# Tubes in $PG(3,q)$

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**Abstract.** A tube (resp. an oval tube) in $PG(3,q)$ is a pair $\mathcal{T} = \{L, \mathcal{L}\}$, where $\{L\} \cup \mathcal{L}$ is a collection of mutually disjoint lines of $PG(3,q)$ such that for each plane $\pi$ of $PG(3,q)$ containing $L$ the intersection of $\pi$ with the lines of $\mathcal{L}$ is a hyperoval (resp. an oval). The line $L$ is called the axis of $\mathcal{T}$. We show that every tube for $q$ even and every oval tube for $q$ odd can be naturally embedded into a regular spread and hence admits a group of automorphisms which fixes every element of $\mathcal{T}$ and acts regularly on each of them. For $q$ odd we obtain a classification of oval tubes up to projective equivalence. Furthermore, we characterize the reguli in $PG(3,q), q$ odd, as oval tubes which admit more than one axis.

## 1. Introduction

A *partial tube* in $PG(3,q)$ is a pair $\mathcal{T} = \{L, \mathcal{L}\}$, where $\{L\} \cup \mathcal{L}$ is a collection of mutually disjoint lines of $PG(3,q)$ such that for each plane $\pi$ of $PG(3,q)$ containing $L$ the intersection of $\pi$ with the lines of $\mathcal{L}$ is an arc. $\mathcal{T}$ is called a *tube* if each of these arcs is complete. It follows that tubes exists only for $q$ even and that $\mathcal{L}$ contains $q+2$ lines if $\mathcal{T}$ is a tube. If $\mathcal{L}$ contains $q+1$ lines then $\mathcal{T}$ is called an *oval tube.* An obvious example of an oval tube is obtained by taking for $\mathcal{L}$ the lines of a regulus and for $L$ any exterior line of the underlying hyperbolic quadric of $\mathcal{L}$. An oval tube of this type is called a quadric tube. If $q$ is even, then $\mathcal{L}$ can be extended by the line $L^{\perp}$ which is the image of $L$ under the polarity associated with $\mathcal{L}$ to form a tube. The line $L$ is called the *axis* of the partial tube $\mathcal{T}$.

Tubes were introduced in [3] in connection with a construction problem for flat $\pi.C_2$ geometries, cp. [9].

Examples of partial tubes can be obtained as follows. Let $L$ be a line of a regular spread in $PG(3,q)$. Choose a plane $\pi_0$ through $L$ and let $\Omega \subset \pi_0 \setminus L$ be an arc. If $\mathcal{L}$ denotes the lines of the regular spread passing through the points of $\Omega$, then $\mathcal{T} = \{L, \mathcal{L}\}$ is a partial tube. If $\Omega$ is a hyperoval or an oval then $\mathcal{T}$ is a tube or an oval tube. We are going to prove that in fact all tubes for $q$ even and all oval tubes for $q$ odd are obtained in this way. Since for odd $q$ all ovals are conics, we obtain a complete classification up to projective equivalence in the odd order case. It turnes out that there are precisely $\frac{3q-1}{4}$ or $\frac{3q-3}{4}$ equivalence classes if $q \equiv 3 \bmod 4$ or $q \equiv 1 \bmod 4$, respectively.

In order to describe these tubes algebraically it seems convenient to introduce coordinates from $GF(q^2)$. We consider the 2-dimensional $GF(q^2)$-vector space $V = GF(q^2) \times$

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$GF(q^2)$ as a 4-dimensional $GF(q)$-vector space. Put $L = \{0\} \times GF(q^2)$ and $L(a,b) = \{(z, az + b\overline{z}) | z \in GF(q^2)\}$ for $a, b \in GF(q^2)$. The sets $L(a,b), a, b \in GF(q^2)$, are precisely the 2-dimensional $GF(q)$-subspaces of $V$ which are complementary to $L$. The set $\mathcal{B} = \{L\} \cup \{L(m,0) | m \in GF(q^2)\}$ is a regular spread. If we take the elements of this spread as points and all reguli contained in it as circles, then we get a model of the Miquelian inversive plane $I(q)$ with pointset $GF(q^2) \cup \{\infty\}$. If we map $L(m,0)$ to $m$ and $L$ to $\infty$, then the reguli contained in $\mathcal{B}$ which do not contain $L$ are mapped to the circles. Here, a circle is a set of the form $\{aw + b | w \in GF(q^2), w\overline{w} = 1\}$ with $a, b \in GF(q^2), a \neq 0$.

We consider $GF(q^2)$ an affine plane over $GF(q)$. Let $\Omega \subset GF(q^2)$ be an arc in this affine plane and put $\mathcal{L} = \{L(a,0) | a \in \Omega\}$. Then $\mathcal{T} = \{L, \mathcal{L}\}$ is a partial tube. $\mathcal{T}$ is a quadric tube if and only if $\Omega$ is a circle.

A partial tube is called central if it admits a group of automorphisms which fixes all elements of $\mathcal{L}$ and acts regularly on each of them.

**Proposition 1.1.** *A partial tube is central if and only if it can be embedded into a regular spread in the way just described.*

*Proof.* This is proved in [3], Theorem 3.2 for tubes, but the argument carries over to partial tubes. □

## 2. Tubes of even order

For any three mutually skew lines $L_1, L_2, L_3$ of $PG(3,q)$ we denote the regulus spanned by them by $\mathcal{R}(L_1, L_2, L_3)$

**Lemma 2.1.** *Let $\mathcal{T} = \{L, \mathcal{L}\}$ be a partial tube and let $\mathcal{L} = \{L_0, \ldots, L_m\}$. Then $\mathcal{A}_n = \cup_{i \neq n} \mathcal{R}(L, L_i, L_n)$ is a partial spread of $PG(3,q)$ for $n = 0, \ldots, m$.*

*Proof.* Let $G$ be any line of $PG(3,q)$ which intersects $L$ and $L_n$ and let $\pi$ be the plane spanned by $L$ and $G$. Since $G$ intersects $L_n$ and the intersection of $\pi$ with the lines of $\mathcal{L}$ is an arc there is at most one $i \in \{0, \ldots, m\} \setminus \{n\}$ with $G \cap L_i \neq \emptyset$. It follows that there is at most one $i \in \{0, \ldots, m\} \setminus \{n\}$ such that $G$ is a transversal of $\mathcal{R}(L, L_i, L_n)$ and hence $\mathcal{A}_n$ is a partial spread. □

**Proposition 2.2.** *Let $\mathcal{T}$ be a tube with $q$ even. Then the partial spread $\mathcal{A}_n$ is a regular spread for $n = 0, \ldots, q+1$ and these spreads all coincide.*

*Proof.* By the preceding lemma $\mathcal{A}_n$ is a partial spread. Since $\mathcal{A}_n$ contains $2 + (q+1)(q-1) = q^2 + 1$ lines it is actually a spread.

Consider the following incidence structure $\mathcal{M}$. Points of $\mathcal{M}$ are the lines of $PG(3,q)$ that intersect $L$ and $L_n$ and circles of $\mathcal{M}$ are the reguli of $PG(3,q)$ that admit $L$ and $L_n$ as transversals. Then it is well-known that $\mathcal{M}$ is isomorphic to the Miquelian Minkowski plane over $GF(q)$, cp. eg. [1: III 4. Satz 5.1].

The reguli opposite to the $\mathcal{R}(L, L_i, L_n), i \in \{0, \ldots, q+1\} \setminus \{n\}$, constitute a flock of this Minkowski plane and $\mathcal{A}_n$ is the spread associated with this flock, cp. [4]. By a result of Thas [10], the flock is linear since $q$ is even and hence $\mathcal{A}_n$ is regular.

Under the Plücker mapping the regular spreads correspond to intersections of the Klein quadric with certain projective spaces of rank 3. It follows that two regular spreads which have 4 common lines that are not contained in a regulus are the same. Since each of the spreads $\mathcal{A}_n, n = 0, \ldots, q+1$ contains the lines $L, L_0, \ldots, L_{q+1}$, they all coincide. $\square$

This result immediately implies the following

**Theorem 2.3.** *Every tube of even order is central.*

## 3. Oval tubes of odd order

For oval tubes of odd order the situation is more complicated. In this case, every partial spread $\mathcal{A}_n$ is associated with a partial flock of a hyperbolic quadric of deficiency one. Partial flocks of this type have been investigated by Johnson [5], and there are examples of such partial flocks which cannot be extended to a flock. In any case, the partial spread $\mathcal{A}_n \setminus \{L, L_n\}$ can be extended to a spread by a collection of transversals to $L$ and $L_n$. The partial flock can be extended to a flock if and only if these transversals form a regulus. There are known counterexamples for $q = 5$ and 9, cp. [6] and [2], (and also for $q = 4$, cp. [5], although we are not interested in the even order case in this section).

In [3] the following description for partial tubes has been given. Let $V$ be the 4-dimensional vector space $GF(q)^2 \times GF(q)^2$ and put $L = \{0\} \times GF(q)^2$. Then every line of $PG(V)$ which is disjoint from $L$ is the graph of a unique linear mapping from $GF(q)^2$ to $GF(q)^2$, which we identify with its matrix. A collection $A_0, \ldots, A_m$ of $2 \times 2$ matrices defines a partial tube with axis $L$ if and only if the following two conditions are satisfied:

(i) $A_i - A_j$ is nonsingular for $i \neq j$

(ii) for any vector $v \neq 0$ and any distinct $i, j, k \in \{0, \ldots, m\}$ the points $A_i v, A_j v, A_k v$ are affine independent.

Note that (ii) is equivalent to $\Omega_v = \{A_0 v, \ldots, A_m v\}$ being an $(m+1)$-arc (in particular an oval if $m = q$) for every $v \neq 0$.

**Lemma 3.1.** *A collection $A_0, \ldots, A_m$ of $2 \times 2$ matrices defines a partial tube with axis $L$ if and only if they satisfy (i) and*

*(ii') For all $(\lambda, \mu) \in GF(q)^2 \setminus \{(0,0)\}$ and all distinct $i, j, k \in \{0, \ldots, m\}$ the matrix $\lambda(A_j - A_i) + \mu(A_k - A_i)$ is nonsingular.*

*Proof.* Condition (ii) is equivalent to the following requirement. For all $(\lambda, \mu) \in GF(q)^2 \setminus \{(0,0)\}$, for all $v \neq 0$ and for all distinct $i, j, k$ there holds $\lambda(A_j - A_i)v + \mu(A_k - A_i)v \neq 0$. But this is also equivalent to (ii'). $\square$

This lemma is in fact equivalent to Lemma 2.1 since it is easy to see that $\{A_n+\lambda(A_i-A_n)|\lambda \in GF(q), i \neq n\}$ is a (partial) matrix spread set for the partial spread $\mathcal{A}_n$. This follows from the fact that the reguli of $PG(V)$ which contain $L$ correspond to the affine lines of the space of $2 \times 2$ matrices, cp. e.g. [8] or [7: Lemma 4.11].

**Lemma 3.2.** *Let $\Omega = \{v_0, \ldots, v_q\} \subset GF(q)^2$ be a conic. Then $\sum_{i=0}^q v_i$ is the midpoint of $\Omega$.*

*Proof.* This is obviously true if the midpoint of $\Omega$ is the origin.

Let $v$ be the midpoint of $\Omega$, then $\Omega - v$ is a conic whose midpoint is the origin. Thus we get

$$0 = \sum_{i=0}^{q}(v_i - v) = (\sum_{i=0}^{q} v_i) - (q+1)v = (\sum_{i=0}^{q} v_i) - v$$

and the result follows. □

**Corollary 3.3.** *The midpoint of $\Omega_v$ is given by $m_v = (\sum_{j=0}^q A_j)v$.*

**Lemma 3.4.** *For every $i \in \{0, \ldots, q\}$ there exists $k = \sigma(i) \in \{0, \ldots, q\}$ such that $A_i + A_{\sigma(i)} = 2(\sum_{j=0}^q A_j)$.*

*Proof.* Given $i \in \{0, \ldots, q\}$ and $v \in GF(q)^2 \setminus \{0\}$ there exists $k \in \{0, \ldots, q\} \setminus \{i\}$ such that $A_i v + A_k v = 2(\sum_{j=0}^q A_j)v$ since $2(\sum_{j=0}^q A_j)v$ is the midpoint of $\Omega v$, but $k$ is dependent on $v$. Since there are $q+1$ mutually linear independent vectors in $GF(q)^2$ but only $q$ choices for $k$, there are linear independent vectors $v$ and $w$ such that $A_i v + A_k v = 2(\sum_{j=0}^q A_j)v$ and $A_i w + A_k w = 2(\sum_{j=0}^q A_j)w$. It follows that $A_i + A_k = 2(\sum_{j=0}^q A_j)$. □

If we replace every $A_i$ by $A_i - (\sum_{j=0}^q A_j)$ we may assume that our collection of matrices is closed unter taking additive inverses and that all conics $\Omega_v$ are centrally symmetric with respect to the origin. Moreover, we may renumber the matrices $A_i$ with indices from $\{\pm 1, \ldots, \pm\frac{q+1}{2}\}$ such that $A_{-i} = -A_i$ for all $i \in \{\pm 1, \ldots, \pm\frac{q+1}{2}\}$.

By identifying $GF(q)^2$ and $GF(q^2)$ we may assume that for each $i \in \{\pm 1, \ldots, \pm\frac{q+1}{2}\}$ there are $a_i, b_i \in GF(q^2)$ such that $A_i v = a_i v + b_i \overline{v}$ for all $v \in GF(q^2)$. We may also assume that $a_1 = 1, b_1 = 0$ and $b_2 = 0$, i.e. that $A_1$ is the identity and that $A_2$ is linear over $GF(q^2)$. Note that the elements of $\mathcal{L}$ are now precisely the sets $L(a_i, b_i), i \in \{\pm 1, \ldots, \pm\frac{q+1}{2}\}$. Since division by non-zero elements of $GF(q^2)$ is linear over $GF(q)$, the sets

$$\Omega_v' = \{\frac{A_i v}{v} | i \in \{\pm 1, \ldots, \pm\frac{q+1}{2}\}\} = \{a_i + b_i \frac{\overline{v}}{v} | i \in \{\pm 1, \ldots, \pm\frac{q+1}{2}\}\}$$

are conics for all $v \in GF(q^2) \setminus \{0\}$. By Hilbert's theorem 90, the elements of the form $\frac{\overline{v}}{v}, v \in GF(q^2)$, are precisely the elements of norm 1. Hence the conics $\Omega_v', v \in GF(q^2) \setminus \{0\}$, coincide with the conics

$$\Omega^*_w = \{a_i + b_i w | i \in \{\pm 1, \dots, \pm \frac{q+1}{2}\}\}, w \in GF(q^2), w\overline{w} = 1.$$

Note that each of these conics passes through the four points $\pm 1$ and $\pm a_2$, while the other points are moving on circles.

**Theorem 3.5.** *Every oval tube of odd order is central.*

*Proof.* The assumptions $b_{\pm 1} = b_{\pm}2 = 0$ imply that $L(a_{\pm 1}, b_{\pm 1})$ and $L(a_{\pm 2}, b_{\pm 2})$ are contained in the regular spread $\mathcal{B} = \{L\} \cup \{L(m, 0) | m \in GF(q^2)\}$. By Proposition 1.1, it is sufficient to show that $b_i = 0$ for $i \in \{\pm 1, \dots, \pm \frac{q+1}{2}\}$. Since a conic is uniquely determined by five points, it is actually sufficient to show that $b_i = 0$ for one $i \in \{\pm 3, \dots, \pm \frac{q+1}{2}\}$.

If $q = 3$ there is nothing to prove.

If $q = 5$, then the union of the six lines connecting any two of the four points $\pm 1, \pm a_2$ contains 21 points. The remaining set of 4 points cannot contain a proper circle and hence $\pm b_3 = 0$.

If $q = 7$, then the union of the six lines connecting any two of the four points $\pm 1, \pm a_2$ contains 33 points. The remaining set of 16 points cannot contain four distinct circles since their union contains at least 20 points.

From now on we assume $q \geq 9$.

Choose $i, j \in \{\pm 3, \dots, \pm \frac{q+1}{2}\}, i \neq \pm j$, and $w \in GF(q^2)$ with $w\overline{w} = 1$. Put $x = a_i + b_i w$ and $y = a_j + b_j w$.

We are going to apply Pascal's theorem to the hexagon $\{1, -1, y, -a_2, a_2, x\}$, which lies on the conic $\Omega^*_w$.

The lines $1 \vee -1$ and $a_2 \vee -a_2$ intersect in the origin.

The intersection point of the lines $1 \vee x$ and $-a_2 \vee y$ is determined by the equation

$$x + t_1(1 - x) = y + t_2(-a_2 - y), t_1, t_2 \in GF(q).$$

This yields

$$t_2 = \frac{x - y + t_1(1 - x)}{-a_2 - y} = \overline{t_2} = \frac{\overline{x} - \overline{y} + t_1(1 - \overline{x})}{-\overline{a_2} - \overline{y}}$$

and hence

$$t_1 = \frac{(\overline{x} - \overline{y})(a_2 + y) - (x - y)(\overline{a_2} + \overline{y})}{(1 - x)(\overline{a_2} + \overline{y}) - (1 - \overline{x})(a_2 + y)}.$$

Similarly, the intersection point of the lines $a_2 \vee x$ and $-1 \vee y$ is determined by the equation

$$x + t_3(a_2 - x) = y + t_4(-1 - y), t_3, t_4 \in GF(q).$$

This yields

$$t_3 = \frac{y - x - t_4(1 + y)}{a_2 - x} = \overline{t_3} = \frac{\overline{y} - \overline{x} - t_4(1 + \overline{y})}{\overline{a_2} - \overline{x}}$$

and hence

$$t_4 = \frac{(\overline{y}-\overline{x})(a_2-x)-(y-x)(\overline{a_2}-\overline{x})}{-(1+y)(\overline{a_2}+\overline{x})+(1+\overline{y})(a_2-x)}.$$

Pascal's theorem now says that the line joining the two points $x+t_1(1-x)$ and $y+t_4(-1-y)$ passes through the origin. This is equivalent to the requirement $\frac{x+t_1(1-x)}{y-t_4(1+y)} \in GF(q)$, which is in turn equivalent to

$$(x+t_1(1-x))\overline{(y-t_4(1+y))} = \overline{(x+t_1(1-x))}(y-t_4(1+y)). \qquad (*)$$

From our computations of $t_1$ and $t_4$ we get

$$\begin{aligned}
x+t_1(1-x) &= \frac{x((1-x)(\overline{a_2}+\overline{y})-(1-\overline{x})(a_2+y))+(1-x)((\overline{x}-\overline{y})(a_2+y)-(x-y)(\overline{a_2}+\overline{y}))}{(1-x)(\overline{a_2}+\overline{y})-(1-\overline{x})(a_2+y)}\\
&= \frac{(x-(x-y))(1-x)(\overline{a_2}+\overline{y})+((1-x)(\overline{x}-\overline{y})-x(1-\overline{x}))(a_2+y)}{(1-x)(\overline{a_2}+\overline{y})-(1-\overline{x})(a_2+y)}\\
&= \frac{y(1-x)(\overline{a_2}+\overline{y})+(\overline{x}-x-(1-x)\overline{y})(a_2+y)}{(1-x)(\overline{a_2}+\overline{y})-(1-\overline{x})(a_2+y)}\\
&= \frac{(\overline{x}-x)(a_2+y)+y(1-x)\overline{a_2}-(1-x)\overline{y}a_2}{(1-x)(\overline{a_2}+\overline{y})-(1-\overline{x})(a_2+y)}
\end{aligned}$$

and

$$\begin{aligned}
y-t_4(1+y) &= \frac{y(-(1+y)(\overline{a_2}-\overline{x})+(1+\overline{y})(a_2-x))-(1+y)((\overline{y}-\overline{x})(a_2-x)-(y-x)(\overline{a_2}-\overline{x}))}{-(1+y)(\overline{a_2}-\overline{x})+(1+\overline{y})(a_2-x)}\\
&= \frac{(-y+(y-x))(1+y)(\overline{a_2}-\overline{x})+(-(1+y)(\overline{y}-\overline{x})+y(1+\overline{y}))(a_2-x)}{-(1+y)(\overline{a_2}-\overline{x})+(1+\overline{y})(a_2-x)}\\
&= \frac{-x(1+y)(\overline{a_2}-\overline{x})+(y-\overline{y}+(1+y)\overline{x})(a_2-x)}{-(1+y)(\overline{a_2}-\overline{x})+(1+\overline{y})(a_2-x)}\\
&= \frac{(y-\overline{y})(a_2-x)-x(1+y)\overline{a_2}+(1+y)\overline{x}a_2}{-(1+y)(\overline{a_2}-\overline{x})+(1+\overline{y})(a_2-x)}
\end{aligned}$$

Note that the denominators of the expressions obtained for the points $y-t_4(1+y)$ and $x+t_1(1-x)$ are both skew with respect to the conjugation mapping and hence cancel in equation $(*)$. This equation now becomes

$$\begin{aligned}
&((\overline{x}-x)(a_2+y)+(x-1)(\overline{y}a_2-y\overline{a_2}))\,((\overline{y}-y)(\overline{a_2}-\overline{x})+(\overline{y}+1)(x\overline{a_2}-\overline{x}a_2))\\
&\quad = ((x-\overline{x})(\overline{a_2}+\overline{y})+(\overline{x}-1)(y\overline{a_2}-\overline{y}a_2))\,((y-\overline{y})(a_2-x)+(y+1)(\overline{x}a_2-x\overline{a_2}))
\end{aligned}$$

which leads to

$$\begin{aligned}
&(\overline{x}-x)(\overline{y}-y)(a_2+y)(\overline{a_2}-\overline{x})+(x-1)(\overline{y}+1)(\overline{y}a_2-y\overline{a_2})(x\overline{a_2}-\overline{x}a_2)\\
&\quad+(\overline{x}-x)(\overline{y}+1)(a_2+y)(x\overline{a_2}-\overline{x}a_2)+(x-1)(\overline{y}-y)(\overline{a_2}-\overline{x})(\overline{y}a_2-y\overline{a_2})\\
=&(x-\overline{x})(y-\overline{y})(\overline{a_2}+\overline{y})(a_2-x)+(\overline{x}-1)(y+1)(y\overline{a_2}-\overline{y}a_2)(\overline{x}a_2-x\overline{a_2})\\
&\quad+(x-\overline{x})(y+1)(\overline{a_2}+\overline{y})(\overline{x}a_2-x\overline{a_2})+(\overline{x}-1)(y-\overline{y})(a_2-x)(y\overline{a_2}-\overline{y}a_2).
\end{aligned}$$

Subtracting the right hand side from the left yields

$$
\begin{aligned}
0 &= (\overline{x}-x)(\overline{y}-y)(y\overline{a_2}-a_2\overline{x}-y\overline{x}-\overline{y}a_2+\overline{a_2}x+\overline{y}x)\\
&\quad+(\overline{y}a_2-y\overline{a_2})(x\overline{a_2}-\overline{x}a_2)(x\overline{y}+x-\overline{y}-\overline{x}y-\overline{x}+y)\\
&\quad+(\overline{x}-x)(x\overline{a_2}-\overline{x}a_2)(\overline{y}a_2+a_2+y-y\overline{a_2}-\overline{a_2}-\overline{y})\\
&\quad+(\overline{y}-y)(\overline{y}a_2-y\overline{a_2})(x\overline{a_2}-\overline{a_2}+\overline{x}-\overline{x}a_2+a_2-x)\\
&= (\overline{x}-x)(\overline{y}-y)(\overline{y}x-y\overline{x})+(\overline{x}-x)(\overline{y}-y)(\overline{a_2}x-a_2\overline{x})+(\overline{x}-x)(\overline{y}-y)(y\overline{a_2}-\overline{y}a_2)\\
&\quad+(\overline{y}a_2-y\overline{a_2})(x\overline{a_2}-\overline{x}a_2)(x\overline{y}-\overline{x}y)+(\overline{y}a_2-y\overline{a_2})(x\overline{a_2}-\overline{x}a_2)(x-\overline{x}+y-\overline{y})\\
&\quad+(\overline{x}-x)(x\overline{a_2}-\overline{x}a_2)(y-\overline{y})+(\overline{x}-x)(x\overline{a_2}-\overline{x}a_2)(\overline{y}a_2-y\overline{a_2})+(\overline{x}-x)(x\overline{a_2}-\overline{x}a_2)(a_2-\overline{a_2})\\
&\quad+(\overline{y}-y)(\overline{y}a_2-y\overline{a_2})(\overline{x}-x)+(\overline{y}-y)(\overline{y}a_2-y\overline{a_2})(x\overline{a_2}-\overline{x}a_2)+(\overline{y}-y)(\overline{y}a_2-y\overline{a_2})(a_2-\overline{a_2})\\
&= (\overline{x}-x)(\overline{y}-y)(\overline{y}x-y\overline{x})+(\overline{y}a_2-y\overline{a_2})(x\overline{a_2}-\overline{x}a_2)(x\overline{y}-\overline{x}y)\\
&\quad+(a_2-\overline{a_2})(\overline{x}-x)(x\overline{a_2}-\overline{x}a_2)+(a_2-\overline{a_2})(\overline{y}-y)(\overline{y}a_2-y\overline{a_2})
\end{aligned}
$$

Since $x=a_i+b_iw$ and $y=a_j+b_jw$ the last expression is a polynomial $P(w,\overline{w})$ of degree 3 in $w$ and $\overline{w}$. Since $w\overline{w}=1$ the term $P(w,\overline{w})w^3$ is a polynomial of degree 6 in $w$. This polynomial has at least the $q+1$ elements $w\in GF(q^2)$ with $w\overline{w}=1$ as zeroes, and so all coefficients are zero since $q\geq 9$. We have

$$
\begin{aligned}
\overline{x}-x &= -b_iw+\overline{a_i}-a_i+\overline{b_i}\overline{w},\\
\overline{y}-y &= -b_jw+\overline{a_j}-a_j+\overline{b_j}\overline{w},\\
\overline{y}x-y\overline{x} &= (\overline{a_j}+\overline{b_j}\overline{w})(a_i+b_iw)-(a_j+b_jw)(\overline{a_i}+\overline{b_i}\overline{w})\\
&= (\overline{a_j}b_i-b_j\overline{a_i})w+\overline{a_j}a_i+\overline{b_j}b_i-a_j\overline{a_i}-b_j\overline{b_i}+(\overline{b_j}a_i-a_j\overline{b_i})\overline{w},\\
x\overline{a_2}-\overline{x}a_2 &= (a_i+b_iw)\overline{a_2}-(\overline{a_i}+\overline{b_i}\overline{w})a_2\\
&= b_i\overline{a_2}w+a_i\overline{a_2}-\overline{a_i}a_2-\overline{b_i}a_2\overline{w},\\
\overline{y}a_2-y\overline{a_2} &= (\overline{a_j}+\overline{b_j}\overline{w})a_2-(a_j+b_jw)\overline{a_2}\\
&= -b_j\overline{a_2}w+\overline{a_j}a_2-a_j\overline{a_2}+\overline{b_j}a_2\overline{w}.
\end{aligned}
$$

For the coefficient of $w^3$ in $P(w,\overline{w})$ we get

$$(-b_i)(-b_j)(\overline{a_j}b_i-b_j\overline{a_i})+(-b_j\overline{a_2})(b_i\overline{a_2})(\overline{a_j}b_i-b_j\overline{a_i})=b_ib_j(\overline{a_j}b_i-b_j\overline{a_i})(1-\overline{a_2}^2).$$

If $b_i=0$ for at least one $i\in\{\pm3,\ldots,\pm\frac{q+1}{2}\}$ the theorem is proved. So we may assume that $b_i\neq 0$ for all $i\in\{\pm3,\ldots,\pm\frac{q+1}{2}\}$. Since $\overline{a_2}^2\neq 1$ this yields

$$\overline{a_j}b_i-b_j\overline{a_i}=0 \quad \text{for all} \quad i,j\in\{\pm3,\ldots,\pm\frac{q+1}{2}\}, i\neq\pm j$$

It follows that $b_i=c\overline{a_i}$ for all $i\in\{\pm3,\ldots,\pm\frac{q+1}{2}\}$ for some constant $c$. For the conics $\Omega_w^*$ this implies

$$\Omega_w^*=\{a_i+c\overline{a_i}w | i\in\{\pm3,\ldots,\pm\frac{q+1}{2}\}\}\cup\{\pm1,\pm a_2\} \quad \text{for all} \quad w\in GF(q^2), w\overline{w}=1.$$

Consider now the $GF(q)$-linear mapping $\lambda_w : GF(q^2) \to GF(q^2)$ which maps $z \in GF(q^2)$ to $z + cw\overline{z}$. The inverse of $\lambda_w$ is given by $\lambda_w^{-1}(z) = \frac{1}{1-c\overline{c}}(z - cw\overline{z})$. For each $w \in GF(q^2), w\overline{w} = 1$ the set

$$\lambda_w^{-1}(\Omega_w^*) = \{a_i | i \in \{\pm 3, \ldots, \pm\frac{q+1}{2}\}\} \cup \{\pm\frac{1-cw}{1-c\overline{c}}, \pm\frac{a_2 - cw\overline{a_2}}{1-c\overline{c}}\}$$

is a conic in the affine plane $GF(q^2)$. Since $q \geq 9$ these conics are all the same and hence $1 - cw$ and $a_2 - cw\overline{a_2}$ are independent of $w$. It follows that $c = 0$ and hence $b_i = 0$ for all $i \in \{\pm 3, \ldots, \pm\frac{q+1}{2}\}$. □

**Theorem 3.6.** *Let $q$ be odd. Let $b \in GF(q^2)$ with $b\overline{b} \neq 1$ and set $\mathcal{L}(b) = \{L(w + b\overline{w}, 0) | w \in GF(q^2), w\overline{w} = 1\}$. Then $\mathcal{T}(b) = \{L, \mathcal{L}(b)\}$ is an oval tube of $PG(V)$. Every oval tube of $PG(3,q)$ is projectively equivalent to some $\mathcal{T}(b)$. The tubes $\mathcal{T}(b_1)$ and $\mathcal{T}(b_2)$ are projectively equivalent if and only $b_2 = b_1c^2, b_2 = b_1^{-1}c^2, b_2 = \overline{b}_1c^2$ or $b_2 = \overline{b}_1^{-1}c^2$ for some $c \in GF(q^2)$ with $c\overline{c} = 1$. If $q \equiv 3 \mod 4$ there are precisely $\frac{3q-1}{4}$ equivalence classes of oval tubes and if $q \equiv 1 \mod 4$ there are $\frac{3q-3}{4}$ equivalence classes. $\mathcal{T}(b)$ is a quadric tube if and only if $b = 0$.*

*Proof.* By Theorem 3.5 and Proposition 1.1 we know that every oval tube is projectively equivalent to an oval tube $\mathcal{T} = \{L, \mathcal{L}\}$ which has axis $L = \{0\} \times GF(q^2)$ and is contained in the regular spread $\mathcal{B} = \{L\} \cup \{L(m, 0) | m \in GF(q^2)\}$. We also know that the set $\Omega = \{m \in GF(q^2) | L(m, 0) \in \mathcal{L}\}$ is a conic in the affine plane $GF(q^2)$, and we may assume that this conic is centered at the origin. It follows that there are $a, b \in GF(q^2)$ with $a\overline{a} \neq b\overline{b}$ such that $\Omega = \Omega(a, b) = \{az + b\overline{z} | z \in GF(q^2), z\overline{z} = 1\}$. Let the corresponding tube be called $\mathcal{T}(a, b)$. Note that $a$ and $b$ are not uniquely determined by $\Omega$, but that $\Omega(a, b) = \Omega(ac, b\overline{c}) = \Omega(b\overline{c}, ac)$, and hence also $\mathcal{T}(a, b) = \mathcal{T}(ac, b\overline{c}) = \mathcal{T}(b\overline{c}, ac)$, for all $c \in GF(q^2), c\overline{c} = 1$.

The tube $\mathcal{T}(a, b)$ is quadric if and only if $\Omega(a, b)$ is a circle and this happens precisely if $a = 0$ or $b = 0$.

From now on we assume that $\mathcal{T} = \mathcal{T}(a, b)$ is not the quadric tube, i.e that $a \neq 0 \neq b$. Then $\mathcal{B}$ is the only regular spread which contains $\mathcal{T}$. It follows that two such tubes are projectively equivalent if and only if one is mapped onto the other by a $GF(q)$-linear mapping of the vector space $GF(q^2)^2$ which fixes the spread $\mathcal{B}$ and the lines $L$ and $L(0, 0)$. These linear mappings are of the form $A(d_1, d_2) : (z, w) \mapsto (d_1z, d_2w)$ or $B(d_1, d_2) : (z, w) \mapsto (d_1\overline{z}, d_2\overline{w}), d_1, d_2 \in GF(q^2), d_1 \neq 0 \neq d_2$. A short calculation shows that $A(d_1, d_2)$ and $B(d_1, d_2)$ map $\mathcal{T}(a, b)$ to $\mathcal{T}(da, db)$ and $\mathcal{T}(d\overline{b}, d\overline{a})$, respectively, where $d = \frac{d_2}{d_1}$. It follows that $\mathcal{T}(a, b)$ is projectively equivalent to $\mathcal{T}(1, \frac{b}{a}) = \mathcal{T}(\frac{b}{a})$.

Assume now that $\mathcal{T}(b_1)$ and $\mathcal{T}(b_2), b_1 \neq 0 \neq b_2$, are projectively equivalent. Then we get that $(1, b_2)$ is equal to one of $(dc, db_1\overline{c}), (db_1\overline{c}, dc), (d\overline{b}_1c, d\overline{c})$ or $(d\overline{c}, d\overline{b}_1c)$ for some $c, d \in GF(q^2), c\overline{c} = 1, d \neq 0$. These four cases lead to $b_2 = b_1\overline{c}^2, b_2 = b_1^{-1}c^2, b_2 = \overline{b}_1^{-1}\overline{c}^2$ or $b_2 = \overline{b}_1c^2$, respectively.

It remains to determine the number of isomorphism classes.

As proved in the previous paragraph, this is essentially the problem of counting the orbits of a group of order $2(q+1)$ acting on the set $M = \{b \in GF(q^2) | 0 \neq b\overline{b} \neq 1\}$, which contains $(q+1)(q-2)$ elements. The group consists of the following mappings

(i) $b \mapsto bc^2, c\overline{c} = 1,$

(ii) $b \mapsto b^{-1}c^2, c\overline{c} = 1,$

(iii) $b \mapsto \overline{b}^{-1}c^2, c\overline{c} = 1,$

(iv) $b \mapsto \overline{b}c^2, c\overline{c} = 1.$

Since $b\overline{b} \neq 1$ the mappings of type (ii) have no fixed points, and so the dihedral group of order $q+1$ which comprises the mappings of type (i) and (ii) acts freely on $M$. So the stabilizer of any point contains at most two elements.

Assume that a mapping of type (iii) fixes $b \in M$, then we get $b\overline{b} = c^2$. Since $1 \neq b\overline{b} \in GF(q)$ and $c\overline{c} = 1$ this yields $b\overline{b} = -1 = c^2$. The equation $c^2 = -1$ has a solution with $c\overline{c} = 1$ if and only if $4|q+1$. In this case there are precisely $q+1$ elements of $M$ which have a mapping of type (iii) in their stabilizer.

Assume now that a mapping of type (iv) fixes a point $b \in M$, then we get $\frac{b}{\overline{b}} = c^2$ with $c\overline{c} = 1$. There are precisely $\frac{(q+1)(q-1)}{2}$ elements $b \in GF(q^2) \setminus \{0\}$ for which this equation has a solution, but those with $b\overline{b} = 1$ are among them. So there are $\frac{(q+1)(q-3)}{2}$ elements of $M$ which have a mapping of type (iv) in their stabilizer.

Now we can count as follows. If $q \equiv 3 \bmod 4$ there are $q+1+\frac{(q+1)(q-3)}{2} = \frac{(q+1)(q-1)}{2}$ elements with a stabilizer of order 2 and hence $(q+1)(q-2) - \frac{(q+1)(q-1)}{2} = \frac{(q+1)(q-3)}{2}$ elements with trivial stabilizer. The number of orbits thus becomes

$$\frac{\frac{(q+1)(q-1)}{2}}{(q+1)} + \frac{\frac{(q+1)(q-3)}{2}}{2(q+1)} = \frac{3q-5}{4}.$$

Taking into account the quadric tube we have to add one to this number and so we have $\frac{3q-1}{4}$ equivalence classes of oval tubes.

If $q \equiv 1 \bmod 4$ a similar computation shows that there are $\frac{3q-3}{4}$ equivalence classes. $\square$

**Remarks 3.7.** a) If $q \equiv 1 \bmod 4$ it is possible to construct a system of representatives for the projective equivalence classes of oval tubes as follows. Choose an element $\mu \in GF(q^2) \setminus GF(q)$ with $\mu^2 \in GF(q)$. Then every oval tube is equivalent to some $\mathcal{T}(b)$ with $b \in GF(q) \setminus \{\pm 1\} \cup \mu \cdot GF(q)$. If $b \in GF(q) \setminus \{0, \pm 1\}$ there are two representatives, namely $b$ and $b^{-1}$, and if $b \in \mu \cdot GF(q) \setminus \{0\}$ there are four, namely $b, -b = \overline{b}, b^{-1}$ and $-b^{-1} = \overline{b}^{-1}$.

If $q \equiv 3 \bmod 4$ there seems to be no obvious choice for a system of representatives.

b) Our classification is in fact equivalent to the classification of the conics in the affine plane under the group of similarities.

c) In principle, it is also possible to classify tubes up to equivalence under $P\Gamma L(4, q)$. The field automorphisms just act in their standard way on $M$.

**Lemma 3.8.** *Let $\Omega$ be a subset of the Miquelian inverse plane $I(q)$, $q$ odd, and assume that there are two distinct points $a, b$ of $I(q) \setminus \Omega$ such that $\Omega$ is a conic in the affine planes $I(q)_a$ and $I(q)_b$. Then $\Omega$ is a circle of $I(q)$.*

*Proof.* We may identify $I(q)_a$ with the affine plane $GF(q)^2$ and we may assume that $b = (0,0)$. The conics in the affine plane which do not pass through $b$ and have size $q+1$ are the sets $\{(x,y) \in GF(q)^2 | a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + b_1x + b_2y + c = 0\}$ for $a_{ij}, b_i, c \in GF(q), c \neq 0, a_{12}^2 - a_{11}a_{22}$ a nonsquare in $GF(q)$.

We may also fix a nonsquare $\varrho$ in $GF(q)$ such that $I(q)$ is the inversive plane associated with the field extension $GF(q)[x]/(x^2 - \varrho) : GF(q)$. Then the circles are precisely the conics with $a_{12} = 0$ and $a_{22} = -\varrho a_{11}$. The mapping $\sigma : I(q) \to I(q)$ which exchanges $a$ and $b$ and maps $(x,y) \neq (0,0)$ to $(\frac{x}{x^2 - \varrho y^2}, \frac{y}{x^2 - \varrho y^2})$ is an involutorial automorphism of $I(q)$.

It is sufficient to show that if $\Omega$ and $\sigma(\Omega)$ are both conics in $GF(q)^2 \setminus \{(0,0)\}$, then $\Omega$ is a circle.

Assume that

$$\Omega = \{(x,y) \in GF(q)^2 | a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + b_1x + b_2y + c = 0\},$$

then we get

$$\sigma(\Omega) = \{(x,y) \in GF(q)^2 | a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + (b_1x + b_2y)(x^2 - \varrho y^2) + c(x^2 - \varrho y^2)^2 = 0\}.$$

This set is a conic if and only if the polynomial

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + (b_1x + b_2y + c(x^2 - \varrho y^2))(x^2 - \varrho y^2)$$

is a product of two polynomials of degree 2. Since this polynomial contains no terms of degree 0 and 1, one of the factors must be the polynomial $a_{11}x^2 + 2a_{12}xy + a_{22}y^2$. It follows that $a_{11}x^2 + 2a_{12}xy + a_{22}y^2$ divides $x^2 - \varrho y^2$ and hence $a_{12} = 0$ and $a_{22} = -\varrho a_{11}$, i.e. $\Omega$ is a circle. $\square$

**Proposition 3.9.** *Let $\mathcal{T} = \{L, \mathcal{L}\}$ be an oval tube of odd order and assume that there exists a line $L' \neq L$ of $PG(3,q)$ such that $\{L', \mathcal{L}\}$ is also an oval tube, then $\mathcal{T}$ is isomorphic to the quadric tube.*

*Proof.* If $\mathcal{T}$ is not the quadric tube then the regular spread containing $\mathcal{L}$ is uniquely determined. It follows that $L'$ is also contained in this spread. Since the lines of this regular spread and the reguli contained in it form a model for the inversive plane $I(q)$ the result now follows from the preceding lemma. $\square$

This proposition yields the following characterization of reguli in $PG(3,q)$.

**Corollary 3.10.** *Let $\mathcal{L}$ be a collection of $q+1$ mutually skew lines in $PG(3,q), q$ odd. Assume that there are two distinct lines $L_1, L_2 \notin \mathcal{L}$ skew with all lines of $\mathcal{L}$ such that each line which intersects $L_1$ or $L_2$ meets at most two distinct lines of $\mathcal{L}$. Then $\mathcal{L}$ is a regulus.*

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