Intransitive geometries

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1 Introduction

Amalgams in group theory have proved their importance in the classification of the finite simple groups (see Sections 28 and 29 of Gorenstein, Lyons, Solomon [12]). Originally one considers the amalgam of the maximal parabolic subgroups of a Chevalley group of rank ≥ 3 in its natural action on the associated building and proves that the universal completion of the amalgam is (some controlled central extension of) the Chevalley group itself, see [8], [26], [28], [29]. In modern terms, see Mühlherr [21], this essentially is implied by the fact that the building and the opposites geometry of the corresponding twin building are simply connected.

Since the mid-1970's there has been interest in other types of amalgams as well, see Phan [19], [20]. Somehow miraculously amalgams of (twisted) Chevalley groups over finite fields were studied that did not come from the action on the building. Aschbacher [3] was the first to realize that Phan's amalgam in [19] arises as a version of the amalgam of rank one and rank two parabolics of the action of $\mathrm{SU}_{n+1}(q^2)$ on the geometry of nondegenerate subspaces of a $(n+1)$-dimensional unitary vector space over $\mathbb{F}_{q^2}$. In order to prove that the universal completion of the amalgam is the group under consideration, one complies to a lemma by Tits [30] saying that this essentially amounts to checking that the geometry is simply connected and residually connected, under the assumption that the geometry is flag-transitive.

Since Phan's papers were a bit vague, there was a demand for a new proof of Phan's result [19]. Das [9] succeeded partially and Bennett, Shpectorov [5] succeeded completely. After preprints of the latter paper were circulated around the 2001 conference in honor of Ernie Shult, things started to develop at a high pace. People finally realized the connection between Mühlherr's [21] new proof of the Curtis-Tits theorem and Aschbacher's [3] geometry for the Phan amalgam. Eventually Hoffman, Shpectorov and the first author [13] constructed a new geometry resulting in the geometric part of a completely new Phan-type theorem. Recently the first author [14] provided the group-theoretic part, a classification of amalgams based on [5], thus completing the new Phan-type theorem.

Later Bennett joined Hoffman, Shpectorov and the first author [4] to develop a theory for this new sort of geometries, called *flipflop geometries*: Take your favorite spherical building and consider it as a twin building à la Tits [31]. The *opposites geometry*, which was used by Mühlherr [21] to re-prove the Curtis-Tits theorem, consists of the pairs of elements of the twin building at codistance one (the neutral element of the associated Weyl group). A *flip* is an involution of that opposites geometry that interchanges the positive and the negative part, flips the distances and preserves the codistance. The flipflop geometry of the opposites geometry with respect to the flip consists of all those elements of the opposites geometry that are stabilized (or rather *flipped*) by the flip.

In case of Aschbacher's geometry for Phan's theorem the building geometry is the projective space corresponding to the group $\mathrm{SL}_{n+1}(q^2)$ and the flip is a nondegenerate unitary polarity. The corresponding flipflip geometry then is the geometry on the nondegenerate subspaces of the projective space with respect to the polarity. Indeed, being opposite means that a subspace and its polar have empty intersection which in turn means that the subspace in question is nondegenerate.

The rank of this geometry is always higher than the one of the associated building, and hence this approach covers more groups. This idea works fine for the unitary groups (see Aschbacher [3], Das [9], Bennett, Shpectorov [5]) and for the symplectic groups (see Das [10] (finite fields, odd characteristic), Das [11] (finite fields, even characteristic), Hoffman, Shpectorov and the first author [13] (finite fields of size at least 8; a by-product of the new geometry), and the first author [16] (all fields)) although, strictly speaking, the symplectic forms do not yield a flipflop geometry. However, for the orthogonal ones over finite fields, we run into problems since the geometry of nondegenerate spaces is, in general, not flag-transitive. The flag-transitive case for forms of Witt index at least one, i.e., over quadratically closed fields has been settled by Altmann [1]. See also Altmann and the first author [2] for the same results and some extensions to real closed fields.

As said before, in order to prove that the universal completion of the amalgam is the group under consideration, one complies to a lemma by Tits [30] saying that this essentially amounts to checking that the geometry is simply connected and residually connected, under the assumption of flag-transitivity. For intransitive geometries one can try to find a flag-transitive subgeometry and to prove that this subgeometry is simply connected and residually connected. However, flag-transitive subgeometries of the geometry of degenerate subspaces of a finite orthogonal classical group are not known to be simply connected, although Hoffman and one of his PhD students are currently trying to establish simple connectivity.

Hence, to overcome these difficulties, one should generalize the theory of amalgams either to non flag-transitive geometries, or to non simply connected ones. Since the former is more realistic (the latter would involve constructing covers of non simply connected geometries), we have chosen to try that. The key idea is to use a theorem by Stroppel [27], which seems not to be so well known, but is very useful in this context. We also discuss the more difficult and more general problem of the amalgam of rank k parabolics in non flag-transitive geometries. It actually turns out that the most natural results occur if one abandons thinking in amalgams of rank k parabolics, but adopts thinking in amalgams of certain *shapes* instead. We then apply our theory to the orthogonal classical groups and give many examples.

In an appendix, we give another example of an amalgam of a rank two Chevalley group, Dickson's group $G_2(q)$, whose universal completion is the Chevalley group itself, by introducing a rank three geometry for it. The reason why we mention this here is to illustrate how wide to range of applications really is: in this case we consider singular points, but nonsingular lines and planes with respect to the classical representation of the associated building, which is a generalized hexagon. In a second appendix we report on recent research by Hoffman and Shpectorov [18] for an interesting amalgam for $G_2(3)$ coming from an intransitive geometry related to the sporadic simple Thompson group.

We conclude this introduction by the remark that in the mid-1980's, using functional analysis and Lie theory, Borovoi [6] and Satarov [23] have obtained related universal completion results for amalgams in compact Lie groups. In this case, however, the geometry acted on is the building, so their results on compact Lie groups follow immediately from the simple connectivity of the building. The classification strategy for amalgams from [5] and [14] was used by the first author in [17] when providing a classification of the amalgams from [6] and [23], yielding a Phan-type theorem for compact Lie groups.

2 Preliminaries

In this section, we define the notions and review the results that we will need to develop our theory. This section has been inspired by [7], [24], [25].

2.1 Coset pregeometries

Definition 2.1 (Pregeometry, geometry) A **pregeometry $\mathcal{G}$ over the set** I is a triple $(X, *, \mathrm{typ})$ consisting of a set X, a symmetric and reflexive **incidence relation** $*$, and a surjective **type function** $\mathrm{typ} : X \to I$, subject to the following condition:

(Pre) If $x * y$ with $\mathrm{typ}(x) = \mathrm{typ}(y)$, then $x = y$.

The set I is usually called the **type set**. A **flag** in X is a set of pairwise incident elements. The **type** of a flag F is the set $\mathrm{typ}(F) := \{\mathrm{typ}(x) : x \in F\}$. A **chamber** is a flag of type I, a **pennant** is a flag of cardinality three. The **rank** of a flag F is $|\mathrm{typ}(F)|$ and the **corank** is equal to $|I \setminus \mathrm{typ}(F)|$.

A **geometry** is a pregeometry with the additional property that

(Geo) every flag is contained in a chamber.

The pregeometry $\mathcal{G}$ is **connected** if the graph $(X, *)$ is connected.

Definition 2.2 (Lounge, hall) Let $\mathcal{G} = (X, *, \mathrm{typ})$ be a pregeometry over I. A subset W of X is called a **lounge** if each subset V of W for which $\mathrm{typ} : V \to I$ is a injection, is a flag. A lounge W with $\mathrm{typ}(W) = I$ is called a **hall**.

Definition 2.3 (Residue) Let F be a flag of $\mathcal{G}$, let us say of type $J \subseteq I$. Then the **residue** $\mathcal{G}_F$ **of** F is the pregeometry

$$(X', *_{|X' \times X'}, \mathrm{typ}_{|I \setminus J})$$

over $I \setminus J$, with

$$X' := \{x \in X : F \cup \{x\} \text{ is a flag of } \mathcal{G} \text{ and } \mathrm{typ}(x) \notin \mathrm{typ}(F)\}.$$

Definition 2.4 (Automorphism) Let $\mathcal{G} = (X, *, \mathrm{typ})$ be a pregeometry over I. An **automorphism** of $\mathcal{G}$ is a permutation σ of X with $\mathrm{typ}(\sigma(x)) = \mathrm{typ}(x)$, for all $x \in X$, and with $x^\sigma * y^\sigma$ if and only if $x * y$, for all $x, y \in X$.

Moreover, a group G of automorphisms of $\mathcal{G}$ is called	if for each pair of flags c, d with
flag-transitive,	$\mathrm{typ}(c) = \mathrm{typ}(d)$,
chamber-transitive,	$\mathrm{typ}(c) = I = \mathrm{typ}(d)$,
pennant-transitive,	$\lvert\mathrm{typ}(c)\rvert = 3 = \lvert\mathrm{typ}(d)\rvert$ and $\mathrm{typ}(c) = \mathrm{typ}(d)$,
incidence-transitive, or	$\lvert\mathrm{typ}(c)\rvert = 2 = \lvert\mathrm{typ}(d)\rvert$ and $\mathrm{typ}(c) = \mathrm{typ}(d)$,
vertex-transitive	$\lvert\mathrm{typ}(c)\rvert = 1 = \lvert\mathrm{typ}(d)\rvert$ and $\mathrm{typ}(c) = \mathrm{typ}(d)$
	there exists a $\sigma \in G$ with $\sigma(c) = d$.

If the group of all automorphisms of $\mathcal{G}$ is flag-transitive, chamber-transitive, incidence-transitive or vertex-transitive, then we say that $\mathcal{G}$ is **flag-transitive**, **chamber-transitive**, **incidence-transitive** or **vertex-transitive**, respectively.

The emphasis of the present paper is on geometries that are not vertex-transitive, and which we will call **intransitive**. Therefore, we first have a look how one can describe such a geometry group-theoretically.

Definition 2.5 (Coset Pregeometry) Let I be a set and let $(T_i)_{i\in I}$ be a family of sets. Also, let G be a group and let $(G^{t,i})_{t\in T_i, i\in I}$ be a family of subgroups of G. Then

$$(\sqcup_{i\in I, t\in T_i} G/G^{t,i}, *, \mathrm{typ})$$

with $\mathrm{typ}(G^{t,i}) = i$ and

(Cos) $gG^{t,i} * hG^{s,j}$ if and only if $gG^{t,i} \cap hG^{s,j} \neq \emptyset$ and either $i \neq j$ or $(t,i) = (s,j)$

is a pregeometry over I, the **coset pregeometry of** G with respect to $(G^{t,i})_{t\in T_i, i\in I}$. Since the type function is completely determined by the indices, we also denote the coset pregeometry of G with respect to $(G^{t,i})_{t\in T_i, i\in I}$ by

$$((G/G^{t,i})_{t\in T_i, i\in I}, *).$$

The family $(G^{t,i})_{t\in T_i, i\in I}$ forms a lounge. If $|T_i| = 1$ for all $i \in I$, then we write G_i instead of $G^{t,i}$. The family $(G_i)_{i\in I}$ forms a chamber of the coset geometry, called the **base chamber**.

Certainly, any coset pregeometry with $|T_i| = 1$ for all $i \in I$, which means nothing else than being vertex-transitive, is incidence-transitive. Indeed, if $gG_i \cap hG_j \neq \emptyset$, then choose $a \in gG_1 \cap hG_j$. It follows $aG_i = gG_i$ and $aG_j = hG_j$ and therefore the automorphism a^{-1} maps the incident pair gG_i, hG_j onto the incident pair G_i, G_j.

Note that the residue of a coset pregeometry in general is not a coset pregeometry. The following lemma describes a situation in which it in fact is a coset pregeometry.

Lemma 2.6 (inspired by Buekenhout/Cohen [7])
The incidence-transitive coset pregeometry $\mathcal{G} = ((G/G_i)_{i\in I}, *)$ of G with respect to $(G_i)_{i\in I}$, satisfies the following properties.

(i) *For each $J \subsetneq I$, there is a natural injective homomorphism*

$$\alpha_J : ((G_J/G_{J\cup\{i\}})_{i\in I\setminus J}, *) \to \mathcal{G}_{\{G_j|j\in J\}}$$

of geometries over $I \setminus J$ given by

$$\alpha_J(aG_{J\cup\{i\}}) = aG_i$$

for $a \in G_J$, $i \in I \setminus J$.

(ii) *Given $J \subsetneq I$, the homomorphism α_J is surjective if and only if, for all $i \in I \setminus J$, we have*

$$\bigcap_{j\in J}(G_jG_i) = G_JG_i;$$

(iii) *Let $J \subsetneq I$. If $\alpha_{J\cup\{i\}}$ is surjective for all $i \in I$, then α_J^{-1} is a homomorphism, i.e., α_J is an isomorphism. In particular, if α_J is surjective for all $J \subsetneq I$, then α_J is an isomorphism for all $J \subsetneq I$.*

Proof. (i) Since $G_{J\cup\{i\}} \subseteq G_i$ and $a \in aG_i \cap G_j$ for all $i \in I \setminus J$, $j \in J$, $a \in G_J$, the map α_J is well defined. Suppose $aG_{J\cup\{i\}} \cap bG_{J\cup\{k\}} \neq \emptyset$. Then also $aG_i \cap bG_k \neq \emptyset$, so α_J is indeed a homomorphism. Suppose that $a, b \in G_J$ satisfy $\alpha_J(aG_{J\cup\{i\}}) = \alpha_J(bG_{J\cup\{i\}})$. Then $aG_i = bG_i$, so that $b^{-1}a \in G_i$. On the other hand, $b^{-1}a \in G_J$, so $b^{-1}a \in G_{J\cup\{i\}}$ whence $aG_{J\cup\{i\}} = bG_{J\cup\{i\}}$. This shows that α_J is injective.

(ii) Suppose that α_J is surjective. If $x \in \bigcap_{j\in J}(G_jG_i)$ for some $i \in I \setminus J$, then xG_i is an element of $\mathcal{G}$ incident to $\{G_j \mid j \in J\}$, so that we can find $x' \in G_J$ with $\alpha_J(x'G_{j\cup\{i\}}) = xG_i$. Then $x'G_i = xG_i$, so $x \in x'G_i \subseteq G_JG_i$, proving $\bigcap_{j\in J}(G_jG_i) = G_JG_i$. The converse is equally straightforward.

(iii) Fix $J \subseteq I$ and suppose that $\alpha_{J\cup\{i\}}$ is surjective for each $i \in I$. We need to show that α_J^{-1} is a homomorphism. If $|I\backslash J|$ has cardinality one, then there is nothing to show. Let xG_i, yG_j, where $i, j \in I \setminus J$, $x, y \in G_J$, be incident elements of the residue $\mathcal{G}_{\{G_k|k\in J\}}$ in $\mathcal{G}$ of the flag $\{G_k \mid k \in J\}$, cf. (i). Then $y^{-1}x \in G_jG_i \cap G_J$. But the surjectivity of $\alpha_{J\cup\{j\}}$ and (ii) yield

$$G_jG_i \cap G_J \subseteq G_jG_i \cap G_JG_i = G_jG_i \cap \bigcap_{k\in J} G_kG_i = \bigcap_{k\in\{j\}\cup J} G_kG_i = G_{\{j\}\cup J}G_i$$

whence

$$G_jG_i \cap G_J \subseteq (G_{\{j\}\cup J}G_i) \cap G_J = G_{\{j\}\cup J}G_{\{i\}\cup J}$$

so that

$$y^{-1}x \in G_{\{j\}\cup J}G_{\{i\}\cup J},$$

proving that $yG_{\{j\}\cup J}$ and $xG_{\{i\}\cup J}$ are incident elements of $((G_J/G_{\{i\}\cup J})_{i\in I\setminus J}, *)$. Hence (iii). $\square$

It is also possible to derive a relation between the transitivity of a coset pregeometry and the fact that its residues are coset pregeometries.

Lemma 2.7 (inspired by Buekenhout/Cohen [7])
Let $\mathcal{G} = ((G/G_i)_{i \in I}, *)$ be an incidence-transitive coset pregeometry of G over I. Let $k \geq 3$ be finite and smaller than or equal to $|I|$. For each $J \subseteq I$ of rank at most k, assume the group G is transitive on the set of flags of $\mathcal{G}$ of type J. Then for each $J \subseteq I$ of rank at most $k-1$ the homomorphism α_J is bijective and for each $J \subseteq I$ of rank at most $k-2$ the homomorphism α_J is an isomorphism.

Proof. Let $J \subseteq I$ be of rank at most $k-1$ and let aG_i be an element of the residue $\mathcal{G}_{\{G_j | j \in J\}}$. Then

$$\{aG_i\} \cup \{G_j \mid j \in J\}$$

is a flag of $\mathcal{G}$ of rank at most k, so by the assumption on the transitivity of G there is

$$g \in G_J = \bigcap_{j \in J} G_j$$

with $g^{-1}a \in G_i$, whence $aG_i = gG_i$. We obtain

$$aG_i = gG_i = \alpha_J(gG_{J \cup \{i\}}).$$

Therefore, α_J is surjective, and hence bijective, cf. Lemma 2.6(i). The claim now follows from Lemma 2.6(iii). □

Similar to the characterizations of vertex-transitivity there exist a large number of group-theoretic characterizations of various geometric properties of coset geometries, see e.g. [7]. The following one, the characterization of connectivity, is an easy but crucial observation for studying amalgams.

Theorem 2.8 (inspired by Buekenhout/Cohen [7])
Let $I \neq \emptyset$. The coset pregeometry $((G/G^{t,i})_{t \in T_i, i \in I}, *)$ is connected if and only if

$$G = \langle G^{t,i} \mid i \in I, t \in T_i \rangle.$$

Proof. Suppose that $\mathcal{G}$ is connected. Take $i \in I$ and $t \in T_i$. If $a \in G$, then there is a path

$$1G^{t,i}, a_0G^{t_0,i_0}, a_1G^{t_1,i_1}, a_2G^{t_2,i_2}, \ldots, a_mG^{t_m,i_m}, aG^{t,i}$$

connecting the elements $1G^{t,i}$ and $aG^{t,i}$ of $\mathcal{G}$. Now

$$a_kG^{t_k,i_k} \cap a_{k+1}G^{t_{k+1},i_{k+1}} \neq \emptyset,$$

so

$$a_k^{-1}a_{k+1} \in G^{t_k,i_k}G^{t_{k+1},i_{k+1}}$$

for $k = 0, \ldots, m-1$. Hence

$$a = (1^{-1}a_0)(a_0^{-1}a_1)\cdots(a_{m-1}^{-1}a_m)(a_m^{-1}a) \in G^{t,i}G^{t_0,i_0}\cdots G^{t_{m-1},i_{m-1}}G^{t_m,i_m}G^{t,i},$$

and so $a \in \langle G^{t,i} \mid i \in I, j \in T_i \rangle$. The converse is obtained by reversing the above argument. The only difficulty that can occur is that $g_1G^{t_1,i}$ and $g_2G^{t_2,i}$ are not incident, even if $g_1G^{t_1,i} \cap g_2G^{t_2,i} \neq$

$\emptyset$. This can be remedied by including some coset $gG^{t,j}$, $j \neq i$, between $g_1 G^{t_1,i}$ and $g_2 G^{t_2,i}$ into the chain of incidences, where $g \in g_1 G^{t_1,i} \cap g_2 G^{t_2,i}$. □

Now we turn to the question which pregeometries actually are coset pregeometries. Stroppel gave the answer in [27]. To this end let us introduce the notion of the sketch of a pregeometry.

Definition 2.9 (Sketch) Let $\mathcal{G} = (X, *, \mathrm{typ})$ be a pregeometry over I, let G be a group of automorphisms of $\mathcal{G}$, and let $W \subset X$ be a set of G-orbit representatives of X. We write

$$W = \bigcup_{i \in I} W_i$$

with $W_i \subseteq \mathrm{typ}^{-1}(i)$. The **sketch of $\mathcal{G}$ with respect to G and W** is the coset geometry

$$((G/G_w)_{w \in W_i, i \in I}, *').$$

Recall that two actions

$$\phi : G \to \mathrm{Aut}\ M \quad \text{and} \quad \phi' : G \to \mathrm{Aut}\ M'$$

are said to be **equivalent** if there is an isomorphism $\psi : M \to M'$ such that $\psi \circ \phi(g) \circ \psi^{-1} = \phi'(g)$ for each $g \in G$ or, equivalently, $\psi \circ \phi(g) = \phi'(g) \circ \psi$ for all $g \in G$. In this case, we shall also say that M and M' are **isomorphic G-sets**.

Theorem 2.10 (Stroppel's reconstruction theorem [27])
Let $\mathcal{G} = (X, *, \mathrm{typ})$ be a pregeometry over I and let G be a group of automorphisms of $\mathcal{G}$. For each $i \in I$ let

$$w_1^i, \ldots, w_{t_i}^i$$

be G-orbit representatives of the elements of type i of $\mathcal{G}$ such that

(i) $W := \bigcup_{i \in I} \{w_1^i, \ldots, w_{t_i}^i\}$ *is a lounge and,*

(ii) *if $V \subseteq W$ is a flag, the action of G on the pregeometry over $\mathrm{typ}(V)$ consisting of all elements of the G-orbits x^G, $x \in V$, is incidence-transitive.*

Then the bijection Φ between the sketch of $\mathcal{G}$ with respect to G and W and the pregeometry $\mathcal{G}$ given by

$$gG_{w_j^i} \mapsto gw_j^i$$

is an isomorphism between pregeometries and an isomorphism between G-sets. □

For a vertex-transitive group G, the previous theorem is just the isomorphism theorem of incidence-transitive pregeometries, see [7].

The geometry consisting of the G-orbits x^G of elements of some fixed maximal flag $V \subseteq W$ as in (ii) of the theorem is called the **orbit geometry for** $(\mathcal{G}, G, V)$.

2.2 Fundamental group and simple connectivity

Definition 2.11 (Fundamental group) Let $\mathcal{G}$ be a connected pregeometry. A path of length k in the geometry is a sequence of elements $(x_0, \ldots, x_k)$ such that x_i and x_{i+1} are incident, $0 \leq i \leq k-1$. A **cycle** based at an element x is a path in which $x_0 = x_k = x$. Two paths based at the same vertex are **homotopically equivalent** if one can be obtained from the other via the following operations (called **elementary homotopies**):

(i) inserting or deleting a repetition (i.e., a cycle of length 1),

(ii) inserting or deleting a return (i.e., a cycle of length 2), or

(iii) inserting or deleting a triangle (i.e., a cycle of length 3).

The equivalence classes of cycles based at an element x form a group under the operation induced by concatenation of cycles. This group is called the **fundamental group** of $\mathcal{G}$ and denoted by $\pi_1(\mathcal{G}, x)$.

A cycle based at x that is homotopically equivalent to the trivial cycle (x) is called **null-homotopic**. Every cycle of length 1, 2, or 3 is null-homotopic.

Definition 2.12 (Covering) Suppose $\mathcal{G}$ and $\widehat{\mathcal{G}}$ are two connected geometries over the same type set and suppose $\phi : \widehat{\mathcal{G}} \to \mathcal{G}$ is a **homomorphism** of geometries, i.e., ϕ preserves the types and sends incident elements to incident elements. A surjective homomorphism ϕ between connected geometries $\widehat{\mathcal{G}}$ and $\mathcal{G}$ is called a **covering** if and only if for every nonempty flag $\widehat{F}$ in $\widehat{\mathcal{G}}$ the mapping ϕ induces an isomorphism between the residue of $\widehat{F}$ in $\widehat{\mathcal{G}}$ and the residue of $F = \phi(\widehat{F})$ in $\mathcal{G}$. Coverings of a geometry correspond to the usual topological coverings of the flag complex. It is well-known and easy to see that a surjective homomorphism ϕ between connected geometries $\widehat{\mathcal{G}}$ and $\mathcal{G}$ is a covering if and only if for every element $\widehat{x}$ in $\widehat{\mathcal{G}}$ the map ϕ induces an isomorphism between the residue of $\widehat{x}$ in $\widehat{\mathcal{G}}$ and the residue of $x = \phi(\widehat{x})$ in $\mathcal{G}$. If ϕ is an isomorphism, then the covering is said to be **trivial**.

Consider the geometry via its colored incidence graph and recall the following results from the theory of simplicial complexes.

Theorem 2.13 (Chapter 8 of Seifert/Threlfall [24])
Let $\mathcal{G}$ be a connected geometry and let x be an element of $\mathcal{G}$. The geometry $\mathcal{G}$ does not admit any nontrivial covering if and only if $\pi_1(\mathcal{G}, x)$ is trivial. □

A geometry satisfying the equivalent conditions in the previous theorem is called **simply connected**.

The following construction can also be found in Chapter 8 of [24].

Definition 2.14 (Fundamental cover) Let Γ be a connected graph and let x be some vertex of Γ. The **fundamental cover $\widehat{\Gamma}$ of Γ based at** x is defined as follows: The vertices of $\widehat{\Gamma}$ are the homotopy classes of paths of Γ based at x where two vertices $[\gamma_1]$ and $[\gamma_2]$ of $\widehat{\Gamma}$ are adjacent if and only if $\left[\gamma_1^{-1}\gamma_2\right] = [t_1 t_2]$ where t_1 is the terminal vertex of γ_1 and t_2 is the terminal vertex of γ_2.

Definition 2.15 (Universal covering) Let Γ and $\widehat{\Gamma}$ be connected graphs and let $x \in \Gamma$, $\widehat{x} \in \widehat{\Gamma}$ be vertices. A covering

$$\pi : \widehat{\Gamma} \to \Gamma$$

mapping $\widehat{x}$ onto x is called **universal** if, for any covering

$$\alpha : \Gamma_1 \to \Gamma \quad \text{and any} \quad x_1 \in \alpha^{-1}(x),$$

there exists a unique covering map

$$\beta : \widehat{\Gamma} \to \Gamma_1$$

with $\pi = \alpha \circ \beta$ and $\beta(\widehat{x}) = x_1$.

$$\begin{tikzcd}
(\widehat{\Gamma}, \widehat{x}) \arrow[r, "\beta"] \arrow[dr, "\pi"'] & (\Gamma_1, x_1) \arrow[d, "\alpha"] \\
 & (\Gamma, x)
\end{tikzcd}$$

Theorem 2.16 (Chapter 8 of Seifert/Threlfall [24])
Let Γ be a connected graph, let x be a vertex of Γ, and let $\widehat{\Gamma}$ be the fundamental cover of Γ based at x. Then the fundamental covering $\pi : \widehat{\Gamma} \to \Gamma$ is universal. □

2.3 Amalgams

Definition 2.17 (Amalgam) An **amalgam of groups** $\mathcal{A}$ over a finite set $I = \{0, 1, \ldots, n\}$ and associated nonempty sets J_i, $i \in I$, is a family of groups $(G_{j,i})_{j \in J_i, i \in I}$ with monomorphisms, called **identifications**,

$$\phi_{j_i,i}^{j_{i+1},i+1} : G_{j_i,i} \to G_{j_{i+1},i+1}$$

for some (j_i, i) and $(j_{i+1}, i+1)$ such that for each $G_{j_i,i}$ there exist identifications whose composition embeds $G_{j_i,i}$ into some $G_{j_n,n}$.

Example 2.18 An amalgam with $I = \{0, 1, 2\}$, $J_0 = \{1, 2\}$, $J_1 = \{1, 2, 3, 4\}$, $J_0 = \{1, 2, 3, 4\}$

can be depicted in the following diagram. The identification maps are given by arrows.

$$\begin{tikzcd}
 & G_{1,1} \arrow[r, "\phi^{1,2}_{1,1}"] \arrow[ddr] & G_{1,2} \\
 & G_{2,1} \arrow[r] \arrow[dr] & G_{2,2} \\
G_{1,0} \arrow[uur, "\phi^{1,1}_{1,0}"] \arrow[ddr] & & G_{3,2} \\
G_{2,0} \arrow[uur, "\phi^{2,1}_{2,0}"] \arrow[r] \arrow[dr] & G_{3,1} \arrow[r] \arrow[uur] & G_{4,2} \\
 & G_{4,1} \arrow[uur, "\phi^{3,2}_{4,1}"] \arrow[ur, "\phi^{4,2}_{4,1}"'] &
\end{tikzcd}$$

Note that the definition of an amalgam does not imply

$$\phi^{3,2}_{2,1} \circ \phi^{2,1}_{2,0} = \phi^{3,2}_{4,1} \circ \phi^{4,1}_{2,0}$$

in the above example.

Two amalgams $\mathcal{A}$ and $\mathcal{B}$ are **similar** if they share the same set I, the same sets J_i and if for all (j_i, i) and $(j_{i+1}, i+1)$ the identification ${}_{\mathcal{A}}\phi^{j_{i+1},i+1}_{j_i,i}$ exists if and only if the identification ${}_{\mathcal{B}}\phi^{j_{i+1},i+1}_{j_i,i}$ exists, i.e., if they can be depicted by the same diagram.

Definition 2.19 (Homomorphism) Let $\mathcal{A} = (G_{j,i})_{j,i}$ and $\mathcal{B} = (H_{j,i})_{j,i}$ be similar amalgams. A map $\psi : \sqcup\mathcal{A} \to \sqcup\mathcal{B}$ will be called an **amalgam homomorphism from $\mathcal{A}$ to $\mathcal{B}$** if

(i) for every $i \in I$ and $j \in J_i$ the restriction of ψ to $G_{j,i}$ is a homomorphism from $G_{j,i}$ to $H_{j,i}$ and

(ii) $\psi \circ {}_{\mathcal{A}}\phi^{j_{i+1},i+1}_{j_i,i} = {}_{\mathcal{B}}\phi^{j_{i+1},i+1}_{j_i,i} \circ \psi_{|G_{j_i,i}}$ in case the respective identifications exist.

If ψ is bijective and its inverse map ψ^{-1} is also an amalgam homomorphism, then ψ is called an **amalgam isomorphism**. An **automorphism** of $\mathcal{A}$ is an isomorphism of $\mathcal{A}$ onto itself. As usual, the automorphisms of $\mathcal{A}$ form the **automorphism group**, $\mathrm{Aut}(\mathcal{A})$.

Definition 2.20 (Quotient, cover) An amalgam $\mathcal{B} = (H_{j,i})_{j,i}$ is a **quotient** of the amalgam $\mathcal{A} = (G_{j,i})_{j,i}$ if there is an amalgam homomorphism π from $\mathcal{A}$ to $\mathcal{B}$ such that the restriction of π to any $G_{j,n}$ maps $G_{j,n}$ onto $H_{j,n}$. The map $\pi : \sqcup\mathcal{A} \to \sqcup\mathcal{B}$ is called a **covering**, $\mathcal{A}$ is called a **cover** of $\mathcal{B}$. Two coverings $(\mathcal{A}_1, \pi_1)$ and $(\mathcal{A}_2, \pi_2)$ of $\mathcal{A}$ are called **equivalent** if there is an isomorphism ψ of $\mathcal{A}_1$ onto $\mathcal{A}_2$, such that $\pi_1 = \pi_2 \circ \psi$.

Notice that a covering $\pi : \sqcup\mathcal{A} \to \sqcup\mathcal{B}$ between amalgams need not map $G_{j,i}$ surjectively onto $H_{j,i}$ for $i \neq n$.

Definition 2.21 (Completion) Let $\mathcal{A}$ be an amalgam. A pair (G, π) consisting of a group G and a map $\pi : \sqcup\mathcal{A} \to G$ is called a **completion** of $\mathcal{A}$, and π is called a **completion map**, if

(i) for all $i \in I$ and $j \in J_i$ the restriction of π to $G_{j,i}$ is a homomorphism of G_i to G;

(ii) $\pi_{|G_{j_{i+1},i+1}} \circ \phi^{j_{i+1},i+1}_{j_i,i} = \pi_{|G_{j_i,i}}$ if the corresponding identification exist; and

(iii) $\pi(\sqcup\mathcal{A})$ generates G.

A completion is called **faithful** if for each $i \in I$ and $j \in J_i$ the restriction of π to $G_{j,i}$ is injective.

Coming back to Example 2.18, the definition of a completion does require that

$$\pi_{|G_{3,2}} \circ \phi^{3,2}_{2,1} \circ \phi^{2,1}_{2,0} = \pi_{|G_{3,2}} \circ \phi^{3,2}_{4,1} \circ \phi^{4,1}_{2,0},$$

although by definition of an amalgam we do not necessarily have

$$\phi^{3,2}_{2,1} \circ \phi^{2,1}_{2,0} = \phi^{3,2}_{4,1} \circ \phi^{4,1}_{2,0}.$$

Proposition 2.22
Let $\mathcal{A} = (G_{j,i})_{j,i}$ be an amalgam of groups, let $F(\mathcal{A}) = \langle (u_g)_{g\in\mathcal{A}} \rangle$ be the free group on the elements of $\mathcal{A}$ and let

$$S_1 = \{u_x u_y = u_z, \text{ whenever } xy = z \text{ in some } G_{j,i}\}$$

and

$$S_2 = \{u_x = u_y, \text{ whenever } \phi(x) = y \text{ for some identification } \phi\}$$

be relations for F. Then for each completion (G, π) of $\mathcal{A}$ there exists a unique group epimorphism

$$\widehat{\pi} : \mathcal{U}(\mathcal{A}) \to G$$

with $\pi = \widehat{\pi} \circ \psi$ where

$$\mathcal{U}(\mathcal{A}) = \langle (u_g)_{g\in\mathcal{A}} \mid S_1, S_2 \rangle \text{ and } \psi : \sqcup\mathcal{A} \to \mathcal{U}(\mathcal{A}) : g \mapsto u_g.$$

$$\begin{tikzcd} \sqcup\mathcal{A} \arrow[r, "\psi"] \arrow[dr, "\pi"'] & \mathcal{U}(\mathcal{A}) \arrow[d, "\widehat{\pi}"] \\ & G \end{tikzcd}$$

Proof. The map $\mathcal{A}$ to $\mathcal{U}(\mathcal{A})$ given by $\psi : g \mapsto u_g$ turns the group $\mathcal{U}(\mathcal{A})$ into a completion of $\mathcal{A}$. If (G, π) is an arbitrary completion of $\mathcal{A}$ then the map

$$\widehat{\pi} : u_g \mapsto \pi(g)$$

leads to a group epimorphism $\widehat{\pi}$ from $\mathcal{U}(\mathcal{A})$ to G because

$$\widehat{\pi}(u_g u_h) = \widehat{\pi}(u_{gh}) = \pi(gh) = \pi(g)\pi(h) = \widehat{\pi}(u_g)\widehat{\pi}(u_h)$$

if u_{gh} exists; otherwise define

$$\widehat{\pi}(u_g u_h) := \pi(g)\pi(h) = \widehat{\pi}(u_g)\widehat{\pi}(u_h).$$

Clearly, $\widehat{\pi}$ is uniquely determined by the requirement

$$\pi(g) = (\widehat{\pi} \circ \psi)(g) = \widehat{\pi}(u_g).$$

$\square$

Definition 2.23 (Universal Completion) Let $\mathcal{A} = (G_{j,i})_{j,i}$ be an amalgam of groups. Then

$$\psi : \sqcup\mathcal{A} \to \mathcal{U}(\mathcal{A}) : g \mapsto u_g$$

for $\mathcal{U}(\mathcal{A})$ as in Proposition 2.22 is called the **universal completion of** $\mathcal{A}$. The amalgam $\mathcal{A}$ **collapses** if $\mathcal{U}(\mathcal{A}) = 1$

Example 2.24 (inspired by [22]) Consider the groups

$$\begin{aligned} G_1 &= \left\langle y, z \mid y^{-1}zy = z^2 \right\rangle, \\ G_2 &= \left\langle z, x \mid z^{-1}xz = x^2 \right\rangle, \\ G_3 &= \left\langle x, y \mid x^{-1}yx = y^2 \right\rangle, \end{aligned}$$

which are nontrivial and pairwise isomorphic. Let $\mathcal{A}$ be the amalgam given by G_1, G_2, G_3 and the intersections

$$\begin{aligned} G_1 \cap G_2 &= \langle z \rangle \cong \mathbb{Z}, \\ G_1 \cap G_3 &= \langle y \rangle \cong \mathbb{Z}, \\ G_2 \cap G_3 &= \langle x \rangle \cong \mathbb{Z} \end{aligned}$$

where the identification maps are given by the inclusion maps. Then $\mathcal{U}(\mathcal{A}) = 1$, so $\mathcal{A}$ collapses.

$$\begin{tikzcd}
 & \langle z \rangle \arrow[r] \arrow[dr] & G_1 \\
1 \arrow[ur, dotted] \arrow[r, dotted] \arrow[dr, dotted] & \langle y \rangle \arrow[ur] \arrow[dr] & G_2 \\
 & \langle x \rangle \arrow[r] \arrow[ur] & G_3
\end{tikzcd}$$

It does not make any difference whether or not we add the identification of the trivial group on the left hand side, as any completion map identifies the different neutral elements of all groups anyway, since the restrictions have to be group homomorphisms.

Notice that if $\mathcal{B}$ is a quotient of $\mathcal{A}$ then $U(\mathcal{B})$ is isomorphic to a factor group of $U(\mathcal{A})$. In particular, if $\mathcal{B}$ does not collapse then neither does $\mathcal{A}$. Also, an amalgam $\mathcal{A}$ admits a faithful completion if and only if its universal completion is faithful.

Definition 2.25 (Amalgams for transitive geometries) Suppose $\mathcal{G}$ is a geometry and $G \leq \mathrm{Aut}\,\mathcal{G}$ is an incidence-transitive group. Corresponding to $\mathcal{G}$ and G and some maximal flag F, there is an amalgam $\mathcal{A} = \mathcal{A}(\mathcal{G}, G, F)$, the **amalgam of parabolics with respect to $\mathcal{G}$, G, F**, defined as the family $(G_E)_{E \subseteq F}$, where G_E denotes the stabilizer of $E \subseteq F$ in G, together with the natural inclusions as identification maps. In case G is flag-transitive, the amalgam $\mathcal{A}$ is independent (up to conjugation) of the choice of F.

For example, let $\mathcal{G}$ be a rank four geometry with a flag p, l, π, Π. Then the amalgam of parabolics looks as follows:

$$\begin{tikzcd}
 & & G_{p,l} \arrow[dr] \arrow[ddr] & \\
 & G_{p,l,\pi} \arrow[ur] \arrow[r] \arrow[dr] & G_{p,\pi} \arrow[r] \arrow[ddr] & G_p \\
G_{p,l,\pi,\Pi} \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & G_{p,l,\Pi} \arrow[uur] \arrow[dr] \arrow[ddr] & G_{l,\pi} \arrow[r] \arrow[dr] & G_l \\
 & G_{p,\pi,\Pi} \arrow[uur] \arrow[r] \arrow[ddr] & G_{p,\Pi} \arrow[uur] \arrow[dr] & G_\pi \\
 & G_{l,\pi,\Pi} \arrow[uur] \arrow[r] \arrow[dr] & G_{l,\Pi} \arrow[uur] \arrow[r] & G_\Pi \\
 & & G_{\pi,\Pi} \arrow[uur] \arrow[ur] &
\end{tikzcd}$$

If $|I| = n$ is finite and $k < n$ the amalgam $\mathcal{A}_{(k)} = \mathcal{A}_{(k)}(\mathcal{G}, G, F)$ is the subamalgam of $\mathcal{A}$ consisting of all parabolics of rank less or equal k. It is called the **amalgam of rank k parabolics**. Of course, $\mathcal{A}_{(n-1)} = \mathcal{A}$.

More generally, for F as above suppose $\mathcal{W} \subseteq 2^F$ such that $2^F \ni U' \supset U \in \mathcal{W}$ implies $U' \in \mathcal{W}$, i.e., $\mathcal{W}$ is a subset of the power set of F that is closed under passing to supersets. A set $\mathcal{W} \subset 2^F$ with those properties is called a **shape**. The **amalgam of shape $\mathcal{W}$ with respect to $\mathcal{G}$, G, F** is the family $(G_U)_{U \in \mathcal{W}}$, where G_U is the stabilizer of $U \in \mathcal{W}$ in G, with the natural inclusion maps as identification maps. It is denoted by $\mathcal{A}_{\mathcal{W}}(\mathcal{G}, G)$.

Definition 2.26 (Amalgams for intransitive geometries) Suppose $\mathcal{G} = (X, *, \mathrm{typ})$ is a geometry over I, the group G is a group of automorphisms of $\mathcal{G}$, and for each $i \in I$ let $w_1^i, \ldots, w_{t_i}^i$ be G-orbit representatives of the elements of type i of $\mathcal{G}$ such that

(i) $W := \bigcup_{i \in I} \{w_1^i, \ldots, w_{t_i}^i\}$ is a lounge and,

(ii) if $V \subseteq W$ is a flag, the action of G on the pregeometry over I consisting of all elements of the G-orbits x^G, $x \in V$, is incidence-transitive.

Then the amalgam $\mathcal{A} = \mathcal{A}(\mathcal{G}, G, W)$ is defined as the family $(G_U)_{U \subseteq W \text{ a flag}}$, where G_U denotes the stabilizer of $U \subseteq W$ in G with the natural inclusion maps as identification maps.

For example, let $\mathcal{G}$ be a rank three geometry with W equal to p, q, l, π. Then the amalgam of parabolics looks as follows:

$$\begin{tikzcd}
 & G_{p,l} \arrow[r] \arrow[ddr] & G_p \\
 & G_{q,l} \arrow[r] \arrow[dr] & G_q \\
G_{p,l,\pi} \arrow[uur] \arrow[r] \arrow[ddr] & G_{p,\pi} \arrow[uur] \arrow[dr] & G_l \\
G_{q,l,\pi} \arrow[uur] \arrow[r] \arrow[dr] & G_{q,\pi} \arrow[uur] \arrow[r] & G_\pi \\
 & G_{l,\pi} \arrow[uur] \arrow[ur] &
\end{tikzcd}$$

If $|I| = n$ is finite and $k < n$ the amalgam $\mathcal{A}_{(k)} = \mathcal{A}_{(k)}(\mathcal{G}, G, W)$ is the subamalgam of $\mathcal{A}$ consisting of all parabolics of rank less or equal k. It is called the **amalgam of rank k parabolics**. Of course, $\mathcal{A}_{(n-1)} = \mathcal{A}$.

More generally, for W as above suppose $\mathcal{W} \subseteq 2^W$ with the properties that each $U \in \mathcal{W}$ is a flag and if $U' \subset W$ is a flag with $U' \supset U \in \mathcal{W}$, then also $U' \in \mathcal{W}$, i.e., $\mathcal{W}$ is a subset of the power set of W consisting of flags that is closed under passing to superflags. A set $\mathcal{W} \subset 2^W$ with those properties is called a **shape**. The **amalgam of shape $\mathcal{W}$ for** $(\mathcal{G}, G, W)$ is defined on the family $(G_U)_{U \in \mathcal{W}}$ with the natural inclusion maps as identification maps. It is denoted by $\mathcal{A}_{\mathcal{W}}(\mathcal{G}, G, W)$.

3 Theory of intransitive flipflop geometries

We now use the foregoing notions, definitions and basic results to develop some theory of intransitive flipflop geometries, that results in criteria to conclude that certain completions of certain amalgams are universal.

Theorem 3.1 (Fundamental theorem of geometric covering theory)
Let $\mathcal{G} = (X, *, \mathrm{typ})$ be a connected geometry over I of rank at least three and let G be a group of automorphisms of $\mathcal{G}$. For each $i \in I$ let

$$w_1^i, \ldots, w_{t_i}^i$$

be G-orbit representatives of the elements of type i of $\mathcal{G}$ such that

(i) *$W := \bigcup_{i \in I} \{w_1^i, \ldots, w_{t_i}^i\}$ is a lounge and,*

(ii) *if $V \subseteq W$ is a flag, the action of G on the pregeometry over $\mathrm{typ}(V)$ consisting of all elements of the G-orbits x^G, $x \in V$, is incidence-transitive and pennant-transitive.*

Let $\mathcal{A} = \mathcal{A}(\mathcal{G}, G, W)$ be the amalgam of parabolics. Then the coset pregeometry

$$\widehat{\mathcal{G}} = ((\mathcal{U}(\mathcal{A})/G_{w_j^i})_{1 \leq j \leq t, i \in I}, *)$$

is a simply connected geometry that admits a universal covering $\pi : \widehat{\mathcal{G}} \to \mathcal{G}$ induced by the natural epimorphism $\mathcal{U}(\mathcal{A}) \to G$. Moreover, $\mathcal{U}(\mathcal{A})$ is of the form $\pi_1(\mathcal{G}).G$.

Proof. The completion

$$\phi : \sqcup \mathcal{A} \to G$$

and, thus, the completion

$$\widehat{\phi} : \sqcup \mathcal{A} \to \mathcal{U}(\mathcal{A})$$

is injective. Therefore the natural epimorphism

$$\psi : \mathcal{U}(\mathcal{A}) \to G$$

induces an isomorphism between the amalgam $\widehat{\phi}(\mathcal{A})$ inside $\mathcal{U}(\mathcal{A})$ and the amalgam $\phi(\mathcal{A})$ inside G. Hence the epimorphism $\psi : \mathcal{U}(\mathcal{A}) \to G$ induces a quotient map between pregeometries

$$\pi : \widehat{\mathcal{G}} = ((\mathcal{U}(\mathcal{A})/G_{w_j^i})_{i \in I, 1 \leq j \leq t_i}, *) \to ((G/G_{w_j^i})_{i \in I, 1 \leq j \leq t_i}, *).$$

The latter coset pregeometry is isomorphic to $\mathcal{G}$ by the Reconstruction Theorem 2.10. Notice that $\mathcal{U}(\mathcal{A})$ acts on $\mathcal{G} \cong ((G/G_{w_j^i})_{i \in I, 1 \leq j \leq t_i}, *)$ via

$$gG_{w_j^i} \mapsto \psi(u) gG_{w_j^i} \quad \text{for} \quad u \in \mathcal{U}(\mathcal{A}).$$

We want to prove that this quotient map actually is a covering map. The pregeometry $\widehat{\mathcal{G}}$ is connected by Theorem 2.8, because $\mathcal{U}(\mathcal{A})$ is generated by $\widehat{\phi}(\mathcal{A})$. Our goal is to apply Lemma 2.7 in order to establish the isomorphism of the residues. By hypothesis (ii) we can assume that $\mathcal{G}$, and hence $\widehat{\mathcal{G}}$, is incidence-transitive. Then the group $\mathcal{U}(\mathcal{A})$ is pennant-transitive on $\widehat{\mathcal{G}}$. For, let (a, b, c) and (x, y, z) be flags of type J for some subset J of I of cardinality three. Then, by incidence-transitivity of $\mathcal{U}(\mathcal{A})$ on $\widehat{\mathcal{G}}$, we can assume $a = x$ and $b = y$. By pennant-transitivity of G on $\mathcal{G}$ there exists an element u of $\mathcal{U}(\mathcal{A})$ mapping $(\pi(a) = \pi(x), \pi(b) = \pi(y), \pi(c))$ onto $(\pi(a) = \pi(x), \pi(b) = \pi(y), \pi(z))$. This element u is contained in $G_a = \psi^{-1}(G_{\pi(a)}) \cong G_{\pi(a)}$. By Lemma 2.7 and using the incidence-transitivity of $\widehat{\mathcal{G}}$ and of $\mathcal{G}$ the map π induces a bijection between the residue $\widehat{\mathcal{G}}_a$ and the residue $\mathcal{G}_{\pi(a)}$, so the element u maps $(a = x, b = y, c)$ onto $(a = x, b = y, z)$. Hence $\mathcal{U}(\mathcal{A})$ is pennant-transitive on $\widehat{\mathcal{G}}$. Another application of Lemma 2.7, this time using the pennant-transitivity of $\widehat{\mathcal{G}}$ and $\mathcal{G}$, implies that $\pi : \widehat{\mathcal{G}} \to \mathcal{G}$ induces isomorphisms between the residues of flags of rank one. So the map $\pi : \widehat{\mathcal{G}} \to \mathcal{G}$ indeed is a covering of pregeometries. Since $\mathcal{G}$ actually is a geometry the pregeometry $\widehat{\mathcal{G}}$ is also a geometry.

Now we want to show that the covering

$$\pi : \widehat{\mathcal{G}} \to \mathcal{G}$$

induced by the canonical map $\mathcal{U}(\mathcal{A}) \to G$ is universal. Denote the fundamental cover of $\mathcal{G}$ at some vertex w_j^i of W by $\mathcal{G}_0$ and let

$$\phi : \mathcal{G}_0 \to \mathcal{G}$$

be the corresponding covering map. If $\widehat{w}^i_j \in \pi^{-1}(w^i_j)$, $\overline{w}^i_j \in \phi^{-1}(w^i_j)$, we will achieve the universality of π by showing that $\pi = \phi \circ \alpha$ for a unique isomorphism

$$\alpha : \widehat{\mathcal{G}} \to \mathcal{G}_0$$

with $\alpha(\widehat{w}^i_j) = \overline{w}^i_j$.

$$\begin{tikzcd}
(\widehat{\mathcal{G}}, \widehat{w}^i_j) \arrow[r, "\pi"] \arrow[d, "\alpha"'] & (\mathcal{G}, w^i_j) \\
(\mathcal{G}_0, \overline{w}^i_j) \arrow[ur, "\phi"'] &
\end{tikzcd}$$

The simple connectivity of $\widehat{\mathcal{G}}$ then is implied by the universal property. For $g \in G_{w^i_j}$ define an automorphism

$$\widehat{g}^{(j,i)} : \mathcal{G}_0 \to \mathcal{G}_0 : \Pi_1(\mathcal{G}, w^i_j) \ni [\gamma] \mapsto [g(\gamma)]\,.$$

The latter is also a homotopy class of paths in $\mathcal{G}$ starting at w^i_j, because $g \in G_{w^i_j}$ stabilizes w^i_j. The fundamental cover $\mathcal{G}_0$ of $\mathcal{G}$ based at w^i_j is isomorphic to the fundamental cover $\mathcal{G}_1$ of $\mathcal{G}$ based at some arbitrary $w^{i'}_{j'} \in W$. Therefore we can define automorphisms on $\mathcal{G}_0$ using the automorphisms on $\mathcal{G}_1$ coming from elements $g \in G_{w^{i'}_{j'}}$. To this end fix a maximal flag $V \subseteq W$ containing w^i_j. Let $y \in V$ be incident to to both w^i_j and $w^{i'}_{j'}$ and for $g \in G_{w^{i'}_{j'}}$ define an automorphism

$$\widehat{g}^{(j',i')} : \mathcal{G}_0 \to \mathcal{G}_0 : ([\gamma]) \mapsto \left[w^i_j, y, w^{i'}_{j'}, g(y), g(\gamma)\right].$$

Since, for a different choice $y' \in V$ incident to both w^i_j and $w^{i'}_{j'}$, the cycles (y, y', w^i_j, y) and $(y, y', w^{i'}_{j'}, y)$ are null-homotopic, the automorphism $\widehat{g}^{(j',i')}$ does not depend on the particular choice of $y \in V$. In particular, if $w^{i'}_{j'} \in V$, we can choose $y = w^{i'}_{j'}$ or $y = w^i_j$.

Also, for incident $w^{i'}_{j'}$ and $w^{i''}_{j''}$, let y be an element of V incident to w^i_j, $w^{i'}_{j'}$ and $w^{i''}_{j''}$. Since the cycles $(y, w^{i'}_{j'}, w^{i''}_{j''}, y)$ and $(g(y), w^{i'}_{j'}, w^{i''}_{j''}, g(y))$ are null-homotopic, for $g \in G_{w^{i'}_{j'}} \cap G_{w^{i''}_{j''}}$ we have

$$\left[w^i_j, y, w^{i'}_{j'}, g(y), g(\gamma)\right] = \left[w^i_j, y, w^{i''}_{j''}, g(y), g(\gamma)\right]$$

and so

$$\widehat{g}^{(j',i')} = \widehat{g}^{(j'',i'')}.$$

Hence

$$\widehat{\ } : \sqcup\mathcal{A} \to \widehat{G} := \left\langle \widehat{\sqcup\mathcal{A}} \right\rangle \leq \mathrm{Aut}\ \mathcal{G}_0$$

is a completion map from $\mathcal{A}$ to $\widehat{G}$. If $\widehat{g}_1^{-1}\widehat{g}_2$ acts trivially on $\widehat{\mathcal{G}}_0$, then $g_1^{-1}g_2$ acts trivially on $\mathcal{G}$, thus $g_1 = g_2$, as G acts faithfully on $\mathcal{G}$. Therefore $\widehat{\ }$ embeds $\mathcal{A}$ in $\widehat{G}$.

The geometry $\mathcal{G}_0$ together with the group $\widehat{G}$ of automorphisms satisfies the hypothesis of the Reconstruction Theorem 2.10, so the geometry $\mathcal{G}_0$ is isomorphic to the coset pregeometry $((\widehat{G}/G_{w^i_j})_{i\in I, 1\leq j\leq t_i}, *)$. The natural epimorphsim $\widehat{G} \to G$ induces a covering map from $\mathcal{G}_0$ onto $\mathcal{G}$. Moreover, the natural epimorphism $\mathcal{U}(\mathcal{A}) \to \widehat{G}$ yields a quotient map $\widehat{\mathcal{G}} \to \mathcal{G}_0$. Since $\mathcal{G}_0$

is universal by Theorem 2.16 and therefore simply connected, this quotient map is a uniquely determined isomorphism. Hence the covering $\pi : \mathcal{G} \to \mathcal{G}$ is universal.

It remains to establish the structure of $\widehat{G} \cong \mathcal{U}(\mathcal{A})$ to be of the form $\pi_1(\mathcal{G}).G$. However, this is evident by Theorem 2.16. $\square$

Corollary 3.2 (Tits' lemma)
Let $\mathcal{G} = (X, *, \mathrm{typ})$ be a geometry over I and let G be a group of automorphisms of $\mathcal{G}$. For each $i \in I$ let

$$w_1^i, \ldots, w_{t_i}^i$$

be G-orbit representatives of the elements of type i of $\mathcal{G}$ such that

(i) *$W := \bigcup_{i \in I} \{w_1^i, \ldots, w_{t_i}^i\}$ is a lounge and,*

(ii) *if $V \subseteq W$ is a flag, the action of G on the pregeometry over $\mathrm{typ}(V)$ consisting of all elements of the G-orbits x^G, $x \in V$, is incidence-transitive and pennant-transitive.*

Let $\mathcal{A}(\mathcal{G}, G, W)$ be the amalgam of parabolics of $\mathcal{G}$ with respect to G and W. The geometry $\mathcal{G}$ is simply connected if and only if the canonical epimorphism

$$\mathcal{U}(\mathcal{A}(\mathcal{G}, G, W)) \to G$$

is an isomorphism. $\square$

Theorem 3.3
Let $\mathcal{G} = (X, *, \mathrm{typ})$ be a geometry over some finite set I and let G be a group of automorphisms of $\mathcal{G}$. For each $i \in I$ let

$$w_1^i, \ldots, w_{t_i}^i$$

be G-orbit representatives of the elements of type i of $\mathcal{G}$ such that

(i) *$W := \bigcup_{i \in I} \{w_1^i, \ldots, w_{t_i}^i\}$ is a lounge and,*

(ii) *if $V \subseteq W$ is a flag, the action of G on the pregeometry over $\mathrm{typ}(V)$ consisting of all elements of the G-orbits x^G, $x \in V$, is flag-transitive.*

Let $\mathcal{W} \subseteq 2^W$ be a shape, assume that for each flag $U \in 2^W \backslash \mathcal{W}$ the residue $\mathcal{G}_U$ is simply connected, and let $\mathcal{A}(\mathcal{G}, G, W)$ and $\mathcal{A}_{\mathcal{W}}(\mathcal{G}, G, W)$ be the amalgam of maximal parabolics respectively the amalgam of shape $\mathcal{W}$ of $\mathcal{G}$ with respect to G and W. Then

$$G = \mathcal{U}(\mathcal{A}_{\mathcal{W}}(\mathcal{G}, G, W)).$$

In particular, if $\emptyset \notin \mathcal{W}$, we have

$$G = \mathcal{U}(\mathcal{A}(\mathcal{G}, G, W)) = \mathcal{U}(\mathcal{A}_{\mathcal{W}}(\mathcal{G}, G, W)).$$

Proof. We will proceed by induction on the number of flags in the set $2^W \backslash \mathcal{W}$. If the set of flags contained in $2^W \backslash \mathcal{W}$ is empty, then $\emptyset \subset \mathcal{W}$, so the amalgam $\mathcal{A}_{\mathcal{W}}(\mathcal{G}, G, W)$ contains the stabilizer in G of the empty flag, i.e., G. Hence $G = \mathcal{U}(\mathcal{A}_{\mathcal{W}}(\mathcal{G}, G, W))$. If there exists a flag in $2^W \backslash \mathcal{W}$, then the empty flag is also contained in $2^W \backslash \mathcal{W}$, because by definition the shape $\mathcal{W}$ is closed under taking superflags. Hence in that case $\mathcal{G}$ is simply connected and by Corollary 3.2 we have $G = \mathcal{U}(\mathcal{A}(\mathcal{G}, G, W))$. We will now prove that $\mathcal{U}(\mathcal{A}(\mathcal{G}, G, W)) = \mathcal{U}(\mathcal{A}_{\mathcal{W}}(\mathcal{G}, G, W))$.

If the empty flag is the only flag contained in $2^W \backslash \mathcal{W}$, then $\mathcal{A}(\mathcal{G}, G, W) = \mathcal{A}_{\mathcal{W}}(\mathcal{G}, G, W)$, so their universal completions coincide. If there exists a nonempty flag in $2^W \backslash \mathcal{W}$, then there also exists a (nonempty) flag U in $2^W \backslash \mathcal{W}$ such that $\mathcal{W}' := \{U\} \cup \mathcal{W}$ is a shape. Then $\mathcal{A}_{\mathcal{W}'}(\mathcal{G}, G, W) = \mathcal{A}_{\mathcal{W}}(\mathcal{G}, G, W) \cup G_U$. By connectivity of $\mathcal{G}_U$, the group G_U is a completion of the amalgam $\mathcal{A}(\mathcal{G}_U, G_U, W_U)$, where

$$W_U := W \cap \operatorname{typ}^{-1}(I \backslash \operatorname{typ}(U)).$$

As $\mathcal{G}_U$ is simply connected, we even have

$$G_U = \mathcal{U}(\mathcal{A}(\mathcal{G}_U, G_U, W_U)).$$

Since $\mathcal{A}(\mathcal{G}_U, G_U, W_U) \subseteq \mathcal{A}_{\mathcal{W}}(\mathcal{G}, G, W)$, we have

$$\begin{aligned} \mathcal{U}(\mathcal{A}_{\mathcal{W}}(\mathcal{G}, G, W)) &= \mathcal{U}(\mathcal{A}_{\mathcal{W}}(\mathcal{G}, G, W) \cup \mathcal{U}(\mathcal{A}(\mathcal{G}_U, G_U, W_U))) \\ &= \mathcal{U}(\mathcal{A}_{\mathcal{W}}(\mathcal{G}, G, W) \cup G_U) \\ &= \mathcal{U}(\mathcal{A}_{\mathcal{W}'}(\mathcal{G}, G, W)). \end{aligned}$$

Hence, by induction, we have $\mathcal{U}(\mathcal{A}_{\mathcal{W}}(\mathcal{G}, G, W)) = \mathcal{U}(\mathcal{A}(\mathcal{G}, G, W))$, finishing the proof. □

Corollary 3.4
Let $\mathcal{G} = (X, *, \operatorname{typ})$ be a geometry over some finite set I, let G be a group of automorphisms of $\mathcal{G}$, for each $i \in I$ let

$$w_1^i, \ldots, w_{t_i}^i$$

be G-orbit representatives of the elements of type i of $\mathcal{G}$ such that

(i) $W := \bigcup_{i \in I} \{w_1^i, \ldots, w_{t_i}^i\}$ *is a lounge and,*

(ii) *if $V \subseteq W$ is a flag, the action of G on the geometry $\operatorname{typ}(W)$ consisting of all elements of the G-orbits x^G, $x \in V$, is flag-transitive.*

Let $k \leq |I|$, assume that all residues of rank greater or equal k with respect to subsets of W are simply connected, and let $\mathcal{A}(\mathcal{G}, G, W)$ and $\mathcal{A}_{(k)}(\mathcal{G}, G, W)$ be the amalgam of maximal parabolics respectively rank k parabolics of $\mathcal{G}$ with respect to G and W. Then

$$G = \mathcal{U}(\mathcal{A}(\mathcal{G}, G, W)) = \mathcal{U}(\mathcal{A}_{(k)}(\mathcal{G}, G, W)).$$

□

4 Intransitive geometries: an example

4.1 Some standard techniques

In this subsection, we collect some general results on simple connectivity and null-homotopic cycles that have been established in recent papers dealing with simple connectivity of flag-transitive geometries.

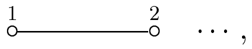
A **geometric cycle** in the geometry $\mathcal{G}$ is a cycle completely contained in the residue $\mathcal{G}_x$ of some element x.

Proposition 4.1 (Lemma 3.2 of [5])
Every geometric cycle is null-homotopic. □

Corollary 4.2 (Lemma 3.3 of [5])
If two cycles based at the same element are obtained from one another by inserting or erasing a geometric cycle then they are homotopic. □

Definition 4.3 (Basic diagram) Let $\mathcal{G}$ be a geometry over the set I. Let $i, j \in I$, then we define $i \sim j$ if there exists a flag f of cotype $\{i, j\}$ such that the residue of f is a geometry containing two elements that are not incident. Then the graph $(I, \sim)$ is called the **basic diagram of** $\mathcal{G}$.

Let $\mathcal{G}$ be a geometry with basic diagram



i.e., the node 1 has a unique neighbor in the basic diagram of $\mathcal{G}$. Then the 1-**graph** (also called the **collinearity graph**) of $\mathcal{G}$ is the graph whose vertices are the elements of type 1, where two such elements are adjacent if they are incident with a common element of type 2.

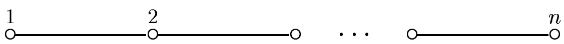
Definition 4.4 (Direct sum of pregeometries) Let $\mathcal{G} = (X, *, \mathrm{typ})$, $\mathcal{G}' = (X', *', \mathrm{typ}')$ be pregeometries over I and I'. The **direct sum**

$$\mathcal{G} \oplus \mathcal{G}'$$

is a pregeometry over $I \sqcup I'$

- whose element set is $X \sqcup X'$,
- whose type function is $\mathrm{typ} \cup \mathrm{typ}'$ and
- whose incidence relation is the symmetric relation $*_\oplus$ with $*_\oplus|_{X \times X} = *$ and $*_\oplus|_{X' \times X'} = *'$ and $*_\oplus|_{X \times X'} = X \times X'$, i.e., elements of X are incident with elements of X'.

Lemma 4.5 (Lemma 5.1 of [13])
Let $\mathcal{G}$ be a geometry of rank $n \geq 3$ with basic diagram



and assume that for each element x of type n the 1*-graph of $\mathcal{G}_x$ is connected. Furthermore, suppose that if the residue $\mathcal{G}_x$ of some element x has a disconnected diagram falling into the two connected components Δ_1 and Δ_2, then $\mathcal{G}_x$ is equal to the direct sum*

$$_{\mathrm{typ}(\Delta_1)}\mathcal{G}_x \oplus {}_{\mathrm{typ}(\Delta_2)}\mathcal{G}_x.$$

Then every cycle of $\mathcal{G}$ based at some element of type 1 *or* 2 *is homotopically equivalent to a cycle passing exclusively through elements of type* 1 *or* 2. □

Lemma 4.6 (Lemma 7.2 of [13])
Assume that $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$ with $\mathcal{G}_1$ connected of rank at least two. Then $\mathcal{G}$ is simply connected.
□

4.2 Generalities about orthogonal spaces

Let $n \geq 1$ and let V be an $(n+1)$-dimensional vector space over some field $\mathbb{F}$ of characteristic distinct from 2 endowed with some nondegenerate symmetric bilinear form $f = (\cdot,\cdot)$. By

$$\mathcal{G}_A^{\mathrm{orth}} = \mathcal{G}_A^{\mathrm{orth}}(n, \mathbb{F}, f)$$

we denote the pregeometry on the proper subspaces of V that are nondegenerate with respect to $(\cdot,\cdot)$ with symmetrized containment as incidence and the vector space dimension as the type.

Arbitrary fields of characteristic not two

We will be using standard terminology. In particular, each finite-dimensional vector space over some finite field admits two isometry classes of nondegenerate quadratic forms, one called **hyperbolic** (also **positive** or **of plus type**), the other called **elliptic** (also **negative** or **of minus type**).

Recall the following rules for determining the type of an orthogonal sum of nondegenerate orthogonal spaces over a finite field:

$$\begin{aligned} + \oplus + &= +, \\ + \oplus - &= -, \\ - \oplus - &= +. \end{aligned}$$

The names *hyperbolic* and *elliptic* are a generalization of the classical usual incidence-theoretic meaning: if a nondegenerate subspace of even dimension $2n \geq 2$ intersects the null-set of a quadratic form in a quadric with Witt index n or $n-1$, respectively, then the subspace is hyperbolic or elliptic, respectively. We extend this as follows. If a one-space takes only square values or non square values, respectively, with respect to the quadratic form, then this one-space is hyperbolic or elliptic, respectively. Now these assignments of hyperbolic and elliptic, together with the above rules, determine the plus/minus type of all nondegenerate subspaces (including the whole space and the zero space).

Theorem 4.7
The pregeometry $\mathcal{G}_A^{\mathrm{orth}}(n, \mathbb{F}, f)$ is a geometry.

Proof. We have to prove that each flag can be embedded in a flag of cardinality n. To this end let $F = \{x_1, \ldots, x_t\}$ be a flag of $\mathcal{G}_A^{\mathrm{orth}}$. We can assume that the nondegenerate subspace x_1 of V has dimension one. Indeed, if it has not, then we can find a nondegenerate one-dimensional subspace x_0 of x_1 and study the flag $F' = F \cup \{x_0\}$ instead. Now observe that the residue of the nondegenerate one-dimensional subspace x_1 is isomorphic to $\mathcal{G}_A^{\mathrm{orth}}(n-1, \mathbb{F}, f')$ for some induced form f' via the map that sends an element U of the residue of x_1 to $U \cap x_1^{\perp}$. Hence induction applies. □

Lemma 4.8
If l is a line and p is a point not on l, then there are at most two points of $\mathcal{G}_A^{\mathrm{orth}}$ on l which are not collinear to p. In other words, if $\mathbb{F}$ is the field $\mathbb{F}_q$ of q elements, there exist at least $q-3$ points on l collinear to p.

Proof. Let U be the 3-space $\langle a, l \rangle$ and let $W = U \cap a^{\perp}$. The space W has rank at least one as U has rank at least two. Hence there are at most two singular points on W and, thus, there are at least $q-1$ nondegenerate lines in U through a. The line l has at most two singular points, so at least $q-3$ of the nondegenerate lines in W through a intersect l is a nonsingular point. □

Proposition 4.9
Let $n \geq 3$ or $n = 2$ and $|\mathbb{F}| \geq 5$. Then the collinearity graph of $\mathcal{G}_A^{\mathrm{orth}}(n, \mathbb{F}, f)$ has diameter two.

Proof. If $n \geq 3$, then the dimension of the vector space V is at least 4. Fix two points $\langle a \rangle$ and $\langle b \rangle$ which are not collinear, i.e., the space $\langle a, b \rangle$ is singular with respect to $(\cdot, \cdot)$. However $\langle a, b \rangle$ is a two-dimensional subspace of V which is not totally singular, because (a, a) and (b, b) are distinct from zero. Therefore the radical of $\langle a, b \rangle$ is a one-dimensional space. The dimension of $\langle a, b \rangle^{\perp}$ is greater or equal to 2. Consequently, the orthogonal complement of $\langle a, b \rangle$ contains a point, say $\langle c \rangle$. Consider the two two-dimensional subspaces $\langle a, c \rangle$ and $\langle b, c \rangle$. Since $\langle a \rangle$ and $\langle b \rangle$ are perpendicular to $\langle c \rangle$, both $\langle a, c \rangle$ and $\langle b, c \rangle$ are lines. The distance between $\langle a \rangle$ and $\langle c \rangle$ is one and so is the distance between $\langle c \rangle$ and $\langle b \rangle$. Thus the distance between $\langle a \rangle$ and $\langle b \rangle$ is two. Certainly $\mathcal{G}_A^{\mathrm{orth}}$ contains a pair of noncollinear points, so we have proved the claim for $n \geq 3$.

If $n = 2$, let $\langle a \rangle$ and $\langle b \rangle$ be two arbitrary points in V. If the space $l = \langle a, b \rangle$ is a line then the distance between $\langle a \rangle$ and $\langle b \rangle$ is one. Otherwise pick a point $\langle \tilde{a} \rangle$ in $\langle a \rangle^{\perp}$. The space $\langle a, \tilde{a} \rangle$ is a line and the point $\langle b \rangle$ is not on $\langle a, \tilde{a} \rangle$. The point $\langle b \rangle$ is collinear with at least two points on $\langle a, \tilde{a} \rangle$ by Lemma 4.8. Pick one of these points, say the point $\langle c \rangle$. We have established that the distance between $\langle a \rangle$ and $\langle b \rangle$ is two. □

Corollary 4.10
Let $n \geq 2$ and $|\mathbb{F}| \geq 5$. Then $\mathcal{G}_A^{\mathrm{orth}}(n, \mathbb{F}, f)$ is residually connected. □

It is shown in [2] that, if $n \geq 3$ and $\mathbb{F}$ not equal to $\mathbb{F}_3$ or $\mathbb{F}_5$, then the geometry $\mathcal{G}_A^{\mathrm{orth}}(n, \mathbb{F}, f)$ is simply connected. If the field $\mathbb{F}$ is quadratically closed, then $\mathcal{G}_A^{\mathrm{orth}}(n, \mathbb{F}, f)$ is flag-transitive and one can apply Corollary 3.2 (Tits' lemma) to obtain presentations of flag-transitive groups of automorphisms of that geometry, see [2]. Also, in some cases like for real closed fields, it is possible to pass to suitable simply connected flag-transitive parts of $\mathcal{G}_A^{\mathrm{orth}}(n,\mathbb{F},f)$ in order to obtain presentations of groups of automorphisms.

Finite fields of characteristic not two

For a finite field $\mathbb{F}$ however, no flag-transitive part of $\mathcal{G}_A^{\mathrm{orth}}(n,\mathbb{F},f)$ is known to be simply connected, so we deal with intransitive geometries instead. The main tool for our proof of simple connectivity is the following lemma. It is clear that it would fail for transitive geometries as, roughly speaking, one loses half the points when passing to a transitive geometry.

Lemma 4.11
Let $n \geq 2$, let $\mathbb{F}$ be a finite field of odd order q, let p be a point of $\mathcal{G}_A^{\mathrm{orth}}(n,\mathbb{F},f)$, let l be an elliptic line such that $\langle p,l\rangle$ is a nondegenerate plane, and let m be a hyperbolic line such that $\langle p,m\rangle$ is a nondegenerate plane. Then there exist at least $\frac{q-1}{2}$ elliptic lines through p intersecting l in a a point of $\mathcal{G}_A^{\mathrm{orth}}(n,\mathbb{F},f)$ and at least $\frac{q-5}{2}$ hyperbolic lines through p intersecting m in a point of $\mathcal{G}_A^{\mathrm{orth}}(n,\mathbb{F},f)$.

Proof. Consider the two-dimensional nondegenerate space $p^\perp \cap \langle p,l\rangle$. It contains $\frac{q+1}{2}$ or $\frac{q-1}{2}$ points of positive type and $\frac{q+1}{2}$ or $\frac{q-1}{2}$ points of $-$ type. Therefore, there exist at least $\frac{q-1}{2}$ elliptic lines through p intersecting $p^\perp \cap \langle p,l\rangle$ and, thus, also l. The claim follows as all points on an elliptic line are nondegenerate.

The number $\frac{q-5}{2} = \frac{q-1}{2} - 2$ of hyperbolic lines through p intersecting m in a point of $\mathcal{G}_A^{\mathrm{orth}}(n,\mathbb{F},f)$ is obtained in exactly the same way plus the observation that two of the hyperbolic lines through p and $p^\perp \cap \langle p,m\rangle$ could intersect m in a singular point. □

4.3 Positive form in dimension at least five

Let q be odd and let V be a vector space over $\mathbb{F}_q$ of dimension $n+1$ at least five endowed with a nondegenerate positive symmetric bilinear form f and let

$$\mathcal{G}_A^{\mathrm{orth}}(n,\mathbb{F}_q,f) = (X,*,\mathrm{typ})$$

be the pregeometry on all nondegenerate subspaces of V. Let

$$W = \{p,p',l,\pi,U,U_1,U_2,\ldots,U_t\}$$

be a lounge where p is a positive point, p' is a negative point, l is a negative line, π is a positive or negative plane, U is a positive four-dimensional subspace of V, and the U_i are arbitrary nondegenerate proper subspaces of V of dimension at least three. Let

$$(\mathcal{G}_A^{\mathrm{orth}})^W = (Y,*_{|Y\times Y},\mathrm{typ}_{|Y})$$

be a pregeometry with

$$Y = \{x \in X \mid \text{ there exists a } g \in \mathrm{SO}_{n+1}(\mathbb{F}_q,f) \text{ with } x \in g(W)\}\,.$$

Proposition 4.12
The pregeometry $(\mathcal{G}_A^{\mathrm{orth}})^W$ is a geometry of rank $|\mathrm{typ}(W)| \geq 3$ with linear diagram and a collinearity graph of diameter two. Moreover, for each element x of maximal type the collinearity graph of the residue $(\mathcal{G}_A^{\mathrm{orth}})^W_x$ is connected. Furthermore, if the residue $(\mathcal{G}_A^{\mathrm{orth}})^W_x$ of some element x has a disconnected diagram falling into the two connected components Δ_1 and Δ_2, then $\mathcal{G}_x$ is equal to the direct sum

$$_{\mathrm{typ}(\Delta_1)}(\mathcal{G}_A^{\mathrm{orth}})^W_x \oplus {}_{\mathrm{typ}(\Delta_2)}(\mathcal{G}_A^{\mathrm{orth}})^W_x .$$

Proof. To prove the statement on the collinearity graph of $(\mathcal{G}_A^{\mathrm{orth}})^W$ let p and p' be points of $(\mathcal{G}_A^{\mathrm{orth}})^W$. Then there exists an elliptic line l through p' with $\langle p, l\rangle$ nondegenerate. By Lemma 4.11 there exist $\frac{q-1}{2}$ elliptic lines through p intersecting l in a point of $(\mathcal{G}_A^{\mathrm{orth}})^W$. Since q is odd, there exists at least one, and the claim is proved. The same argument implies that the collinearity graph of the residue of an element x of maximal type, which is at least four, is connected. □

The preceding proposition allows us to apply Lemma 4.5, so we can study the collinearity graph of $(\mathcal{G}_A^{\mathrm{orth}})^W$ in order to establish the simple connectivity of $(\mathcal{G}_A^{\mathrm{orth}})^W$.

Lemma 4.13
Let $q > 7$. Then any triangle in the collinearity graph of $(\mathcal{G}_A^{\mathrm{orth}})^W$ is homotopically trivial.

Proof. Let a, b, c denote the points of a triangle. If $\langle a, b, c\rangle$ is nondegenerate, then its polar $\langle a, b, c\rangle^\perp$ contains a nondegenerate two-dimensional subspace of V and, thus, points of positive type and of negative type. Choosing a positive point p of that line if $\langle a, b, c\rangle$ is positive and choosing a negative point p of that line if $\langle a, b, c\rangle$ is negative, we obtain a positive space $\langle a, b, c, p\rangle$ containing the triangle a, b, c. Therefore that triangle is geometric, whence null-homotopic by Proposition 4.1.

Now suppose the triangle a, b, c spans a degenerate space $\langle a, b, c\rangle$ with one-dimensional radical x. Notice first that any line not passing through x is elliptic. If a, b, c are all of positive type consider an arbitrary nondegenerate four-dimensional subspace of V containing $\langle a, b, c\rangle$. That four-dimensional space necessarily is of negative type, so its polar contains a negative point p. But $\langle a, p\rangle$, $\langle b, p\rangle$, $\langle c, p\rangle$ then are elliptic lines and the three-dimensional spaces $\langle a, b, p\rangle$, $\langle b, c, p\rangle$, $\langle a, c, p\rangle$ are nondegenerate, so the original triangle a, b, c is null-homotopic. If all of a, b, c are negative points, then we can choose any positive point p on the line $\langle b, c\rangle$ such that $\langle a, p\rangle$ does not contain x. Then $\langle a, p\rangle$ is an elliptic line and we have decomposed the triangle a, b, c into two triangles in which positive points occur. If b and c are of negative type and a is of positive type we can again choose any positive point p on the line $\langle b, c\rangle$ such that $\langle a, p\rangle$ does not contain x, decomposing the triangle a, b, c into two triangles with one negative point and two positive points.

We are left with the case of one negative point, say a, and two positive points, say b and c. If neither b nor c are perpendicular to a, we can choose the point b' on $\langle a, b\rangle$ perpendicular to a, which is a positive point as it is perpendicular to the negative point on the elliptic (negative) line $\langle a, b\rangle$. Since c is not perpendicular to a, the line $\langle b', c\rangle$ does not pass through x and, thus, is elliptic. The triangle b, b', c consists of positive points only and hence is null-homotopic, so we can assume $a \perp b$ in our original triangle. The space $\langle a, b, c\rangle$ is contained in a four-dimensional nondegenerate negative space which is in turn contained in a five-dimensional nondegenerate positive space W (which may be equal to V). The space $U := \langle b, c\rangle^\perp \cap W$ is a three-dimensional negative space. As $b \perp a$ the space $\langle a, U\rangle$ equals $b^\perp \cap W$, which is a nondegenerate four-dimensional positive space. Through a there are $q+1$ tangent planes of $\langle a, U\rangle$. Moreover, in U there are $q+1$ tangent lines. If all tangent planes through a would pass through a tangent line of U, we would have that a equals the projection of c onto $\langle a, U\rangle$ with respect to the direct decomposition $\langle b\rangle \oplus \langle a, U\rangle$ of W, which would imply that a, b, c are linearly dependent. So there exists a nondegenerate plane of $\langle a, U\rangle$ through a that intersects U in a tangent line of U. Since U is a negative space tangent lines of U contain q negative points besides the radical. We have to find a point p among those q points that spans an elliptic line together with a and nondegenerate three-dimensional spaces with $\langle a, b\rangle$ and $\langle a, c\rangle$. Since $b \perp a$ and $b \perp p$, the space $\langle a, b, p\rangle$ is automatically nondegenerate if $\langle a, p\rangle$ is an elliptic line. The space $\langle a, c, p\rangle$ has a Gram matrix of the form

$$\begin{pmatrix} * & * & \alpha \\ * & * & 0 \\ \alpha & 0 & c \end{pmatrix}$$

with respect to the basis a, c, p for a nonzero constant c and a variable α. Hence there are at most two choices of p for which $\langle a, c, p\rangle$ is degenerate. Hence there exist $q - 2 - 2 - \frac{q-1}{2}$ points p on a common elliptic line with a. Indeed, there are q negative points, two of which might give rise to a nondegenerate space $\langle a, c, p\rangle$, two of which might give rise to a nongenerate space $\langle a, p\rangle$ and $\frac{q-1}{2}$ of which might span hyperbolic lines together with a. This number is positive since $q > 7$. □

Lemma 4.14
Let $q > 3$. Then any quadrangle of the collinearity graph of $(\mathcal{G}_A^{\mathrm{orth}})^W$ is homotopically trivial.

Proof. Let a, b, c, d be a quadrangle and let $l := ab$ and $m := cd$. If l and m intersect in a point e, then the quadrangle a, b, c, d decomposes into two triangles a, d, e and b, c, e.

Therefore we can assume $\langle l, m\rangle$ is four-dimensional. Our goal is to prove that the point line geometry consisting of the points of l and m and the elliptic lines in $\langle l, m\rangle$ intersecting l and m is connected. The fact that a, b, c, d is null-homotopic then follows, as any path from a to b via points on l and m and elliptic lines intersecting both l and m decomposes the quadrangle a, b, c, d into triangles. We have to consider the following five cases of possible structure for $\langle l, m\rangle$: (i) two-dimensional radical, elliptic line as complement; (ii) two-dimensional radical, hyperbolic line as complement; (iii) one-dimensional radical; (iv) nondegenerate negative space; (v) nondegenerate positive space. In the first case any line not through the radical is elliptic and there is nothing to prove. The second case cannot occur as the lines l and m are elliptic. In the third case let x denote the radical of $\langle l, m\rangle$. The planes $\langle l, x\rangle$ and $\langle m, x\rangle$ intersect in a line, s say. Denote the intersection of l and s by y and the intersection of m and s by z. All lines in $\langle l, x\rangle$ through z except s are elliptic, whence z is in the same connected component as any point on l distinct from y. By symmetry, y is in the same connected component as any point on m distinct from z. Now let p be any point on l distinct from y and consider the plane $\langle p, m\rangle$. This plane is a complement in $\langle l, m\rangle$ of x, so it is nondegenerate. By Lemma 4.11 there exist $\frac{q-1}{2}$ elliptic lines through p in $\langle p, m\rangle$. This is at least two if q is larger than three, so there exists an elliptic line through p intersecting m in a point distinct from z and, thus, the geometry consisting of the points of l and m and the elliptic lines of $\langle l, m\rangle$ intersecting l and m is connected. In case four we can apply the same argument as above by using tangent planes of the elliptic quadric containing l or m. In the fifth case the space $\langle l, m\rangle$ is an object of the geometry $(\mathcal{G}_A^{\mathrm{orth}})^W$, so the quadrangle a, b, c, d is geometric and hence, by Lemma 4.1, null-homotopic. $\square$

Lemma 4.15
Any pentagon of the collinearity graph of $(\mathcal{G}_A^{\mathrm{orth}})^W$ is homotopically trivial.

Proof. Let a, b, c, d, e be a pentagon and let $l := cd$. If $\langle a, l\rangle$ is nondegenerate, then there exist $\frac{q-1}{2}$ elliptic lines through a intersecting l, which is at least one, and if $\langle a, l\rangle$ is degenerate, then there exist q elliptic lines through a intersecting l, as in $\langle a, l\rangle$ each complement of the radical is an elliptic line. In both cases we have decomposed the pentagon a, b, c, d, e into two quadrangles. $\square$

By Proposition 4.12, the three lemmas we have proved the following theorem.

Theorem 4.16
Let $q \geq 9$. Then the geometry $(\mathcal{G}_A^{\mathrm{orth}})^W$ is simply connected. $\square$

Theorem 4.17
Let $q \geq 9$ be odd, let $n \geq 4$, let V be an $(n+1)$-dimensional vector space over $\mathbb{F}_q$ endowed with a nondegenerate positive symmetric bilinear form f. Let $\mathcal{G} = (\mathcal{G}_A^{\mathrm{orth}})^W$, let $G = \mathrm{SO}_{n+1}(\mathbb{F}_q, f)$ and let $\mathcal{A} = \mathcal{A}(\mathcal{G}, G, W)$ be the amalgam of maximal parabolics of $(\mathcal{G}_A^{\mathrm{orth}})^W$. Then $\mathcal{U}(\mathcal{A}) = \mathrm{SO}_{n+1}(\mathbb{F}_q, f)$.

Proof. This follows by Theorem 4.16 and Corollary 3.2. $\square$

4.4 Negative form in dimension at least five

Let q be odd and let V be a vector space over $\mathbb{F}_q$ of dimension $n+1$ at least five endowed with a nondegenerate negative symmetric bilinear form f and let

$$\mathcal{G}_A^{\mathrm{orth}}(n, \mathbb{F}_q, f) = (X, *, \mathrm{typ})$$

be the pregeometry on all nondegenerate subspaces of V. Let

$$W = \left\{p, p', l, \pi, U, U_1, U_2, \ldots, U_t\right\}$$

be a lounge where p is a positive point, p' is a negative point, l is a negative line, π is a positive or negative plane, U is a positive four-dimensional subspace of V, and the U_i are arbitrary nondegenerate proper subspaces of V of dimension at least three. Let

$$(\mathcal{G}_A^{\mathrm{orth}})^W = (Y, *_{|Y \times Y}, \mathrm{typ}_{|Y})$$

be a pregeometry with

$$Y = \{x \in X \mid \text{ there exists a } g \in \mathrm{SO}_{n+1}(\mathbb{F}_q, f) \text{ with } x \in g(W)\}.$$

Theorem 4.18
Let $q \geq 9$. Then the geometry $(\mathcal{G}_A^{\mathrm{orth}})^W$ is simply connected.

Proof. The proof is almost the same as the proof of Theorem 4.16, i.e., it follows by versions of Lemmas 4.13, 4.14 and 4.15. The crucial step is finding a version of the proof of Lemma 4.13 that works. This, however, simply amounts to interchanging the words *positive* and *negative* in a suitable way. The other two lemmas can be copied literally. □

Theorem 4.19
Let $q \geq 9$ be odd, let $n \geq 4$, let V be an $(n+1)$-dimensional vector space over $\mathbb{F}_q$ endowed with a nondegenerate negative symmetric bilinear form f. Let $\mathcal{G} = (\mathcal{G}_A^{\mathrm{orth}})^W$, let $G = \mathrm{SO}_{n+1}(\mathbb{F}_q, f)$ and let $\mathcal{A} = \mathcal{A}(\mathcal{G}, G, W)$ be the amalgam of maximal parabolics of $(\mathcal{G}_A^{\mathrm{orth}})^W$. Then $\mathcal{U}(\mathcal{A}) = \mathrm{SO}_{n+1}(\mathbb{F}_q, f)$. □

4.5 Negative form in dimension four

Let q be odd and let V be a vector space over $\mathbb{F}_q$ of dimension four endowed with a nondegenerate negative symmetric bilinear form f and let

$$\mathcal{G}_A^{\mathrm{orth}}(n, \mathbb{F}_q, f) = (X, *, \mathrm{typ})$$

be the pregeometry on all nondegenerate subspaces of V. Let

$$W = \{p, p', l, \pi, \pi'\}$$

be a lounge where p is a positive point, p' is a negative point, l is a negative line, π is a positive plane, and π' is a negative plane. Let

$$(\mathcal{G}_A^{\mathrm{orth}})^W = (Y, *_{|Y \times Y}, \mathrm{typ}_{|Y})$$

be a pregeometry with

$$Y = \{x \in X \mid \text{ there exists a } g \in \mathrm{SO}_{n+1}(\mathbb{F}_q, f) \text{ with } x \in g(W)\}.$$

Lemma 4.20
Let $q \geq 7$. Then any triangle in the collinearity graph of $(\mathcal{G}_A^{\mathrm{orth}})^W$ is homotopically trivial.

Proof. Let a, b, c be a triangle in a degenerate plane with one-dimensional radical p. Let π be a nondegenerate plane through ab. There are two degenerate planes through bc, namely $\langle a, b, c\rangle$ and some plane π_{bc}; likewise there are two degenerate planes $\langle a, b, c\rangle$ and π_{ac} through ac. The planes π_{ac} and π_{bc} meet π in two lines l_{ac} and l_{bc}, respectively, through a and b. Since, in π, there are at least $\frac{q-1}{2}$ elliptic lines through any nonsingular point, we find two elliptic lines l_a and l_b through a and b, respectively, distinct from l_{ac}, l_{bc}, and $\langle ab\rangle$. Let d be the intersection of l_a with l_b. The plane $\langle c, d, p\rangle$ is nondegenerate since the only degenerate plane through the tangent line cp is $\langle a, b, c\rangle$. Hence there is some point c' on cp with the property that $c'd$ is elliptic. It is now clear that, since all triangles a, b, d and a, c', d and b, c', d are contained in nondegenerate planes, that a, b, c' is null-homotopic. But the automorphism group of the quadric contains a group of order $q-1$ fixing ab pointwise, fixing p and acting transitively on the points of pc except for p and the intersection $pc \cap ab$. So we conclude that also a, b, c is null-homotopic. $\square$

Theorem 4.21
Let $q > 7$. Then $(\mathcal{G}_A^{\mathrm{orth}})^W$ is simply connected.

Proof. Case (iv) of Lemma 4.14 shows that any quadrangle of $(\mathcal{G}_A^{\mathrm{orth}})^W$ is null-homotopic and Lemma 4.15 shows that any pentagon of $(\mathcal{G}_A^{\mathrm{orth}})^W$ is null-homotopic. $\square$

Theorem 4.22
Let $q \geq 9$ be odd, let V be a four-dimensional vector space over $\mathbb{F}_q$ endowed with a positive nondegenerate form f. Let $\mathcal{G} = (\mathcal{G}_A^{\mathrm{orth}})^W$, let $G = \mathrm{SO}_4(\mathbb{F}_q, f)$ and let $\mathcal{A} = \mathcal{A}(\mathcal{G}, G, W)$ be the amalgam of maximal parabolics of $(\mathcal{G}_A^{\mathrm{orth}})^W$. Then $\mathcal{U}(\mathcal{A}) = \mathrm{SO}_4(\mathbb{F}_q, f)$. $\square$

4.6 Smaller amalgams

Theorem 4.23
Let $q \geq 9$ be odd, let $n \geq 6$, let V be an $(n+1)$-dimensional vector space over $\mathbb{F}_q$ endowed with a nondegenerate positive symmetric bilinear form f. Assume that W is a lounge containing positive and negative hyperplanes, negative hyperlines, a positive or negative codimension three space and a positive codimension four space. Let $\mathcal{G} = (\mathcal{G}_A^{\mathrm{orth}})^W$, let $G = \mathrm{SO}_{n+1}(\mathbb{F}_q, f)$ and let $\mathcal{A}_{n-2} = \mathcal{A}_{n-2}(\mathcal{G}, G, W)$ be the amalgam of rank $n-2$ parabolics of $(\mathcal{G}_A^{\mathrm{orth}})^W$. Then

$$\mathcal{U}(\mathcal{A}_{n-2}) = \mathrm{SO}_{n+1}(\mathbb{F}_q, f).$$

Proof. In view of Theorem 4.16 in order to apply Corollary 3.4, we have to prove that all residues of flags of rank one are simply connected. If the flag x of rank one is not a point of $(\mathcal{G}_A^{\mathrm{orth}})^W$, then the simple connectivity of $(\mathcal{G}_A^{\mathrm{orth}})^W_x$ follows from Theorem 4.16 or Lemma 4.6 according to whether x is a hyperplane or not. So assume x is a point. If it is a positive point, then the dual of the residue $(\mathcal{G}_A^{\mathrm{orth}})^W_x$ is simply connected by Theorem 4.16 and hence also $(\mathcal{G}_A^{\mathrm{orth}})^W_x$ is simply connected. If x is a negative point, then the hyperline in the residue $(\mathcal{G}_A^{\mathrm{orth}})^W_x$ becomes a positive hyperline, while the codimension four subspace of $(\mathcal{G}_A^{\mathrm{orth}})^W_x$ becomes negative. After dualizing $(\mathcal{G}_A^{\mathrm{orth}})^W_x$, the simple connectivity of $(\mathcal{G}_A^{\mathrm{orth}})^W_x$ follows by Theorem 4.18. $\square$

In principle, the theorem would also work for $n = 5$, but then by assumption W would have to contain a negative line and a positive codimension four space, which would be a positive line. But this would contradict the fact, that W contains a positive and a negative point, because the connecting line between those two points cannot be both positive and negative.

Theorem 4.24
Let $q \geq 9$ be odd, let $n \geq 4$, let V be an $(n+1)$-dimensional vector space over $\mathbb{F}_q$ endowed with a nondegenerate positive symmetric bilinear form f. Let

$$\mathcal{G}_A^{\mathrm{orth}}(n, \mathbb{F}_q, f) = (X, *, \mathrm{typ})$$

be the pregeometry on all nondegenerate subspaces of V. Let

$$W = \{p, p', l, \pi, U, U_1, U_2, \dots, U_t\}$$

be a lounge where p is a positive point, p' is a negative point, l is a negative line, π is a positive or negative plane, U is a positive four-dimensional subspace of V, and the U_i are arbitrary nondegenerate proper subspaces of V of dimension at least three. Let $G = \mathrm{SO}_{n+1}(\mathbb{F}_q, f)$ and let

$$(\mathcal{G}_A^{\mathrm{orth}})^W = (Y, *_{|Y\times Y}, \mathrm{typ}_{|Y})$$

be a pregeometry with

$$Y = \{x \in X \mid \text{ there exists a } g \in G \text{ with } x \in g(W)\}.$$

Let $\mathcal{W} \subset 2^W$ be a shape containing p, p', every flag of corank two, and the flag consisting of all elements of type greater or equal four. Then

$$G = \mathcal{U}(\mathcal{A}_{\mathcal{W}}(\mathcal{G}, G, W).$$

Proof. This follows from Theorem 3.4 plus Theorem 4.16 and Lemmas 3.2 and 4.6. □

A Appendix: A transitive geometry for $G_2(q)$

In the sequel we study a rank three geometry related to the split Cayley hexagon over a finite field. Its simple connectivity can be proven with methods dealing with finite quadrics as before. To be precise, we consider the group $G_2(q)$. Let $\mathsf{H}(q)$ be the associated generalized hexagon. This hexagons can be represented on a projective nondegenerate quadric $Q(6,q)$ in projective 6-space $\mathrm{PG}(6,q)$. An **ideal line** of $\mathsf{H}(q)$ is a line of $Q(6,q)$ that is not a line of $\mathsf{H}(q)$. An **ideal plane** of $\mathsf{H}(q)$ is a plane of $Q(6,q)$ that does not contain any line of $\mathsf{H}(q)$. Ideal lines and planes can also be defined only using the geometry of the hexagon $\mathsf{H}(q)$, see [32]. The rank 3 geometry that we will consider consists of the points of the split Cayley hexagon $\mathsf{H}(q)$, the ideal lines, and the ideal planes, with natural incidence. The planes of the quadric $Q(6,q)$ that contain a line pencil of the hexagon will be referred to as "degenerate" planes. Every ideal line lies in a unique degenerate plane, and in every such plane there is a unique point with the property that every line through that point in that plane is a hexagon line. We call that point the **ideal center** of the ideal line.

We will apply Lemma 4.5 in order to study cycles in the collinearity graph. Therefore, we need that the residue of an ideal plane is connected. This is true since an ideal plane in the rank 3 geometry is just a projective plane.

Lemma A.1
Let a, b, c, d be a quadrangle with the property that no two consecutive sides have ideal centers incident with the same hexagon line. Then a, b, c, d is null-homotopic, provided $q \geq 4$.

Proof. Indeed, the space $\{a,b,c\}^\perp$ is three dimensional and meets the quadric in a quadratic cone C with vertex b. Our assumptions imply that d is not collinear with b on the quadric. Hence $d^\perp$ meets C in a nondegenerate conic. Let e be any point of that conic, chosen in such a way that none of the planes abe, bce, cde, ade are degenerate (this is possible since there is a unique degenerate plane through every ideal line, and since $q \geq 4$). Then e is, on the quadric, collinear with all of a,b,c,d. If one of the lines ae, be, ce, de were a hexagon line, say ae, then ae would contain the ideal centers of ab and ad, a contradiction. Hence all lines ae, be, ce, de are ideal and all of the triangles a,b,e and b,c,e and c,d,e and a,d,e are contained in an ideal plane. Hence all these triangles are null homotopic and the claim follows. □

Two ideal lines the ideal centers of which are not incident with the same hexagon line will be called **in general position**.

Lemma A.2
Let a,b,c be a triangle in a degenerate plane. Then a,b,c is null-homotopic.

Proof. Indeed, choose a point d at hexagon-distance 4 from both a and b and opposite c. This is possible by the following argument. The points a and b are contained in a trace, say in the trace of some point d. If c is contained in the trace of d as well, then $c \in ab$ by the 2-regularity of the hexagon. In that case, however, the triangle a, b, c is geometric and hence, by Proposition 4.1, null-homotopic. Therefore we can assume that c is not contained in the trace of d, whence it is opposite d. Then the lines ad and bd are ideal and the triangle a,b,d is null-homotopic. Choose a hexagon line l through d at hexagon-distance 5 from a and b. Choose a hexagon line l' through c at hexagon-distance 5 from a (and hence also from b) and opposite l. Finally, let e be a point at hexagon-distance 3 from both l and l', and at hexagon-distance 4 from both c and d. Then the ideal lines ce and de are in general position, and so are the ideal lines de and bd; bd and bc; de and ad; ad and ac; ac and ce; bc and ce. By Lemma A.1 above, the quadrangles a,c,e,d and b,c,e,d are null-homotopic, which implies that the quadrangle a,c,b,d is null-homotopic. Since the triangle a,b,d is null-homotopic, we conclude that also the triangle a,b,c is null-homotopic. □

Lemma A.3
Every quadrangle a,b,c,d is null-homotopic.

Proof. Suppose first that the pairs $\{a,c\}$ and $\{b,d\}$ are opposite pairs of points (in the hexagon). Then the proof of Lemma A.1 applies, taking into account that we now do not have the restriction of e to be chosen such that abe, etc., is nondegenerate, but instead, we require that e is such that ae, be, ce nor de is a hexagon line. This can be achieved since this is so for at most two choices of e. Indeed, if there exist points e, e' in $\langle a,b,c,d\rangle^\perp$ such that ae and be' are hexagon lines, then $\langle a,b,e\rangle$ and $\langle a,b,e'\rangle$ are planes of the quadric, Since both planes contain hexagon lines, they are both degenerate. However, both $\langle a,b,e\rangle$ and $\langle a,b,e'\rangle$ contain the ideal line ab, which in turn is contained in a unique degenerate plane. Hence $\langle a,b,e\rangle = \langle a,b,e'\rangle$ and, thus, $e = e'$.

Hence we may assume that a and c are collinear on the quadric. If ac is ideal, then we are done by the fact that all triangles are now null-homotopic. Hence we may assume that ac is a hexagon line. Clearly, we may assume that b and d are not collinear on the quadric as otherwise a, b, c, d lie in a plane of the quadric and then ad meets bc in some point e. The triangles a, b, e and c, d, e are null-homotopic by Lemma A.2, hence the result.

Let x be a point at hexagon-distance 5 from ac and opposite all of a, b, c and d. This can be chosen as follows: consider a line l at hexagon-distance 3 from both b and d, but different from ac. Consider any point x' on l, at hexagon-distance 4 from both b and d, and then one can choose x suitable but collinear with x' (and using $q \geq 4$). Choose two lines m and n through x opposite ac. Let e and f be incident with m and n, respectively, and at hexagon-distance 4 from a and c, respectively. We claim that the pentagon a, b, c, f, e is null-homotopic. Indeed, b is, on the quadric, collinear to some point of the ideal line ef, but in the hexagon not collinear to any point of ef (as otherwise x and b are not opposite). If both eb and fb are ideal lines, then we have the null-homotopic triangles a, b, e and b, e, f and b, c, f. If e is opposite b and bf is an ideal line, then b, c, f is null-homotopic, but also a, b, f, e is null-homotopic because f is clearly opposite a, and b is opposite e by assumption; so we may apply the previous paragraph in our present proof. If both e and f are opposite b, then, likewise, we have the null-homotopic quadrangles a, b, g, e and b, c, f, g, with g a point on ef at hexagon-distance 4 from b (or, equivalently, collinear on the quadric with b). Now the pentagon a, b, c, f, e is null-homotopic Similarly, the pentagon a, d, c, f, e is null-homotopic. But this now implies that the quadrangle a, b, c, d is null-homotopic. □

Lemma A.4
Every pentagon a, b, c, d, e is null-homotopic.

Proof. Certainly, there is a point f on cd collinear on the quadric with a. If af is an ideal line, then we have subdivided our pentagon into either two null-homotopic quadrangles, or one null homotopic quadrangle and a null-homotopic triangle. So we may assume that af is a hexagon line. First we suppose that $c \neq f \neq d$. If the ideal center of cd is incident with af, then acd is a degenerate plane and hence we can find a point g in that plane such that ag, cg and dg are ideal lines. We then have subdivided our pentagon onto the null-homotopic circuits a, b, c, g and c, d, g and a, e, d, g. So we may assume that the ideal center x of cd is off af. We consider any point h on the line fx, with $f \neq h \neq x$. Then ah, dh and ch are ideal lines and we have subdivided our pentagon onto the null homotopic circuits a, b, c, h and c, d, h and a, e, d, h.

Hence we may at last assume that $f = c$ and so that ac is a hexagon line. Similarly as above, we may also assume that the ideal center x of cd is not incident with ac. We choose an arbitrary point k on cx, $c \neq k \neq x$. Then dk and ak are ideal lines. Inside the degenerate plane acx, we can easily find a point m such that am, cm and km are ideal lines. We have now subdivided our pentagon into the quadrangles a, b, c, m and c, d, k, m and a, e, d, k, and the triangle a, k, m, which are all null-homotopic.

The result follows. □

Lemma A.5
Every hexagon a, b, c, d, e, f is null-homotopic.

Proof. This is similar to the proof of Lemma A.4. We just have to add one to the girth of every circuit we considered containing e (it now also contains f). □

Theorem A.6
Let $q \geq 4$. Then the geometry $\mathcal{G}$ consisting of the points, the ideal lines and the ideal planes of the split Cayley hexagon $\mathsf{H}(\mathsf{q})$ is simply connected.

Proof. Let $a_1, a_2, \ldots, a_n$ be a circuit, with $n \geq 6$ in view of the foregoing lemmas. We prove the assertion by means of induction on n. For $n = 6$, this is the previous lemma. Now let $n \geq 7$. On the ideal line a_4a_5 there is at least one point b collinear on the quadric with a_1. If a_1b is an ideal line, then we apply induction on the circuit $a_1, b, a_5, a_6, \ldots, a_n$ (where possibly $b = a_4$), and, together with the fact that a_1, a_2, a_3, a_4, b (with possibly $a_4 = b$) is null-homotopic, this implies the result.

So we may assume that a_1b is a hexagon line. Then there is a point c with a_1c and bc ideal lines, and we can apply induction on the circuit $a_1, c, b, a_5, a_6, \ldots, a_n$ (with possibly $a_5 = b$), which, together with the fact that a_1, a_2, a_3, a_4, b, c is null-homotopic by the previous lemma, implies the assertion. □

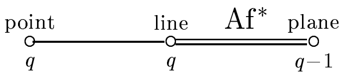
Theorem A.7
Let $q \geq 4$, let $G = G_2(q)$ and let $\mathcal{G}$ be the geometry consisting of the points, the ideal lines and the ideal planes of the split Cayley hexagon $\mathsf{H}(\mathsf{q})$. Let F be a maximal flag of $\mathcal{G}$. Then

$$G \cong \mathcal{U}(\mathcal{A}(\mathcal{G}, G, F).$$

Proof. This follows by Theorem A.6 and Lemma 3.2. □

We now describe the amalgam in more detail. The stabilizer of a point is a parabolic subgroup $G_1 := q^5 : \mathrm{GL}_2(q)$. The stabilizer of an ideal line is a group $G_2 := q^3 : \mathrm{GL}_2(q)$, and the stabilizer of an ideal plane is $G_3 := \mathrm{SL}_3(q)$. The amalgam is defined in such a way that $G_{1,2}$, which comes from the intersection $G_1 \cap G_2$ in $G_2(q)$, is isomorphic to a group of order $q^4(q-1)^2$; the other two groups $G_{2,3}$ and $G_{1,3}$ are the line and point stabilizer, respectively, in $\mathrm{SL}_3(q)$ in the natural action on a projective plane of order q. The group $G_{1,2,3}$ is a flag stabilizer in the latter.

Note that the geometry $\mathcal{G}$ has a linear diagram of the form



,

where the Af* denotes the dual of an affine generalized quadrangle. In our case, we delete a line, all points on it, and all lines concurrent with it from an orthogonal quadrangle $Q(4, q)$.

B Appendix: An intransitive geometry for $G_2(3)$

Here is another application of our new theory. In [18] Hoffman and Shpectorov study an amalgam of maximal subgroups of $\widehat{G} = \mathrm{Aut}(G_2(3))$ given by a certain choice of subgroups

$$\begin{aligned} \widehat{L} &= 2^3 \cdot \mathrm{L}_3(2) : 2, \\ \widehat{N} &= 2^{1+4}.(S_3 \times S_3), \\ M &= G_2(2) = U_3(3) : 2 \end{aligned}$$

which corresponds to an amalgam of subgroups of $G = G_2(3)$ given by

$$\begin{aligned} L &= \widehat{L} \cap G = 2^3 \cdot \mathrm{L}_3(2), \\ N &= \widehat{N} \cap G = 2^{1+4}.(3 \times 3).2, \\ M &= G_2(2) = U_3(3) : 2, \\ K &= eMe^{-1} \quad \text{for } e \in O_2(\widehat{L}) \backslash O_2(L) \end{aligned}$$

where $O_2(\widehat{L})$ denotes the largest normal subgroup of $\widehat{L}$ that is a 2-group. The groups

$$\begin{aligned} \widehat{G}_1 &= \widehat{L}, \\ \widehat{G}_2 &= \widehat{N}, \\ \widehat{G}_3 &= M \end{aligned}$$

define a flag-transitive coset geometry $\mathcal{G}$ of rank three for $\widehat{G} = \mathrm{Aut}(G_2(3))$, which is simply connected by [18]. The subgroup $G = G_2(3)$ of $\widehat{G}$ does not act flag-transitively on $\mathcal{G}$. Nevertheless, the groups

$$\begin{aligned} G^{1,1} &= L, \\ G^{1,2} &= N, \\ G^{1,3} &= M, \\ G^{2,3} &= K \end{aligned}$$

define an intransitive coset geometry of rank three for $G = G_2(3)$, which is isomorphic to $\mathcal{G}$ by [18] and, hence, simply connected. Corollary 3.2 implies that $\widehat{G}$ is the universal completion of the amalgam given by $\widehat{L}$, $\widehat{N}$ and M and their intersections as indicated in Definition 2.25 and that G is the universal completion of the amalgam given by L, N, M and K and their intersections excluding $M \cap K$ as indicated in Definition 2.26.

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