A topological Maslov index for 3-graded Lie groups

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Abstract. Motivated by the generalization of the Maslov index to tube domains and by numerous applications of related index function in infinite-dimensional situations, we describe in this paper a topologically oriented approach to an index function generalizing the Maslov index for bounded symmetric domains of tube type to a variety of infinite-dimensional situations containing in particular the class of all bounded symmetric domains of tube type in Banach spaces. The framework is that of 3-graded Banach–Lie groups and corresponding Jordan triple systems.

Introduction

Let $\mathcal{D}$ be a finite-dimensional bounded symmetric domain of tube type and S its Shilov boundary. In [CØ01] and [Cl04] J. L. Clerc and the second author have defined a function

$$\mu: S^3 \to \mathbb{Z}$$

called the *Maslov index* which is invariant under the action of the identity component $H := \mathrm{Aut}(\mathcal{D})_0$ on the set S^3 of triples in the Shilov boundary. Their index function generalizes in particular the classical Maslov index, which is obtained if $\mathcal{D}$ is the open unit ball in the space $\mathrm{Sym}_n(\mathbb{C})$ of complex symmetric matrices and $\mathrm{Aut}(\mathcal{D})_0 = \mathrm{Sp}_{2n}(\mathbb{R})$ is the symplectic group. In this case S can be identified with the set of Lagrangian subspaces of a $2n$-dimensional symplectic vector space W and the Maslov index is an integer $\tau(L_1, L_2, L_3)$ defined for L_1, L_2, and $L_3 \in S$. For the applications to boundary value problems for differential operators and corresponding index theories, it is important to allow W to be infinite-dimensional; but also for $W = \mathbb{R}^{2n}$ with the standard symplectic form, the Maslov index plays a non-trivial role, and our approach offers new insight in this case as well. In the classical situation, this means we can identify S with the set of unitary symmetric matrices.

Motivated by the generalization of the Maslov index to tube domains and by numerous applications of related index function in infinite-dimensional situations (cf. [CLM94]), we describe in this paper a topologically oriented approach to an index function generalizing the Maslov index for bounded symmetric domains of tube type to a variety of infinite-dimensional situations containing in particular the class of all bounded symmetric domains of tube type in Banach spaces.

We start with the following group theoretic setup. We consider a Banach–Lie group G endowed with an involution τ and whose Lie algebra $\mathfrak{g}$ is endowed with a 3-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ arising as the eigenspace decomposition of some $\mathrm{ad}\, E$, $E \in \mathfrak{g}_0$, and reversed by τ. We then call $(G, \mathrm{ad}\, E, \tau)$ an involutive 3-graded Lie group.

We have subgroups $G^\pm$ and G^0 of G corresponding to $\mathfrak{g}_\pm$ and $\mathfrak{g}_0$, and we thus obtain a homogeneous manifold $X := G/G^0G^-$ into which we embed the Banach space $V := \mathfrak{g}_1$ by the map $x \mapsto \exp xG^0G^-$. The involution τ and the 3-grading provide on V the structure of a Jordan triple by

$$\{x, y, z\} := \tfrac{1}{2}[[x, \tau.y], z].$$

If the operator $Q(x): y \mapsto \{x, y, x\}$ on V is invertible, we call the element x invertible and we say that $e \in V$ is a tripotent if $\{e, e, e\} = e$. We now write S for the set of invertible tripotents

in V. If S is non-empty, then τ induces an involution τ_X on X such that $S = V \cap X^\tau$ is the set of τ_X-fixed points in the open subset V of X. We make the **assumptions**
(A1) $H := G_0^\tau \subseteq G^+G^0G^-$ (where G_0^τ denotes the identity component of G^τ), and that
(A2) S is invariant under the action of H on X.

A pair $(z, w) \in V^2$ is called *quasi-invertible* if $\exp(-\tau.w)\exp z \in G^+G^0G^-$ (this can also be expressed directly in Jordan theoretic terms). For a quasi-invertible pair we defined $B_G(z, w) \in G^0$ by $\exp(-\tau.w)\exp z \in G^+B_G(z, w)^{-1}G^-$. We write $V^3_\top$ for the set of all quasi-invertible triples in V and consider the function

$$d_G\colon V^3_\top \to G^0, \quad (x, y, z) \mapsto B_G(x, y)B_G(z, y)^{-1}B_G(z, x)B_G(y, x)^{-1}B_G(y, z)B_G(x, z)^{-1}.$$

For $S^3_\top := S^3 \cap V^3_\top$ we show that $d_G(S^3_\top) \subseteq Z(G_0)^\tau$ and that the assumption
(A3) $d_G(S^3_\top) = \{\mathbf{1}\}$
is always satisfied for a quotient of the identity component G_0 of G by a discrete central elementary abelian 2-subgroup. For the group $\mathrm{GL}_2(A)$ over a hermitian Banach-$*$-algebra $(A, *)$ we only have to factor the subgroup $\{\pm\mathbf{1}\}$ (see Section II). The main goal of Section I is the definition of an index map

$$\mu_G\colon S^3_\top \to \pi_1(G^0)$$

assigning to a quasi-invertible triple in S a homotopy class of a loop in the group G^0. This map is obtained by showing that $[0, 1] \to V^3, t \mapsto (ts_1, ts_2, ts_3)$ is a path in $V^3_\top$, so that composing it with d_G yields a loop in G^0 whose homotopy class is defined to be $\mu_G(s_1, s_2, s_3)$.

We show in Section II that all infinite-dimensional bounded symmetric domains $\mathcal{D}$ of tube type are covered by our setup, where S is the corresponding "Shilov boundary". This observation builds heavily on results of W. Kaup and H. Upmeier (cf. [Up85]). If, in addition, $\mathcal{D}$ is finite-dimensional, then we can compose d_G with the determinant function $\det\colon \mathrm{GL}(V) \to \mathbb{C}^\times$ and the natural representation $\rho_V\colon G^0 \to \mathrm{GL}(V)$ to obtain a map $\det \circ \rho_V \circ d_G\colon V^3_\top \to \mathbb{C}^\times$ which leads to a map

$$\widetilde{\mu}_G\colon S^3_\top \to \pi_1(\mathbb{C}^\times) \cong \mathbb{Z}.$$

Up to a constant factor, this map is the Maslov index defined in [CØ01].

From its definition it is almost obvious that μ_G is constant on the connected components of $S^3_\top$, and in Section III we show that these connected components coincide with the orbits of H on $S^3_\top$. We further show that each orbit contains a triple of the form $(e, -e, \sigma)$ with $Q(e)\sigma = -\sigma$. In Section IV we then turn to the calculation of the index function. This is eventually reduced to the case of the group $\mathrm{SL}_2(\mathbb{C})/\{\pm\mathbf{1}\}$ by observing that $\mathrm{span}_\mathbb{R}\{e, \sigma\}$ is a Jordan sub-triple of V isomorphic to $\mathbb{C}$ with $\{x, y, z\} = x\overline{y}z$ and then using functorial properties of the index map. The outcome is the interesting result that

$$\mu_G(e, -e, \sigma) = [\chi_\sigma] \quad \text{with} \quad \chi_\sigma \in \mathrm{Hom}(\mathbb{T}, G^0), \quad \chi_\sigma(t + \mathbb{Z}) = \exp_G(\pi t[\tau.e, \sigma]).$$

In the last Section V we calculate the Maslov index for several classes of examples. If $V = A$ is a hermitian Banach-$*$-algebra and $S = U(A)$ its unitary group, then a triple $(s_1, s_2, s_3) \in S^3$ is quasi-invertible if and only if all differences $s_j - s_k$ are invertible. So our index function assigns to each such triple a loop in the group $G^0 \cong (A^\times \times A^\times)/\{\pm\mathbf{1}\}$ whose homotopy class is invariant under the action of the group $H = U_{1,1}(A, *)_0$, and each triple is conjugate to one of the form $(\mathbf{1}, -\mathbf{1}, i(\mathbf{1} - 2p))$, where p is a hermitian projection in A. Therefore the index map leads to a map

$$\pi_0(\mathrm{Idem}(A, *)) \to \pi_1(G^0), \quad [p] \mapsto [\gamma_p], \quad \text{where} \quad \mathrm{Idem}(A, *) := \{p \in A\colon p = p^2 = p^*\}$$

and $[\gamma_p]$ denotes the homotopy class of the projection loop defined by $\gamma_p(t + \mathbb{Z}) = e^{2\pi i t p}$ in $U(A)$. In this case $\mathcal{D} = U_{1,1}(A, *).0$ is the unit ball for the largest C^*-seminorm on A. This is a symmetric Banach manifold, but it is bounded if and only if A is a C^*-algebra. For complex Banach algebras the projection loop construction leads to the Bott map

$$\beta\colon K_0(A) \to K_2(A) = \varinjlim \pi_1(\mathrm{GL}_n(A)), \quad [p] \mapsto [\gamma_p]$$

and the main point in Bott periodicity is that this map is an isomorphism (cf. [Kar78]). It would be very interesting to see if there are deeper connections between our index function μ_G and topological K-theory for Banach algebras, in particular for real Banach algebras.

It is remarkable that our setup never needs that G is a complex group or that V is a complex vector space. All the results in the present paper remain valid in the real setting, hence in particular for the "Shilov boundaries" of real bounded symmetric domains, but the geometric implications for this setting will be investigated in a future paper.

Our approach to the index function μ_G via involutive 3-graded Lie groups is closely related to the geometry of inner 3-filtrations and 3-gradings developed in [BN04a], from where we use several results. To keep this paper reasonably self-contained, we included an appendix on basic results on Jordan triples used throughout and also a second appendix on the basic notions concerning inner 3-filtrations of Lie algebras. The theory in [BN04a] is algebraic, it even works over fields of positive characteristic $\neq 2,3$. Thinking of the index μ_G as a Jordan algebra version of the Bott map, it would be interesting to see if there is an algebraic variant of μ_G which is related to the Laurent polynomial constructions in the algebraic K-theory of rings.

I. The index function for quasi-invertible triples

In this section we introduce involutive 3-graded Banach–Lie groups and discuss the assumptions (A1-3) mentioned in the introduction. We shall use Cayley transforms associated to invertible tripotents to show that for each quasi-invertible triple $(s_1, s_2, s_3) \in S^3_\top$ the line segment connecting it to $(0,0,0)$ consists of quasi-invertible triples. With this information we can define the index function $\mu_G\colon S^3_\top \to \pi_1(G^0)$.

Three graded involutive Lie groups

Definition I.1. An *inner* 3*-grading* of a Lie algebra $\mathfrak{g}$ is a 3-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ for which the derivation $D \in \mathrm{der}(\mathfrak{g})$ defined by $\mathfrak{g}_j = \ker(D - j\,\mathrm{id}_\mathfrak{g})$ for $j = 1, 0, -1$, is inner. Then the elements $E \in \mathfrak{g}_0$ with $D = \mathrm{ad}\, E$ are called *grading elements.* Note that $\mathfrak{g}_{\pm 2} = \{0\}$ implies in particular that the spaces $\mathfrak{g}_\pm := \mathfrak{g}_{\pm 1}$ are abelian subalgebras of $\mathfrak{g}$.

A pair (G, D) of a Banach–Lie group G and an inner derivation $D \in \mathrm{ad}\,\mathfrak{g}$ is called a 3*-graded Lie group* if the eigenspaces $\mathfrak{g}_j := \ker(D - j\,\mathrm{id}_\mathfrak{g})$, $j = -1, 0, 1$, define a 3-grading.

A triple (G, D, τ) consisting of a 3-graded Banach–Lie group (G, D) and an involutive automorphism τ of G whose differential $\mathbf{L}(\tau)$ reverses the grading, i.e., $\mathbf{L}(\tau).\mathfrak{g}_j = \mathfrak{g}_{-j}$ for $j = -1, 0, 1$, is called an *involutive* 3*-graded Lie group.* ∎

Proposition I.2. *Let (G, D) be a 3-graded Banach–Lie group. The subgroups*

$$G^\pm := \exp \mathfrak{g}_\pm, \quad G^0 := \{g \in G\colon (\forall j)\,\mathrm{Ad}(g)\mathfrak{g}_j = \mathfrak{g}_j\} = \{g \in G\colon \mathrm{Ad}(g)D = D\,\mathrm{Ad}(g)\}$$

and $P^\pm := G^\pm G^0$ have the following properties:

(1) *$P^+ \cap P^- = G^0$, $P^\pm \cap G^\mp = \{\mathbf{1}\}$ and $P^\pm \cong G^\pm \rtimes G^0$. All these groups are complemented Lie subgroups of G.*

(2) *The multiplication map $G^+ \times G^0 \times G^- \to G$, $(x, y, z) \mapsto xyz$ is a diffeomorphism onto an open subset of G.*

(3) *$X := G/P^-$ is a homogeneous Banach manifold and the map $\mathfrak{g}_1 \to X, x \mapsto \exp xP^-$ is a diffeomorphism onto an open subset.*

(4) *The orbits of the identity component G_0 of G coincide with the connected components of X.*

(5) *For the inner 3-filtrations $\mathfrak{f}_\pm = (\mathfrak{g}_\pm, \mathfrak{g}_\pm + \mathfrak{g}_0)$ of $\mathfrak{g}$ we have $G_{\mathfrak{f}_\pm} = P^\pm$ and hence an embedding*

$$X \to \mathcal{F}, \quad gP^- \mapsto g.\mathfrak{f}_- \tag{1.1}$$

of X into the set $\mathcal{F}$ of inner 3-filtrations of $\mathfrak{g}$.

Proof. (1) Since G^0 preserves the grading of $\mathfrak{g}$, it normalizes the subgroups $G^{\pm}$, so that $P^{\pm}$ are groups.

We consider the two inner 3-filtrations

$$\mathfrak{f}_+ := (\mathfrak{g}_+, \mathfrak{g}_+ + \mathfrak{g}_0) \quad \text{and } \mathfrak{f}_- := (\mathfrak{g}_-, \mathfrak{g}_- + \mathfrak{g}_0)$$

defined by the 3-grading of $\mathfrak{g}$ (cf. Appendix B for the definitions concerning inner 3-filtrations). For a 3-filtration $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_0)$ let

$$G_{\mathfrak{f}} := \{g \in G \colon \mathrm{Ad}(g).\mathfrak{f}_0 = \mathfrak{f}_0, \mathrm{Ad}(g).\mathfrak{f}_1 = \mathfrak{f}_1\}$$

denote its stabilizer subgroup in G. Then we clearly have $P^{\pm} \subseteq G_{\mathfrak{f}_{\pm}}$.

On the other hand each element $g \in G_{\mathfrak{f}_+}$ also stabilizes the subset $\mathfrak{f}_+^{\top} = \{\mathfrak{e} \in \mathcal{F} \colon \mathfrak{e} \top \mathfrak{f}_+\}$ of all inner 3-filtrations of $\mathfrak{g}$ transversal to $\mathfrak{f}_+$. According to [BN04a, Th. 1.6(2)], the group G^+ acts transitively on the set $\mathfrak{f}_+^{\top}$ containing $\mathfrak{f}_-$. Hence there exists an element $g_+ \in G^+$ with $g.\mathfrak{f}_- = g_+.\mathfrak{f}_-$. Then $g_+^{-1}g.\mathfrak{f}_{\pm} = \mathfrak{f}_{\pm}$ implies that $g_+^{-1}g$ also preserves the 3-grading given by

$$\mathfrak{g}_+ = \mathfrak{f}_{+,1}, \quad \mathfrak{g}_- = \mathfrak{f}_{-,1} \quad \text{and} \quad \mathfrak{g}_0 = \mathfrak{f}_{+,0} \cap \mathfrak{f}_{-,0}.$$

Therefore $g_+^{-1}g \in G^0$, so that $g \in g_+G^0 \subseteq P^+$. This shows that $P^+ = G_{\mathfrak{f}_+}$ and likewise we get $P^- = G_{\mathfrak{f}_-}$. From that we obtain

$$P^+ \cap P^- = G_{\mathfrak{f}_+} \cap G_{\mathfrak{f}_-} = G^0.$$

Let $E \in \mathfrak{g}_0$ be a grading element, i.e., $\mathfrak{g}_j$ is the j-eigenspace of $\mathrm{ad}\, E$. Then we have for $x \in \mathfrak{g}_+$ the relation

$$\mathrm{Ad}(\exp x).E = e^{\mathrm{ad}\, x}.E = E - [x, E] = E + x.$$

Since this element is contained in $\mathfrak{g}_- + \mathfrak{g}_0 = \mathfrak{f}_{-,0}$ if and only if $x = 0$, we get

$$G^+ \cap P^- = G^+ \cap G_{\mathfrak{f}_-} = \{\mathbf{1}\},$$

and likewise $G^- \cap P^+ = \{\mathbf{1}\}$.

From $P^{\pm} = G_{\mathfrak{f}_{\pm}}$ we derive in particular that $P^{\pm}$ and G^0 are Lie subgroups of $\mathfrak{g}$ with the Lie algebras $\mathfrak{p}^{\pm} = \mathfrak{g}_+ + \mathfrak{g}_0$ which are the normalizers of the flags $\mathfrak{f}_{\pm}$ on the Lie algebra level ([Ne04, Lemmas IV.11, IV.12]). Clearly the Lie algebras of all these subgroups have closed complements because

$$\mathfrak{g} = \mathfrak{p}^+ \oplus \mathfrak{g}_- = \mathfrak{p}^- \oplus \mathfrak{g}_+ = \mathfrak{g}_0 \oplus (\mathfrak{g}_+ + \mathfrak{g}_-).$$

This means that they are complemented Lie subgroups.

(2) follows immediately from (1), the Inverse Function Theorem, and the fact that the map

$$(G^+ \rtimes G_0) \times G^- \to G, \quad (x, y, z) \mapsto xyz^{-1}$$

is an orbit map for a smooth action of the group $(G^+ \rtimes G_0) \times G^-$ on G.

(3) follows from (1) and (2).

(4) We know from (3) that the orbit of the base point in X under G^+ is open. Hence the orbit of a point gP^- under the group gG^+g^{-1} is open, and since all subgroups gG^+g^{-1} are contained in G_0, all orbits of G_0 in X are open. This implies that the G_0-orbits in X are the connected components.

(5) follow from the proof of (1). ∎

Lemma I.3. *For $v \in \mathfrak{g}_1$ and $w \in \mathfrak{g}_{-1}$ the following are equivalent*
(1) $\exp w \exp v \in G^+G^0G^-$.
(2) *The operators*

$$B_+(v,w) := \mathrm{id}_{\mathfrak{g}_1} + \mathrm{ad}\, v\, \mathrm{ad}\, w + \frac{1}{4}(\mathrm{ad}\, v)^2(\mathrm{ad}\, w)^2 \in \mathrm{End}(\mathfrak{g}_1)$$

and

$$B_-(w,v) := \mathrm{id}_{\mathfrak{g}_{-1}} + \mathrm{ad}\, w\, \mathrm{ad}\, v + \frac{1}{4}(\mathrm{ad}\, w)^2(\mathrm{ad}\, v)^2 \in \mathrm{End}(\mathfrak{g}_{-1})$$

are invertible.

Proof. Consider the map $\eta\colon G \to X, g \mapsto gP^-$ and identify $\mathfrak{g}_1$ with the open subset $G^+.P^- \subseteq X$. Then $\eta^{-1}(\mathfrak{g}_1) = G^+G^0G^-$. Therefore $\exp w \exp v \in G^+G^0G^-$ is equivalent to $(\exp w).v \in \mathfrak{g}_1$, and the assertion follows from [BN04a, Cor. 1.10]. ■

Definition I.4. Let (G, D, τ) be an involutive 3-graded Banach–Lie group. We also write τ for its derivative on the Lie algebra $\mathfrak{g}$. Then $\tau(\mathfrak{g}_j) = \mathfrak{g}_{-j}, j = -1, 0, 1$, and the space $V := \mathfrak{g}_+$ carries a Jordan triple structure given by

$$\{x, y, z\} := \tfrac{1}{2}[[x, \tau.y], z]$$

(Theorem A.5). Using Proposition I.2(3), we think of V as an open subset of the homogeneous space X and view X as a *conformal completion of the Jordan triple* V.

We call an element $x \in V$ *invertible* if the operator

$$Q(x)\colon V \to V, \quad y \mapsto Q(x)(y) := \{x, y, x\}$$

is invertible and write $V^\times$ for the set of invertible elements in V. For $x \in V^\times$ the *(Jordan triple) inverse* is defined by

$$x^\sharp := Q(x)^{-1}.x.$$

The elements of the set

$$S := \{x \in V^\times : x^\sharp = x\} = \{x \in V^\times : \{x, x, x\} = x\}$$

are called *involutions* or *invertible tripotents* (cf. Definition A.1). ■

Definition I.5. (a) We have seen above that the multiplication map $G^+ \times G^0 \times G^- \to G$ is a diffeomorphism onto an open subset of the group G. Therefore we have smooth maps

$$p_j\colon G^+G^0G^- \to G^j \quad \text{with} \quad g = p_+(g)p_0(g)p_-(g) \quad \text{for} \quad g \in G^+G^0G^-.$$

For $z \in \mathfrak{g}_1$ and $g \in G$ with $g \exp z \in G^+G^0G^-$ we define

$$J_G(g, z) := p_0(g \exp z) \in G^0.$$

The function J_G is called the *universal automorphy factor* of G.

(b) For $g \in G$ we put $g^* := \tau(g)^{-1}$ and for $x \in \mathfrak{g}$ we put $x^* := -\tau.x$. For $w \in \mathfrak{g}_1$ and $g = (\exp w)^* = \exp w^* \in G^-$ we then set

$$B_G(z, w) := J_G\big((\exp w)^*, z\big)^{-1} = p_0\big((\exp w)^* \exp z\big)^{-1} \in G_0$$

whenever $\exp w^* \exp z \in G^+G^0G^-$. According to Lemma I.3, this happens if and only if the *Bergman operators*

$$\begin{aligned} B(v, w) &:= B_+(v, w^*) = \mathrm{id}_V + \mathrm{ad}\, v\, \mathrm{ad}\, w^* + \frac{1}{4}(\mathrm{ad}\, v)^2(\mathrm{ad}\, w^*)^2 \\ &= \mathrm{id}_V + \mathrm{ad}\, v\, \mathrm{ad}\, w^* + \frac{1}{4}(\mathrm{ad}\, v)^2 \circ \tau \circ (\mathrm{ad}\, w)^2 \circ \tau = \mathrm{id}_V - 2v\Box w + Q(v)Q(w) \end{aligned}$$

and $B(w,v)$ are invertible. In this case the pair $(v,w) \in V^2$ is called *quasi-invertible* and we write $v \top w$ to denote quasi-invertibility. This notation is motivated by the fact that, in terms of Appendix B, quasi-invertibility of (v,w) is equivalent to $(\exp(-\tau.w)\exp v.\mathfrak{f}_-)\top\mathfrak{f}_+$, which means that the 3-filtration $\exp v.\mathfrak{f}_-$ is transversal to the 3-filtration $\exp(\tau.w).\mathfrak{f}_+ = \tau_X(\exp w.\mathfrak{f}_-)$.

(c) We write

$$V^2_\top := \{(x,y) \in V^2 : B(x,y), B(y,x) \in \mathrm{GL}(V)\}$$

for the set of *quasi-invertible pairs* in V, and $V^3_\top := \{(x,y,z) \in V^3 : (x,y),(y,z),(x,z) \in V^2_\top\}$ for the set of quasi-invertible triples. For the set S of involutions in V we put $S^2_\top := S^2 \cap V^2_\top$ and $S^3_\top := S^3 \cap V^3_\top$. We then consider the functions

$$c_G : V^3_\top \to G^0, \quad c_G(x,y,z) := B_G(x,y)B_G(z,y)^{-1}B_G(z,x)$$

and $d_G : V^3_\top \to G^0, (x,y,z) \mapsto c_G(x,y,z)c_G(x,z,y)^{-1}$ with

$$d_G(x,y,z) = B_G(x,y)B_G(z,y)^{-1}B_G(z,x)B_G(y,x)^{-1}B_G(y,z)B_G(x,z)^{-1}. \qquad \blacksquare$$

Lemma I.6. *For a quasi-invertible pair (v,w) in V and the adjoint representation $\rho_V : G^0 \to \mathrm{GL}(V)$ of G^0 on $\mathfrak{g}_1 = V$ we have $B(v,w) = \rho_V(B_G(v,w))$.*

Proof. This follows from the proof of Theorem 2.10 in [BN04a]. $\blacksquare$

Lemma I.7. *The functions B_G, J_G and d_G have the following properties:*

(1) *For $z \in V$ and $g, g' \in G$ with $g'.z, gg'.z \in V$ we have $J_G(gg',z) = J_G(g, g'.z)J_G(g',z)$. In particular $J_G(g^{-1}, g.z) = J_G(g,z)^{-1}$ for $z \in V$ and $g.z \in V$.*

(2) *If $g.z, \tau(g).w \in V$, then $B_G(g.z, \tau(g).w) = J_G(g,z)B_G(z,w)J_G(\tau(g),w)^*$.*

(3) *$B_G(w,z) = B_G(z,w)^*$ for $\exp w^* \exp z \in G^+G^0G^-$.*

(4) *$d_G(z_1,z_3,z_2) = d_G(z_1,z_2,z_3)^{-1}$.*

(5) *$d_G(z_1,z_2,z_3) = B_G(z_1,z_2)B_G(z_3,z_2)^{-1}d_G(z_3,z_1,z_2)B_G(z_3,z_2)B_G(z_1,z_2)^{-1}$.*

(6) *$d_G(g.z_1, g.z_2, g.z_3) = J_G(g,z_1)d_G(z_1,z_2,z_3)J_G(g,z_1)^{-1}$ for $g \in G^\tau$ with $g.z_j \in V$ for $j = 1,2,3$.*

(7) *For $g \in G^\tau$, $(v,w) \in V^2_\top$ and $g.(v,w) \in V^2$ we have $g.(v,w) \in V^2_\top$.*

Proof. The elementary proof of (1)-(3) can be found in [Ne99, Lemma XII.1.9].

(4) follows from

$$\begin{aligned} d_G(z_1,z_3,z_2) &= c_G(z_1,z_3,z_2)c_G(z_1,z_2,z_3)^{-1} = \big(c_G(z_1,z_2,z_3)c_G(z_1,z_3,z_2)^{-1}\big)^{-1} \\ &= d_G(z_1,z_2,z_3)^{-1}. \end{aligned}$$

(5) follows from

$$\begin{aligned} d_G(z_1,z_2,z_3) &= B_G(z_1,z_2)B_G(z_3,z_2)^{-1}B_G(z_3,z_1)B_G(z_2,z_1)^{-1}B_G(z_2,z_3)B_G(z_1,z_3)^{-1} \\ &= B_G(z_1,z_2)B_G(z_3,z_2)^{-1}\big(B_G(z_3,z_1)B_G(z_2,z_1)^{-1}B_G(z_2,z_3) \\ &\quad B_G(z_1,z_3)^{-1}B_G(z_1,z_2)B_G(z_3,z_2)^{-1}\big)B_G(z_3,z_2)B_G(z_1,z_2)^{-1} \\ &= B_G(z_1,z_2)B_G(z_3,z_2)^{-1}d_G(z_3,z_1,z_2)B_G(z_3,z_2)B_G(z_1,z_2)^{-1}. \end{aligned}$$

(6) follows from (2).

(7) From (2) we derive $B_G(g.z, .gw) = J_G(g,z)B_G(z,w)J_G(g,w)^*$, and therefore

$$B(g.z, g.w) = \rho_V(J_G(g,z))B(z,w)\rho_V(J_G(g,w)^*)$$

is invertible (Lemma I.6). $\blacksquare$

Proposition I.8. *If $S \neq \emptyset$, then the involution τ induces an involution τ_X on the homogeneous space X, and the following assertions hold:*

(1) *The fixed point set $X^\tau := \{x \in X : \tau_X(x) = x\}$ is a submanifold of X.*
(2) *With respect to the embedding $V \hookrightarrow X$ we have $S = X^\tau \cap V$.*
(3) *The group G^τ preserves the subset $X^\tau \subseteq X$ and the orbits of its identity component $H := G_0^\tau$ are the connected components of the manifold X^τ.*
(4) *For the transversality relation $\top$ of inner 3-filtrations and $\mathfrak{f} \in X^\tau$ the subgroup $\exp(\mathfrak{f}_1^\tau)$ of H acts transitively on the set $X^\tau \cap \mathfrak{f}^\top$.*

Proof. (1) In the proof of Proposition I.2 we have seen that $P^\pm$ coincide with the stabilizers of the 3-filtrations $\mathfrak{f}_\pm := (\mathfrak{g}_\pm, \mathfrak{g}_\pm + \mathfrak{g}_0)$, so that we obtain an embedding of X into the set $\mathcal{F}$ of inner 3-filtrations of $\mathfrak{g}$ by $X \to \mathcal{F}, gP^- \mapsto g.\mathfrak{f}_-$ ([BN04a, Th. 1.12]).

Suppose now that $\mathfrak{e} \in X^\tau$ and that $\mathfrak{f} \in \mathfrak{e}^\top$. Then $\tau(\mathfrak{e}_1) = \mathfrak{e}_1$, and from $\tau_X(e^{\mathrm{ad}\, x}.\mathfrak{f}) = e^{\mathrm{ad}\, \tau.x}.(\tau_X.\mathfrak{f})$ for $x \in \mathfrak{e}_1$ it follows that τ acts on the affine space $\mathfrak{e}^\top$ by an affine involution. Therefore it has a fixed point $\mathfrak{f}$. Then the affine space $\mathfrak{f}^\top \subseteq X$ is an open subset containing $\mathfrak{e}$, and on this open set, the map τ_X corresponds to the restriction $\tau|_{\mathfrak{f}_1}$. This shows that

$$X^\tau \cap \mathfrak{f}^\top = e^{\mathrm{ad}\, \mathfrak{f}_1^\tau}.\mathfrak{e}, \tag{1.2}$$

which is an affine subspace of the affine space $\mathfrak{f}^\top$. Hence X^τ carries a natural manifold structure given by the affine charts of the form $X^\tau \cap \mathfrak{f}^\top \cong \mathfrak{f}_1^\tau$.

(2) If τ_X denotes the restriction of the involution τ to X, considered as a subspace of the set $\mathcal{F}$ of inner 3-filtrations of $\mathfrak{g}$, then Proposition B.2 implies $\tau_X^{-1}(V) \cap V = V^\times$ with $\tau_X(v) = v^\sharp$ for $v \in V^\times$. From that we immediately get $X^\tau \cap V = S$.

(3) It is clear that the restriction of the action of the subgroup G^τ of G on X preserves the set X^τ. For $\mathfrak{e} \in X^\tau$ we have seen in (1) that there exists some τ-invariant $\mathfrak{f} \in \mathfrak{e}^\top$ such that $e^{\mathrm{ad}\, \mathfrak{f}_1^\tau}.\mathfrak{e}$ is a neighborhoof of $\mathfrak{e}$. Since $\exp(\mathfrak{f}_1^\tau) \subseteq H$, all orbits of H in X^τ are open, hence coincide with the connected components.

(4) is an immediate consequence of (1.2) in the proof of (1). ∎

Tripotents and partial Cayley transforms

In this subsection we introduce the partial Cayley transform C_e associated to a Jordan tripotent, following the definition of O. Loos in [Lo77].

Definition I.9. (a) Let $e \in V$ be a tripotent, $f := \tau(e)$, $h := [e, f]$ and $\mathfrak{g}_e := \mathrm{span}_{\mathbb{R}}\{h, e, f\}$. Then

$$[h, e] = 2\{e, e, e\} = 2e \quad \text{and} \quad [h, f] = \tau[\tau h, e] = -\tau[h, e] = -2\tau e = -2f,$$

so that $\mathfrak{g}_e \cong \mathfrak{sl}_2(\mathbb{R})$ is a 3-dimensional subalgebra of $\mathfrak{g}$ with $\mathfrak{g}_e^\tau = \mathbb{R}(e + f)$.

Write $p_{\mathrm{SL}_2(\mathbb{R})} : \widetilde{\mathrm{SL}}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ for the universal covering morphism of $\mathrm{SL}_2(\mathbb{R})$ and let $\widetilde{\eta}_e^G : \widetilde{\mathrm{SL}}_2(\mathbb{R}) \to G$ denote the unique homomorphism with

$$\mathbf{L}(\widetilde{\eta}_e^G)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e, \quad \mathbf{L}(\widetilde{\eta}_e^G)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = f \quad \text{and} \quad \mathbf{L}(\widetilde{\eta}_e^G)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h.$$

The kernel of $p_{\mathrm{SL}_2(\mathbb{R})}$ is annihilated by every homomorphism of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ into the unit group $B^\times$ of some Banach algebra B because it factors through a homomorphism $\mathrm{SL}_2(\mathbb{C}) \to (B_{\mathbb{C}})^\times$, where $B_{\mathbb{C}}$ is the complexification of B. Therefore the homomorphism $\mathrm{Ad} \circ \widetilde{\eta}_e^G$ factors through a homomorphism $\eta_e^G : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{Aut}(\mathfrak{g})$ with $\eta_e^G \circ p_{\mathrm{SL}_2(\mathbb{R})} = \mathrm{Ad} \circ \widetilde{\eta}_e^G$.

From

$$\mathbf{L}(\eta_e^G) \circ \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tau \circ \mathbf{L}(\eta_e^G)(\cdot) \circ \tau$$

we derive on the group level that

$$\eta_e^G\Big(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Big) = \tau\eta_e^G(g)\tau \quad \text{for} \quad g \in \mathrm{SL}_2(\mathbb{R}).$$

(b) We define the *partial Cayley transform* by

$$C_e := \eta_e^G\Big(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\Big) = \exp\Big(\frac{\pi}{4}\operatorname{ad}(e-f)\Big) \in \operatorname{Aut}(\mathfrak{g}).$$

If, in addition, e is invertible, we call C_e the associated *Cayley transform.* ∎

Remark I.10. We keep the notation of the preceding definition and write $V = V_2 \oplus V_1 \oplus V_0$ for the eigenspace decomposition of V with respect to $2(e\Box e)$ (cf. Lemma C.1).

(a) Let $v \in V_2$ and $w := \tau(v)$. Then $[h,v] = 2v$ implies that $[h,w] = \tau.[-h,v] = -2w$. From that it easily follows that

$$M := \operatorname{span}_{\mathbb{R}}\{w, [e,w], [e,[e,w]]\}$$

is a $\mathfrak{g}_e$-submodule of $\mathfrak{g}$ equivalent to the adjoint module ([Bou90, Ch. VIII, §1, no. 2, Prop. 1]).

(b) According to Lemma C.1, the tripotent e is invertible if and only if $V = V_2$. Suppose this is the case. Then $Q(e)^2 = 2(e\Box e)^2 - e\Box e = \mathrm{id}_V$ (Lemma A.2(4)), so that $(V, e, Q(e))$ is an involutive unital Jordan algebra (Proposition A.5). Moreover, $\frac{1}{2}h \in \mathfrak{g}_0$ is a grading element by Proposition C.4(1). We conclude that $\operatorname{ad} h$ is diagonalizable on $\mathfrak{g}$, and since $\operatorname{ad} e$ and $\operatorname{ad} f$ are nilpotent, the Lie algebra $\mathfrak{g}$ is a locally finite $\mathfrak{g}_e$-module, hence semisimple by Weyl's Theorem. Since the only eigenvalues of $\operatorname{ad} h$ on $\mathfrak{g}$ are $\{0, \pm 2\}$, the Lie algebra $\mathfrak{g}$ is a direct sum of trivial and 3-dimensional $\mathfrak{g}_e$-modules. ∎

In the following lemma we collect some crucial properties of the partial Cayley transform C_e.

Lemma I.11. *For the partial Cayley transform associated to the tripotent $e \in V$ the following assertions hold:*

(1) $C_e^8 = \mathrm{id}_{\mathfrak{g}}$ *and if e is invertible, then* $C_e^4 = \mathrm{id}_{\mathfrak{g}}$.

(2) *Identifying V with a subset of X, for $v \in V$ the condition $C_e(v) \in V$ is equivalent to the quasi-invertibility of (e,v). For an element $v \in V_2$ this means that $e - v$ is invertible in the unital Jordan algebra (V_2, e), and then*

$$C_e(v) = (e+v)(e-v)^{-1}.$$

(3) $C_e(-e) = 0$, $C_e(0) = C_e(\mathfrak{f}_-) = e$, $C_e(e) = \mathfrak{f}_+$ *and* $C_e(\mathfrak{f}_+) = -e$.

(4) *On the subspace $V_2 \subseteq V$ we have* $C_e^2 \circ \tau = -Q(e)$.

(5) $\tau C_e \tau = C_e^{-1}$.

Proof. (1) For $I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ the matrix $\frac{1}{\sqrt{2}}(\mathbf{1} + I) \in \mathrm{SL}_2(\mathbb{R})$ is of order 8 and its square is I. Therefore the order of C_e is at most 8 and we have

$$C_e^2 = \eta_e^G\Big(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Big) = \exp\Big(\frac{\pi}{2}\operatorname{ad}(e-f)\Big).$$

If e is invertible, then $\mathfrak{g}$ is a direct sum of trivial and 3-dimensional simple $\mathfrak{sl}_2(\mathbb{R})$-modules (Remark I.10(b)). On both types of modules the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ acts like an involution, so that C_e^2 is an involution and therefore $C_e^4 = \mathrm{id}_{\mathfrak{g}}$.

(2) From the decomposition

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

in $\mathrm{SL}_2(\mathbb{R})$ we derive in $\mathrm{Aut}(\mathfrak{g})$ the decomposition

$$C_e = \exp(\operatorname{ad} e)\exp\big((\log\sqrt{2})\operatorname{ad} h\big)\exp(-\operatorname{ad} f). \tag{2.1}$$

Since $\exp(\operatorname{ad} e)\exp\big((\log\sqrt{2})\operatorname{ad} h\big) \in \mathrm{Ad}(P^+)$ acts as an affine map on $V \subseteq X$, we see that $C_e(v) \in V$ is equivalent to $\exp(-\tau.e).v = \exp(-f).v \in V$, which means that (e, v) is quasi-invertible (Definition I.5, Lemma I.3). If this is the case, then

$$\exp(-f).v = B(v,e)^{-1}.(v - Q(v).e)$$

([BN04a, 2.8]). In the Jordan algebra $V^{(e)}$ we have $Q(v).e = Q(v)Q(e).e = P(v).e = v^2$ and

$$B(v,e) = \mathrm{id}_V - 2L(v) + P(v),$$

and in the unital Jordan algebra $V^{(e)} \times \mathbb{R}$ with the identity $\mathbf{1} := (0,1)$ we have

$$\mathbf{1} - 2L(x) + P(x) = P(\mathbf{1},\mathbf{1}) - 2P(\mathbf{1},x) + P(x,x) = P(\mathbf{1} - x),$$

i.e., the quasi-invertibility of (x, e) is equivalent to the quasi-invertibility of x in the Jordan algebra $V^{(e)}$. In this algebra we have for any quasi-invertible pair (v, e):

$$\exp(-f).v = P(\mathbf{1} - v)^{-1}.(v - v^2) = (\mathbf{1} - v)^{-1}v.$$

For any element v in the unital Jordan algebra (V_2, e), the Cayley transform therefore takes the form

$$C_e(v) = e + 2(e - v)^{-1}v = (e - v + 2v)(e - v)^{-1} = (e + v)(e - v)^{-1}.$$

(3) We have $C_e(-e) = (e - e)(e - (-e))^{-1} = 0$ and $C_e(0) = e$.

We further have in V, as a subset of X, the relation $\tau_X(e) = e^\sharp = e$ (Proposition B.2), which leads to

$$\exp(-\operatorname{ad} f).e = \exp(-\operatorname{ad}\tau.e).e = \tau_X\exp(-\operatorname{ad} e)\tau_X.e = \tau_X\exp(-\operatorname{ad} e).e = \tau_X.0 = \tau_X.\mathfrak{f}_- = \mathfrak{f}_+,$$

so that

$$C_e.e = \exp(\operatorname{ad} e)\exp(\log\sqrt{2}\operatorname{ad} h)\exp(-\operatorname{ad} f).e = \exp(\operatorname{ad} e)\exp(\log\sqrt{2}\operatorname{ad} h).\mathfrak{f}_+ = \mathfrak{f}_+.$$

Moreover,

$$\begin{aligned}\exp(-\operatorname{ad} f).\mathfrak{f}_+ &= \tau_X\exp(-\operatorname{ad} e)\tau_X.\mathfrak{f}_+ = \tau_X\exp(-\operatorname{ad} e).\mathfrak{f}_- = \tau_X\exp(-\operatorname{ad} e).0\\ &= \tau_X.(-e) = (-e)^\sharp = -e,\end{aligned}$$

and hence

$$\begin{aligned}C_e.\mathfrak{f}_+ &= \exp(\operatorname{ad} e)\exp(\log\sqrt{2}\operatorname{ad} h)\exp(-\operatorname{ad} f).\mathfrak{f}_+\\ &= \exp(\operatorname{ad} e)\exp(\log\sqrt{2}\operatorname{ad} h).(-e) = e + 2(-e) = -e.\end{aligned}$$

(4) Let $v \in V_2$. According to Remark I.10(a), for $w := \tau.v$ the space $M := \mathrm{span}_{\mathbb{R}}\{w, [e,w], [e,[e,w]]\}$ is a $\mathfrak{g}_e$-submodule of $\mathfrak{g}$ isomorphic to $\mathfrak{g}_e$ with the adjoint representation. From the relation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

we obtain

$$\mathrm{Ad}(\exp(e - f)) \circ \tfrac{1}{2}(\operatorname{ad} e)^2.f = \mathrm{Ad}(\exp(e - f)).(-e) = f,$$

and this leads to $C_e^2\big(\frac{1}{2}(\operatorname{ad} e)^2\big)\tau.v = -C_e^2 Q(e).v = \tau.v$.

(5) follows immediately from $\tau(e - f) = \tau(e - \tau(e)) = \tau(e) - e = f - e$ and $C_e \in \exp(\mathbb{R}(e - f))$. ■

Proposition I.12. *For any tripotent $e \in V$ we have $\exp(\mathfrak{g}_e^\tau).0 =]-1,1[\cdot e$ in V, considered as a subset of X. In particular we have $]-1,1[\cdot S \subseteq H.0$*

Proof. We have seen above that (e,h,f) is an $\mathfrak{sl}_2$-triple, so that $e+\tau(e)$ corresponds to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and e to the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. To calculate $\exp(t(e+\tau(e)).0$ in $V \subseteq X$, we observe that

$$\exp\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \in \exp(\mathbb{R}f + \mathbb{R}h)\cdot \begin{pmatrix} 1 & \tanh t \\ 0 & 1 \end{pmatrix},$$

which leads to $\exp(t(e+\tau(e))).0 = \tanh t \cdot e$, and from that the assertion follows. ■

Consider the following **assumptions on the involutive 3-graded group** G:
(A1) $\mathcal{D} := H.0 \subseteq V$, i.e., $H \subseteq G^+G^0G^-$.
(A2) $H.S \subseteq V$.
(A3) $d_G(S^3_\top) = \{\mathbf{1}\}$.

Condition (A1) is well-known from the setting of groups of Harish-Chandra type. In view of Proposition I.8, condition (A2) is equivalent to the invariance of the subset $X^\tau \cap V$ under the action of the group H.

Proposition I.13. $\rho_V \circ d_G(S^3_\top) = \{\mathbf{1}\}$. *In particular,* (A3) *is satisfied if G^0 acts faithfully on V.*

Proof. For $(x,y,z) \in S^3_\top$ we derive from Lemma A.10(2) the relation

$$B(x,y) = B(x,y^\sharp) = Q(x-y)Q(y)^{-1},$$

so that we get with Lemma I.6

$$\begin{aligned} \rho_V(d_G(x,y,z)) &= B(x,y)B(z,y)^{-1}B(z,x)B(y,x)^{-1}B(y,z)B(x,z)^{-1} \\ &= B(x,y^{-1})B(z,y^{-1})^{-1}B(z,x^{-1})B(y,x^{-1})^{-1}B(y,z^{-1})B(x,z^{-1})^{-1} \\ &= Q(x-y)Q(z-y)^{-1}Q(z-x)Q(y-x)^{-1}Q(y-z)Q(x-z)^{-1} \\ &= Q(y-x)Q(z-y)^{-1}Q(x-z)Q(y-x)^{-1}Q(z-y)Q(x-z)^{-1} = \mathbf{1}, \end{aligned}$$

where the last equality follows from Proposition A.7. ■

In Proposition IV.4 below we shall use the results of Section III on H-orbits in $S^3_\top$ to see that the preceding result can be sharpened considerably to the observation that $d_G(S^3_\top) \subseteq Z(G_0)$.

In the following we shall also see interesting examples where (A3) is satisfied and G^0 does not act faithfully on V. This holds in particular for the group $G = \mathrm{GL}_2(A)/\{\pm\mathbf{1}\}$, where A is a hermitian Banach-$*$-algebra (cf. Example II.6 below).

Lemma I.14. *If* (A1) *is satisfied, then for each $v \in V$ with $H.v \subseteq V$ we have $\mathcal{D} \times H.v \subseteq V^2_\top$. If, in addition,* (A2) *holds, then $\mathcal{D} \times (\mathcal{D} \cup S) \subseteq V^2_\top$.*

Proof. Suppose that (A1) is satisfied, i.e. $\mathcal{D} = H.0 \subseteq V$ and let $v \in V$ with $H.v \subseteq V$. For $h_1, h_2 \in H$ and $h_1.0 \in \mathcal{D}$ it now follows that $(h_1.0, h_2.v)$ is quasi-invertible because $(0, h_1^{-1}h_2.v)$ is quasi-invertible (Lemma I.7(7)).

If, in addition, $H.S \subseteq V$, then the preceding argument applies with $v = 0$ or $v \in S$, and the assertion follows. ■

Definition I.15. Suppose that (A1-3) hold. For $(x,y,z) \in S^3_\top$ we consider the continuous curve

$$\alpha_{x,y,z}\colon [0,1] \to V^3, \quad t \mapsto (tx,ty,tz),$$

starting at $(0,0,0)$ and ending at $(x,y,z) \in S^3_\top$. Proposition I.12 and Lemma I.14 now implies that $\mathrm{im}(\alpha_{x,y,z})$ is contained in the open subset $V^3_\top$ of V^3, so that the curve

$$d_G \circ \alpha_{x,y,z}\colon [0,1] \to G^0, \quad t \mapsto d_G(tx,ty,tz),$$

is defined. Since $d_G(0,0,0) = \mathbf{1}$ and $d_G(x,y,z) = \mathbf{1}$ by (A3), this curve is a loop in G^0.

We thus obtain a map

$$\mu_G\colon S^3_\top \to \pi_1(G^0), \quad (x,y,z) \mapsto [d_G \circ \alpha_{x,y,z}].$$

For reasons to be explained later, we call this map the *topological (Maslov) index.* Since the path $\alpha_{x,y,z}$ depends continuously on the triple (x,y,z), this map is constant on the connected components of $S^3_\top$, hence induces a map $\pi_0(S^3_\top) \to \pi_1(G^0)$. ∎

We shall see in Example IV.2 below that for the case where $\mathcal{D} \subseteq V$ is a finite-dimensional irreducible bounded symmetric domain of tube type, the index map μ_G can be used to obtain the Maslov index by composing with the homomorphism $\det \circ \rho_V\colon G^0 \to \mathbb{C}^\times$ to obtain a map

$$\pi_1(\det \circ \rho_V) \circ \mu_G\colon S^3_\top \to \pi_1(\mathbb{C}^\times) \cong \mathbb{Z}.$$

Proposition I.16. *The index map $\mu_G\colon S^3_\top \to \pi_1(G^0)$ is an alternating function with values in the abelian group $\pi_1(G^0)$, i.e.*

$$\mu_G(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = \mu_G(x_1,x_2,x_3)^{\mathrm{sgn}(\sigma)} \quad \textit{for} \quad (x_1,x_2,x_3) \in S^3_\top, \sigma \in S_3.$$

Proof. From Lemma I.7(4) we immediately derive that $[\alpha_{x,y,z}] = [\alpha^{-1}_{x,z,x}] = [\alpha_{x,z,x}]^{-1}$.

We further get from Lemma I.7(5) a continuous path $\beta\colon [0,1] \to G^0$ with

$$\alpha_{y,z,x} = \beta \cdot \alpha_{x,y,z} \cdot \beta^{-1},$$

and this loop in G^0 is homotopic to the loop $\alpha_{x,y,z}$, which leads to $[\alpha_{y,z,x}] = [\alpha_{x,y,z}]$. Since the symmetric group S_3 is generated by the cycle $(1\ 2\ 3)$ and the transposition $(2\ 3)$, the assertion follows. ∎

II. Bounded symmetric domains and hermitian Banach-$*$-algebras

In this section we discuss two large classes of groups for which our assumptions (A1-3) are satisfied. The groups of the first class are the complexifications G of the identity component $\mathrm{Aut}(\mathcal{D})_0$ of the group of biholomorphic maps of a bounded symmetric domain $\mathcal{D}$ in a Banach space, and the second class contains the groups $\mathrm{GL}_2(A)/\{\pm\mathbf{1}\}$ for a hermitian unital Banach-$*$-algebra A. In this case the corresponding domain $\mathcal{D}$ is bounded if and only if A is a C^*-algebra.

Bounded symmetric domains in Banach spaces

Let V be a complex Banach space and $\mathcal{D} \subseteq V$ be a *bounded symmetric domain*, i.e., a bounded open connected subset such that for each $z \in \mathcal{D}$ there exists an involution $j_z \in \mathrm{Aut}(\mathcal{D})$, the group of biholomorphic mappings of $\mathcal{D}$, such that z is an isolated fixed point of j_z. The group $\mathrm{Aut}(\mathcal{D})$ carries a natural Banach–Lie group structure such that the transitive action on $\mathcal{D}$ is real analytic ([Up85, Th. 13.14]). According to Kaup's Riemann Mapping Theorem ([Ka83], [Up85, Th. 20.23]), there is a norm on the space V such that $\mathcal{D}$ is biholomorphic to the open unit ball in V. Therefore we assume from now on that

$$\mathcal{D} = \{z \in V\colon \|z\| < 1\}.$$

The identity component $H := \mathrm{Aut}(\mathcal{D})_0$ of $\mathrm{Aut}(\mathcal{D})$ carries a natural Banach–Lie group structure such that the transitive action of H on $\mathcal{D}$ is real analytic.

We think of $\mathbf{L}(H)$ as a Lie algebra of holomorphic vector fields on the domain $\mathcal{D} \subseteq V$. It is shown in [Up85, Th. 18.17] that the elements of $\mathbf{L}(H)$ are polynomial vector fields of degree at most 2 and that

$$\mathfrak{g} := \mathbf{L}(H) + i\,\mathbf{L}(H)$$

carries a natural structure of a centerfree 3-graded Banach–Lie algebra on which there is a grading reversing antilinear involution τ for which $\mathbf{L}(H) = \mathfrak{g}^\tau$. The grading is given by the degree of vector fields, where $\mathfrak{g}_j$ consists of vector fields of degree $1-j$. Since the unit ball $\mathcal{D}$ is in particular cicular, $\mathfrak{g}$ contains the Euler vector field corresponding to the function $E(z) = z$ on V, which defines the grading of $\mathfrak{g}$. We conclude that the grading of $\mathfrak{g}$ is inner.

We then consider the complex Banach–Lie group

$$G := \mathrm{Aut}(\mathfrak{g})_0.$$

Then $\mathbf{L}(G) = \mathrm{der}\,\mathfrak{g} = \mathrm{ad}\,\mathfrak{g} \cong \mathfrak{g}$ (cf. [Up85, Lemma 9.9]) and the involution τ on $\mathfrak{g}$ induces by conjugation an involution, also denoted τ, on G. We thus obtain a situation as discussed in Section I, where we considered a Banach–Lie group G endowed with an involution τ reversing an inner 3-grading on $\mathfrak{g}$. Clearly $H = \mathrm{Aut}(\mathcal{D})_0 = G_0^\tau$ follows from the equality of the Lie algebras of both subgroups of G.

In this case the orbit $H.0$ of the base point $0 \in V \cong \mathfrak{g}_1$ in the homogeneous space $X = G/P^-$ coincides with the bounded symmetric domain $\mathcal{D}$ ([Up85, Th. 20.20]). Therefore our assumption (A1) is satisfied.

Theorem II.1. *The closure $\overline{\mathcal{D}}$ of $\mathcal{D}$ in V also is a closed subset of X.*

Proof. Since $X = G/P^-$ is a quotient space and the inverse image of $\overline{\mathcal{D}}$ in G is the product set $\exp(\overline{\mathcal{D}})P^- = \exp(\overline{\mathcal{D}})G^0G^-$, it suffices to show that $Y := \exp(\overline{\mathcal{D}})G^0G^-$ is a closed subset of G.

Let $U \subseteq G$ be an open identity neighborhood with $U \cdot U$ contained in the open subset $G^+G^0G^-$. If $0 \in V$ is identified with the base point P^- of the homogeneous space $X = G/P^-$, then this implies that $UU.0 \subseteq V$.

Since $\mathcal{D} \subseteq V$ is a bounded subset and $\mathrm{ad}\,E|_{\mathfrak{g}_1} = \mathrm{id}_{\mathfrak{g}_1}$, there exists a $t > 0$ with

$$\exp(-tE).\mathcal{D} \subseteq U.0.$$

For the identity neighborhood $U' := \exp(tE)U\exp(-tE)$ of G we then obtain

$$U'.\mathcal{D} = \exp(tE)U\exp(-tE).\mathcal{D} \subseteq \exp(tE)UU.0 \subseteq V,$$

i.e., $U'\exp(\mathcal{D})G^0G^- \subseteq G^+G^0G^-$, so that

$$\overline{\exp(\mathcal{D})G^0G^-} \subseteq U'\exp(\mathcal{D})G^0G^- \subseteq G^+G^0G^-.$$

Since the open subset $G^+G^0G^-$ is homeomorphic to the topological product $G^+ \times G^0 \times G^-$, it follows that

$$\overline{\exp(\mathcal{D})G^0G^-} = (\exp\overline{\mathcal{D}})G^0G^-$$

is the closure of $(\exp\mathcal{D})G^0G^-$ in G. ■

By continuity we now obtain immediately

Corollary II.2. *$H.\overline{\mathcal{D}} \subseteq \overline{\mathcal{D}} \subseteq V$ and in particular $H.S \subseteq V$.* ■

Proposition II.3. *If $S \neq \emptyset$, i.e., $\mathcal{D}$ is a bounded symmetric domain of tube type, then the assumptions* (A1-3) *are satisfied for the involutive 3-graded group $(G, \mathrm{ad}\,E, \tau)$.*

Proof. Assumption (A1) follows from the realization of $\mathcal{D}$ as a bounded domain in $V \subseteq G/P^-$. The preceding corollary implies that (A2) is satisfied. Further (A3) will follow from the fact that the representation of G^0 on V is faithful (Proposition I.13). To verify that this representation is faithful, let $g \in G^0$ act trivially on V. Then the adjoint action, which corresponds to the action of g on a set of vector fields on V, is trivial. Therefore $G \subseteq \mathrm{Aut}(\mathfrak{g})$ implies $g = \mathbf{1}$. This proves that (A1-3) are satisfied. ■

Hermitian Banach-$*$-algebras

Definition II.4. A *Banach-$*$-algebra* is a pair $(A,*)$ of a complex Banach algebra together with an antilinear isometric antiisomorphism $*$. It is called *hermitian* if the spectra of hermitian elements are real. ■

The following simple lemma will be helpfull in evaluating $d_G(S^3_\top)$ for the group $\mathrm{GL}_2(A)$.

Lemma II.5. *Let (R,e) be a unital ring and $a,b,c \in R$ with $a+b+c=0$ and $b \in R^\times$. Then*

$$ab^{-1}c = cb^{-1}a.$$

Proof. The relation $a+b+c=0$ implies that $ab^{-1}+cb^{-1}=-e$, so that ab^{-1} and cb^{-1} commute, and the assertion follows from $ab^{-1}cb^{-1}=cb^{-1}ab^{-1}$ by multiplying with b from the right. ■

Example II.6. Let $(A,*)$ be a hermitian Banach-$*$-algebra. First we consider $G := \mathrm{GL}_2(A)$ with the involution τ given by

$$\tau\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & -c^* \\ -b^* & d^* \end{pmatrix}^{-1}$$

and whose fixed point set is denoted $U_{1,1}(A,*) := \mathrm{GL}_2(A)^\tau$. Its Lie algebra $\mathfrak{g} = \mathfrak{gl}_2(A)$ is 3-graded with

$$\mathfrak{g}_+ = \begin{pmatrix} \mathbf{0} & A \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \mathfrak{g}_0 = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix} \quad \text{and} \quad \mathfrak{g}_- = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ A & \mathbf{0} \end{pmatrix}.$$

Since $E := \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$ is a grading element, the grading is inner. On the Lie algebra level we have

$$\tau\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a^* & c^* \\ b^* & -d^* \end{pmatrix},$$

showing that τ reverses the grading. The corresponding Jordan triple product in $A \cong \mathfrak{g}_+$ is given by

$$\{x,y,z\} = \tfrac{1}{2}(xy^*z + zy^*x).$$

On the group level we have

$$\mathrm{GL}_2(A)^+ = \begin{pmatrix} \mathbf{1} & A \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \quad \mathrm{GL}_2(A)^0 = \begin{pmatrix} A^\times & \mathbf{0} \\ \mathbf{0} & A^\times \end{pmatrix} \quad \text{and} \quad \mathrm{GL}_2(A)^- = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ A & \mathbf{1} \end{pmatrix}.$$

Then

$$\mathrm{GL}_2(A)^+\,\mathrm{GL}_2(A)^0\,\mathrm{GL}_2(A)^- = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) \colon d \in A^\times \right\},$$

and any matrix in this set decomposes as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathbf{1} & bd^{-1} \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} a-bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ d^{-1}c & \mathbf{1} \end{pmatrix}.$$

From

$$\begin{pmatrix} \mathbf{1} & 0 \\ -w^* & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & z \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & z \\ -w^* & \mathbf{1}-w^*z \end{pmatrix}$$

we obtain

$$B_G(z,w) = \begin{pmatrix} \mathbf{1} - z(\mathbf{1} - w^*z)^{-1}(-w^*) & 0 \\ 0 & \mathbf{1} - w^*z \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{1} - zw^* & 0 \\ 0 & (\mathbf{1} - w^*z)^{-1} \end{pmatrix}.$$

Next we calculate d_G on quasi-invertible unitary triples (s_1, s_2, s_3). For unitary elements $z, w \in S$ quasi-invertibility means that $\mathbf{1} - w^*z = \mathbf{1} - w^{-1}z$ is invertible, which means that $w - z$ is invertible. Therefore all differences $s_j - s_k$, $j \neq k$, are invertible. Since

$$(s_1 - s_2) + (s_2 - s_3) + (s_3 - s_1) = 0,$$

Lemma II.5 leads to

$$\begin{aligned} &(\mathbf{1} - s_1 s_2^*)(\mathbf{1} - s_3 s_2^*)^{-1}(\mathbf{1} - s_3 s_1^*)(\mathbf{1} - s_2 s_1^*)^{-1}(\mathbf{1} - s_2 s_3^*)(\mathbf{1} - s_1 s_3^*)^{-1} \\ &= (\mathbf{1} - s_1 s_2^{-1})(\mathbf{1} - s_3 s_2^{-1})^{-1}(\mathbf{1} - s_3 s_1^{-1})(\mathbf{1} - s_2 s_1^{-1})^{-1}(\mathbf{1} - s_2 s_3^{-1})(\mathbf{1} - s_1 s_3^{-1})^{-1} \\ &= (s_2 - s_1)(s_2 - s_3)^{-1}(s_1 - s_3)(s_1 - s_2)^{-1}(s_3 - s_2)(s_3 - s_1)^{-1} \\ &= -(s_1 - s_2)(s_2 - s_3)^{-1}(s_3 - s_1)(s_1 - s_2)^{-1}(s_2 - s_3)(s_3 - s_1)^{-1} = -\mathbf{1} \end{aligned}$$

and we likewise get

$$\begin{aligned} &(\mathbf{1} - s_2^* s_1)^{-1}(\mathbf{1} - s_2^* s_3)(\mathbf{1} - s_1^* s_3)^{-1}(\mathbf{1} - s_1^* s_2)(\mathbf{1} - s_3^* s_2)^{-1}(\mathbf{1} - s_3^* s_1) \\ &= (\mathbf{1} - s_2^{-1} s_1)^{-1}(\mathbf{1} - s_2^{-1} s_3)(\mathbf{1} - s_1^{-1} s_3)^{-1}(\mathbf{1} - s_1^{-1} s_2)(\mathbf{1} - s_3^{-1} s_2)^{-1}(\mathbf{1} - s_3^{-1} s_1) \\ &= (s_2 - s_1)(s_2 - s_3)^{-1}(s_1 - s_3)(s_1 - s_2)^{-1}(s_3 - s_2)(s_3 - s_1)^{-1} = -\mathbf{1}. \end{aligned}$$

This shows that

$$d_G(s_1, s_2, s_3) = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$

Let σ_{C^*} denote the largest C^*-seminorm on A, i.e., $\sigma_{C^*}(a) = \|\eta(a)\|$ if $\eta: A \to C^*(A)$ is the universal map into the universal enveloping C^*-algebra $C^*(A)$ of A. From [Bi04, Lemma 8.2.7] we know that the orbit of $H = G_0^\tau$ in X is contained in A and coincides with the convex open set

$$\mathcal{D} = \{a \in A: \sigma_{C^*}(a) < 1\}.$$

For the invertible tripotent $e := \mathbf{1} \in A$ we have $Q(e)a = a^*$, so that

$$S = U(A) = \{a \in A^\times : a^* = a^{-1}\}.$$

We claim that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_{1,1}(A, *)$ and $z \in A$ with $\sigma_{C^*}(z) \leq 1$, then $cz + d \in A^\times$, which implies that $g.z = (az + b)(cz + d)^{-1}$ is contained in $V = A$, and hence that (A1) and (A2) are satisfied.

If A is a C^*-algebra, then $\mathcal{D}$is the open unit ball in A, and the transitivity of the holomorphic action of H on $\mathcal{D}$ implies that it is a bounded symmetric domain. From Corollary II.2 above we know that in this case the closure of $\mathcal{D}$ in X coincides with the closure of $\mathcal{D}$ in V which is invariant under the action of H.

This argument can be carried over to a general hermitian Banach $*$-algebra as follows. Since η induces homomorphisms

$$\mathrm{GL}_2(A) \to \mathrm{GL}_2(C^*(A)) \quad \text{and} \quad U_{1,1}(A, *) \to U_{1,1}(C^*(A), *),$$

we conclude from the case of C^*-algebras that $\eta(cz + d) = \eta(c)\eta(z) + \eta(d)$ is invertible in $C^*(A)$, which in turn implies that $cz + d$ is invertible in A because the property $\eta^{-1}(C^*(A)^\times) = A^\times$ characterizes hermitian Banach $*$-algebras (cf. [Bi04, Prop. 2.7.5], see also [Pt70/72] for the Banach version of Biller's results).

The domain $\mathcal{D}$ is bounded if and only if the natural homomorphism $\eta: A \to C^*(A)$ is an embedding, i.e., if and only if A is a C^*-algebra. ∎

As an immediate consequence of the discussion in Example II.6, we obtain the following theorem:

Theorem II.7. *If $(A, *)$ is a hermitian Banach-$*$-algebra, then*

$$\tau\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & -c^* \\ -b^* & d^* \end{pmatrix}^{-1}$$

and $E := \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$ *define an involutive 3-graded Banach–Lie group* $(\mathrm{GL}_2(A), \mathrm{ad}\, E, \tau)$ *satisfying* (A1/2). *If we write* $\mathbf{1} \in M_2(A)$ *for the identity matrix, then the group* $G := \mathrm{GL}_2(A)/\{\pm\mathbf{1}\}$ *satisfies* (A1-3) *with respect to the induced involution.* ∎

III. Connected components and H-orbits in $S^3_\top$

We have already seen that the group H acts on X^τ in such a way that its orbits are the connected components (Proposition I.8). Under the assumption (A2), the set S is a union of such H-orbits. In the following we shall use this correspondence to get a better description of the connected components in $S^3_\top$. In particular, we shall see that they coincide with the orbits of H in $S^3_\top$ and that each orbit contains a triple of the form $(e, -e, \sigma)$ with $\sigma^* = Q(e)\sigma = -\sigma$ in the unital involutive Jordan algebra $(V, e, Q(e))$. Since σ is an invertible tripotent, the latter condition implies that

$$\sigma = Q(\sigma)\sigma = -Q(\sigma)Q(e)\sigma = -P(\sigma)\sigma = -\sigma^3$$

and therefore $\sigma^2 = -e$. In the following we put $V_\pm := \{v \in V : v^* = Q(e)v = \pm v\}$.

Lemma III.1. *Let $e \in S$ and $C_e \in \mathrm{Aut}(\mathfrak{g})$ denote the corresponding Cayley transform. For $v \in V$ and $v^* = Q(e)v$ we have*

$$\tau(C_e.v) = -C_e.v^* \quad \text{and} \quad \tau(C_e^{-1}.v) = -C_e^{-1}.v^*.$$

In particular $C_e.v, C_e^{-1}.v \in \mathfrak{g}^\tau$ if $v^* = -v$, where $C_e.v$ refers to the linear action of C_e on $\mathfrak{g}$. The corresponding element $g := \exp(C_e^{-1}.v) \in H$ satisfies

$$g.(-e) = C_e^{-1}(v) = (v - e)(e + v)^{-1}.$$

Moreover, $e + v$ is invertible whenever $v^* = -v$.

Proof. The first equality follows from

$$\tau \circ C_e = C_e^{-1} \circ \tau = C_e^3 \circ \tau = -C_e \circ Q(e)$$

on V (Lemma I.11(1),(4),(5)), and we likewise obtain on V the relation $\tau \circ C_e^{-1} = C_e \circ \tau = -C_e^{-1}Q(e)$.

From Lemma I.11(3) we know that $C_e(-e) = 0$ for the action of C_e on X, so that we obtain for $g = \exp(C_e^{-1}.v) \in H$ that

$$C_e^{-1}(v) = C_e^{-1}e^{\mathrm{ad}\, v}.0 = C_e^{-1}e^{\mathrm{ad}\, v}C_e.(-e) = \exp(C_e^{-1}.v).(-e) = g.(-e) \in S.$$

In particular $e + v$ is invertible (Lemma I.11(2)). ∎

Lemma III.2.
(1) *The action of H on $\mathcal{D} \cup S$ preserves quasi-invertibility.*
(2) *If $e \in S$, then the stabilizer H_e of e in H acts transitively on $\{f \in S : e \top f\}$.*
(3) *If $\mathfrak{g}$ is complex and τ is antilinear, then $(e, f) \in S^2_\top$ implies $f \in H.e$.*
(4) *For $(e, f) \in S^2_\top$ we have $H.(e, f) = \{(a, b) \in S^2_\top : a \in H.e\}$.*

(5) *The* H*-orbit of* $(e,x,y) \in S^3_\top$ *contains an element of the form* $(e,-e,z)$ *and*

$$\{z \in S : z\top \pm e\} = C_e(V_- \cap V^\times).$$

Proof. (1) follows from Lemma I.14.

(2) According to Proposition I.8, we have $S = X^\tau \cap V$. Further $(z,w) \in V^2_\top$ is equivalent to the transversality of the 3-filtrations $\exp z.\mathfrak{f}_-$ and $\tau(\exp w.\mathfrak{f}_-)$ (cf. Definition I.5(b)). For $z,w \in S \subseteq X^\tau$ this is equivalent to the quasi-invertibility of (z,w). Hence

$$\{f \in S : e\top f\} \subseteq (\exp e.\mathfrak{f}_-)^\top,$$

and Proposition I.8(2) implies that $H_e = H_{\exp e.\mathfrak{f}_-}$ acts transitively on $(\exp e.\mathfrak{f}_-)^\top$.

We also give a second proof of (2) which is more direct and uses (1): The quasi-invertibility of (e,f) implies that $e-f$ is invertible in the unital involutive Jordan algebra $(V,e,Q(e))$, so that $x := C_e(f) \in X$ is an element of V (Lemma I.11(2)). We have

$$x^* = C_e(f)^* = ((e+f)(e-f)^{-1})^* = (e+f^*)(e-f^*)^{-1} = C_e(f^*) = C_e(f^{-1}) = -C_e(f) = -x$$

(Lemma A.11), so that $g := \exp(C_e^{-1}.x) \in H$ satisfies $g.(-e) = C_e^{-1}.x = f$ (Lemma II.1). We further get with Lemma I.11(3) in $X \subseteq \mathcal{F}$:

$$g.e = C_e^{-1} e^{\mathrm{ad}\, x} C_e(e) = C_e^{-1} e^{\mathrm{ad}\, x}.\mathfrak{f}_+ = C_e^{-1}.\mathfrak{f}_+ = e.$$

(3) For $e \in S$ we consider the 3-dimensional subalgebra $\mathfrak{g}_e = \mathrm{span}_{\mathbb{C}}\{e, \tau(e), [e,\tau(e)]\} \subseteq \mathfrak{g}$. Then $E := \frac{1}{2}[e,\tau(e)]$ is a grading element with $\tau(E) = -E$ (Proposition C.4), and $\tau(iE) = iE$ implies that $\mathbb{T} \cong \exp(i\mathbb{R}E) \subseteq H$. We therefore obtain $-e \in \exp(i\mathbb{R}E).e \subseteq H.e$, and the assertion follows from (2) and $e\top -e$.

(4) In view of (1), each element $(a,b) \in S^2$ of the form $(g.e, g.f)$ satisfies $a \in H.e$ and $b\top a$.

If, conversely, $a = g.e$ and $b\top a$, then $(g^{-1}.b, g^{-1}.a) = (g^{-1}.b, e)$, so that (2) implies the existence of $h \in H_e$ with $h.f = g^{-1}.b$, and then $h.(e,f) = (e, g^{-1}.b) = g^{-1}.(a,b)$ implies $(a,b) \in H.(e,f)$.

(5) From (2) it follows that the H-orbit of (e,x,y) contains an element of the form $(e,-e,z)$. Then z is a unitary element in the involutive unital Jordan algebra $(V,e,Q(e))$ with involution $v^* := Q(e)v$. The quasi-invertibility of $(z,\pm e)$ is equivalent to the invertibilty of $z \pm e$ in the Jordan triple V (Lemma A.9) and hence in the unital Jordan algebra (V,e). Therefore $e-(-z) = e+z$ is invertible, and we put $v := -C_e(-z) = C_e^{-1}(z)$ to obtain an element $v \in V$ with $C_e(v) = z$. We further obtain with Lemma A.11(1):

$$\begin{aligned} v^* &= (-C_e(-z))^* = -C_e(-z)^* = -C_e(-z^*) = -C_e(-z^{-1}) \\ &= -C_e((-z)^{-1}) = -(-C_e(-z)) = C_e(-z) = -v, \end{aligned}$$

so that $v \in V_-$. If, conversely, $v \in V_-^\times$, then $e-v$ is invertible (Lemma III.1) and $z := C_e(v) \in S$ is a unitary element for which $z+e$ is invertible. Since v is invertible, Lemma A.11 implies that $z = C_e(v)$ lies in the domain $V^\times + e$ of C_e, so that also $e-z$ is invertible, and hence $(e,-e,z) \in S^3_\top$. ∎

Remark III.3. (a) The preceding lemma shows in particular that $(e,f) \in S^2_\top$ implies that $f \in S_{-e} = -S_e$, where S_e denotes the connected component of S containing e. For $(e,f,g) \in S^3_\top$ we even conclude that $S_e = S_{-g} = S_f = S_{-e} = S_g$. This leads to the disjoint decomposition

$$S^3_\top = \bigcup_e (S_e)^3_\top,$$

so that it is no loss of generality if we consider only a fixed connected component S_e of the set S and study the index map on the subset $(S_e)^3_\top$ of $S^3_\top$.

(b) For $G = \mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$ with grading derivation

$$D = \mathrm{ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \tau\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -c \\ -b & d \end{pmatrix}^{-1}$$

we have

$$V \cong \mathbb{R} \quad \text{with} \quad \{x,y,z\} = xyz \quad \text{and} \quad S = \{\pm 1\}.$$

Here $S^2_\top = \{(1,-1),(-1,1)\}$ and the connected group H acts trivially. In this case we have $S_1 = \{1\} \neq \{-1\} = S_{-1}$. ∎

Lemma III.4. *Fix $e \in S$ and consider the associated Cayley transform $C := C_e \in \mathrm{Aut}(\mathfrak{g})$. Then the involution $\tau^C := C\tau C^{-1} \in \mathrm{Aut}(\mathfrak{g})$ satisfies:*

(1) *τ^C preserves the 3-grading of $\mathfrak{g}$.*

(2) *$\tau^C = \tau C^2$, where τ and C^2 are commuting involutions of $\mathfrak{g}$.*

(3) *The Lie subalgebra $\mathfrak{l} := C(\mathfrak{g}^\tau) = \mathfrak{g}^{\tau^C}$ is adapted to the 3-grading of $\mathfrak{g}$ and τ-invariant.*

(4) *$\tau^C|_V = -Q(e)$.*

(5) *For the stabilizer group $H_{e,-e}$, the identity component $L := (G^{\tau^C})_0$ and $C^G = \exp\left(\frac{\pi}{4}(e-f)\right) \in G$ we have*

$$\mathrm{Ad}(C^G) = C \quad \text{and} \quad L^0 := L \cap G^0 = C^G \cdot H_{e,-e} \cdot (C^G)^{-1}.$$

Proof. (1) With Lemma I.11 we get in $X \subseteq \mathcal{F}$:

$$\tau^C(\mathfrak{f}_-) = \tau^C(0) = C\tau_X(-e) = C(-e) = 0 = \mathfrak{f}_- \quad \text{and} \quad \tau^C(\mathfrak{f}_+) = C\tau_X(e) = C(e) = \mathfrak{f}_+.$$

Therefore τ^C fixes the two filtrations $\mathfrak{f}_\pm$ and hence the corresponding 3-grading of $\mathfrak{g}$.

(2) With Lemma I.11 we get $\tau^C = C\tau C^{-1} = C^2\tau = \tau C^{-2} = \tau C^2$, so that the two involutions τ and C^2 commute.

(3) That $\mathfrak{l}$ is adapted to the 3-grading of $\mathfrak{g}$ follows directly from (1). Since τ and τ^C commutes by (2), $\mathfrak{l}$ is τ-invariant.

(4) follows from Lemma I.11(4).

(5) The relation $\mathrm{Ad}(C^G) = C$ is immediate from the definitions. Further $C(\pm e) = \mathfrak{f}_\pm$ and $C^G H (C^G)^{-1} = L$ lead to $L^0 = L \cap G_{\mathfrak{f}_\pm} = C^G \cdot H_{e,-e} \cdot (C^G)^{-1}$. ■

Proposition III.5. *Let $\sigma: G \times M \to M$ be a smooth action of the Banach–Lie group G on the Banach manifold M and $T\sigma: TG \times TM \to TM$ its tangent map. If $\mathfrak{g}.p := T\sigma(\mathbf{1}, p)(\mathfrak{g} \times \{0\}) = T_p(M)$, then the orbit $G.p$ of p is open.*

Proof. For a smooth map $f: N \to M$ between Banach manifold for which the differential $df(x): T_x(N) \to T_{f(x)}(M)$ is surjective, the image of f is a neighborhood of $f(x)$ ([De85, Cor. 15.2]).

The condition $\mathfrak{g}.p = T_p(M)$ means that the differential of the orbit map $G \to M, g \mapsto g.p$ in $g = \mathbf{1}$ is surjective, so that the aforementioned fact implies that the orbit $G.p$ is a neighborhood of p. This implies that $G.p$ is open. ■

Proposition III.6. *All orbits of $(L^0)_0$ in $V_-^\times := V_- \cap V^\times$ are open.*

Proof. From Lemma III.4(4) we immediately get $V_- \subseteq \mathfrak{l}_1$. Let $v \in V_-^\times \subseteq \mathfrak{l}_1$ be an invertible element. Then $v^{-1} \in V_-^\times$ and

$$v^\sharp = Q(v)^{-1}v = Q(e)v^{-1} = -v^{-1} \in V_- \subseteq \mathfrak{l}_1.$$

Therefore $V_- \Box v^\sharp = [V_-, \tau(v^\sharp)] \subseteq [\mathfrak{l}_1, \mathfrak{l}_{-1}] \subseteq \mathfrak{l}_0$. Since the map

$$V_- \to V_-, \quad x \mapsto (x \Box v^\sharp).v = \{v, v^\sharp, x\} = (v \Box v^\sharp).x = x$$

is bijective (cf. Lemma A.4(1)), the orbit map $\mathfrak{l}_0 \to V_-, x \mapsto x.v$ is surjective, and Proposition III.5 implies that the orbit $L_0^0.v$ in V_- is open. ■

In general the group L^0, resp., $H_{e,-e}$ is not connected, so that the orbits of this group may also be unions of several connected components in $V_-^\times$. If, f.i. $G = \mathrm{GL}_2(A)/\{\pm\mathbf{1}\}$ for a hermitian Banach-$*$-algebra A, then

$$C = \mathrm{Ad}\left(\frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ -\mathbf{1} & \mathbf{1} \end{pmatrix}\right) \quad \text{and} \quad C^2 = \mathrm{Ad}\left(\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}\right)$$

lead to

$$\tau^C\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}\begin{pmatrix} -a^* & c^* \\ b^* & -d^* \end{pmatrix}\begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} -d^* & -b^* \\ -c^* & -a^* \end{pmatrix},$$

so that

$$L^0 = (G^0)^{\tau^C} = \left\{\begin{pmatrix} a & 0 \\ 0 & \pm a^{-*} \end{pmatrix} : a \in A^\times\right\}/\{\pm\mathbf{1}\} \cong (A^\times/\{\pm\mathbf{1}\}) \rtimes \left\{\pm\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}\right\},$$

which is not connected if $A^\times$ is not connected.

Proposition III.7. *The orbits of H in S, $S^2_\top$ and $S^3_\top$ are open, hence coincide with the connected components.*

Proof. That the orbits of H in S are open follows from Proposition I.8.

For $(e,f) \in S^2_\top$, Lemma III.2(4) implies $H.(e,f) = S^2_\top \cap (H.e \times S)$, and since $H.e$ is open in S, it follows that $H.(e,f)$ is open in S^2.

Let $(e,g,f) \in S^3_\top$. In view of Lemma III.2(5), we may assume that $f = -e$. So it remains to see that if $f \top \pm e$, then $H.(e,-e,f)$ is open in S^3.

Conjugating everything with the Cayley transform $C = C_e$, we are lead to the quasi-invertible triple $(C(e), C(-e), C(f)) = (\mathfrak{f}_-, \mathfrak{f}_+, z)$ with $z \in V_-^\times$ (Lemma III.2, Corollary B.3)

We have to show that the orbit of the group L^0 in $S^C := C(S) \subseteq X^{\tau^C}$ is open. The Lie algebra $\mathfrak{l} = C(\mathfrak{h})$ is adapted to the grading of $\mathfrak{g}$ (Lemma III.4), so that

$$\mathfrak{l}_{\mathfrak{f}_\pm} = \mathfrak{l}_\pm \oplus \mathfrak{l}_0 \quad \text{and} \quad \mathfrak{l}_{\mathfrak{f}_+, \mathfrak{f}_-} = \mathfrak{l}_0.$$

The argument in the proof of Proposition III.6 shows that the map $\mathfrak{l}_0 \to V, x \mapsto x.z$ is surjective, and since $\mathfrak{l}_0$ is the kernel of the surjective map

$$\mathfrak{l} \to T_{\mathfrak{f}_+}(S^C) \times T_{\mathfrak{f}_-}(S^C) = \mathfrak{l}_1 \oplus \mathfrak{l}_{-1}, \quad x \mapsto x.(e,-e) = (x_+, x_-),$$

we see that the map $\mathfrak{l} \to T_{\mathfrak{f}_+}(S^C) \times T_{\mathfrak{f}_-}(S^C) \times T_z(S^C)$ is surjective. In view of Proposition III.5, this implies that the L-orbit of $(\mathfrak{f}_+, \mathfrak{f}_-, z)$ in $(S^C)^3$ is open and therefore that the H-orbit of $(e,-e,f)$ in S^3 is open. ∎

So far we have seen that the H-orbits in $S^3_\top$ coincide with the connected components and that each such orbit contains an element of the form $(e,-e,C(v))$ for some $v \in C(V_-^\times)$. With the aid of the following lemma, we shall be able to reduce this further to the case where $v^2 = -e$.

Lemma III.8. *Let $(A,e,*)$ be a real unital involutive Banach algebra and $z \in A_-$ such that $\lambda z + e$ is invertible for each $\lambda \in \mathbb{R}$. If, in addition, z is invertible, then there exists a hermitian element $x = x^* \in A$ with $-z^2 = e^x$. Then $\sigma := ze^{-\frac{1}{2}x} \in A_-^\times$ satisfies $\sigma^2 = -\mathbf{1}$ and σ lies in the same connected component of $A_-^\times$ as z.*

Proof. The assumption $e + \lambda z \in A^\times$ for $\lambda \in \mathbb{R}^\times$ implies that $(z - \lambda e)(z + \lambda e) = z^2 - \lambda^2 e$ is invertible, so that $\mathrm{Spec}(-z^2) \cap]-\infty, 0[= \emptyset$.

Let $A_\mathbb{C}$ denote the complexification of $(A,*)$, endowed with the antilinear involution given by $(x+iy)^* := x^* - iy^*$. On the open subset

$$\Omega := \{w \in A_\mathbb{C} \colon \mathrm{Spec}(w) \cap]-\infty, 0] = \emptyset\}$$

we then have a holomorphic logarithm function

$$\log\colon \Omega \to A_\mathbb{C}, \quad \log(w) = \frac{1}{2\pi i} \oint_\gamma \log(\zeta)(\zeta\mathbf{1} - w)^{-1}\, d\zeta,$$

where γ is a piecewise smooth cycle in $\mathbb{C} \backslash]-\infty, 0]$ with winding number 1 in each point of $\mathrm{Spec}(w)$ ([Ru73, Ths. 10.20, 10.38]). In view of $\mathrm{Spec}(w^*) = \overline{\mathrm{Spec}(w)}$, the domain Ω is invariant under the involution, and we have

$$\log(w)^* = -\frac{1}{2\pi i} \oint_\gamma \log(\overline{\zeta})(\overline{\zeta}\mathbf{1} - w^*)^{-1}\, d\overline{\zeta}.$$

Since the winding number of $\overline{\gamma}$ in each point of $\mathrm{Spec}(w)$ is -1, we obtain

$$\log(w)^* = \frac{1}{2\pi i} \oint_\gamma \log(\zeta)(\zeta\mathbf{1} - w^*)^{-1}\, d\zeta = \log(w^*).$$

Therefore $x := \log(-z^2)$ is a hermitian element of $A_\mathbb{C}$ lying in the commutant of z. A similar argument applies to the antilinear involution $\tau\colon A_\mathbb{C} \to A_\mathbb{C}$ with $A = \{a \in A_\mathbb{C} \colon \tau(a) = a\}$ and shows that $\tau(\log w) = \log \tau(w)$ for $w \in \Omega$, hence in particular $x \in A_\mathbb{C}^\tau = A$. We clearly have $e^x = -z^2$.

For $\sigma_t := e^{-t\frac{1}{2}x} z = ze^{-t\frac{1}{2}x}$ we obtain

$$\sigma_t^* = z^* e^{-t\frac{1}{2}x} = -ze^{-t\frac{1}{2}x} = -\sigma_t \quad \text{and} \quad \sigma_1^2 = e^{-x}z^2 = -e.$$

For each $t \in \mathbb{R}$ the element $e^{-tx}z$ lies in $A_-^\times$, so that z and σ_1 lie in the same connected component of $A_-^\times$. ∎

Theorem III.9. *If the involutive 3-graded Lie group (G, D, τ) satisfies* (A1/2), *then each connected component of $S^3_\top$ contains an element of the form $(e, -e, \sigma)$ with $\sigma^* = -\sigma$ and $\sigma^2 = -e$.*

Proof. From Lemma III.2(5) we know that each connected component of $S^3_\top$ contains an element of the form $(e, -e, C(v))$ with $v \in V_-^\times$. Let $A \subseteq (V, e)$ denote the closed unital Jordan subalgebra generated by v and v^{-1}. In view of [Jac68, Ch. I, Sect. 11, Th. 13], A is a commutative associative algebra, hence a commutative Banach algebra in which v is invertible. Further $v^* = -v$ implies that A is invariant under the involution, hence an involutive Banach algebra.

We now consider the analytic map

$$\eta: \mathbb{R} \to V, \quad \lambda \mapsto (e - \lambda v)^{-1}.$$

There exists an $\varepsilon > 0$ such that the Neumann series $\sum_{n=0}^\infty \lambda^n v^n$ converges to $(e - \lambda v)^{-1}$ for $|\lambda| < \varepsilon$. This implies that $\eta(\lambda) \in A$ for all these λ. Since η is analytic and A is a closed subspace of V, we conclude with the Principle of Analytic Continuation that $\mathrm{im}(\eta) \subseteq A$, hence that $e - \lambda v$ is invertible in A for all $\lambda \in \mathbb{R}$.

Now Lemma III.8 applies to the element $v \in A$, and we find an element $\sigma \in A_-^\times$ in the same connected component as v, satisfying $\sigma^2 = -e$. Eventually Lemma III.2(5) implies that $(e, -e, C(\sigma))$ lies in the same connected component of $S^3_\top$ as $(e, -e, C(v))$. Further $\sigma^2 = -e$ leads to $\sigma(e - \sigma) = \sigma - \sigma^2 = \sigma + e$, which means that $C(\sigma) = (e + \sigma)(e - \sigma)^{-1} = \sigma$. This completes the proof. ∎

IV. Evaluating the index map

In the preceding section we have reduced the problem to calculate the index function $\mu_G: S^3_\top \to \pi_1(G^0)$ to triples of the form $(e, -e, \sigma)$ with $\sigma^2 = -e$ in the unital Jordan algebra (V, e). The next step is to calculate the index function on these triples explicitly by showing that $\mu_G(e, -e, \sigma)$ is represented by the group homomorphism

$$\chi_\sigma: \mathbb{T} \cong \mathbb{R}/\mathbb{Z} \to G^0, \quad t + \mathbb{Z} \mapsto \exp_G(\pi t[\sigma, \tau.e]).$$

Applying the representation $\rho_V: G^0 \to \mathrm{GL}(V)$, this leads to the loop

$$\mathbb{T} \cong \mathbb{R}/\mathbb{Z} \to G^0, \quad t + \mathbb{Z} \mapsto e^{\pi t 2L(\sigma)} = P(e^{\pi t \sigma}).$$

To obtain the explicit formula for the index, we first investigate functoriality properties of the index and then calculate it explicitly for the group $\mathrm{SL}_2(\mathbb{C})/\{\pm\mathbf{1}\}$.

Remark IV.1. (a) Let U and G be 3-graded Lie groups and $\varphi: U \to G$ a homomorphism of Lie groups compatible with the 3-grading.

We then have

$$\varphi(U^\pm) = \varphi(\exp \mathfrak{u}_\pm) = \exp \mathbf{L}(\varphi)\mathfrak{u}_\pm \subseteq \exp \mathfrak{g}_\pm = G^\pm \quad \text{and} \quad \varphi(U^0_0) \subseteq G^0_0.$$

For a subset $M \subseteq G$ we write $C_G(M)$ for the centralizer of M in G and for a subset $M \subseteq \mathrm{Aut}(\mathfrak{g})$ we write $C_G(M) := \mathrm{Ad}^{-1}(C_{\mathrm{Aut}(\mathfrak{g})}(M))$ for the set of all those elements $g \in G$ for which $\mathrm{Ad}(g)$ commutes with M. This means that for a grading element $E \in \mathfrak{g}_0$ we have $G^0 = C_G(\mathrm{ad}\, E)$. If there is a grading element $E_U \in \mathfrak{u}_0$ for which $E_G := \mathbf{L}(\varphi)E_U$ is a grading element of $\mathfrak{g}$, then we thus obtain

$$\varphi(U^0) = \varphi(C_U(\mathrm{ad}\, E_U)) \subseteq C_G(\mathrm{ad}\, E_G) = G^0.$$

Then φ induces a map $U^+U^0U^- \to G^+G^0G^-$ compatible with the projection maps $p_j^G: G^+G^0G^- \to G^j$ in the sense that

$$p_j^G \circ \varphi = \varphi \circ p_j^U, \quad j = +, 0, -.$$

For $z \in \mathfrak{u}_+$ and $w \in \mathfrak{u}_-$ the condition $\exp w \exp z \in U^+U^0U^-$ therefore implies

$$(\exp \mathbf{L}(\varphi)w)(\exp \mathbf{L}(\varphi)z) \in G^+G^0G^-,$$

which shows that $\mathbf{L}(\varphi)$ preserves quasi-invertibility, and for such pairs we have

$$\varphi \circ p_0^U(\exp w \exp z) = p_0^G((\exp \mathbf{L}(\varphi)w)(\exp \mathbf{L}(\varphi)z)).$$

(b) Now suppose, in addition, that U and G are involutive 3-graded Lie groups and that $\varphi \circ \tau_U = \tau_G \circ \varphi$. Then we conclude that for quasi-invertible pairs $(z, w) \in \mathfrak{u}_+$ the pair $(\mathbf{L}(\varphi).z, \mathbf{L}(\varphi).w))$ is quasi-invertible with

$$\varphi(B_U(z, w)) = B_G(\mathbf{L}(\varphi)z, \mathbf{L}(\varphi)w).$$

This relation leads to

$$\varphi(d_U(z_1, z_2, z_3)) = d_G(\mathbf{L}(\varphi)z_1, \mathbf{L}(\varphi)z_2, \mathbf{L}(\varphi)z_3)$$

for quasi-invertible triples $(z_1, z_2, z_3) \in (V_U)^3_\top$.

If U and G satisfy (A1-3), then we further get $\mathbf{L}(\varphi)(\mathcal{D}_U)$. To see that $\mathbf{L}(\varphi)$ also maps S_U into S_G, we first observe that we have an induced map

$$\varphi_X: X_U := U/U^0U^- \to X_G := G/G^0G^-$$

satisfying $\varphi_X \circ \tau_X^U = \tau_X^G \circ \varphi_X$ for the corresponding involutions τ_X^U on X_U and τ_X^G on X_G. Therefore φ_X maps the fixed point set of τ_U^X into the fixed point set of τ_G^X. On the open subset $V_U \subseteq X_U$ the map φ_X coincides with $\mathbf{L}(\varphi)$, and since $S_U = V_U \cap (X^U)^{\tau_U^X}$, we see that

$$\mathbf{L}(\varphi)S_U \subseteq S_G.$$

Eventually this leads to the important relation

$$\pi_1(\varphi|_{U^0}) \circ \mu_U(s_1, s_2, s_3) = \mu_G(\mathbf{L}(\varphi)s_1, \mathbf{L}(\varphi)s_2, \mathbf{L}(\varphi)s_3) \tag{4.1}$$

for quasi-invertible triples $(s_1, s_2, s_3) \in (S_U)^3_\top$. ∎

In the following we shall use the preceding remark as a tool to calculate the index of special triples in $S^3_\top$.

Lemma IV.2. *Let $e \in S$ and consider the corresponding unital involutive Jordan algebra $(V, e, Q(e))$. Suppose that $\sigma \in V_- \cap S$ is an element with $\sigma^2 = -e$. Then $E := \mathbb{R}e + \mathbb{R}\sigma$ is a real involutive Jordan subalgebra of V isomorphic to $(\mathbb{C}, 1)$ with the involution $z^* = \overline{z}$ and*

$$\mathfrak{g}_E := E + \tau(E) + [E, \tau(E)] \cong \mathfrak{sl}_2(\mathbb{C})$$

with the 3-grading defined by the grading element $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the antilinear involution

$$\tau\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} -\overline{a} & \overline{c} \\ \overline{b} & \overline{a} \end{pmatrix}.$$

There is a unique morphism $\eta_\sigma^{\mathfrak g}\colon \mathfrak{sl}_2(\mathbb C) \to \mathfrak g$ *of involutive Lie algebras with*

$$\eta_\sigma^{\mathfrak g}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e \quad \textit{and} \quad \eta_\sigma^{\mathfrak g}\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} = \sigma.$$

Proof. Clearly the map $\eta_E\colon \mathbb C \to V, x+iy \mapsto xe + y\sigma$ is a morphism of involutive unital Jordan algebras, where the involution on $\mathbb C$ is complex conjugation.

We recall from Theorem A.8 that $\mathfrak g_E$ is a Lie subalgebra of $\mathfrak g$. Since $\mathfrak g_E$ is generated by E and $\tau(E)$, its center $\mathfrak z_E$ coincides with the centralizer $\mathfrak z_E$ of $E + \tau(E)$, and the quotient $\mathfrak g_E' := \mathfrak g_E/\mathfrak z_E$ is an involutive 3-graded Lie algebra whose 0-component has a faithful representation on E. From that it easily follows that $\mathfrak g_E'$ is isomorphic to the Tits-Kantor-Koecher Lie algebra $\mathrm{TKK}(E) = \mathrm{TKK}(\mathbb C) \cong \mathfrak{sl}_2(\mathbb C)$ of the unital Jordan algebra $\mathbb C$ because it is an A_1-graded Lie algebra (cf. [Ne03, Ex. I.9(a),(c) for more details). Since all central extensions of the simple Lie algebra $\mathfrak{sl}_2(\mathbb C)$ are trivial, we conclude that $\mathfrak z_E \cap [\mathfrak g_E, \mathfrak g_E] = \{0\}$, so that $\mathfrak g_E \cap \mathfrak g_0 = [E, \tau(E)]$ implies $\mathfrak z_E = \{0\}$ and therefore $\mathfrak g_E \cong \mathfrak{sl}_2(\mathbb C)$.

In Definition I.9 we have seen that the Lie algebra $\mathfrak g_e = \mathrm{span}\{e, \tau(e), [e,\tau(e)]\}$ with 1-dimensional grading spaces is isomorphic to $\mathfrak{sl}_2(\mathbb R)$ with the involution

$$\tau_e\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & c \\ b & -d \end{pmatrix}.$$

Since the grading spaces $\mathfrak g_E \cap \mathfrak g_j$ are complex one-dimensional, it follows that $\mathfrak g_e$ is a real form of the complex Lie algebra $\mathfrak g_E$.

Next we determine the involution τ_E on $\mathfrak g_E \cong \mathfrak{sl}_2(\mathbb C)$ corresponding to the restriction of τ to $\mathfrak g_E$. Since the centroid

$$\mathrm{Cent}(\mathfrak g_E) = \{\varphi \in \mathrm{End}(\mathfrak g)\colon (\forall x \in \mathfrak g_E)\ [\varphi, \mathrm{ad}\, x] = 0\}$$

is isomorphic to $\mathbb C$ as an associative algebra, the involution τ induces a field isomorphism τ' on $\mathrm{Cent}(\mathfrak g_E)$. The involution τ_E is complex linear if this isomorphism is trivial and it is antilinear otherwise. We denote the scalar multiplication with i on $\mathfrak{sl}_2(\mathbb C)$ by i, which is considered as an element of $\mathrm{Cent}(\mathfrak g_E)$. Then $\sigma = i.e$ leads to $\tau.\sigma = \tau'(i)\tau(e)$. From

$$\begin{aligned} -ie = -\sigma = Q(e)\sigma = \tfrac12[[e,\tau.\sigma],e] &= -\tfrac12(\mathrm{ad}\, e)^2\tau.\sigma = -\tfrac12(\mathrm{ad}\, e)^2\tau'(i)\tau(e) \\ &= -\tau'(i)\tfrac12(\mathrm{ad}\, e)^2\tau(e) = \tau'(i)Q(e)e = \tau'(i)e \end{aligned}$$

we derive $\tau'(i) = -i$ and hence that τ_E is antilinear.

Therefore τ_E is determined by its restriction to the real form $\mathfrak g_e$, and hence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} -\overline a & \overline c \\ \overline b & -\overline d \end{pmatrix}$$

is the involution on $\mathfrak{sl}_2(\mathbb C)$ for which $\eta_\sigma^{\mathfrak g}$ is a morphism of involutive Lie algebras. ■

Lemma IV.3. *If G satisfies* (A3), *then the homomorphism* $\widetilde\eta_\sigma^G\colon \mathrm{SL}_2(\mathbb C) \to G$ *integrating* $\eta_\sigma^{\mathfrak g}$ *maps* $-\mathbf 1$ *to* $\mathbf 1$.

Proof. Since $A := \mathbb C$ is a hermitian Banach-$*$-algebra with respect to $z^* := \overline z$, the discussion of the special case of hermitian Banach-$*$-algebras in Example II.6 implies that

$$d_{\mathrm{SL}_2(\mathbb C)}(1,-1,i) = -\mathbf 1 \in \mathrm{SL}_2(\mathbb C).$$

Applying Remark III.1 to $\widetilde\eta_\sigma^G$, we conclude that $d_{\mathrm{SL}_2(\mathbb C)}(1,-1,i)$ is mapped to $d_G(e,-e,\sigma) = \mathbf 1$. ■

Proposition IV.4. *Suppose that the involutive* 3*-graded Lie group* G *satisfies* (A1/2). *Then* $d_G(S^3_\top)$ *is contained in* $Z(G_0)^\tau$ *and generates an elementary abelian* 2*-group* Γ *which is discrete. The group* G_0/Γ *satisfies* (A1-3).

Proof. Since the connected components of $S^3_\top$ coincide with the H-orbits, Theorem III.9 implies that for each quasi-invertible triple $(s_1, s_2, s_3) \in S^3_\top$ there exists an element $g \in H$ and a triple of the form $(e, -e, \sigma)$ with $Q(e)\sigma = -\sigma$ such that $(s_1, s_2, s_3) = g.(e, -e, \sigma)$. Then Lemma I.7(6) implies that

$$d_G(s_1, s_2, s_3) = J_G(g, z_1) d_G(e, -e, \sigma) J_G(g, z_1)^{-1}.$$

To see that $d_G(s_1, s_2, s_3) \in Z(G)^\tau$ is an involution, we may therefore assume w.l.o.g. that $(s_1, s_2, s_3) = (e, -e, \sigma)$ with $Q(e)\sigma = -\sigma$.

Let $\eta^{\mathfrak g}_\sigma: \mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak g$ denote the corresponding homomorphism of 3-graded Lie algebras constructed in Lemma IV.2. From Example II.6 we know that

$$d_{\mathrm{SL}_2(\mathbb{C})}(1, -1, i) = -\mathbf{1} \in \mathrm{SL}_2(\mathbb{C}).$$

Applying Remark III.1 to the homomorphism $\widetilde\eta^G_\sigma: \mathrm{SL}_2(\mathbb{C}) \to G$ integrating $\eta^{\mathfrak g}_\sigma$, we conclude that

$$d_G(e, -e, \sigma) = \widetilde\eta^G_\sigma(d_{\mathrm{SL}_2(\mathbb{C})}(1, -1, i)) = \widetilde\eta^G_\sigma(-\mathbf{1}).$$

The involution on $\mathrm{SL}_2(\mathbb{C})$ fixes $-\mathbf{1}$, which leads to $d_G(e, -e, \sigma) \in G^\tau$. Since $\mathfrak g$ decomposes as a direct sum of $\mathfrak{sl}_2(\mathbb{R})$-modules isomorphic to the trivial and the adjoint modules (Remark I.10(b)), we have $\mathrm{Ad}(\widetilde\eta^G_\sigma(-\mathbf{1})) = \mathbf{1}$, so that $\widetilde\eta^G_\sigma(-\mathbf{1}) \in Z(G_0)$. Therefore $d_G(e, -e, \sigma)$ is a central τ-invariant involution in G_0.

Further Lemma I.7 implies that the map $d_G: S^3_\top \to Z(G_0)^\tau$ is constant on the H-orbits and alternating.

The image of d_G consists of central involutions, hence the group Γ it generates is an elementary abelian 2-group. Since the Banach–Lie group G contains no small subgroups, there exists an identity neighborhood $U \subseteq G$ with $U \cap \Gamma = \{\mathbf{1}\}$, so that Γ is discrete.

We conclude that $\widehat G := G_0/\Gamma$ is a Lie group with the same Lie algebra $\mathfrak g$, and since Γ is τ-invariant, this Lie group is involutive. Clearly (A1/2) also holds for this quotient group, and $d_G(S^3_\top) \subseteq \Gamma$ leads to $d_{\widehat G}(S^3_\top) = \{\mathbf{1}\}$ in $\widehat G^0 = G^0_0/\Gamma$. ■

Definition IV.5. In the following we write $\eta^G_\sigma: \mathrm{SL}_2(\mathbb{C})/\{\pm\mathbf{1}\} \to G$ for the unique morphism of 3-graded involutive Lie groups with $\mathbf{L}(\eta^G_\sigma) = \eta^{\mathfrak g}_\sigma$ whose existence follows from the simple connectedness of $\mathrm{SL}_2(\mathbb{C})$ and Lemma IV.3. ■

According to Remark IV.1, we have

$$\mu_G(e, -e, \sigma) = \pi_1(\eta^G_\sigma)\mu_{\mathrm{SL}_2(\mathbb{C})/\{\pm\mathbf{1}\}}(1, -1, i).$$

Therefore the calculation of the index map is essentially reduced to the calculation of the single case $\mu_{\mathrm{SL}_2(\mathbb{C})/\{\pm\mathbf{1}\}}(1, -1, i)$.

The next proposition provides the index function for $\mathrm{SL}_2(\mathbb{C})/\{\pm\mathbf{1}\}$.

Proposition IV.6. *We consider the* 3*-graded involutive Lie group* $G := \mathrm{SL}_2(\mathbb{C})/\{\pm\mathbf{1}\}$ *which satisfies* (A1-3) *by* Theorem II.7. *We have an isomorphism*

$$\rho: G^0 = \left\{ \pm \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{C}^\times \right\} \to \mathbb{C}^\times, \quad \pm \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \mapsto z^2$$

and identify $\pi_1(G^0)$ *accordingly with* $\pi_1(\mathbb{C}^\times) \cong \mathbb{Z}$, *where we use* $p_{\mathbb{C}^\times}: \mathbb{C} \to \mathbb{C}^\times, z \mapsto e^{2\pi i z}$ *as the universal covering map. In these terms we have*

$$\mu_G(1, -1, \pm i) = \mp 1.$$

Proof. In the following we shall use the explicit formulas from the discussion of hermitian Banach algebras in Example II.6. We have

$$B_{\mathrm{SL}_2(\mathbb{C})}(z,w) = \begin{pmatrix} 1 - z\overline{w} & 0 \\ 0 & (1 - z\overline{w})^{-1} \end{pmatrix},$$

which leads to

$$B_G(z,w) = (1 - z\overline{w})^2$$

in terms of our identification of G^0 with $\mathbb{C}^\times$. From that we further obtain for quasi-invertible triples (z_1, z_2, z_3):

$$\begin{aligned} d_G(z_1,z_2,z_3) &= (1 - z_1\overline{z_2})^2(1 - z_3\overline{z_2})^{-2}(1 - z_3\overline{z_1})^2(1 - z_2\overline{z_1})^{-2}(1 - z_2\overline{z_3})^2(1 - z_1\overline{z_3})^{-2} \\ &= \Big(\frac{1 - z_1\overline{z_2}}{1 - z_2\overline{z_1}}\Big)^2 \Big(\frac{1 - z_3\overline{z_1}}{1 - z_1\overline{z_3}}\Big)^2 \Big(\frac{1 - z_2\overline{z_3}}{1 - z_3\overline{z_2}}\Big)^2. \end{aligned}$$

We obtain in particular

$$d_G(z_1,z_2,0) = \Big(\frac{1 - z_1\overline{z_2}}{1 - z_2\overline{z_1}}\Big)^2 \quad \text{and} \quad d_G(1,-1,z_3) = \Big(\frac{1 - z_3}{1 - \overline{z_3}}\Big)^2 \Big(\frac{1 + \overline{z_3}}{1 + z_3}\Big)^2.$$

For the curve

$$\alpha_1\colon [0,1] \to \mathbb{C}^3_\top, \quad t \mapsto (t, -t, 0)$$

from $(0,0,0)$ to $(1,-1,0)$ this leads to $d_G(\alpha_1(t)) = (1 - t^2)(1 - t^2)^{-1} = 1$. For the path

$$\alpha_2\colon [0,1] \to \mathbb{C}^3_\top, \quad t \mapsto (1, -1, \pm ti)$$

from $(1,-1,0)$ to $(1,-1,\pm i)$ we obtain

$$d_G(\alpha_2(t)) = \Big(\frac{1 \mp it}{1 \pm it}\Big)^2 \Big(\frac{1 \mp it}{1 \pm it}\Big)^2 = \Big(\frac{1 \mp it}{1 \pm it}\Big)^4 = e^{8i\arg(1 \mp it)}.$$

This curve describes a loop in $\mathbb{C}^\times$ corresponding to the element $\mp 1 \in \mathbb{Z} \cong \pi_1(\mathbb{C}^\times)$.

Concatenating the two paths α_1 and α_2, we obtain a path from $(0,0,0)$ to $(1,-1,\pm i)$ which lies in the contractible set

$$\overline{\mathcal{D}}^3_\top = \{(z_1,z_2,z_3) \in \mathbb{C}^3 \colon (\forall j \neq k)\ |z_j| \leq 1, z_j\overline{z_k} \neq 1\}.$$

We conclude that this path is homotopic to the path

$$\alpha_3\colon [0,1] \to \mathbb{C}^3_\top, \quad t \mapsto (t, -t, \pm ti),$$

and this implies the assertion. ∎

Theorem IV.7. *Let $e \in S$ and $\sigma \in S$ with $Q(e)\sigma = \sigma^* = -\sigma$. Then the index of $(e,-e,\sigma)$ is represented by the homomorphism*

$$\chi_\sigma\colon \mathbb{T} = \mathbb{R}/\mathbb{Z} \to G^0, \quad t \mapsto \exp_G(-\pi t[\sigma, \tau.e])$$

and composing with the representation ρ_V on V leads to the homomorphism

$$\rho_V \circ \chi_\sigma\colon \mathbb{T} = \mathbb{R}/\mathbb{Z} \to \mathrm{GL}(V), \quad t \mapsto P(e^{-\pi t\sigma}).$$

Proof. In terms of the Lie group structure, the index of $(1,-1,i)$ for $\mathrm{SL}_2(\mathbb{C})/\{\pm\mathbf{1}\}$ is represented by the loop

$$[0,1] \to \mathrm{SL}_2(\mathbb{C})^0/\{\pm\mathbf{1}\}, \quad t \mapsto \exp\begin{pmatrix} -\pi it & 0 \\ 0 & \pi it \end{pmatrix}.$$

In view of Remark IV.1, $\mu_G(e,-e,\sigma)$ can be represented by the homomorphism

$$\mathbb{R}/\mathbb{Z} \to G, \quad t + \mathbb{Z} \mapsto \eta^G_\sigma\Big(\exp\begin{pmatrix} -\pi it & 0 \\ 0 & \pi it \end{pmatrix}\Big) = \exp_G(-\pi t[\sigma, \tau.e])$$

because $h = [e, \tau.e]$ implies that $ih = [ie, \tau.e] = [\sigma, \tau.e]$ (cf. Definition I.9). Applying the representation ρ_V, we get the loop

$$\mathbb{R}/\mathbb{Z} \to \mathrm{GL}(V), \quad t + \mathbb{Z} \mapsto e^{-2\pi t(\sigma \Box e)} = e^{-2\pi t L(\sigma)} = P(e^{-\pi t\sigma})$$

in the unital Jordan algebra (V,e). Here we use the relation $P(e^x) = e^{2L(x)}$ which holds in every Banach–Jordan algebra (cf. [FK94, Prop. II.3.4]). ∎

Proposition IV.8. *Suppose that $\mathfrak{g}$ is a complex Lie algebra and that τ is antilinear. Then $(V, e, Q(e))$ is a complex unital Jordan algebra and the involution $Q(e)v = v^*$ is antilinear. For each hermitian projection $p = p^* = p^2 \in V_+$ let*

$$\gamma_p : \mathbb{R}/\mathbb{Z} \to G^0, \quad t + \mathbb{Z} \mapsto \exp(2\pi i t[p, \tau.p])$$

denote the corresponding projection loop, which is a group homomorphism. We then have for the involution $\sigma = e - 2p$ the projection loop formula

$$\mu_G(e, -e, -i\sigma) = \mu_G(e, -e, -ie) - [\gamma_p]$$

Proof. We have $V_- = iV_+$, so that every unitary element in V_- is of the form $i\sigma$, where $\sigma \in V_+$ is a hermitian involution. Then $p := \frac{1}{2}(e - \sigma)$ is a hermitian idempotent in the Jordan algebra (V, e) with $\sigma = e - 2p$.

The index $\mu_G(e, -e, -i\sigma)$ can be calculated directly from the real τ-invariant subalgebra generated by e and $-i\sigma$, which is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. As we have seen in Theorem IV.6, this leads to the one-parameter subgroup $\mathbb{T} \to G^0$ corresponding to the element

$$\pi[i\sigma, \tau.e] \in \exp^{-1}(\mathbf{1}).$$

In particular, the index $\mu_G(e, -e, -ie)$ corresponds to the element

$$\pi[ie, \tau.e] = \pi i[e, \tau.e] \in \exp^{-1}(\mathbf{1}),$$

and the difference is the element

$$\pi[ie - i\sigma, \tau.e] = \pi i[e - \sigma, \tau.e] = 2\pi i[p, \tau.e] = 2\pi i[p, \tau.p], \tag{4.2}$$

which belongs to the Lie algebra $\mathfrak{g}_p := \mathrm{span}_{\mathbb{C}}\{p, \tau.p, [p, \tau.p]\} \cong \mathfrak{sl}_2(\mathbb{C})$ (cf. Definition I.9), where $h := [p, \tau.p]$ corresponds to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$ which satisfies $\exp(2\pi i h) = \mathbf{1}$. From (4.2) we now derive the projection loop formula because $[e, \tau.e]$ is central in $\mathfrak{g}_0$ (Remark I.11). ■

V. The Maslov index for some examples

In this section we give more concrete formulas for the index function for several classes of hermitian Banach-$*$-algebras and discuss the case of finite-dimensional bounded symmetric domains.

Example V.1. We take a closer look at the index function for the case $G = \mathrm{GL}_2(A)/\{\pm\mathbf{1}\}$ for a hermitian Banach-$*$-algebra.

Then $H = U_{1,1}(A, *)_0$ and $G^0 = (A^\times \times A^\times)/\{\pm\mathbf{1}\}$. Note that $-\mathbf{1} \in A_0^\times$ follows from the connectedness of $\mathbb{C}^\times\mathbf{1}$. Therefore $G_0^0 \cong (A_0^\times \times A_0^\times)/\{\pm\mathbf{1}\}$, and the covering map $A_0^\times \times A_0^\times \to G_0^0$ leads to an exact sequence

$$\pi_1(A^\times) \times \pi_1(A^\times) \hookrightarrow \pi_1(G^0) \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}. \tag{5.1}$$

The exactness of this sequence follows from the long exact homotopy sequence of the covering. We can also think of $\pi_1(G^0)$ as the set of homotopy classes of paths $\gamma: [0,1] \to \mathrm{GL}_2(A)$ starting in $\mathbf{1}$ and ending either in $\mathbf{1}$ or $-\mathbf{1}$.

The Maslov index of a triple $(e, -e, -i\sigma)$, where σ is a hermitian involution, is given by the loop

$$\chi_\sigma : \mathbb{R}/\mathbb{Z} \to G^0, \quad t + \mathbb{Z} \mapsto= \exp_G(\pi i t[\sigma, \tau.e]).$$

More explicitly we have

$$[\sigma, \tau.e] = \left[\begin{pmatrix} 0 & \sigma \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix} \right] = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix},$$

and since σ is an involution, we have $\exp_G \left(\pi i \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} \right) = \mathbf{1}$.

Writing σ as $\mathbf{1} - 2p$ for a hermitian projection p, we get the decomposition

$$\begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} - 2 \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix},$$

and the latter element already leads to a loop in the group $\mathrm{GL}_2(A)$. In this sense we get

$$[\chi_\sigma] = [\chi_{\mathbf{1}}] - ([\gamma_p], -[\gamma_p]),$$

where γ_p is the projection loop defined by p in A, where we consider the pair $([\gamma_p], -[\gamma_p])$ as an element of $\pi_1(A^\times) \times \pi_1(A^\times)$ according to (5.1). ∎

Example V.2. For the special case $A = C(X, \mathbb{C})$ we have $A^\times = C(X, \mathbb{C}^\times)$, and the exponential map

$$\exp_A : C(X, \mathbb{C}) \to C(X, \mathbb{C}^\times), \quad f \mapsto e^{2\pi i f}$$

is the universal covering of the identity component $A_0^\times$, consisting of all maps $X \to \mathbb{C}^\times$ homotopic to a constant map. This shows that

$$\pi_1(A^\times) \cong \ker \exp = C(X, \mathbb{Z}).$$

On the other hand each hermitian projection $p \in A$ is a continuous function $X \to \{0, 1\}$, so that the index of $(\mathbf{1}, -\mathbf{1}, -i\sigma)$ is of the form

$$[\chi_{\mathbf{1}}] + (p, -p) \in [\chi_{\mathbf{1}}] + (C(X, \mathbb{Z}) \times C(X, \mathbb{Z})) \subseteq \pi_1(G^0).$$

In this case $S = U(A) = C(X, \mathbb{T})$ and

$$\pi_0(S) \cong \pi_0(C(X, \mathbb{T})) \cong [X, \mathbb{T}] \cong \check{H}^1(X, \mathbb{Z})$$

is the set of homotopy classes of continuous maps $X \to \mathbb{T}$, resp., the first Čech cohomology group. ∎

Example V.3. If, moreover, X is a finite set, so that $A := C(X, \mathbb{C}) \cong \mathbb{C}^n$ for $n := |X|$, then $C(X, \mathbb{Z}) \cong \mathbb{Z}^n$ and

$$G^0 \cong (\mathbb{C}^\times)^n \times (\mathbb{C}^\times)/\{\pm \mathbf{1}\} \cong \mathbb{C}^{2n}/(2\pi i \mathbb{Z}^{2n} + \pi i (1, \ldots, 1)) \cong (\mathbb{C}^\times)^{2n}.$$

Here we see in particular that $\pi_1(G^0)$ is a free group, so that the sequence (5.1) does not split. ∎

Example V.4. Fix $q \in [1, \infty]$ and let $\mathcal{H}$ be an infinite-dimensional Hilbert space. We consider the hermitian Banach-$*$ algebra $A := B_q(\mathcal{H}) + \mathbb{C}\mathbf{1}$, where $B_q(\mathcal{H})$ is the ideal of $B(\mathcal{H})$ consisting of all operators of Schatten class q. For $q = \infty$ the ideal $B_\infty(\mathcal{H})$ coincides with the space of compact operators on $\mathcal{H}$.

We write

$$\mathrm{GL}_q(\mathcal{H}) := (B_q(\mathcal{H}) + \mathbf{1}) \cap \mathrm{GL}(\mathcal{H})$$

for the group of all invertible operators in $\mathbf{1} + B_q(\mathcal{H})$ and recall that

$$\pi_1(\mathrm{GL}_q(\mathcal{H})) \cong \varinjlim \pi_1(\mathrm{GL}_n(\mathbb{C})) \cong \mathbb{Z}$$

(cf. [Ne04, Ths. A.10/11]). Each projection loop corresponding to a 1-dimensional subspace of $\mathcal{H}$ generates this group.

Since $B_q(\mathcal{H})$ is an ideal of A complemented by $\mathbb{C}\mathbf{1}$, we have

$$A^\times \cong \mathrm{GL}_q(\mathcal{H}) \times \mathbb{C}^\times$$

and therefore

$$\pi_1(A^\times) \cong \pi_1(\mathrm{GL}_q(\mathcal{H})) \times \mathbb{Z} \cong \mathbb{Z}^2.$$

Accordingly we write $A^\times \times A^\times \cong \mathrm{GL}_q(\mathcal{H})^2 \times (\mathbb{C}^\times)^2$ and

$$G^0 \cong \mathrm{GL}_q(\mathcal{H})^2 \times \Big((\mathbb{C}^\times \times \mathbb{C}^\times)/\{\pm(1,1)\}\Big)$$

with

$$\pi_1(G^0) \cong \mathbb{Z}^2 \times \{(n,m) \in \tfrac{1}{2}\mathbb{Z}^2 : n - m \in 2\mathbb{Z}\}. \cong \mathbb{Z}^2 \times (\mathbb{Z}^2 + \mathbb{Z}\tfrac{1}{2}(1,1)).$$

If $p \in A$ is a hermitian projection, then either p or $\mathbf{1} - p$ has finite rank. If p has finite rank, then the corresponding projection loop γ_p satisfies

$$[\gamma_p] = \operatorname{tr} p = \dim(p.\mathcal{H}) \in \mathbb{Z} \cong \pi_1(\mathrm{GL}_q(\mathcal{H})).$$

If $\mathbf{1} - p$ has finite rank, then $p = (p - \mathbf{1}) + \mathbf{1}$ leads to

$$[\gamma_p] = (\operatorname{tr}(p - \mathbf{1}), 1) \in \mathbb{Z}^2 \cong \pi_1(\mathrm{GL}_q(\mathcal{H})) \times \pi_1(\mathbb{C}^\times).$$

Therefore the index of $(\mathbf{1}, -\mathbf{1}, -i(\mathbf{1} - 2p))$ is given by

$$\mu_G(\mathbf{1}, -\mathbf{1}, -i(\mathbf{1} - 2p)) = \begin{cases} (-\operatorname{tr} p, \operatorname{tr} p, (\frac{1}{2}, \frac{1}{2})) & \text{for } \operatorname{rk} p < \infty \\ (-\operatorname{tr}(p - \mathbf{1}), \operatorname{tr}(p - \mathbf{1}), (\frac{1}{2}, \frac{1}{2}) - (1, -1)) & \text{for } \operatorname{rk} p = \infty \end{cases}$$

■

Example V.5. For von Neumann algebras, one has refined information on the relation between projections and loops in $A^\times$ (cf. [ASS71]): Let $\mathcal{H}$ be a separable Hilbert space and $A \subseteq B(\mathcal{H})$ a von Neumann algebra. Then the following assertions hold:

(a) For two projections $p, q \in \mathrm{Idem}(A, *)$ the condition $p \sim q$ and $\mathbf{1} - p \sim \mathbf{1} - q$ is equivalent to lying in the same path component of $\mathrm{Idem}(A, *)$.

(b) $\pi_1(A^\times)$ is generated by $\mathrm{Hom}(\mathbb{T}, A^\times)$ and hence by the projection loops.

(c) If A is a factor of infinite type, then $A^\times$ is simply connected.

(d) If A is a factor of type II_1, then $\pi_1(A^\times) \cong \mathbb{R}$, where $\pi_1(Z(A^\times)$ corresponds to $\mathbb{Z}$. For a projection $p \in \mathrm{Idem}(A, *)$ the projection loop γ_p then corresponds to the element $\operatorname{tr} p \in [0, 1] \subseteq \mathbb{R} \cong \pi_1(A^\times)$. In particular we have $[\gamma_p] = [\gamma_q]$ if and only if $\operatorname{tr} p = \operatorname{tr} q$ ([ASS71, Th. 3.3]) ■

Example V.6. If $\mathcal{D}$ is a finite-dimensional bounded symmetric domain of tube type and $H = \mathrm{Aut}(\mathcal{D})_0$, then the corresponding Jordan triple V contains invertible tripotents. We assume that $\mathcal{D}$ is irreducible of rank r, i.e., H is a simple Lie group of real rank r and $G = H_{\mathbb{C}}$.

Let us fix $e \in S$, so that $(V, e, Q(e))$ is a unital involutive Jordan algebra. The real form $V_+ := \{v \in V : v^* = v\}$ is a euclidean Jordan algebra. Therefore the set $V_+^\times$ of invertible hermitian elements and its connected components contain the involutions of the form $e - 2p$, where p is a hermitian projection whose rank lies in $\{0, 1, 2, \ldots, r\}$. It follows in particular that there are $r + 1$ connected components (cf. [FK94]). In this case the index function is determined by its values on the triples $(e, -e, -i(e - 2p))$, where p is a fixed hermitian projection of rank k.

Since in this case the representation ρ_V is faithful, we have already seen in Remark IV.7 that the homotopy class $\mu_G(e, -e, -i(e - 2p)) \in \pi_1(G^0)$ is represented by the loop

$$\mathbb{T} = \mathbb{R}/\mathbb{Z} \to \mathrm{GL}(V), \quad t + \mathbb{Z} \mapsto e^{2\pi i t} \cdot e^{-\pi i t 4(p \Box p)} = e^{2\pi i t(\mathrm{id}_V - 2(p \Box p))}.$$

In view of the Pierce decomposition of V, the operator $2p\Box p = 2p\Box e = 2L(p)$ is diagonalizable with possible eigenvalues $\{0,1,2\}$, so that the formula above defines indeed a loop. We further have

$$e^{2\pi it(\mathrm{id}_V - 2(p\Box p))} = e^{2\pi itL(e-2p)} = P(e^{\pi it(e-2p)}).$$

For the determinant function $\det\colon \mathrm{GL}(V) \to \mathbb{C}^\times$ and a linear endomorhism $D \in \mathrm{End}(V)$ with integral eigenvalues, composition of the loop $t \mapsto e^{2\pi itD}$ with det leads to the loop $e^{2\pi it \operatorname{tr} D}$ in $\mathbb{C}^\times$, which corresponds to the element $\operatorname{tr} D \in \mathbb{Z} \cong \pi_1(\mathbb{C}^\times)$. For $n := \dim V$ we therefore get the function

$$\pi_1(\det) \circ \mu_G \colon S^3_\top \to \pi_1(\mathbb{C}^\times) \cong \mathbb{Z}$$

with

$$(e, -e, -i(e-2p)) \mapsto \operatorname{tr} L(e-2p) = n - 2\operatorname{tr} L(p) = n - 2k\frac{n}{r} = \frac{n}{r}(r-2k),$$

which is, up to the factor $\frac{n}{r}$, the Maslov index defined in [CØ01]. ■

Problem V. (a) Is μ_G a cocycle in the sense that

$$\mu_G(z_1,z_2,z_3) = \mu_G(z_1,z_2,z_4) + \mu_G(z_2,z_3,z_4) + \mu_G(z_1,z_4,z_3)?$$

(b) Is the index function invariant under the full group G^τ? This would follow if G^0 acts trivially on $\pi_1(G^0)$, but this is certainly not always the case because G may be of the form $G = G_1 \times G_2$ with $G_2 \subseteq G^0$ and G_2 can be any Lie group.

If A is a hermitian Banach-$*$-algebra, then $\mathrm{GL}_2(A)^0 \cong A^\times \times A^\times$. In this case the problem from above leads to the question whether $\pi_0(A^\times)$ act trivially on $\pi_1(A^\times)$. This is not always the case, as we see for $A = M_2(\mathbb{R})$ with the involution $a \mapsto a^\top$. In this case $\pi_0(A^\times) = \pi_0(\mathrm{GL}_2(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(A^\times) = \pi_1(\mathrm{GL}_2(\mathbb{R})) = \pi_1(\mathrm{SL}_2(\mathbb{R})) \cong \mathbb{Z}$, where the group $\pi_0(A^\times)$ acts by inversion on $\pi_1(A^\times)$. ■

Appendix A. Jordan triple systems and Jordan algebras

In this appendix we collect some basic facts on Jordan algebras and Jordan triples over a field $\mathbb{K}$ with $2,3 \in \mathbb{K}^\times$.

Definition A.1. (a) A vector space V over a field $\mathbb{K}$ is said to be a *Jordan triple system* (JTS) if it is endowed with a trilinear map $\{\cdot\}\colon V \times V \times V \to V$ satisfying:
(JT1) $\{x,y,z\} = \{z,y,x\}$.
(JT2) $\{a,b,\{x,y,z\}\} = \{\{a,b,x\},y,z\} - \{x,\{b,a,y\},z\} + \{x,y,\{a,b,z\}\}$ for all $a,b,x,y,z \in V$.
For $x,y \in V$ we define operators $x\Box y$, $Q(x)$ and $Q(x,z)$ on V by

$$(x\Box y).z := \{x,y,z\}, \quad Q(x)(y) := \{x,y,x\}, \quad Q(x,z)(y) := \{x,y,z\}.$$

The *Bergman operator of* V is defined by

$$B(x,y) := \mathbf{1} - 2x\Box y + Q(x)Q(y).$$

We define the set of *invertible elements of* V by $V^\times := \{v \in V \colon Q(v) \in \mathrm{GL}(V)\}$ and the *inversion map* by $V^\times \to V^\times, v \mapsto v^\sharp := Q(v)^{-1}.v$. The elements of the set

$$S := \{v \in V^\times \colon v^\sharp = v\} = \{v \in V^\times \colon \{v,v,v\} = v\}$$

are called *involutions*, resp., *invertible tripotents.* ■

Lemma A.2. *If $3 \in \mathbb{K}^\times$ and $(V, \{\cdot,\cdot,\cdot\})$ is a Jordan triple system, then the following formulas hold for $x, y, z \in V$:*
(1) $Q(x).\{y,x,z\} = \{Q(x).y, z, x\} = \{x, y, Q(x).z\}$.
(2) $Q(x)(y\Box x) = (x\Box y)Q(x) = Q(Q(x).y, x)$.
(3) $[Q(x)Q(y), x\Box y] = 0$.
(4) $2(x\Box y)^2 - Q(x)Q(y) = x\Box(Q(y)x) = (Q(x)y)\Box y$.
(5) $Q(x, Q(z)y) = 2(z\Box y)Q(x,z) - Q(z)(y\Box x)$.
(6) $Q(Q(x)y) = Q(x)Q(y)Q(x)$.
(7) *For $x \in V^\times$ we have $Q(x)^{-1}$ and $(x^\sharp)^\sharp = x$.*
(8) $B(x,y)Q(x) = Q(x - Q(x).y)$.
(9) $B(x,y)Q(z)B(y,x) = Q(B(x,y).z)$.

Proof. (1)-(5) can be found in [Ro00, Prop. I.2.1], (6) is [Ro00, Prop. I.4.1], and (8),(9) are [Ro00, Props. I.5.1/2].

(1) From (JT2) we derive with (JT1)

$$\begin{aligned}\{x,y,\{x,z,x\}\} &= \{\{x,y,x\},z,x\} - \{x,\{y,x,z\},x\} + \{x,z,\{x,y,x\}\}\\ &= 2\{\{x,y,x\},z,x\} - \{x,\{y,x,z\},x\}\\ &= 2\{x,y,\{x,z,x\}\} - 2\{x,\{y,x,z\},x\} + 2\{\{x,z,x\},y,x\} - \{x,\{y,x,z\},x\}\\ &= 4\{x,y,\{x,z,x\}\} - 3\{x,\{y,x,z\},x\}.\end{aligned}$$

This implies

$$3\{x,y,\{x,z,x\}\} = 3\{x,\{y,x,z\},x\},$$

so that $3 \in \mathbb{K}^\times$ leads to

$$\{x,y,\{x,z,x\}\} = \{x,\{y,x,z\},x\}.$$

This proves that the first and third term are equal. The equality of the first and the second term now follows from (JT1).

(2) follows directly from (1).

(3) is an immediate consequence of (2).

(4) First we observe that

$$\{x,y,\{x,y,z\}\} = \{\{x,y,x\},y,z\} - \{x,\{y,x,y\},z\} + \{x,y,\{x,y,z\}\}$$

implies that

$$(Q(x)y)\Box y = x\Box(Q(y)x).$$

We further have

$$\begin{aligned}\{\{x,y,x\},y,z\} = \{z,y,\{x,y,x\}\} &= \{\{z,y,x\},y,x\} - \{x,\{y,z,y\},x\} + \{x,y,\{z,y,x\}\}\\ &= 2\{\{z,y,x\},y,x\} - \{x,\{y,z,y\},x\}\end{aligned}$$

which leads to

$$(Q(x)y)\Box y = 2(x\Box y)^2 - Q(x)Q(y).$$

(5) is the identity

$$\{x,a,\{z,y,z\}\} = 2\{\{x,a,z\},y,z\} - \{z,\{a,x,y\},z\}.$$

(6) From (5) we obtain by replacing z by x and x by $Q(x)y$ with (2) the relation

$$\begin{aligned}Q(Q(x)y) = Q(Q(x)y, Q(x)y) &= 2(x\Box y)Q(Q(x)y, x) - Q(x)(y\Box Q(x)y)\\ &= 2(x\Box y)(x\Box y)Q(x) - Q(x)(2(y\Box x)^2 - Q(y)Q(x))\\ &= 2(x\Box y)^2Q(x) - 2Q(x)(y\Box x)^2 + Q(x)Q(y)Q(x)\\ &= Q(x)Q(y)Q(x).\end{aligned}$$

(7) Using (6), we get

$$Q(x^\sharp) = Q(Q(x)^{-1}.x) = Q(x)^{-1}Q(x)Q(x)^{-1} = Q(x)^{-1},$$

and further

$$(x^\sharp)^\sharp = Q(x^\sharp).x^\sharp = Q(x)^{-1}.x^\sharp = x.$$

(8) From $B(x,y) = \mathbf{1} - 2x\Box y + Q(x)Q(y)$ we derive with (2) that

$$\begin{aligned}B(x,y)Q(x) &= Q(x) - 2(x\Box y)Q(x) + Q(x)Q(y)Q(x) = Q(x) - 2Q(Q(x).y,x) + Q(x)Q(y)Q(x)\\ &= Q(x - Q(x).y).\end{aligned}$$

(9) [Ro00, p.444] ∎

Theorem A.3. *Suppose that* $2,3 \in \mathbb{K}^\times$ *.*

(a) *If* J *is a Jordan algebra, then* J *is a Jordan triple system with respect to*

$$\{x,y,z\} = (xy)z + x(yz) - y(xz), \qquad i.e., \quad x\Box y = L(xy) + [L(x), L(y)],$$

where we write $L(x)y := xy$ *for the left multiplications in* J*. We have*

$$Q(x) = P(x) := 2L(x)^2 - L(x^2).$$

(b) *If* V *is a Jordan triple system and* $a \in V$*, then*

$$x \cdot_a y := \{x, a, y\}$$

defines on V *the structure of a Jordan algebra whose quadratic representation is given by*

$$P(v) := 2L(v)^2 - L(v^2) = Q(v)Q(a).$$

The Jordan triple structure determined by the Jordan product $\cdot_a$ *is given by*

$$\{x,y,z\}_a = \{x, \{a,y,a\}, z\} = \{x, Q(a).y, z\}.$$

It coincides with the original one if $Q(a) = \mathbf{1}$*.*

Proof. (cf. [Jac68, Ch. I, Sects. 8,12]) This is proved in [Ne03, Theorem C.4], up to the formula for the quadratic representation, which follows from

$$P(v) = 2L(v)^2 - L(v^2) = 2(v\Box a)^2 - (Q(v)a)\Box a = Q(v)Q(a)$$

(Lemma A.2(4)). ∎

Lemma A.4. *In a Jordan triple system* V *the following assertions hold:*

(1) $x\Box x^\sharp = \mathrm{id}_V$ *for each* $x \in V^\times$*.*

(2) $S = \{x \in V : x\Box x = \mathrm{id}_V\}$*.*

(3) $Q(e)^2 = \mathrm{id}_V$ *holds for each* $e \in S$*.*

Proof. (1) In view of Lemma A.2(2), we have

$$Q(x) = Q(x,x) = Q(x, Q(x)x^\sharp) = (x\Box x^\sharp)Q(x),$$

so that the invertibility of $Q(x)$ implies (1).

(2), (3) If $e \in S$, then $e = e^\sharp$ and (1) imply $e\Box e = \mathrm{id}_V$.

If, conversely, $e\Box e = \mathrm{id}_V$, then $Q(e)e = \{e,e,e\} = e$. Further Lemma A.2(4) implies

$$2\,\mathrm{id}_V - Q(e)^2 = e\Box Q(e)e = e\Box e = \mathrm{id}_V,$$

which leads to $Q(e)^2 = \mathrm{id}_V$. Hence e is invertible and $e^\sharp = Q(e)^{-1}e = Q(e)e = e$. ∎

Proposition A.5. (a) *Let* $(V,\{\cdot,\cdot,\cdot\})$ *be a Jordan triple system and* $e \in S$ *an invertible tripotent. Then*

$$ab := \{a,e,b\}, \quad a^* := \{e,a,e\}$$

defines on V *the structure of an involutive Jordan algebra and the Jordan triple structure can be reconstructed from* $(V,e,*)$ *by*

$$\{x,y,z\} = (xy^*)z + x(y^*z) - y^*(xz), \quad x,y,z \in V.$$

The set S *of involutions of the Jordan triple* V *coincides with the set*

$$S = \{v \in V : v^* = v^{-1}\}$$

of unitary elements of the unital involutive Jordan algebra $(V,e,*)$.

(b) *If* $(V,e,*)$ *is a unital involutive Jordan algebra, then*

$$\{x,y,z\} := (xy^*)z + x(y^*z) - y^*(xz), \quad x,y,z \in V$$

defines a Jordan triple structure on V *with*

$$ab = \{a,e,b\} \quad \textit{and} \quad a^* = \{e,a,e\}.$$

Proof. (a) It follows from Theorem A.3 that $ab := \{a,e,b\}$ defines on V a Jordan algebra structure with multiplication maps $L(a) = a\Box e$. In particular $L(e) = e\Box e = \mathrm{id}_V$, so that e is an identity of V. Moreover, $Q(e)^2 = \mathrm{id}_V$ follows from Lemma A.4(3). Next

$$(a^2)^* = Q(e)Q(a)e = Q(e)Q(a)Q(e)e = Q(Q(e)a)e = Q(a^*)e = (a^*)^2,$$

and polarization leads to $(ab)^* = a^*b^*$ for $a,b \in V$.

Finally Theorem A.3(b) entails

$$(xy^*)z + x(y^*z) - y^*(xz) = \{x, Q(e)y^*, z\} = \{x, Q(e)^2y, z\} = \{x,y,z\}.$$

The condition $z \in S$ means $z^\sharp = z$, so that the description of the set S in terms of the involutive Jordan algebra follows from $(z^\sharp)^* = Q(e)Q(z)^{-1}z = P(z)^{-1}z = z^{-1}$. ■

Remark A.6. If $a \in V^\times$ is invertible, then $xy := \{x,a^\sharp,y\}$ defines on V the structure of a Jordan algebra with identity a because $L(a) = a\Box a^\sharp = \mathrm{id}_V$ (Lemma A.4). ■

Proposition A.7. *Let* V *be a Jordan triple and* $a,b,c \in V$ *with* $c \in V^\times$ *and* $a+b+c=0$. *Then*

$$Q(a)Q(c)^{-1}Q(b) = Q(b)Q(c)^{-1}Q(a).$$

Proof. We consider the unital Jordan algebra (V,c) with the product $xy := \{x,c^\sharp,y\}$ (Remark A.6). Then the quadratic representation of this Jordan algebra is given by

$$P(v) = Q(v)Q(c^\sharp) = Q(v)Q(c)^{-1}.$$

Therefore it suffices to show that $P(a)P(b) = P(b)P(a)$. As $b = -c-a$ and c is the identity element, we have

$$P(b) = P(-c-a) = P(c+a) = P(c) + 2P(c,a) + P(a) = \mathrm{id}_V + 2L(a) + P(a),$$

and this operator commutes with $P(a)$ because $L(a)$ commutes with $L(a^2)$. ■

Theorem A.8. *If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ is a 3-graded Lie algebra with an involutive automorphism τ satisfying $\tau(\mathfrak{g}_j) = \mathfrak{g}_{-j}$ for $j = 0, \pm 1$, then $V := \mathfrak{g}_1$ is a Jordan triple system with respect to $\{x, y, z\} := \frac{1}{2}\big[[x, \tau.y], z\big]$.*

If $E \subseteq V$ is a Jordan subtriple, then $\mathfrak{g}_E := E + \tau(E) + [E, \tau(E)] \subseteq \mathfrak{g}$ is a τ-invariant 3-graded subalgebra.

Proof. The first part is contained in [Ne03, Theorem C.3].

For the second part, let $E \subseteq V$ be a Jordan subtriple. Then the elements $[v, \tau w] \in \mathfrak{g}_0$, $v, w \in E$, act on V as the operators $v \Box w$, hence preserve the Jordan subtriple E. We conclude that $[[E, \tau(E)], E] \subseteq E$, and by applying τ, we also obtain $[[E, \tau(E)], \tau(E)] \subseteq \tau(E)$. We further have

$$[[v, \tau w], [v', \tau w']] = [[[v, \tau w], v'], \tau w'] + [v', [[v, \tau w], \tau w']],$$

showing that $[E, \tau(E)]$ is a subalgebra of $\mathfrak{g}_0$. Therefore $\mathfrak{g}_E$ is a subalgebra of $\mathfrak{g}$. ∎

Lemma A.9. *In a unital Jordan algebra (V, e) we have for invertible elements $v, w \in V^\times$ the relations*

$$L(v^{-1}) = P(v)^{-1} L(v) = L(v) P(v)^{-1} \quad \text{and} \quad P(v^{-1} + w^{-1}) = P(w)^{-1} P(v + w) P(v)^{-1}.$$

Proof. First we observe that the canonical Jordan triple structure on V turns it into a Jordan triple system with $Q(x) = P(x)$ for all $x \in V$ and $L(x) = x \Box e$ (Theorem A.3). Putting $x = e$, $y = v$ and $z = v^{-1}$ in Lemma A.2(5), we get with Lemma A.4:

$$\begin{aligned} L(v^{-1}) &= v^{-1} \Box e = (Q(v)^{-1}.v) \Box e = Q(e, Q(v^{-1}).v) = 2(v^{-1} \Box v) Q(e, v^{-1}) - Q(v^{-1})(v \Box e) \\ &= 2Q(e, v^{-1}) - Q(v^{-1}) L(v) = 2L(v^{-1}) - Q(v^{-1}) L(v), \end{aligned}$$

and therefore $L(v^{-1}) = Q(v)^{-1} L(v)$ (cf. [Jac68, Ch. I, Sect. 11, Th. 13]). Note that the Jordan identity $[L(v), L(v^2)] = 0$ means that $Q(v) = P(v)$ commutes with $L(v)$.

To derive the second identity, we first calculate

$$\begin{aligned} P(e + v^{-1}) P(v) &= \big(P(e) + 2P(e, v^{-1}) + P(v^{-1})\big) P(v) = P(v) + 2L(v^{-1}) P(v) + \mathrm{id}_V \\ &= P(v) + 2L(v) + \mathrm{id}_V = P(e + v). \end{aligned}$$

Now we consider the unital Jordan algebra (V, w) with the isotopic product $a *_w b := \{a, w^{-1}, b\}$ and the quadratic representation

$$\widetilde{P}(v) = Q(v) Q(w)^{-1} = P(v) P(w)^{-1}$$

(Theorem A.3, Lemma A.4). Then we obtain with the formula in the preceding paragraph and

$$P(w).v^{-1} = P(w) P(v)^{-1}.v = \widetilde{P}(v)^{-1}.v$$

the relation

$$\begin{aligned} P(v^{-1} + w^{-1}) &= P\big(P(w)^{-1}.(P(w).v^{-1} + w)\big) = P(w)^{-1} P(P(w).v^{-1} + w) P(w)^{-1} \\ &= P(w)^{-1} \widetilde{P}(w + P(w).v^{-1}) = P(w)^{-1} \widetilde{P}(w + \widetilde{P}(v)^{-1}.v) \\ &= P(w)^{-1} \widetilde{P}(w + v) \widetilde{P}(v)^{-1} = P(w)^{-1} P(w + v) P(w)^{-1} P(w) P(v)^{-1} \\ &= P(w)^{-1} P(w + v) P(v)^{-1}. \end{aligned}$$

∎

Lemma A.10. *For invertible elements x, y in the Jordan triple V we have*
(1) $Q(x)Q(x^\sharp + y^\sharp)Q(y) = Q(x+y)$ *and*
(2) $B(x, y^\sharp) = Q(x-y)Q(y)^{-1}$.

Proof. (1) We consider on V the unital Jordan algebra structure defined by $ab := \{a, x^\sharp, b\}$ with unit x (Remark A.6). Then the quadratic representation of the unital Jordan algebra (V, x) is given by $P(v) = Q(v)Q(x)^{-1}$ and the Jordan inversion by $v^{-1} = P(v)^{-1}.v = Q(x)v^\sharp$ (Theorem A.3). Hence Lemma A.9 leads to

$$\begin{aligned} Q(x)Q(x^\sharp + y^\sharp)Q(x) &= Q(Q(x)x^\sharp + Q(x)y^\sharp) = Q(x + y^\sharp) = P(x+y^\sharp)Q(x) \\ &= P(x+y)P(y^\sharp)Q(x) = P(x+y)P(y)^{-1}Q(x) = Q(x+y)Q(y)^{-1}Q(x). \end{aligned}$$

This completes the proof.

(2) In view of Lemma A.2(8) and (1), assertion (2) follows from

$$\begin{aligned} B(x, y^\sharp) &= Q(x - Q(x)y^\sharp)Q(x)^{-1} = Q(x)Q(x^\sharp - y^\sharp)Q(x)Q(x)^{-1} \\ &= Q(x)Q(x^\sharp - y^\sharp) = Q(x-y)Q(y)^{-1}. \end{aligned}$$

∎

Lemma A.11. *Let (V, e) be a unital Jordan algebra and $V^\times$ the set of invertible elements in V. Then the Cayley transform*

$$C\colon V^\times + e \to V^\times - e, \quad z \mapsto (e+z)(e-z)^{-1}$$

is a bijective map with $C^{-1}(z) = -C(-z)$ which further satisfies
(1) $C(z)^{-1} = C(-z)$ *for* $z \pm e \in V^\times$ *and* $C^2(z) = -z^{-1}$ *if* $z, e - z \in V^\times$.
(2) $P(C(z)) = P(e+z)P(e-z)^{-1}$ *for* $z - e \in V^\times$.
(3) $c(e, -e, z) = 4P(C(z))^{-1}$ *for* $z \pm e \in V^\times$.
(4) $d(e, -e, z) = P(C(z))^{-1}P(C(z))^*$ *for* $z \pm e \in V^\times$.

Proof. (1) From

$$C(z) + e = (e + z + e - z)(e-z)^{-1} = 2(e-z)^{-1} \in V^\times$$

we see that $C(-C(z))$ is defined, and an easy calculation leads to $C(-C(z)) = -z$ for $z \in V^\times + e$. This implies that $-C$ is an involution of the subset $V^\times - e$ of V and that

$$C^{-1}(z) = -C(-z) = -(e-z)(e+z)^{-1} = (z-e)(z+e)^{-1}.$$

Moreover, if $C(z)$ is invertible, then we have

$$C(z)^{-1} = (e-z)(e+z)^{-1} = C(-z),$$

showing also that this happens if and only if $e \pm z$ are invertible. If z and $z - e$ are invertible, then $z^{-1} - e$ is invertible and we get

$$C(z^{-1}) = (e + z^{-1})(e - z^{-1})^{-1} = (z+e)(z-e)^{-1} = -C(z),$$

showing that $C^2(z^{-1}) = C(-C(z)) = -z$ and therefore $C^2(z) = -z^{-1}$.

(2) For $z - e \in V^\times$ we get with Lemma A.9:

$$\begin{aligned} P(C(z)) &= P(C(z) + e - e) = P(2(e-z)^{-1} - e) \\ &= P(e - 2(e-z)^{-1}) = P(e - \tfrac{1}{2}(e-z))P(-\tfrac{1}{2}(e-z))^{-1} \\ &= P(\tfrac{1}{2}(e+z))P(-\tfrac{1}{2}(e-z))^{-1} = P(e+z)P(e-z)^{-1}. \end{aligned}$$

(3) In view of Lemma A.10(2), we have

$$\begin{aligned} c(e, -e, z) &= B(e, -e)B(z, -e)^{-1}B(z, e) = B(e, -e^{-1})B(z, -e^{-1})^{-1}B(z, e^{-1}) \\ &= Q(e+e)Q(z+e)^{-1}Q(z-e)Q(e) = P(2e)P(z+e)^{-1}P(z-e)P(e) \\ &= 4P(z-e)P(z+e)^{-1} = 4P(C(z))^{-1}. \end{aligned}$$

(4) is an immediate consequence of (3) and

$$d(e, -e, z) = c(e, -e, z)c(e, z, -e)^{-1} = c(e, -e, z)(c(e, -e, z)^{-1})^*$$

∎

Appendix B. Transversality of 3-filtrations

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$ not of characteristic 2 or 3. In this appendix we shall explain some general fact on inner 3-filtrations of Lie algebras. We shall closely follow the setup in [BN04a], from which we shall refine one result that is crucial for the present paper.

Our basic objects are on the one hand 3*-graded Lie algebras*, i.e., Lie algebras of the form $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ satisfying the relations $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ for $\alpha, \beta \in \{-1, 0, 1\}$, and on the other hand 3*-filtered Lie algebras*, i.e., Lie algebras $\mathfrak{g}$ with a flag $\mathfrak{f} : \{0\} = \mathfrak{f}_2 \subset \mathfrak{f}_1 \subset \mathfrak{f}_0 \subset \mathfrak{g}$ of subalgebras such that $[\mathfrak{f}_\alpha, \mathfrak{f}_\beta] \subset \mathfrak{f}_{\alpha+\beta}$. For simplicity we shall also write these flags as pairs $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_0)$. If $\mathfrak{g}$ is 3-graded, then the 3-grading is the eigenspace decomposition for a unique derivation $D \in \mathrm{der}(\mathfrak{g})$ with $D(X) = iX$ for $X \in \mathfrak{g}_i$. The derivation D is called the *characteristic element of the grading*, and if $D = \mathrm{ad}(E)$, E will be called an *Euler operator*. For a 3-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with corresponding derivation D there are *two* naturally associated filtrations $\mathfrak{f}_+ := \mathfrak{f}_+(D) := (\mathfrak{g}_1, \mathfrak{g}_1 \oplus \mathfrak{g}_0)$ and $\mathfrak{f}_- := \mathfrak{f}_-(D) := (\mathfrak{g}_{-1}, \mathfrak{g}_{-1} \oplus \mathfrak{g}_0)$. We write

$$\mathcal{F} = \{\mathfrak{f}_+(D) : D \in \mathcal{G}\}$$

for the *space of inner* 3*-filtrations of* $\mathfrak{g}$. The space $\mathcal{F}$ carries an interesting geometric structure. First we have a transversality relation $\top$ on $\mathcal{F} \times \mathcal{F}$ defined by

$$\mathfrak{e} = (\mathfrak{e}_1, \mathfrak{e}_0) \top \mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_0) \quad \Leftrightarrow \quad \mathfrak{g} = \mathfrak{e}_1 \oplus \mathfrak{f}_0 = \mathfrak{f}_1 \oplus \mathfrak{e}_0.$$

A key result on the structure of 3-graded Lie algebras ([BN04a, Th. 1.6]) asserts that the set of transversal pairs in $\mathcal{F}$ corresponds to the set of inner 3-gradings of $\mathfrak{g}$, where the 3-grading associated to the pair $(\mathfrak{e}, \mathfrak{f})$ is determined by

$$(B.1) \qquad \mathfrak{g}_1 = \mathfrak{e}_1, \quad \mathfrak{g}_0 = \mathfrak{e}_0 \cap \mathfrak{f}_0 \quad \text{and} \quad \mathfrak{g}_{-1} = \mathfrak{f}_1.$$

For $\mathfrak{e} \in \mathcal{F}$ we write

$$\mathfrak{e}^\top := \{\mathfrak{f} \in \mathcal{F} : \mathfrak{e} \top \mathfrak{f}\}$$

for the set of filtrations transversal to $\mathfrak{e}$.

The group $\mathrm{Aut}(\mathfrak{g})$ acts naturally on $\mathcal{F}$ by $g.(\mathfrak{e}_1, \mathfrak{e}_0) := (g.\mathfrak{e}_1, g.\mathfrak{e}_0)$, preserving the transversality relation, and it also acts on $\mathcal{G}$. For any inner 3-filtration $\mathfrak{e}$ and $x \in \mathfrak{e}_1$ we have $(\mathrm{ad}\, x)^3 = 0$ because $\mathrm{ad}\, x(\mathfrak{g}) \subseteq \mathfrak{e}_0$ and $(\mathrm{ad}\, x)^2(\mathfrak{g}) \subseteq \mathfrak{e}_1$. Since 2 and 3 are invertible in $\mathbb{K}$,

$$e^{\mathrm{ad}\, x} := \mathbf{1} + \mathrm{ad}\, x + \tfrac{1}{2}(\mathrm{ad}\, x)^2$$

defines an automorphism of $\mathfrak{g}$. In [BN04a] we show that the set $\mathfrak{e}^\top$ of filtrations transversal to a given filtration $\mathfrak{e}$ carries a natural structure of an affine space over $\mathbb{K}$ with translation group $e^{\mathrm{ad}\, \mathfrak{e}_1} \cong (\mathfrak{e}_1, +)$ which acts as a subgroup of $\mathrm{Aut}(\mathfrak{g})$ on $\mathcal{F}$.

We fix an inner 3-grading $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ of $\mathfrak{g}$ and consider an involutive automorphism τ of $\mathfrak{g}$ with $\tau(\mathfrak{g}_i) = \mathfrak{g}_{-i}$ for $i = -1, 0, 1$. For the associated flags $\mathfrak{f}_+ = (\mathfrak{g}_1, \mathfrak{g}_0 + \mathfrak{g}_1)$ and $\mathfrak{f}_- = (\mathfrak{g}_{-1}, \mathfrak{g}_0 + \mathfrak{g}_{-1})$ this means that $\tau.\mathfrak{f}_\pm = \mathfrak{f}_\mp$ in $\mathcal{F}$. Hence the involution $g \mapsto \tau.g := \tau g \tau$ of $\mathrm{Aut}(\mathfrak{g})$ preserves $G(\mathfrak{f}_+, \mathfrak{f}_-)$. We also write $g^\tau := \tau g \tau$ to simplify the notation.

Definition B.1. On the vector space $V := \mathfrak{g}_1$ we define by $\{x, y, z\} := \frac{1}{2}[[x, \tau.y], z]$ the structure of a Jordan triple system (Theorem A.5). In terms of Lie triple systems we then have on V the relations

$$Q(x).y = -\tfrac{1}{2}(\mathrm{ad}\, x)^2 \circ \tau \quad \text{and} \quad x \Box y = \tfrac{1}{2}\,\mathrm{ad}[x, \tau.y]|_V = \tfrac{1}{2}\,\mathrm{ad}\, x\, \mathrm{ad}(\tau.y)\,|_V$$

which shows in particular that the set $V^\times$ of invertible elements in the Jordan triple V does not depend on the involution τ. The corresponding *Bergman operator* is given by

$$B(x, y) = \mathbf{1} - 2x \Box y + Q(x)Q(y) = \mathbf{1} - \mathrm{ad}\, x\, \mathrm{ad}\, \tau.y + \frac{1}{4}(\mathrm{ad}\, x)^2(\mathrm{ad}\, \tau.y)^2. \qquad \blacksquare$$

The following proposition is a slight refinement of [BN04a, 5.2].

Proposition B.2. *Let τ be an involution of $\mathfrak{g}$ with $\tau(\mathfrak{g}_i) = \mathfrak{g}_{-i}$ for $i = -1, 0, 1$ and $\mathfrak{f}_\pm$ the corresponding two 3-filtrations. We identify the Jordan triple $V = \mathfrak{g}_1$ with the subset $e^{\operatorname{ad}\mathfrak{f}_+}.\mathfrak{f}_-$ of $\mathcal{F}$ via the map $v \mapsto e^{\operatorname{ad} v}.\mathfrak{f}_-$. Let $\tau_{\mathcal{F}}$ denote the involution of the set $\mathcal{F}$ induced by the involution τ. Then*

$$\tau_{\mathcal{F}}^{-1}(V) \cap V = V^\times$$

is the set of invertible elements in V, and for $v \in V^\times$ we have

$$\tau_{\mathcal{F}}(v) = v^\sharp = Q(v)^{-1}.v.$$

Proof. With respect to the 3-grading of $\mathfrak{g}$, we write each automorphism $g \in \operatorname{Aut}(\mathfrak{g})$ as a matrix $g = (g_{ij})$ with $g_{ij} \in \operatorname{Hom}(\mathfrak{g}_j, \mathfrak{g}_i)$. Let $E \in \mathfrak{g}_0$ be such that $\operatorname{ad} E$ is a derivation defining the grading of $\mathfrak{g}$. For $x \in V$ we define

$$d_g(x) := (e^{-\operatorname{ad} x} g^{-1})_{11}, \quad c_g(x) := (g e^{\operatorname{ad} x})_{-1,-1} \quad \text{and} \quad n_g(x) := (e^{-\operatorname{ad} x} g^{-1} E)_1.$$

In view of [BN04, Cor. 1.10, Th. 2.8], $g.x \in V$ is equivalent to the invertibility of $d_g(x)$ and $c_g(x)$, and in this case

$$g.x = d_g(x)^{-1} n_g(x).$$

For $g := \tau$ we have $g_{ij} = 0$ for $i \neq -j$, and therefore

$$d_\tau(x) := (e^{-\operatorname{ad} x}\tau)_{11} = \tfrac{1}{2}(\operatorname{ad} x)^2 \tau|_{\mathfrak{g}_1} = -Q(x).$$

Further

$$c_\tau(x) := (\tau e^{\operatorname{ad} x})_{-1,-1} = \tau \tfrac{1}{2}(\operatorname{ad} x)^2|_{\mathfrak{g}_{-1}} = -\tau Q(x) \tau|_{\mathfrak{g}_{-1}}.$$

This shows that $\tau_{\mathcal{F}}.x \in V$ is equivalent to $x \in V^\times$. Eventually the fact that τ reverses the grading implies $\tau.E + E \in \mathfrak{z}(\mathfrak{g})$, so that

$$n_\tau(x) = (e^{-\operatorname{ad} x}\tau.E)_1 = (e^{-\operatorname{ad} x}.(-E))_1 = [x, E] = -x.$$

We conclude that

$$\tau_{\mathcal{F}}(x) = d_\tau(x)^{-1} n_\tau(x) = -Q(x)^{-1}.(-x) = Q(x)^{-1}.x = x^\sharp. \qquad \blacksquare$$

Remark B.3. Let $x \in V = \mathfrak{g}_1$. Then the pairs $(\mathfrak{f}_+, \mathfrak{f}_-)$ and $(\mathfrak{f}_+, e^{\operatorname{ad} x}.\mathfrak{f}_-)$ are transversal, so that the triple $(\mathfrak{f}_+, \mathfrak{f}_-, e^{\operatorname{ad} x}.\mathfrak{f}_-)$ is transversal if and only if $e^{\operatorname{ad} x}.\mathfrak{f}_-$ is transversal to $\mathfrak{f}_-$, i.e., $\tau_{\mathcal{F}}(e^{\operatorname{ad} x}.\mathfrak{f}_-)$ is transversal to $\mathfrak{f}_+ = \tau_{\mathcal{F}}(\mathfrak{f}_-)$. In view of Proposition B.2, this is equivalent to $x \in V^\times$. ■

Appendix C. Tripotents and the Peirce decomposition

In this appendix we briefly discuss the Peirce decomposition of a Jordan triple with respect to a tripotent e and the representation of the corresponding $\mathfrak{sl}_2$-subalgebra on $\mathfrak{g}$.

Lemma C.1. (Peirce decomposition) *For each tripotent $e \in V$ the operator $2e\Box e$ is diagonalizability with eigenvalues in $\{0, 1, 2\}$, and for the corresponding eigenspaces V_α we have*

$$\{V_\alpha, V_\beta, V_\gamma\} \subseteq V_{\alpha-\beta+\gamma}. \tag{C.1}$$

The tripotent e is invertible if and only if $V = V_2$.

Proof. We put $D := e\Box e$. First Lemma A.2(4) leads to the relation

$$2(e\Box e)^2 - Q(e)^2 = e\Box(Q(e)e) = e\Box e,$$

and hence to

$$2D^2 - D = Q(e)^2. \tag{C.2}$$

On the other hand, Lemma A.2(2) yields $Q(e) = DQ(e) = Q(e)D$, so that multiplication of (C.2) with D entails

$$2D^3 - D^2 = DQ(e)^2 = Q(e)^2 = 2D^2 - D,$$

and further

$$0 = 2D^3 - 3D^2 + D = D(D - \mathbf{1})(2D - \mathbf{1}).$$

Since the three roots of this polynomial are different, D is diagonalizable with eigenvalues in $\{0, \frac{1}{2}, 1\}$. The relation (C.1) is a consequence of the fact that D is a Lie triple derivation by (JT2).

If e is invertible, then $Q(e)^2 = 2D^2 - D = D(2D - \mathbf{1})$ is invertible, so that $D = \mathrm{id}_V$, i.e., $V = V_2$. If, conversely, $V = V_2$, i.e., $D = \mathbf{1}$, then $Q(e)^2 = \mathrm{id}_V$ implies that $Q(e)$ is an involution, hence invertible. ■

In the following $\mathfrak{g}$ denotes a 3-graded Lie algebra with involution τ reversing the grading and $V = \mathfrak{g}_1$ carries the Jordan triple structure from Theorem A.8.

Definition C.2. A triple (e, h, f) of elements of $\mathfrak{g}$ is called an $\mathfrak{sl}_2$*-triple* if

$$[e, f] = h, \quad [h, e] = 2e \quad \text{and} \quad [h, f] = -2f.$$

It is called a *graded* $\mathfrak{sl}_2$*-triple if* $e \in \mathfrak{g}_1$ *and* $f \in \mathfrak{g}_{-1}$. ■

Lemma C.3. *If* $e \in V = \mathfrak{g}_1$ *is a tripotent, then* $(e, [e, \tau.e], \tau.e)$ *is a graded* $\mathfrak{sl}_2$*-triple.*

Proof. We have $[h, e] = 2\{e, e, e\} = 2e$ and $[h, f] = \tau[\tau h, e] = -\tau[h, e] = -2\tau e = -2f$. ■

Proposition C.4. *Let* $\mathfrak{g}$ *be a* 3*-graded Lie algebra.*

(1) *If* $x \in \mathfrak{g}_1$ *is such that the linear map* $(\mathrm{ad}\, x)^2 \colon \mathfrak{g}_{-1} \to \mathfrak{g}_1$ *is bijective, then there exist unique elements* $y \in \mathfrak{g}_{-1}$ *and* h *in* $\mathfrak{g}$ *such that* (x, h, y) *is a graded* $\mathfrak{sl}_2$*-triple. In this case* $\frac{1}{2}h \in \mathfrak{g}_0$ *is a grading element.*

(2) *If* (x, h, y) *is a graded* $\mathfrak{sl}_2$*-triple such that* $\frac{1}{2}h \in \mathfrak{g}_0$ *is a grading element, then* $(\mathrm{ad}\, x)^2 \colon \mathfrak{g}_{-1} \to \mathfrak{g}_1$ *is bijective.*

Proof. (1) Our assumption implies that there exists a unqiue element $y \in \mathfrak{g}_{-1}$ with $-\frac{1}{2}(\mathrm{ad}\, x)^2.y = x$. This implies already the uniqueness assertion. To prove existence, we put $h := [x, y] \in \mathfrak{g}_0$. The definition of y then implies that

$$[h, x] = -(\mathrm{ad}\, x)^2.y = 2x.$$

Further

$$[x, [h, y]] = [[x, h], y] + [h, [x, y]] = [-2x, y] = -2h$$

leads to

$$-\frac{1}{2}(\mathrm{ad}\, x)^2.[h, y] = [x, h] = -2x,$$

and hence to $[h, y] = -2y$ by the injectivity of $(\mathrm{ad}\, x)^2$ on $\mathfrak{g}_{-1}$.

We recall the following formulas from elementary $\mathfrak{sl}_2$-theory ([Bou90, Ch. VIII, §1, no. 1, Lemma 1]):

$$[\mathrm{ad}\, h, (\mathrm{ad}\, x)^n] = 2n(\mathrm{ad}\, x)^n, \quad [\mathrm{ad}\, h, (\mathrm{ad}\, y)^n] = -2n(\mathrm{ad}\, y)^n$$

and

$$[\mathrm{ad}\, y, (\mathrm{ad}\, x)^n] = -n(\mathrm{ad}\, x)^{n-1}\big(\mathrm{ad}\, h + (n-1)\,\mathrm{id}\big) = -n\big(\mathrm{ad}\, h - (n-1)\,\mathrm{id}\big)(\mathrm{ad}\, x)^{n-1}. \tag{C.3}$$

For $w \in \mathfrak{g}_{-1}$ we have $(\operatorname{ad} x)^3.w = 0$ and therefore

$$0 = \operatorname{ad} y (\operatorname{ad} x)^3.w = [\operatorname{ad} y, (\operatorname{ad} x)^3].w = -3(\operatorname{ad} x)^2\big(\operatorname{ad} h + 2\mathbf{1}\big).w.$$

Since $(\operatorname{ad} x)^2 \mid_{\mathfrak{g}_{-1}}$ is injective, we get $[h, w] = -2w$. This further leads to $[h, (\operatorname{ad} x)^2.w] = 2(\operatorname{ad} x)^2.w$ and hence to $[h, v] = 2v$ for all $v \in \mathfrak{g}_1$.

This implies that $[h, [\mathfrak{g}_1, \mathfrak{g}_{-1}]] = \{0\}$ and in particular $[h, [x, \mathfrak{g}_{-1}]] = \{0\}$. Since the map $(\operatorname{ad} x)^2 \colon \mathfrak{g}_{-1} \to \mathfrak{g}_1$ is bijective,

$$\operatorname{ad} x\,|_{[x,\mathfrak{g}_{-1}]} \colon [x, \mathfrak{g}_{-1}] \to \mathfrak{g}_1$$

also is bijective. Thus $\operatorname{ad} x([x, \mathfrak{g}_{-1}]) = \mathfrak{g}_1$ and hence

$$\mathfrak{g}_0 = [x, \mathfrak{g}_{-1}] \oplus (\ker \operatorname{ad} x \cap \mathfrak{g}_0).$$

For $z \in \mathfrak{g}_0 \cap \ker \operatorname{ad} x$ the operators $\operatorname{ad} z$ and $\operatorname{ad} x$ commutes, so that

$$(\operatorname{ad} x)^2([y, z]) \in -\operatorname{ad} z (\operatorname{ad} x)^2.y = -2 \operatorname{ad} z.x = 0,$$

and therefore $[y, z] = 0$. This also implies that $[h, z] = 0$, and we conclude that $h \in \mathfrak{z}(\mathfrak{g}_0)$. Hence $\frac{1}{2}h$ is a grading element.

(2) For $w \in \mathfrak{g}_{-1}$ the relations $[y, w] = 0$ and $[h, w] = -2w$ imply that w generates an at most 3-dimensional submodule for the Lie subalgebra $\mathfrak{g}_x := \operatorname{span}_{\mathbb{K}}\{x, y, h\}$ ([Bou90, Ch. VIII, §1, no. 1, Lemma 1]).

For $n = 2$ we get with (C.3) and $[y, w] = 0$:

$$\operatorname{ad} y (\operatorname{ad} x)^2.w = [\operatorname{ad} y, (\operatorname{ad} x)^2].w = -2(\operatorname{ad} x)\big(\operatorname{ad} h + \mathrm{id}\big).w = 2 \operatorname{ad} x.w.$$

Therefore $(\operatorname{ad} x)^2.w = 0$ implies $[x, w] = 0$, so that $0 = [h, w] = -\frac{1}{2}w$. We conclude that $(\operatorname{ad} x)^2\,|_{\mathfrak{g}_{-1}}$ is injective.

For $w \in \mathfrak{g}_1$ the relations $[h, w] = 2w$ and $[x, w] = 0$, together with the relation

$$(C.4) \qquad [\operatorname{ad} x, (\operatorname{ad} y)^n] = n(\operatorname{ad} y)^{n-1}\big(\operatorname{ad} h - (n-1)\operatorname{id}\big) = n\big(\operatorname{ad} h + (n-1)\operatorname{id}\big)(\operatorname{ad} y)^{n-1}$$

leads to

$$(\operatorname{ad} x)^2(\operatorname{ad} y)^2.w = (\operatorname{ad} x).[\operatorname{ad} x, (\operatorname{ad} y)^2].w = (\operatorname{ad} x).(2 \operatorname{ad} y.w) = 2 \operatorname{ad}[x, y].w = 4w,$$

and hence to $w \in (\operatorname{ad} x)^2(\mathfrak{g}_{-1})$. ■

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