On the Use of Hypotheses in Cumulative Type Theory

Peter Zahn

ABSTRACT: Given a language of ramified cumulative type theory as introduced in (Zahn 2004). We shall construct and investigate an extension, $\mathcal{L}$, of it, which is a language of the same sort, but also containes sentences which express that certain sentences of $\mathcal{L}$ are deducible from others (hypotheses) by given rules. To this we introduce 'names' of terms and formulas of $\mathcal{L}$ and include them in $\mathcal{L}$. So in $\mathcal{L}$ we can not only use but also 'speak about' sentences of that language. Especially, by means of first order sentences we can speak about higher order sentences. Despite this possibility of 'reduction' of order, all sentences of $\mathcal{L}$ are non-circular. The considered deducibility-relations of sentences from others correspond to systems of labelled modal logic of types K_4 and G.

Motivation: In everyday speech and in empirical sciences one does necessarily not only assert established facts but also uses universal hypotheses or conjectures, which often do not even get cited. If A is the conjunction of all current hypotheses, we could use (assert) any sentence B as short for $A \to B$. Note that, for every admissible inference rule $\mathcal{B}_1, \ldots, \mathcal{B}_n \Rightarrow \mathcal{B}$, the rule $A \to \mathcal{B}_1, \ldots, A \to \mathcal{B}_n \Rightarrow A \to \mathcal{B}$ is also admissible. (Here, the $\mathcal{B}_i$'s and $\mathcal{B}$ represent sentence schemes, and '$\Rightarrow$' indicates steps of successive assertions.) But as soon as A becomes rejected, it becomes obviously unserviceable to assert sentences of the form $A \to B$ (or abbreviations for them). Accordingly, if A contains (probably) untrue hypotheses (such as simplifications of conjectures) we can instead of $A \to B$ better use the statement that B has been *deduced* from A and already justly asserted sentences of a given class, K, by the rules of classical logic (e.g.). This statement reminds of *necessity*, say "B is *necessary* with respect to (A, K)". (The set K should be chosen considering particular purposes. It might be a set of physical or medical sentences, e.g., that can possibly be verified.) Then the sets S_i of all sentences that are deducible at successive times t_i ($i = 0, 1, 2, \ldots$) form a monotonic increasing sequence $S_0, S_1, S_2, \ldots$.

1 A Language of Cumulative Type Theory

At first we incompletely sketch the construction of the language of a particular ramified cumulative type theory, which has been investigated in (Zahn 2004).

Assume that we already dispose of certain **elementary formulas** and terms, which are said to be **original terms**. All variables that occur in those formulas or terms are said to be of order 0. Let

$\mathcal{V}_0$ = set of all variables of order 0
$\mathcal{T}_{\text{Or}}$ = set of all original terms, $\mathcal{V}_0 \subset \mathcal{T}_{\text{Or}}$
$\mathcal{E}$ = set of all elementary formulas (to be considered).
$\mathcal{V}_0$ is permitted to contain variables of several sorts. (Of course, $\mathcal{V}_0$ is supposed to contain denumerably many variables of every of those sorts. Also $\mathcal{T}_{\text{Or}}$ and $\mathcal{E}$ are supposed to satisfy certain conditions.) Let **constants / sentences** be closed terms / formulas, respectively (i.e. without free occurring variables).

We shall introduce sets (or, if you like, 'properties' and 'relations-in-intension', given by certain constants) of order 1, whose elements are (tuples of) constants of order 0 (or objects denoted by them), sets of order 2, whose elements are (tuples of) constants of order 0 or 1, etc. So a set of order n contains only elements that have orders $< n$. However, a set of order n will also be said to have any order larger than n.

To this end we introduce the following sets of higher order terms and formulas:
$\mathcal{T}_n$ = set of all terms of order n,
$\mathcal{F}_n$ = set of all formulas of order n.
Here and in the following, m, n range over (signs of) ordinal numbers belonging to a given set Ω with $\mathbb{N} \subseteq \Omega$. We define
$\mathcal{C}_n \rightleftharpoons$ set of all constants belonging to $\mathcal{T}_n$,
$\overline{\mathcal{C}}_n \rightleftharpoons \bigcup_{j\in\mathbb{N}^+} \mathcal{C}_n^j$,
which is the set of all j-tuples $(c_1, \ldots, c_j)$ of constants $c_i \in \mathcal{C}_n$ with arbitrary length $j \in \mathbb{N}^+ \rightleftharpoons \mathbb{N} \setminus \{0\}$. Let also be given two disjunct denumerable sets $\mathcal{V}$ and $\overline{\mathcal{V}}$ of 'new' variables which do not occur in elements of $\mathcal{T}_{\text{Or}} \cup \mathcal{E}$. We use the elements of $\mathcal{V}$ as variables for elements of $\mathcal{C} \rightleftharpoons \bigcup_{n\in\Omega} \mathcal{C}_n$, i.e. for constants of arbitrary order, and the elements of $\overline{\mathcal{V}}$ as variables for elements of $\overline{\mathcal{C}} \rightleftharpoons \bigcup_{n\in\Omega} \overline{\mathcal{C}}_n$, i.e. for arbitrary tuples of constants. - Moreover, let
$\overline{\mathcal{T}}_n \rightleftharpoons \bigcup_{j\in\mathbb{N}^+} \mathcal{T}_n^j \cup \overline{\mathcal{V}}$.
So $\overline{\mathcal{C}}_n$ is the set of all closed elements of $\overline{\mathcal{T}}_n$.

We shall also use the following abbreviations: $\mathcal{F} \rightleftharpoons \bigcup_{n\in\Omega} \mathcal{F}_n$, $\mathcal{T} \rightleftharpoons \bigcup_{n\in\Omega} \mathcal{T}_n$, $\overline{\mathcal{T}} \rightleftharpoons \bigcup_{n\in\Omega} \overline{\mathcal{T}}_n$, and $\mathcal{A} \rightleftharpoons$ set of all sentences of $\mathcal{F}$.

As signs of the object language for $\mathcal{C}_n, \overline{\mathcal{C}}_n$, and $\in$ we use $C_n, \overline{C}_n$, and ε, respectively. For the present, $x, x_1, x_2, \ldots$ range over variables of $\mathcal{V}_0 \cup \mathcal{V}$, and $\overline{x}, \overline{y}$ over variables of $\overline{\mathcal{V}}$.

All elements of $\mathcal{C}_n \setminus \mathcal{C}_0$ are to be introduced as (signs of) subsets of $\bigcup_{m<n} \overline{\mathcal{C}}_m$. A constant of the form $\{\overline{x} \;\varepsilon\; \overline{C}_m : A(\overline{x})\}$ is to denote the set of all elements $c \in \overline{\mathcal{C}}_m$ satisfying $A(c)$. A sentence of the form $\exists x \;\varepsilon\; C_m . A(x)$ is to mean that there exists a constant c of order m satisfying $A(c)$. By this means, j-ary relations ($j \in \mathbb{N}^+$) can be described in the form

$$\{(x_1, \ldots, x_j) \;\varepsilon\; C_m^j : A(x_1, \ldots, x_j)\} \rightleftharpoons$$
$$\{\overline{x} \;\varepsilon\; \overline{C}_m : \exists x_1 \;\varepsilon\; C_m . \ldots \exists x_j \;\varepsilon\; C_m . (\overline{x} =_m (x_1, \ldots, x_j) \wedge A(x_1, \ldots, x_j))\}.$$

(To this end, the sign '$=_m$' must previously be introduced suitably.) - So we at first demand that

$$\begin{array}{rll} t \in \mathcal{T}_n & \text{if} & t \in \mathcal{T}_{\text{or}} \cup \mathcal{V}, \\ \{\overline{x}\ \varepsilon\ \overline{C}_m : F\} \in \mathcal{T}_n & \text{if} & F \in \mathcal{F}_n,\ m < n, \\ E \in \mathcal{F}_n & \text{if} & E \in \mathcal{E}, \\ (F \wedge G), (F \vee G) \in \mathcal{F}_n & \text{if} & F, G \in \mathcal{F}_n, \\ (\neg F) \in \mathcal{F}_n & \text{if} & F \in \mathcal{F}_n, \\ (\exists x\ \varepsilon\ C_m .\, F) \in \mathcal{F}_n & \text{if} & F \in \mathcal{F}_n,\ m < n, \\ (s\ \varepsilon\ t) \in \mathcal{F}_n & \text{if} & s \in \overline{\mathcal{T}}_n,\ t \in \mathcal{T}_n. \end{array}$$

Note that we need not deal with complicated types that include information about 'arities' of relations. So we may simply identify types with orders.

The latter and certain subsequently adduced demands can be formulated as formal rules - called $\mathcal{T}, \mathcal{F}$-rules - to construct terms and formulas of order n. But we need also 'semantical' stipulations. Accordingly, in (Zahn 2004) there is also introduced an assertion game, which contains certain '*primary rules*' to restrict assertions of sentences of arbitrary order. All inference rules of classical logic can be shown to be admissible in the '*classical game*' of assertion which is given by the agreement that a sentence may be asserted in this game if and only if the assertion of its double negation would not violate a primary rule. - Note that, for purposes of classical reasoning, the particles $\rightarrow$, $\leftrightarrow$, and $\forall$ can be defined by means of $\wedge, \neg$ and $\exists$.

For mathematical purposes we want also to dispose of sequences R of relations $R(0), R(1), R(2), \ldots \in \mathcal{C}_n$ satisfying

$$(\underline{c}, k)\ \varepsilon\ R(l) \leftrightarrow (\underline{c})\ \varepsilon\ \overline{C}_m\ \wedge\ k < l\ \wedge\ A((\underline{c}), k, R(k))$$

for all tuples $(\underline{c}) \equiv (c_1, \ldots, c_j)$ of constants and all $k, l \in \Omega$, if any formula $A(\overline{x}, \mu, z) \in \mathcal{F}_n$ and any ordinal $m < n$ are given. By this 'recursive characterization', $R(l)$ depends upon the relations $R(k)$ with numbers $k < l$ only. - We designate R by $(\mathrm{J}\overline{x}\ \varepsilon\ \overline{C}_m, \mu, z \colon A(\overline{x}, \mu, z))$. Accordingly, we demand:

$$(\mathrm{J}\overline{x}\ \varepsilon\ \overline{C}_m, \mu, z \colon F)(q) \in \mathcal{T}_n \quad \text{if} \quad F \in \mathcal{F}_n,\ q \in \mathcal{T}(\Omega),\ m < n,\ \mu \in \mathcal{V}(\Omega), z \in \mathcal{V}$$

where $\mathcal{T}(\Omega)$ ($\subseteq \mathcal{T}_{\text{or}}$) is a given set of terms whose substitution instances are elements of Ω, and $\mathcal{V}(\Omega) = \mathcal{V}_0 \cap \mathcal{T}(\Omega)$ is a set of variables for elements of Ω. ('J' is an 'induction operator') - Then it can be shown that if two formulas $A(\overline{x}), B(\overline{x}, \mu, z) \in \mathcal{F}_n$ and an order $m < n$ are given, there also exists a sequence of relations $S_0, S_1, S_2, \ldots \in \mathcal{C}_n$ satisfying

$$\begin{array}{rcl} c\ \varepsilon\ S_0 & \leftrightarrow & c\ \varepsilon\ \overline{C}_m\ \wedge\ A(c) \\ c\ \varepsilon\ S_{k+1} & \leftrightarrow & c\ \varepsilon\ \overline{C}_m\ \wedge\ B(c, k, S_k) \end{array}$$

for all $c \in \overline{\mathcal{C}}$ and all $k \in \mathbb{N}$.

We want to introduce equations $x = y$ such that all formulas $A(x)$ of arbitrary order are invariant under $(=)$, i.e. satisfy $c = d \wedge A(c) \rightarrow A(d)$ for all constants c, d.

To this end, equal constants must especially have the same orders, and equal sets must contain the same elements:

$$\begin{aligned} c = d \;&\rightarrow\; \forall\mu\,\varepsilon\, C_0.\,(c\,\varepsilon\, C_\mu \;\leftrightarrow\; d\,\varepsilon\, C_\mu) \\ c = d \;\wedge\; \neg\,(c\,\varepsilon\, C_0) \;&\rightarrow\; c \subseteq d \;\wedge\; d \subseteq c \end{aligned}$$

where $\mu \in \mathcal{V}(\Omega)$ (again), and $c \subseteq d$ means that c is a subset of d (see below). Since the formulas $c\,\varepsilon\, C_\mu$ and $c \subseteq d$ should belong to the object language considered, we demand and define the following (where $\exists\overline{x}\,\varepsilon\, t.\, F$ is to be read as "For some $\overline{x}$, $\overline{x}\,\varepsilon\, t$ and F"):

$$\begin{aligned} (t\,\varepsilon\, C_q) \in \mathcal{F}_n \quad &\text{if} \quad t \in \mathcal{T}_n,\; q \in \mathcal{T}(\Omega) \\ (\exists\overline{x}\,\varepsilon\, t.\, F) \in \mathcal{F}_n \quad &\text{if} \quad t \in \mathcal{T}_n,\; F \in \mathcal{F}_n \\ \forall\overline{x}\,\varepsilon\, s.\, F \quad &\rightleftharpoons \quad \neg\,\exists\overline{x}\,\varepsilon\, s.\,\neg F \\ s \subseteq t \quad &\rightleftharpoons \quad \forall\overline{x}\,\varepsilon\, s.\,\overline{x}\,\varepsilon\, t \;\wedge\; \neg\,(s\,\varepsilon\, C_0) \;\wedge\; \neg\,(t\,\varepsilon\, C_0). \end{aligned}$$

Notice, however, that if q (is or) contains a variable, we do not rank C_q with the terms of $\mathcal{T}$.

Now we presuppose: Let $(=_0)$ be an equivalence relation on $\mathcal{C}_0$ (which has already been introduced and is suitable for certain purposes). Assume that all terms of $\mathcal{T}_{\mathrm{or}}$ and all formulas of $\mathcal{E}$ are invariant under $(=_0)$. For terms s, t of any order we define

$$\begin{aligned} s \sim t \;&\rightleftharpoons\; \forall\mu\,\varepsilon\, C_0.\,(s\,\varepsilon\, C_\mu \;\leftrightarrow\; t\,\varepsilon\, C_\mu) \\ s = t \;&\rightleftharpoons\; s =_0 t \;\vee\; (s \subseteq t \;\wedge\; t \subseteq s \;\wedge\; s \sim t). \end{aligned}$$

Of course, we demand that

$$(s =_0 t) \in \mathcal{F}_n \quad \text{if} \quad s, t \in \mathcal{T}_n.$$

Then it can be shown that all formulas of $\mathcal{F}$ are invariant under $(=)$.

The 'type-free' relations $(\subseteq)$, $(\sim)$, and $(=)$ are definable in our object language but they are neither elements of $\mathcal{C}$ nor elements of elements of $\mathcal{C}$.

Given a formula $A(x)$, a tuple $c \equiv (c_1, \ldots, c_j) \in \overline{\mathcal{C}}_m$ of constants, and some $i = 1, \ldots, j$. Then $A(c_i)$ means that the i^{th} component of c satisfies $A(x)$. Since our object language also contains variables $\overline{y}$ for such tuples c of constants, we postulate, in addition, that the object language contains a formula expressing that the i^{th} component of any given value of $\overline{y}$ belongs to $\mathcal{C}_m$ and satisfies $A(x)$. For that formula we take $\exists x\,\varepsilon\,\pi_m(\overline{y}, i).\, A(x)$ (with π for "projection"). Generalizing we demand

$$(\exists x\,\varepsilon\,\pi_m(s, p).\, F) \in \mathcal{F}_n \;\text{ if }\; m < n,\; s\,\varepsilon\,\overline{\mathcal{T}}_n,\; p \in \mathcal{T}(\mathbb{N}^+),\; F \in \mathcal{F}_n$$

where $\mathcal{T}(\mathbb{N}^+)$ $(\subseteq \mathcal{T}_{\mathrm{or}})$ is a given set of terms (inclusive of variables) whose substitution instances are elements of $\mathbb{N}^+$. Then all sentences of the form

$$\exists x\,\varepsilon\,\pi_m((c_1, \ldots, c_j), i).\, A(x) \;\leftrightarrow\; c_i\,\varepsilon\, C_m \wedge A(c_i)$$

$(i = 1, \ldots, j)$ may be asserted in the correspondingly stipulated classsical game.

In the above definition of j-ary relations we have already used the following definition: For $s, t \in \overline{\mathcal{T}}$,

$$s =_m t \;\; \rightleftharpoons \;\; s, t \,\varepsilon\, \overline{C}_m \\ \wedge\, \forall \kappa \,\varepsilon\, C_0.\, \forall x \,\varepsilon\, C_m.\, \big(\exists y \,\varepsilon\, \pi_m(s, \kappa).\, x = y \;\leftrightarrow\; \exists z \,\varepsilon\, \pi_m(t, \kappa).\, x = z\big).$$

where $\kappa \in \mathcal{V}_0$ is a variable for elements of $\mathbb{N}^+$, and $x, y, z \in \mathcal{V}$ are different variables that do not occur in s or t. - For all $a \equiv (a_1, \ldots, a_j)$ and $b \equiv (b_1, \ldots, b_j)$ we obtain:

$$a =_m b \;\leftrightarrow\; a_1 = b_1 \,\varepsilon\, C_m \;\wedge\; \ldots \;\wedge\; a_j = b_j \,\varepsilon\, C_m.$$

2 Deducibility of sentences from hypotheses considered modal-logically

We have just sketched a comprehensive language of cumulative type theory. We shall construct and investigate an extension, $\mathcal{L}$, of it, which is a language of the same sort, but also contains sentences which express that certain sentences of $\mathcal{L}$ are deducible from others by given rules. To this we shall introduce 'names' of sentences of $\mathcal{L}$ and include them in $\mathcal{L}$. So in $\mathcal{L}$ we can not only use but also 'speak about' sentences of that language.

But in this section we only deal with the deducibility of sentences from 'hypotheses' by given axioms and rules. (Later we shall show how we can formulate that deducibility in $\mathcal{L}$.) Assume that $A \equiv A_1 \wedge \ldots \wedge A_j$ is the conjunction of all 'current hypotheses'. We shall introduce sentences of the form $A \triangleright B$ which are to mean that B is *deducible* from A and certain additional axioms by certain rules. The system of those axioms and rules will be denoted by $\mathcal{S}$.

Let now be given a language of cumulative type theory as described in section 1. Define: $\mathcal{W} \rightleftharpoons \mathcal{V}_0 \cup \mathcal{V}$ and $\overline{\mathcal{W}} \rightleftharpoons \mathcal{W} \cup \overline{\mathcal{V}}$. We extend $\mathcal{F}$ as follows: Let $\mathcal{F}^+$ $(\supset \mathcal{F})$ be the set of all formulas constructible by the following six rules (where '$\Rightarrow$' indicates the steps of construction):

$$\begin{array}{rcll} & \Rightarrow & F, & \text{if } F \in \mathcal{F} \\ F & \Rightarrow & (\neg F),\ (\exists x\, F), & \text{if } x \in \overline{\mathcal{W}} \\ F, G & \Rightarrow & (F \wedge G),\ (F \vee G),\ (F \triangleright G). & \end{array}$$

In the following, x, y, z range over $\overline{\mathcal{W}}$, $\overline{x}$ over $\overline{\mathcal{V}}$, and $\underline{y}, \underline{z}$ over all lists $z_1, \ldots, z_k$ of variables $z_i \in \overline{\mathcal{W}}$ with arbitrary length $k \in \mathbb{N}$; t ranges over $\mathcal{T} \cup \overline{\mathcal{T}}$; m over Ω; F, G, H over $\mathcal{F}^+$; and A, B, C over $\mathcal{A}^+$, i.e. the set of all sentences belonging to $\mathcal{F}^+$. - Let $\forall z_1, \ldots, z_k\, F$ stand for $\forall z_1 \ldots \forall z_k\, F$ (or, in case $k = 0$, for F). Accordingly, we let $\forall \underline{y}$ and $\forall \underline{z}$ range over all prefices of the form $\forall z_1 \ldots \forall z_k$ with $k \geq 0$.

Note that the quantifications in $\exists x\, F$ and in $\forall \underline{z}\, F$ are not restricted to any order, and that formulas of $\mathcal{F}^+ \setminus \mathcal{F}$ do not occur in terms of $\mathcal{T}$.

Now we assign the **axioms** of $\mathcal{S}$ under 1. - 4.:

1. Let *PL* be the 'propositional language' whose formulas are as usual composed of 'propositional variables' and $\perp$ ($\rightleftharpoons\ 0 = 1$) by means of $\wedge, \vee, \neg, \exists$, and $(,)$. Let *TAU* be a particular finite set of tautologies that are formulated in *PL*. *TAU* with the rule of *modus ponens* is assumed to be 'complete'. As axioms of $\mathcal{S}$ we take all formulas of the shape $\forall \underline{z}\, F$ where F is a 'substitution instance' of an element of *TAU*.

Explanations. By a 'substitution instance' of a formula, p, of *PL* we understand a *formula* that results from p by replacing all occurrences of propositional variables with *formulas* of $\mathcal{F}^+$. But by a sustitution instance of a formula, F, of $\mathcal{F}^+$ we (as usual) understand a *sentence* that results from F by replacing all free occurrences of variables with values of them, which are constants. - The axioms of $\mathcal{S}$, which have the shape $\forall \underline{z}\, F$, are permitted to contain further free variables. (This will in section 3 be convenient for including these axioms in $\mathcal{F}$.) In the following, $\mathrm{Fr}(t, x, F)$ is to mean that t is free for x in F, and $\mathrm{N}(y, G)$ is to mean that y does not occur free in G.

2. Let all formulas of the following shapes be axioms of $\mathcal{S}$:

$$\begin{array}{rcll}
 & & \forall \underline{z}\,(t = t) & \text{with } t \in \mathcal{T}; \\
\forall \underline{z}\,(x = t & \rightarrow & (F \leftrightarrow F^x_t)) & \text{with } \mathrm{Fr}(t, x, F),\ t \in \mathcal{T},\ x \in \mathcal{W}; \\
\forall \underline{z}\,(F^x_t & \rightarrow & \exists x\, F) & \text{with } \mathrm{Fr}(t, x, F); \\
\forall \underline{z}\,(\forall y\,(F^x_y \rightarrow H) & \rightarrow & (\exists x\, F \rightarrow H)) & \text{with } \mathrm{Fr}(y, x, F),\ \mathrm{N}(y, (\exists x\, F \rightarrow H)); \\
\forall \underline{z}\,(\exists x\, \varepsilon\, C_m.\, F & \leftrightarrow & \exists x\,(x\, \varepsilon\, C_m \wedge F)) & \text{with } x \in \mathcal{W},\ F \in \mathcal{F}; \\
\forall \underline{z}\,(\exists \overline{x}\, \varepsilon\, t.\, F & \leftrightarrow & \exists \overline{x}\,(\overline{x}\, \varepsilon\, t \ \wedge\ F)) & \text{with } t \in \mathcal{T},\ F \in \mathcal{F}; \\
\forall \underline{z}\,(s\, \varepsilon\, \{\overline{x}\, \varepsilon\, \overline{C}_m \colon F\} & \leftrightarrow & s\, \varepsilon\, \overline{C}_m \ \wedge\ F^{\overline{x}}_s) & \text{with } \mathrm{Fr}(s, \overline{x}, F),\ s \in \overline{\mathcal{T}},\ F \in \mathcal{F};
\end{array}$$

$$\forall \underline{z}\,\big(\, (\underline{s}, p)\, \varepsilon\, T(q) \ \leftrightarrow\ (\underline{s})\, \varepsilon\, \overline{C}_m \ \wedge\ p < q \ \wedge\ F((\underline{s}), p, T(p))\,\big)$$

with $(\underline{s}) \in \overline{\mathcal{T}}$, $p, q \in \mathcal{T}(\Omega)$, $T \equiv (\mathrm{J}\overline{x}\, \varepsilon\, \overline{C}_m, \mu, z :\ F(\overline{x}, \mu, z))$, $\mu \in \mathcal{V}(\Omega)$, $z \in \mathcal{V}$, $\mathrm{Fr}((\underline{s}), \overline{x}, F(\ldots))$, and $\mathrm{Fr}(p, \mu, F(\ldots))$;

$$\begin{array}{l}
\forall \underline{z}\,(\exists x\, \varepsilon\, \pi_m((t_1, \ldots, t_j), i).\, F \ \leftrightarrow\ t_i\, \varepsilon\, C_m \ \wedge\ F^x_{t_i}), \\
\forall \underline{z}\,(\exists x\, \varepsilon\, \pi_m((t_1, \ldots, t_j), p).\, F \ \rightarrow\ p = 1 \ \vee\ \ldots\ \vee\ p = j)
\end{array}$$

with $i = 1, \ldots, j$ and $p \in \mathcal{T}(\mathbb{N}^+)$, respectively, $x \in \mathcal{W}$, $t_1, \ldots, t_j \in \mathcal{T}$, and $F \in \mathcal{F}$.

3. The following axiom schemes, which we include in $\mathcal{S}$, concern the connective $\triangleright$:

$$\begin{array}{cr}
\forall \underline{z}\,(F \triangleright F) & [1] \\
\forall \underline{z}\,(F \triangleright G \ \wedge\ G \triangleright H \ \rightarrow\ F \triangleright H) & [2] \\
\forall \underline{z}\,(F \triangleright \forall \underline{y}\,(G \rightarrow H) \ \rightarrow\ (F \triangleright \forall \underline{y}\, G \ \rightarrow\ F \triangleright \forall \underline{y}\, H)) & [3] \\
\forall \underline{z}\,(F \triangleright \forall \underline{y}\, H \ \rightarrow\ F \triangleright \forall \underline{y}\,(G \triangleright H)) & [4].
\end{array}$$

Note. [3] and [4] remind of the following axiom schemes of labelled modal logic: $\boxed{\text{i}}\,(A \to B) \to (\boxed{\text{i}}\,A \to \boxed{\text{i}}\,B)$ and $\boxed{\text{i}}\,A \to \boxed{\text{i}}\,\boxed{\text{j}}\,A$, respectively, which are in case $i = j$ (or without labelles i, j) usually designated by (K) and (4) (cf. (Popkorn 1994), chap. 2).

4. As axioms of $\mathcal{S}$ we can (for certain purposes) also take other formulas whose substitution instances may be asserted due to certain rules of assertion, especially formulas of the shape $\forall \underline{z}\,(E_1 \wedge \ldots \wedge E_n \to E)$ with $E_1, \ldots, E_n, E \in \mathcal{E}$, where $E_1, \ldots, E_n \Rightarrow E$ (with metavariables for certain elements of $\mathcal{C}_0$ in place of variables) is an agreed rule of assertion (cf. (Zahn 2004, section 0)).

As **rules** of $\mathcal{S}$ we take

$$\begin{array}{rcll} \forall \underline{z}\, F,\ \forall \underline{z}\,(F \to G) & \Rightarrow & \forall \underline{z}\, G & (\textit{modus ponens}) \\ \forall \underline{z}\, H & \Rightarrow & \forall \underline{z}\,(G \rhd H) & (\textit{necessitation}). \end{array}$$

(Special cases of these rules are $F, F \to G \Rightarrow G$ and $H \Rightarrow G \rhd H$.)

Let $\mathcal{S} \vdash B$ be short for "B is deducible in $\mathcal{S}$ (i.e. from the axioms of $\mathcal{S}$ by the rules of $\mathcal{S}$)", and $\mathcal{S}(A) \vdash B$ for "B is deducible in $\mathcal{S}(A)$ (i.e. from A and the axioms of $\mathcal{S}$ by the rules of $\mathcal{S}$)." We now interprete $A \rhd B$ as $\mathcal{S}(A) \vdash B$, i.e. we fix the 'primary rule' (cf. section 1): Assert $A \rhd B$ only if $\mathcal{S}(A) \vdash B$ has been asserted. (This rule is invertible, since we do not restrict the assertion of $A \rhd B$ by other rules.) But all sentences of $\mathcal{A}^+$ are to be understood classically, i.e. with respect to the classical game of assertion (mentioned in section 1).

Notes. We have: $\mathcal{S} \vdash \forall \underline{z}\, F$ if and only if $\mathcal{S} \vdash F$. This can be shown by induction on $\mathcal{S}$ (i.e. on the number of corresponding deduction steps). The same also holds for $\mathcal{S}(A)$ instead of $\mathcal{S}$. Moreover, we have $\mathcal{S} \vdash \forall \underline{z}\, \big(F \rhd \forall x\, G \to \forall x\,(F \rhd G)\big)$; this reminds of the inverse Barcan formula, $\boxed{\text{i}}\,\forall x\, G \to \forall x\,\boxed{\text{i}}\,G$.

2.1 Proposition: If $\mathcal{S} \vdash F$, then all substitution instances of F are true (assertible).

Proof (by a well-known model, see (Smullyan 1987), chap. 26, proof of Theorem 1, e.g.): At first we show that all substitution instances of the axioms [1] - [4] are true. To this, we consider any substitution instances A, B, and C of F, G, and H, respectively.
Ad [1]: Let A be said to be deducible from itself. So $\mathcal{S}(A) \vdash A$.
Ad [2]: If $\mathcal{S}(A) \vdash B$ and $\mathcal{S}(B) \vdash C$, then $\mathcal{S}(A) \vdash C$.
Ad [3]: Let $A \rhd \forall \underline{y}\,(B\underline{y} \to C\underline{y})$ be a substitution instance of $F \rhd \forall \underline{y}\,(G \to H)$.
If $\mathcal{S}(A) \vdash \forall \underline{y}\,(B\underline{y} \to C\underline{y})$ and $\mathcal{S}(A) \vdash \forall \underline{y}\, B\underline{y}$, then, by *modus ponens*, $\mathcal{S}(A) \vdash \forall \underline{y}\, C\underline{y}$.
Ad [4]: If $\mathcal{S}(A) \vdash \forall \underline{y}\, C\underline{y}$, then, by *necessitation*, $\mathcal{S}(A) \vdash \forall \underline{y}\,(B\underline{y} \rhd C\underline{y})$. □

Also all substitution instances of the residual axioms of $\mathcal{S}$ are true. Now we easily obtain 2.1 by induction on $\mathcal{S}$. To this note the following: To obtain $\mathcal{S} \vdash \forall \underline{z}\,(G \rhd H)$ by *necessitation*, we must previously have $\mathcal{S} \vdash \forall \underline{z}\, H$. But then, for any substitution instance $B \rhd C$ of $G \rhd H$, C is deducible in $\mathcal{S}$ and so in $\mathcal{S}(B)$ so that $B \rhd C$ is true. Thus every substitution instance of $\forall \underline{z}\,(G \rhd H)$ is true. □

2.2 Proposition: For all $A, B \in \mathcal{A}^+$, $\mathcal{S}(A) \vdash B$ if and only if $\mathcal{S} \vdash A \triangleright B$.

Proof: Let $A \in \mathcal{A}^+$. Due to 2.1 it suffices to prove that, for any $H \in \mathcal{F}^+$, if $\mathcal{S}(A) \vdash H$, then $\mathcal{S} \vdash A \triangleright H$. We do this by induction on $\mathcal{S}(A)$. Let $\mathcal{S}(A) \vdash H$. If H is an axiom of $\mathcal{S}$, then $\mathcal{S} \vdash A \triangleright H$ by *necessitation.* If $H \equiv A$, then $\mathcal{S} \vdash A \triangleright H$ by axiom [1]. - If $H \equiv \forall \underline{z}\, G$ has been deduced in $\mathcal{S}(A)$ by applying *modus ponens* from the premises $\forall \underline{z}\, F$ and $\forall \underline{z}\, (F \to G)$, say, then we may use the induction hypotheses that $\mathcal{S} \vdash A \triangleright \forall \underline{z}\, F$ and $\mathcal{S} \vdash A \triangleright \forall \underline{z}\, (F \to G)$. Then, by axiom [3] and *modus ponens*, $\mathcal{S} \vdash A \triangleright \forall \underline{z}\, G$. - If $H \equiv \forall \underline{z}\, (F \triangleright G)$ has been deduced in $\mathcal{S}(A)$ by applying *necessitation* from the premise $\forall \underline{z}\, G$, then, by induction hypothesis, we have $\mathcal{S} \vdash A \triangleright \forall \underline{z}\, G$ and so, by [4] and *modus ponens*, $\mathcal{S} \vdash A \triangleright \forall \underline{z}\, (F \triangleright G)$. ▭

3 A language of cumulative type theory with quotation marks

In the following we construct a language of cumulative type theory that contains 'names' of sentences. By means of first order sentences of that language we can also speak about higher order sentences of it. Despite this possibility of 'reduction' of order, all sentences of that language are non-circular.

In the context of section 1 we say that the 'language' $\mathcal{T}, \overline{\mathcal{T}}, \mathcal{F}$ results from $\mathcal{T}_{\mathrm{Or}}, \mathcal{E}, \mathcal{V}_0, \mathcal{V}, \overline{\mathcal{V}}$ by the $\mathcal{T}, \mathcal{F}$-rules. Now we presuppose that a given language $\mathcal{T}^\circ, \overline{\mathcal{T}}^\circ, \mathcal{F}^\circ$ results from $\mathcal{T}^\circ_{\mathrm{Or}}, \mathcal{E}^\circ, \mathcal{V}^\circ_0, \mathcal{V}, \overline{\mathcal{V}}$ by those rules. We shall construct extensions $\mathcal{T}_{\mathrm{Or}} \supset \mathcal{T}^\circ_{\mathrm{Or}}$, $\mathcal{E} \supseteq \mathcal{E}^\circ$, and $\mathcal{V}_0 \supset \mathcal{V}^\circ_0$ such that the language $\mathcal{L}$ (i.e. $\mathcal{T}, \overline{\mathcal{T}}, \mathcal{F}$) which results from $\mathcal{T}_{\mathrm{Or}}, \mathcal{E}, \mathcal{V}_0, \mathcal{V}, \overline{\mathcal{V}}$ contains sentences expressing that $\mathcal{S}(A) \vdash B$, for $\mathcal{S}$ as above and any sentences A, B of $\mathcal{A}^+$. *Note that $\mathcal{L}$ is a language of cumulative type theory.*

Let $\mathcal{V}_{\mathcal{N}}$ be a denumerable set of 'new variables' that do not occur in the elements of $\mathcal{T}^\circ \cup \overline{\mathcal{T}}^\circ \cup \mathcal{F}^\circ$. ($\mathcal{N}$ will be defined below.) Let the set Σ° contain all atomic symbols occurring in elements of $\mathcal{T}^\circ \cup \overline{\mathcal{T}}^\circ \cup \mathcal{F}^\circ \cup \mathcal{V}_{\mathcal{N}}$, and the additional symbols $\triangleright$, N, Fr, and Sub. Let the symbol set Σ result from Σ° by adding the new symbols $\lceil\alpha\rceil, \lceil\lceil\alpha\rceil\rceil, \lceil\lceil\lceil\alpha\rceil\rceil\rceil, \ldots$, for every $\alpha \in \Sigma^\circ$. The symbols $\lceil, \rceil$ are supposed not to belong to Σ°. We do also *not* include them in Σ. So we may consider all elements of Σ as *atomic* symbols.

Let Σ^* be the set of all strings $\alpha_1 \ldots \alpha_j$ ($j \geq 0$) of symbols $\alpha_i \in \Sigma$. So, especially, the 'empty word' belongs to Σ^* (case $j = 0$). For $\alpha_1, \ldots, \alpha_j \in \Sigma$ ($j \geq 0$) we define

$$\lceil\alpha_1\alpha_2 \ldots \alpha_j\rceil \rightleftharpoons \lceil\alpha_1\rceil\lceil\alpha_2\rceil \ldots \lceil\alpha_j\rceil.$$

Let this figure be said to be the **name** of $\alpha_1\alpha_2 \ldots \alpha_j$. Especially, $\lceil\,\rceil$ stands for the empty word, which is its own name. Let $\mathcal{N}$ be the set of all such names of elements of Σ^*. We shall use the elements of $\mathcal{V}_{\mathcal{N}}$ as variables for (all or particular) elements of $\mathcal{N}$. All variables occurring in an element of $\mathcal{N}$ are considered to be *bound* (by the 'quotation marks' $\lceil, \rceil$).

Let $\mathcal{T}(\mathcal{N})$ be the set of all figures of the shape

$$\lceil S_0 \rceil X_{11} \ldots X_{1k_1} \lceil S_1 \rceil X_{21} \ldots X_{2k_2} \lceil S_2 \rceil \ldots X_{j1} \ldots X_{jk_j} \lceil S_j \rceil$$

with $S_i \in \Sigma^* \backslash \{\lceil \, \rceil\}$, variables $X_{ik} \in \mathcal{V}_\mathcal{N}$, $j \geq 0$, and $k_1, k_2, \ldots, k_j \geq 1$, - and of all figures which result from them by omitting $\lceil S_0 \rceil$ or $\lceil S_j \rceil$ or both.

Note. We have $\mathcal{N} \cup \mathcal{V}_\mathcal{N} \subseteq \mathcal{T}(\mathcal{N})$. If we replace all free occurrences of a variable in an element of $\mathcal{T}(\mathcal{N})$ by an element of $\mathcal{T}(\mathcal{N})$ - or, especially, by the empty word -, then we again obtain an element of $\mathcal{T}(\mathcal{N})$.

$\mathcal{V}_0^\circ$ ($\subseteq \mathcal{T}_{\mathrm{or}}^\circ$, see above) is assumed to be a set of variables (of several given sorts) for certain constants belonging to $\mathcal{T}_{\mathrm{or}}^\circ$. Let $\mathcal{V}_0 \rightleftharpoons \mathcal{V}_0^\circ \cup \mathcal{V}_\mathcal{N}$, $\mathcal{T}_{\mathrm{or}} \rightleftharpoons \mathcal{T}_{\mathrm{or}}^\circ \cup \mathcal{T}(\mathcal{N})$, and let $\mathcal{E}$ contain all elements of $\mathcal{E}^\circ$ and certain further formulas, which we shall specify below. Let, as announced, $\mathcal{T}, \overline{\mathcal{T}}, \mathcal{F}$ result from $\mathcal{T}_{\mathrm{or}}, \mathcal{E}, \mathcal{V}_0, \mathcal{V}, \overline{\mathcal{V}}$ by the $\mathcal{T}, \mathcal{F}$-rules.

Examples of tuples containing the empty word are: $(\,)$, $(t,\,)$, $(\,,\,)$, $(\,,\,,t)$, for any $t \in \mathcal{T}$. Such tuples are particular elements of $\overline{\mathcal{T}}$.

Given a system $\mathcal{S}$ of axioms and rules as indicated in section 2. We want to define sets $R_0, R_1, R_2, \ldots \in \mathcal{C}_1$ such that, for all $n \in \mathbb{N}$, R_n is the set of all names of sentences that are deducible in $\mathcal{S}$ by $\leq n$ steps of deduction.

For any $\mathcal{U} \subseteq \Sigma^*$ let $\mathcal{U}^{\lceil,\rceil}$ be the set of all names $\lceil u \rceil$ of elements u of $\mathcal{U}$. (So we have $\mathcal{U}^{\lceil,\rceil} \subseteq \Sigma^{*\lceil,\rceil} = \mathcal{N}$.) The sign '$=_0$' between elements of $\mathcal{N}$ is to mean their literal equality.

So long we have used the letters $x, y, \overline{x}, s, t, m, F, G, H, \ldots$ as metavariables. However, to make the following definitions easier to understand, we now use these and some other letters to indicate particular variables of $\mathcal{V}_\mathcal{N}$ that range over certain subsets of $\mathcal{N}$. That is, we provisionally let w ($\in \mathcal{V}_\mathcal{N}$) range over $\mathcal{W}^{\lceil,\rceil}$, x, y over $\overline{\mathcal{W}}^{\lceil,\rceil}$, $\overline{x}$ over $\overline{\mathcal{V}}^{\lceil,\rceil}$, r over $\mathcal{T}^{\lceil,\rceil}$, s over $\overline{\mathcal{T}}^{\lceil,\rceil}$, t over $\mathcal{T}^{\lceil,\rceil} \cup \overline{\mathcal{T}}^{\lceil,\rceil}$, m over $\Omega^{\lceil,\rceil}$, F, G, H, X over $\mathcal{F}^{+\lceil,\rceil}$, P, Q over $\mathcal{F}^{\lceil,\rceil}$ only, and η, ζ over names $\lceil \forall z_1 \ldots \forall z_k \rceil$ of prefices with variables $z_i \in \overline{\mathcal{W}}$ and length $k \geq 0$. (Of course, we presuppose that $\mathcal{V}_\mathcal{N}$ contains denumerably many variables of each of those sorts.) We write $\lceil \ldots \check{x} \text{- - -} \rceil$ for $\lceil \ldots \rceil x \lceil \text{- - -} \rceil$ (wherein x occurs free), $\lceil \ldots \check{x}\check{F} \text{- - -} \rceil$ for $\lceil \ldots \rceil x F \lceil \text{- - -} \rceil$, e.g., $\lceil \ldots \forall \underline{\check{y}} \text{- - -} \rceil$ for $\lceil \ldots \rceil \eta \lceil \text{- - -} \rceil$, and $\lceil \forall \underline{\check{z}} \text{- - -} \rceil$ for $\zeta \lceil \text{- - -} \rceil$.

The following formula 'Axiom(X)' of $\mathcal{F}_1$ can be read as "X is the name of an axiom of $\mathcal{S}$":

$$
\begin{aligned}
\mathrm{Axiom}(X) \quad \rightleftharpoons \quad & \exists\, F, G, H, P, Q, w, x, y, \overline{x}, \eta, \zeta, r, s, t, m \;\varepsilon\; C_0.\ \big(X =_0 \lceil \forall \underline{\check z}\, (\check F \triangleright \check F)\rceil \\
& \vee\ X =_0 \lceil \forall \underline{\check z}\, (\check F \triangleright \check G \ \wedge\ \check G \triangleright \check H \ \rightarrow\ \check F \triangleright \check H)\rceil \\
& \vee\ X =_0 \lceil \forall \underline{\check z}\, (\check F \triangleright \forall \underline{\check y}\, (\check G \rightarrow \check H) \rightarrow (\check F \triangleright \forall \underline{\check y}\, \check G \rightarrow \check F \triangleright \forall \underline{\check y}\, \check H))\rceil \\
& \vee\ X =_0 \lceil \forall \underline{\check z}\, (\check F \triangleright \forall \underline{\check y}\, \check H \ \rightarrow\ \check F \triangleright \forall \underline{\check y}\, (\check G \triangleright \check{\overline{H}}))\rceil \\
& \vee\ \ldots \\
& \vee\ \big(X =_0 \lceil \forall \underline{\check z}\, (\check G \rightarrow \exists \check x\, \check F)\rceil \ \wedge\ \mathrm{Sub}(G, F, t, x) \ \wedge\ \mathrm{Fr}(t, x, F)\big) \\
& \vee\ \big(X =_0 \lceil \forall \underline{\check z}\, (\forall \check y\, (\check G \rightarrow \check H) \rightarrow (\exists \check x\, \check F \rightarrow \check H))\rceil \\
& \qquad \wedge \mathrm{Sub}(G, F, y, x) \ \wedge\ \mathrm{Fr}(y, x, F) \ \wedge\ \mathrm{N}(y, \lceil \exists \check x\, \check F \rightarrow \check H\rceil)\big) \\
& \vee\ X =_0 \lceil \forall \underline{\check z}\, (\exists \check w \;\varepsilon\; C_{\check m}.\, \check P \ \leftrightarrow\ \exists \check w\, (\check w \;\varepsilon\; C_{\check m} \wedge \check P))\rceil \\
& \vee\ X =_0 \lceil \forall \underline{\check z}\, (\exists \check{\overline{x}} \;\varepsilon\; \check r.\, \check P \ \leftrightarrow\ \exists \check{\overline{x}}\, (\check{\overline{x}} \;\varepsilon\; \check r \wedge \check P))\rceil \\
& \vee\ \big(X =_0 \lceil \forall \underline{\check z}\, (\check s \;\varepsilon\; \{\check{\overline{x}} \;\varepsilon\; \overline{C}_{\check m} \colon \check P\} \ \leftrightarrow\ \check s \;\varepsilon\; \overline{C}_{\check m} \ \wedge\ \check Q)\rceil \\
& \qquad \wedge \mathrm{Sub}(Q, P, s, \overline{x}) \ \wedge\ \mathrm{Fr}(s, \overline{x}, P)\big) \\
& \vee\ \ldots\big).
\end{aligned}
$$

(We have ommited several brackets here.) Of course, a *sentence* of the shape $\mathrm{Fr}(t, x, F)$ with *names* t, x, F *(in place of variables)* is to mean that t' is free for x' in F' where t' is the term denoted by t, x' is the variable denoted by x, and F' is the formula denoted by F. Similarly, $\mathrm{N}(y, G)$ is to mean that y' does not occur free in G', and $\mathrm{Sub}(G, F, t, x)$ is to mean that G' results from F' by substituting t' for x'. We include all *formulas* of those shapes in $\mathcal{E}$. To formulate this in more detail, we at first define: For $\mathcal{U} \subseteq \Sigma^*$ let $\mathcal{T}(\mathcal{U}^{\lceil,\rceil})$ be the set of all elements of $\mathcal{T}(\mathcal{N})$ whose substitution instances are elements of $\mathcal{U}^{\lceil,\rceil}$. Now let $\mathcal{E}$ contain all elements of $\mathcal{E}^\circ$ and all formulas $\mathrm{Fr}(t, x, F)$, $\mathrm{N}(y, G)$, and $\mathrm{Sub}(G, F, t, x)$ with $x, y \in \mathcal{T}(\overline{\mathcal{W}}^{\lceil,\rceil})$; $F, G \in \mathcal{T}(\mathcal{F}^{+\lceil,\rceil})$, and $t \in \mathcal{T}(\mathcal{T}^{\lceil,\rceil} \cup \overline{\mathcal{T}}^{\lceil,\rceil})$.

We now recursively define $R_0, R_1, R_2, \ldots$:

$$
\begin{aligned}
R_0 \;&=\; \big\{X \;\varepsilon\; C_0 \colon \mathrm{Axiom}(X)\big\} \\
R_{n+1} \;&=\; \big\{X \;\varepsilon\; C_0 \colon\ X \;\varepsilon\; R_n \\
&\qquad \vee \exists\, F, G, \zeta \;\varepsilon\; C_0.\, (\lceil \forall \underline{\check z}\, \check F\rceil \;\varepsilon\; R_n \ \wedge\ \lceil \forall \underline{\check z}\, (\check F \rightarrow \check G)\rceil \;\varepsilon\; R_n \ \wedge\ X =_0 \lceil \forall \underline{\check z}\, \check G\rceil) \\
&\qquad \vee \exists\, G, H, \zeta \;\varepsilon\; C_0.\, (\lceil \forall \underline{\check z}\, \check H\rceil \;\varepsilon\; R_n \ \wedge\ X =_0 \lceil \forall \underline{\check z}\, (\check G \triangleright \check H)\rceil)\big\}.
\end{aligned}
$$

It is easy to see that R_n is the set of all names of sentences that are deducible in $\mathcal{S}$ by $\leq n$ steps (cf. (Zahn 1993, p. 425f.)). For any $\nu \in \mathcal{V}(\Omega)$, R_ν can also be defined as an element of $\mathcal{T}_1$ (cf. section 1). Let $R \rightleftharpoons \bigcup_{\nu \,\varepsilon\, \mathbb{N}} R_\nu$. So $R \in \mathcal{C}_1$, and for any $B \in \mathcal{A}^+$, the sentence $\lceil B\rceil \;\varepsilon\; R$ belongs to $\mathcal{A}_1$ and means that B is deducible in $\mathcal{S}$. So, by 2.2, $\lceil A \triangleright B\rceil \;\varepsilon\; R$ means that B is deducible in $\mathcal{S}(A)$ (i.e. 'from A in $\mathcal{S}$').

Remarks: In the above definition of $R_0, R_1, R_2, \ldots$ it has been convenient to use several sorts of variables belonging to $\mathcal{V}_\mathcal{N}$, namely for every set $\mathcal{U} \in \{\mathcal{W}, \overline{\mathcal{V}}, \overline{\mathcal{W}}, \mathcal{T}, \overline{\mathcal{T}},\ \mathcal{T} \cup \overline{\mathcal{T}}, \Omega, \mathcal{F}, \mathcal{F}^+\}$ variables ranging over $\mathcal{U}^{\lceil,\rceil}$, and variables ranging over names $\lceil \forall z_1 \ldots \forall z_k\rceil$ of prefices with $z_i \in \overline{\mathcal{W}}$ and $k \geq 0$. To this use we must previously have introduced such variables. But instead of them we need only *one* sort of variables, namely variables ranging over $\mathcal{N}$. Then we have to reformulate Axiom(X) thus:

$$\exists F,\ldots,\zeta,\ldots \;\varepsilon\; C_0.\big(\, F \;\varepsilon\; \mathcal{F}^{+\lceil,\rceil} \;\wedge\; \ldots \;\wedge\; \zeta \;\varepsilon\; \Pi^{\lceil,\rceil} \;\wedge\; \ldots \;\wedge\; (X =_0 \zeta \lceil(\check{F} \rhd \check{F})\rceil \;\vee\; \ldots)\big)$$

where $X, F, \zeta, \ldots$ are elements of $\mathcal{V}(\mathcal{N})$ that range over $\mathcal{N}$. Here we have added the clauses $F \;\varepsilon\; \mathcal{F}^{+\lceil,\rceil}$, $\zeta \;\varepsilon\; \Pi^{\lceil,\rceil}, \ldots$, where Π denotes the set of the above mentioned prefices. (To avoid misunderstandings we can replace '$\mathcal{F}^{+}$' by a new sign in this context.) We can effect that the latter clauses are in $\mathcal{F}_1$ - provided that $\mathcal{T}^\circ_{\text{or}}, \mathcal{E}^\circ, \mathcal{V}^\circ_0, \mathcal{V}$, and $\overline{\mathcal{V}}$, are recursively enumerable (i.e. constructible by formal rules). Indeed, in this case also the sets $\mathcal{T}, \overline{\mathcal{T}}, \mathcal{F}, \mathcal{F}^+, \Pi, \ldots$ are recursively enumerable so that the corresponding sets of names for elements of those sets can be introduced as elements of $\mathcal{C}_1$ (namely on the model of the above introduction of $R \rightleftharpoons \bigcup_{\nu \,\varepsilon\, \mathbb{N}} R_\nu$). - Complete reformulations of 'Axiom(X)' and the definition of R_{n+1} are left to the reader.

The predicates N$(\cdot,\cdot)$, Fr$(\cdot,\cdot,\cdot)$, and Sub$(.,.,.)$ have recursively enumerable extents and can, therefore, be defined to be elements of $\mathcal{C}_1$. So it suffices to take $\mathcal{E}$ to be $\mathcal{E}^\circ$. (Recall that, by a demand given in section 1, we have $(s =_0 t) \in \mathcal{F}_0$ for all $s, t \in \mathcal{T}(\mathcal{N})$.) We may also omit the signs N, Fr, and Sub from Σ°. - We shall, however, not employ these reductions of basic means of the object language.

When we say that a sentence B is deducible in $\mathcal{S}(A)$, we do not *use* the sentences A and B, we only *refer* to them. To indicate this fact we can put them in quotation marks. Accordingly, it would be adequate to understand $A \rhd B$ as a shorthand of $\lceil A\rceil \rhd \lceil B\rceil$. But then the definiens of 'Axiom(X)' turns in

$$\exists F,\ldots,\zeta,\ldots \;\varepsilon\; C_0.\big(\, X =_0 \lceil \forall \underline{\check{z}}\,(F \rhd F)\rceil \;\vee\; \ldots\,\big),$$

where several occurrences of F are bound by $\lceil,\rceil$, which misses the intended meaning. We do no further discuss that matter.

4 A version of the Theorem of Löb

Modifying an idea of Craig (see (Smullyan 1987), chap. 26, e.g.) we now extend $\mathcal{F}^+$ by the following rule: For all $F, G \in \mathcal{F}^+$ let $\Delta(F, G)$ be a formula of $\mathcal{E}$ ($\subset \mathcal{F} \subset \mathcal{F}^+$). For all $A, B \in \mathcal{A}^+$ let

$$\Delta(A, B) \text{ mean that } \mathcal{S}(A) \vdash (\Delta(A, B) \to B).$$

(Note that the latter deducibility relation does not depend on the meaning of $\Delta(A, B)$.) So for all $F, G \in \mathcal{F}^+$, all substitution instances of

$$\forall \underline{z}\,\big(\,\Delta(F, G) \;\leftrightarrow\; F \rhd (\Delta(F, G) \to G)\,\big)$$

are true. We now take all formulas of this form as additional axioms of $\mathcal{S}$. (These axioms can easily be enclosed in 'Axiom(X)'.) So all formulas of the following form are deducible in $\mathcal{S}$:

$$\forall \underline{z} \left\{ (\Delta(F,G) \to G) \; \leftrightarrow \; [\, F \rhd (\Delta(F,G) \to G) \to G] \right\}.$$

Writing H for $(\Delta(F,G) \to G)$ we obtain this version of the

Diagonal Lemma: For all $F, G \in \mathcal{F}^+$ there is an $H \in \mathcal{F}^+$ satisfying $\mathcal{S} \vdash \forall \underline{z} \left\{ H \; \leftrightarrow \; (F \rhd H \to G) \right\}$.

The special case with $\mathcal{A}^+$ instead of $\mathcal{F}^+$ implies the following version of the

Theorem of Löb: For all $B, C \in \mathcal{A}^+$, if $\mathcal{S} \vdash (B \rhd C \to C)$, then $\mathcal{S} \vdash C$.

The proof given in (Boolos 1989), p.187, can easily be transformed into a proof of this version of Löb's theorem. By another well known theorem of modal logic, this version yields

$$\mathcal{S} \vdash (B \rhd (B \rhd C \to C) \to B \rhd C),$$

which reminds of the modal scheme $\boxed{\text{i}}(\boxed{\text{i}}C \to C) \to \boxed{\text{i}}C)$. Obviously, all results of this section also hold for $\mathcal{S}(A)$ instead of $\mathcal{S}$.

Notes: 1. Modal systems satisfying the latter scheme together with (K) and (4) are said to be of type G.
2. Let $\top \rightleftharpoons \neg\bot$, e.g. Because of $\mathcal{S} \not\vdash \bot$, Löb's theorem especially implies $\mathcal{S} \not\vdash \neg(\top \rhd \bot)$ (cf. Gödel's second incompleteness theorem).

REFERENCES

Blau, U. 1993: *Zur natürlichen Logik der Unbestimmten und Paradoxien.* In: H. Stachowiak (Hg.): Pragmatik IV. Meiner, Hamburg, 353-380.

Boolos, G. 1980: *On systems of modal logic with provability interpretations.* Theoria 46 (1980), no. 1, 7-18.

Boolos, G. & Jeffrey, R.C. 1989: *Computability and Logic.* Third edition. Cambridge Univ. Press.

Boolos, G. 1995: *Quotational ambiguity.* On Quine (San Marino, 1990), 283-296, Cambridge Univ. Press.

Popkorn, S. 1994: *First Steps in Modal Logic.* Cambridge Univ. Press.

Smullyan, R. 1987: *Forever undecided. A puzzle Guide to Gödel.* New York

Wray, D.O. 1987: *Logic in Quotes.* Journal of Philosophical Logic 16, 77-144.

Zahn, P. 2004: *A Normative Model of Classical Reasoning in Higher Order Languages.* Synt 128

Zahn, P. 1993: *Gedanken zur pragmatischen Begründung von Logik und Mathematik.* In: H. Stachowiak (Hg.): Pragmatik IV. Meiner, Hamburg, 424-455.