H^∞-CALCULUS FOR PRODUCTS OF NON-COMMUTING OPERATORS

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ABSTRACT. It is shown that the product of two sectorial operators A and B admits a bounded H^∞-calculus on a Banach space X provided suitable commutator estimates and Kalton-Weis type assumptions on A and B are satisfied.

1. Introduction

The question of maximal L^p-regularity for partial differential equations has attracted much attention in the last decade. One reason for this is that, by linearization techniques, one obtains a powerful approach to many nonlinear parabolic problems.

Starting from the fundamental paper of Da Prato and Grisvard [DPG75], the so-called method of operator sums was further developed by Dore and Venni [DV87] and more recently by Kalton and Weis [KW01]. They proved, roughly speaking, that the sum $A+B$ of two commuting operators A and B, equipped with its natural domain, has similar properties as A and B. The most important examples fitting in this framework are of course the time derivative and differential operators with respect to the space variable.

Whereas problems of this kind for commuting operators may be regarded as fairly well understood, the situation is less clear in the non-commuting context. A first result in this direction was given by Monniaux and Prüss [MP97], who proved a theorem of Dore-Venni type, assuming the Labbas-Terreni commutator condition (see [LT87]). Very recently, Prüss and Simonett were able to prove a non-commutative version of the Kalton-Weis theorem for both Da Prato-Grisvard and Labbas-Terreni commutator conditions, see [PS04]. Applications of this result to parabolic equations on wedges and cones yield optimal regularity results for the solution of these equations.

For many applications it is essential to have results of this kind not only for the sum of A and B but also for products AB. Indeed, recent developments in free boundary value problemes with moving contact lines show that regularity results on products of non commuting operators are very helpful in this context.

First results on products of non commuting sectorial operators under certain commutator estimates were obtained by Weber and Štrkalj. Indeed, a Dore-Venni type result for products was first obtained by Weber [Web98]. Štrkalj [Štr01] proved that the product AB of A and B is sectorial provided the underlying space is B-convex and assumptions of Kalton-Weis type are satisfied.

It is the aim of this paper to study the remaining question in this context: existence of an H^∞-calculus for the product AB of non commuting operators A and B under suitable commutator and Kalton-Weis type assumptions. In the following Theorems 3.1 and 3.2 we give an affirmative answer to this question.

2. Preliminaries

In this section we introduce the notation being used throughout this article and collect certain properties of sectorial operators and operators with a bounded H^∞-calculus.

If X and Y are Banach spaces, $\mathcal{L}(X,Y)$ denotes the space of all bounded, linear operators from X to Y; moreover, $\mathcal{L}(X) := \mathcal{L}(X,X)$. The spectrum of a linear operator A in X is denoted by

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$\sigma(A)$, its resolvent set by $\varrho(A)$. As usual domain and range of an operator A are denoted by $D(A)$ and $R(A)$, respectively.

Let X be a complex Banach space, and A be a closed linear operator in X. Then A is called *sectorial* if $\overline{D(A)} = X$, $\overline{R(A)} = X$, $(-\infty, 0) \subseteq \varrho(A)$ and

$$\|t(t+A)^{-1}\| \leq M, \qquad t > 0,$$

for some $M < \infty$. We denote the class of sectorial operators in X by $\mathcal{S}(X)$. $\Sigma_\theta \subseteq \mathbb{C}$ means the open sector

$$\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \theta\}.$$

If $A \in \mathcal{S}(X)$ then $\varrho(-A) \supseteq \Sigma_\theta$ and $\sup\{\|\lambda(\lambda + A)^{-1}\| : \ |\arg\lambda| < \theta\} < \infty$ for some $\theta > 0$.

We thus define the *spectral angle* φ_A of $A \in \mathcal{S}(X)$ by

$$\varphi_A = \inf\{\phi : \varrho(-A) \supseteq \Sigma_{\pi-\phi}, \ \sup_{\lambda \in \Sigma_{\pi-\phi}} \|\lambda(\lambda + A)^{-1}\| < \infty\}.$$

Evidently, we have $\varphi_A \in [0, \pi)$ and $\varphi_A \geq \sup\{|\arg\lambda| : \lambda \in \sigma(A)\}$. For $\phi \in (0, \pi]$ we define the space of holomorphic functions on Σ_ϕ by $H(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic}\}$, and

$$H^\infty(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic and bounded}\}.$$

The space $H^\infty(\Sigma_\phi)$ with norm $\|f\|_\infty^\phi = \sup\{|f(\lambda)| : \ |\arg\lambda| < \phi\}$ forms a Banach algebra. We also set $H_0^\infty(\Sigma_\phi) := \bigcup_{\alpha,\beta<0} H_{\alpha,\beta}(\Sigma_\phi)$, where $H_{\alpha,\beta}(\Sigma_\phi) := \{f \in H(\Sigma_\phi) : \ \|f\|_{\alpha,\beta}^\phi < \infty\}$, and $\|f\|_{\alpha,\beta}^\phi := \sup_{|\lambda|\leq 1} |\lambda^\alpha f(\lambda)| + \sup_{|\lambda|\geq 1} |\lambda^{-\beta} f(\lambda)|$. Given $A \in \mathcal{S}(X)$, fix any $\phi \in (\varphi_A, \pi]$ and let $\Gamma_\psi = -(-\infty, 0]e^{i\psi} \cup [0, \infty)e^{-i\psi}$ with $\varphi_A < \psi < \phi$. Then

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma_\psi} f(\lambda)(\lambda - A)^{-1} \, \mathrm{d}\lambda, \quad f \in H_0^\infty(\Sigma_\phi),$$

defines via $\Phi_A(f) = f(A)$ a functional calculus $\Phi_A : H_0^\infty(\Sigma_\phi) \to \mathcal{L}(X)$ which is an algebra homomorphism. Following McIntosh [McI86], we say that a sectorial operator A admits a *bounded H^∞-calculus* if there are $\phi > \varphi_A$ and a constant $K_\phi < \infty$ such that

$$\|f(A)\| \leq K_\phi \|f\|_\infty^\phi, \qquad \text{for all } f \in H_0^\infty(\Sigma_\phi). \tag{2.1}$$

The class of sectorial operators A which admit a bounded H^∞-calculus will be denoted by $\mathcal{H}^\infty(X)$ and the *$\mathcal{H}^\infty$-angle* of A is defined by

$$\varphi_A^\infty = \inf\{\phi > \varphi_A : \ (2.1) \text{ is valid}\}.$$

If this is the case, the functional calculus for A on $H_0^\infty(\Sigma_\phi)$ extends uniquely to $H^\infty(\Sigma_\phi)$.

We consider next another subclass of $\mathcal{S}(X)$, namely operators with bounded imaginary powers. More precisely, a sectorial operator A in X is said to admit *bounded imaginary powers* if $A^{is} \in \mathcal{L}(X)$ for each $s \in \mathbb{R}$ and there is a constant $C > 0$ such that $\|A^{is}\| \leq C$ for $|s| \leq 1$. The class of such operators will be denoted by $\mathcal{BIP}(X)$. We call

$$\varphi_A^{\mathrm{BIP}} = \overline{\lim}_{|s|\to\infty} \frac{1}{|s|} \log \|A^{is}\|$$

the *power angle* of A. Since the functions f_s defined by $f_s(z) = z^{is}$ belong to $H^\infty(\Sigma_\phi)$, for any $s \in \mathbb{R}$ and $\phi \in (0, \pi)$, we obviously have the inclusions

$$\mathcal{H}^\infty(X) \subseteq \mathcal{BIP}(X) \subseteq \mathcal{S}(X),$$

and the inequalities

$$\varphi_A^\infty \geq \varphi_A^{\mathrm{BIP}} \geq \varphi_A \geq \sup\{|\arg\lambda| : \ \lambda \in \sigma(A)\}.$$

Let Y be another Banach space. A family of operators $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ is called *$\mathcal{R}$-bounded*, if there is a constant $C > 0$ and $p \in [1, \infty)$, such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and for all independent, symmetric, $\{-1,1\}$-valued random variables ε_j on a probability space $(\Omega, \mathcal{M}, \mu)$ the inequality

$$\left\| \sum_{j=1}^{N} \varepsilon_j T_j x_j \right\|_{L_p(\Omega;Y)} \leq C \left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\|_{L_p(\Omega;X)}$$

is valid. The smallest such C is called $\mathcal{R}$-*bound* of $\mathcal{T}$, we denote it by $\mathcal{R}(\mathcal{T})$. Observe that the concept of $\mathcal{R}$-boundedness does not depend on p, however $\mathcal{R}(\mathcal{T})$ does, see [CdPSW00], [Wei01], [KW01], [DHP03].

The concept of $\mathcal{R}$-bounded families of operators leads immediately to the notion of $\mathcal{R}$-sectorial operators. Indeed, a sectorial operator is called $\mathcal{R}$-*sectorial* (see [CP01]) if

$$\mathcal{R}_A(0) := \mathcal{R}(\{t(t+A)^{-1} : t > 0\}) < \infty.$$

The $\mathcal{R}$-*angle* $\varphi_A^{\mathcal{R}}$ of A is defined by means of

$$\varphi_A^{\mathcal{R}} := \inf\{\theta \in (0,\pi) : \ \mathcal{R}_A(\pi - \theta) < \infty\},$$

where

$$\mathcal{R}_A(\theta) := \mathcal{R}(\{\lambda(\lambda + A)^{-1} : |\arg \lambda| \leq \theta\}).$$

Finally, we say that $A \in \mathcal{H}^\infty(X)$ admits an $\mathcal{R}$-bounded H^∞-calculus, if the set

$$\{f(A) : f \in H^\infty(\Sigma_\phi), \ \|f\|_\infty^\phi \leq 1\}$$

is $\mathcal{R}$-bounded for some $\phi \in (0,\pi)$. As above, the infimum $\varphi_A^{\mathcal{R}\infty}$ of such ϕ is called the $\mathcal{RH}^\infty$-angle of A. The class of such operators is denoted by $\mathcal{RH}^\infty(X)$.

Assume that the underlying space X satisfies the so-called *property* (α), see [CdPSW00, Definition 3.11]. Then Kalton and Weis [KW01, Theorem 5.3] proved that every operator $A \in \mathcal{H}^\infty(X)$ already admits an $\mathcal{R}$-bounded H^∞-calculus. More precisely, we have

$$\mathcal{H}^\infty(X) = \mathcal{RH}^\infty(X) \qquad \text{with} \qquad \varphi_A^{\mathcal{R}\infty} = \varphi_A^\infty. \tag{2.2}$$

It is well known that L^p-spaces with $1 < p < \infty$ possess the property (α).

We are now able to state the Kalton-Weis theorem which gives a sufficient condition for the existence of an operator-valued H^∞-calculus.

Theorem 2.1 ([KW01])**.** *Let X be a Banach space. Assume that $A \in \mathcal{H}^\infty(X)$, $F \in H^\infty(\Sigma_\phi; \mathcal{L}(X))$ such that*

$$F(\lambda)(\mu - A)^{-1} = (\mu - A)^{-1}F(\lambda), \quad \mu \in \rho(A), \lambda \in \Sigma_\phi,$$

and that $\phi > \varphi_A^\infty$ and $\mathcal{R}(F(\Sigma_\phi)) < \infty$. Then there exists a constant C independent of F such that $F(A) \in \mathcal{L}(X)$ and

$$\|F(A)\|_{\mathcal{L}(X)} \leq C\mathcal{R}(F(\Sigma_\phi)).$$

Consider for two sectorial operators A and B in X their product AB defined by

$$(AB)x := ABx, \qquad D(AB) := \{x \in D(B) : Bx \in D(A)\}.$$

We then observe that AB is closed as soon as A is invertible or B is bounded. The Kalton-Weis theorem moreover implies for commuting operators A and B the following result.

Corollary 2.2. *Let X be a Banach space and assume that A and B are sectorial operators in X which commute in the sense of resolvents. Suppose that $0 \in \rho(A)$, $A \in \mathcal{H}^\infty(X)$, $B \in \mathcal{RS}(X)$ and that $\varphi_A^\infty + \varphi_B^{\mathcal{R}} < \pi$.*

a) *Then AB is sectorial and $\varphi_{AB} \leq \varphi_A^\infty + \varphi_B^{\mathcal{R}}$.*
b) *If in addition $B \in \mathcal{RH}^\infty(X)$ with $\varphi_A^\infty + \varphi_B^{\mathcal{R}\infty} < \pi$, then $AB \in \mathcal{H}^\infty(X)$ and $\varphi_{AB}^\infty \leq \varphi_A^\infty + \varphi_B^{\mathcal{R}\infty}$.*

We say that a Banach space X belongs to the class $\mathcal{HT}$, if the Hilbert transform is bounded on $L^p(\mathbb{R};X)$ for some (and then all) $p \in (1,\infty)$. Here the Hilbert transform H of a function $f \in \mathcal{S}(\mathbb{R};X)$, the Schwartz space of rapidly decreasing X-valued functions, is defined by

$$Hf := \frac{1}{\pi} PV(\frac{1}{t}) * f$$

These spaces are also called UMD Banach spaces, where the UMD stands for *unconditional martingale difference property.* It is a well known theorem that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces; see e.g. [Bur86].

Throughout this paper, for $\psi \in (0,\pi)$ and $r \geq 0$ we denote by Γ_ψ^r the path given by

$$\Gamma_\psi^r := -(-\infty,-r]e^{i\psi} \cup re^{-i[-\psi,\psi]} \cup [r,\infty)e^{-i\psi},$$

and we write $\Gamma_\psi := \Gamma_\psi^0$.

We remark that by C, M and c we denote various constants which may differ from line to line but which are always independent of the free variables.

3. The main result

Is the product of sectorial operators again sectorial? Let us recall that the first result in this direction for non-commuting operators was proved by Weber [Web98]. He showed that in UMD spaces, $\nu + AB$ with natural domain $D(AB) = \{x \in D(B) : Bx \in D(A)\}$ is sectorial provided A and B have bounded imaginary powers of suitable power angles and certain commutator estimates are fulfilled. This result was later on extended by Strkalj [Štr01] to Kalton-Weis type assumptions for operators defined in B-convex Banach lattices.

We start with a generalization of the latter result to arbitrary Banach spaces.

Theorem 3.1. *Let X be a Banach space. Assume that $A \in \mathcal{H}^\infty(X)$ and $B \in \mathcal{RS}(X)$ with $0 \in \varrho(A)$ fulfill the following properties:*

a) $(\mu - B)^{-1}D(A) \subseteq D(A)$ *for some (all)* $\mu \in \varrho(B)$,
b) *there are* $\theta_A > \varphi_A^\infty$ *and* $\theta_B > \varphi_B^{\mathcal{R}}$, *such that* $\theta_A + \theta_B < \pi$ *and there exist constants* $c, \alpha \geq 0$ *and* $\beta > 0$ *with* $\alpha + \beta < 1$, *such that*

$$\|[A,(\mu+B)^{-1}](\lambda+A)^{-1}\| \leq \frac{c}{(1+|\lambda|)^{1-\alpha}|\mu|^{1+\beta}}$$

for all $\lambda \in \Sigma_{\pi-\theta_A}$ *and* $\mu \in \Sigma_{\pi-\theta_B}$.

Then there exists $\nu \geq 0$ such that the operator $\nu + AB$ with domain $D(AB)$ is sectorial with $\varphi_{\nu+AB} \leq \theta_A + \theta_B$.

The following main result of the paper states that in the above situation we even have $AB \in \mathcal{H}^\infty(X)$ provided $B \in \mathcal{RH}^\infty(X)$.

Theorem 3.2. *Let X be a Banach space. Assume that $A \in \mathcal{H}^\infty(X)$ and $B \in \mathcal{RH}^\infty(X)$ with $0 \in \varrho(A)$ fulfill the following properties:*

a) $(\mu - B)^{-1}D(A) \subseteq D(A)$ *for some (all)* $\mu \in \varrho(B)$,
b) *there are* $\theta_A > \varphi_A^\infty$ *and* $\theta_B > \varphi_B^{\mathcal{R}\infty}$, *such that* $\theta_A + \theta_B < \pi$ *and there exist constants* $c, \alpha \geq 0$ *and* $\beta > 0$ *with* $\alpha + \beta < 1$, *such that*

$$\|[A,(\mu+B)^{-1}](\lambda+A)^{-1}\| \leq \frac{c}{(1+|\lambda|)^{1-\alpha}|\mu|^{1+\beta}}$$

for all $\lambda \in \Sigma_{\pi-\theta_A}$ *and* $\mu \in \Sigma_{\pi-\theta_B}$.

Then there exists $\nu \geq 0$, such that the operator $\nu + AB$ with domain $D(AB)$ has a bounded H^∞-calculus with $\varphi_{\nu+AB}^\infty \leq \theta_A + \theta_B$.

Remark 3.3. a) As both theorems are not symmetric in the roles of A and B, it is worthwile to note that the same results hold true if the properties of A and B are interchanged. The proofs stay the same.

b) If X has property (α), the classes $\mathcal{H}^\infty(X)$ and $\mathcal{RH}^\infty(X)$ coincide; see (2.2). Thus Theorem 3.2 may be formulated in this case with $B \in \mathcal{H}^\infty(X)$.

c) It will become apparent in the proof of the two theorems, that the amount of the shift ν is determined mainly by the constant c in the commutator estimate.

Finally we observe, that the invertibility of A implies $\lambda \in \varrho(A)$ for all $|\lambda| < \|A^{-1}\|^{-1}$ and $\|(\lambda + A)^{-1}\| \leq \|A^{-1}\|(1 - |\lambda|\|A^{-1}\|)^{-1}$ for all these λ. Thus the commutator estimate can be extended to all $\lambda \in \Sigma_{\pi-\theta_A} \cup \{z \in \mathbb{C} : |z| < r\}$ whenever $r < \|A^{-1}\|^{-1}$.

4. Sectoriality of $\nu + AB$

In this section we give a proof of Theorem 3.1. Our method is inspired by the work of Weber [Web98] and Prüss and Simonett [PS04]. It is heavily based on properties of the families of operators S_μ and T_μ which are defined as follows.

We fix angles $\gamma \in (0, \pi - \theta_A - \theta_B)$ and $\varphi \in (\theta_A, \pi - \gamma - \theta_B)$, as well as a number $r \in (0, \|A^{-1}\|^{-1})$. Now let $\mu \in \Sigma_\gamma$. Then for all $z \in \Gamma_\varphi^r$, we have $\mu/z \in \varrho(-B)$. Furthermore, by the choices of φ and r, the inclusion $\Gamma_\varphi^r \subseteq \varrho(A)$ holds true. We then define for $x \in D(A)$

$$\begin{aligned} S_\mu x &:= \frac{1}{2\pi i} \int_{\Gamma_\varphi^r} \frac{1}{z^2} \left(\frac{\mu}{z} + B\right)^{-1} A(z-A)^{-1} x \, \mathrm{d}z, \\ T_\mu x &:= \frac{1}{2\pi i} \int_{\Gamma_\varphi^r} \frac{1}{z^2} A(z-A)^{-1} \left(\frac{\mu}{z} + B\right)^{-1} x \, \mathrm{d}z. \end{aligned}$$

The operators S_μ are clearly bounded from $D(A)$ to X. For T_μ this follows by the commutator estimate. Indeed for $x \in D(A)$, we have $(\mu/z + B)^{-1}x \in D(A)$ and

$$\left\| A \left(\frac{\mu}{z} + B\right)^{-1} x \right\| = \left\| \left(\left(\frac{\mu}{z} + B\right)^{-1} + \left[A, \left(\frac{\mu}{z} + B\right)^{-1}\right] A^{-1} \right) Ax \right\| \leq \left(\frac{|z|}{|\mu|} + \frac{C|z|^{1+\beta}}{|\mu|^{1+\beta}} \right) \|Ax\|.$$

The operators S_μ and T_μ are even bounded on X. In order to show this, we introduce the following lemma due to Kalton and Weis [KW01, Lemma 4.1]. Further proofs may be found also in [DDH+04].

Lemma 4.1. *Suppose $A \in \mathcal{H}^\infty(X)$, $\phi > \varphi_A^\infty$ and $h \in H_0^\infty(\Sigma_\phi)$. Then there is a constant $C > 0$, such that*

$$\left\| \sum_{k \in \mathbb{Z}} \alpha_k h(2^k t A) \right\|_{\mathcal{L}(X)} \leq C \sup_{k \in \mathbb{Z}} |\alpha_k|$$

for all $\alpha_k \in \mathbb{C}$ and $t > 0$.

The above Lemma 4.1 enables us to prove that S_μ and T_μ are bounded on X. More precisely, we have the following.

Lemma 4.2. *Let $\gamma \in (0, \pi - \theta_A - \theta_B)$. The operators S_μ and T_μ have unique bounded extensions on X for every $\mu \in \Sigma_\gamma$ and there is a constant C_γ, such that*

$$\|S_\mu\|_{\mathcal{L}(X)} + \|T_\mu\|_{\mathcal{L}(X)} \leq \frac{C_\gamma}{|\mu|} \left(1 + \frac{1}{|\mu|^\beta}\right).$$

Proof. By Cauchy's Theorem we may rewrite S_μ for an arbitrary number $a \in (0,1)$ as

$$S_\mu x = \frac{1}{2\pi i} \int_{\Gamma_\varphi^r} \frac{1}{z^{1+a}} \left(\frac{\mu}{z} + B\right)^{-1} A^a (z-A)^{-1} x \, \mathrm{d}z.$$

In a second step we commute some part of $A^a(z-A)^{-1}$ with the resolvent of B and get for every $b \in (0,1)$

$$\begin{aligned} S_\mu x &= \frac{1}{2\pi i}\int_{\Gamma_\varphi^r} \frac{1}{z^a}\big(A^a(z-A)^{-1}\big)^b(\mu+zB)^{-1}\big(A^a(z-A)^{-1}\big)^{1-b}x \,\mathrm{d}z \\ &\quad + \frac{1}{2\pi i}\int_{\Gamma_\varphi^r} \frac{1}{z^a}\Big[(\mu+zB)^{-1},\big(A^a(z-A)^{-1}\big)^b\Big]\big(A^a(z-A)^{-1}\big)^{1-b}x \,\mathrm{d}z \\ &=: I_1 + I_2. \end{aligned}$$

The right values of a and b will be chosen later on.

Analogously we get

$$\begin{aligned} T_\mu x &= \frac{1}{2\pi i}\int_{\Gamma_\varphi^r} \frac{1}{z^a}\big(A^a(z-A)^{-1}\big)^{1-b}(\mu+zB)^{-1}\big(A^a(z-A)^{-1}\big)^{b}x \,\mathrm{d}z \\ &\quad - \frac{1}{2\pi i}\int_{\Gamma_\varphi^r} \frac{1}{z^a}\big(A^a(z-A)^{-1}\big)^{1-b}\Big[(\mu+zB)^{-1},\big(A^a(z-A)^{-1}\big)^b\Big]x \,\mathrm{d}z, \end{aligned}$$

so that the proofs for S_μ and T_μ are basically the same and we will concentrate on S_μ.

The main step to estimate I_2 is to use the bounded H^∞-calculus of A and write $(A^a(z-A)^{-1})^b$ as a contour integral. To this end, we define the function $g_z(\zeta) := (\zeta^a(z-\zeta)^{-1})^b$ and choose angles ω and ϑ with $\theta_A < \omega < \vartheta < \varphi$ as well as numbers $\tilde r$ and $\hat r$ satisfying $r < \tilde r < \hat r < \|A^{-1}\|^{-1}$. Then $g_z \in H_0^\infty(\Sigma_\vartheta)$ and we have

$$\big(A^a(z-A)^{-1}\big)^b = \frac{1}{2\pi i}\int_{\Gamma_\omega^{\hat r}} g_z(\zeta)(\zeta-A)^{-1}\,\mathrm{d}\zeta.$$

Using this, we can rewrite the commutator in I_2 as

$$\Big[(\mu+zB)^{-1},\big(A^a(z-A)^{-1}\big)^b\Big] = \frac{1}{2\pi i}\int_{\Gamma_\omega^{\hat r}} \frac{\zeta^{ab}}{(z-\zeta)^b}\Big[(\mu+zB)^{-1},(\zeta-A)^{-1}\Big]\,\mathrm{d}\zeta.$$

A closer look at the remaining commutator yields

$$\Big[(\mu+zB)^{-1},(\zeta-A)^{-1}\Big] = \frac{1}{z}(\zeta-A)^{-1}\left[\Big(\frac{\mu}{z}+B\Big)^{-1},A\right](\zeta-A)^{-1}$$

and, using the commutator estimates, that leads to

$$\begin{aligned} &\left\|\Big[(\mu+zB)^{-1},\big(A^a(z-A)^{-1}\big)^b\Big]\right\| \\ &\le C\frac{1}{|z|}\int_{\Gamma_\omega^{\hat r}} \frac{|\zeta|^{ab}}{|z-\zeta|^b}\|(\zeta-A)^{-1}\|\left\|\left[\Big(\frac{\mu}{z}+B\Big)^{-1},A\right](\zeta-A)^{-1}\right\|\,\mathrm{d}|\zeta| \\ &\le C|z|^{\beta-b}\int_{\Gamma_\omega^{\hat r}} \frac{|z|^b|\zeta|^{ab}}{|z-\zeta|^b(1+|\zeta|)^{2-\alpha}|\mu|^{1+\beta}}\,\mathrm{d}|\zeta|. \end{aligned} \tag{4.1}$$

In order to estimate this integral, we first observe, that there is a constant $C>0$, such that for all $z \in \Gamma_\varphi^r$ and all $\zeta \in \Gamma_\omega^{\hat r}$ we have

$$|z-\zeta| \ge C(|z|+|\zeta|), \tag{4.2}$$

at least if we have $\varphi - \omega < \pi/2$, but this can be guaranteed by a suitable choice of ω above. Thus (4.1) may be estimated further by

$$C\frac{|z|^{\beta-b}}{|\mu|^{1+\beta}}\int_{\Gamma_\omega^{\hat r}}\left(\frac{|z|}{|z|+|\zeta|}\right)^b\frac{|\zeta|^{ab}}{(1+|\zeta|)^{2-\alpha}}\,\mathrm{d}|\zeta| \le C\frac{|z|^{\beta-b}}{|\mu|^{1+\beta}}\int_{\Gamma_\omega^{\hat r}}\frac{|\zeta|^{ab}}{(1+|\zeta|)^{2-\alpha}}\,\mathrm{d}|\zeta|.$$

This last integral is convergent if a and b satisfy $\alpha < 1 - ab$.

Using this result, we estimate I_2 as

$$\|I_2\| \le \frac{C}{|\mu|^{1+\beta}}\int_{\Gamma_\varphi^r}|z|^{\beta-a-b}\left\|\big(A^a(z-A)^{-1}\big)^{1-b}\right\|\|x\|\,\mathrm{d}|z|.$$

As the function $x \mapsto x^a/(c+x)$, $x \in [0,\infty)$ takes its maximum at $x = ca/(1-a)$, we see by the bounded H^∞-calculus of A and (4.2), that

$$\left\| \left(A^a(z-A)^{-1}\right)^{1-b} \right\| \leq C \sup_{x\in[0,\infty)} \left(\frac{x^a}{|z|+x}\right)^{1-b} = C|z|^{(a-1)(1-b)}.$$

This finally yields the estimate

$$\|I_2\| \leq \frac{C}{|\mu|^{1+\beta}} \int_{\Gamma_\varphi^r} |z|^{\beta-ab-1} \,\mathrm{d}|z| \,\|x\|,$$

that gives us a second condition for a and b to make this integral converge: $\beta < ab$. If we choose $a, b \in (0,1)$, such that $\alpha+\beta < \alpha + ab < 1$, which is possible thanks to $\alpha+\beta<1$, both conditions are satisfied simultaneously and we get $\|I_2\| \leq C/|\mu|^{1+\beta}\|x\|$.

We now turn our attention to I_1. As we have seen, the integrals defining I_2 and $S_\mu x$ converge absolutely. Thus the integral defining I_1 also converges absolutely. Since there is no singularity of the integrand for small $|z|$, we only look at $|z| \geq 2^{n_0}$, where $n_0 \in \mathbb{Z}$ is a fixed number, such that $2^{n_0} > r$. For this part of I_1 we may write thanks to the absolute convergence

$$\lim_{N\to\infty} \int_{\Gamma_\varphi^{r,N}} \frac{1}{z^a}\left(A^a(z-A)^{-1}\right)^b (\mu+zB)^{-1}\left(A^a(z-A)^{-1}\right)^{1-b} x \,\mathrm{d}z,$$

where $\Gamma_\varphi^{r,N} := \{\lambda \in \Gamma_\varphi^r : 2^{n_0} \leq |\lambda| \leq 2^N\}$ for $N > n_0$.

For one of these integrals we have

$$\begin{aligned} K_N &:= \int_{2^{n_0}}^{2^N} \frac{1}{(te^{i\varphi})^a}\left(A^a(te^{i\varphi}-A)^{-1}\right)^b(\mu+te^{i\varphi}B)^{-1}\left(A^a(te^{i\varphi}-A)^{-1}\right)^{1-b} x e^{i\varphi}\,\mathrm{d}t \\ &= e^{i\varphi(1-a)} \sum_{k=n_0}^{N-1} \int_{2^k}^{2^{k+1}} \frac{1}{t}\left(\left(\frac{A}{t}\right)^a\left(e^{i\varphi}-\frac{A}{t}\right)^{-1}\right)^b (\mu+te^{i\varphi}B)^{-1} \\ &\qquad \cdot \left(\left(\frac{A}{t}\right)^a\left(e^{i\varphi}-\frac{A}{t}\right)^{-1}\right)^{1-b} x\,\mathrm{d}t. \end{aligned}$$

Substituting $t = 2^k s$ and setting $\tilde{g}_z := g_z^{(1-b)/b}$, we obtain

$$K_N = e^{i\varphi(1-a)} \sum_{k=n_0}^{N-1} \int_1^2 g_{e^{i\varphi}}\left(\frac{A}{2^k s}\right)(\mu+2^k s e^{i\varphi}B)^{-1}\,\tilde{g}_{e^{i\varphi}}\left(\frac{A}{2^k s}\right) x\,\frac{\mathrm{d}s}{s}.$$

In order to estimate the norm of this expression, we choose by the Hahn-Banach theorem $x^* \in X'$ with $\|x^*\| = 1$ and

$$\begin{aligned} \|K_N\| &= \left|\left\langle \sum_{k=n_0}^{N-1} \int_1^2 g_{e^{i\varphi}}\left(\frac{A}{2^k s}\right)(\mu+2^k s e^{i\varphi}B)^{-1}\,\tilde{g}_{e^{i\varphi}}\left(\frac{A}{2^k s}\right) x\,\frac{\mathrm{d}s}{s}, x^*\right\rangle\right| \\ &\leq \int_1^2 \left|\sum_{k=n_0}^{N-1} \left\langle g_{e^{i\varphi}}\left(\frac{A}{2^k s}\right)(\mu+2^k s e^{i\varphi}B)^{-1}\,\tilde{g}_{e^{i\varphi}}\left(\frac{A}{2^k s}\right) x, x^*\right\rangle\right| \frac{\mathrm{d}s}{s}. \end{aligned}$$

Having in mind the $\mathcal{R}$-sectoriality of B, we plug in independent, symmetric, $\{-1,1\}$-valued random variables $\varepsilon_{n_0}, \ldots, \varepsilon_{N-1}$ on some probability space Ω.

$$= \int_1^2 \left|\int_\Omega \sum_{k=n_0}^{N-1} \varepsilon_k^2(\omega)\left\langle g_{e^{i\varphi}}\left(\frac{A}{2^k s}\right)(\mu+2^k s e^{i\varphi}B)^{-1}\,\tilde{g}_{e^{i\varphi}}\left(\frac{A}{2^k s}\right) x, x^*\right\rangle \mathrm{d}\omega\right| \frac{\mathrm{d}s}{s}.$$

As these random variables are independent, we may write

$$= \int_1^2 \left|\int_\Omega \left\langle \sum_{k=n_0}^{N-1} \varepsilon_k(\omega)(\mu+2^k s e^{i\varphi}B)^{-1}\,\tilde{g}_{e^{i\varphi}}\left(\frac{A}{2^k s}\right) x, \sum_{k=n_0}^{N-1} \varepsilon_k(\omega)\overline{g_{e^{i\varphi}}}\left(\frac{A^*}{2^k s}\right) x^*\right\rangle \mathrm{d}\omega\right| \frac{\mathrm{d}s}{s}.$$

Note that $A^* \in \mathcal{H}^\infty(X')$ with $\varphi_A^\infty = \varphi_{A^*}^\infty$ in case $D(A^*)$ is dense in X'. If this is not the case, we may use the sun-dual $A^\odot$ on $X^\odot$ instead (c.f. [vN92, Chapter 1.3]). By the Cauchy-Schwarz inequality we see

$$\begin{aligned}
\leq\ & \int_1^2 \left\| \sum_{k=n_0}^{N-1} \varepsilon_k (\mu + 2^k s e^{i\varphi} B)^{-1}\ \tilde{g}_{e^{i\varphi}} \left(\frac{A}{2^k s}\right) x \right\|_{L^2(\Omega;X)} \left\| \sum_{k=n_0}^{N-1} \varepsilon_k \overline{g_{e^{i\varphi}}} \left(\frac{A^*}{2^k s}\right) x^* \right\|_{L^2(\Omega;X')} \frac{\mathrm{d}s}{s} \\
=\ & \int_1^2 \frac{1}{|\mu|} \left\| \sum_{k=n_0}^{N-1} \varepsilon_k \frac{\mu}{2^k s e^{i\varphi}} \left(\frac{\mu}{2^k s e^{i\varphi}} + B\right)^{-1}\ \tilde{g}_{e^{i\varphi}} \left(\frac{A}{2^k s}\right) x \right\|_{L^2(\Omega;X)} \\
& \cdot \left\| \sum_{k=n_0}^{N-1} \varepsilon_k \overline{g_{e^{i\varphi}}} \left(\frac{A^*}{2^k s}\right) x^* \right\|_{L^2(\Omega;X')} \frac{\mathrm{d}s}{s}.
\end{aligned}$$

As B is $\mathcal{R}$-sectorial, and $|\arg(\mu/(2^k s e^{i\varphi}))| < \pi - \theta_B$ we can estimate the latter term by

$$\leq C_R \int_1^2 \frac{1}{|\mu|} \left\| \sum_{k=n_0}^{N-1} \varepsilon_k\ \tilde{g}_{e^{i\varphi}} \left(\frac{A}{2^k s}\right) x \right\|_{L^2(\Omega;X)} \left\| \sum_{k=n_0}^{N-1} \varepsilon_k \overline{g_{e^{i\varphi}}} \left(\frac{A^*}{2^k s}\right) x^* \right\|_{L^2(\Omega;X')} \frac{\mathrm{d}s}{s},$$

where $C_R := \mathcal{R}\left(\{\lambda(\lambda + B)^{-1} : \lambda \in \Sigma_{\pi-\theta_B}\}\right)$.

Finally, we apply Lemma 4.1 to the two remaining norms. This yields

$$\|K_N\| \leq \frac{C}{|\mu|} \int_1^2 \left(\sup_{k=n_0}^{N-1} |\varepsilon_k|\right)^2 \|x\|\ \frac{\mathrm{d}s}{s} = \frac{C}{|\mu|}\|x\|.$$

Summing up our considerations, we finally get for all $x \in D(A)$

$$\|S_\mu x\| \leq \|I_1\| + \|I_2\| \leq C\frac{1}{|\mu|}\left(1 + \frac{1}{|\mu|^\beta}\right)\|x\|.$$

Thus, by density, we have the same estimate on all of X, finishing the proof. □

The above lemma is now the basis for the construction of a right inverse for $\mu + AB$. Also, we show that $\nu + AB$ is injective.

To this end, let $x \in D(A) \cap R(A)$. Then, by the closedness of B, we even have $S_\mu x \in D(B)$ and we can calculate with the help of Cauchy's Theorem

$$\begin{aligned}
BS_\mu x &= \frac{1}{2\pi i} \int_{\Gamma_\varphi^r} \frac{1}{z^2} B \left(\frac{\mu}{z} + B\right)^{-1} A(z-A)^{-1} x\ \mathrm{d}z \\
&= \frac{1}{2\pi i} \int_{\Gamma_\varphi^r} \frac{1}{z^2} A(z-A)^{-1} x\ \mathrm{d}z - \frac{\mu}{2\pi i} \int_{\Gamma_\varphi^r} \frac{1}{z^3} \left(\frac{\mu}{z} + B\right)^{-1} A(z-A)^{-1} x\ \mathrm{d}z \qquad (4.3)\\
&= A^{-1}x - \frac{\mu}{2\pi i} \int_{\Gamma_\varphi^r} \frac{1}{z^2} \left(\frac{\mu}{z} + B\right)^{-1} (z-A)^{-1} x\ \mathrm{d}z.
\end{aligned}$$

This yields

$$\|BS_\mu x\| \leq \|A^{-1}\|\|x\| + C|\mu| \int_{\Gamma_\varphi^r} \frac{1}{|z|^2} \frac{|z|}{|\mu|} \frac{1}{1+|z|}\ \mathrm{d}|z| \|x\| \leq C\|x\|$$

for every $x \in D(A) \cap R(A)$. By density of $D(A) \cap R(A)$ and the closedness of B this shows that we even have $S_\mu \in \mathcal{L}(X, D(B))$.

Looking again at (4.3), we see, that for every $x \in D(A) \cap R(A)$ the integral on the right hand side is in $D(A)$. Thus we get

$$BS_\mu x = A^{-1}\left(x - \frac{\mu}{2\pi i} \int_{\Gamma_\varphi^r} \frac{1}{z^2} A \left(\frac{\mu}{z} + B\right)^{-1} (z-A)^{-1} x\ \mathrm{d}z\right) = A^{-1}\left(x - \mu S_\mu x + Q_\mu x\right),$$

where

$$Q_\mu := \frac{\mu}{2\pi i}\int_{\Gamma_\varphi^r} \frac{1}{z^2}\left[\left(\frac{\mu}{z}+B\right)^{-1}, A\right](z-A)^{-1}\,\mathrm{d}z$$

is in $\mathcal{L}(X)$ with

$$\|Q_\mu\|_{\mathcal{L}(X)} \le C\frac{c}{|\mu|^\beta}, \tag{4.4}$$

thanks to the commutator estimate. As BS_μ is a bounded operator on X and A is closed, this means, that even for every $x \in X$ we have $BS_\mu x \in D(A)$ and $ABS_\mu x = (1+Q_\mu)x - \mu S_\mu x$. This implies $S_\mu x \in D(AB)$ for every $x \in X$ and

$$(\mu + AB)S_\mu = 1 + Q_\mu$$

in $\mathcal{L}(X)$. Choosing $\nu \ge 0$, such that $\|Q_\mu\| \le \delta < 1$ whenever $|\mu| \ge \nu$, we get the right inverse $R_\mu := S_\mu(1+Q_\mu)^{-1}$ of $\mu + AB$. Summarizing we proved the following result.

Lemma 4.3. *There exists $\nu \ge 0$ and for every $\gamma \in (0, \pi - \theta_A - \theta_B)$ there is a constant $C_\gamma \ge 0$ such that for every $\mu \in \Sigma_\gamma$ with $|\mu| \ge \nu$ the operator $\mu + AB$ is surjective with right inverse R_μ and*

$$\|R_\mu\| \le \frac{C_\gamma}{|\mu|}\left(1 + \frac{1}{|\mu|^\beta}\right).$$

Before we start to prove that $\mu + AB$ is injective, we show that $D(AB) \cap D(A)$ is dense in X. We will need this later on, but on the other hand this also implies the density of $D(AB)$ in X, that we need to prove sectoriality of $\nu + AB$. In order to do so, fix $\lambda \in \varrho(B)$ and note that, thanks to $(\lambda - B)^{-1}D(A) \subseteq D(A)$ we have $(\lambda - B)^{-1}D(A) \subseteq D(AB) \cap D(A)$. Let $x \in X$. By density of $D(B)$, we approximate x in X by $(w_n) \subseteq D(B)$ and set $y_n := (\lambda - B)w_n$. Since $D(A)$ is also dense in X, for every $n \in \mathbb{N}$, there is a $z_n \in D(A)$, such that $\|z_n - y_n\| \le 1/n$. Now, $(\lambda - B)^{-1}z_n \in D(AB) \cap D(A)$ for every $n \in \mathbb{N}$ and

$$\|(\lambda - B)^{-1}z_n - x\| \le \|(\lambda - B)^{-1}\|\|z_n - y_n\| + \|w_n - x\| \le \frac{C}{n} + \|w_n - x\| \longrightarrow 0 \qquad (n \to \infty).$$

Lemma 4.4. *There is a constant $\nu \ge 0$ such that the operator $\mu + AB$ is injective for all $\mu \in \Sigma_{\pi-\theta_A-\theta_B}$ with $|\mu| \ge \nu$.*

Proof. The beginning of the proof is very similar to the construction of the right inverse, we just look at $T_\mu B$ instead of BS_μ. Let $x \in D(AB) \cap D(A)$. Then we have $Bx \in D(A)$ and

$$\begin{aligned} T_\mu Bx &= \frac{1}{2\pi i}\int_{\Gamma_\varphi^r} \frac{1}{z^2}A(z-A)^{-1}\left(\frac{\mu}{z}+B\right)^{-1} Bx\,\mathrm{d}z \\ &= A^{-1}x - \frac{\mu}{2\pi i}\int_{\Gamma_\varphi^r} \frac{1}{z^2}(z-A)^{-1}\left(\frac{\mu}{z}+B\right)^{-1} x\,\mathrm{d}z, \end{aligned}$$

analogously to (4.3). As x is supposed to be in $D(A)$, this again implies $T_\mu Bx = A^{-1}(x - \mu T_\mu x)$. We deduce as before, that $AT_\mu B \in \mathcal{L}(X)$ and $AT_\mu B = I - \mu T_\mu$ thanks to the density of $D(AB) \cap D(A)$ in X.

Since $x \in D(AB)$ we may commute A and T_μ and we see, that

$$T_\mu(\mu + AB)x = x - [T_\mu, A]Bx, \qquad x \in D(AB).$$

In order to prove injectivity of $\mu + AB$, choose $x \in D(AB)$ with $(\mu + AB)x = 0$. Then our considerations above yield $x = [T_\mu, A]Bx = \mu[T_\mu, A]A^{-1}x$. Choosing $a \in (\beta, 1-\alpha)$ and applying

A^{-a} to this identity, we obtain

$$\begin{aligned} A^{-a}x &= \frac{\mu}{2\pi i}\int_{\Gamma_\varphi^r} \frac{1}{z^2} A^{1-a}(z-A)^{-1}\left[\Big(\frac{\mu}{z}+B\Big)^{-1}, A\right] A^{-1+a}A^{-a}x \,\mathrm{d}z \\ &= -\frac{\mu}{4\pi}\int_{\Gamma_\varphi^r}\int_{\Gamma_\varphi^r} \frac{A^{1-a}(z-A)^{-1}}{z^2\zeta^{1-a}}\left[\Big(\frac{\mu}{z}+B\Big)^{-1}, A\right](\zeta-A)^{-1}A^{-a}x \,\mathrm{d}\zeta\,\mathrm{d}z, \end{aligned}$$

by the Dunford calculus. The commutator estimate and $\|A^{1-a}(z-A)^{-1}\| \le C|z|^{-a}$ then yields

$$\|A^{-a}x\| \le C\frac{c}{|\mu|^\beta}\int_{\Gamma_\varphi^r}\frac{1}{|z|^{a-\beta}(1+|z|)}\,\mathrm{d}|z|\int_{\Gamma_\varphi^r}\frac{1}{|\zeta|^{1-a}(1+|\zeta|)^{1-\alpha}}\,\mathrm{d}|\zeta|\|A^{-a}x\|.$$

By the choice of a, these two integrals converge, so we end up with

$$\|A^{-a}x\| \le C\frac{c}{|\mu|^\beta}\|A^{-a}x\|.$$

If we choose ν so big, that $Cc/|\mu|^\beta < 1$, this implies $A^{-a}x = 0$, and hence $x = 0$, finishing the proof. □

Summarizing, we have proved Theorem 3.1: In fact, if we choose $\nu \ge 0$ big enough, then, for all $\mu \in \Sigma_{\pi-\theta_A-\theta_B} \setminus B(0,\nu)$ the operator $\mu + AB$ is surjective by Lemma 4.3 and injective by Lemma 4.4. Furthermore $D(AB)$ and $R(AB)$ are dense in X and for every $\gamma \in (0, \pi-\theta_A-\theta_B)$ we have the resolvent estimate

$$\|(\mu+AB)^{-1}\| = \|R_\mu\| \le \frac{C_\gamma}{|\mu|}\left(1+\frac{1}{|\mu|^\beta}\right), \qquad \mu \in \Sigma_\gamma \setminus B(0,\nu).$$

This implies, that $\varphi_{\nu+AB} \le \theta_A + \theta_B$.

5. Bounded H^∞-calculus for $\nu + AB$

In order to prove a bounded H^∞-calculus for $\nu + AB$ with $\varphi_{\nu+AB} \le \theta_A + \theta_B$, we choose angles ϕ and η with $\theta_A + \theta_B < \phi < \eta < \pi$ and $f \in H_0^\infty(\Sigma_\eta)$. We have to prove the estimate

$$\left\| \int_{\Gamma_\phi} f(\lambda)\big(\lambda - (\nu+AB)\big)^{-1}\,\mathrm{d}\lambda \right\| \le C\|f\|_\infty^\eta. \tag{5.1}$$

Here, we choose $\nu \ge 0$ in such a way, that by Theorem 3.1 the operator $\nu + AB$ is sectorial. Thus the resolvent in the above integral exists and can be respresented as $(\lambda - (\nu+AB))^{-1} = -S_{\nu-\lambda}(1+Q_{\nu-\lambda})^{-1}$. Note that, as $|\arg(-\lambda)| = \pi - \phi < \pi - \theta_A - \theta_B$, we have $-\lambda \in \Sigma_\gamma$ for some $\gamma \in (\pi-\phi, \pi-\theta_A-\theta_B)$, so that $S_{\nu-\lambda}$ and $Q_{\nu-\lambda}$ are well defined.

As we have $S_\mu(1+Q_\mu)^{-1} = S_\mu - S_\mu Q_\mu(1+Q_\mu)^{-1} =: S_\mu + P_\mu$, we may split the integral for (5.1) into two parts, namely

$$\int_{\Gamma_\phi} f(\lambda)\big(\lambda-(\nu+AB)\big)^{-1}\,\mathrm{d}\lambda = -\int_{\Gamma_\phi} f(\lambda)S_{\nu-\lambda}\,\mathrm{d}\lambda - \int_{\Gamma_\phi} f(\lambda)P_{\nu-\lambda}\,\mathrm{d}\lambda =: J_1 + J_2.$$

The easy part is to estimate J_2, as we may simply take the norm into the integral. Using Lemma 4.2 and (4.4), we get

$$\begin{aligned} \left\|\int_{\Gamma_\phi} f(\lambda)P_{\nu-\lambda}\,\mathrm{d}\lambda\right\| &\le \|f\|_\infty^\eta \int_{\Gamma_\phi}\|S_{\nu-\lambda}Q_{\nu-\lambda}(1+Q_{\nu-\lambda})^{-1}\|\,\mathrm{d}|\lambda| \\ &\le C\|f\|_\infty^\eta\int_{\Gamma_\phi}\left(1+\frac{1}{|\nu-\lambda|^\beta}\right)\frac{1}{|\nu-\lambda|^{1+\beta}}\,\mathrm{d}|\lambda| \le C\|f\|_\infty^\eta, \end{aligned}$$

thanks to $\beta > 0$.

In order to estimate J_1, we calculate for $x \in D(A)$, using the definition of S_μ

$$\begin{aligned}\int_{\Gamma_\phi} f(\lambda) S_{\nu-\lambda} x \, \mathrm{d}\lambda &= \frac{1}{2\pi i}\int_{\Gamma_\phi} f(\lambda) \int_{\Gamma_\varphi^r} \frac{1}{z^2}\left(\frac{\nu-\lambda}{z}+B\right)^{-1} A(z-A)^{-1}x \, \mathrm{d}z \, \mathrm{d}\lambda \\ &= -\frac{1}{2\pi i}\int_{\Gamma_\varphi^r} \frac{1}{z}\int_{\Gamma_\phi} f(\lambda)\big(\lambda-(\nu+zB)\big)^{-1} \, \mathrm{d}\lambda \, A(z-A)^{-1}x \, \mathrm{d}z.\end{aligned}$$

As $B \in \mathcal{H}^\infty(X)$, by permanence properties of the bounded H^∞-calculus (see e.g. [DHP03, Proposition 2.11]), the operator $\nu + zB$ also is in $\mathcal{H}^\infty(X)$ with $\varphi^\infty_{\nu+zB} \le \varphi^\infty_B + |\arg(z)| < \theta_B + \phi - \theta_B = \phi$. Thus we may write

$$J_1 = -\int_{\Gamma_\phi} f(\lambda) S_{\nu-\lambda}x \, \mathrm{d}\lambda = \int_{\Gamma_\varphi^r} \frac{1}{z} f(\nu+zB) A(z-A)^{-1}x \, \mathrm{d}z.$$

By Cauchy's Theorem we may rewrite this for an arbitrary number $a \in (0,1)$ as

$$\int_{\Gamma_\varphi^r} \frac{1}{z^a} f(\nu+zB) A^a(z-A)^{-1}x \, \mathrm{d}z,$$

in the same manner as in the proof of Lemma 4.2. The resulting integral now also looks very much like the integral we started the proof of this lemma with (we just have $f(\nu+zB)$ instead of $(\nu+zB)^{-1}$). Indeed, the method here will be exactly the same as in the proof of Lemma 4.2, so we will freely use the notations introduced there and only indicate the differences in the proof.

After commuting $(A^a(z-A)^{-1})^b$ with $f(\nu+zB)$ we now end up with

$$\begin{aligned} J_1 &= -\int_{\Gamma_\varphi^r} \frac{1}{z^a}\big(A^a(z-A)^{-1}\big)^b f(\nu+zB)\big(A^a(z-A)^{-1}\big)^{1-b}x \, \mathrm{d}z \\ &\quad -\int_{\Gamma_\varphi^r} \frac{1}{z^a}\left[f(\nu+zB), \big(A^a(z-A)^{-1}\big)^b\right]\big(A^a(z-A)^{-1}\big)^{1-b}x \, \mathrm{d}z \\ &=: J_{11} + J_{12}. \end{aligned}$$

Evaluating the commutator in J_{12} this time leads to a double integral:

$$\left[f(\nu+zB), \big(A^a(z-A)^{-1}\big)^b\right] = \frac{1}{4\pi^2}\int_{\Gamma_\phi}\int_{\Gamma_\omega^{\hat r}} f(\lambda)\frac{\zeta^{ab}}{(z-\zeta)^b}\left[\big(\lambda-(\nu+zB)\big)^{-1}, (\zeta-A)^{-1}\right] \mathrm{d}\zeta \, \mathrm{d}\lambda.$$

As we may again apply our commutator estimates, thanks to

$$\left[\big(\lambda-(\nu+zB)\big)^{-1}, (\zeta-A)^{-1}\right] = \frac{1}{z}(\zeta-A)^{-1}\left[\left(\frac{\nu-\lambda}{z}+B\right)^{-1}, A\right](\zeta-A)^{-1},$$

we find

$$\begin{aligned} &\left\|\left[f(\nu+zB), \big(A^a(z-A)^{-1}\big)^b\right]\right\| \\ &\le C\|f\|_\infty^\eta \frac{1}{|z|}\int_{\Gamma_\phi}\int_{\Gamma_\omega^{\hat r}} \frac{|\zeta|^{ab}}{|z-\zeta|^b}\|(\zeta-A)^{-1}\|\left\|\left[\left(\frac{\nu-\lambda}{z}+B\right)^{-1}, A\right](\zeta-A)^{-1}\right\| \mathrm{d}|\zeta| \, \mathrm{d}|\lambda| \\ &\le C\|f\|_\infty^\eta |z|^{\beta-b}\int_{\Gamma_\phi}\frac{1}{|\nu-\lambda|^{1+\beta}}\,\mathrm{d}|\lambda| \int_{\Gamma_\omega^{\hat r}} \frac{|z|^b|\zeta|^{ab}}{|z-\zeta|^b(1+|\zeta|)^{2-\alpha}}\,\mathrm{d}|\zeta| \;\le\; C\|f\|_\infty^\eta |z|^{\beta-b}. \end{aligned}$$

Here the first integral converges thanks to $\beta > 0$ and the second is of exactly the same form as in (4.1). Now we can conclude $\|J_{12}\| \le C\|f\|_\infty^\eta \|x\|$ as before.

Also for J_{11} we may do the same calculations as we did for I_1 in the proof of Lemma 4.2 until we reach the line

$$\int_1^2 \left\|\sum_{k=n_0}^{N-1} \varepsilon_k f(\nu + 2^k s e^{i\varphi}B)\, \tilde g_{e^{i\varphi}}\left(\frac{A}{2^k s}\right)x\right\|_{L^2(\Omega;X)} \left\|\sum_{k=n_0}^{N-1} \varepsilon_k \overline{g_{e^{i\varphi}}}\left(\frac{A^*}{2^k s}\right)x^*\right\|_{L^2(\Omega;X')} \frac{\mathrm{d}s}{s},$$

where we used the $\mathcal{R}$-sectoriality of B before. Evidently, now the $\mathcal{R}$-bounded H^∞-calculus of B does the job. Thus we end up with

$$C_R \|f\|_\infty^\eta \int_1^2 \left\| \sum_{k=n_0}^{N-1} \varepsilon_k \, \tilde{g}_{e^{i\varphi}} \left(\frac{A}{2^k s} \right) x \right\|_{L^2(\Omega;X)} \left\| \sum_{k=n_0}^{N-1} \varepsilon_k \overline{g_{e^{i\varphi}}} \left(\frac{A^*}{2^k s} \right) x^* \right\|_{L^2(\Omega;X')} \frac{\mathrm{d}s}{s},$$

where $C_R := \mathcal{R}\left(\left\{ f(\nu + 2^k s e^{i\varphi} B) : f \in H_0^\infty(\Sigma_\eta), \ \|f\|_\infty^\eta \le 1 \right\} \right)$. As

$$\left\{ f(\nu + 2^k s e^{i\varphi} B) : f \in H_0^\infty(\Sigma_\eta), \ \|f\|_\infty^\eta \le 1 \right\} \subseteq \{ f(B) : f \in H_0^\infty(\Sigma_{\theta_B}), \ \|f\|_\infty^\eta \le 1 \}$$

and $\theta_B > \varphi_B^{\mathcal{R}\infty}$, this $\mathcal{R}$-bound is finite. Now we estimate the two remaining norms as before, getting $\|J_1\| \le C \|f\|_\infty^\eta \|x\|$ for all $x \in D(A)$ in the end. A density argument again finishes the proof. □

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