$L^p - L^q$-ESTIMATES FOR PARABOLIC SYSTEMS IN NON-DIVERGENCE FORM WITH VMO-COEFFICIENTS

ROBERT HALLER-DINTELMANN, HORST HECK, MATTHIAS HIEBER

Abstract. Consider a parabolic $N \times N$-system of order m on $\mathbb{R}^n$ with top-order coefficients $a_\alpha \in \mathrm{VMO} \cap L^\infty$. Let $1 < p, q < \infty$ and let ω be a Muckenhoupt weight. It is proved that systems of this kind possess a unique solution u satifying

$$\|u'\|_{L^q(J;L^p_\omega(\mathbb{R}^n)^N)} + \|\mathcal{A}u\|_{L^q(J;L^p_\omega(\mathbb{R}^n)^N)} \leq C\|f\|_{L^q(J;L^p_\omega(\mathbb{R}^n)^N)},$$

where $\mathcal{A}u = \sum_{|\alpha|\leq m} a_\alpha D^\alpha u$ and $J = [0, \infty)$. In particular, chosing $\omega = 1$, the realization of $\mathcal{A}$ in $L^p(\mathbb{R}^n)^N$ has maximal $L^p - L^q$-regularity.

1. Introduction

$L^p - L^q$-estimates for parabolic differential equations are of particular interest since, combined with interpolation theory, they provide a powerful tool for many nonlinear problems. Whereas $L^p - L^q$-regularity properties for a single second-order equation in divergence form are fairly well understood (see e.g. the recent monograph of Auscher [Aus04]), the situation is far from clear for systems in divergence form, for higher order operators or for equations (or systems) having a non-divergence structure.

In this paper we consider higher order parabolic $N \times N$-systems on $\mathbb{R}^n$ in non-divergence form, i.e. systems of the form

$$u_t - \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha u = f, \tag{1.1}$$

where the coefficients a_α for $|\alpha| = m$ belong to the class $\mathrm{VMO}(\mathbb{R}^n; \mathbb{C}^{N\times N})$. Here $\mathrm{VMO}(\mathbb{R}^n; \mathbb{C}^{N\times N})$ denotes the Sarason space of all functions on $\mathbb{R}^n$ with values in $\mathbb{C}^{N\times N}$ having vanishing mean oscillation. Our aim is to prove that equation (1.1) admits a unique solution which satisfies an estimate of the form

$$\|u'\|_{L^q(J;L^p_\omega(\mathbb{R}^n)^N)} + \|\mathcal{A}u\|_{L^q(J;L^p_\omega(\mathbb{R}^n)^N)} \leq C\|f\|_{L^q(J;L^p_\omega(\mathbb{R}^n)^N)}, \tag{1.2}$$

where $1 < p, q < \infty$, $\mathcal{A}u = \sum_{|\alpha|\leq m} a_\alpha D^\alpha u$, $J = [0, \infty)$ and $\omega \in A_p$ is a Muckenhoupt weight.

Note that in particular the Sobolev space $W^{\theta,n/\theta}(\mathbb{R}^n)$ for $\theta \in (0, 1]$ is contained in $\mathrm{VMO}(\mathbb{R}^n)$; thus our approach allows to treat parabolic systems with *not necessarily continuous* coefficients.

The elliptic problem $\mathcal{A}u = f$ for a single differential operator $\mathcal{A}$ of order $2m$ in non-divergence form subject to general boundary conditions was solved in the L^p-setting in a classical paper by Agmon, Douglis and Nirenberg [ADN59], provided the top-order coefficients of A are bounded and uniformly continuous. The corresponding result for parabolic equations (and even systems) was proved recently by Denk, Hieber and Prüss in [DHP03]. For previous results dealing with the case of $\mathbb{R}^n$, second order operators with particular boundary conditions or with additional regularity assumptions on the coefficients, we refer to [LSU68], [Duo90], [PS93], [AHS94], [Ama95], [HP97] and [DS97].

2000 *Mathematics Subject Classification.* 35J45, 35K55.

Key words and phrases. VMO, parabolic systems, $L^p - L^q$-estimates, Muckenhoupt weights.

Supported by the Deutsche Forschungsgemeinschaft (DFG) by the project "Regularity properties of elliptic and parabolic equations".

The study of linear and quasilinear *elliptic* equations with VMO-coefficients started with the pioneering work of Chiarenza, Frasca and Longo [CFL91], [CFL93]. They proved $W^{2,p}$-estimates for the solution u of the Dirichlet problem associated to

$$\sum_{i,j=1}^{n} a_{ij} D_i D_j u = f,$$

provided $1 < p < \infty$, $f \in L^p$ and $a_{ij} \in \mathrm{VMO} \cap L^\infty$. Their proof was based on parameter-dependent Calderón-Zygmund theory. The techniques were lateron generalized by Palagachev, Di Fazio, Maugeri and Softova [Pal95], [DP96], [MP98], [MPS00] to quasilinear elliptic equations and to the oblique derivative problem.

The corresponding result for certain *elliptic systems* was proved recently by Ragusa [Rag02].

The first result for a *scalar, second order parabolic equation* is due to Bramanti and Cerutti [BC93]. They extended the technique developed in [CFL93] to the parabolic situation. The oblique derivative problem in the parabolic situation was solved more recently by Softova [Sof00]. Using wavelet techniques, Duong and Yan [DY02] proved that the operators associated to these equations admit a bounded H^∞-calculus on $L^p(\mathbb{R}^n)$; see also previous work by Angeletti, Mazet and Tchamitchian [AMT97]. Combining their result with the Dore-Venni theorem one obtains also estimates of the form (1.2) for second order operators. For a result on elliptic boundary value problems with coefficients in VMO see Guidetti [Gui02].

We already mentioned that $L^p - L^q$-estimates for *parabolic systems in non-divergence form* have been considered so far only for coefficients which are bounded and uniformly continuous (see [DHP03]). The approach there was to localize and to consider perturbations which are small in L^∞. Of course, this method needs uniformly continuous coefficients.

Our approach to parabolic systems with VMO-coefficients is very different. It is based on a particular representation of the highest order derivatives of the solution u of $\lambda u + Au = f$. This representation allows to estimate $D^\nu u$ for $|\nu| = m$ by parameter-depending Calderón-Zygmund theory and commutator techniques. More precisely, we are aiming for a weighted a priori estimate of the form

$$\sum_{|\nu| \le m} \|\lambda^{1-\frac{|\nu|}{m}} D^\nu u\|_{p,\omega} \le C \|(\lambda + \mathcal{A})u\|_{p,\omega},$$

where $\omega \in A_p$ is a Muckenhoupt weight, $u \in W^{m,p}_\omega(\mathbb{R}^n)^N$, $\lambda \in \mathbb{C}$ belongs to a suitable sector of the complex plane and the coefficients of $\mathcal{A}$ are small in BMO.

The reason for introducing Muckenhoupt weights in this context is the following: combining the characterization theorem of maximal L^p-regularity due to Weis [Wei01] with results due to Rubio de Francia [Rub80] one sees that estimate (1.2) is implied by weighted estimates of the form

$$\|\lambda(\lambda + A)^{-1} f\|_{p,\omega} \le C \|f\|_{p,\omega}, \qquad \omega \in A_p,\, f \in L^p_\omega(\mathbb{R}^n)^N,$$

where A is the realization of $\mathcal{A}$ given by $\mathcal{A}u = \sum_{|\alpha| \le m} a_\alpha D^\alpha u$ in $L^p_\omega(\mathbb{R}^n)^N$, λ lies in a suitable sector of the complex plane and the constant C is allowed to depend on the A_p-constant of the weight ω only. For details see the following section and [HHH03]. This method was already successfully used in [HH03] for scalar parabolic equations.

The problem of determining the "minimal" regularity assumption on the top-order coefficients a_α of $\mathcal{A}$ such that (1.1) admits a unique solution satisfying (1.2) is still far from being solved. It is quite surprising that the answer also depends on the space dimension n. Indeed, it is shown in [HW04] that for scalar second order elliptic operators with L^∞-coefficients the estimate (1.2) holds true provided $n = 2$, $1 < q < \infty$ and $p \in (1, 2]$ is close to 2. This is no longer true for $n > 2$, even for $p = 2$, as the following example due to Talenti [Tal65] shows: let $n > 2$ and let Ω be the unit ball in $\mathbb{R}^n$. Set

$$\mathcal{A} = \sum_{i,j=1}^{n} \left(\frac{x_i x_j}{|x|^2}(1 - cn) + \delta_{ij} c\right) D_{ij} =: \sum_{i,j=1}^{n} a_{ij} D_{ij}$$

for $0 < c < \frac{n-2}{n(n-1)}$. Then $\mathcal{A} : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ is not an isomorphism. We refer also to Maugeri, Palagachev and Softova [MPS00] and the refences therein. Observe that in the above case $a_{ij} \in W^{1,n-\varepsilon}(\Omega)$ for all $\varepsilon > 0$. Thus parabolic equations associated to operators with coefficients $a_{ij} \in W^{1,n-\varepsilon}(\Omega)$ for $n \geq 3$ cannot fulfill an estimate of the form (1.2) in general. On the other hand, if $a_{ij} \in W^{1,n+\varepsilon}(\Omega)$, then by Morrey's theorem $a_{ij} \in C^\alpha(\overline{\Omega})$ and estimate (1.2) is well known in this case; see e.g. [HP97] or [DDHPV04]. Since $W^{1,n}(\mathbb{R}^n)$ is a subspace of $\mathrm{VMO}(\mathbb{R}^n)$, our regularity assumptions allow in particular to treat the limiting case above.

This paper is organized as follows. After collecting certain results on Muckenhoupt weights and fundamental solutions of parabolic systems in Section 2, we state our main result in section 3. Section 4 deals with singular integrals, parameter dependening Calderón-Zygmund kernels and commutator estimates in weighted L^p-spaces. These results are fundamental for the a priori estimate given in Section 5. Finally, in Section 6 we give a proof of our main result.

2. Preliminaries

We start this section with the definition of the Muckenhoupt class A_p for $1 < p < \infty$. More precisely, a function $0 \leq \omega \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is called an *A_p-weight in the sense of Muckenhoupt*, if there is a constant $C > 0$ such that

$$\left(\frac{1}{|Q|}\int_Q \omega \, \mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q \omega^{-\frac{1}{p-1}} \, \mathrm{d}x\right)^{p-1} \leq C,$$

for all cubes $Q \subseteq \mathbb{R}^n$ with sides parallel to the axes. The smallest such C is called the *A_p-constant* of ω. We call a constant $C = C(\omega)$ to be *A_p-consistent* if it depends on the A_p-constant of ω only. For operators acting in weighted L^p-spaces with a weight belonging to A_p the following *extrapolation theorem* (see [GR85, Theorem IV.5.19] or [Ste93, V.6.17]) is true. Let $1 < p, q < \infty$ and $\mathcal{T}$ be a family of operators such that for all $\omega \in A_p$ there exists a constant C, depending only on the A_p-constant of ω, such that

$$\|Tf\|_{p,\omega} \leq C\|f\|_{p,\omega}, \qquad T \in \mathcal{T}.$$

Then it follows that the same inequality, with p replaced by q, holds for all $\omega \in A_q$. This extrapolation theorem allows to give the following sufficient criterion for maximal $L^p - L^q$-regularity, and so (1.2), on L^p-spaces, see [HHH03]. The proof combines a result due to Garcia-Cuerva and Rubio de Francia [GR85, Theorem V.6.4] with a recent Fourier multiplier theorem due to Weis [Wei01].

Proposition 2.1 ([HHH03]). *Let $1 < p, q < \infty$ and assume that A is a sectorial operator in $L^p_\omega(\mathbb{R}^n)^N$ of angle $\varphi < \frac{\pi}{2}$. Suppose that a weighted estimate of the form*

$$\|s(is + A)^{-1}f\|_{p,\omega} \leq C\|f\|_{p,\omega}, \qquad \omega \in A_p,\ f \in L^p_\omega(\mathbb{R}^n)^N, \tag{2.1}$$

holds, where the constant C depends only on the A_p-constant of the weight ω. Then there exists a constant $M \geq 0$, such that

$$\|u'\|_{L^q(J;L^p_\omega(\mathbb{R}^n)^N)} + \|Au\|_{L^q(J;L^p_\omega(\mathbb{R}^n)^N)} \leq M\|f\|_{L^q(J;L^p_\omega(\mathbb{R}^n)^N)}.$$

In the following, we consider systems of differential operators of the form $\mathcal{A} = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$, where $D = -i(\partial_1, \ldots, \partial_n)$ and $a_\alpha \in L^\infty(\mathbb{R}^n; \mathbb{C}^{N\times N})$. We will assume that $\mathcal{A}$ is (M,θ)-elliptic; this means that there exist constants $\theta \in [0,\pi)$ and $M > 0$, such that the principal part $\mathcal{A}_\#(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ of the symbol of $\mathcal{A}$ satisfies the following conditions:

$$\begin{array}{rcll} \sigma(\mathcal{A}_\#(x,\xi)) & \subset & \overline{\Sigma}_\theta & \text{and} \\ \|\mathcal{A}_\#(x,\xi)^{-1}\| & \leq & M & \text{for all } \xi \in \mathbb{R}^n,\ |\xi| = 1, \end{array}$$

for almost all $x \in \mathbb{R}^n$. The set of all such x will be denoted by E. Here Σ_θ denotes the sector in the complex plane defined by $\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \theta\}$ and the spectrum of an $N \times N$-matrix M is denoted by $\sigma(M)$.

Let $A_{p,\omega}$ be the realization of the differential operator $\mathcal{A}$ in $L^p_\omega(\mathbb{R}^n)^N$, that is defined by

$$\begin{aligned} A_{p,\omega}u &:= \mathcal{A}u, \\ D(A_{p,\omega}) &:= W^{m,p}_\omega(\mathbb{R}^n)^N. \end{aligned} \tag{2.2}$$

For the time being, we freeze the elliptic operator $\mathcal{A}$ in $x_0 \in E$ and consider the fundamental solution $\gamma_\lambda^{x_0}$ of $\lambda + \mathcal{A}_{x_0}$ for $\lambda \in \Sigma_{\pi-\phi}$, where $\phi \in (\theta, \pi)$. Then by [DHP03] we have the following pointwise estimates for the fundamental solution $\gamma_\lambda^{x_0}$:

$$\|D^\nu \gamma_\lambda^{x_0}(x)\| \le C_{\phi,k}|\lambda|^{\frac{n+k}{m}-1} p^n_{m,k}(c_\phi |\lambda|^{\frac{1}{m}}|x|), \qquad x \in \mathbb{R}^n,\ k := |\nu|, \tag{2.3}$$

where

$$\begin{aligned} p^n_{m,k}(r) &= \int_0^\infty \frac{s^{n-2}}{(1+s)^{m-k-1}} e^{-r(1+s)}\,\mathrm{d}s, \qquad n \ge 2, \\ p^1_{m,k}(r) &= \int_0^\infty \frac{1}{(1+s)^{m-k}} e^{-r(1+s)}\,\mathrm{d}s. \end{aligned}$$

This means that $D^\nu \gamma_\lambda^{x_0}$ is integrable for all ν with $|\nu| < m$.

We next consider the spaces of functions with *bounded mean oscillation* $\mathrm{BMO}(\mathbb{R}^n)$ and *vanishing mean oscillation* $\mathrm{VMO}(\mathbb{R}^n)$, respectively. For this it is useful to introduce the following notation. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{C}^{N\times N})$ and $G \subseteq \mathbb{R}^n$ open and bounded we write

$$f_G := \fint_G f(x)\,\mathrm{d}x := \frac{1}{|G|}\int_G f(x)\,\mathrm{d}x.$$

for the mean value of f over G. We then say that a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{C}^{N\times N})$ has *bounded mean oscillation*, or $f \in \mathrm{BMO}(\mathbb{R}^n;\mathbb{C}^{N\times N})$, if

$$\|f\|_* := \sup_{B\in\mathcal{B}} \fint_B \|f(x) - f_B\|\,\mathrm{d}x = \sup_{B\in\mathcal{B}} \|f - f_B\|_B < \infty,$$

where $\mathcal{B}$ denotes the set of all balls in $\mathbb{R}^n$.

For given $r > 0$ we write $\mathcal{B}_r$ for the set of all balls in $\mathbb{R}^n$ with radius less than r. We then further define the VMO-modulus η_f of a function $f \in \mathrm{BMO}(\mathbb{R}^n;\mathbb{C}^{N\times N})$ via

$$\eta_f(r) = \sup_{B\in\mathcal{B}_r} \fint_B \|f(x) - f_B\|\,\mathrm{d}x, \qquad r > 0.$$

and we say that f has *vanishing mean oscillation*, or $f \in \mathrm{VMO}(\mathbb{R}^n;\mathbb{C}^{N\times N})$, if

$$\lim_{r\to 0+} \eta_f(r) = 0.$$

If $N = 1$ we will shortly write $\mathrm{BMO}(\mathbb{R}^n)$ or $\mathrm{VMO}(\mathbb{R}^n)$ for $\mathrm{BMO}(\mathbb{R}^n;\mathbb{C}^{N\times N})$ or $\mathrm{VMO}(\mathbb{R}^n;\mathbb{C}^{N\times N})$, respectively. Note that a function $f = (f_{j,k})_{j,k=1}^N$ belongs to $\mathrm{BMO}(\mathbb{R}^n;\mathbb{C}^{N\times N})$ or $\mathrm{VMO}(\mathbb{R}^n;\mathbb{C}^{N\times N})$, if and only if every component $f_{j,k}$ of f belongs to $\mathrm{BMO}(\mathbb{R}^n)$ or $\mathrm{VMO}(\mathbb{R}^n)$, respectively.

Next we introduce the sharp function $f^\#$ of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$f^\#(x) := \sup_{B\in\mathcal{B}, B\ni x} \fint_B |f(y) - f_B|\,\mathrm{d}y.$$

Then the following holds.

Lemma 2.2. *Let $1 < p < \infty$ and $\omega \in A_p$. Then there exists an A_p-consistent constant C such that*

$$\|f\|_{p,\omega} \le C\|f^\#\|_{p,\omega}, \qquad f \in L^p_\omega(\mathbb{R}^n).$$

For a proof of this fact see e.g. Proposition 5.4 in [HH03].

The following inequality will be often useful to have.

$$\fint_I |f(x) - f_I|\,\mathrm{d}x \le 2\fint_I |f(x) - c|\,\mathrm{d}x, \tag{2.4}$$

for any constant $c \in \mathbb{R}$ and any measurable set $I \subseteq \mathbb{R}^n$ with positive measure.

Furthermore, we recall the well known John-Nirenberg inequality (see e.g. [Ste93, IV.1.3]). Denote by I a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes and let $1 \leq p < \infty$. Then

$$\left(\fint_I |f(x) - f_I|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \leq C_p \|f\|_*. \tag{2.5}$$

Moreover, let $2^j I$, $j \in \mathbb{N}$, be the cube with the same center as I but with sidelength $2^j d$, where d is the sidelength of I. Then

$$\left(\fint_{2^j I} |f(x) - f_I|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \leq C_p (j+1) \|f\|_*. \tag{2.6}$$

We note that (2.6) follows from (2.5) by induction.

3. Main Results

We are now in the position to state the main result of this paper.

Theorem 3.1. *Let $n \geq 2$, $1 < p < \infty$, $\omega \in A_p$, $\phi_0 \in (0, \pi)$, $\phi > \phi_0$ and $M > 0$. Assume that $\mathcal{A} = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ is an (M, ϕ_0)-elliptic operator in $L^p_\omega(\mathbb{R}^n)^N$ with coefficients a_α satisfying*

a) *$a_\alpha \in L^\infty(\mathbb{R}^n; \mathbb{C}^{N \times N}) \cap \mathrm{VMO}(\mathbb{R}^n; \mathbb{C}^{N \times N})$ for $|\alpha| = m$,*
b) *$a_\alpha \in L^\infty(\mathbb{R}^n; \mathbb{C}^{N \times N})$ for $|\alpha| < m$.*

Then there are A_p-consistent constants $\lambda_0, C \geq 0$, such that

$$\|(\lambda + \lambda_0 + A_{p,\omega})^{-1}\|_{\mathcal{L}(L^p_\omega(\mathbb{R}^n)^N)} \leq \frac{C}{|\lambda|}, \qquad \lambda \in \Sigma_{\pi - \phi}.$$

The following corollary follows immediately from Proposition 2.1 and Theorem 3.1.

Corollary 3.2. *Let $1 < p, q < \infty$. Assume that $\phi_0 < \frac{\pi}{2}$. Then there exist constants $M, \mu \geq 0$ such that*

$$\|u'\|_{L^q(J; L^p_\omega(\mathbb{R}^n)^N)} + \|(\mu + A_{p,\omega})u\|_{L^q(J; L^p_\omega(\mathbb{R}^n)^N)} \leq M \|f\|_{L^q(J; L^p_\omega(\mathbb{R}^n)^N)},$$

In particular, $-(\mu + A_{p,\omega})$ generates an analytic semigroup on $L^p_\omega(\mathbb{R}^n)^N$.

Remark 3.3. Obviously, choosing $\omega = 1$, the above assertions hold also in the unweighted space $L^p(\mathbb{R}^n)^N$.

4. Singular Integrals and Commutators

In this section we consider integral operators of Calderón-Zygmund type and related commutators with BMO-functions. More precisely, we call a function $K \in L^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^{N \times N})$ a *Calderón-Zygmund kernel*, if for a constant $C > 0$ the following three conditions are satisfied:

a) $\|\mathcal{F}K\|_\infty \leq C$,
b) $\|K(x)\| \leq C|x|^{-n}$ for all $x \in \mathbb{R}^n \setminus \{0\}$,
c) $\|K(x-y) - K(x)\| \leq C|y||x|^{-(n+1)}$ for all $x, y \in \mathbb{R}^n$, where $|x| > 2|y| > 0$.

Here we denote by $\mathcal{F}$ the Fourier transform.

Remark 4.1. It is clear that the entries $K_{j,k}$ of $K = (K_{j,k})_{j,k=1}^N$ are scalar Calderón-Zygmund kernels if and only if K is a Calderón-Zygmund kernel.

The following result on scalar-valued kernels was proved in [HH03, Propositions 5.1 and 5.4]. It extends in particular a classical commutator result due to Coifman, Rochberg and Weiss [CRW76] to the weighted situation.

Proposition 4.2. *Let $1 < p < \infty$, $\omega \in A_p$, $a \in \mathrm{BMO}(\mathbb{R}^n)$ and $k \in L^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ be a (scalar-valued) Calderón-Zygmund kernel. Then*

$$Tf := k * f \quad \text{and} \quad [T,a]f := T(af) - aTf, \qquad f \in \mathcal{S}(\mathbb{R}^n),$$

define bounded linear operators on $L^p_\omega(\mathbb{R}^n)$ and there is an A_p-consistent constant C such that

$$\|Tf\|_{p,\omega} \le C\|f\|_{p,\omega} \quad \text{and} \quad \|[T,a]f\|_{p,\omega} \le C\|a\|_* \|f\|_{p,\omega}.$$

We next give a result analog to Proposition 4.2 but now for kernels having two variables. We assume that the kernel is homogeneous in the second variable. By S^{n-1} we denote the unit sphere in $\mathbb{R}^n$.

Proposition 4.3. *Let $1 < p < \infty$, $\omega \in A_p$ and $a \in \mathrm{BMO}(\mathbb{R}^n)$. Further let $k : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{C}$ be measurable, such that*

a) *the function $k(x,\cdot)$ is homogeneous of degree $-n$ for a.e. $x \in \mathbb{R}^n$,*
b) $\int_{S^{n-1}} k(x,y)\, \mathrm{d}y = 0$,
c) $\|D^\alpha_y k(x,y)\|_{L^\infty(\mathbb{R}^n \times S^{n-1})} \le M, \qquad |\alpha| \le 2n.$

Consider the operators T and $[T,a]$ given by

$$\begin{aligned} Tf(x) &= \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} k(x,x-y) f(y)\, \mathrm{d}y = \text{p.v.} \int_{\mathbb{R}^n} k(x,x-y) f(y)\, \mathrm{d}y, \\ [T,a]\, f(x) &= \text{p.v.} \int_{\mathbb{R}^n} k(x,x-y)\big(a(x) - a(y)\big) f(y)\, \mathrm{d}y. \end{aligned}$$

Then T and $[T,a]$ are bounded on $L^p_\omega(\mathbb{R}^n)$ and there is an A_p-consistent constant C such that

$$\|Tf\|_{p,\omega} \le C\|f\|_{p,\omega}, \qquad \|[T,a]f\|_{p,\omega} \le C\|a\|_* \|f\|_{p,\omega}.$$

The proof of this proposition is given in [HH03, Proposition 5.5] and uses a representation of $|y|^n k(x,y)$ by spherical harmonics. As we need this method in the following, we shortly describe it in the next proposition.

Proposition 4.4. *Let k be as in Proposition 4.3. Then for $j \in \mathbb{N}$ there exist $b_j \in L^\infty(\mathbb{R}^n)$ and $Y_j \in L^\infty(\mathbb{R}^n \setminus \{0\})$ such that*

a) $k(x,y) = \displaystyle\sum_{j=1}^\infty b_j(x) \frac{Y_j(y)}{|y|^n}$,
b) *$Y_j(y)/|y|^n$ is a Calderón-Zygmund kernel,*
c) *the norms of the associated operators on $L^p_\omega(\mathbb{R}^n)$ are uniformly bounded in $j \in \mathbb{N}$,*
d) $\sum_{j=1}^\infty \|b_j\|_\infty < \infty$.

The following result is a consequence of (2.3), see e.g. [HH03, Lemma 6.1].

Proposition 4.5. *Let $\mathcal{B}$ be an (M,ϕ_0)-elliptic differential operator, homogenous of degree m, with constant coefficients. Further, let $\phi > \phi_0$ and let γ_λ be the fundamental solution of $\lambda + \mathcal{B}$ for $\lambda \in \Sigma_{\pi-\phi}$. Then, for every $\lambda \in \Sigma_{\pi-\phi}$ and $|\nu| \le m$, the functions*

$$\lambda^{1-|\nu|/m} D^\nu \gamma_\lambda$$

are Calderón-Zygmund kernels. Furthermore, the constant C, that appears in the definition of Calderón-Zygmund kernels, can be chosen independently of λ.

Remark 4.6. In particular, the kernels $\gamma_\lambda^{x_0}$, $x_0 \in E$, fulfill the conditions of this proposition. Note that the inequalities in (2.3) do not depend on x_0. Therefore, the constant C can also be chosen independently of $x_0 \in E$.

The next result ensures the existence of a certain integral kernel.

Lemma 4.7. *Let $h \in C^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^{N\times N})$ be homogeneous of degree 0 and $T \in \mathcal{L}(L^2(\mathbb{R}^n)^N)$ be defined as $Tf = \mathcal{F}^{-1}h\mathcal{F}f$. Then there exists a function $k \in C^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^{N\times N})$ which is homogeneous of degree $-n$ and satisfies $\int_{S^{n-1}} k(x)\,\mathrm{d}x = 0$. Furthermore,*

$$Tf = c \cdot f + \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} k(y)f(\cdot - y)\,\mathrm{d}y, \qquad f \in L^2(\mathbb{R}^n)^N,$$

where $c = \int_{S^{n-1}} h(x)\,\mathrm{d}x$ and

$$\|D^\alpha k\|_{L^\infty(S^{n-1})} \le C \sum_{|\beta|\le r} \|D^\beta h\|_{L^\infty(S^{n-1})}$$

for some $r \in \mathbb{N}$ depending only on n and $|\alpha|$.

Proof. The proof follows easily from the scalar-valued case given in [HH03]. In fact, set $h = (h_{j,\ell})_{j,\ell=1}^N$. Then $h_{j,\ell} \in C^\infty(\mathbb{R}^n \setminus \{0\})$ for all $j, \ell \in \{1, \dots, N\}$. Furthermore, for $x \neq 0$ and $\lambda \in \mathbb{R}$ we have

$$h_{j,\ell}(\lambda x) = e_j^T h(\lambda x)e_\ell = e_j^T h(x)e_\ell = h_{j,\ell}(x),$$

which implies the homogeneity of each component $h_{j,\ell}$. By [HH03, Lemma 6.2] we obtain scalar kernels $k_{j,\ell} \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and $c_{j,\ell}$ with the desired properties. Now $k := (k_{j,\ell})_{j,\ell=1}^N$ and $c := (c_{j,\ell})_{j,\ell=1}^N$ is the function and the constant we were looking for. □

5. A Priori Estimates

The main result of this section is the following a priori estimate. We denote by $\mathcal{A}$ always a homogeneous, (M, ϕ_0)-elliptic differential operator of order m, where $\phi_0 \in (0, \pi)$. We also assume that $a_\alpha \in L^\infty(\mathbb{R}^n; \mathbb{C}^{N\times N})$ for all α with $|\alpha| = m$.

Theorem 5.1. *Let $1 < p < \infty$, $\omega \in A_p$ and $\phi > \phi_0$. Then there exist A_p-consistent constants $C, \eta > 0$ and $\lambda_0 \ge 0$, such that for all $u \in W^{m,p}_\omega(\mathbb{R}^n)^N$ and all $\lambda \in \Sigma_{\pi-\phi}$ with $|\lambda| > \lambda_0$*

$$\sum_{|\nu|\le m} \|\lambda^{1-\frac{|\nu|}{m}} D^\nu u\|_{p,\omega} \le C\|(\lambda + \mathcal{A})u\|_{p,\omega},$$

provided $\|a_\alpha\|_* \le \eta$ for all $|\alpha| = m$.

The proof of the above theorem is based on a representation of $D^\nu u$ in terms of kernels $\gamma_\lambda^{x_0}$ and operators $\lambda + \mathcal{A}_{x_0}$. More precisely, we have the following representation formula.

Lemma 5.2. *Let $u \in \mathcal{D}(\mathbb{R}^n)^N$ and $x \in E$. If $|\nu| < m$, then*

$$D^\nu u(x) = \int_{\mathbb{R}^n} D^\nu \gamma_\lambda^x(x - y)(\lambda + \mathcal{A}_x)u(y)\,\mathrm{d}y.$$

Moreover, for $|\nu| = m$

$$\begin{aligned} D^\nu u(x) \;&=\; \mathrm{p.v.}\int_{\mathbb{R}^n} K_\nu(x, x - y)(\lambda + \mathcal{A}_x)u(y)\,\mathrm{d}y + c_\nu(x)(\lambda + \mathcal{A})u(x) \\ &\quad - \int_{\mathbb{R}^n} \lambda\gamma_\lambda^x(x - y)\left(\mathrm{p.v.}\int_{\mathbb{R}^n} K_\nu(x, y - z)(\lambda + \mathcal{A}_x)u(z)\,\mathrm{d}z\right)\mathrm{d}y \\ &\quad - c_\nu(x)\int_{\mathbb{R}^n} \lambda\gamma_\lambda^x(x - y)(\lambda + \mathcal{A}_x)u(y)\,\mathrm{d}y. \end{aligned}$$

Here $c_\nu \in L^\infty(\mathbb{R}^n)$ and K_ν is a matrix-valued integral kernel, whose entries fulfill the assumptions of Proposition 4.3.

Proof. For every $x_0 \in E$ we have $\mathcal{F}u = (\lambda + A_\#(x_0,\xi))^{-1}\mathcal{F}(\lambda + \mathcal{A}_{x_0})u$. The first part of the assertion follows easily by setting $x_0 = x$ since by the estimates (2.3) $D^\nu \gamma_\lambda^{x_0}$ is in $L^1(\mathbb{R}^n)^N$ for every $|\nu| < m$.

Hence we turn to the case where $|\nu| = m$ and see that

$$\begin{aligned}
\mathcal{F}D^\nu u(\xi) &= \xi^\nu \big(\lambda + A_\#(x_0,\xi)\big)^{-1} \big(\mathcal{F}(\lambda + \mathcal{A}_{x_0})u\big)(\xi) \\
&= \xi^\nu (A_\#(x_0,\xi))^{-1} \big(\mathcal{F}(\lambda + \mathcal{A}_{x_0})u\big)(\xi) \\
&\quad + \Big[\xi^\nu \big(\lambda + A_\#(x_0,\xi)\big)^{-1} - \xi^\nu (A_\#(x_0,\xi))^{-1}\Big] \big(\mathcal{F}(\lambda + \mathcal{A}_{x_0})u\big)(\xi).
\end{aligned}$$

We may now replace $(\lambda + A_\#(x_0,\xi))^{-1} - (A_\#(x_0,\xi))^{-1}$ by $-\lambda\big(\lambda + A_\#(x_0,\xi)\big)^{-1}(A_\#(x_0,\xi))^{-1}$, since the two matrices commute. Hence we obtain

$$\begin{aligned}
\mathcal{F}D^\nu u(\xi) &= \xi^\nu (A_\#(x_0,\xi))^{-1} \big(\mathcal{F}(\lambda + \mathcal{A}_{x_0})u\big)(\xi) \\
&\quad - \lambda\big(\lambda + A_\#(x_0,\xi)\big)^{-1} \xi^\nu (A_\#(x_0,\xi))^{-1} \big(\mathcal{F}(\lambda + \mathcal{A}_{x_0})u\big)(\xi).
\end{aligned}$$

As $|\nu| = m$ and $\mathcal{A}$ is an (M, ϕ_0)-elliptic operator, the symbol $h_{x_0}(\xi) := \xi^\nu (A_\#(x_0,\xi))^{-1}$ is a smooth homogeneous function of degree 0.

By Lemma 4.7 there exist kernels $K_\nu(x_0,\cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^{N\times N})$, that are homogeneous of degree $-n$ with $\int_{S^{n-1}} K_\nu(x_0,y)\,\mathrm{d}y = 0$. Furthermore, the ellipticity of $\mathcal{A}$ yields the estimate $\|D^\beta h_{x_0}\|_{L^\infty(S^{n-1})} \le C_k$ uniformly in $x_0 \in E$ and for all $|\beta| \le k$. This implies, that the derivatives of $K_\nu(x_0,\cdot)$ are bounded on the unit sphere independently of $x_0 \in E$. By this we finally obtain, that the entries of K_ν fulfill the assumptions of Proposition 4.3.

For the constants $c_\nu(x_0)$, which we obtain from Lemma 4.7, ellipticity of $\mathcal{A}$ yields

$$|c_\nu(x_0)| \le \int_{S^{n-1}} |\xi^\nu (A_\#(x_0,\xi))^{-1}|\,\mathrm{d}\xi \le \int_{S^{n-1}} M\,\mathrm{d}\xi \le CM.$$

Thus $c_\nu \in L^\infty(\mathbb{R}^n)$.

Inserting these kernels and the constants c_ν into the above equation and applying $\mathcal{F}^{-1}$ we get

$$\begin{aligned}
D^\nu u(x) &= \text{p.v.} \int_{\mathbb{R}^n} K_\nu(x_0, x-y)(\lambda + \mathcal{A}_{x_0})u(y)\,\mathrm{d}y + c_\nu(x_0)(\lambda + \mathcal{A}_{x_0})u(x) \\
&\quad - \int_{\mathbb{R}^n} \lambda\gamma_\lambda^{x_0}(x-y) \left(\text{p.v.} \int_{\mathbb{R}^n} K_\nu(x_0, y-z)(\lambda + \mathcal{A}_{x_0})u(z)\,\mathrm{d}z\right) \mathrm{d}y \\
&\quad - c_\nu(x_0) \int_{\mathbb{R}^n} \lambda\gamma_\lambda^{x_0}(x-y)(\lambda + \mathcal{A}_{x_0})u(y)\,\mathrm{d}y
\end{aligned}$$

for every $x_0 \in E$. The assertion follows again by setting $x_0 = x$. □

Before proving Theorem 5.1 some comments on our notation are in order. The entries of the matrices $a_\alpha(x)$ and $\gamma_\lambda^{x_0}$ are denoted by $a_{j,k}^{(\alpha)}(x)$ and $g_{j,k}^{\lambda,x_0}$ for $j,k = 1,\dots,N$, respectively. Furthermore, we write u_j, $j = 1,\dots,N$, for the components of u.

By Lemma 5.2 we get for $|\nu| < m$

$$\begin{aligned}
\lambda^{1-\frac{|\nu|}{m}} D^\nu u(x) &= \int_{\mathbb{R}^n} \lambda^{1-\frac{|\nu|}{m}} D^\nu \gamma_\lambda^x(x-y)(\lambda + \mathcal{A}_x)u(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \lambda^{1-\frac{|\nu|}{m}} D^\nu \gamma_\lambda^x(x-y)(\lambda + \mathcal{A})u(y)\,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^n} \lambda^{1-\frac{|\nu|}{m}} D^\nu \gamma_\lambda^x(x-y)(\mathcal{A}_x - \mathcal{A})u(y)\,\mathrm{d}y.
\end{aligned}$$

Looking at one component one obtains

$$
\begin{aligned}
\lambda^{1-\frac{|\nu|}{m}} D^\nu u_j(x) &= \sum_{k=1}^N \int_{\mathbb{R}^n} \lambda^{1-\frac{|\nu|}{m}} D^\nu g_{j,k}^{\lambda,x}(x-y)\big((\lambda+\mathcal{A})u(y)\big)_k \,\mathrm{d}y \\
&\quad + \sum_{\ell=1}^N \sum_{k=1}^N \int_{\mathbb{R}^n} \lambda^{1-\frac{|\nu|}{m}} D^\nu g_{j,k}^{\lambda,x}(x-y) \sum_{|\alpha|=m} \big(a_{k,l}^{(\alpha)}(x) - a_{k,l}^{(\alpha)}(y)\big) D^\alpha u_\ell(y) \,\mathrm{d}y \qquad (5.1)\\
&= \sum_{k=1}^N \big(T_{j,k}^{3,\lambda,\nu}((\lambda+\mathcal{A})u)_k\big)(x) + \sum_{\ell=1}^N \sum_{k=1}^N \sum_{|\alpha|=m} \big([T_{j,k}^{3,\lambda,\nu}, a_{k,\ell}^{(\alpha)}] D^\alpha u_\ell\big)(x),
\end{aligned}
$$

where

$$
\begin{aligned}
T_{j,k}^{3,\lambda,\nu} f(x) &= \int_{\mathbb{R}^n} \lambda^{1-\frac{|\nu|}{m}} D^\nu g_{j,k}^{\lambda,x}(x-y) f(y) \,\mathrm{d}y, \\
[T_{j,k}^{3,\lambda,\nu}, a] f(x) &= \int_{\mathbb{R}^n} \lambda^{1-\frac{|\nu|}{m}} D^\nu g_{j,k}^{\lambda,x}(x-y)\big(a(x)-a(y)\big) f(y) \,\mathrm{d}y
\end{aligned}
$$

for $j,k = 1,\ldots,N$, $|\nu| < m$, $\lambda \in \Sigma_{\pi-\phi}$ and $a \in \mathrm{BMO}(\mathbb{R}^n)$. By the same calculations, we derive from the representation formula in Lemma 5.2 for $|\nu| = m$

$$
\begin{aligned}
D^\nu u(x) &= \text{p.v.} \int_{\mathbb{R}^n} K_\nu(x, x-y)(\lambda+\mathcal{A})u(y) \,\mathrm{d}y + \text{p.v.} \int_{\mathbb{R}^n} K_\nu(x, x-y)(\mathcal{A}_x - \mathcal{A})u(y) \,\mathrm{d}y \\
&\quad + c_\nu(x)(\lambda+\mathcal{A})u(x) - \int_{\mathbb{R}^n} \lambda \gamma_\lambda^x(x-y) \left(\text{p.v.} \int_{\mathbb{R}^n} K_\nu(x, y-z)(\lambda+\mathcal{A})u(z) \,\mathrm{d}z\right) \mathrm{d}y \\
&\quad - \int_{\mathbb{R}^n} \lambda \gamma_\lambda^x(x-y) \left(\text{p.v.} \int_{\mathbb{R}^n} K_\nu(x, y-z)(\mathcal{A}_x - \mathcal{A})u(z) \,\mathrm{d}z\right) \mathrm{d}y \\
&\quad - c_\nu(x) \int_{\mathbb{R}^n} \lambda \gamma_\lambda^x(x-y)(\lambda+\mathcal{A})u(y) \,\mathrm{d}y - c_\nu(x) \int_{\mathbb{R}^n} \lambda \gamma_\lambda^x(x-y)(\mathcal{A}_x - \mathcal{A})u(y) \,\mathrm{d}y.
\end{aligned}
$$

For the j-th component this means

$$
\begin{aligned}
D^\nu u_j(x) &= \sum_{k=1}^N \text{p.v.} \int_{\mathbb{R}^n} K_{j,k}^\nu(x, x-y)\big((\lambda+\mathcal{A})u(y)\big)_k \,\mathrm{d}y \\
&\quad + \sum_{|\alpha|=m} \sum_{\ell=1}^N \sum_{k=1}^N \text{p.v.} \int_{\mathbb{R}^n} K_{j,k}^\nu(x, x-y)\big(a_{k,\ell}^{(\alpha)}(x) - a_{k,\ell}^{(\alpha)}(y)\big) D^\alpha u_\ell(y) \,\mathrm{d}y \\
&\quad + \sum_{k=1}^N c_{j,k}^\nu(x)\big((\lambda+\mathcal{A})u(x)\big)_k \\
&\quad - \sum_{\ell=1}^N \sum_{k=1}^N \int_{\mathbb{R}^n} \lambda g_{j,k}^{\lambda,x}(x-y) \ \text{p.v.} \int_{\mathbb{R}^n} K_{k,\ell}^\nu(x, y-z)\big((\lambda+\mathcal{A})u(z)\big)_\ell \,\mathrm{d}z \,\mathrm{d}y \\
&\quad - \sum_{|\alpha|=m} \sum_{r=1}^N \sum_{\ell=1}^N \sum_{k=1}^N \int_{\mathbb{R}^n} \lambda g_{j,k}^{\lambda,x}(x-y) \\
&\qquad\qquad \cdot \text{p.v.} \int_{\mathbb{R}^n} K_{k,\ell}^\nu(x, y-z)\big(a_{\ell,r}^{(\alpha)}(x) - a_{\ell,r}^{(\alpha)}(z)\big) D^\alpha u_r(z) \,\mathrm{d}z \,\mathrm{d}y \\
&\quad - \sum_{\ell=1}^N \sum_{k=1}^N c_{j,k}^\nu(x) \int_{\mathbb{R}^n} \lambda g_{k,\ell}^{\lambda,x}(x-y)\big((\lambda+\mathcal{A})u(y)\big)_\ell \,\mathrm{d}y \\
&\quad - \sum_{|\alpha|=m} \sum_{r=1}^N \sum_{\ell=1}^N \sum_{k=1}^N c_{j,k}^\nu(x) \int_{\mathbb{R}^n} \lambda g_{k,\ell}^{\lambda,x}(x-y)\big(a_{\ell,r}^{(\alpha)}(x) - a_{\ell,r}^{(\alpha)}(y)\big) D^\alpha u_r(y) \,\mathrm{d}y,
\end{aligned}
$$

where $K^{\nu}_{j,k}(x,x-y)$ and $c^{\nu}_{j,k}(x)$ are the entries of the matrices $K_\nu(x,x-y)$ and $c_\nu(x)$ respectively.

In order to get a notational grip on this formula we introduce the following operators. Let $j,k,\ell \in \{1,\dots,N\}$, $|\nu| = m$, $\lambda \in \Sigma_{\pi-\phi}$ and let $a \in \mathrm{BMO}(\mathbb{R}^n)$ and $f \in L^p_\omega(\mathbb{R}^n)$. Then we define

$$\begin{aligned}
T^{1,\nu}_{j,k} f(x) &= \text{p.v.} \int_{\mathbb{R}^n} K^{\nu}_{j,k}(x,x-y) f(y)\,\mathrm{d}y,\\
[T^{1,\nu}_{j,k}, a] f(x) &= \text{p.v.} \int_{\mathbb{R}^n} K^{\nu}_{j,k}(x,x-y) \big(a(x)-a(y)\big) f(y)\,\mathrm{d}y,\\
T^{2,\lambda,\nu}_{j,k,\ell} f(x) &= \int_{\mathbb{R}^n} \lambda g^{\lambda,x}_{j,k}(x-y)\ \text{p.v.} \int_{\mathbb{R}^n} K^{\nu}_{k,\ell}(x,y-z) f(z)\,\mathrm{d}z\,\mathrm{d}y,\\
[T^{2,\lambda,\nu}_{j,k,\ell}, a] f(x) &= \int_{\mathbb{R}^n} \lambda g^{\lambda,x}_{j,k}(x-y)\ \text{p.v.} \int_{\mathbb{R}^n} K^{\nu}_{k,\ell}(x,y-z)\big(a(x)-a(z)\big) f(z)\,\mathrm{d}z\,\mathrm{d}y.
\end{aligned}$$

Using this notation we finally write

$$\begin{aligned}
D^\nu u_j(x) \;=\;& \sum_{k=1}^N T^{1,\nu}_{j,k}\big((\lambda+\mathcal{A})u\big)_k(x) + \sum_{|\alpha|=m}\sum_{\ell=1}^N\sum_{k=1}^N \Big[T^{1,\nu}_{j,k}, a^{(\alpha)}_{k,\ell}\Big] D^\alpha u_\ell(x)\\
&+\sum_{k=1}^N c^{\nu}_{j,k}(x)\big((\lambda+\mathcal{A})u(x)\big)_k - \sum_{\ell=1}^N\sum_{k=1}^N T^{2,\lambda,\nu}_{j,k,\ell}\big((\lambda+\mathcal{A})u\big)_\ell(x)\\
&-\sum_{|\alpha|=m}\sum_{r=1}^N\sum_{\ell=1}^N\sum_{k=1}^N \Big[T^{2,\lambda,\nu}_{j,k,\ell}, a^{(\alpha)}_{\ell,r}\Big] D^\alpha u_r(x) - \sum_{\ell=1}^N\sum_{k=1}^N c^{\nu}_{j,k}(x) T^{3,\lambda,0}_{k,\ell}\big((\lambda+\mathcal{A})u\big)_\ell(x)\\
&-\sum_{|\alpha|=m}\sum_{r=1}^N\sum_{\ell=1}^N\sum_{k=1}^N c^{\nu}_{j,k}(x)\Big[T^{3,\lambda,0}_{k,\ell}, a^{(\alpha)}_{\ell,r}\Big] D^\alpha u_r(x).
\end{aligned}\tag{5.2}$$

Note, that the operator $[T^{2,\lambda,\nu}_{j,k,\ell}, a]$ is not a commutator. As we can handle it by the same methods as the commutators, we however use this notation.

We now prove the following mapping properties of the above operators.

Lemma 5.3. *Let $j,k,\ell \in \{1,\dots,N\}$, $|\nu| = m$, $|\mu| < m$, $\lambda \in \Sigma_{\pi-\phi}$, $a \in L^\infty(\mathbb{R}^n)$, $1 < p < \infty$, $\omega \in A_p$ and $f \in L^p_\omega(\mathbb{R}^n)$. Then*

$$\begin{aligned}
&\|T^{1,\nu}_{j,k} f\|_{p,\omega} \le C\|f\|_{p,\omega}, && \Big\|\Big[T^{1,\nu}_{j,k}, a\Big] f\Big\|_{p,\omega} \le C\|a\|_*\|f\|_{p,\omega},\\
&\|T^{2,\lambda,\nu}_{j,k,\ell} f\|_{p,\omega} \le C\|f\|_{p,\omega}, && \Big\|[T^{2,\lambda,\nu}_{j,k,\ell}, a] f\Big\|_{p,\omega} \le C A_*(1+\|a\|_\infty)\|f\|_{p,\omega},\\
&\|T^{3,\lambda,\mu}_{j,k} f\|_{p,\omega} \le C\|f\|_{p,\omega}, && \Big\|[T^{3,\lambda,\mu}_{j,k}, a] f\Big\|_{p,\omega} \le C A_*(1+\|a\|_\infty)\|f\|_{p,\omega},
\end{aligned}$$

where

$$A_* := \max\Big\{\|a\|_*, \max_{|\beta|=m}\max_{r,s=1}^N \|a^{(\beta)}_{r,s}\|_*\Big\}.$$

Proof. By Lemma 5.2 the kernels $K^{\nu}_{j,k}$ fulfill the hypotheses of Proposition 4.3. Thus the assertion concerning the T^1-operators follows.

We turn to the operators containing T^3. Using (2.3), we have

$$\|D^\mu \gamma^x_\lambda(z)\| \le C|\lambda|^{\frac{n+|\mu|}{m}-1} p^n_{m,|\mu|}(c|\lambda|^{\frac{1}{m}}|z|)$$

for all $|\mu| < m$, $\lambda \in \Sigma_{\pi-\phi}$ and $x,z \in \mathbb{R}^n$. Consequently, we have the same estimate for the entries $g^{\lambda,x}_{j,k}$ and the boundedness of $T^{3,\lambda,\mu}_{j,k}$ follows as in [HH03, Proposition 6.5].

Dealing with the commutator of $T^{3,\lambda,\mu}_{j,k}$ with BMO-functions is much harder. We first show a point-wise estimate for the sharp function of $[T^{3,\lambda,\mu}_{j,k}, a]f$.

Let I be an axis-parallel cube in $\mathbb{R}^n$ and $x \in I$. Then

$$\begin{aligned}[T_{j,k}^{3,\lambda,\mu}, a]f(z) &= (a(z)-a_I)T_{j,k}^{3,\lambda,\mu}f(z) - T_{j,k}^{3,\lambda,\mu}\big((a-a_I)\mathbf{1}_{2I}f\big)(z) - T_{j,k}^{3,\lambda,\mu}\big((a-a_I)\mathbf{1}_{(2I)^c}f\big)(z) \\ &=: A(z)+B(z)+C(z).\end{aligned}$$

We estimate $A^{\#}(x)$, $B^{\#}(x)$, and $C^{\#}(x)$ separately. For A and B we choose $c=0$ in (2.4) and look at $⨍_I |A(z)|\,\mathrm{d}z$ and $⨍_I |B(z)|\,\mathrm{d}z$.

The Hölder inequality yields for all $1<r<\infty$

$$⨍_I |A(z)|\,\mathrm{d}z \le C\left(⨍_I |a(z)-a_I|^{r'}\,\mathrm{d}z\right)^{\frac{1}{r'}}\left(⨍_I |T_{j,k}^{3,\lambda,\mu}f(z)|^r\,\mathrm{d}z\right)^{\frac{1}{r}}.$$

For the first factor we use the John Nirenberg inequality (2.5), and we estimate the second one by the supremum over all cubes that contain x, and thus by the maximal operator. This yields

$$⨍_I |A(z)|\,\mathrm{d}z \le C\|a\|_*\big(M|T_{j,k}^{3,\lambda,\mu}f|^r\big)^{\frac{1}{r}}(x).$$

For estimating $B^{\#}$, we first apply Jensen's inequality:

$$⨍_I |B(z)|\,\mathrm{d}z \le \left(⨍_I |T_{j,k}^{3,\lambda,\mu}\big((a-a_I)f\mathbf{1}_{2I}\big)(z)|^q\,\mathrm{d}z\right)^{\frac{1}{q}} \le \left(\frac{1}{|I|}\right)^{\frac{1}{q}}\|T_{j,k}^{3,\lambda,\mu}\big((a-a_I)f\mathbf{1}_{2I}\big)\|_q,$$

where $1<q<\infty$. Now we use the boundedness of $T_{j,k}^{3,\lambda,\mu}$ in $L^q(\mathbb{R}^n)$ and then again Hölder's inequality as above. Doing so, we get for every $u\in(1,\infty)$ and every $1<r=qu<\infty$:

$$⨍_I |B(z)|\,\mathrm{d}z \le C\left(\frac{1}{|I|}\int_{2I}|a(z)-a_I|^{qu'}\,\mathrm{d}z\right)^{\frac{1}{qu'}}\left(\frac{1}{|I|}\int_{2I}|f(z)|^{qu}\,\mathrm{d}z\right)^{\frac{1}{qu}} \le C\|a\|_*(M|f|^r)^{\frac{1}{r}}(x),$$

where we again estimated the first factor by the John Nirenberg inequality and the second one by the maximal operator.

The third part $C^{\#}(x)$ is more involved. Here we have to use the structure of our kernels and we cannot choose $c=0$ in (2.4). We define the averaged operator

$$\mathcal{A}_I := \sum_{|\alpha|=m} ⨍_I a_\alpha(x)\,\mathrm{d}x D^\alpha,$$

which is a homogeneous differential operator with constant coefficients. Thus there exist suitable constants $\tilde\lambda_0, K\ge 0$, such that the operator $\tilde\lambda_0+\mathcal{A}_I$ is $(K,\phi_0/2)$-elliptic. By Proposition 4.5 we see, that $(\lambda+\mathcal{A}_I)^{-1}$ is given by a kernel $K_{I,\lambda}$ for all $\lambda\in\tilde\lambda_0+\Sigma_{\pi-\phi_0/2}$. Choosing $\lambda_0\ge\tilde\lambda_0$ so big, that $\Sigma_{\pi-\phi_0}\setminus B(0,\lambda_0)\subseteq\tilde\lambda_0+\Sigma_{\pi-\phi_0/2}$, we get for all $\phi>\phi_0$ and $\lambda\in\Sigma_{\pi-\phi}$ satisfying $|\lambda|>\lambda_0$ the estimate

$$\|D^\beta K_{I,\lambda}(x)\| \le C|\lambda|^{\frac{n+|\beta|}{m}-1}p^n_{m,|\beta|}(c|\lambda|^{\frac{1}{m}}|x|),$$

as in (2.3). Thus for all these λ and for all $|\beta|\le m$ the function $\lambda^{1-|\beta|/m}D^\beta K_{I,\lambda}$ is a Calderón-Zygmund kernel and the associated operators are bounded in $L^p_\omega(\mathbb{R}^n)$ uniformly in $\lambda\in\Sigma_{\pi-\phi}\setminus B(0,\lambda_0)$.

With this preparations in hand, we now estimate $C^{\#}(x)$. To this end we choose the constant c in (2.4) as

$$c := \int_{\mathbb{R}^n}\lambda^{1-\frac{|\mu|}{m}}D^\mu K_{j,k}^{I,\lambda}(x_I-y)(a(y)-a_I)\mathbf{1}_{(2I)^c}(y)f(y)\,\mathrm{d}y,$$

where x_I is the center of the cube I. Setting $g(y) := (a(y) - a_I)\mathbf{1}_{(2I)^c}(y)f(y)$ we obtain

$$\begin{aligned}
⨍_I |C(z) - c| \,\mathrm{d}z &= ⨍_I \left| \int_{\mathbb{R}^n} \lambda^{1-\frac{|\mu|}{m}} D^\mu \left(g_{j,k}^{\lambda,z}(z-y) - K_{j,k}^{I,\lambda}(x_I - y) \right) g(y) \,\mathrm{d}y \right| \mathrm{d}z \\
&= ⨍_I \left| \int_{\mathbb{R}^n} \lambda^{1-\frac{|\mu|}{m}} D^\mu \left(g_{j,k}^{\lambda,z}(z-y) - K_{j,k}^{I,\lambda}(z - y) \right) g(y) \,\mathrm{d}y \right| \mathrm{d}z \\
&\quad + ⨍_I \left| \int_{\mathbb{R}^n} \lambda^{1-\frac{|\mu|}{m}} D^\mu \left(K_{j,k}^{I,\lambda}(z-y) - K_{j,k}^{I,\lambda}(x_I - y) \right) g(y) \,\mathrm{d}y \right| \mathrm{d}z \\
&=: I_1 + I_2.
\end{aligned}$$

Therefore we need information about the differences $g_{j,k}^{\lambda,z} - K_{j,k}^{I,\lambda}$ and $K_{j,k}^{I,\lambda}(z - \cdot) - K_{j,k}^{I,\lambda}(x_I - \cdot)$. To handle the first difference we calculate for the corresponding matrix-valued kernels γ_λ^z and $K_{I,\lambda}$ and for every $\varphi \in \mathcal{D}(\mathbb{R}^n)^N$

$$\begin{aligned}
(\gamma_\lambda^z - K_{I,\lambda}) * \varphi &= \mathcal{F}^{-1}\left((\lambda + A_\#(z,\xi))^{-1} - (\lambda + A_{I,\#}(\xi))^{-1} \right) \mathcal{F}\varphi \\
&= \mathcal{F}^{-1}(\lambda + A_\#(z,\xi))^{-1}\big(A_{I,\#}(\xi) - A_\#(z,\xi)\big)(\lambda + A_{I,\#}(\xi))^{-1}\mathcal{F}\varphi \\
&= \sum_{|\beta|=m} \mathcal{F}^{-1}(\lambda + A_\#(z,\xi))^{-1}\big((a_\beta)_I - a_\beta(z)\big)\xi^\beta(\lambda + A_{I,\#}(\xi))^{-1}\mathcal{F}\varphi \\
&= \sum_{|\beta|=m} \gamma_\lambda^z * \left[\big((a_\beta)_I - a_\beta(z)\big) D^\beta K_{I,\lambda} * \varphi \right].
\end{aligned}$$

For one component this identity means, that for every $f \in \mathcal{D}(\mathbb{R}^n)$

$$\begin{aligned}
&\left[\lambda^{1-\frac{|\mu|}{m}} D^\mu \big(g_{j,k}^{\lambda,z} - K_{j,k}^{I,\lambda} \big) \right] * f(z) \\
&= \sum_{|\beta|=m} \sum_{\ell=1}^N \sum_{s=1}^N \big((a_{\ell,s}^{(\beta)})_I - a_{\ell,s}^{(\beta)}(z)\big) \int_{\mathbb{R}^n} \lambda^{1-\frac{|\mu|}{m}} D^\mu g_{j,\ell}^{\lambda,z}(z-y) \int_{\mathbb{R}^n} D^\beta K_{s,k}^{I,\lambda}(y-u) f(u) \,\mathrm{d}u \,\mathrm{d}y \\
&=: \sum_{|\beta|=m} \sum_{\ell=1}^N \sum_{s=1}^N \big((a_{\ell,s}^{(\beta)})_I - a_{\ell,s}^{(\beta)}(z)\big) S_{j,k,\ell,s}^{\beta,\lambda,\mu} f(z),
\end{aligned}$$

where $S_{j,k,\ell,s}^{\beta,\lambda,\mu}$ is a bounded operator on $L_\omega^p(\mathbb{R}^n)$, as it is a composition of two convolutions with Calderón-Zygmund kernels. Propositions 4.2 and 4.5 even allow us to estimate the norm of this operator A_p-consistently and independently of λ.

Using this to estimate I_1, we get

$$\begin{aligned}
I_1 \le \sum_{|\beta|=m} \sum_{\ell=1}^N \sum_{s=1}^N &\left[⨍_I \left| (a_{\ell,s}^{(\beta)})_I - a_{\ell,s}^{(\beta)}(z) \right| \left| \left[S_{j,k,\ell,s}^{\beta,\lambda,\mu}(a\mathbf{1}_{(2I)^c} f) \right](z) \right| \mathrm{d}z \right. \\
&\left. + ⨍_I \left| (a_{\ell,s}^{(\beta)})_I - a_{\ell,s}^{(\beta)}(z) \right| \left| \left[S_{j,k,\ell,s}^{\beta,\lambda,\mu}(a_I\mathbf{1}_{(2I)^c} f) \right](z) \right| \mathrm{d}z \right] =: I_{11} + I_{12}.
\end{aligned}$$

In order to treat I_{11} we write $\mathbf{1}_{(2I)^c} f = f - \mathbf{1}_{2I} f$. The triangle inequality then yields for every $1 < r < \infty$

$$\begin{aligned}
I_{11} &\le \sum_{|\beta|=m} \sum_{\ell=1}^N \sum_{s=1}^N \left[⨍_I \left| (a_{\ell,s}^{(\beta)})_I - a_{\ell,s}^{(\beta)}(z) \right| \left| \left[S_{j,k,\ell,s}^{\beta,\lambda,\mu}(af) \right](z) \right| \mathrm{d}z \right. \\
&\qquad \left. + ⨍_I \left| (a_{\ell,s}^{(\beta)})_I - a_{\ell,s}^{(\beta)}(z) \right| \left| \left[S_{j,k,\ell,s}^{\beta,\lambda,\mu}(a\mathbf{1}_{2I} f) \right](z) \right| \mathrm{d}z \right] \\
&\le C \sum_{|\beta|=m} \sum_{\ell=1}^N \sum_{s=1}^N \left\| a_{\ell,s}^{(\beta)} \right\|_* \big((M|S_{j,k,\ell,s}^{\beta,\lambda,\mu} af|^r)^{\frac{1}{r}}(x) + \|a\|_\infty (M|f|^r)^{\frac{1}{r}}(x) \big),
\end{aligned}$$

in the same way as we estimated $\fint_I |A(z)|\,\mathrm{d}z$ and $\fint_I |B(z)|\,\mathrm{d}z$. Analogously

$$\begin{aligned} I_{12} &\le C \sum_{|\beta|=m} \sum_{\ell=1}^{N} \sum_{s=1}^{N} \left\| a_{\ell,s}^{(\beta)} \right\|_* \big((M|S_{j,k,\ell,s}^{\beta,\lambda,\mu} a_I f|^r)^{\frac{1}{r}}(x) + \|a\|_\infty (M|f|^r)^{\frac{1}{r}}(x) \big) \\ &\le C\|a\|_\infty \sum_{|\beta|=m} \sum_{\ell=1}^{N} \sum_{s=1}^{N} \left\| a_{\ell,s}^{(\beta)} \right\|_* \big((M|S_{j,k,\ell,s}^{\beta,\lambda,\mu} f|^r)^{\frac{1}{r}}(x) + (M|f|^r)^{\frac{1}{r}}(x) \big). \end{aligned}$$

For I_2 we use that every entry of $\lambda^{1-|\mu|/m} D^\mu K_{I,\lambda}$ is a Calderón-Zygmund kernel:

$$\begin{aligned} I_2 &= \fint_I \left| \int_{\mathbb{R}^n} \lambda^{1-\frac{|\mu|}{m}} D^\mu \left(K_{j,k}^{I,\lambda}(z-y) - K_{j,k}^{I,\lambda}(x_I - y) \right) g(y)\,\mathrm{d}y \right| \mathrm{d}z \\ &\le \fint_I \int_{\mathbb{R}^n \setminus 2I} \left| \lambda^{1-\frac{|\mu|}{m}} D^\mu K_{j,k}^{I,\lambda}\big(x_I - y - (x_I - z)\big) - \lambda^{1-\frac{|\mu|}{m}} D^\mu K_{j,k}^{I,\lambda}(x_I - y) \right| \\ &\qquad \cdot |a(y) - a_I||f(y)|\,\mathrm{d}y\,\mathrm{d}z \\ &\le C \fint_I \int_{\mathbb{R}^n \setminus 2I} \frac{|x_I - z|}{|x_I - y|^{n+1}} |a(y) - a_I||f(y)|\,\mathrm{d}y\,\mathrm{d}z. \end{aligned}$$

This estimate is possible, as $|x_I - y| > 2|x_I - z|$ is valid. Applying the Hölder inequality

$$I_2 \le C \fint_I \left(\int_{\mathbb{R}^n \setminus 2I} \frac{|x_I - z|}{|x_I - y|^{n+1}} |a(y) - a_I|^{r'}\,\mathrm{d}y \right)^{\frac{1}{r'}} \left(\int_{\mathbb{R}^n \setminus 2I} \frac{|x_I - z|}{|x_I - y|^{n+1}} |f(y)|^r\,\mathrm{d}y \right)^{\frac{1}{r}} \mathrm{d}z$$

follows for every $1 < r < \infty$. Denoting the side-length of I by d, we estimate the first factor:

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus 2I} \frac{|x_I - z|}{|x_I - y|^{n+1}} |a(y) - a_I|^{r'}\,\mathrm{d}y \\ &= \sum_{\ell=2}^{\infty} \int_{2^\ell I \setminus 2^{\ell-1} I} \frac{|x_I - z|}{|x_I - y|^{n+1}} |a(y) - a_I|^{r'}\,\mathrm{d}y \;\le\; \sum_{\ell=2}^{\infty} \frac{d}{(2^{\ell-2} d)^{n+1}} \int_{2^\ell I} |a(y) - a_I|^{r'}\,\mathrm{d}y \\ &= 4^{n+1} \sum_{\ell=2}^{\infty} 2^{-\ell} \fint_{2^\ell I} |a(y) - a_I|^{r'}\,\mathrm{d}y \;\le\; C \sum_{\ell=2}^{\infty} 2^{-\ell} (\ell+1)^{r'} \|a\|_*^{r'} \;\le\; C\|a\|_*^{r'} \end{aligned}$$

by (2.6) and since $(2^{\ell-2} d)^{n+1} = 2^\ell 4^{-(n+1)} d |2^\ell I|$.

Analogously we get for the second factor

$$\int_{\mathbb{R}^n \setminus 2I} \frac{|x_I - z|}{|x_I - y|^{n+1}} |f(y)|^r\,\mathrm{d}y \le C \sum_{\ell=2}^{\infty} 2^{-\ell} \fint_{2^\ell I} |f(y)|^r\,\mathrm{d}y \le CM|f|^r(x).$$

Altogether we obtain for I_2

$$I_2 \le C\|a\|_* \big(M|f|^r\big)^{\frac{1}{r}}(x).$$

Combining this with the estimate for I_1 we see that

$$\begin{aligned} \fint_I |C(z) - c|\,\mathrm{d}z \;\le\; C\Bigg[&\|a\|_* \big(M|f|^r\big)^{\frac{1}{r}}(x) + \sum_{|\beta|=m} \sum_{\ell=1}^{N} \sum_{s=1}^{N} \left\| a_{\ell,s}^{(\beta)} \right\|_* \Big(\big(M|S_{j,k,\ell,s}^{\beta,\lambda,\mu} a f|^r\big)^{\frac{1}{r}}(x) \\ &+ \|a\|_\infty \big(M|S_{j,k,\ell,s}^{\beta,\lambda,\mu} f|^r\big)^{\frac{1}{r}}(x) + \|a\|_\infty \big(M|f|^r\big)^{\frac{1}{r}}(x) \Big) \Bigg]. \end{aligned}$$

Collecting the results for A, B and C, we have for every $1 < r < \infty$

$$\begin{aligned}
\Big([T^{3,\lambda,\mu}_{j,k}, a]f\Big)^{\#}(x) &\leq A^{\#}(x) + B^{\#}(x) + C^{\#}(x) \\
&\leq C\|a\|_* \Big(\big(M|T^{3,\lambda,\mu}_{j,k} f|^r\big)^{\frac{1}{r}}(x) + \big(M|f|^r\big)^{\frac{1}{r}}(x)\Big) \\
&\quad + C \sum_{|\beta|=m} \sum_{\ell=1}^{N} \sum_{s=1}^{N} \Big\|a^{(\beta)}_{\ell,s}\Big\|_* \Big(\big(M|S^{\beta,\lambda,\mu}_{j,k,\ell,s} af|^r\big)^{\frac{1}{r}}(x) \\
&\quad + \|a\|_\infty \big(M|S^{\beta,\lambda,\mu}_{j,k,\ell,s} f|^r\big)^{\frac{1}{r}}(x) + \|a\|_\infty \big(M|f|^r\big)^{\frac{1}{r}}(x)\Big).
\end{aligned}$$

As in the third step of the proof of Proposition 5.4 in [HH03] there exists an $r > 1$, such that

$$\|\big(M|f|^r\big)^{\frac{1}{r}}\|_{p,\omega} \leq C\|f\|_{p,\omega}, \qquad f \in L^p_\omega(\mathbb{R}^n),$$

with an A_p-consistent constant C. This and Lemma 2.2 now yield

$$\begin{aligned}
&\Big\|[T^{3,\lambda,\mu}_{j,k}, a]f\Big\|_{p,\omega} \leq C \Big\|\Big([T^{3,\lambda,\mu}_{j,k}, a]f\Big)^{\#}\Big\|_{p,\omega} \\
&\leq \; CA_* \Big[\|T^{3,\lambda,\mu}_{j,k} f\|_{p,\omega} + \|f\|_{p,\omega} \\
&\qquad + \sum_{|\beta|=m} \sum_{\ell=1}^{N} \sum_{s=1}^{N} \Big(\|S^{\beta,\lambda,\mu}_{j,k,\ell,s} af\|_{p,\omega} + \|a\|_\infty \big(\|S^{\beta,\lambda,\mu}_{j,k,\ell,s} f\|_{p,\omega} + \|f\|_{p,\omega}\big)\Big)\Big] \\
&\leq \; CA_* \Big[\|f\|_{p,\omega} + N^{m+2}\big(\|af\|_{p,\omega} + \|a\|_\infty \|f\|_{p,\omega}\big)\Big] \;\leq\; CA_*(1 + \|a\|_\infty)\|f\|_{p,\omega},
\end{aligned}$$

where C is an A_p-consistent constant.

We finally show the boundedness of the operators $T^{2,\lambda,\nu}_{j,k,\ell}$ and $[T^{2,\lambda,\nu}_{j,k,\ell}, a]$. For this purpose, we use Proposition 4.4 to expand the kernel of the principal value integral in the definition of $T^{2,\lambda,\nu}_{j,k,\ell}$. This yields

$$T^{2,\lambda,\nu}_{j,k,\ell} f(x) = \sum_{s=1}^{\infty} b^{\nu,k,\ell}_s(x) \int_{\mathbb{R}^n} \lambda g^{\lambda,x}_{j,k}(x-y) \underbrace{\text{p.v.}\int_{\mathbb{R}^n} \frac{Y^{\nu,k,\ell}_s(y-z)}{|y-z|^n} f(z)\,\mathrm{d}z}_{=:R^{\nu,k,\ell}_s f(y)} \,\mathrm{d}y,$$

where $R^{\nu,k,\ell}_s$ is a Calderón-Zygmund operator on $L^p_\omega(\mathbb{R}^n)$, whose norm can be bounded independently of s. Hence

$$\begin{aligned}
\|T^{2,\lambda,\nu}_{j,k,\ell} f\|_{p,\omega} &= \Big\|\sum_{s=1}^{\infty} b^{\nu,k,\ell}_s \Big(T^{3,\lambda,0}_{j,k} R^{\nu,k,\ell}_s f\Big)\Big\|_{p,\omega} \\
&\leq \sum_{s=1}^{\infty} \|b^{\nu,k,\ell}_s\|_\infty \|T^{3,\lambda,0}_{j,k} R^{\nu,k,\ell}_s f\|_{p,\omega} \;\leq\; C \sum_{s=1}^{\infty} \|b^{\nu,k,\ell}_s\|_\infty \|f\|_{p,\omega} \leq C\|f\|_{p,\omega}
\end{aligned}$$

with an A_p-consistent constant C.

In order to bound the operator $[T^{2,\lambda,\nu}_{j,k,\ell}, a]$ we write

$$\begin{aligned}
[T^{2,\lambda,\nu}_{j,k,\ell}, a]f(x) &= \int_{\mathbb{R}^n} \lambda g^{\lambda,x}_{j,k}(x-y)\,\text{p.v.}\int_{\mathbb{R}^n} K^\nu_{k,\ell}(x, y-z)\big(a(x) - a(z)\big) f(z)\,\mathrm{d}z\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \lambda g^{\lambda,x}_{j,k}(x-y)\big(a(x) - a(y)\big)\,\text{p.v.}\int_{\mathbb{R}^n} K^\nu_{k,\ell}(x, y-z) f(z)\,\mathrm{d}z\,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^n} \lambda g^{\lambda,x}_{j,k}(x-y)\,\text{p.v.}\int_{\mathbb{R}^n} K^\nu_{k,\ell}(x, y-z)\big(a(y) - a(z)\big) f(z)\,\mathrm{d}z\,\mathrm{d}y.
\end{aligned}$$

Applying once more Proposition 4.4 to expand the kernel $K^{\nu}_{k,\ell}$, we get

$$[T^{2,\lambda,\nu}_{j,k,\ell}, a]f(x) = \sum_{s=1}^{\infty} b_s^{\nu,k,\ell}(x)\Big(\big([T^{3,\lambda,0}_{j,k}, a]R_s^{\nu,k,\ell}f\big)(x) + T^{3,\lambda,0}_{j,k}\big([R_s^{\nu,k,\ell}, a]f\big)(x)\Big).$$

As we have already bounded the T^3-operators, we finally conclude with the help of Propositions 4.3 and 4.4

$$\begin{aligned}\Big\|[T^{2,\lambda,\nu}_{j,k,\ell}, a]f\Big\|_{p,\omega} &\le C\sum_{s=1}^{\infty}\|b_s^{\nu,k,\ell}\|_\infty\big(A_*(1+\|a\|_\infty)\|f\|_{p,\omega} + \|a\|_*\|f\|_{p,\omega}\big)\\ &\le CA_*(1+\|a\|_\infty)\|f\|_{p,\omega}.\end{aligned}$$

□

Proof of Theorem 5.1: For every $|\nu| < m$ and all $j = 1,\dots,N$ we have by (5.1)

$$\lambda^{1-\frac{|\nu|}{m}}D^\nu u_j(x) = \sum_{k=1}^N\big(T^{3,\lambda,\nu}_{j,k}((\lambda+\mathcal{A})u)_k\big)(x) + \sum_{\ell=1}^N\sum_{k=1}^N\sum_{|\alpha|=m}\big([T^{3,\lambda,\nu}_{j,k}, a^{(\alpha)}_{k,\ell}]D^\alpha u_\ell\big)(x).$$

By the preceeding lemma we see that

$$\|\lambda^{1-\frac{|\nu|}{m}}D^\nu u_j\|_{p,\omega} \le C\Big(\sum_{k=1}^N\|\big((\lambda+\mathcal{A})u\big)_k\|_{p,\omega} + a_*a_\infty\sum_{\ell=1}^N\sum_{|\alpha|=m}\|D^\alpha u_\ell\|_{p,\omega}\Big),$$

where

$$a_* := \max_{|\alpha|=m}\max_{k,\ell=1}^N\|a^{(\alpha)}_{k,\ell}\|_* \quad\text{and}\quad a_\infty := 1 + \max_{|\alpha|=m}\max_{k,\ell=1}^N\|a^{(\alpha)}_{k,\ell}\|_\infty.$$

Analogously we get by (5.2) for all $|\nu| = m$ and all $j = 1,\dots,N$

$$\begin{aligned}\|D^\nu u_j\|_{p,\omega} \le\ & C\Big(\sum_{k=1}^N\|\big((\lambda+\mathcal{A})u\big)_k\|_{p,\omega} + a_*\sum_{k=1}^N\sum_{|\alpha|=m}\|D^\alpha u_k\|_{p,\omega} + \sum_{k=1}^N\|\big((\lambda+\mathcal{A})u\big)_k\|_{p,\omega}\\ &+\sum_{k=1}^N\|\big((\lambda+\mathcal{A})u\big)_k\|_{p,\omega} + a_*a_\infty\sum_{k=1}^N\sum_{|\alpha|=m}\|D^\alpha u_k\|_{p,\omega}\\ &+\sum_{k=1}^N\|\big((\lambda+\mathcal{A})u\big)_k\|_{p,\omega} + a_*a_\infty\sum_{k=1}^N\sum_{|\alpha|=m}\|D^\alpha u_k\|_{p,\omega}\Big).\end{aligned}$$

Combinig these two estimates, we conclude that

$$\begin{aligned}\sum_{|\nu|\le m}\Big\|\lambda^{1-\frac{|\nu|}{m}}D^\nu u\Big\|_{p,\omega} &\le C\sum_{j=1}^N\sum_{|\nu|\le m}\Big\|\lambda^{1-\frac{|\nu|}{m}}D^\nu u_j\Big\|_{p,\omega}\\ &\le C\Big(\sum_{k=1}^N\|\big((\lambda+\mathcal{A})u\big)_k\|_{p,\omega} + a_*a_\infty\sum_{k=1}^N\sum_{|\alpha|=m}\|D^\alpha u_k\|_{p,\omega}\Big)\end{aligned}$$

with an A_p-consistent constant C. If $\eta < (Ca_\infty)^{-1}$, we have $Ca_*a_\infty \le C\eta a_\infty < 1$. Thus we may bring the second term on the left hand side to the right hand side and therefore finish the proof.

□

6. Proof of the Main Result

Due to the a priori estimate given in the previous section, it remains to prove that the operator $\lambda + \mathcal{A}$ from $W^{m,p}_\omega(\mathbb{R}^n)$ onto $L^p_\omega(\mathbb{R}^n)^N$ is surjective. This follows from the continuity method in the same way as in [HH03]. We just refer to Theorem 6.7 in [HH03]. We thus have the following result.

Proposition 6.1. *Let $n \geq 2$, $1 < p < \infty$, $\phi_0 \in (0, \pi)$, $\phi > \phi_0$, $\omega \in A_p$. Let $\mathcal{A}$ be as above. Then there are A_p-consistent constants $\eta > 0$ und $\lambda_0 \geq 0$, such that for every $f \in L^p_\omega(\mathbb{R}^n)^N$ and every $\lambda \in \Sigma_{\pi-\phi}$ there exists a unique function $u \in W^{m,p}_\omega(\mathbb{R}^n)^N$ such that*

$$(\lambda + \lambda_0 + \mathcal{A})u = f,$$

whenever $\max_{|\alpha|=m} \|a_\alpha\|_* \leq \eta$.

Corollary 6.2. *Under the assumptions of Proposition 6.1, there exist A_p-consistent constants $C, \lambda \geq 0$ such that*

$$\|(\lambda + \lambda_0 + A_{p,\omega})^{-1}\|_{p,\omega} \leq \frac{C}{|\lambda|}, \quad \lambda \in \Sigma_{\pi-\phi}.$$

If $\phi_0 < \pi/2$, then $-(\lambda_0 + A_{p,\omega})$ generates an analytic semigroup on $L^p_\omega(\mathbb{R}^n)$. Moreover, $\lambda_0 + A_{p,\omega}$ satisfies the estimate (1.2).

Proof. Theorem 5.1 implies that $\lambda + \lambda_0 + A_{p,\omega}$ is one to one as well as the resolvent estimate. Surjectivity of $\lambda + \lambda_0 + A_{p,\omega}$ follows from Proposition 6.1. The L^p-L^q-estimate is now a consequence of Proposition 2.1. □

In order to apply a localization procedure to differential operators with VMO coefficients, we need the following lemma.

Lemma 6.3. *Let $f \in \mathrm{VMO}(\mathbb{R}^n; \mathbb{C}^{N\times N})$. Then for every $\varepsilon > 0$, there is a radius $r = r(\varepsilon, f) > 0$, such that for all $x_0 \in \mathbb{R}^n$, there exists a function $g \in \mathrm{VMO}(\mathbb{R}^n; \mathbb{C}^{N\times N})$ with $\|g\|_* \leq \varepsilon$ and $f(x) = g(x)$ for all $x \in B(x_0, r)$.*

Proof. The corresponding result for $N = 1$ is in [HH03, Lemma 3.2]. By this, for every entry $f_{j,k}$, $j, k = 1, \ldots, N$, of f we get a function $g_{j,k}$ having the above properties. Setting $g = (g_{j,k})_{j,k=1}^N$ yields the result. □

Proof of Theorem 3.1. In order to apply the usual localization procedure, we have to define local operators A_j, $j \in \mathbb{N}$, that are equal to $\mathcal{A}$ on balls $B(x_j, r) \subseteq \mathbb{R}^n$, but whose coefficients have BMO norm smaller than η, where η is the constant from Theorem 5.1.

In order to do so, we choose for every $|\alpha| = m$ a radius $r_\alpha > 0$ given by Lemma 6.3 with $\varepsilon = \eta$. We set $r := \min_{|\alpha|=m} r_\alpha$. By Lemma 6.3 there exist functions $a_{\alpha,j} \in \mathrm{BMO}(\mathbb{R}^n; \mathbb{C}^{N\times N})$ with $\|a_{\alpha,j}\|_* \leq \eta$ and $a_{\alpha,j} = a_\alpha$ on $B(x_j, r)$. Taking $a_{\alpha,j}$ as coefficients for suitable differential operators A_j the result follows by the localization method used in [AHS94] and [HH03]. □

References

[ADN59] Agmon, S., Douglis, A., Nirenberg, L.: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I *Comm. Pure Appl. Math.* **12** (1959), 623–727.

[Ama95] Amann, H.: *Linear and Quasilinear Parabolic Problems, Vol. I. Abstract Linear Theory.* Monographs in Mathematics **89**, Birkhäuser, Basel, 1995.

[AHS94] Amann, H., Hieber, M., Simonett, G.: Bounded H^∞-calculus for elliptic operators. *Differential Integral Equations* **7** (1994), 613–653.

[AMT97] Angeletti, J.-M., Mazet, S., Tchamitchian, Ph.: Analysis of second order elliptic operators without boundary conditions and with VMO or Hölderian coefficients. In: *Multiscale Wavelet Methods for PDEs*, W. Dahmen, A. Kurdilla, P. Oswald (eds.), Academic Press 1997, 495-539.

[Aus04] Auscher, P.: On necessary and sufficient conditions for L^p estimates of Riesz transforms associated to elliptic operators on $\mathbb{R}^n$ and related estimates. *Memoirs Amer. Math. Soc.*, to appear.

[BC93] Bramanti, M., Cerutti, M.C.: $W_p^{1,2}$ solvability for the Cauchy Dirichlet problem for parabolic equations with VMO coefficients. *Commun. Partial Differ. Equations* **18** (1993), 1735-1763.

[CFL91] Chiarenza, F., Frasca, M., Longo, P.: Interior $W^{2,p}$ estimates for nondivergence elliptic equations with discontinuous coefficients. *Ricerche di Mat.* **40** (1991), 149-168.

[CFL93] Chiarenza, F., Frasca, M., Longo, P.: $W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients. *Trans. Amer. Math. Soc.* **336** (1993), 841-853.

[CRW76] Coifman, R., Rochberg, R., Weiss, G.: Factorization theorems for Hardy spaces in several variables. *Ann. of Math.* **103** (1976), 611-635.

[DDHPV04] Denk, R., Dore, G., Hieber, M., Prüss, J., Venni, A.: New Thoughts on old theorems of R.T. Seeley. *Math. Ann.* **328** 2004, 545–583.

[DHP03] Denk, R., Hieber, M., Prüss, J.: $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Memoirs Amer. Math. Soc.*, **166** (2003).

[DP96] Di Fazio, G., Palagachev, D.K.: Oblique derivative problem for elliptic equations in non-divergence form with VMO coefficients. *Comment. Math. Univ. Carolin.* **37** (1996), 537–556.

[DV87] Dore, G., Venni, A.: On the closedness of the sum of two closed operators. *Math. Z.* **196** (1987), 189-201.

[Duo90] Duong, X.T.: H_∞ Functional calculus of elliptic operators with C^∞ coefficients on L^p spaces on smooth domains. *J. Austral. Math. Soc. Ser. A* **48** (1990), 113-123.

[DS97] Duong, X.T., Simonett, G.: H^∞-calculus for elliptic operators with nonsmooth coefficients. *Differential Integral Equations* **10** (1997), 201–217.

[DY02] Duong, X.T., Yan, L.X.: Bounded holomorphic functional calculus for non-divergence form differential operators. *Differential Integral Equations* **15** (2002), 709–730.

[GR85] Garcia-Cuerva, J., Rubio de Francia, J.L.: *Weighted Norm Inequalities and Related Topics*, North-Holland, 1985.

[Gui02] Guidetti, D.: General linear boundary value problems for elliptic operators with VMO coefficients. *Math. Nachr.* **237** (2002), 62-88.

[HHH03] Haller, R., Heck, H., Hieber, M.: Muckenhoupt weights and maximal regularity. *Arch. Math.* **81** (2003), 422–430.

[HH03] Heck, H., Hieber, M.: Maximal L^p-regularity for elliptic operators with VMO-coefficients. *J. Evol. Equ.* **3** (2003), 332–359.

[HP97] Hieber, M., Prüss, J.: Heat kernels and maximal L^p-L^q estimates for parabolic evolution equations. *Comm. Partial Differential Equations* **22** (1997), 1647–1669.

[HW04] Hieber, M. Wood, I.: The Dirichlet Problem in Convex Bounded Domains for Operators with L^∞-Coefficients. Preprint, TU Darmstadt (2004).

[LSU68] Ladyzenskaja, O., Solonnikov, V.A., Uralceva, N.N.: *Linear and Quasilinear Equations of Parabolic Type.* Amer. Math. Soc. Transl. Math. Monographs, Providence, R.I., 1968.

[MP98] Maugeri, A., Palagachev, D.K.: Boundary value problems with an oblique derivative for uniformly elliptic operators with discontinuous coefficients. *Forum Math.* **10** (1998), 393-405.

[MPS00] Maugeri, A., Palagachev, D., Softova, L.: *Elliptic and Parabolic Equations with Discontinuous Coefficients.* Wiley, 2000.

[Pal95] Palagachev, D.K.: Quasilinear elliptic equations with VMO coefficients. *Trans. Amer. Math. Soc.* **347** (1995), 2481–2493.

[PS93] Prüss, J., Sohr, H.: Imaginary powers of elliptic second order differential operators in L^p-spaces. *Hiroshima Math. J.* **23** (1993), 161-192.

[Rag02] Ragusa, M.A.: Local Hölder regularity for solutions of elliptic systems. *Duke Math. J.* **113** (2002), 385–397.

[Rub80] Rubio de Francia, J.L.: Vector valued inequalities for operators on L^p-spaces. *Bull. London Math. Soc.* **12** (1980), 211-215.

[Sar75] Sarason, D.: Functions of vanishing mean oscillation. *Trans. Amer. Math. Soc.* **207** (1975), 391-405.

[Sof00] Softova, L.: Oblique derivative problem for parabolic operators with VMO coefficients, *Manuscripta Math.* **103** (2000), 203–220.

[Ste93] Stein, E.M.: *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals.* Princeton University Press, Princeton, 1993.

[SW71] Stein, E.M., Weiss, G.: *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton University Press, Princeton, 1971.

[Tal65] Talenti, G.: Sopra una classe di equazioni ellittichi a coefficienti misurabili. *Ann. Mat. Pura Appl.* **69** (1965), 285-304.

[Tor86] Torchinsky, A.: *Real-variable Methods in Harmonic Analysis.* Academic Press, New York, 1986.

[Wei01] Weis, L.: Operator-valued Fourier multiplier theorems and maximal L_p-regularity, *Math. Ann.*, **319** (2001), 735-758.

Technische Universität Darmstadt, Fachbereich Mathematik, Schlossgartenstr. 7, D-64289 Darmstadt, Germany

E-mail address: haller@mathematik.tu-darmstadt.de

Technische Universität Darmstadt, Fachbereich Mathematik, Schlossgartenstr. 7, D-64289 Darmstadt, Germany

E-mail address: heck@mathematik.tu-darmstadt.de

Technische Universität Darmstadt, Fachbereich Mathematik, Schlossgartenstr. 7, D-64289 Darmstadt, Germany

E-mail address: hieber@mathematik.tu-darmstadt.de