Derivations of locally simple Lie algebras

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Abstract. Let $\mathfrak{g}$ be a locally finite Lie algebra over a field of characteristic zero which is a direct limit of finite-dimensional simple ones. In this note we study the Lie algebra of derivations of $\mathfrak{g}$. The main results are that each invariant symmetric bilinear form on $\mathfrak{g}$ is invariant under all derivations and that each such form defines a natural embedding $\operatorname{der}\mathfrak{g} \hookrightarrow \mathfrak{g}^*$. The latter embedding is used to determine $\operatorname{der}\mathfrak{g}$ explicitly for all locally finite split simple Lie algebras.
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Introduction

Let $\mathbb{F}$ be a field of characteristic zero and $\mathfrak{g}$ an $\mathbb{F}$-Lie algebra which is a directed union of simple finite-dimensional Lie algebras. This means that $\mathfrak{g} = \varinjlim \mathfrak{g}_j$ is the direct limit of a family $(\mathfrak{g}_j)_{j\in J}$ of finite-dimensional simple Lie algebras $\mathfrak{g}_j$ which are subalgebras of $\mathfrak{g}$ and the directed order $\leq$ on the index set J is given by $j \leq k$ if $\mathfrak{g}_j \leq \mathfrak{g}_k$.

In this note we study the Lie algebra of derivations of $\mathfrak{g}$. The main results are that each invariant symmetric bilinear form on $\mathfrak{g}$ is invariant under all derivations and that each such form defines a natural embedding $\operatorname{der}\mathfrak{g} \hookrightarrow \mathfrak{g}^*$. The latter embedding is used to determine $\operatorname{der}\mathfrak{g}$ explicitly for all locally finite split simple Lie algebras. According to [NS01], every such Lie algebra is isomorphic to a Lie algebra of the form $\mathfrak{sl}_I(\mathbb{F})$, $\mathfrak{o}_{I,I}(\mathbb{F})$ or $\mathfrak{sp}_I(\mathbb{F})$, which are defined as follows.

Let I be a set. We write $M_I(\mathbb{F}) \cong \mathbb{F}^{I\times I}$ for the set of all $I \times I$-matrices with entries in $\mathbb{F}$, $M_I(\mathbb{F})_{\rm rc-fin} \subseteq M_I(\mathbb{F})$ for the set of all $I \times I$-matrices with at most finitely many non-zero entries in each row and each column, and $\mathfrak{gl}_I(\mathbb{F})$ for the subspace consisting of all matrices with at most finitely many non-zero entries. Note that the column-finite matrices correspond to the linear endomorphisms of the free vector space $\mathbb{F}^{(I)}$ over I with respect to the canonical basis. The additional requirement of row-finiteness means that also the transpose matrix defines an endomorphism of $\mathbb{F}^{(I)}$, i.e., the adjoint endomorphism of the dual space $\mathbb{F}^I$ preserves the subspace $\mathbb{F}^{(I)}$.

The matrix product xy is defined if at least one factor is in $\mathfrak{gl}_I(\mathbb{F})$ and the other is in $M_I(\mathbb{F})$. In particular $\mathfrak{gl}_I(\mathbb{F})$ thus inherits the structure of locally finite Lie algebra via $[x,y] := xy - yx$ and

$$\mathfrak{sl}_I(\mathbb{F}) := \{x \in \mathfrak{gl}_I(\mathbb{F}) \colon \operatorname{tr} x = 0\}$$

is a hyperplane ideal which is a simple Lie algebra.

To define the Lie algebras $\mathfrak{o}_{I,I}(\mathbb{F})$ and $\mathfrak{sp}_I(\mathbb{F})$, we put $2I := I \dot\cup -I$, where $-I$ denotes a copy of I whose elements are denoted by $-i$, $i \in I$, and consider the $2I \times 2I$-matrices

$$Q_1 := \sum_{i\in I} E_{i,-i} + E_{-i,i} \quad \text{and} \quad Q_2 = \sum_{i\in I} E_{i,-i} - E_{-i,i}.$$

We then define

$$\mathfrak{o}_{I,I}(\mathbb{F}) := \{x \in \mathfrak{gl}_{2I}(\mathbb{F}) : x^\top Q_1 + Q_1 x = 0\} \quad \text{and} \quad \mathfrak{sp}_I(\mathbb{F}) := \{x \in \mathfrak{gl}_{2I}(\mathbb{F}) : x^\top Q_2 + Q_2 x = 0\}.$$

For these Lie algebras we show that

$$\mathrm{der}(\mathfrak{sl}_I(\mathbb{F})) \cong M_I(\mathbb{F})_{\mathrm{rc-fin}}/\mathbb{F}\mathbf{1},$$

$$\mathrm{der}(\mathfrak{o}_{I,I}(\mathbb{F})) \cong \{A \in M_I(\mathbb{F})_{\mathrm{rc-fin}} : x^\top Q_1 + Q_1 x = 0\},$$

and

$$\mathrm{der}(\mathfrak{sp}_I(\mathbb{F})) \cong \{A \in M_I(\mathbb{F})_{\mathrm{rc-fin}} : x^\top Q_2 + Q_2 x = 0\}.$$

We are grateful to Y. Yoshii whose question concerning the derivations of locally finite split simple Lie algebras inspired the present paper. In [MY05] only those derivations commuting with the standard Cartan subalgebras have been considered, and it has been shown that they can be written as brackets with infinite diagonal matrices. The result above describes all derivations.

The description of $\mathrm{der}\,\mathfrak{g}$ given in the present paper complements the description by H. Strade in [Str99, Th. 2.1]. It provides additional information that leads for the split case to the aforementioned description by infinite matrices.

There is also a description of the automorphisms of the infinite-dimensional locally finite split simple Lie algebras due to N. Stumme ([Stu01]) which is formally quite similar to our description of the derivations.

I. Derivations and projective limits

Lemma I.1. *Let $D \in \mathrm{der}\,\mathfrak{g}$. Then there exists for each $j \in J$ a unique element $x_j \in [\mathfrak{g}_j, \mathfrak{g}]$ with $D|_{\mathfrak{g}_j} = \mathrm{ad}\, x_j|_{\mathfrak{g}_j}$.*

Proof. Fix $j \in J$. First we observe that $\mathfrak{g}$ is a locally finite $\mathfrak{g}_j$-module, hence semisimple. It follows in particular that

$$\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}_j) \oplus [\mathfrak{g}_j, \mathfrak{g}] \tag{1.1}$$

and that $H^1(\mathfrak{g}_j, \mathfrak{g}) = \{0\}$ ([Ne03, Lemma A.3]). Since $D|_{\mathfrak{g}_j}$ is a 1-cocycle in $Z^1(\mathfrak{g}_j, \mathfrak{g})$, we see that there exists an $x \in \mathfrak{g}$ with $D|_{\mathfrak{g}_j} = -d_{\mathfrak{g}_j} x = \mathrm{ad}\, x|_{\mathfrak{g}_j}$. Clearly x is determined by this property up to an element of the centralizer $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}_j)$ of $\mathfrak{g}_j$ in $\mathfrak{g}$, so that (1.1) shows that x is unique if we require it to be contained in the complement $[\mathfrak{g}_j, \mathfrak{g}]$ of $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}_j)$. ∎

For $\mathfrak{g}_j \subseteq \mathfrak{g}_k$ we have $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}_k) \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}_j)$ and $[\mathfrak{g}_j, \mathfrak{g}] \subseteq [\mathfrak{g}_k, \mathfrak{g}]$. Let

$$p_{jk} : [\mathfrak{g}_k, \mathfrak{g}] \to [\mathfrak{g}_j, \mathfrak{g}]$$

denote the linear projection with kernel $\mathfrak{z}_{[\mathfrak{g}_k, \mathfrak{g}]}(\mathfrak{g}_j)$. Then the uniqueness assertion of Lemma I.1 implies that

$$p_{jk}(x_k) = x_j,$$

which leads to

$$\mathrm{der}(\mathfrak{g}) \cong \lim_{\longleftarrow} [\mathfrak{g}_j, \mathfrak{g}] = \Big\{(x_j)_{j\in J} \in \prod_{j\in J} [\mathfrak{g}_j, \mathfrak{g}] : p_{jk}(x_k) = x_j \quad \text{for} \quad j \leq k\Big\}. \tag{1.2}$$

Here we associate to $(x_j)_{j\in J} \in \lim\limits_{\longleftarrow} [\mathfrak{g}_j, \mathfrak{g}]$ the unique derivation D with

$$D|_{\mathfrak{g}_j} = \mathrm{ad}\, x_j|_{\mathfrak{g}_j}, \quad j \in J.$$

Proposition I.2. *Every invariant symmetric bilinear form κ on $\mathfrak{g}$ is invariant under all derivations of $\mathfrak{g}$.*

Proof. Let $D \in \mathrm{der}\,\mathfrak{g}$ and $x, y \in \mathfrak{g}$. Pick a subalgebra $\mathfrak{g}_j$ containing x, y and an element $z \in \mathfrak{g}$ with $D|_{\mathfrak{g}_j} = \mathrm{ad}\, z|_{\mathfrak{g}_j}$ (Lemma I.1). Then

$$\kappa(D.x, y) + \kappa(x, D.y) = \kappa([z, x], y) + \kappa(x, [z, y]) = 0.$$ ∎

We fix an non-degenerate invariant symmetric bilinear form κ on $\mathfrak{g}$ and write

$$\eta\colon \mathfrak{g} \to \mathfrak{g}^*, \quad \eta(x)(y) := \kappa(x,y)$$

for the corresponding equivariant embedding of $\mathfrak{g}$ into $\mathfrak{g}^*$.

For each $j \in J$ the decomposition $\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}_j) \oplus [\mathfrak{g}_j, \mathfrak{g}]$ is orthogonal because for $x \in \mathfrak{g}_j$, $y \in \mathfrak{g}$ and $z \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}_j)$ we have

$$\kappa([x,y],z) = -\kappa(y,[x,z]) = 0.$$

From $\mathfrak{g}_j \subseteq [\mathfrak{g}_j, \mathfrak{g}]$ we thus derive that for each element $x_k \in [\mathfrak{g}_k, \mathfrak{g}]$ with $p_{jk}(x_k) = x_j$ we have

$$\eta(x_k)\,|_{\mathfrak{g}_j} = \eta(x_j)\,|_{\mathfrak{g}_j}.$$

This shows that each tuple $(x_j)_{j\in J} \in \varprojlim\, [\mathfrak{g}_j, \mathfrak{g}]$ defines an element $\psi((x_j)) \in \mathfrak{g}^*$ satisfying

$$\psi((x_j))\,|_{\mathfrak{g}_j} = \eta(x_j)\,|_{\mathfrak{g}_j} \quad \text{for each} \quad j \in J.$$

Combining this observation with the isomorphism $\operatorname{der}\mathfrak{g} \cong \varprojlim\, [\mathfrak{g}_j, \mathfrak{g}]$, we see that for each derivation $D \in \operatorname{der}\mathfrak{g}$ there exists a unique element $\alpha_D \in \mathfrak{g}^*$ satisfying

$$\eta(D.x)(y) = \kappa(D.x,y) = \kappa([x_j,x],y) = \kappa(x_j,[x,y]) = \eta(x_j)([x,y]) = \alpha_D([x,y])$$

whenever $x \in \mathfrak{g}_j$. We conclude that

$$\eta(D.x) = \alpha_D \circ \operatorname{ad} x \quad \text{for all} \quad x \in \mathfrak{g}.$$

Theorem I.3. *Let I be a set, $M_I(\mathbb{F})_{\mathrm{rc-fin}}$ the Lie algebra of row- and column-finite $I\times I$-matrices and $\mathbf{1} = (\delta_{ij})$ the identity matrix. Then*

$$\operatorname{der}(\mathfrak{sl}_I(\mathbb{F})) \cong M_I(\mathbb{F})_{\mathrm{rc-fin}}/\mathbb{F}\mathbf{1},$$
$$\operatorname{der}(\mathfrak{o}_{I,I}(\mathbb{F})) \cong \{A \in M_I(\mathbb{F})_{\mathrm{rc-fin}} \colon x^\top Q_1 + Q_1 x = 0\},$$

and

$$\operatorname{der}(\mathfrak{sp}_I(\mathbb{F})) \cong \{A \in M_I(\mathbb{F})_{\mathrm{rc-fin}} \colon x^\top Q_2 + Q_2 x = 0\}.$$

Proof. First we consider $\mathfrak{g} := \mathfrak{sl}_I(\mathbb{F})$. Then $\kappa(x,y) := \operatorname{tr}(xy)$ is a non-degenerate invariant bilinear form on $\mathfrak{g}$ and the larger Lie algebra $\mathfrak{gl}_I(\mathbb{F})$ of all finite $I\times I$-matrices. From the trace form we obtain the isomorphism $\mathfrak{gl}_I(\mathbb{F})^* \cong \mathbb{F}^{I\times I} = M_I(\mathbb{F})$, and therefore

$$\mathfrak{g}^* \cong M_I(\mathbb{F})/\mathbb{F}\mathbf{1}.$$

Let $D \in \operatorname{der}\mathfrak{g}$. Then the linear functional $\alpha_D \in \mathfrak{g}^*$ can be written as $\alpha_D(x) = \operatorname{tr}(Ax)$ for a matrix $A \in M_I(\mathbb{F})$ which is unique up to $\mathbb{F}\mathbf{1}$. Then we have for $x,y \in \mathfrak{g}$:

$$\operatorname{tr}(D.x\cdot y) = \eta(D.x)(y) = \alpha_D([x,y]) = \operatorname{tr}(A[x,y]) = \operatorname{tr}([A,x]y). \tag{1.3}$$

Here we use that for each $x \in \mathfrak{gl}_I(\mathbb{F})$ and each matrix $A \in M_I(\mathbb{F})$ the commutator $[A,x] := Ax - xA$ is a well-defined element of $M_I(\mathbb{F})$ with $\operatorname{tr}(A[x,y]) = \operatorname{tr}([A,x]y)$, which has to be verified only for matrix units $E_{ij} \in \mathfrak{gl}_I(\mathbb{F})$:

$$\operatorname{tr}(A[E_{ij},E_{kl}]) = \operatorname{tr}(A(\delta_{jk}E_{il} - \delta_{li}E_{kj})) = \delta_{jk}a_{li} - \delta_{li}a_{jk} = \operatorname{tr}([A,E_{ij}]E_{kl}).$$

Equation (1.3) shows that

$$D.x = [A,x] \quad \text{for all} \quad x \in \mathfrak{sl}_I(\mathbb{F}). \tag{1.4}$$

The condition $[A, \mathfrak{sl}_I(\mathbb{F})] \subseteq \mathfrak{sl}_I(\mathbb{F})$ implies that for $i \neq j$ fixed $\delta_{jk}a_{li} - \delta_{li}a_{jk}$ is non-zero for only finitely many pairs (k,l), which implies that $A \in M_I(\mathbb{F})_{\mathrm{rc-fin}}$.

If, conversely, $A \in M_I(\mathbb{F})_{\mathrm{rc-fin}}$, then $[A, \mathfrak{gl}_I(\mathbb{F})] \subseteq \mathfrak{gl}_I(\mathbb{F})$, and

$$\operatorname{tr}([A,E_{ij}]) = a_{ji} - a_{ji} = 0$$

further implies that $[A, \mathfrak{gl}_I(\mathbb{F})] \subseteq \mathfrak{sl}_I(\mathbb{F})$. We conclude that $\operatorname{der}\mathfrak{g}$ can be identified with the quotient $M_I(\mathbb{F})_{\mathrm{rc-fin}}/\mathbb{F}\mathbf{1}$.

Now let $\mathfrak{g} \in \{\mathfrak{o}_{I,I}(\mathbb{F}), \mathfrak{sp}_I(\mathbb{F})\}$ and recall that this Lie algebra can be written as

$$\mathfrak{g} = \{x \in \mathfrak{gl}_{2I}(\mathbb{F}) \colon x^\top Q + Qx = 0\} = \{x \in \mathfrak{gl}_{2I}(\mathbb{F}) \colon x^\top = -QxQ^{-1}\}$$

for some $Q \in M_{2I}(\mathbb{F})$ with $Q^{-1} \in \{\pm Q\}$. Then we obtain with the trace form

$$\mathfrak{g}^* \cong \{x \in M_{2I}(\mathbb{F}) \colon x^\top = -QxQ^{-1}\} = \{x \in M_{2I}(\mathbb{F}) \colon x^\top Q + Qx = 0\},$$

and from that

$$\operatorname{der}(\mathfrak{g}) \cong \{x \in M_{2I}(\mathbb{F})_{\mathrm{rc-fin}} \colon x^\top Q + Qx = 0\}. \qquad \blacksquare$$

II. Invariant bilinear forms

In the preceding section we have used an invariant symmetric bilinear form κ on $\mathfrak{g}$ to embedd $\mathrm{der}\,\mathfrak{g}$ into $\mathfrak{g}^*$. In the present section we show that such a form always exists.

Lemma II.1. *There exists a non-degenerate invariant symmetric bilinear form* $\kappa\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{F}$.

Proof. Fix $j_0\in J$ and let κ_{j_0} denote the Cartan–Killing form of the simple Lie algebra $\mathfrak{g}_{j_0}$. Since κ_{j_0} is non-degenerate and in particular non-zero, there exists an element $x_0\in\mathfrak{g}_{j_0}$ with $\kappa_{j_0}(x_0,x_0)\neq 0$. For $j_0\leq j$ the adjoint representation of $\mathfrak{g}_{j_0}$ on $\mathfrak{g}_j$ is faithful, and the restriction of the Cartan–Killing form κ_j of $\mathfrak{g}_j$ to $\mathfrak{g}_{j_0}$ is the corresponding trace form, hence non-zero by Cartan's Criterion because $\mathfrak{g}_{j_0}$ is simple.

Let $\overline{\mathbb{F}}$ denote the algebraic closure of the field $\mathbb{F}$. We recall from the finite-dimensional theory that the form $\kappa_{j_0}^{\overline{\mathbb{F}}}$ obtained by scalar extension is the Killing form of the $\overline{\mathbb{F}}$-Lie algebra $\mathfrak{g}_{j_0}^{\overline{\mathbb{F}}}:=\overline{\mathbb{F}}\otimes_{\mathbb{F}}\mathfrak{g}_{j_0}$ and that any other invariant symmetric bilinear form with values in $\overline{\mathbb{F}}$ is a scalar multiple of this form. It follows that for $j\geq j_0$ the restriction of $\kappa_j^{\overline{\mathbb{F}}}$ to $\mathfrak{g}_{j_0}^{\overline{\mathbb{F}}}$ is a non-zero scalar multiple of $\kappa_{j_0}^{\overline{\mathbb{F}}}$, which implies that $\kappa_j(x_0,x_0)=\kappa_j^{\overline{\mathbb{F}}}(x_0,x_0)\neq 0$.

For $\mu_j:=\kappa_{j_0}(x_0,x_0)\kappa_j(x_0,x_0)^{-1}\in\mathbb{F}$ we thus obtain a unique non-degenerate invariant bilinear form $\mu_j\kappa_j$ on $\mathfrak{g}_j$ whose restriction to $\mathfrak{g}_{j_0}$ coincides with κ_{j_0}.

For $j\leq k$ we then have $\mu_j\kappa_j^{\overline{\mathbb{F}}}\,|_{\mathfrak{g}_{j_0}}=\mu_k\kappa_k^{\overline{\mathbb{F}}}\,|_{\mathfrak{g}_{j_0}}$, so that the uniqueness assertion on $\mathfrak{g}_j^{\overline{\mathbb{F}}}$ implies $\mu_j\kappa_j^{\overline{\mathbb{F}}}=\mu_k\kappa_k^{\overline{\mathbb{F}}}|_{\mathfrak{g}_j}$ and therefore $\mu_j\kappa_j=\mu_k\kappa_k\,|_{\mathfrak{g}_j}$. We conclude that the collection of the forms $(\mu_j\kappa_j)_{j\geq j_0}$ defines a symmetric invariant bilinear form κ on $\mathfrak{g}$. ∎

Lemma II.2. *If all the Lie algebras $\mathfrak{g}_j$ are split over $\mathbb{F}$, then κ is unique up to a scalar factor in $\mathbb{F}$. The same conclusion holds if $\mathbb{F}$ is algebraically closed.*

Proof. Let κ' be an $\mathbb{F}$-valued invariant symmetric bilinear form on $\mathfrak{g}$. Then there exists a $j\geq j_0$ (notation as in the proof of Lemma II.1) such that the restriction κ_j' of κ' to $\mathfrak{g}_j$ is non-zero, hence non-degenerate.

Since $\mathfrak{g}_j$ is split, its centroid $\mathrm{End}_{\mathrm{ad}\,\mathfrak{g}_j}(\mathfrak{g}_j)$ is $\mathbb{F}\mathrm{id}_{\mathfrak{g}_j}$, which implies that there exists a $\nu_j\in\mathbb{F}$ with $\kappa_j'=\nu_j\kappa$ on $\mathfrak{g}_j$. If $\mathbb{F}$ is algebraically closed, Schur's lemma also implies that the centroid of $\mathfrak{g}_j$ is $\mathbb{F}$, and the same conclusion holds.

Then, for each $k\geq j$ the two forms $\nu_j\kappa$ and κ' are non-zero invariant and symmetric on $\mathfrak{g}_k$, so that the assumption that $\mathfrak{g}_k$ is split implies that they coincide. We conclude that $\kappa'=\nu_j\kappa$. ∎

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