdot \mathbb{E}||B^H||_\infty^2 \cdot s^2.
$$

Note that $\mathbb{E}||X||_\infty^2 < \infty$ by (2) and in particular $\mathbb{E}||B^H||_\infty^2 < \infty$. Thus, we finally obtain

$$
|\sigma(t)| - c \cdot s^{1-H} \leq \frac{1}{s^H} \cdot (\mathbb{E}|X(t+s)-X(t)|^2)^{1/2} \leq |\sigma(t)| + c \cdot s^{1-H},
$$

which completes the proof.

5.2. **Preliminaries for the Proof of Theorem 1.** Let

$$
0 = t_0 < t_1 < \ldots < t_n = 1
$$

be a discretization of $[0,1]$ and put $\Delta = \max_{i=1,\ldots,n} |t_i - t_{i-1}|$. We will use the notations

$$
Z(t) = \int_0^t a(\tau, X(\tau))\, d\tau, \quad F(t) = \int_0^t \sigma(\tau)\, dB^H(\tau), \qquad t \in [0,1],
$$

and

$$
\begin{aligned}
\widetilde{Z}(t) &= \int_0^t \sum_{i=0}^{n-1} a(t_i, X(t_i)) \cdot 1_{[t_i,t_{i+1})}(\tau)\, d\tau, \qquad t \in [0,1],\\
\widetilde{F}(t) &= \int_0^t \sum_{i=0}^{n-1} \sigma(t_i) \cdot 1_{[t_i,t_{i+1})}(\tau)\, dB^H(\tau), \qquad t \in [0,1].
\end{aligned}
$$

Moreover, let

$$
\phi(s,t) = H(2H-1)|s-t|^{2H-2}, \qquad s,t \in [0,1].
$$

Lemma 1. *It holds*

$$\sup_{t\in[0,1]} \mathbb{E}|F(t) - \widetilde{F}(t)|^2 \le c \cdot \Delta^2.$$

Proof: We have

$$F(t) - \widetilde{F}(t) = \int_0^t \sum_{i=0}^{n-1} (\sigma(\tau) - \sigma(t_i)) \cdot 1_{[t_i,t_{i+1})}(\tau)\, dB^H(\tau).$$

Using the isometry for integrals with respect to fractional Brownian motion with deterministic integrands, see, e.g., Lemma 2.1 in Duncan *et al.* (2000), we obtain

$$\mathbb{E}|F(t) - \widetilde{F}(t)|^2$$
$$= \int_0^t \int_0^t \sum_{i,j=0}^{n-1} (\sigma(\tau_1) - \sigma(t_i))(\sigma(\tau_2) - \sigma(t_j))\phi(\tau_1,\tau_2) \cdot 1_{[t_i,t_{i+1})\times[t_j,t_{j+1})}(\tau_1,\tau_2) d\tau_1 d\tau_2.$$

So we get by assumption (B)

$$\begin{aligned}\mathbb{E}|F(t) - \widetilde{F}(t)|^2 &\le c^2 \cdot \Delta^2 \int_0^t \int_0^t \sum_{i,j=0}^{n-1} \phi(\tau_1,\tau_2) \cdot 1_{[t_i,t_{i+1})\times[t_j,t_{j+1})}(\tau_1,\tau_2)\, d\tau_1 d\tau_2 \\ &= c^2 \cdot \Delta^2 \int_0^t \int_0^t \phi(\tau_1,\tau_2)\, d\tau_1 d\tau_2 = c^2 t^{2H} \cdot \Delta^2.\end{aligned}$$

□

Recall that almost all sample paths of the solution X of equation (1) are Hölder continuous of any order $\lambda < H$. Hence, if $g \in C^1(\mathbb{R})$, the Riemann-Stieltjes integrals

$$\int_0^t g(X(s))\, dB^H(s), \quad t \in [0,1],$$

exist almost surely. Compare, e.g., Theorem 4.2.1 in Zähle (1998). We will use the following change-of-variable formula, which follows straightforward from Theorem 4.3.1 and 4.4.2 in Zähle (1998).

Lemma 2. *Let $g \in C^2(\mathbb{R})$. It holds*

$$g(X(t)) = g(x_0) + \int_0^t g'(X(s))a(s,X(s))\, ds + \int_0^t g'(X(s))\sigma(s)\, dB^H(s), \quad t \in [0,1],$$

almost surely.

In the following, we will also apply the Malliavin calculus for fractional Brownian motion. For an overview on this topic, see, e.g., Alòs and Nualart (2003).

In particular, we will require the Malliavin derivative $D_sX(t)$, $s,t \in [0,1]$ of the solution X. The following Lemma can be obtained by a slightly modification of Proposition 7 in Ferrante and Rovira (2004) or Theorem 5.4.1 in Nourdin (2004).

Lemma 3. *We have*

$$D_sX(t) = \sigma(s) \exp\left(\int_s^t a_x(\tau, X(\tau))\, d\tau\right) \cdot 1_{[0,t]}(s), \qquad s,t \in [0,1].$$

Next we analyze the approximation $\widetilde{Z}$ of Z, using Lemma 2 and 3.

Lemma 4. *We have*

$$\sup_{t\in[0,1]} \mathbb{E}|Z(t)-\widetilde{Z}(t)|^2 \leq c\cdot\Delta^2.$$

Proof: We have

$$\begin{aligned}\mathbb{E}|Z(t)-\widetilde{Z}(t)|^2 \leq 2\mathbb{E}&\left|\int_0^t \sum_{i=0}^{n-1}(a(\tau,X(\tau))-a(t_i,X(\tau)))\cdot 1_{[t_i,t_{i+1})}(\tau)\,d\tau\right|^2\\ &+2\mathbb{E}\left|\int_0^t \sum_{i=0}^{n-1}(a(t_i,X(\tau))-a(t_i,X(t_i)))\cdot 1_{[t_i,t_{i+1})}(\tau)\,d\tau\right|^2.\end{aligned}$$

Since $|a(\tau_1,x)-a(\tau_2,x)|\leq K_3\cdot(1+|x|)\cdot|\tau_1-\tau_2|$ due to Assumption (A) we get for the first summand

$$\mathbb{E}\left|\int_0^t \sum_{i=0}^{n-1}(a(\tau,X(\tau))-a(t_i,X(\tau)))\cdot 1_{[t_i,t_{i+1})}(\tau)\,d\tau\right|^2 \leq c\cdot\mathbb{E}(1+||X||_\infty)^2\cdot\Delta^2.$$

For the second summand we have

$$\begin{aligned}\mathbb{E}&\left|\int_0^t \sum_{i=0}^{n-1}(a(t_i,X(\tau))-a(t_i,X(t_i)))\cdot 1_{[t_i,t_{i+1})}(\tau)\,d\tau\right|^2\\ &\leq \sum_{i,j=0}^{n-1}\int_{t_j}^{t_{j+1}}\int_{t_i}^{t_{i+1}}|R(t_i,t_j,\tau_1,\tau_2)|\,d\tau_1\,d\tau_2,\end{aligned}$$

where

$$R(t_i,t_j,\tau_1,\tau_2)=\mathbb{E}[a(t_i,X(\tau_1))-a(t_i,X(t_i))]\,[a(t_j,X(\tau_2))-a(t_j,X(t_j))]$$

for $i,j=0,\ldots,n-1$ and $\tau_1,\tau_2\in[0,1]$.

Now fix t_i and consider the process $a(t_i,X(t)), t\in[0,1]$. By Lemma 2 we get

$$\begin{aligned}a(t_i,X(t))-a(t_i,X(t_i))=&\int_{t_i}^t a_x(t_i,X(u))a(u,X(u))\,du\\ &+\int_{t_i}^t a_x(t_i,X(u))\sigma(u)dB^H(u),\quad t\in[0,1],\end{aligned}$$

almost surely. Moreover, by the chain rule for the Malliavin derivative we have

$$D_s[\sigma(t)a_x(t_i,X(t))]=\sigma(t)a_{xx}(t_i,X(t))D_sX(t),\qquad s,t\in[0,1].$$

Since

$$\sup_{t\in[0,1]}|\sigma(t)a_x(t_i,X(t))|\leq\|\sigma\|_\infty\cdot K_1, \tag{5}$$

$$\sup_{s,t\in[0,1]}|D_s[\sigma(t)a_x(t_i,X(t))]|\leq\|\sigma\|_\infty^2\cdot K_2\exp(K_1), \tag{6}$$

the process $\sigma(t)a(t_i,X(t))$, $t\in[0,1]$, is Skorohod integrable, see, e.g., Lemma 2 in Ferrante and Rovira (2004). Moreover, by the relation between the Riemann-Stieltjes integral and the Skorohod integral for fractional Brownian motion, see, e.g., Section 2.1 in Nualart *et al.* (2003), we obtain

$$\begin{aligned}a(t_i, X(t)) - a(t_i, X(t_i)) = &\int_{t_i}^{t} a_x(t_i, X(u))a(u, X(u))\,du \\ &+ \int_{t_i}^{t} a_x(t_i, X(u))\sigma(u)\delta B^H(u) \\ &+ \int_{t_i}^{t}\int_0^1 D_s[\sigma(u)a_x(t_i, X(u))]\phi(s,u)\,ds\,du \quad a.s.,\end{aligned}$$

where the integral with respect to δB^H denotes the Skorohod integral. Since

$$\sup_{s\in[0,1]}\int_0^1 \phi(\tau, s)\,d\tau \le 2H, \tag{7}$$

it follows by ($\tilde{\mathrm{A}}1$), (5) and (6)

$$\begin{aligned}|R(t_i, t_j, \tau_1, \tau_2)| \le c\cdot \mathbb{E}(1+\|X\|_\infty)^2\cdot\Delta^2 \\ + \left|\mathbb{E}\int_{t_i}^{\tau_1} a_x(t_i, X(u))\sigma(u)\delta B^H(u)\int_{t_j}^{\tau_2} a_x(t_j, X(u))\sigma(u)\delta B^H(u)\right|.\end{aligned}$$

By the isometry for Skorohod integrals, see, e.g., Lemma 5 in Nualart *et al.* (2003), we have moreover

$$\begin{aligned}&\mathbb{E}\int_{t_i}^{\tau_1} a_x(t_i, X(u))\sigma(u)\delta B^H(u)\int_{t_j}^{\tau_2} a_x(t_j, X(u))\sigma(u)\delta B^H(u) \\ &= \mathbb{E}\int_{t_j}^{\tau_2}\int_{t_i}^{\tau_1} a_x(t_i, X(u_1))\sigma(u_1)a_x(t_j, X(u_2))\sigma(u_2)\phi(u_1,u_2)\,du_1du_2 \\ &\quad + \mathbb{E}\int_{t_j}^{\tau_2}\int_{t_i}^{\tau_1}\int_0^1\int_0^1 D_{v_1}[\sigma(u_1)a_x(t_i, X(u_1))]D_{v_2}[\sigma(u_2)a_x(t_j, X(u_2))] \\ &\qquad\qquad \cdot\phi(v_1,u_2)\phi(v_2,u_1)\ dv_1\,dv_2\,du_1\,du_2.\end{aligned}$$

Hence it follows by (5), (6) and (7)

$$\begin{aligned}&\left|\mathbb{E}\int_{t_i}^{\tau_1} a_x(t_i, X(u))\sigma(u)\delta B^H(u)\int_{t_j}^{\tau_2} a_x(t_j, X(u))\sigma(u)\delta B^H(u)\right| \\ &\qquad\le c\int_{t_j}^{\tau_2}\int_{t_i}^{\tau_1}\phi(u_1,u_2)\,du_1du_2 + c\cdot|\tau_1 - t_i||\tau_2 - t_j|\end{aligned}$$

and therefore

$$|R(t_i, t_j, \tau_1, \tau_2)| \le c\cdot\mathbb{E}(1+\|X\|_\infty)^2\cdot\Delta^2 + c\int_{t_j}^{t_{j+1}}\int_{t_i}^{t_{i+1}}\phi(u_1,u_2)\,du_1\,du_2$$

for $(\tau_1, \tau_2) \in [t_i, t_{i+1}]\times[t_j, t_{j+1}]$. So we finally obtain

$$\begin{aligned}&\mathbb{E}\left|\int_0^t\sum_{i=0}^{n-1}(a(t_i, X(\tau)) - a(t_i, X(t_i)))\cdot 1_{[t_i,t_{i+1})}(\tau)\,d\tau\right|^2 \\ &\qquad\le c\cdot\mathbb{E}(1+\|X\|_\infty)^2\cdot\Delta^2 + c\cdot\Delta^2\sum_{i,j=0}^{n-1}\int_{t_j}^{t_{j+1}}\int_{t_i}^{t_{i+1}}\phi(u_1,u_2)\,du_1\,du_2 \le c\cdot\Delta^2.\end{aligned}$$

$\square$

To analyze the error of the Euler approximation $\widehat{X}^E$ in the discretization points, we will use the Euler process $\widetilde{X}^E$ given by

$$\widetilde{X}^E(t) = \widehat{X}^E(t_j) + a(t_j, \widehat{X}^E(t_j)) \cdot (t - t_j) + \sigma(t_j) \cdot (B^H(t) - B^H(t_j))$$

for $t \in [t_j, t_{j+1})$. Clearly, we have $\widehat{X}^E(t_j) = \widetilde{X}^E(t_j)$ for $j = 0, 1, \ldots, n$. Note that the Euler process requires complete knowledge of the trajectories of B^H.

Lemma 5. *It holds*

$$\sup_{t\in[0,1]} \mathbb{E}|X(t) - \widetilde{X}^E(t)|^2 \leq c \cdot \Delta^2.$$

Proof: We have

$$\begin{aligned} X(t) - \widetilde{X}^E(t) &= Z(t) - \widetilde{Z}(t) + F(t) - \widetilde{F}(t) \\ &\quad + \int_0^t \sum_{i=0}^{n-1} (a(t_i, X(t_i)) - a(t_i, \widetilde{X}^E(t_i))) \cdot 1_{[t_i, t_{i+1})}(\tau)\, d\tau. \end{aligned}$$

By Lemma 1 and 4 and we get

$$\mathbb{E}|X(t) - \widetilde{X}^E(t)|^2 \leq \mathbb{E}\left| \int_0^t \sum_{i=0}^{n-1} (a(t_i, X(t_i)) - a(t_i, \widetilde{X}^E(t_i))) \cdot 1_{[t_i, t_{i+1})}(\tau)\, d\tau \right|^2 + c \cdot \Delta^2,$$

Moreover, by the Hölder inequality and (Ã2) it follows

$$\mathbb{E}|X(t) - \widetilde{X}^E(t)|^2 \leq c \int_0^t \sum_{i=0}^{n-1} \mathbb{E}|X(t_i) - \widetilde{X}^E(t_i)|^2 \cdot 1_{[t_i, t_{i+1})}(\tau)\, d\tau + c \cdot \Delta^2,$$

and

$$\sup_{0\leq s\leq t} \mathbb{E}|X(s) - \widetilde{X}^E(s)|^2 \leq c \int_0^t \sup_{0\leq s\leq \tau} \mathbb{E}|X(s) - \widetilde{X}^E(s)|^2\, d\tau + c \cdot \Delta^2,$$

respectively. Consequently, an application of Gronwalls lemma completes the proof.

$\square$

5.3. **Proof of Theorem 1.** By $X^{\text{lin}}_{h,n}$ we denote the piecewise linear interpolation of X based on the discretization $0 = t_{0,n} < t_{1,n} < \ldots < t_{n,n} = 1$ generated by the density function h, i.e.,

$$X^{\text{lin}}_{h,n}(t) = \frac{t_{j+1,n} - t}{t_{j+1,n} - t_{j,n}} \cdot X(t_{j,n}) + \frac{t - t_{j,n}}{t_{j+1,n} - t_{j,n}} \cdot X(t_{j+1,n})$$

for $t \in [t_{j,n}, t_{j+1,n}]$. We have

$$\sup_{n\in\mathbb{N}} \max_{i=1,\ldots,n} |t_{i,n} - t_{i-1,n}| \leq \|1/h\|_\infty \cdot n^{-1}.$$

Note that $\|1/h\|_\infty < \infty$, since the density function h is strictly positive. Hence it follows by Lemma 5

$$\left(\mathbb{E} \int_0^1 |X^{\text{lin}}_{h,n}(t) - \widehat{X}^E_{h,n}(t)|^2\, dt \right)^{1/2} \leq c \cdot \|1/h\|_\infty \cdot n^{-1}. \tag{8}$$

Furthermore we obtain due to Theorem 1 in Seleznjev (2000) and Proposition 1

$$\lim_{n\to\infty} n^H \cdot \left(\mathbb{E}\int_0^1 |X(t) - X^{\mathrm{lin}}_{h,n}(t)|^2\, dt\right)^{1/2} = \beta_H \cdot \left(\int_0^1 \sigma(t)^2 h^{-2H}(t)\, dt\right)^{1/2}.$$

Hence the assertion follows.

5.4. **Preliminaries for the Proof of Theorem 2.** Let

$$Y(t) = x_0 + \int_0^t a(\tau, X(\tau))\, d\tau - \int_0^t \sigma'(\tau) B^H(\tau)\, d\tau = X(t) - \sigma(t) B^H(t), \qquad t \in [0,1].$$

Moreover, define for a discretization $0 = t_0 < t_1 < \ldots < t_n = 1$ an approximation $\widehat{Y}$ of Y by

$$\widehat{Y}(t) = \widehat{X}^E(t) - \sigma(t_j) B^H(t_j) \frac{t_{j+1} - t}{t_{j+1} - t_j} - \sigma(t_{j+1}) B^H(t_{j+1}) \frac{t - t_j}{t_{j+1} - t_j} \tag{9}$$

for $t \in [t_j, t_{j+1}]$.

The asymptotic behavior of the eigenvalues λ_k, $k = 1, 2, \ldots$, of the Karhunen-Loéve expansion of $\sigma(t) B^H(t)$, $t \in [0,1]$, is given by

$$\lim_{k\to\infty} k^{2H+1} \cdot \lambda_k = ||\sigma||^2_{1/(H+1/2)} \cdot \frac{\Gamma(2H+1)\sin(\pi H)}{\pi^{1+2H}}.$$

See Propositions 2.2 and 2.3 in Nazarov and Nikitin (2003). Note that

$$\lim_{n\to\infty} n^{2H} \sum_{k>n} \lambda_k = ||\sigma||^2_{1/(H+1/2)} \cdot \frac{\Gamma(2H)\sin(\pi H)}{\pi^{1+2H}}. \tag{10}$$

5.5. **Proof of Theorem 2.** (i) We first establish the lower bound. Let $\widehat{X}_n$, $n = 1, 2, \ldots$, be an arbitrary sequence of approximations methods. Moreover fix $H < \alpha < 1$ and denote by $\widehat{Y}_n$ the approximation of Y given by (9), based on the discretization

$$t_{i,\lceil n^\alpha \rceil} = \frac{i}{\lceil n^\alpha \rceil}, \quad i = 0, 1, \ldots, \lceil n^\alpha \rceil. \tag{11}$$

Define

$$\widehat{V}_n = \widehat{X}_n - \widehat{Y}_n.$$

Hence we have

$$\left(\int_0^1 \mathbb{E}|X(t) - \widehat{X}_n(t)|^2\, dt\right)^{1/2} \geq \left(\int_0^1 \mathbb{E}|\sigma(t) B^H(t) - \widehat{V}_n(t)|^2\, dt\right)^{1/2} - A_n$$

with

$$A_n = \left(\int_0^1 \mathbb{E}|Y(t) - \widehat{Y}_n(t)|^2\, dt\right)^{1/2}.$$

Denoting by Y_n^{lin} the linear interpolation of Y based on the discretization (11), we get

$$A_n \leq \left(\int_0^1 \mathbb{E}|Y(t) - Y_n^{\mathrm{lin}}(t)|^2\, dt\right)^{1/2} + \left(\int_0^1 \mathbb{E}|Y_n^{\mathrm{lin}}(t) - \widehat{Y}_n(t)|^2\, dt\right)^{1/2}.$$

Since

$$Y_n^{\mathrm{lin}} - \widehat{Y}_n = X^{\mathrm{lin}}_{\mathrm{id},\lceil n_\alpha \rceil} - \widehat{X}^E_{\mathrm{id},\lceil n_\alpha \rceil}$$

and

$$\mathbb{E}|Y(t) - Y(s)|^2 \leq c \cdot |t-s|^2$$

for $s, t \in [0,1]$, it follows by (8)

$$A_n \leq c \cdot n^{-\alpha}.$$

Hence we obtain

$$\liminf_{n\to\infty} n^H \cdot \left(\int_0^1 \mathbb{E}|X(t) - \widehat{X}_n(t)|^2 \, dt \right)^{1/2}$$
$$\geq \liminf_{n\to\infty} n^H \cdot \left(\int_0^1 \mathbb{E}|\sigma(t) B^H(t) - \widehat{V}_n(t)|^2 \, dt \right)^{1/2}.$$

Setting

$$\widehat{V}_n^* = \widehat{V}_n / \sigma,$$

it remains to show that

$$\liminf_{n\to\infty} n^H \cdot \left(\int_0^1 \mathbb{E}|B^H(t) - \widehat{V}_n^*(t)|^2 \cdot \sigma(t)^2 \, dt \right)^{1/2} \geq \gamma_H \cdot ||\sigma||_{1/(H+1/2)}.$$

Note that $\widehat{V}_n^*$ is an approximation of B^H using at most $m(n) = n + \lceil n^\alpha \rceil$ bounded linear functionals that are applied to B^H. Moreover, approximating B^H in the mean square weighted L^2-norm with weight function σ^2 from finitely many bounded linear functionals, which are applied to B^H, defines a linear problem with a Gaussian measure in the sense of Traub *et al.* (1988), Chapter 6.5. Therefore sequential selection of the functionals does not help and it holds

$$\int_0^1 \mathbb{E}|B^H(t) - \widehat{V}_n^*(t)|^2 \cdot \sigma(t)^2 \, dt \geq \sum_{k>m(n)} \lambda_k,$$

see Traub *et al.* (1988), Chapter 6.5, and the references therein. Since $\lim_{n\to\infty} m(n)/n = 1$, the proof of the lower bound is completed by (10).

(ii) We have

$$\int_0^1 \mathbb{E}|B^H(t) - \widehat{V}_n^\dagger(t)|^2 \cdot \sigma(t)^2 \, dt = \sum_{k>n} \lambda_k,$$

for

$$\widehat{V}_n^\dagger = \sum_{k=1}^n \int_0^1 B^H(\tau)\sigma(\tau)\xi_k(\tau) \, d\tau \cdot \frac{\xi_k}{\sigma},$$

where $\xi_1, \xi_2, \ldots$ denote an orthonormal set of eigenfunctions corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots$ of the Karhunen-Loeve expansion of $\sigma(t)B^H(t)$, $t \in [0,1]$. Fix $H < \alpha < 1$ and set

$$\widehat{X}_n^\dagger = \widehat{Y}_n + \widehat{V}_{m^*(n)}^\dagger, \quad n = 1, 2, \ldots,$$

with $\widehat{Y}_n$ given as in (i) and $m^*(n) = n - \lceil n^\alpha \rceil$. For this sequence of approximations it follows

$$\lim_{n\to\infty} n^H \cdot e(\widehat{X}_n^\dagger) = \gamma_H \cdot \|\sigma\|_{1/(H+1/2)},$$

which completes the proof.

5.6. **Discussion of the Proof of Theorem 2.** The lower bound is established by reducing the approximation problem for the stochastic differential equation to a weighted approximation problem for B^H, for which the minimal error is strongly asymptotic equivalent to

$$\nu_n = \gamma_H \cdot \|\sigma\|_{1/(H+1/2)} \cdot n^{-H}$$

Since ν_n^2 is a convex sequence, i.e.,

$$\nu_n^2 \leq \frac{\nu_{n-1}^2 + \nu_{n+1}^2}{2},$$

and ν_n satisfies

$$\lim_{n\to\infty} \frac{\nu_n}{\nu_{n+1}} = 1,$$

varying cardinality does not help for the approximation of B^H. See Traub *et al.* (1988), Chapter 6.5, and the references therein. Thus, the lower bound in Theorem 2 also holds, if n sequentially selected bounded linear functionals on average are allowed.

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