Solutions to a Model with Nonuniformly Parabolic Terms for Phase Evolution Driven by Configurational Forces

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Abstract

We prove existence of solutions global in time to an initial-boundary value problem for a system of partial differential equations, which consists of the equations of linear elasticity and a nonlinear, non-uniformly parabolic equation of second order. This problem models the behavior of material phases, whose evolution in time is driven by configurational forces. The model is obtained by inserting a parabolic, regularizing term into an original model with hyperbolic character, which is derived in [2, 3] by transforming a well known sharp interface model for the evolution of a surface of strain discontinuity. Our existence proof, which contributes to the verification of the model, is only valid in one space dimension.

1 Introduction

In [3] a model has been derived for the behavior of materials with phase transistions, for which the evolution of the interfacial regions is driven by configurational forces. The model consists of the partial differential equations of linear elasticity coupled to a quasilinear, non-uniformly parabolic equation of second order. To verify the validity of a new model investigations are necessary, in which not only simulations must be carried out but also the analytical properties of the model must be determined. Here we contribute to the verification of the new model by showing that in the case of one space dimension an initial-boundary value problem to this model has solutions global in time. We first formulate this initial-boundary value problem in the three-dimensional case, give a short sketch of the derivation of the model

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and conclude this introduction by reducing the model to the one-dimensional case and by stating our main result.

Let $\Omega \subset \mathbb{R}^3$ be an open set. It represents the material points of a solid body. The different phases are characterized by the order parameter $S(t,x) \in \mathbb{R}$. A value of $S(t,x)$ near to zero indicates that the material is in the matrix phase at the point $x \in \Omega$ at time t, a value near to one indicates that the material is in the second phase. The other unknowns are the displacement $u(t,x) \in \mathbb{R}^3$ of the material point x at time t and the Cauchy stress tensor $T(t,x) \in \mathcal{S}^3$, where $\mathcal{S}^3$ denotes the set of symmetric 3×3-matrices. The unknowns must satisfy the quasi-static equations

$$-\operatorname{div}_x T(t,x) = b(t,x), \tag{1.1}$$

$$T(t,x) = D\big(\varepsilon(\nabla_x u(t,x)) - \bar{\varepsilon} S(t,x)\big), \tag{1.2}$$

$$S_t(t,x) = -c\Big(\psi_S(\varepsilon(\nabla_x u(t,x)), S(t,x)) - \nu \Delta_x S(t,x)\Big)|\nabla_x S(t,x)| \tag{1.3}$$

for $(t,x) \in (0,\infty) \times \Omega$. The boundary and initial conditions are

$$u(t,x) = \gamma(t,x), \qquad S(t,x) = 0, \qquad (t,x) \in [0,\infty) \times \partial\Omega, \tag{1.4}$$

$$S(0,x) = S_0(x), \qquad x \in \Omega. \tag{1.5}$$

Here $\nabla_x u$ denotes the 3×3-matrix of first order derivatives of u, the deformation gradient, $(\varepsilon(\nabla_x u))^T$ denotes the transposed matrix and

$$\varepsilon(\nabla_x u) = \frac{1}{2}\left(\nabla_x u + (\nabla_x u)^T\right)$$

is the strain tensor. $\bar{\varepsilon} \in \mathcal{S}^3$ is a given matrix, the misfit strain, and $D : \mathcal{S}^3 \to \mathcal{S}^3$ is a linear, symmetric, positive definite mapping, the elasticity tensor. In the free energy

$$\psi(\varepsilon, S) = \frac{1}{2}\big(D(\varepsilon - \bar{\varepsilon}S)\big) \cdot (\varepsilon - \bar{\varepsilon}S) + \hat{\psi}(S) \tag{1.6}$$

we choose for $\hat{\psi} \in C^2(\mathbb{R}, [0,\infty))$ a double well potential with minima at $S = 0$ and $S = 1$. Also, c is a positive constant and ν is a small nonnegative constant. Given are the volume force $b : [0,\infty) \times \Omega \to \mathbb{R}^3$ and the boundary and initial data $\gamma : [0,\infty) \times \partial\Omega \to \mathbb{R}^3$, $S_0 : \Omega \to \mathbb{R}$.

This completes the formulation of the initial-boundary value problem. The equations (1.1) and (1.2) differ from the system of linear elasticity only by the term $\bar{\varepsilon}S$, which couples this system to equation (1.3). The evolution equation (1.3) for the order parameter S is non-uniformly parabolic because of the term $\nu \Delta S |\nabla_x S|$.

We sketch the derivation of the model and explain the interest in the present investigations: Material phases with moving phase interfaces driven by configurational forces can for example consist of regions in the material with differing crystal structures. In the different regions the lattice constants of the crystal differ slightly. The phase interface is therefore a surface of strain discontinuity. This strain discontinuity causes configurational forces, which by some process can transform the material at the phase interface from one phase to the other and thus move the phase interface. A well known sharp interface model for moving surfaces of strain discontinuity has been formulated in [1]; applications of this model can for example be found in [7, 13, 15, 16]. Under a large number of publications on configurational forces we only mention [9, 12].

For varies reasons it is advantages to work with a phase field model instead of a sharp interface model. Therefore in [2, 3] the sharp interface model from [1] has been transformed into the phase field model (1.1) – (1.5). This transformation runs along the following lines: In [2] it has been observed that the equation for the normal speed of the interface in the sharp interface model can be reformulated as a partial differential equation allowing smooth and distributional solutions. In particular, if $x \mapsto S(t, x) : \Omega \to \{0, 1\}$ is the characteristic function of the region in Ω, which at time t forms the second phase, and if (u, T, S) solves the equations (1.1), (1.2), then (u, T, S) is a distributional solution of this partial differential equation. On the other hand, if (u, T, S) is a smooth solution of the equations (1.1), (1.2) and of the distributional partial differential equation, then this partial differential equation simplifies and becomes the Hamilton-Jacobi transport equation

$$S_t = -c\psi_S(\varepsilon(\nabla_x u), S)\,|\nabla_x S|. \tag{1.7}$$

The idea suggests itself to approximate the solution of the sharp interface model by smooth solutions (u, T, S) of the system (1.1), (1.2), (1.7). Yet, examples show that in general the function S in such a smooth solution develops a jump after finite time. From that time on the equation (1.7) can no longer be used to govern the evolution of S. To avoid this problem and to force solutions to stay smooth, (1.7) has been replaced by the equation (1.3), which contains the regularizing term $\nu|\nabla_x S|\Delta_x S$ with the small positive parameter ν. This yields the model (1.1) – (1.5) first stated in [3].

To regularize (1.7) one could also try the equation

$$S_t(t, x) = -c\psi_S(\varepsilon(\nabla_x u), S)\,|\nabla_x S| + \nu\Delta S. \tag{1.8}$$

However, in contrast to (1.1), (1.2), (1.8), the system (1.1) – (1.3) satisfies the second law of thermodynamics with the free energy (1.6) replaced by

$$\psi^*(\varepsilon, S) = \frac{1}{2}\big(D(\varepsilon - \bar{\varepsilon}S)\big)\cdot(\varepsilon - \bar{\varepsilon}S) + \hat{\psi}(S) + \frac{\nu}{2}|\nabla_x S|^2, \tag{1.9}$$

cf. [3]. One expects that this is an advantage, which is confirmed by our mathematical investigations. They indicate that the system (1.1) – (1.3) has better mathematical properties than the system (1.1), (1.2), (1.8), though (1.3) is seemingly more singular than (1.8).

Thus, to verify that (1.1) – (1.5) is indeed a phase field model regularizing the sharp interface model in [1], it must be shown that the initial-boundary value problem (1.1) – (1.5) with positive ν has solutions which exist global in time, and that these solutions tend to solutions of the sharp interface model for $\nu \to 0$. In this article we contribute to the first part of this program and show that in one space dimension the initial-boundary value problem has solutions. Whether solutions in three space dimensions exist and whether these solutions converge to a solution of the sharp interface model for $\nu \to 0$ is an open problem not investigated here.

We mention some related investigations: The evolution equation (1.3) is of parabolic type, but it originates from the hyperbolic equation (1.7) by regularization. Indeed, in the sharp interface model the movement of interfaces is a transport process. A corresponding model with partial differential equations should therefore have hyperbolic character. On the other hand, if the movement of interfaces is not predominantly resulting from configurational forces but from a diffusion process, the corresponding model should naturally have parabolic character. This is the case for models consisting of the Cahn-Allen or Cahn-Hilliard equations coupled with the equations of elasticity, which are therefore the appropriate models for such processes. They have recently been studied in [5, 6, 8]. Another related article is [4], where the second law of thermodynamics for phase transition models with free energies of the type (1.9) is studied.

Statement of the main result. We now assume that all functions only depend on the variables x_1 and t, and, to simplify the notation, denote x_1 by x. The set $\Omega = (a, d)$ is a bounded open interval with constants $a < d$. We write $Q_{T_e} := (0, T_e) \times \Omega$, where T_e is a positive constant, and define

$$(v, \varphi)_Z = \int_Z v(y)\varphi(y)\, dy\,,$$

for $Z = \Omega$ or $Z = Q_{T_e}$. If v is a function defined on Q_{T_e} we denote the mapping $x \to v(t, x)$ by $v(t)$. If no confusion is possible we sometimes drop the argument t and write $v = v(t)$. We still allow that the material points can be displaced in three directions, hence $u(t, x) \in \mathbb{R}^3$, $T(t, x) \in \mathcal{S}^3$ and $S \in \mathbb{R}$. If we denote the first column of the matrix $T(t, x)$ by $T_1(t, x)$ and set

$$\varepsilon(u_x) = \frac{1}{2}((u_x, 0, 0) + (u_x, 0, 0)^T) \in \mathcal{S}^3,$$

then with these definitions the equations (1.1) – (1.3) in the case of one space dimension can be written in the form

$$\begin{aligned} -T_{1x} &= b, && (1.10)\\ T &= D(\varepsilon(u_x) - \bar{\varepsilon}S), && (1.11)\\ S_t &= c\Big(T \cdot \bar{\varepsilon} - \hat{\psi}'(S) + \nu S_{xx}\Big)\,|S_x|, && (1.12) \end{aligned}$$

which must be satisfied in Q_{T_e}. Here we have inserted $\psi_S(\varepsilon, S) = -T \cdot \bar{\varepsilon} + \hat{\psi}'(S)$. Since the equations (1.10), (1.11) are linear, the inhomogeneous Dirichlet boundary condition for u can be reduced in the standard way to the homogeneous condition. For simplicity we thus assume that $\gamma = 0$. The initial and boundary conditions therefore are

$$u(t,x) = 0, \qquad (t,x) \in (0,T_e) \times \partial\Omega, \tag{1.13}$$
$$S(t,x) = 0, \qquad (t,x) \in (0,T_e) \times \partial\Omega, \tag{1.14}$$
$$S(0,x) = S_0(x), \qquad x \in \Omega. \tag{1.15}$$

To define weak solutions of this initial-boundary value problem we note that because of $\frac{1}{2}(|y|y)' = |y|$ equation (1.12) is equivalent to

$$S_t - c\nu \frac{1}{2}(|S_x|S_x)_x - c\left(T \cdot \bar{\varepsilon} - \hat{\psi}'(S)\right)|S_x| = 0. \tag{1.16}$$

Definition 1.1. *Let $b \in L^\infty(0,T_e,L^2(\Omega))$, $S_0 \in L^\infty(\Omega)$. A function (u,T,S) with*

$$u \in L^\infty(0,T_e;W_0^{1,\infty}(\Omega)), \tag{1.17}$$
$$T \in L^\infty(Q_{T_e}), \tag{1.18}$$
$$S \in L^\infty(Q_{T_e}) \cap L^\infty(0,T_e,H_0^1(\Omega)), \tag{1.19}$$

is a weak solution to the problem (1.10) – (1.15), if the equations (1.10), (1.11), (1.13) are satisfied weakly and if for all $\varphi \in C_0^\infty((-\infty,T_e) \times \Omega)$

$$(S,\varphi_t)_{Q_{T_e}} - c\nu\frac{1}{2}(|S_x|S_x,\, \varphi_x)_{Q_{T_e}} + c\left(\left(T \cdot \bar{\varepsilon} - \hat{\psi}'(S)\right)|S_x|,\, \varphi\right)_{Q_{T_e}} + (S_0,\varphi(0))_\Omega = 0. \tag{1.20}$$

The main result of this article is

Theorem 1.1 *To all $S_0 \in H_0^1(\Omega)$ and $b \in C(\overline{Q}_{T_e})$ with $b_t \in C(\overline{Q}_{T_e})$ there exists a weak solution (u,T,S) of the problem (1.10) – (1.15), which in addition to (1.17) – (1.20) satisfies*

$$S_t \in L^{\frac{4}{3}}(Q_{T_e}), \quad S_x \in L^{\frac{8}{3}}(0,T_e;L^q(\Omega)), \quad \text{for any } 1 < q < \infty \tag{1.21}$$

and

$$(|S_x|S_x)_x \in L^{\frac{4}{3}}(Q_{T_e}), \quad S_{xt} \in L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega)). \tag{1.22}$$

The remaining sections are devoted to the proof of this theorem. The main difficulty in the proof stems from the fact that the coefficient $\nu|S_x|$ of the highest order derivative S_{xx} in the equation (1.12) is not bounded away from zero and that it is not differentiable with respect to S_x.

To prove Theorem 1.1 we therefore consider in Section 2 a modified initial-boundary value problem which consists of (1.10), (1.11), (1.13) – (1.15) and the equation

$$S_t - (c\nu|S_x|_\kappa + \kappa)S_{xx} - c\left(T \cdot \bar{\varepsilon} - \hat{\psi}'(S)\right)|S_x|_\kappa = 0,\ x \in \Omega,\ t > 0 \tag{1.23}$$

with a constant $\kappa > 0$. Here we use the notation

$$|p|_\kappa := \frac{|p|^2}{\sqrt{\kappa^2 + |p|^2}}. \tag{1.24}$$

Since (1.23) is a uniformly parabolic equation we can use a standard theorem to conclude that the modified initial-boundary value problem has a sufficiently smooth solution $(u^\kappa, T^\kappa, S^\kappa)$. For this solution we derive in Section 3 a-priori estimates independent of κ.

To select a subsequence converging to a solution for $\kappa \to 0$ we need a compactness result. However, our a-priori estimates are not strong enough to show that the sequence S_x^κ is compact; instead, we can only show that the sequence $|S_x^\kappa| S_x^\kappa$, or more precisely, an approximation to this sequence, has bounded derivatives and thus is compact. It turns out that this is enough to prove existence of a solution. For the compactness proof in Section 4 we use the Aubin-Lions Lemma; since one of our a-priori estimates for derivatives of the approximating sequence is only valid in $L^1(0, T_e; H^{-2}(\Omega))$, we must use the generalized form of this lemma given by Roubíců̌ek [14], which is valid in L^1.

For the a-priori estimates it is crucial that the term $|S_x| S_{xx}$ in (1.12) can be written in the form $\frac{1}{2}(|S_x| S_x)_x$. In the higher dimensional case the corresponding term $|\nabla_x S| \Delta_x S$ cannot be rewritten in this way. This is an important reason why our proof is not valid for higher space dimensions. We surmise that existence of solutions in two space dimensions can be proved using the formula

$$|\nabla_x S| \Delta_x S = \frac{1}{2}(S_r |S_r|)_r + S_r |S_r| \Gamma,$$

where $\frac{d}{dr}$ denotes the spatial derivative in the direction of $\nabla_x S(t, x)$ and Γ is the curvature of the curve $S = \text{Const} = S(t, x)$. Yet, this problem is open.

2 Existence of solutions to the modified problem

In this section, we study the modified initial-boundary value problem and show that it has a Hölder continuous classical solution. To formulate this problem, let $\chi \in C_0^\infty(\mathbb{R}, [0, \infty))$ satisfy $\int_{-\infty}^{\infty} \chi(t) dt = 1$. For $\kappa > 0$, we set

$$\chi_\kappa(t) := \frac{1}{\kappa} \chi\left(\frac{t}{\kappa}\right),$$

and for $S \in L^\infty(Q_{T_e}, \mathbb{R})$ we define

$$(\chi_\kappa * S)(t, x) = \int_0^{T_e} \chi_\kappa(t - s) S(s, x) ds. \tag{2.1}$$

The modified initial-boundary value problem consists of the equations

$$-T_{1x} = b, \tag{2.2}$$

$$T = D(\varepsilon(u_x) - \bar{\varepsilon} \chi_\kappa * S), \tag{2.3}$$

$$S_t = (c\nu |S_x|_\kappa + \kappa) S_{xx} + c(T \cdot \bar{\varepsilon} - \hat{\psi}'(S)) |S_x|_\kappa\,, \tag{2.4}$$

which must hold in Q_{T_e}, and of the boundary and initial conditions

$$u(t,x) = 0, \quad (t,x) \in (0,T_e) \times \partial\Omega, \tag{2.5}$$
$$S(t,x) = 0, \quad (t,x) \in (0,T_e) \times \partial\Omega, \tag{2.6}$$
$$S(0,x) = S_0(x), \quad x \in \Omega. \tag{2.7}$$

To formulate an existence theorem for this problem we need some function spaces: For nonnegative integers m, n and a real number $\alpha \in (0,1)$ we denote by $C^{m+\alpha}(\overline{\Omega})$ the space of $m-$times differentiable functions on $\overline{\Omega}$, whose $m-$th derivative is Hölder continuous with exponent α. The space $C^{\alpha,\alpha/2}(\overline{Q}_{T_e})$ consists of all functions on $\overline{Q}_{T_e}$, which are Hölder continuous in the parabolic distance

$$\mathrm{d}((t,x),(s,y)) := \sqrt{|t-s| + |x-y|^2}.$$

$C^{m,n}(\overline{Q}_{T_e})$ and $C^{m+\alpha,n+\alpha/2}(\overline{Q}_{T_e})$, respectively, are the spaces of functions, whose x–derivatives up to order m and t–derivatives up to order n belong to $C(\overline{Q}_{T_e})$ or to $C^{\alpha,\alpha/2}(\overline{Q}_{T_e})$, respectively.

Theorem 2.1 *Let $\nu, \kappa > 0$, $T_e > 0$, suppose that the function $b \in C(\overline{Q}_{T_e})$ has the derivative $b_t \in C(\overline{Q}_{T_e})$ and that the initial data $S_0 \in C^{2+\alpha}(\overline{\Omega})$ satisfy $S_0|_{\partial\Omega} = S_{0,x}|_{\partial\Omega} = S_{0,xx}|_{\partial\Omega} = 0$. Then there is a solution*

$$(u,T,S) \in C^{2,1}(\overline{Q}_{T_e}) \times C^{1,1}(\overline{Q}_{T_e}) \times C^{2+\alpha,1+\alpha/2}(\overline{Q}_{T_e})$$

of the modified initial-boundary value problem (2.2) – (2.7). This solution satisfies $S_{tx} \in L^2(Q_{T_e})$ and

$$\max_{\overline{Q}_{T_e}} |S| \le \max_{\overline{\Omega}} |S_0|. \tag{2.8}$$

Proof. Note first that if S is given then for every t the equations (2.2), (2.3), (2.5) form a linear elliptic boundary value problem for the unknown function $x \mapsto (u(t,x), T(t,x))$. In [3] it is shown that the unique solution is given by

$$u(t,x) = u^* \left(\int_a^x (\chi_\kappa * S)(t,y)dy - \frac{x-a}{d-a} \int_a^d (\chi_\kappa * S)(t,y)dy \right) + w(t,x), \tag{2.9}$$
$$T(t,x) = D(\varepsilon^* - \bar{\varepsilon})(\chi_\kappa * S)(t,x) - \frac{D\varepsilon^*}{d-a} \int_a^d (\chi_\kappa * S)(t,y)dy + \sigma(t,x), \tag{2.10}$$

where $u^* \in \mathbb{R}^3, \varepsilon^* \in \mathcal{S}^3$ are suitable constants only depending on $\bar{\varepsilon}$ and D, and where for every $t \in [0,T_e]$ the function $(w(t), \sigma(t)) : \Omega \to \mathbb{R}^3 \times \mathcal{S}^3$ is the solution to the boundary value problem

$$-\sigma_{1x}(t) = b(t),$$
$$\sigma(t) = D\varepsilon(w_x(t)),$$
$$w(t)_{|\partial\Omega} = 0.$$

Since by assumption b and b_t belong to $C(\bar{Q}_{T_e})$, it follows that $(w, \sigma) \in C^{2,1}(\bar{Q}_{T_e}) \times C^{1,1}(\bar{Q}_{T_e})$. We insert (2.10) into (2.4) and obtain the equation

$$S_t = a_1(S_x)S_{xx} + a_2\left(t, x, S, S_x, \chi_\kappa * S, \frac{1}{d-a}\int_a^d (\chi_\kappa * S)(t, y)dy\right) \tag{2.11}$$

in Q_{T_e}, where

$$a_1(p) = c\nu|p|_\kappa + \kappa$$

and

$$a_2(t, x, S, p, r, s) = c\left(\bar{\varepsilon} \cdot D(\varepsilon^* - \bar{\varepsilon})r - \bar{\varepsilon} \cdot D\varepsilon^* s + \bar{\varepsilon} \cdot \sigma(t, x) - \hat{\psi}'(S)\right)|p|_\kappa .$$

The equations (2.11), (2.6) and (2.7) form an initial-boundary value problem with nonlocal terms, which is equivalent to the problem (2.2) – (2.7). To prove Theorem 2.1 it therefore suffices to show that this initial-boundary value problem is solvable. This follows from

Theorem 2.2 *Let $T_e > 0$, $M > 0$ and suppose that the coefficient functions $a_1 \in C^1(\mathbb{R}, [0, \infty))$ and $a_2 \in C^1(\overline{Q}_{T_e} \times [-M, M] \times \mathbb{R} \times [-M, M]^2, \mathbb{R})$ satisfy the equations and inequalities*

$$a_2(t, x, S, 0, r, s) = 0, \tag{2.12}$$

$$\mu_1(1 + |p|)^{m-2} \le a_1(p) \le \mu_2(1 + |p|)^{m-2}, \tag{2.13}$$

$$\left|\frac{\partial a_1}{\partial p}\right|(1 + |p|)^3 + \left|\frac{\partial a_2}{\partial p}\right|(1 + |p|) + |a_2| \le \mu_3(1 + |p|)^m, \tag{2.14}$$

$$\left|\frac{\partial a_2}{\partial x}\right| \le (\mu_4 + P(|p|))(1 + |p|)^{m+1}, \tag{2.15}$$

$$\max\left(\frac{\partial a_2}{\partial S}, \frac{\partial a_2}{\partial r}, \frac{\partial a_2}{\partial s}\right) \le (\mu_4 + P(|p|))(1 + |p|)^m, \tag{2.16}$$

where $P(\rho)$ is a nonnegative continuous function that tends to zero for $\rho \to \infty$, $\mu_1, \cdots, \mu_4$ are positive constants and m is an arbitrary number.

If the number μ_4 is sufficiently small, depending on the numbers M, μ_1, μ_2, μ_3 and $\hat{P} = \max_{\rho \ge 0} P(\rho)$, and if the initial data $S_0 \in C^{2+\alpha}(\bar{\Omega}, [-M, M])$ satisfy the compatibility conditions $S_0|_{\partial\Omega} = 0$ and

$$a_1(S_{0,x}(x))S_{0,xx}(x) + a_2(0, x, S_0(x), S_{0,x}(x), r, s) = 0 \tag{2.17}$$

for all $x \in \partial\Omega$ and for all $-M \le r, s \le M$, then there is a solution $S \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_{T_e})$ of the problem (2.11), (2.6) and (2.7). This solution has derivatives $S_{tx} \in L^2(Q_{T_e})$.

A proof of Theorem 2.2 is obtained by modification of the proof of the analogous Theorem 5.2 in [10, p.564], which is valid for the quasilinear parabolic initial boundary value problem

$$\begin{aligned} S_t &= a_1(S_x)S_{xx} + a_2(t,x,S,S_x), \\ S(t,x) &= 0, \qquad (t,x) \in (0,T_e) \times \partial\Omega, \\ S(0,x) &= S_0(x), \qquad x \in \Omega, \end{aligned}$$

which does not contain nonlocal terms. The theorem in [10] states that if the coefficient functions satisfy the conditions (2.12) – (2.16) and if the initial data satisfy compatibility and regularity conditions analogous the ones given above, then this initial-boundary value problem has a solution S with the regularity stated in Theorem 2.2. Actually, in [10] more general coefficient functions are considered. The proof is based on the Leray-Schauder fixed point theorem. We leave the modification, which is technical, to the reader.

End of the proof of Theorem 2.1: It is immediately seen that the coefficients a_1 and a_2 in (2.11) satisfy the relations (2.12) – (2.16) with $m = 3$. In particular, we can choose μ_1, μ_2, μ_3, $\hat{P}$ such that the inequalities (2.15), (2.16) hold for every $\mu_4 > 0$, with a suitable function P depending on μ_4. Moreover, from the assumption $S_0|_{\partial\Omega} = S_{0,x}|_{\partial\Omega} = S_{0,xx}|_{\partial\Omega} = 0$ together with (2.12) it follows that the compatibility condition (2.17) holds. Thus, Theorem 2.2 asserts that a solution $S \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_{T_e})$ of (2.11), (2.6), (2.7) exists with $S_{xt} \in L^2(Q_{T_e})$. The functions T and u with the regularity stated in Theorem 2.1 are obtained from (2.10), (2.9). Finally, since $a_1(0) = \kappa > 0$ and $a_2(t,x,S,0,r,s) = 0$, we can apply [10, Theorem 2.9, p.23] to (2.11) and conclude that the estimate (2.8) holds.

3 A priori estimates

In this section we establish a-priori estimates for solutions of the modified problem, which are uniform with respect to κ. We remark that the estimates in Lemma 3.1 and Corollary 3.1, though stated in the one-dimensional case, can be generalized to higher space dimensions.

In what follows we assume that

$$0 < \kappa \leq 1, \tag{3.1}$$

since we consider the limit $\kappa \to 0$. The $L^2(\Omega)$-norm is denoted by $\|\cdot\|$, and the letter C stands for varies positive constants independent of κ. Supplementing (1.24) we also use the notation

$$[p]_\kappa := \frac{p|p|}{\sqrt{\kappa^2 + p^2}}. \tag{3.2}$$

We start by constructing a family of approximate solutions to the modified problem. To this end let T_e be a fixed positive number and choose for every κ a function

$S_0^\kappa \in C_0^\infty(\Omega)$ such that

$$\|S_0^\kappa - S_0\|_{H_0^1(\Omega)} \to 0, \quad \kappa \to 0, \tag{3.3}$$

where $S_0 \in H_0^1(\Omega)$ are the initial data given in Theorem 1.1. We insert for S_0 in (2.7) the function S_0^κ and choose for b in (2.2) the function given in Theorem 1.1. These functions satisfy the assumptions of Theorem 2.1, hence there is a solution $(u^\kappa, T^\kappa, S^\kappa)$ of the modified problem (2.2) – (2.7), which exists in Q_{T_e}. The inequality (2.8) and Sobolev's embedding theorem yield for this solution

$$\sup_{0<\kappa\leq 1} \|S^\kappa\|_{L^\infty(Q_{T_e})} \leq \sup_{0<\kappa\leq 1} \|S_0^\kappa\|_{L^\infty(\Omega)} \leq C. \tag{3.4}$$

Remembering that σ in (2.10) belongs to $C^{1,1}(\bar{Q}_{T_e})$, we conclude from (3.4) that also

$$\max_{\overline{Q}_{T_e}} |c(T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'(S^\kappa))| \leq C. \tag{3.5}$$

Lemma 3.1 *There holds for any $t \in [0, T_e]$*

$$\|S_x^\kappa(t)\|^2 + \int_0^t \int_\Omega (\nu |S_x^\kappa|_\kappa + 2\kappa) \, |S_{xx}^\kappa|^2 dx d\tau \leq C. \tag{3.6}$$

Proof. Observe first that $S_{tx}^\kappa \in L^2(Q_{T_e})$, by Theorem 2.1, which yields that for almost all t

$$\frac{1}{2} \frac{d}{dt} \|S_x^\kappa(t)\|^2 = \int_\Omega S_x^\kappa(t) S_{xt}^\kappa(t) dx.$$

Using this relation and (3.5) we obtain by multiplication of (2.4) by $-S_{xx}^\kappa$ and integration by parts with respect to x, where we take the boundary condition (2.6) into account, that for almost all t

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|S_x^\kappa\|^2 + \int_\Omega (\nu |S_x^\kappa|_\kappa + \kappa) \, |S_{xx}^\kappa|^2 dx = \int_\Omega c(\hat{\psi}'(S^\kappa) - T^\kappa \cdot \bar{\varepsilon}) |S_x^\kappa|_\kappa S_{xx}^\kappa dx \\
&\leq C \int_\Omega |S_x^\kappa|_\kappa |S_{xx}^\kappa| dx = C \int_\Omega |S_x^\kappa|_\kappa^{\frac{1}{2}} |S_x^\kappa|_\kappa^{\frac{1}{2}} |S_{xx}^\kappa| dx \\
&\leq \frac{\nu}{2} \int_\Omega |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx + \frac{2C^2}{\nu} \int_\Omega (|S_x^\kappa|_\kappa)^2 dx.
\end{aligned} \tag{3.7}$$

We subtract the term $\frac{\nu}{2} \int_\Omega |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx$ on both sides of this inequality and use Gronwall's Lemma to derive (3.6) from the resulting estimate, noting also (3.3).

Corollary 3.1 There holds for any $t \in [0, T_e]$

$$\int_0^t \int_\Omega (|S_x^\kappa|_\kappa |S_{xx}^\kappa|)^{\frac{4}{3}} \, dx d\tau \leq C. \tag{3.8}$$

Proof. By Hölder's inequality, we have for some $2 > p \geq 1, q = \frac{2}{p}$ and $\frac{1}{q} + \frac{1}{q'} = 1$ that

$$\begin{aligned}
&\int_0^t \int_\Omega (|S_x^\kappa|_\kappa |S_{xx}^\kappa|)^p \, dx d\tau \\
= \;& \int_0^t \int_\Omega (|S_x^\kappa|_\kappa)^{\frac{p}{2}} \left((|S_x^\kappa|_\kappa)^{\frac{p}{2}} \, |S_{xx}^\kappa|^p \right) dx d\tau \\
\leq \;& \left(\int_0^t \int_\Omega (|S_x^\kappa|_\kappa)^{\frac{pq'}{2}} \, dx d\tau \right)^{\frac{1}{q'}} \left(\int_0^t \int_\Omega (|S_x^\kappa|_\kappa)^{\frac{pq}{2}} \, |S_{xx}^\kappa|^{pq} dx d\tau \right)^{\frac{1}{q}} \\
\leq \;& \left(\int_0^t \int_\Omega (|S_x^\kappa|_\kappa)^{\frac{p}{2-p}} \, dx d\tau \right)^{\frac{2-p}{2}} \left(\int_0^t \int_\Omega |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx d\tau \right)^{\frac{p}{2}} . \qquad (3.9)
\end{aligned}$$

Inequality (3.6) implies for $\frac{p}{2-p} \leq 2$, i.e. $p \leq \frac{4}{3}$, that the right hand side of (3.9) is bounded.

Lemma 3.2 *There hold*

$$\int_0^t \int_\Omega \left| ([S_x^\kappa]_\kappa \, |S_x^\kappa|_\kappa)_x \right|^{\frac{4}{3}} dx d\tau \leq 2^{\frac{8}{3}} \int_0^t \int_\Omega \left| |S_x^\kappa|_\kappa \, S_{xx}^\kappa \right|^{\frac{4}{3}} dx d\tau \leq C, \qquad (3.10)$$

$$\int_0^t \| |S_x^\kappa|_\kappa \|_{L^\infty(\Omega)}^{\frac{8}{3}} \, d\tau = \int_0^t \| [S_x^\kappa]_\kappa \, |S_x^\kappa|_\kappa \|_{L^\infty(\Omega)}^{\frac{4}{3}} \, d\tau \leq C. \qquad (3.11)$$

Proof. We first show that (3.11) is a consequence of (3.10). Equation (3.10) and the Poincaré inequality imply

$$\begin{aligned}
\int_0^t \| \, [S_x^\kappa]_\kappa \, |S_x^\kappa|_\kappa - \overline{[S_x^\kappa]_\kappa \, |S_x^\kappa|_\kappa} \|_{L^{\frac{4}{3}}}^{\frac{4}{3}} d\tau \;&\leq\; C \int_0^t \int_\Omega \left| ([S_x^\kappa]_\kappa \, |S_x^\kappa|_\kappa)_x \right|^{\frac{4}{3}} dx d\tau \\
&\leq\; C, \qquad (3.12)
\end{aligned}$$

where for the function $f = f(t,x)$ we have used the notation

$$\bar{f}(t) = \frac{1}{|\Omega|} \int_\Omega f(t,x) dx.$$

$|\Omega|$ is the volume of the domain Ω. Equations (1.24) and (3.2) imply $|S_x^\kappa|_\kappa \leq |S_x^\kappa|$ and $| \, [S_x^\kappa]_\kappa \, | \leq |S_x^\kappa|$. From Lemma 3.1 we thus conclude

$$\begin{aligned}
\int_0^t \int_\Omega \left| \overline{[S_x^\kappa]_\kappa \, |S_x^\kappa|_\kappa} \right|^{\frac{4}{3}} dx d\tau \;&\leq\; \frac{1}{|\Omega|^{\frac{4}{3}}} \int_0^t \int_\Omega \left(\int_\Omega |S_x^\kappa|^2 dx \right)^{\frac{4}{3}} dx d\tau \\
&=\; \frac{1}{|\Omega|^{\frac{1}{3}}} \int_0^t \|S_x^\kappa\|^{\frac{8}{3}} d\tau \leq \int_0^t C d\tau \leq Ct. \qquad (3.13)
\end{aligned}$$

Combination of the above two inequalities yields

$$\int_0^t \| [S_x^\kappa]_\kappa \, |S_x^\kappa|_\kappa \|_{L^{\frac{4}{3}}(\Omega)}^{\frac{4}{3}} \, d\tau \leq C. \qquad (3.14)$$

Invoking (3.10) we assert that

$$[S_x^\kappa]_\kappa\,|S_x^\kappa|_\kappa \in L^{\frac{4}{3}}(0,T_e;W^{1,\frac{4}{3}}(\Omega)),$$

whence by the Sobolev imbedding theorem we obtain

$$\int_0^t \|\,[S_x^\kappa]_\kappa\,|S_x^\kappa|_\kappa\,\|_{L^\infty(\Omega)}^{\frac{4}{3}}d\tau \le C\int_0^t \|\,[S_x^\kappa]_\kappa\,|S_x^\kappa|_\kappa\,\|_{W^{1,\frac{4}{3}}(\Omega)}^{\frac{4}{3}}d\tau \le C. \tag{3.15}$$

Thus (3.11) is proved, and it remains to verify (3.10).

To simplify the notation in the following computation we write $y = S_x^\kappa$. Using that

$$\begin{aligned}([y]_\kappa\,|y|_\kappa)_x &= \left(\frac{y^3|y|}{\kappa^2+y^2}\right)_x = \frac{2|y|^3(2\kappa^2+y^2)}{(\kappa^2+y^2)^2}y_x\\ &= |y|_\kappa\frac{2|y|(2\kappa^2+y^2)}{(\kappa^2+y^2)^{\frac{3}{2}}}y_x,\end{aligned} \tag{3.16}$$

we obtain from Young's inequality that

$$\begin{aligned}\left|([y]_\kappa\,|y|_\kappa)_x\right| &= |y|_\kappa\frac{2|y|(\kappa^2+y^2)}{(\kappa^2+y^2)^{\frac{3}{2}}}|y_x|\\ &\le 2|y|_\kappa\frac{\frac{1}{3}|y|^3+\frac{2}{3}(\kappa^2+y^2)^{\frac{3}{2}}}{(\kappa^2+y^2)^{\frac{3}{2}}}|y_x|\\ &= 2|y|_\kappa\frac{\frac{1}{3}|y|^{2\frac{3}{2}}+\frac{2}{3}(\kappa^2+y^2)^{\frac{3}{2}}}{(\kappa^2+y^2)^{\frac{3}{2}}}|y_x|\\ &\le 2|\,|y|_\kappa\,y_x| = 2|\,|S_x^\kappa|_\kappa\,S_{xx}^\kappa|.\end{aligned} \tag{3.17}$$

Therefore, from (3.8) we have

$$\int_0^t\int_\Omega \left|([S_x^\kappa]_\kappa\,|S_x^\kappa|_\kappa)_x\right|^{\frac{4}{3}}dxd\tau \le \int_0^t\int_\Omega |2\,|S_x^\kappa|_\kappa\,S_{xx}^\kappa|^{\frac{4}{3}}dxd\tau \le 2^{\frac{4}{3}}C, \tag{3.18}$$

which is (3.10) and completes the proof of this lemma.

Lemma 3.3 *The function S_t^κ belongs to $L^{\frac{4}{3}}(Q_{T_e})$ and we have the estimates*

$$\|S_t^\kappa\|_{L^{4/3}(Q_{T_e})} \le C\,, \tag{3.19}$$

$$\|S_x^\kappa S_{xt}^\kappa\|_{L^1(0,T_e;H^{-2}(\Omega))} \le C\,, \tag{3.20}$$

$$\|\,([S_x^\kappa]_\kappa|S_x^\kappa|_\kappa)_t\,\|_{L^1(0,T_e;H^{-2}(\Omega))} \le C\,. \tag{3.21}$$

Proof. From the equation (2.4) and the estimates (3.6), (3.5) and (3.8) we immediately see that $S_t^\kappa \in L^{\frac{4}{3}}(Q_{T_e})$ and that (3.19 holds. Therefore we only need to prove the remaining two estimates.

To prove the first one we show that there is a constant C, which is independent of κ, such that

$$\left|(S_x^\kappa S_{xt}^\kappa, \varphi)_{Q_{T_e}}\right| \le C\|\varphi\|_{L^\infty(0,T_e;H_0^2(\Omega))} \tag{3.22}$$

for all $\varphi \in L^\infty(0,T_e;H_0^2(\Omega))$. This estimate implies (3.20), since $L^1(0,T_e;H^{-2}(\Omega))$ is isometrically imbedded into the dual space of $L^\infty(0,T_e;H_0^2(\Omega))$.

For the proof of (3.22) recall first that $S_{xt}^\kappa \in L^2(Q_{T_e})$, which implies that the right-hand side is well defined. We integrate by parts to get

$$(S_x^\kappa S_{xt}^\kappa, \varphi)_{Q_{T_e}} = (S_t^\kappa, -S_{xx}^\kappa \varphi)_{Q_{T_e}} + (S_t^\kappa, -S_x^\kappa \varphi_x)_{Q_{T_e}} =: I_1 + I_2. \tag{3.23}$$

To estimate I_1 we apply (2.4) and obtain

$$(S_t^\kappa, -S_{xx}^\kappa \varphi)_{Q_{T_e}} = \left((c\nu|S_x^\kappa|_\kappa + \kappa)S_{xx}^\kappa + c\left(T\cdot\bar\varepsilon' - \hat\psi'(S^\kappa)\right)|S_x^\kappa|_\kappa, -S_{xx}^\kappa\varphi\right)_{Q_{T_e}}. \tag{3.24}$$

We estimate the right hand side of this equation term by term. For the first term we obtain from Lemma 3.1

$$\begin{aligned}\left|((c\nu|S_x^\kappa|_\kappa + \kappa)S_{xx}^\kappa, -S_{xx}^\kappa\varphi)_{Q_{T_e}}\right| &\le \|\varphi\|_{L^\infty(Q_{T_e})}\int_{Q_{T_e}}(c\nu|S_x^\kappa|_\kappa+\kappa)|S_{xx}^\kappa|^2 d(x,\tau)\\ &\le C\|\varphi\|_{L^\infty(Q_{T_e})} \le C\|\varphi\|_{L^\infty(0,T_e;H_0^2(\Omega))}. \end{aligned}\tag{3.25}$$

For the second term it follows from (3.5) and (3.8) that

$$\begin{aligned}\left|\left(c\left(T\cdot\bar\varepsilon' - \hat\psi'(S^\kappa)\right)|S_x^\kappa|_\kappa, -S_{xx}^\kappa\varphi\right)_{Q_{T_e}}\right| &\le C\|\varphi\|_{L^\infty(Q_{T_e})}\int_0^{T_e}\|\,|S_x^\kappa|_\kappa S_{xx}^\kappa\|_{L^1}d\tau\\ &\le C\|\varphi\|_{L^\infty(0,T_e;H_0^2(\Omega))}. \end{aligned}\tag{3.26}$$

The estimates (3.25) and (3.26) together yield

$$|I_1| \le C\|\varphi\|_{L^\infty(0,T_e;H_0^2(\Omega))}. \tag{3.27}$$

Now we estimate I_2. From (2.4) and (3.5) we have

$$\begin{aligned} I_2 &= \left|(S_t^\kappa, S_x^\kappa\varphi_x)_{Q_{T_e}}\right|\\ &= \left|\left((c\nu|S_x^\kappa|_\kappa+\kappa)S_{xx}^\kappa + c\left(T\cdot\bar\varepsilon' - \hat\psi'(S^\kappa)\right)|S_x^\kappa|_\kappa, -S_x^\kappa\varphi_x\right)_{Q_{T_e}}\right|\\ &\le C\int_{Q_{T_e}}\left(|S_x^\kappa|_\kappa|S_{xx}^\kappa| + \kappa|S_{xx}^\kappa| + |S_x^\kappa|_\kappa\right)|S_x^\kappa\varphi_x|d(x,t)\\ &=: C(I_{2,1}+I_{2,2}+I_{2,3}). \end{aligned}\tag{3.28}$$

We are now going to deal with $I_{2,1}, I_{2,2}$ and $I_{2,3}$. Using the Cauchy-Schwarz inequality and invoking the estimates (3.6), (3.8) and (3.11), we arrive at

$$\begin{aligned}
I_{2,1} &= \int_{Q_{T_e}} |S_x^\kappa|_\kappa |S_{xx}^\kappa S_x^\kappa \varphi_x| d(x,t) \\
&\le C \int_0^{T_e} \| |S_x^\kappa|_\kappa \|_{L^\infty(\Omega)}^{\frac{1}{2}} \|\varphi_x\|_{L^\infty(\Omega)} \int_\Omega \left(|S_x^\kappa|_\kappa\right)^{\frac{1}{2}} |S_{xx}^\kappa S_x^\kappa| dx d\tau \\
&\le C \int_0^{T_e} \| |S_x^\kappa|_\kappa \|_{L^\infty(\Omega)}^{\frac{1}{2}} \|\varphi_x\|_{L^\infty(\Omega)} \left(\int_\Omega |S_x^\kappa|^2 dx \right)^{\frac{1}{2}} \left(\int_\Omega \left(|S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 \right) dx \right)^{\frac{1}{2}} d\tau \\
&\le C \|\varphi\|_{L^\infty(0,T;H_0^2(\Omega))} \int_0^{T_e} \| |S_x^\kappa|_\kappa \|_{L^\infty(\Omega)}^{\frac{1}{2}} \left\| (|S_x^\kappa|_\kappa)^{\frac{1}{2}} S_{xx}^\kappa \right\|_{L^2} d\tau \\
&\le C \|\varphi\|_{L^\infty(0,T;H_0^2(\Omega))} \left(\int_0^{T_e} \| |S_x^\kappa|_\kappa \|_{L^\infty(\Omega)} d\tau \right)^{\frac{1}{2}} \left(\int_0^{T_e} \left\| (|S_x^\kappa|_\kappa)^{\frac{1}{2}} S_{xx}^\kappa \right\|_{L^2(\Omega)}^2 d\tau \right)^{\frac{1}{2}} \\
&\le C \|\varphi\|_{L^\infty(0,T;H_0^2(\Omega))}. \qquad (3.29)
\end{aligned}$$

The other terms are easier to handle. It follows from the estimate (3.6) and the assumption $0 < \kappa \le 1$ that

$$\begin{aligned}
I_{2,2} &= \int_{Q_{T_e}} \kappa |S_{xx}^\kappa S_x^\kappa \varphi_x| d(x,\tau) \\
&\le C\kappa^{\frac{1}{2}} \|\varphi_x\|_{L^\infty(Q_T)} \int_{Q_{T_e}} \kappa^{\frac{1}{2}} |S_{xx}^\kappa| |S_x^\kappa| d(x,\tau) \\
&\le C\kappa^{\frac{1}{2}} \|\varphi_x\|_{L^\infty(Q_T)} \left(\int_{Q_{T_e}} \kappa |S_{xx}^\kappa|^2 dx \right)^{\frac{1}{2}} \left(\int_{Q_{T_e}} |S_x^\kappa|^2 d(x,\tau) \right)^{\frac{1}{2}} \\
&\le C \|\varphi\|_{L^\infty(0,T;H_0^2(\Omega))}. \qquad (3.30)
\end{aligned}$$

Finally, (3.6) and the fact that $|S_x^\kappa|_\kappa \le |S_x^\kappa|$ imply

$$\begin{aligned}
I_{2,3} &= \int_{Q_{T_e}} |S_x^\kappa|_\kappa |S_x^\kappa \varphi_x| d(x,t) \\
&\le C \|\varphi_x\|_{L^\infty(Q_{T_e})} \int_{Q_{T_e}} |S_x^\kappa|^2 d(x,t) \\
&\le C \|\varphi\|_{L^\infty(0,T_e;H_0^2(\Omega))}. \qquad (3.31)
\end{aligned}$$

The estimates (3.28) – (3.31) yield

$$|I_2| \le C \|\varphi\|_{L^\infty(0,T;H_0^2(\Omega))}.$$

This inequality and (3.23), (3.27) together yield the desired estimate (3.22), hence (3.20) follows.

To prove the third statement of the lemma we define

$$\mathcal{R}_\kappa := (c\nu |S_x^\kappa|_\kappa + \kappa) S_{xx}^\kappa + c \left(T \cdot \bar{\varepsilon}' - \hat{\psi}'(S^\kappa) \right) |S_x^\kappa|_\kappa \qquad (3.32)$$

and set $y = S^\kappa_x$. Remembering that $S^\kappa_{xt} \in L^2(Q_{T_e})$ we obtain as in (3.16) that

$$\left([y]_\kappa\,|y|_\kappa\right)_t = \frac{2|y|^3(2\kappa^2+y^2)}{(\kappa^2+y^2)^2}y_t. \tag{3.33}$$

Multiply the equation (2.4) by

$$\left(S^\kappa_x\varphi\frac{y|y|(2\kappa^2+y^2)}{(\kappa^2+y^2)^2}\right)_x = \left(\varphi\frac{|y|^3(2\kappa^2+y^2)}{(\kappa^2+y^2)^2}\right)_x,$$

integrate the resulting equation with respect to (x,t) over Q_{T_e} and note (3.33) to obtain

$$\begin{aligned}
0 &= \left(S^\kappa_t - \mathcal{R}_\kappa, \left(S^\kappa_x\varphi\frac{y|y|(2\kappa^2+y^2)}{(\kappa^2+y^2)^2}\right)_x\right)_{Q_{T_e}} \\
&= -\left(S^\kappa_{xt}, S^\kappa_x\varphi\frac{y|y|(2\kappa^2+y^2)}{(\kappa^2+y^2)^2}\right)_{Q_{T_e}} - \left(\mathcal{R}_\kappa, \left(S^\kappa_x\varphi\frac{y|y|(2\kappa^2+y^2)}{(\kappa^2+y^2)^2}\right)_x\right)_{Q_{T_e}} \\
&= -\frac{1}{2}\left(\left([S^\kappa_x]_\kappa\,|S^\kappa_x|_\kappa\right)_t, \varphi\right)_{Q_{T_e}} - \left(\mathcal{R}_\kappa, \left(S^\kappa_x\varphi\frac{y|y|(2\kappa^2+y^2)}{(\kappa^2+y^2)^2}\right)_x\right)_{Q_{T_e}} \\
&= -\frac{1}{2}\left(\left([S^\kappa_x]_\kappa\,|S^\kappa_x|_\kappa\right)_t, \varphi\right)_{Q_{T_e}} - \left(\mathcal{R}_\kappa, \left(S^\kappa_{xx}\varphi + S^\kappa_x\varphi_x\right)\frac{y|y|(2\kappa^2+y^2)}{(\kappa^2+y^2)^2}\right)_{Q_{T_e}} \\
&\qquad - \left(\mathcal{R}_\kappa, S^\kappa_x\varphi\left(\frac{y|y|(2\kappa^2+y^2)}{(\kappa^2+y^2)^2}\right)_x\right)_{Q_{T_e}}.
\end{aligned} \tag{3.34}$$

To estimate the last two terms on the right-hand side of this inequality we note that

$$\left(\frac{y|y|(2\kappa^2+y^2)}{(\kappa^2+y^2)^2}\right)_x = \frac{4|y|\kappa^4}{(\kappa^2+y^2)^3}y_x.$$

Thus we have the following inequalities

$$\left|\frac{y|y|(2\kappa^2+y^2)}{(y^2+\kappa^2)^2}\right| \le \frac{(y^2+\kappa^2)^2}{(\kappa^2+y^2)^2} = 1$$

and

$$\left|y\left(\frac{y|y|(2\kappa^2+y^2)}{(\kappa^2+y^2)^2}\right)_x\right| = \frac{4y^2\kappa^4}{(\kappa^2+y^2)^3}|y_x| \le \frac{4}{3}\frac{(y^2+\kappa^2)^3}{(\kappa^2+y^2)^3}|y_x| = \frac{4}{3}|S^\kappa_{xx}|,$$

which yield the estimates

$$\begin{aligned}
&\left|\left(\mathcal{R}_\kappa, \left(S^\kappa_{xx}\varphi + S^\kappa_x\varphi_x\right)\frac{y|y|(2\kappa^2+y^2)}{(\kappa^2+y^2)^2}\right)_{Q_{T_e}}\right| \\
&\le C\int_{Q_{T_e}} |\mathcal{R}_\kappa|\left(|S^\kappa_{xx}\varphi| + |S^\kappa_x\varphi_x|\right)d(x,\tau)
\end{aligned} \tag{3.35}$$

and

$$\left|\left(\mathcal{R}_\kappa, S^\kappa_x\varphi\left(\frac{y|y|(2\kappa^2+y^2)}{(\kappa^2+y^2)^2}\right)_x\right)_{Q_{T_e}}\right| \le C\int_{Q_{T_e}} |\mathcal{R}_\kappa S^\kappa_{xx}\varphi| d(x,\tau). \qquad (3.36)$$

The term $\int_{Q_{T_e}} |\mathcal{R}_\kappa S^\kappa_{xx}\varphi| d(x,\tau)$ coincides with the right-hand side of (3.24), which was estimated in (3.25) – (3.27) by $C\|\varphi\|_{L^\infty(0,T;H^2_0(\Omega))}$. The term

$$\int_{Q_{T_e}} |\mathcal{R}_\kappa S^\kappa_x\varphi_x| d(x,\tau) = \int_{Q_{T_e}} |S^\kappa_t S^\kappa_x\varphi_x| d(x,\tau)$$

was estimated in (3.28) – (3.31) by $C\|\varphi\|_{L^\infty(0,T;H^2_0(\Omega))}$. These results and (3.34) – (3.36) yield

$$\left|\left(([S^\kappa_x]_\kappa|S^\kappa_x|_\kappa)_t\,,\varphi\right)_{Q_{T_e}}\right| \le C\|\varphi\|_{L^\infty(0,T_e;H^2_0(\Omega))}$$

which implies (3.21).

4 Existence of solutions to the phase field model

In this section we use the a priori estimates established in the previous section to study the convergence of $(u^\kappa, T^\kappa, S^\kappa)$ as $\kappa \to 0$. We shall show that there is a subsequence, which converges to a weak solution of the initial-boundary value problem (1.10) – (1.15), thereby proving Theorem 1.1.

Note first that the estimates (3.6), (3.19), the fact that $S^\kappa(t,x) = 0$ for all $(t,x) \in [0,T_e]\times\partial\Omega$ and Poincaré's inequality imply

$$\|S^\kappa\|_{W^{1,4/3}(Q_{T_e})} \le C\,, \qquad (4.1)$$

for a constant C independent of κ. Hence, we can select a sequence $\kappa_n \to 0$ and a function $S \in W^{1,4/3}(Q_{T_e})$, such that the sequence S^{κ_n}, which we again denote by S^κ, satisfies

$$\|S^\kappa - S\|_{L^{4/3}(Q_{T_e})} \to 0, \qquad S^\kappa_x \rightharpoonup S_x\,, \qquad S^\kappa_t \rightharpoonup S_t\,, \qquad (4.2)$$

where the weak convergence is in $L^{4/3}(Q_{T_e})$.

As usual, since the equation (1.12) is nonlinear, the weak convergence of S^κ_x is not enough to prove that the limit function solves this equation. In the following lemma we therefore show that S^κ_x converges pointwise almost everywhere:

Lemma 4.1 *There exists a subsequence of S^κ_x, we still denote it by S^κ_x, such that*

$$S^\kappa_x \to S_x, \qquad \text{a.e. in } Q_{T_e}, \qquad (4.3)$$

$$[S^\kappa_x]_\kappa \to S_x, \quad |S^\kappa_x|_\kappa \to |S_x|, \qquad \text{a.e. in } Q_{T_e}, \qquad (4.4)$$

$$|S^\kappa_x|_\kappa \rightharpoonup |S_x|, \quad [S^\kappa_x]_\kappa \rightharpoonup S_x, \qquad \text{weakly in } L^{\frac43}(Q_{T_e}), \qquad (4.5)$$

$$[S^\kappa_x]_\kappa|S^\kappa_x|_\kappa \to S_x|S_x|, \qquad \text{strongly in } L^{\frac43}(0,T_e;L^2(\Omega)), \qquad (4.6)$$

as $\kappa \to 0$.

The proof is based on the following two results:

Theorem 4.1 *Let B_0 be a normed linear space imbedded compactly into another normed linear space B which is continuously imbedded into a Hausdorff locally convex space B_1, and $1 \le p < +\infty$. If $v, v_i \in L^p(0,T_e;B_0), i \in \mathbb{N}$, the sequence $\{v_i\}_{i\in\mathbb{N}}$ converges weakly to v in $L^p(0,T_e;B_0)$, and $\{\frac{\partial v_i}{\partial t}\}_{i\in\mathbb{N}}$ is bounded in $L^1(0,T_e;B_1)$, then v_i converges to v strongly in $L^p(0,T_e;B)$.*

Lemma 4.2 *Let $(0,T_e)\times\Omega$ be an open set in $\mathbb{R}^+\times\mathbb{R}^n$. Suppose functions g_n, g are in $L^q((0,T_e)\times\Omega)$ for any given $1 < q < \infty$, which satisfy*

$$\|g_n\|_{L^q((0,T_e)\times\Omega)} \le C, \quad g_n \to g \text{ almost everywhere in } (0,T_e)\times\Omega.$$

Then g_n converges to g weakly in $L^q((0,T_e)\times\Omega)$.

Theorem 4.1 is a general version of Aubin-Lions lemma valid under the weak assumption $\partial_t v_i \in L^1(0,T_e;B_1)$. This version, which we need here, is proved in [14]. A proof of Lemma 4.2 can be found in [11, p.12].

Proof of Lemma 4.1: We choose $p = \frac{4}{3}$ and

$$B_0 = W^{1,\frac{4}{3}}(\Omega), \quad B = L^2(\Omega), \quad B_1 = H^{-2}(\Omega).$$

These spaces satisfy the assumptions of the theorem. Since the estimates (3.10) and (3.21) imply that the sequence $([S_x^\kappa]_\kappa|S_x^\kappa|_\kappa)$ is uniformly bounded in $L^p(0,T_e;B_0)$ for $\kappa\to 0$ and $([S_x^\kappa]_\kappa|S_x^\kappa|_\kappa)_t$ is uniformly bounded in $L^1(0,T_e;B_1)$, it follows from Theorem 4.1 that there is a subsequence, still denoted by $([S_x^\kappa]_\kappa|S_x^\kappa|_\kappa)$, which converges strongly in $L^p(0,T_e;B) = L^{\frac{4}{3}}(0,T_e;L^2(\Omega))$ to a limit function $G \in L^{\frac{4}{3}}(0,T_e;L^2(\Omega))$. Consequently, from this sequence we can select another subsequence, denoted in the same way, which converges almost everywhere in Q_{T_e}. Using that the mapping $y \mapsto f(y) := y|y|$ has a continuous inverse $f^{-1} : \mathbb{R}\to\mathbb{R}$, we infer that also the sequence $[S_x^\kappa]_\kappa = f^{-1}([S_x^\kappa]_\kappa|S_x^\kappa|_\kappa)$ converges pointwise almost everywhere in Q_{T_e}.

From this we deduce that also the sequence S_x^κ converges pointwise almost everywhere. For, let $y_\kappa = S_x^\kappa$, $v_\kappa = [S_x^\kappa]_\kappa$ and $v = \lim_{\kappa\to 0} v_\kappa$. From

$$y_\kappa^4 = v_\kappa^2(\kappa^2 + y_\kappa^2) = v_\kappa^2\kappa^2 + v_\kappa^2 y_\kappa^2$$

we conclude

$$y_\kappa^4 - v_\kappa^2\kappa^2 - v_\kappa^2 y_\kappa^2 = 0, \tag{4.7}$$

hence

$$y_\kappa^2 = \frac{v_\kappa^2 + \sqrt{v_\kappa^4 + 4v_\kappa^2\kappa^2}}{2},$$

since the second solution of (4.7) is negative. Therefore, for $\kappa\to 0$,

$$y_\kappa^2 = \frac{v_\kappa^2 + \sqrt{v_\kappa^4 + 4v_\kappa^2\kappa^2}}{2} \to \frac{v^2 + \sqrt{v^4}}{2} = v^2.$$

From the fact that $\text{sign}(v_\kappa) = \text{sign}(y_\kappa)$ we thus obtain

$$\begin{aligned} |y_\kappa - v_\kappa|^2 &= y_\kappa^2 - 2y_\kappa v_\kappa + v_\kappa^2 \\ &= y_\kappa^2 - 2|y_\kappa||v_\kappa| + v_\kappa^2 \to v^2 - 2|v||v| + |v|^2 = 0, \end{aligned} \tag{4.8}$$

hence

$$\lim_{\kappa\to 0} S_x^\kappa = \lim_{\kappa\to 0} y_\kappa = \lim_{\kappa\to 0} v_\kappa = v = \lim_{\kappa\to 0} [S_x^\kappa]_\kappa.$$

Therefore S_x^κ converges pointwise almost everywhere in Q_{T_e}. Since $S_x^\kappa \rightharpoonup S_x$ weakly in $L^{\frac{4}{3}}(Q_{T_e})$, we conclude from Lemma 4.2 that $S_x^\kappa \to S_x$ and $[S_x]^\kappa \to S_x$ almost everywhere in Q_{T_e}. This proves (4.3) and (4.4). Relation (4.4) yields $[S_x^\kappa]_\kappa |S_x^\kappa|_\kappa \to S_x|S_x|$ a.e in Q_{T_e}, which implies that the limit function G of $[S_x^\kappa]_\kappa |S_x^\kappa|_\kappa$ is equal to $S_x|S_x|$. This prove (4.6).

To prove (4.5) we note that the estimate $|[S_x^\kappa]_\kappa| = |S_x^\kappa|_\kappa \le |S_x^\kappa|$ and the inequality (4.1) together imply that the sequences $[S_x^\kappa]_\kappa$ and $|S_x^\kappa|_\kappa$ are uniformly bounded in $L^{\frac{4}{3}}(Q_{T_e})$. Thus, (4.5) is a consequence of (4.4) and Lemma 4.2.

Proof of Theorem 1.1: Define the functions u and T by

$$\begin{aligned} u(t,x) &= u^* \left(\int_a^x S(t,y)dy - \frac{x-a}{d-a} \int_a^d S(t,y)dy \right) + w(t,x), \\ T(t,x) &= D(\varepsilon^* - \bar{\varepsilon})S - D\varepsilon^* \frac{1}{d-a} \int_a^d S(t,y)dy + \sigma(t,x), \end{aligned} \tag{4.9}$$

where for S we insert the limit function of the sequence S^κ given in (4.2), and where $u^* \in \mathbb{R}^3$, $\varepsilon^* \in \mathcal{S}^3$ and (w,σ) are the same constants and functions as in (2.9) and (2.10). We prove that (u,T,S) is a weak solution of problem (1.10) – (1.15).

To this end note that (3.4) and (4.2) imply $S \in L^\infty(Q_{T_e})$. From this relation, from the above definition of u and T and from $(w,\sigma) \in C^{2,1}(\bar{Q}_{T_e}) \times C^{1,1}(\bar{Q}_{T_e})$ we immediately see that u and T satisfy (1.17) and (1.18). Observe next that $\|S^\kappa\|_{L^\infty(0,T_e;H_0^1(\Omega))} \le C$, by (3.6). This implies $S \in L^\infty(0,T_e;H_0^1(\Omega))$, since we can select a subsequence of S^κ which converges weakly to S in thus space. Thus, S satisfies (1.19).

It is shown in [3] that the functions u and T defined in this way satisfy the equations (1.10), (1.11) and (1.14). We remarked this previously. It therefore suffices to show that the equations (1.12) and (1.15) are fulfilled in the weak sense. By definition, these equations are satisfied in the weak sense if the relation (1.20) holds. To verify (1.20) we use that by construction (T^κ, S^κ) solves (2.4), (2.6) and (2.7). If we multiply equation (2.4) by a test function $\varphi \in C_0^\infty((-\infty,T_e)\times\Omega)$ and integrate the resulting equation over Q_{T_e} we obtain

$$\begin{aligned} 0 &= (S_t^\kappa, \varphi)_{Q_{T_e}} + \left(-(c\nu|S_x^\kappa|_\kappa + \kappa) S_{xx}^\kappa - c\left(T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'(S^\kappa) \right) |S_x^\kappa|_\kappa, \varphi \right)_{Q_T} \\ &= -(S_0^\kappa, \varphi(0))_\Omega - (S^\kappa, \varphi_t)_{Q_{T_e}} + \left(c\nu \int_0^{S_x^\kappa} |y|_\kappa dy + \kappa S_x^\kappa, \varphi_x \right)_{Q_{T_e}} \\ &\quad + \left(c\left(T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'(S^\kappa) \right) |S_x^\kappa|_\kappa, \varphi \right)_{Q_{T_e}}. \end{aligned}$$

Equation (1.20) follows from this relation if we show that

$$(S_0^\kappa, \varphi(0))_\Omega \to (S_0, \varphi(0))_\Omega, \tag{4.10}$$

$$(S^\kappa, \varphi_t)_{Q_{T_e}} \to (S, \varphi_t)_{Q_{T_e}}, \tag{4.11}$$

$$\left(\int_0^{S_x^\kappa} |y|_\kappa dy, \varphi_x\right)_{Q_{T_e}} \to \left(\frac{1}{2}|S_x|S_x, \varphi_x\right)_{Q_{T_e}}, \tag{4.12}$$

$$\left(\left(T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'(S^\kappa)\right)|S_x^\kappa|_\kappa, \varphi\right)_{Q_{T_e}} \to \left(\left(T \cdot \bar{\varepsilon} - \hat{\psi}'(S)\right)|S_x|, \varphi\right)_{Q_{T_e}}, \tag{4.13}$$

$$(\kappa S_x^\kappa, \varphi_x)_{Q_{T_e}} \to 0, \tag{4.14}$$

for $\kappa \to 0$. Now, the relation (4.10) follows from (3.3), the relation (4.11) is a consequence of (4.2), and the relation (4.14) is obtained from (4.1). To prove (4.12) we use that

$$\begin{aligned}\int_0^{S_x^\kappa} |y|_\kappa dy - \frac{1}{2}S_x|S_x| &= \left(\int_0^{S_x^\kappa} |y|_\kappa dy - \frac{1}{2}[S_x]_\kappa|S_x|_\kappa\right) + \frac{1}{2}\left([S_x]_\kappa|S_x|_\kappa - S_x|S_x|\right) \\ &=: I_1 + I_2.\end{aligned} \tag{4.15}$$

The relation (4.6) implies

$$\|I_2\|_{L^{\frac{4}{3}}(0,T_e;L^2(\Omega))} \to 0 \tag{4.16}$$

for $\kappa \to 0$. Moreover,

$$\begin{aligned}|I_1| &= \left|\int_0^{S_x^\kappa} |y|_\kappa dy - \int_0^{S_x^\kappa} |y| dy\right| = \left|\int_0^{S_x^\kappa} \left(\frac{y^2}{\sqrt{\kappa^2 + y^2}} - |y|\right) dy\right| \\ &\le \int_0^{|S_x^\kappa|} \frac{|y|}{\sqrt{\kappa^2 + y^2}} \left|\sqrt{\kappa^2 + y^2} - |y|\right| dy \le \int_0^{|S_x^\kappa|} \kappa dy = \kappa|S_x^\kappa|,\end{aligned}$$

whence (3.6) implies

$$\|I_1\|_{L^{\frac{4}{3}}(0,T_e;L^2(\Omega))} \le C\|I_1\|_{L^2(Q_{T_e})} \le C\kappa \to 0$$

for $\kappa \to 0$. From this relation and from (4.15), (4.16) we obtain

$$\left\|\int_0^{S_x^\kappa} |y|_\kappa dy - \frac{1}{2}S_x|S_x|\right\|_{L^{\frac{4}{3}}(0,T_e;L^2(\Omega))} \to 0,$$

which implies (4.12). To verify (4.13) we note that (2.10) and (4.9) yield

$$\begin{aligned} & T^\kappa(t,x) - T(t,x) \\ = \; & D(\varepsilon^* - \bar{\varepsilon})(\chi_\kappa * S^\kappa - S)(t,x) - \frac{D\varepsilon^*}{d-a}\int_a^d (\chi_\kappa * S^\kappa - S)(t,y)dy. \end{aligned} \quad (4.17)$$

From (2.1) and (4.2) we conclude that

$$\begin{aligned} \|\chi_\kappa * S^\kappa - S\|_{L^{\frac{4}{3}}(Q_{T_e})} &\leq \|\chi_\kappa * (S^\kappa - S)\|_{L^{\frac{4}{3}}(Q_{T_e})} + \|(S - \chi_\kappa * S)\|_{L^{\frac{4}{3}}(Q_{T_e})} \\ &\leq \|(S - \chi_\kappa * S)\|_{L^{\frac{4}{3}}(Q_{T_e})} + \|S^\kappa - S\|_{L^{\frac{4}{3}}(Q_{T_e})} \to 0 \end{aligned}$$

for $\kappa \to 0$. Since ε^* and $\bar{\varepsilon}$ are constants, we infer from this relation and from (4.17) that

$$\|T - T^\kappa\|_{L^{\frac{4}{3}}(Q_{T_e})} \to 0$$

for $\kappa \to 0$. Thus, after selecting a subsequence we have $T^\kappa \to T$ a.e in Q_{T_e}. Together with (4.3) and (4.4) we see that $(T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'(S^\kappa))|S_x^\kappa|_\kappa$ tends to $(T \cdot \bar{\varepsilon} - \hat{\psi}'(S))|S_x|$, almost everywhere in Q_{T_e}. Since (3.6) and (3.5) imply that $(T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'(S^\kappa))|S_x^\kappa|_\kappa$ is uniformly bounded in $L^2(Q_{T_e})$, we deduce from Lemma 4.2 that

$$(T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'(S^\kappa))|S_x^\kappa|_\kappa \rightharpoonup (T \cdot \bar{\varepsilon} - \hat{\psi}'(S))|S_x|,$$

weakly in $L^2(Q_{T_e})$, which implies (4.13). Consequently (1.20) holds.

It remains to prove that the solution has the regularity properties stated in (1.21) and (1.22). The relation $S_t \in L^{\frac{4}{3}}(Q_{T_e})$ is implied by (4.2). To verify the second assertion in (1.21), we use estimate (3.11) to get

$$\int_0^{T_e} \|[S_x^\kappa]_\kappa\|_{L^q(\Omega)}^{\frac{8}{3}} dt \leq C$$

for any $1 < q < \infty$, since Ω is bounded. Using this estimate and (4.4) we infer from Lemma 4.2 that $[S_x^\kappa]_\kappa \rightharpoonup S_x$ in $L^{\frac{8}{3}}(0,T_e;L^q(\Omega))$, whence $S_x \in L^{\frac{8}{3}}(0,T_e;L^q(\Omega))$ follows.

To prove (1.22), we recall that $[S_x^\kappa]_\kappa|S_x^\kappa|_\kappa$ converges to $|S_x|S_x$ strongly in the space $L^{\frac{4}{3}}(0,T_e;L^2(\Omega)) \subset L^{\frac{4}{3}}(Q_{T_e})$ and that $([S_x^\kappa]_\kappa|S_x^\kappa|_\kappa)_x$ is uniformly bounded in $L^{\frac{4}{3}}(Q_{T_e})$ for $\kappa \to 0$, by (3.10). This together implies that $(|S_x|S_x)_x \in L^{\frac{4}{3}}(Q_{T_e})$. Finally, to prove the second assertion of (1.22) we choose a test function $\varphi \in L^4(0,T_e,W_0^{1,4}(\Omega))$, multiply equation (2.4) by $-\varphi_x$ and integrate the resulting equation over Q_{T_e} to obtain

$$0 = (S_t^\kappa - \mathcal{R}_\kappa, -\varphi_x)_{Q_{T_e}} = (S_{xt}^\kappa, \varphi)_{Q_{T_e}} + (\mathcal{R}_\kappa, \varphi_x)_{Q_{T_e}}, \quad (4.18)$$

with $\mathcal{R}_\kappa$ defined in (3.32). Invoking the estimates (3.6), (3.5) and (3.8) we deduce that

$$\|\mathcal{R}_\kappa\|_{L^{\frac{4}{3}}(Q_{T_e})} \leq C,$$

hence (4.18) yields

$$(S^\kappa_{xt},\varphi)_{Q_{T_e}} \le \|\mathcal{R}_\kappa\|_{L^{\frac{4}{3}}(Q_{T_e})}\|\varphi_x\|_{L^4(Q_{T_e})} \le C\|\varphi\|_{L^4(0,T_e;W^{1,4}_0(\Omega))}\,,$$

and this means that S^κ_{xt} is uniformly bounded in $L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega))$. From this estimate and from $S^\kappa_t \rightharpoonup S_t$ in $L^{\frac{4}{3}}(Q_{T_e})$ we deduce easily that S_{xt} belongs to the dual space of $L^4(0,T_e;W^{1,4}_0(\Omega))$, which is $L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega))$.

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