Defining amalgams of compact Lie groups

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Abstract

For $n \geq 2$ let Δ be a Dynkin diagram of rank n and let $I = \{1, \ldots, n\}$ be the set of labels of Δ. A group G admits a *weak Phan system of type* Δ *over* $\mathbb{C}$ if G is generated by subgroups U_i, $i \in I$, which are central quotients of simply connected compact semisimple Lie groups of rank one, and contains subgroups $U_{i,j} = \langle U_i, U_j \rangle$, $i \neq j \in I$, which are central quotients of simply connected compact semisimple Lie groups of rank two such that U_i and U_j are rank one subgroups of $U_{i,j}$ corresponding to a choice of a maximal torus and a fundamental system of roots for $U_{i,j}$. It is shown in this article that G then is a central quotient of the simply connected compact semisimple Lie group whose complexification is the simply connected complex semisimple Lie group of type Δ.

1 Introduction

In 1977 Kok-Wee Phan [27] gave a method for identifying a group G as a quotient of the finite unitary group $\mathrm{SU}_{n+1}(q^2)$ by finding a generating configuration of subgroups

$$\mathrm{SU}_3(q^2) \quad \text{and} \quad \mathrm{SU}_2(q^2) \times \mathrm{SU}_2(q^2)$$

in G. We begin by looking at the configuration of subgroups in $\mathrm{SU}_{n+1}(q^2)$ to motivate our later definition. Suppose $n \geq 2$ and suppose q is a prime power. Consider $G = \mathrm{SU}_{n+1}(q^2)$ acting as matrices on a Hermitian $(n+1)$-dimensional vector space over $\mathbb{F}_{q^2}$ with respect to an orthonormal basis and let $U_i \cong \mathrm{SU}_2(q^2)$, $i = 1, 2, \ldots, n$, be the subgroups of G, represented as matrix groups with respect to the chosen orthonormal basis, corresponding to the (2×2)-blocks along the main diagonal. Let T_i be the diagonal subgroup in U_i, which is a maximal torus of U_i of size $q+1$. When $q \neq 2$ the following hold for $1 \leq i, j \leq n$:

(P1) if $|i-j| > 1$, then $[x, y] = 1$ for all $x \in U_i$ and $y \in U_j$;

(P2) if $|i-j| = 1$, then $\langle U_i, U_j \rangle$ is isomorphic to $\mathrm{SU}_3(q^2)$; moreover $[x, y] = 1$ for all $x \in T_i$ and $y \in T_j$; and

(P3) the subgroups U_i, $1 \leq i \leq n$, generate G.

Suppose now G is an arbitrary group containing a system of subgroups $U_i \cong \mathrm{SU}_2(q^2)$, and suppose a maximal torus T_i of size $q+1$ is chosen in each U_i. If the conditions (P1)–(P3) above hold for G, we will say that G contains a *Phan system of type* A_n *over* $\mathbb{F}_{q^2}$. Aschbacher called this configuration a generating system of type I in [1].

In [27] Kok-Wee Phan proved the following result:

Phan's Theorem:
Let $q \geq 5$ and let $n \geq 3$. If G contains a Phan system of type A_n over $\mathbb{F}_{q^2}$, then G is isomorphic to a central quotient of $\mathrm{SU}_{n+1}(q^2)$.

In [28] Phan proved similar results for finite groups corresponding to all simply laced Dynkin diagrams. For the second-generation proof of the classification of the finite simple groups [11], [12], [13], [14], [15] the question was raised whether one could generalize and unify Phan's results. After a number of partially successful attempts by several people of reproving Phan's theorems (see, e.g., [9]), the program described in [2] led to new proofs of some of Phan's old results, see [3], [19], and to new unexpected Phan-type theorems, see [16], [17].

The purpose of the present article is to apply the methods from the program [2], which have originally been developed for finite groups, to compact Lie groups, yielding a generalization of a result by Borovoi [4] on generators and relations in compact Lie groups. The methods and ideas used in this paper have been adopted from [3], [17], [18].

To be able to properly state the result, we have to fix the setting and to define some notions. Let G be a simply connected compact semisimple Lie group of rank two, i.e., G is isomorphic to $\mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C})$ or $\mathrm{SU}_3(\mathbb{C})$ or $\mathrm{Spin}_5(\mathbb{R}) \cong \mathrm{U}_2(\mathbb{H})$ or $G_{2,-14}$ by [21], see also 94.33 of [31]. Let T be a maximal torus of G, let $\Sigma = \Sigma(G_{\mathbb{C}}, T_{\mathbb{C}})$ be its root system, and let $\{\alpha, \beta\}$ be a fundamental system of roots of Σ, cf. [5] or [24]. To the simple roots α, β corresponds a pair of semisimple subgroups G_α and G_β of G normalized by T and isomorphic to $\mathrm{SU}_2(\mathbb{C}) \cong \mathrm{Spin}_3(\mathbb{R}) \cong \mathrm{U}_1(\mathbb{H})$, which is called a *standard pair* of G. If α and β have different length, then the standard pair (G_α, G_β) is not conjugate to the standard pair (G_β, G_α), so, by convention, we assume that in a standard pair (G_α, G_β) the root α is shorter than the root β if they have different lengths. A standard pair in a central quotient of G is defined as the image of a standard pair of G under the natural homomorphism. Note that the images of a standard pair in the quotient have the same isomorphism types as in G modulo some central subgroups.

Moreover, for $n \geq 2$ let Δ be a Dynkin diagram of rank n (see [6] for a complete list) and let $I = \{1, \ldots, n\}$ be the set of labels of Δ. A group G admits a *weak Phan system of type Δ over* $\mathbb{C}$ if G is generated by subgroups U_i, $i \in I$, which are central quotients of simply connected compact semisimple Lie groups of rank one, and contains subgroups $U_{i,j} = \langle U_i, U_j \rangle$, $i \neq j \in I$, which are central quotients of simply connected compact semisimple Lie groups of rank two such that (U_i, U_j) or (U_j, U_i) forms a standard pair in $U_{i,j}$. In particular the groups U_i and $U_{i,j}$ have the following isomorphism types:

(1) $U_i \cong \mathrm{SU}_2(\mathbb{C})$ or $U_i \cong \mathrm{SO}_3(\mathbb{R})$ for all $1 \leq i \leq n$;

(2) $$\langle U_i, U_j \rangle \cong \begin{cases} (U_i \times U_j)/Z, & \text{in case } \underset{i}{\circ} \qquad \underset{j}{\circ}, \text{ where } Z \text{ is a central subgroup of } U_i \times U_j, \\ \mathrm{SU}_3(\mathbb{C}) \text{ or } \mathrm{PSU}_3(\mathbb{C}), & \text{in case } \underset{i}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{j}{\circ}, \\ \mathrm{U}_2(\mathbb{H}) \text{ or } \mathrm{SO}_5(\mathbb{R}), & \text{in case } \underset{i}{\circ}\!\overset{<}{=\!=\!=}\!\underset{j}{\circ} \text{ or } \underset{i}{\circ}\!\overset{>}{=\!=\!=}\!\underset{j}{\circ}, \\ G_{2,-14}, & \text{in case } \underset{i}{\circ}\!\overset{<}{\equiv\!\equiv\!\equiv}\!\underset{j}{\circ} \text{ or } \underset{i}{\circ}\!\overset{>}{\equiv\!\equiv\!\equiv}\!\underset{j}{\circ}. \end{cases}$$

Main Theorem.
Let Δ be a Dynkin diagram and let G be a group admitting a weak Phan system of type Δ over $\mathbb{C}$. Then G is a central quotient of the simply connected compact semisimple Lie group whose complexification is the simply connected complex semisimple Lie group of type Δ. In particular, for irreducible Dynkin diagrams, the group G is a central quotient of

- $\mathrm{SU}_{n+1}(\mathbb{C})$, *if* $\Delta = A_n$,
- $\mathrm{Spin}_{2n+1}(\mathbb{R})$, *if* $\Delta = B_n$,
- $\mathrm{U}_n(\mathbb{H})$, *if* $\Delta = C_n$,
- $\mathrm{Spin}_{2n}(\mathbb{R})$, *if* $\Delta = D_n$,
- $E_{6,-78}$, *if* $\Delta = E_6$,
- $E_{7,-133}$, *if* $\Delta = E_7$,
- $E_{8,-248}$, *if* $\Delta = E_8$,
- $F_{4,-52}$, *if* $\Delta = F_4$.

While the theorem is true for all Dynkin diagrams, it is a tautology for Dynkin diagrams of rank at most two. In particular, the theorem does not yield an interesting characterization of the group $G_{2,-14}$.

This article is organized as follows. In Section 2 we remind the reader of the definition of a geometry and an amalgam and state some important lemmas. In Section 3 we recall the result by Borovoi [4] and give an alternative proof using geometric covering theory. In Section 4 we study Phan systems and Phan amalgams, indicate how to pass from one concept to the other and, moreover, prove a result on uniqueness of covers of Phan amalgams. In Section 5, finally, we classify the unique covers of Phan amalgams from Section 4 and prove the Main Theorem. In Section 6 we point out a question for further research.

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2 Geometries, amalgams and some lemmas

In this section we collect relevant definitions and results from incidence geometry and the theory of amalgams. See [20] for a short introduction to the topic. A thorough introduction to incidence geometry can be found in [8].

Geometries

Definition 2.1 A **pregeometry** $\mathcal{G}$ **over the set** I is a triple $(X, *, \mathrm{typ})$ consisting of a set X, a symmetric and reflexive **incidence relation** $*$, and a surjective **type function** $\mathrm{typ} : X \to I$, subject to the following condition:

(Pre) If $x * y$ with $\mathrm{typ}(x) = \mathrm{typ}(y)$, then $x = y$.

The set I is usually called the **type set**. A **flag** in X is a set of pairwise incident elements. The **type** of a flag F is the set $\mathrm{typ}(F) := \{\mathrm{typ}(x) : x \in F\}$. A **chamber** is a flag of type I. The **rank** of a flag F is $|\mathrm{typ}(F)|$ and the **corank** is equal to $|I \setminus \mathrm{typ}(F)|$. The cardinality of I is called the **rank** of $\mathcal{G}$. The pregeometry $\mathcal{G}$ is **connected** if the graph $(X, *)$ is connected.

A **geometry** is a pregeometry with the additional property that

(Geo) every flag is contained in a chamber.

Let $\mathcal{G} = (X, *, \mathrm{typ})$ be a pregeometry over I. An **automorphism** of $\mathcal{G}$ is a permutation σ of X with $\mathrm{typ}(\sigma(x)) = \mathrm{typ}(x)$, for all $x \in X$, and with $\sigma(x) * \sigma(y)$ if and only if $x * y$, for all $x, y \in X$. A group G of automorphisms of $\mathcal{G}$ is called **flag-transitive** if for each pair F, F' of flags of $\mathcal{G}$ with $\mathrm{typ}(F) = \mathrm{typ}(F')$ there exists a $g \in G$ with $g(F) = F'$. A group G of automorphisms of $\mathcal{G}$ is called **chamber-transitive** if for each pair F, F' of flags of $\mathcal{G}$ with $\mathrm{typ}(F) = I = \mathrm{typ}(F')$ there exists a $g \in G$ with $g(F) = F'$. Flag-transitivity implies chamber-transitivity, for a geometry flag-transitivity and chamber-transitivity coincide, and a flag-transitive pregeometry containing a chamber automatically is a geometry, cf. [8].

Let F be a flag of $\mathcal{G}$, say of type $J \subseteq I$. Then the **residue $\mathcal{G}_F$ of** F is the pregeometry

$$(X', *_{|X' \times X'}, \mathrm{typ}_{|I \setminus J})$$

over $I \backslash J$, with

$$X' := \{x \in X : F \cup \{x\} \text{ is a flag of } \mathcal{G} \text{ and } \mathrm{typ}(x) \notin \mathrm{typ}(F)\}.$$

Definition 2.2 Let $\mathcal{G}$ and $\widehat{\mathcal{G}}$ be connected geometries over the same type set and let $\phi : \widehat{\mathcal{G}} \to \mathcal{G}$ be a **homomorphism** of geometries, i.e., ϕ preserves the types and sends incident elements to incident elements. A surjective homomorphism ϕ between connected geometries $\widehat{\mathcal{G}}$ and $\mathcal{G}$ is called a **covering** if and only if for every nonempty flag $\widehat{F}$ in $\widehat{\mathcal{G}}$ the map ϕ induces an isomorphism between the residue of $\widehat{F}$ in $\widehat{\mathcal{G}}$ and the residue of $F = \phi(\widehat{F})$ in $\mathcal{G}$. Coverings of a geometry correspond to the usual topological coverings of the flag complex. If ϕ is an isomorphism, then the covering is said to be **trivial**. A connected geometry $\mathcal{G}$ is called **simply connected** if any covering $\widehat{\mathcal{G}} \to \mathcal{G}$ of that geometry is trivial.

Definition 2.3 Let I be a set, let G be a group and let $(G_i)_{i \in I}$ be a family of subgroups of G. Then $(\sqcup_{i \in I} G/G_i, *, \mathrm{typ})$ with $\mathrm{typ}(G_i) = i$ and

(Cos) $gG_i * hG_j$ if and only if $gG_i \cap hG_j \neq \emptyset$

is a pregeometry over I, the **coset pregeometry of** G with respect to $(G_i)_{i \in I}$. Since the type function is completely determined by the indices, we also denote the coset pregeometry of G with respect to $(G_i)_{i \in I}$ by

$$((G/G_i)_{i \in I}, *).$$

The family $(G_i)_{i \in I}$ forms a chamber. A coset pregeometry that is a geometry is called a **coset geometry**.

Definition 2.4 A **building geometry** is a coset geometry $((G/G_i)_{i \in I}, *)$ where G is a Chevalley group, I is the set of labels of the corresponding Dynkin diagram and $(G_i)_{i \in I}$ is the collection of the maximal parabolic subgroups of G, cf. [36] or [37]. The concept of building geometries is equivalent to the concept of Tits buildings, see [7] or [8].

By Theorem IV.5.2 of [7] or by Theorem 13.32 of [37], a building geometry of rank at least three is simply connected. In the present paper, we are interested in building geometries coming from simply connected complex semisimple Lie groups. For example, the building geometry of the group $\mathrm{SL}_{n+1}(\mathbb{C})$ is isomorphic to the complex projective geometry $\mathbb{P}(\mathbb{C}^{n+1})$. The building geometries of the groups $\mathrm{Spin}_{2n+1}(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$, $\mathrm{Spin}_{2n}(\mathbb{C})$ are isomorphic to the respective polar geometries, i.e., the incidence geometries of the totally isotropic subspaces of nondegenerate symmetric bilinear, respectively alternating bilinear forms of Witt index n over $\mathbb{C}$.

Amalgams

Definition 2.5 An **amalgam** $\mathcal{A}$ of groups is a set with a partial operation of multiplication and a collection of subsets $(H_i)_{i\in I}$, for some index set I, such that the following conditions hold:

(1) $\mathcal{A} = \bigcup_{i\in I} H_i$;

(2) the product ab is defined if and only if $a, b \in H_i$ for some $i \in I$;

(3) the restriction of the multiplication to each H_i turns H_i into a group; and

(4) $H_i \cap H_j$ is a subgroup in both H_i and H_j for all $i, j \in I$.

It follows that the groups H_i share the same identity element, which is then the only identity element in $\mathcal{A}$, and that $a^{-1} \in \mathcal{A}$ is well-defined for every $a \in \mathcal{A}$. Notice that the above definition of an amalgam of groups fits well into the general concept of an amalgam of groups, see [35].

An amalgam $\mathcal{B} = \bigcup_{i\in I} H_i$ is a **quotient** of the amalgam $\mathcal{A} = \bigcup_{i\in I} G_i$ if there is a homomorphism π from $\mathcal{A}$ to $\mathcal{B}$ such that the restriction of π to every G_i maps G_i onto H_i. The amalgam $\mathcal{A}$ together with the homomorphism π is called a **cover** of the amalgam $\mathcal{B}$. Two covers $(\mathcal{A}_1, \pi_1)$ and $(\mathcal{A}_2, \pi_2)$ of $\mathcal{A}$ are called **equivalent** if there is an isomorphism ϕ of $\mathcal{A}_1$ onto $\mathcal{A}_2$, such that $\pi_1 = \pi_2 \circ \phi$.

Definition 2.6 A group H is called a **completion** of an amalgam $\mathcal{A}$ if there exists a map $\pi : \mathcal{A} \to H$ such that

(1) for all $i \in I$ the restriction of π to H_i is a homomorphism of H_i to H; and

(2) $\pi(\mathcal{A})$ generates H.

Among all completions of $\mathcal{A}$ there is a largest one which can be defined as the group having the following presentation:

$$\mathcal{U}(\mathcal{A}) = \langle t_h \mid h \in \mathcal{A},\ t_x t_y = t_z,\ \text{whenever } xy = z \text{ in } \mathcal{A}\rangle.$$

Obviously, $\mathcal{U}(\mathcal{A})$ is a completion of $\mathcal{A}$ since one can take π to be the mapping $h \mapsto t_h$. Every completion of $\mathcal{A}$ is isomorphic to a quotient of $\mathcal{U}(\mathcal{A})$, and because of that $\mathcal{U}(\mathcal{A})$ is called the **universal completion**. An amalgam $\mathcal{A}$ **collapses** if $\mathcal{U}(\mathcal{A}) = 1$.

Example 2.7 Consider the groups

$$\begin{aligned} G_1 &= \langle y, z \mid y^{-1}zy = z^2\rangle, \\ G_2 &= \langle z, x \mid z^{-1}xz = x^2\rangle, \\ G_3 &= \langle x, y \mid x^{-1}yx = y^2\rangle, \end{aligned}$$

which are nontrivial and pairwise isomorphic. Let $\mathcal{A}$ be the amalgam given by G_1, G_2, G_3 and the intersections

$$\begin{aligned} G_1 \cap G_2 &= \langle z \rangle \cong \mathbb{Z}, \\ G_1 \cap G_3 &= \langle y \rangle \cong \mathbb{Z}, \\ G_2 \cap G_3 &= \langle x \rangle \cong \mathbb{Z}. \end{aligned}$$

Then $\mathcal{U}(\mathcal{A}) = 1$ by Exercises 2.2.7 and 2.2.10 of [29], so $\mathcal{A}$ collapses.

$$\begin{tikzcd} & \langle z \rangle \arrow[r] \arrow[dr] & G_1 \\ 1 \arrow[ur] \arrow[r] \arrow[dr] & \langle y \rangle \arrow[ur] \arrow[dr] & G_2 \\ & \langle x \rangle \arrow[r] \arrow[ur] & G_3 \end{tikzcd}$$

Some lemmas

Lemma 2.8 (Tits' Lemma) *Let $\mathcal{G}$ be a connected geometry over I of rank at least three, let G be a flag-transitive group of automorphisms of $\mathcal{G}$, and let F be a maximal flag of $\mathcal{G}$. Let $\mathcal{A}(\mathcal{G}, G, F)$ be the amalgam of stabilizers in G of the elements of F. The geometry $\mathcal{G}$ is simply connected if and only if the canonical epimorphism $\mathcal{U}(\mathcal{A}(\mathcal{G}, G, F)) \to G$ is an isomorphism.*

Proof. See Corollary 1.4.6 of [20] or Corollary 1 of [38]. □

Definition 2.9 Let $\mathcal{A} = P_1 \cup P_2$ and $\mathcal{A}' = P_1' \cup P_2'$ be amalgams of rank two. The amalgams $\mathcal{A}$ and $\mathcal{A}'$ are of the same **type** if there exist isomorphisms $\phi_i : P_i \to P_i'$ such that $\phi_i(P_1 \cap P_2) = P_1' \cap P_2'$ for $i = 1, 2$.

Lemma 2.10 (Goldschmidt's Lemma) *Let $\mathcal{A} = (P_1, P_2, P_1 \cap P_2)$ be an amalgam of rank two, let $A_i = \mathrm{Stab}_{\mathrm{Aut}\,(P_i)}(P_1 \cap P_2)$ for $i = 1, 2$, and let $\alpha_i : A_i \to \mathrm{Aut}\,(P_1 \cap P_2)$ be homomorphisms mapping $a \in A_i$ onto its restriction to $P_1 \cap P_2$. Then there is a one-to-one correspondence between isomorphism classes of amalgams of the same type as $\mathcal{A}$ and $\alpha_2(A_2)$-$\alpha_1(A_1)$ double cosets in $\mathrm{Aut}\,(P_1 \cap P_2)$. In other words, there is a one-to-one correspondence between the different isomorphism types of amalgams $P_1 \hookleftarrow (P_1 \cap P_2) \hookrightarrow P_2$ and the double cosets $\alpha_2(A_2) \backslash \mathrm{Aut}\,(P_1 \cap P_2) / \alpha_2(A_1)$.*

Proof. See Lemma 2.7 of [10] or Proposition 8.3.2 of [20]. □

Definition 2.11 Let $\mathcal{A} = (H_i)_{i \in I}$ be an amalgam. A completion G of $\mathcal{A}$ is called **characteristic** if and only if every automorphism of $\mathcal{A}$ extends to an automorphism of G.

Notice that, since G is generated by the image of $\mathcal{A}$ under the corresponding completion map, this extension of an automorphism is unique. Clearly, the universal completion is always characteristic as is the trivial completion.

Lemma 2.12 (Bennett-Shpectorov Lemma) *For $i = 1, 2$, let $\mathcal{A}_i$ be an amalgam and let G_i be a completion of $\mathcal{A}_i$ with completion map π_i. Suppose there exist isomorphisms $\psi : \mathcal{A}_1 \to \mathcal{A}_2$ and $\phi : G_1 \to G_2$ such that $\phi \circ \pi_1 = \pi_2 \circ \psi$. If G_1 is a characteristic completion of $\mathcal{A}_1$, then for any isomorphism $\psi' : \mathcal{A}_1 \to \mathcal{A}_2$ there exists a unique isomorphism $\phi' : G_1 \to G_2$ such that $\phi' \circ \pi_1 = \pi_2 \circ \psi'$.*

$$\begin{tikzcd} \mathcal{A}_1 \arrow[r, "\pi_1"] \arrow[d, bend right, "\psi"'] \arrow[d, bend left, "\psi'"] & G_1 \arrow[d, bend right, "\phi"'] \arrow[d, bend left, dotted, "\phi'"] \\ \mathcal{A}_2 \arrow[r, "\pi_2"] & G_2 \end{tikzcd}$$

Proof. See Lemma 6.4 of [3]. □

3 Generators and relations

Let us recall here the results by Borovoi [4]. Let G be a simply connected compact semisimple Lie group, let T be a maximal torus of G, let $\Sigma = \Sigma(G_\mathbb{C}, T_\mathbb{C})$ be its root system, and let Π be a system of fundamental roots of Σ. To each root $\alpha \in \Pi$ corresponds some semisimple group $G_\alpha \leq G$ of rank one such that T normalizes G_α. For simple roots α, β, we denote by $G_{\alpha\beta}$ the group generated by the groups G_α and G_β, and by $\Sigma_{\alpha\beta}$ its root system relative to the torus $T_{\alpha\beta} = T \cap G_{\alpha\beta}$. The group $G_{\alpha\beta}$ is a semisimple group of rank two and $\{\alpha, \beta\}$ is a fundamental system of $\Sigma_{\alpha\beta}$.

Then the following assertion holds:

Theorem 3.1 (Theorem of Borovoi [4]) *Let G be a simply connected compact semisimple Lie group, let T be a maximal torus of G, let $\Sigma = \Sigma(G_\mathbb{C}, T_\mathbb{C})$ be its root system, and let Π be a system of fundamental roots of Σ. Then the natural epimorphism $\mathcal{U}(\mathcal{A}) \to G$ is an isomorphism where $\mathcal{A} = (G_{\alpha\beta})_{\alpha,\beta\in\Pi}$ is the amalgam of rank one and rank two subgroups of G.*

Borovoi's proof consists of computations of reduced words in the group $\mathcal{U}(\mathcal{A})$ given by generators and relations. Using the theory of Tits buildings and geometric covering theory one gets the following alternative proof:

Geometric proof of Theorem 3.1. For rank at most two there is nothing to show, so we can assume that the rank is at least three. By the Iwasawa decomposition (see Theorem VI.5.1 of [21] or Theorem III.6.32 of [22]) the group G acts chamber-transitively on the building geometry $\mathcal{G}$ of type Π corresponding to $G_\mathbb{C}$. Let F be a chamber of $\mathcal{G}$ stabilized by the torus T of G, so that the stabilizers of subflags of corank one and two of F with respect to the natural action of G on $\mathcal{G}$ are exactly the groups $G_\alpha T$ and $G_{\alpha\beta}T$. By the simple connectedness of building geometries of rank at least three (cf. Theorem IV.5.2 of [7] or Theorem 13.32 of [37]) plus Tits' Lemma (Lemma 2.8) the group G equals the universal completion of the amalgam $(G_{\alpha\beta}T)_{\alpha,\beta\in\Pi}$. Finally, by Lemma 29.3 of [12] (or by a reduction argument as in the proof of Theorem 2 of [16] or in the proof of Theorem 4.3.6 of [18]) the torus T can be reconstructed from the rank two tori $T_{\alpha\beta}$, $\alpha, \beta \in \Pi$, and so the group G actually equals the universal completion of the amalgam $(G_{\alpha\beta})_{\alpha,\beta\in\Pi}$. □

Proposition 3.2 *Let $n \geq 2$ and let G be a simply connected compact semisimple Lie group. Then the group G is a characteristic completion of the amalgam $(G_{\alpha\beta})_{\alpha\beta\in\Pi}$ of rank one and rank two subgroups.*

Proof. By Theorem 3.1 the group G is the universal completion of the amalgam $(G_{\alpha\beta})_{\alpha\beta\in\Pi}$. Therefore any automorphism of the amalgam extends to G, making G a characteristic completion. $\square$

A result similar to Theorem 3.1 has been proved by Satarov [32] for special unitary groups over quadratic extensions of real closed fields. This case has already been covered by Borovoi's remark after his Theorem in [4]. Here, too, the group acts chamber-transitively on the building geometry, so our proof above applies as well.

4 Phan systems and Phan amalgams

Definition 4.1 Let G be a simply connected compact semisimple Lie group of rank two, i.e., G is isomorphic to $\mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C})$ or $\mathrm{SU}_3(\mathbb{C})$ or $\mathrm{Spin}_5(\mathbb{R}) \cong \mathrm{U}_2(\mathbb{H})$ or $G_{2,-14}$ by [21], see also 94.33 of [31]. Let T be a maximal torus of G, let $\Sigma = \Sigma(G_{\mathbb{C}}, T_{\mathbb{C}})$ be its root system, and let $\{\alpha, \beta\}$ be a fundamental root system of Σ. To the simple roots α, β corresponds a pair of semisimple subgroups G_α and G_β of G normalized by T and isomorphic to $\mathrm{SU}_2(\mathbb{C}) \cong \mathrm{Spin}_3(\mathbb{R}) \cong \mathrm{U}_1(\mathbb{H})$ called a **standard pair** of G. If α and β have different length, then the standard pair (G_α, G_β) is not conjugate to the standard pair (G_β, G_α), so when speaking of a standard pair (G_α, G_β), we assume α to be shorter than β if the roots have different lengths.

A standard pair in a central quotient of G is defined as the image of a standard pair of G under the natural homomorphism. Note that the images of a standard pair in the quotient are isomorphic to $\mathrm{SU}_2(\mathbb{C})$ or to $\mathrm{SO}_3(\mathbb{R})$.

Lemma 4.2 *Standard pairs are conjugate.*

Proof. This follows immediately from the fact that maximal tori are conjugate, cf. Theorem 6.25 of [24], and the fact that, if $\alpha, \beta \in \Pi$ and $\alpha_1, \beta_1 \in \Sigma$ have the same lengths and the same angle, there exists an element w of the Weyl group with $w(\alpha_1) = \alpha$ and $w(\beta_1) = \beta$, cf. [6]. $\square$

Definition 4.3 Let $n \geq 2$, let Δ be a Dynkin diagram of rank n (see [6] for a complete list) and let $I = \{1, \ldots, n\}$ be the set of labels of Δ. A group G admits a **weak Phan system of type Δ over $\mathbb{C}$** if G is generated by subgroups $U_i \cong \mathrm{SU}_2(\mathbb{C})$ or $U_i \cong \mathrm{SO}_3(\mathbb{R})$, $i \in I$, which are central quotients of simply connected compact semisimple Lie groups of rank one, and contains subgroups $U_{i,j} = \langle U_i, U_j \rangle$, $i \neq j \in I$, which are central quotients of simply connected compact semisimple Lie groups of rank two such that (U_i, U_j) or (U_j, U_i) forms a standard pair in $U_{i,j}$. In particular, any $U_{i,j}$ is isomorphic to a central quotient of $\mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C})$ or to $\mathrm{SU}_3(\mathbb{C})$ or $\mathrm{PSU}_3(\mathbb{C})$ or $\mathrm{U}_2(\mathbb{H}) \cong \mathrm{Spin}_5(\mathbb{R})$ or $\mathrm{SO}_5(\mathbb{R})$ or $G_{2,-14}$ depending on the subdiagram of Δ induced on i and j.

The paramount examples for groups with a weak Phan system are the simply connected compact semisimple Lie groups together with the amalgam $(G_{\alpha\beta})_{\alpha\beta\in\Pi}$ of rank one and rank two subgroups. Any central quotient of such a group of rank at least two also admits a weak Phan system.

Definition 4.4 A **Phan amalgam** is an amalgam $\mathcal{A} = (L_{\alpha\beta})_{\alpha,\beta\in\Pi}$, where $L_{\alpha\beta}$ is a group isomorphic to a central quotient of $G_{\alpha\beta}$ where it is required that L_α and L_β are the images of G_α, respectively G_β under the natural epimorphism from $G_{\alpha\beta}$ onto $L_{\alpha\beta}$. A Phan amalgam is called **irreducible** if it is obtained from the natural amalgam $(G_{\alpha\beta})_{\alpha,\beta\in\Pi}$ of a simply connected compact almost simple Lie group, i.e., if the Dynkin diagram of that group is connected or, equivalently, if the corresponding root system is irreducible, cf. [6]. A complete list of the compact almost simple Lie groups can be found in [21] or [31]. A Phan amalgam is called **strongly noncollapsing** if there exists a completion $\pi : \mathcal{A} \to G$ such that the kernel of the restriction $\pi_{|L_{\alpha_i}}$ is central for each $i \in I$. The **rank** of a Phan amalgam is defined to be the rank of the corresponding fundamental system Π. The amalgam $(G_{\alpha\beta})_{\alpha,\beta\in\Pi}$ is called a **standard Phan amalgam**.

If a group G contains a weak Phan system $U_1, \dots, U_n$, then $\mathcal{A} = (U_{i,j})_{i,j\in I}$ is a strongly noncollapsing Phan amalgam. The converse is also true: a Phan amalgam admitting a faithful completion G turns the group G into a group with a weak Phan system of the respective type.

Definition 4.5 A Phan amalgam $(L_{\alpha\beta})_{\alpha,\beta\in\Pi}$ is called **unambiguous** if every $L_{\alpha\beta}$ is isomorphic to the corresponding $G_{\alpha\beta}$.

Proposition 4.6 *Every Phan amalgam $\mathcal{A}$ has an unambiguous covering $\widehat{\mathcal{A}}$ that is unique up to equivalence of coverings. Furthermore, every (strongly) noncollapsing Phan amalgam $\mathcal{A}$ has a unique (up to equivalence of coverings) unambiguous (strongly) noncollapsing covering $\widehat{\mathcal{A}}$.*

Proof. We will proceed by induction on $|S|$, where S is a subset of $\binom{\Pi}{1} \cup \binom{\Pi}{2}$ which is closed under taking subsets and $\mathcal{A} = (L_J)_{J\in S}$. Our basis is the case $S = \emptyset$ which vacuously yields an unambiguous amalgam. Suppose now that S is non-empty, and that for every subset $S' \subsetneq S$ the claim holds. Let J be an element of S which is maximal with respect to inclusion and define $S' = S \setminus \{J\}$ and $\mathcal{A}' = (L_{J'})_{J'\in S'}$. Then S' is closed under taking subsets, and $\mathcal{A}'$ is a subamalgam in $\mathcal{A}$.

By the inductive assumption, there is a unique unambiguous covering amalgam $(\widehat{\mathcal{A}}' = (\widehat{L}_{J'})_{J'\in S'}, \pi')$ of $\mathcal{A}'$. We will find an unambiguous covering $(\widehat{\mathcal{A}}, \pi)$ of $\mathcal{A}$ by gluing a copy of G_J to $\widehat{\mathcal{A}}'$ and by extending π' to the new member of the amalgam. To glue G_J to the amalgam $\widehat{\mathcal{A}}'$, we need to construct an isomorphism from the subamalgam $\widehat{\mathcal{L}} = (\widehat{L}_{J'})_{J'\subsetneq J}$ of $\widehat{\mathcal{A}}'$ onto the corresponding amalgam $\mathcal{G} = (G_{J'})_{J'\subsetneq J}$ of subgroups of G_J. By the definition of a Phan amalgam, there is a homomorphism ψ from G_J onto L_J mapping $\mathcal{G}$ onto $\mathcal{L} = (L_{J'})_{J'\subsetneq J}$. Note that $(\widehat{\mathcal{L}}, \pi'|_{\widehat{\mathcal{L}}})$ and $(\mathcal{G}, \psi|_{\mathcal{G}})$ are two unambiguous coverings of $\mathcal{L}$. By induction, the uniqueness of the unambiguous covering holds so that there is an amalgam isomorphism ϕ from $\widehat{\mathcal{L}}$ onto $\mathcal{G}$ such that $\psi \circ \phi = \pi'|_{\widehat{\mathcal{L}}}$. Clearly, ϕ tells us how to glue G_J to $\widehat{\mathcal{A}}'$ to produce $\widehat{\mathcal{A}}$ and, furthermore, as π we can take the union of ψ and π'. The condition $\psi \circ \phi = \pi'|_{\widehat{\mathcal{L}}}$ guarantees that ψ and π' agree on the intersection $\widehat{\mathcal{L}} \overset{\phi}{\cong} \mathcal{G}$. Finally, notice that $\widehat{\mathcal{A}}$ is an unambiguous Phan amalgam, so $(\widehat{\mathcal{A}}, \pi)$ is an unambiguous covering of $\mathcal{A}$. This completes the proof of the existence of an unambiguous covering $\widehat{\mathcal{A}}$.

Now we will prove the uniqueness. Suppose we have two such coverings $\widehat{\mathcal{B}} = (B_J)_{J\in S}$ and $\widehat{\mathcal{C}} = (C_J)_{J\in S}$ with corresponding amalgam homomorphism π_1 and π_2 onto $\mathcal{A}$. Select J as an element of S which is maximal with respect to inclusion, and define $S' = S \setminus \{J\}$. Let $\mathcal{A}'$, $\widehat{\mathcal{B}}'$ and $\widehat{\mathcal{C}}'$ be the subamalgams of shape S' in $\mathcal{A}$, $\widehat{\mathcal{B}}$ and $\widehat{\mathcal{C}}$, respectively. By induction, there exists an isomorphism ϕ from $\widehat{\mathcal{B}}'$ onto $\widehat{\mathcal{C}}'$ such that $\pi_1|_{\widehat{\mathcal{B}}'} = \pi_2 \circ \phi$. It suffices to extend ϕ to B_J.

We have to deal with two cases: First, let us assume that $J = \{\alpha, \beta\}$ where α and β are orthogonal roots. In this case, $B_{\alpha\beta} \cong C_{\alpha\beta} \cong G_{\alpha\beta}$ is isomorphic to a direct product of $B_\alpha \cong C_\alpha \cong G_\alpha$ and $B_\beta \cong C_\beta \cong G_\beta$. Clearly ϕ is already known on B_α and B_β, and so ϕ extends uniquely to $B_{\alpha\beta}$. This extension, also denoted ϕ, is a well-defined amalgam isomorphism from $\mathcal{B}$ to $\mathcal{C}$, and furthermore, $\pi_1 = \pi_2 \circ \phi$ holds.

In the second case, $B_J \cong C_J \cong G_J$ is isomorphic to a simply connected compact almost simple Lie group of rank one or two. By the universality of the covering $\pi_1 : B_J \to L_J$, as B_J is simply connected, there exists a unique isomorphism $\psi : B_J \to C_J$ such that $\pi_1 = \pi_2 \circ \psi$.

$$\begin{tikzcd} C_J \arrow[dr, "\pi_2"'] & B_J \arrow[l, "\psi"'] \arrow[d, "\pi_1"] \\ & L_J \end{tikzcd}$$

Consider a mapping α from L_J to L_J defined as follows: For $u \in L_J$, let $\alpha(u) = (\pi_2 \circ \psi \circ \pi_1^{-1})(u)$. Notice that α is a well-defined automorphism of L_J, because the cosets of the kernel of π_1 are mapped by ψ to cosets of the kernel of π_2. Every automorphism of L_J lifts to a unique automorphism of C_J. Indeed, both L_J and C_J are perfect by a corollary of Gotô's Commutator Theorem (see Corollary 6.56 of [24]) and, by Theorem 2.1 of [30], the group C_J, which is isomorphic to $\mathrm{SU}_2(\mathbb{C}) \cong \mathrm{Spin}_3(\mathbb{R}) \cong \mathrm{U}_1(\mathbb{H})$ or to $\mathrm{SU}_3(\mathbb{C})$ or to $\mathrm{Spin}_5(\mathbb{R}) \cong \mathrm{U}_2(\mathbb{H})$, is the universal perfect central extension of L_J, cf. [25] or [33], [34]. Alternatively, one can argue as follows: Every automorphism of L_J is continuous by Corollary 6.56 of [24] and van der Waerden's Continuity Theorem (cf. Theorem 5.64 of [24]), which lifts to a unique continuous automorphism of C_J by [26], see also [23]. Finally, this lift in fact is the unique abstract lift of α, as any automorphism of C_J again is continuous.

Thus, there is a unique automorphism β of C_J such that $\pi_2 \circ \beta = \alpha \circ \pi_2$. Define $\theta : B_J \to C_J : \theta(b) = (\beta^{-1} \circ \psi)(b)$. First of all, by definition we have $\pi_1|_{B_J} = \pi_2 \circ \theta$, as

$$\begin{aligned} \pi_2 \circ \theta &= \pi_2 \circ \beta^{-1} \circ \psi \\ &= \alpha^{-1} \circ \pi_2 \circ \psi \\ &= \pi_1|_{B_J} \circ \psi^{-1} \circ \pi_2^{-1}|_{L_J} \circ \pi_2 \circ \psi \\ &= \pi_1|_{B_J}. \end{aligned}$$

Second, for every $J' \subset J$ we have that $\theta^{-1} \circ \phi_{|B_{J'}}$ is a lifting to $B_{J'}$ of the identity automorphism of $L_{J'}$ and, by the above, it is the identity. For $\theta^{-1} \circ \phi_{|B_{J'}} = \psi^{-1} \circ \beta \circ \phi_{|B_{J'}}$ and, the following considered on $B_{J'} / \ker(\pi_{1|B_{J'}})$,

$$\begin{aligned} \psi^{-1} \circ \pi_2{}^{-1}_{|C_{J'}} \circ \alpha \circ \pi_2 \circ \phi_{|B_{J'}} &= \psi^{-1} \circ \pi_2{}^{-1}_{|C_{J'}} \circ \pi_2 \circ \psi \circ \pi_1{}^{-1}_{|B_{J'}} \circ \pi_2 \circ \phi_{|B_{J'}} \\ &= \pi_1{}^{-1}_{|B_{J'}} \circ \pi_2 \circ \phi_{|B_{J'}} \\ &= \mathrm{id}. \end{aligned}$$

This shows that ϕ and θ agree on every subgroup $B_{J'}$, which allows us to extend ϕ to the entire $\widehat{\mathcal{B}}$ by defining it on B_J as θ. Finally, if $\mathcal{A}$ is (strongly) noncollapsing, so is its unambiguous covering $\widehat{\mathcal{A}}$, finishing the proof. □

5 Uniqueness of unambiguous amalgams

Let $\mathcal{A} = (L_{I \backslash \{i,j\}})_{(i,j) \in I}$ be an unambiguous strongly noncollapsing irreducible Phan amalgam of rank at least two. We will establish the uniqueness of the respective amalgams $\mathcal{A}$ up to isomorphism in a series of lemmas. The amalgams of rank two are unique by definition.

Rank three

Assume the rank of $\mathcal{A}$ to be three. Since $\mathcal{A}$ is unambiguous, each subgroup $L_{I\setminus\{i\}}$ coincides with $L_{I\setminus\{i,j\}} \cap L_{I\setminus\{i,k\}}$ for $\{i,j,k\} = \{1,2,3\}$. We want to prove the uniqueness of the amalgam $\mathcal{A} = (L_{I\setminus\{i,j\}})_{i,j\in\{1,2,3\}}$.

For A_3, i.e., for the diagram $\underset{L_{I\setminus\{1\}}}{\circ}\!\!\frac{\quad\quad}{}\!\!\underset{L_{I\setminus\{2\}}}{\circ}\!\!\frac{\quad\quad}{}\!\!\underset{L_{I\setminus\{3\}}}{\circ}$, recall the isomorphisms

$$\begin{aligned}
L_{I\setminus\{2,3\}} &\cong \mathrm{SU}_3(\mathbb{C}),\\
L_{I\setminus\{1,3\}} &\cong \mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C}),\\
L_{I\setminus\{1,2\}} &\cong \mathrm{SU}_3(\mathbb{C}),\\
L_{I\setminus\{3\}} = L_{I\setminus\{2,3\}} \cap L_{I\setminus\{1,3\}} &\cong \mathrm{SU}_2(\mathbb{C}),\\
L_{I\setminus\{2\}} = L_{I\setminus\{2,3\}} \cap L_{I\setminus\{1,2\}} &\cong \mathrm{SU}_2(\mathbb{C}),\\
L_{I\setminus\{1\}} = L_{I\setminus\{1,3\}} \cap L_{I\setminus\{1,2\}} &\cong \mathrm{SU}_2(\mathbb{C}).
\end{aligned}$$

For B_3, i.e., for the diagram $\underset{L_{I\setminus\{1\}}}{\circ}\!\!\frac{\quad\quad}{}\!\!\underset{L_{I\setminus\{2\}}}{\circ}\!\!\overset{>}{=\!=\!=}\!\!\underset{L_{I\setminus\{3\}}}{\circ}$, recall the isomorphisms

$$\begin{aligned}
L_{I\setminus\{2,3\}} &\cong \mathrm{Spin}_5(\mathbb{R}),\\
L_{I\setminus\{1,3\}} &\cong \mathrm{SU}_2(\mathbb{C}) \times \mathrm{Spin}_3(\mathbb{R}),\\
L_{I\setminus\{1,2\}} &\cong \mathrm{SU}_3(\mathbb{C}),\\
L_{I\setminus\{3\}} = L_{I\setminus\{2,3\}} \cap L_{I\setminus\{1,3\}} &\cong \mathrm{Spin}_3(\mathbb{R}),\\
L_{I\setminus\{2\}} = L_{I\setminus\{2,3\}} \cap L_{I\setminus\{1,2\}} &\cong \mathrm{SU}_2(\mathbb{C}),\\
L_{I\setminus\{1\}} = L_{I\setminus\{1,3\}} \cap L_{I\setminus\{1,2\}} &\cong \mathrm{SU}_2(\mathbb{C}).
\end{aligned}$$

For C_3, i.e., for the diagram $\underset{L_{I\setminus\{1\}}}{\circ}\!\!\frac{\quad\quad}{}\!\!\underset{L_{I\setminus\{2\}}}{\circ}\!\!\overset{<}{=\!=\!=}\!\!\underset{L_{I\setminus\{3\}}}{\circ}$, recall the isomorphisms

$$\begin{aligned}
L_{I\setminus\{2,3\}} &\cong \mathrm{U}_2(\mathbb{H}),\\
L_{I\setminus\{1,3\}} &\cong \mathrm{SU}_2(\mathbb{C}) \times \mathrm{U}_1(\mathbb{H}),\\
L_{I\setminus\{1,2\}} &\cong \mathrm{SU}_3(\mathbb{C}),\\
L_{I\setminus\{3\}} = L_{I\setminus\{2,3\}} \cap L_{I\setminus\{1,3\}} &\cong \mathrm{U}_1(\mathbb{H}),\\
L_{I\setminus\{2\}} = L_{I\setminus\{2,3\}} \cap L_{I\setminus\{1,2\}} &\cong \mathrm{SU}_2(\mathbb{C}),\\
L_{I\setminus\{1\}} = L_{I\setminus\{1,3\}} \cap L_{I\setminus\{1,2\}} &\cong \mathrm{SU}_2(\mathbb{C}).
\end{aligned}$$

Assume there exists another amalgam $\mathcal{A}' = (L'_{I\setminus\{i,j\}})_{i,j\in\{1,2,3\}}$. According to Goldschmidt's Lemma (Lemma 2.10) the amalgams $\mathcal{B} = (L_{I\setminus\{2,3\}}, L_{I\setminus\{1,2\}}, L_{I\setminus\{2\}})$ and $\mathcal{B}' = (L'_{I\setminus\{2,3\}}, L'_{I\setminus\{1,2\}}, L'_{I\setminus\{2\}})$ are isomorphic via some amalgam isomorphism ψ, because every automorphism of the group $L_{I\setminus\{2\}} \cong \mathrm{SU}_2(\mathbb{C})$ is induced by some automorphism of the group $L_{I\setminus\{1,2\}} \cong \mathrm{SU}_3(\mathbb{C})$. Indeed, $L_{I\setminus\{2\}}$ is embedded as the stabilizer of a vector of length one of the natural module of $L_{I\setminus\{1,2\}}$. Clearly, $\psi(L_{I\setminus\{2\}}) = \psi(L_{I\setminus\{2,3\}} \cap L_{I\setminus\{1,2\}}) = L'_{I\setminus\{2,3\}} \cap L'_{I\setminus\{1,2\}} = L'_{I\setminus\{2\}}$. The groups $L_{I\setminus\{1\}}$ and $L_{I\setminus\{2\}}$ form a standard pair in $L_{I\setminus\{1,2\}}$, and hence $\psi(L_{I\setminus\{1\}})$ and $L'_{I\setminus\{2\}} = \psi(L_{I\setminus\{2\}})$ form a standard pair in $L'_{I\setminus\{1,2\}} = \psi(L_{I\setminus\{1,2\}})$. Certainly also $L'_{I\setminus\{1\}}$ and $L'_{I\setminus\{2\}}$ form a standard pair in $L'_{I\setminus\{1,2\}}$. Therefore, by Lemma 4.2, there exists an automorphism of $L'_{I\setminus\{1,2\}}$ that maps $\psi(L_{I\setminus\{1\}})$ onto $L'_{I\setminus\{1\}}$ and that normalizes $L'_{I\setminus\{2\}}$. Thus, we can assume $\psi(L_{I\setminus\{1\}}) = L'_{I\setminus\{1\}}$.

Before we can continue we have to study the amalgam $\mathcal{A}$ a bit more carefully. Define

$$D_1 = N_{L_{I\setminus\{1\}}}(L_{I\setminus\{2\}}) \quad \text{and} \quad D_3 = N_{L_{I\setminus\{3\}}}(L_{I\setminus\{2\}})$$

where the groups $L_{I\setminus\{2\}}$, $L_{I\setminus\{1\}}$ are considered as subgroups of $L_{I\setminus\{1,2\}}$ and the groups $L_{I\setminus\{3\}}$, $L_{I\setminus\{2\}}$ are considered as subgroups of $L_{I\setminus\{2.3\}}$. Since $L_{I\setminus\{2\}}$ and $L_{I\setminus\{1\}}$ form a standard pair in $L_{I\setminus\{1,2\}}$, it follows that D_1 is a maximal torus in $L_{I\setminus\{1\}} \cong \mathrm{SU}_2(\mathbb{C})$. Similarly, D_3 is a maximal torus in $L_{I\setminus\{3\}}$. We also define

$$D_2^1 = N_{L_{I\setminus\{2\}}}(L_{I\setminus\{1\}}) \quad \text{and} \quad D_2^3 = N_{L_{I\setminus\{2\}}}(L_{I\setminus\{3\}}).$$

Again, these are two maximal tori in $L_{I\setminus\{2\}} \cong \mathrm{SU}_2(\mathbb{C})$. The following lemma gives us an extra condition on $\mathcal{A}$ that holds because $\mathcal{A}$ is strongly noncollapsing.

Lemma 5.1 $D_2^1 = D_2^3$.

Proof. Let G be a nontrivial completion of $\mathcal{A}$ and let π be the corresponding map from $\mathcal{A}$ to G. Since $\mathcal{A}$ is assumed to be strongly noncollapsing, we may assume that π is injective on every $L_{I\setminus\{i\}}$. Observe that $D_2^i = C_{L_{\{1,3\}}}(D_i)$ for $i = 1, 3$. Thus, $\pi(D_2^i) = C_{\pi(L_{I\setminus\{2\}})}(\pi(D_i))$. Since D_1 and D_3 commute elementwise in $L_{I\setminus\{1,3\}}$, we have that $\pi(D_1)$ and $\pi(D_3)$ commute elementwise as well. Since $L_{I\setminus\{2\}}$ is invariant under $D_1 = N_{L_{I\setminus\{1\}}}(L_{I\setminus\{2\}})$ (in $L_{I\setminus\{1,2\}}$) and since π is injective on $L_{I\setminus\{2\}}$, it follows that $D_2^3 = C_{L_{I\setminus\{2\}}}(D_3)$ is invariant under D_1 (again as subgroups of $L_{I\setminus\{1,2\}}$) and $\pi(D_2^3) = C_{\pi(L_{I\setminus\{2\}})}(\pi(D_3))$ is invariant under $\pi(D_1)$. Here, injectivity of π is needed for the following argument. D_1 and D_3 commute as subgroups of $L_{I\setminus\{1,3\}}$. The group $L_{I\setminus\{2\}}$ is invariant under D_1 as a subgroup of $L_{I\setminus\{1,2\}}$. Since $L_{I\setminus\{1,3\}}$ and $L_{I\setminus\{1,2\}}$ are not contained in a common group of the amalgam $\mathcal{A}$, we cannot conclude that D_1 leaves D_2^3 invariant. However, in G, since $L_{I\setminus\{2\}}$, D_1, D_3, D_2^3 are embedded via π, we can draw that conclusion.

But now the maximal torus D_1 of $L_{I\setminus\{1\}} \cong \mathrm{SU}_2(\mathbb{C})$ leaves invariant the maximal tori D_2^1 and D_2^3 of $L_{I\setminus\{2\}} \cong \mathrm{SU}_2(\mathbb{C})$. Analysis of the group $L_{I\setminus\{1,2\}} \cong \mathrm{SU}_3(\mathbb{C})$ shows that $D_2^1 = D_2^3$. $\square$

In view of this lemma we can use the notation

$$D_2 = D_2^1 = D_2^3.$$

Since $N_{L_{I\setminus\{2\}}}(L_{I\setminus\{1\}}) = D_2^1 = D_2 = D_2^3 = N_{L_{I\setminus\{2\}}}(L_{I\setminus\{3\}})$, the considerations made before Lemma 5.1 imply $\psi(D_2) = D_2'$. Let d be a nontrivial element of D_2' of order distinct from two. Denote by W the natural three-dimensional module of $L'_{I\setminus\{1,2\}}$, and recall that $L'_{I\setminus\{2\}}$ and $L'_{I\setminus\{3\}}$ form a standard pair of $L'_{I\setminus\{2,3\}}$. As $D_2' \leq L'_{I\setminus\{2\}}$, the group D_2' fixes a non-isotropic vector u of length one of W fixed by $L'_{I\setminus\{2\}}$. Since D_2' normalizes $L'_{I\setminus\{3\}}$, it also stabilizes $\langle v\rangle$, where v is a non-isotropic vector of length one of W fixed by $L'_{I\setminus\{3\}}$. Moreover, since $L'_{I\setminus\{2\}}$ and $L'_{I\setminus\{3\}}$ form a standard pair, u is perpendicular to v in W. Let $\langle w\rangle$ be the one-dimensional subspace of W that is perpendicular to both u and v and assume w has length one. Then u, v, w is an orthonormal basis of W, and d acts diagonally with respect to that basis via $\mathrm{diag}(1, a, a^{-1})$. Since the order of d is distinct from two, we have $a \neq a^{-1}$, so the one-dimensional subspaces of W stabilized by d are precisely $\langle u\rangle$, $\langle v\rangle$, $\langle w\rangle$. It follows, since $D_2' = \psi(D_2) = N_{\psi(L_{I\setminus\{2\}})}(\psi(L_{I\setminus\{3\}})) = N_{L'_{I\setminus\{2\}}}(\psi(L_{I\setminus\{3\}}))$, that $\psi(L_{I\setminus\{3\}})$ is the stabilizer of either v or w.

In the former case we have $\psi(L_{I\setminus\{3\}}) = L'_{I\setminus\{3\}}$, and we have proved $\mathcal{A} \cong \mathcal{A}'$, since $L_{I\setminus\{1,3\}} = L_{I\setminus\{3\}} \times L_{I\setminus\{1\}}$ and $L'_{I\setminus\{1,3\}} = L'_{I\setminus\{3\}} \times L'_{I\setminus\{1\}}$.

In the latter case consider the element g of $L'_{I\backslash\{2\}}$ whose matrix with respect to the orthonormal basis u, v, w has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Conjugation with g induces the action of the contragredient automorphism on $L'_{I\backslash\{2\}}$. By the defining relation

$$A^{-1} = \bar{A}^T$$

of unitary matrices the action of the contragredient automorphism of $L'_{I\backslash\{2\}}$ coincides with the field involution. Therefore, we can define an automorphism α of $\mathcal{B}'$ that acts trivially on $L'_{I\backslash\{1,2\}}$ and as the composition of the field automorphism and conjugation by g on $L'_{I\backslash\{2,3\}}$, since by the above this automorphism acts trivially on $L'_{I\backslash\{2\}} = L'_{I\backslash\{2,3\}} \cap L'_{I\backslash\{1,2\}}$. Moreover, α interchanges $\langle v\rangle$ and $\langle w\rangle$, so it maps $\psi(L_{I\backslash\{3\}})$ onto $L'_{I\backslash\{3\}}$.

We have proved the following.

Proposition 5.2 *Let $\mathcal{A}$ be a strongly noncollapsing unambiguous irreducible Phan amalgam of rank three. Then $\mathcal{A}$ is unique up to isomorphism, i.e., $\mathcal{A}$ is isomorphic to a standard Phan amalgam.* □

Rank at least four

Let $\mathcal{A} = (L_{I\backslash\{i,j\}})_{1\leq i<j\leq n}$ be a strongly noncollapsing unambiguous irreducible Phan amalgam of rank at least four. We complete the proof of the uniqueness of $\mathcal{A}$ by induction, the case of rank three from Proposition 5.2 being the basis of induction.

Lemma 5.3 *Let $n \geq 4$ and let $\mathcal{A}$ be a strongly noncollapsing unambiguous irreducible Phan amalgam of rank n. Then there exists a unique amalgam*

$$\begin{aligned} \mathcal{B}_{\mathcal{A}} &= \mathcal{A} \cup H_1 \cup H_2 \\ &\textit{with} \\ H_1 &= \langle L_{I\backslash\{i,j\}} \mid 1 \leq i < j \leq n-1\rangle \textit{ and} \\ H_2 &= \langle L_{I\backslash\{i,j\}} \mid 2 \leq i < j \leq n\rangle. \end{aligned}$$

The group H_1 is isomorphic to $\mathrm{SU}_n(\mathbb{C})$ unless the case of the Dynkin diagram F_4, where H_1 is isomorphic to $\mathrm{Spin}_7(\mathbb{R})$, while the group H_2 is isomorphic to

$$\begin{array}{rl} \mathrm{SU}_n(\mathbb{C}) & \textit{for the diagram } A_n, \\ \mathrm{Spin}_{2n-1}(\mathbb{R}) & \textit{for the diagram } B_n, \\ \mathrm{U}_{n-1}(\mathbb{H}) & \textit{for the diagram } C_n, \\ \mathrm{Spin}_{2n-2}(\mathbb{R}) & \textit{for the diagram } D_n, \\ \mathrm{Spin}_{10}(\mathbb{R}) & \textit{for the diagram } E_6, \\ \mathrm{Spin}_{12}(\mathbb{R}) & \textit{for the diagram } E_7, \\ \mathrm{Spin}_{14}(\mathbb{R}) & \textit{for the diagram } E_8, \\ \mathrm{U}_3(\mathbb{H}) & \textit{for the diagram } F_4. \end{array}$$

Proof. Let

$$\begin{aligned}
\mathcal{B}_1 &:= (L_{I\setminus\{i,j\}})_{1\le i<j\le n-1}, \\
\mathcal{B}_2 &:= (L_{I\setminus\{i,j\}})_{2\le i<j\le n}, \quad \text{and} \\
\mathcal{C} &:= \mathcal{B}_1 \cap \mathcal{B}_2.
\end{aligned}$$

By the inductive assumption, both $\mathcal{B}_1$ and $\mathcal{B}_2$ are isomorphic to some standard Phan amalgam and hence there exist faithful completions $\pi_i : \mathcal{B}_i \to H_i$ where the isomorphism types of H_1 and H_2 are given as in the hypothesis. We want to glue H_1 and H_2 to the amalgam $\mathcal{A}$ via π_1 and π_2. Let $K_i := \langle \pi_i(\mathcal{C})\rangle$. Since, again by the inductive assumption, the amalgam $\mathcal{C}$ is isomorphic to a standard Phan amalgam, we have $K_i \cong \mathrm{SU}_{n-1}(\mathbb{C})$ or, in case of the diagram F_4, we have $K_i \cong \mathrm{Spin}_5(\mathbb{R}) \cong \mathrm{U}_2(\mathbb{H})$. By Proposition 3.2 the group K_i is a characteristic completion of the amalgam $\mathcal{C}$, so there exists an isomorphism $\phi : K_1 \to K_2$ that takes $\pi_1(\mathcal{C})$ to $\pi_2(\mathcal{C})$. Let ψ be the restriction of ϕ to $\pi_1(\mathcal{C})$. Applying the Bennett-Shpectorov Lemma (Lemma 2.12) with $\phi : K_1 \to K_2$ and $\psi : \pi_1(\mathcal{C}) \to \pi_2(\mathcal{C})$ as above and $\psi' : \pi_1(\mathcal{C}) \to \pi_2(\mathcal{C})$ with $\psi' = \pi_2 \circ \pi_1{}_{|\mathcal{C}}^{-1}$, there exists a unique isomorphism $\phi' : K_1 \to K_2$ such that $\phi'_{|\pi_1(\mathcal{C})} = \psi'$. Thus, $\phi' \circ \pi_1{}_{|\mathcal{C}} = \pi_2|_{\mathcal{C}}$. Identifying K_1 with K_2 via ϕ' we obtain our unique amalgam $\mathcal{B}$. □

Let us now turn to the uniqueness of the amalgam $\mathcal{A}$. Suppose we have strongly noncollapsing unambiguous irreducible Phan amalgams $\mathcal{A}$ and $\mathcal{A}'$ corresponding to the same diagram. Extend $\mathcal{A}$ and $\mathcal{A}'$ to amalgams $\mathcal{B}_{\mathcal{A}} = \mathcal{A} \cup H_1 \cup H_2$ and $\mathcal{B}'_{\mathcal{A}'} = \mathcal{A}' \cup H'_1 \cup H'_2$ as in Lemma 5.3. By Goldschmidt's Lemma (Lemma 2.10) there exists an isomorphism ϕ from $H_1 \cup H_2$ onto $H'_1 \cup H'_2$. By the inductive assumption $(L_{I\setminus\{i,j\}})_{1<i<j<n}$ is isomorphic to a standard Phan amalgam embedded in $H_1 \cap H_2$. Similarly $(L'_{I\setminus\{i,j\}})_{1<i<j<n}$ and $\phi(L_{I\setminus\{i,j\}})_{1<i<j<n}$ are isomorphic to standard Phan amalgams embedded in $H'_1 \cap H'_2$. These two amalgams correspond to two choices of a maximal torus of $H'_1 \cap H'_2$, which are conjugate by Theorem 6.27 of [24]. So, correcting ϕ if necessary by an inner automorphism of $H'_1 \cap H'_2$, we may assume that $\phi(L_{I\setminus\{i\}}) = L'_{I\setminus\{i\}}$ for $1 < i < n$ and $\phi(L_{I\setminus\{i,j\}}) = L'_{I\setminus\{i,j\}}$ for $1 < i < j < n$. Also, by studying the standard Phan amalgam inside H'_1, we have

$$\begin{aligned}
\phi\left(L_{I\setminus\{1\}}\right) &= \phi\left(C_{H_1}\left(\langle L_{I\setminus\{3\}}, \ldots, L_{I\setminus\{n-1\}}\rangle\right)\right) \\
&= C_{\phi(H_1)}\left(\phi\left(\langle L_{I\setminus\{3\}}, \ldots, L_{I\setminus\{n-1\}}\rangle\right)\right) \\
&= C_{H'_1}\left(\langle L'_{I\setminus\{3\}}, \ldots, L'_{I\setminus\{n-1\}}\rangle\right) \\
&= L'_{I\setminus\{1\}}.
\end{aligned}$$

By a similar argument, $\phi(L_{I\setminus\{n\}}) = L'_{I\setminus\{n\}}$. Therefore ϕ extends to an isomorphism from $\mathcal{A}$ to $\mathcal{A}'$. Indeed, ϕ is already defined on all $L_{I\setminus\{i,j\}}$ with $2 \le i < j \le n-1$. Also, inside the standard Phan amalgam of H'_1 we see that $\phi(L_{I\setminus\{1,i\}}) = L'_{I\setminus\{1,i\}}$ for $i < n$, since $L_{I\setminus\{1,i\}} = \langle L_{I\setminus\{1\}}, L_{I\setminus\{i\}}\rangle$. Similarly, in the standard Phan amalgam of H'_2 we see that $\phi(L_{I\setminus\{i,n\}}) = L'_{I\setminus\{i,n\}}$ for $1 < i$. It remains to realize that $L_{I\setminus\{1,n\}}$ is the direct product of $L_{I\setminus\{1\}}$ and $L_{I\setminus\{n\}}$, so that ϕ extends to an isomorphism of $\mathcal{A}$ to $\mathcal{A}'$.

Thus we have shown:

Proposition 5.4 *Let $n \ge 4$, and let $\mathcal{A}$ be a strongly noncollapsing unambiguous irreducible Phan amalgam of rank n. Then $\mathcal{A}$ is unique up to isomorphism, i.e., $\mathcal{A}$ is isomorphic to a standard Phan amalgam.* □

Proof of the Main Theorem. The weak Phan system of G gives rise to a strongly noncollapsing Phan amalgam $\mathcal{A}$, which by Proposition 4.6 is covered by a unique strongly noncollapsing unambiguous Phan amalgam $\widehat{\mathcal{A}}$. This strongly noncollapsing unambiguous Phan amalgam $\widehat{\mathcal{A}}$ is isomorphic to a standard Phan amalgam by Propositions 5.2 and 5.4 applied to the irreducible components of Δ of rank at least three and by Definition 4.4 applied to the irreducible components of Δ of rank at most two. Finally, the first claim follows by Theorem 3.1. The second claim follows immediately from the first claim by the classification of irreducible Dynkin diagrams, see [6], and by [21] or by 94.33 of [31]. $\square$

6 Open problem

Let $G_{\mathbb{C}}$ be a simply connected complex semisimple Lie group and let G be its split real form. Then, by the Iwasawa decomposition (see Theorem VI.5.1 of [21] or Theorem III.6.32 of [22]), the maximal compact subgroup K of G acts flag-transitively on the building geometrry of G. Our proof of Theorem 3.1 implies that G equals the universal completion of certain split real forms of complex Lie groups of rank one and two.

The question is whether it is possible to classify the occuring amalgams in order to obtain a result like our Main Theorem.

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