A geometrically exact Cosserat shell-model including size effects, avoiding degeneracy in the thin shell limit. Rigourous justification via Γ-convergence for the elastic plate.

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Abstract

We are concerned with the derivation of the Γ-limit to a three-dimensional geometrically exact Cosserat model as the relative thickness $h > 0$ of a flat domain tends to zero. The Cosserat bulk model involves already exact rotations as a second independent field. It is shown that the Γ-limit based on a natural scaling assumption consists of a membrane like energy contribution and a homogenized transverse shear energy both scaling with h, augmented by an additional curvature stiffness due to the underlying Cosserat bulk formulation, also scaling with h. No specific bending term appears in the dimensional homogenization process. The formulation exhibits an internal length scale L_c which survives the homogenization process. A major technical difficulty, which we encounter in applying the Γ-convergence arguments, is to establish equi-coercivity of the sequence of functionals as the relative thickness h tends to zero. Usually, equi-coercivity follows from a local coerciveness assumption. While the three-dimensional problem is well-posed for the Cosserat couple modulus $\mu_c \geq 0$, equi-coercivity forces us to assume a strictly positive Cosserat couple modulus $\mu_c > 0$. The Γ-limit model determines the midsurface deformation $m \in H^{1,2}(\omega, \mathbb{R}^3)$. For the case of zero Cosserat couple modulus $\mu_c = 0$ we obtain an estimate of the $\Gamma - \liminf$ and $\Gamma - \limsup$, without equi-coercivity which is then strenghtened to a Γ-convergence result for zero Cosserat couple modulus. The classical linear Reissner-Mindlin model is "almost" the linearization of the Γ-limit for $\mu_c = 0$ apart from a stabilizing shear energy term.

Key words: shells, plates, membranes, thin films, polar materials, non-simple materials, Γ-convergence, homogenization, transverse shear, shear correction factor.
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Contents

1 Introduction

1.1 Aspects of shell theory

The dimensional reduction of a given continuum-mechanical model is already an old and mature subject and it has seen many "solutions". The different approaches toward elastic shell theory proposed in the literature and relevant references thereof are, therefore, too numerous to list here. One possible way to proceed is the so called **derivation approach**, i.e., reducing a given three-dimensional model via physically reasonable constitutive assumptions on the kinematics to a two-dimensional model. This is opposed to either the **intrinsic approach** which views the shell from the onset as a two-dimensional surface and invokes concepts from differential geometry or the **asymptotic methods** which try to establish two-dimensional equations by formal expansion of the three-dimensional solution in power series in terms of a small (thickness) parameter. The intrinsic approach is closely related to the **direct approach** which takes the shell to be a two-dimensional medium with additional **extrinsic directors** in the sense of a **restricted Cosserat surface** [19].[1] There, two-dimensional equilibrium in appropriate new resultant stress and strain variables is postulated ab-initio more or less independent of three-dimensional considerations, cf. [2, 37, 25, 17, 16, 18, 60].

A detailed presentation of the different approaches in classical shell theories can be found in the monograph [47]. A thorough mathematical analysis of linear, infinitesimal-displacement shell theory, based on asymptotic methods is to be found in [13] and the extensive references therein, see also [12, 14, 2, 20, 22, 32, 3]. Excellent reviews and insightful discussions of the modelling and finite element implementation may be found in [64, 61, 63, 38, 39, 75, 7, 11] and in the series of papers [65, 67, 68, 70, 69, 66]. Properly invariant, geometrically exact, elastic plate theories are derived by formal asymptotic methods in [27]. This formal derivation is extended to curvilinear shells in [43, 41]. Apart from the pure bending case [30, 31], which is rigourously justified as the **Γ-limit** of the three-dimensional model and which can be shown to be intrinsically well-posed, the obtained finite-strain models have not yet been shown to be well-posed. Indeed, the membrane energy contribution is notoriously not Legendre-Hadamard elliptic. The different membrane model formally justified in [24] by Γ-convergence is geometrically exact and automatically quasiconvex/elliptic but unfortunately does not coincide upon linearization with the otherwise well-established infinitesimal-displacement membrane model. Moreover, this model does not describe the detailed geometry of deformation in compression but reduces to a tension-field theory [71].

There is no place here to comment further on the relative merits of each alternative approach. The "rational" of descend from three to two dimensions should in any case be complemented by an investigation of the intrinsic mathematical properties of the obtained reduced models. Today, the need to simulate the mechanical response of highly flexible thin structures allowing easily for finite rotations excludes the use of classical infinitesimal-displacement models, either

[1] Restricted, since no material length scale usually enters the direct approach, only the **relative thickness** h appears in the model. In terminology it is useful to distinguish between a "true" Cosserat model operating on $SO(3,\mathbb{R})$ and theories with any number of directors.

of Reissner-Mindlin (14.11) or Kichhoff-Love type (14.14). Also, certain "intermediary" models allowing in principle for buckling like the "nonlinear" von Kármán plates (see [12, p.403], justified by means of Γ-convergence in [29] as a very low energy limit of three-dimensional elasticity) and penalized "nonlinear" Reissner-Mindlin models [21][2] or "semilinear" Kirchhoff-Love plate models [46] are not geometrically exact (not frame-indifferent). Nevertheless, the nonlinear von Kármán plate has been succesfully applied to the delamination problem of thin films [55, 34, 33].

Mielke [44] established in the infinitesimal-displacement context that by using more than five ansatz-functions in a director model it is possible to obtain exponential decay estimates for the boundary layer and to establish therefore a St.Venant principle for linearized plates. While it is not clear how his methods can be transferred to the finite-strain case, they provide, independent of mechanical/physical considerations, a strong motivation to use a director ansatz also in the finite-strain case in order to better capture the boundary layer phenomena.

Indeed, so called shear-deformable theories with independent directors are usually preferred in the engineering community. In view of an efficient finite element implementation one considers a hyperelastic, variationally based formulation with second-order Euler-Lagrange equations and uses standard C^0-conforming elements. The prototype examples are models based on the Reissner-Mindlin kinematical assumption. There are numerous proposals in the engineering literature for a finite-strain, geometrically exact plate formulation, see e.g. [28, 64, 62, 63, 75, 7, 11]. In many cases the need has been felt to devote specific attention to proper rotations $R \in \mathrm{SO}(3, \mathbb{R})$, since finite rotations are the dominant deformation mode of a flexible structure. This has led to the so called **drill-rotation formulation** which means that proper rotations either appear in the formulation as independent fields (leading to a restricted Cosserat surface) or they are an intermediary ingredient in the numerical treatment (constraint Cosserat surface, only continuum rotations matter finally). While the computational merit of this approach is well documented, a mathematical analysis for such a family of finite-strain plate models is yet missing, both for the Cosserat surface with independent rotations and the constraint model. It may be speculated that those restricted Cosserat plates (obtained from classical non-polar bulk models or from direct modelling) though geometrically exact and allowing for transverse shear and the description of boundary layers, might not be well posed for certain membrane strain measures either, notably if Green-strains: $F^TF - 1\!\!1$ or Hencky-strains: $\ln F^TF$ are used. Another drawback from a modelling point of view is that the inclusion of drill-rotations is most often done in an ad-hoc fashion.

Addressing partly this problem, in [53] a geometrically exact, viscoelastic membrane formulation has been proposed by the first author, where the viscoelastic effect, operative through an independent local field of rotations, is driven by transverse shear. This formulation has been shown to be locally well-posed [51].

It is also observed experimentally that **very thin structures** behave **comparably stiffer** than absolutely thicker structures while both have the **same relative thickness**. These **non-classical size effects** cannot be neglected for very thin structures [15]. Such effects are not accounted for neither in classical theories nor in the viscoelastic case.

In addition, classical infinitesimal-displacement or finite-strain shell models predict unrealistically high levels of smoothness, typically $m \in W^{1,4}(\omega, \mathbb{R}^3)$ for the midsurface m in both finite-strain Kirchhoff-Love and Reissner-Mindlin models and $m \in H^2(\omega, \mathbb{R}^3)$ in the finite-strain pure bending problem [30] and the von Kármán model. This implies at least $C^{0,\alpha}(\omega)$ for the midsurface m, which rule out the description of boundary layer effects and possible failure along asymptotic lines of the surface.

[2]Conceptually a von Kármán plate with one independent director $\vec{d} \in \mathbb{S}^2$ and addition of a penalisation term $\mu_c\,\left(\langle \vec{d}, \partial_x m\rangle^2 + \langle \vec{d}, \partial_y m\rangle^2\right)$, $\mu_c \to \infty$, with $m : \mathbb{R}^2 \mapsto \mathbb{R}^3$ the sought midsurface deformation.

1.2 Scope of study and outline of this contribution

In [49] the first author has proposed a new shell model for very thin almost rigid materials which should remedy some of the aforementioned limitations with a view towards a subsequent stringent mathematical analysis and possible stable finite element implementation. It was the goal to provide a model which is both theoretically and physically sound, such that its numerical implementation can concentrate on real convergence issues.

The formal derivation of the new plate model, summarized in Section 8, however, still gave rise to questions as far as the asymptotic correctness and convergence is concerned. In this paper we want to address this point by showing, that the Γ-limit of the Cosserat bulk model (under certain natural scaling assumptions) is given by the corresponding formal derivation, if energy contributions scaling with h are retained and if the coefficient of the transverse shear energy is slightly modified. Given that the information provided by the formal Γ-limit hinges also on these scaling assumptions, we think that our present result is a justification of the formal derivation and the employed kinematical ansatz.

Central to our development is therefore the notion of **Γ-convergence**, a powerful theory originally initiated by De Giorgi [35, 36] and especially suited for a variational framework on which in turn the numerical treatment with finite elements is based. This approach has thus far provided the only known convergence theorems for justifying lower dimensional nonlinear, frame-indifferent theories of elastic bodies.

In this contribution, after presenting the notation, we introduce in Section 2 the underlying "parent" three-dimensional finite-strain frame-indifferent Cosserat model with **size effects** and already appearing **independent microrotations** $\overline{R}$, i.e.a **triad of rigid directors** $(\overline{R}_1|\overline{R}_2|\overline{R}_3) = \overline{R} \in \mathrm{SO}(3,\mathbb{R})$ and we recall the obtained existence results for this Cosserat bulk model. We then provide in Section 3 the restriction of the bulk model to a thin domain and introduce the scaling to a fixed reference domain Ω_1 with constant thickness on which the Γ-convergence procedure is based.

In Section 4 we recapitulate briefly the relevant topics from Γ-convergence theory and we introduce the Γ-limit for the rescaled formulation with respect to the two independent fields $(\varphi, \overline{R})$ of deformations and microrotations in Section 5. Two limit cases, $\mu_c = 0$ and $\mu_c = \infty$ deserve additional attention. Following we provide the analytical proof for the statements in Section 6. Section 7 provides an estimate of the $\Gamma - \liminf$ and $\Gamma - \limsup$ in case of zero Cosserat couple modulus which is then strengthened to a full Γ-convergence statement.

In order to put the Γ-limit formulation into the proper framework, we provide in Section 8 the Cosserat plate model originally derived by means of a formal ansatz. It is seen that both formulations, within the same scaling assumptions, differ only by the coefficient of the transverse shear energy. Therefore in Section 9 we shortly review the form of the transverse shear energy given in the literature and discuss the role of the **shear correction factor** κ in light of our development and disclose its intimate connection with the Cosserat couple modulus μ_c. In Section 10 we are able to draw an interesting consequence for the numerical value of the Cosserat couple modulus μ_c, already for the bulk model. Section 11 schematically summarizes the relations between the discussed models.

In the Appendix we derive an upper bound for the $\Gamma - \limsup$ of classical linear elasticity and it is shown that a linearization of the geometrically exact Cosserat Γ-limit model turns into the linear membrane plate which coincides with this $\Gamma - \limsup$ upper bound. For the exposition to be sufficiently self-contained we also relate the new finite-strain Cosserat plate model based on a formal ansatz to classical approaches. Notably, we show that a **linearization** of the new "formal" plate model **with zero Cosserat couple modulus** $\mu_c = 0$ **results in** the classical infinitesimal-displacement **Reissner-Mindlin** model (without extra size effects and therefore without drill-rotations) and shear correction factor $\kappa = 1$. However, weaker boundary conditions for the increment of the director in the linearized infinitesimal-displacement Reissner-Mindlin model (14.11) are motivated. Nevertheless, this new boundary condition reduces to the classical condition on the increment of the normal in the linearized Kirchhoff-Love model

(14.14). Finally, the possible treatment of external loads is given.

1.3 Notation

1.3.1 Notation for bulk material

Let $\Omega \subset \mathbb{R}^3$ be a bounded open domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3\times 3}$. The identity tensor on $\mathbb{M}^{3\times3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[X] = \langle X, 1\!\!1\rangle$ and $\operatorname{tr}[X]^2 = \langle X, 1\!\!1\rangle^2$. We let Sym and PSym denote the symmetric and positive definite symmetric tensors respectively. We adopt the usual abbreviations of Lie-group theory, i.e., $\mathrm{GL}(3,\mathbb{R}) := \{X \in \mathbb{M}^{3\times3} \;|\det[X] \neq 0\}$ the general linear group, $\mathrm{SL}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \;|\det[X] = 1\}$, $\mathrm{O}(3) := \{X \in \mathrm{GL}(3,\mathbb{R}) \mid X^TX = 1\!\!1\}$, $\mathrm{SO}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \;|X^TX = 1\!\!1,\ \det[X] = 1\}$ with corresponding Lie-algebras $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times3} \;|X^T = -X\}$ of skew symmetric tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{M}^{3\times3} \;|\operatorname{tr}[X] = 0\}$ of traceless tensors. With $\operatorname{Adj} X$ we denote the tensor of transposed cofactors $\mathrm{Cof(X)}$ such that $\operatorname{Adj} X = \det[X]\, X^{-1} = \mathrm{Cof(X)^T}$ if $X \in \mathrm{GL}(3,\mathbb{R})$. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$. For $X \in \mathbb{M}^{3\times3}$ we set for the deviatoric part $\operatorname{dev} X = X - \frac{1}{3}\operatorname{tr}[X]\,1\!\!1 \in \mathfrak{sl}(3)$ and for vectors $\xi, \eta \in \mathbb{R}^n$ we have the tensor product $(\xi\otimes\eta)_{ij} = \xi_i\,\eta_j$.

We write the polar decomposition in the form $F = RU = \operatorname{polar}(F)U$ with $R = \operatorname{polar}(F)$ the orthogonal part of F. For a second order tensor X we define the third order tensor $\mathfrak{h} = \mathrm{D_x} X(x) = (\nabla(X(x).e_1), \nabla(X(x).e_2), \nabla(X(x).e_3)) = (\mathfrak{h}^1, \mathfrak{h}^2, \mathfrak{h}^3) \in \mathbb{M}^{3\times3}\times\mathbb{M}^{3\times3}\times\mathbb{M}^{3\times3} \cong \mathfrak{T}(3)$. For third order tensors $\mathfrak{h} \in \mathfrak{T}(3)$ we set $\|\mathfrak{h}\|^2 = \sum_{i=1}^3 \|\mathfrak{h}^i\|^2$ together with $\operatorname{sym}(\mathfrak{h}) := (\operatorname{sym}\mathfrak{h}^1, \operatorname{sym}\mathfrak{h}^2, \operatorname{sym}\mathfrak{h}^3)$ and $\operatorname{tr}[\mathfrak{h}] := (\operatorname{tr}\left[\mathfrak{h}^1\right], \operatorname{tr}\left[\mathfrak{h}^2\right], \operatorname{tr}\left[\mathfrak{h}^3\right]) \in \mathbb{R}^3$. Moreover, for any second order tensor X we define $X\cdot\mathfrak{h} := (X\mathfrak{h}^1, X\mathfrak{h}^2, X\mathfrak{h}^3)$ and $\mathfrak{h}\cdot X$, correspondingly. Quantities with a bar, e.g. the micropolar rotation $\overline{R}$, represent the micropolar replacement of the corresponding classical continuum rotation R. In general we work in the context of nonlinear, finite-strain elasticity. For the total deformation $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^3)$ we have the deformation gradient $F = \nabla\varphi \in C(\overline{\Omega}, \mathbb{M}^{3\times3})$. Furthermore, $S_1(F) = D_F W(F)$ and $S_2(F) = F^{-1}D_F W(F)$ denote the first and second Piola Kirchhoff stress tensors, respectively. Total time derivatives are written $\frac{\mathrm{d}}{\mathrm{dt}}X(t) = \dot{X}$. The first and second differential of a scalar valued function $W(F)$ are written $D_F W(F).H$ and $D_F^2 W(F).(H,H)$, respectively. We employ the standard notation of Sobolev spaces, i.e. $L^2(\Omega), H^{1,2}(\Omega), H_\circ^{1,2}(\Omega), W^{1,q}(\Omega)$, which we use indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions. The set $W^{1,q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$ denotes orthogonal tensors whose components are in $W^{1,q}(\Omega)$. Moreover, we set $\|X\|_\infty = \sup_{x\in\Omega}\|X(x)\|$. For $A \in C^1(\overline{\Omega}, \mathbb{M}^{3\times3})$ we define $\operatorname{Curl} A(x)$ as the operation curl applied row wise. We define $H_\circ^{1,2}(\Omega, \Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|_\Gamma} = 0\}$, where $\phi_{|_\Gamma} = 0$ is to be understood in the sense of traces and by $C_0^\infty(\Omega)$ we denote infinitely differentiable functions with compact support in Ω. We use capital letters to denote possibly large positive constants, e.g. C^+, K and lower case letters to denote possibly small positive constants, e.g. c^+, d^+. The smallest eigenvalue of a positive definite symmetric tensor P is abbreviated by $\lambda_{\min}(P)$.

1.3.2 Notation for plates and shells

Let $\omega \subset \mathbb{R}^2$ be a bounded open domain with Lipschitz boundary $\partial\omega$ and let γ_0 be a smooth subset of $\partial\omega$ with non-vanishing 1-dimensional Hausdorff measure. The thickness of the plate is taken to be $h > 0$ with dimension length (contrary to Ciarlet's definition of the thickness to be 2ε, which difference leads only to various different constants in the resulting formulas). We denote by $\mathbb{M}^{n\times m}$ the set of matrices mapping $\mathbb{R}^n \mapsto \mathbb{R}^m$. For $H \in \mathbb{M}^{2\times3}$ and $\xi \in \mathbb{R}^3$ we employ also the notation $(H|\xi) \in \mathbb{M}^{3\times3}$ to denote the matrix composed of H and the column

ξ. Likewise $(v|\xi|\eta)$ is the matrix composed of the columns v, ξ, η. This allows us to write for $\varphi \in C^1\mathbb{R}^3, \mathbb{R}^3) : \nabla\varphi = (\varphi_x|\varphi_y|\varphi_z) = (\partial_x\varphi|\partial_y\varphi|\partial_z\varphi)$. The identity tensor on $\mathbb{M}^{2\times 2}$ will be denoted by $1\!\!1_2$. The mapping $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ is the deformation of the midsurface, ∇m is the corresponding deformation gradient and $\vec{n}_m$ is the outer unit normal on m. A matrix $X \in \mathbb{M}^{3\times 3}$ can now be written as $X = (X.e_2|X.e_2|X.e_3) = (X_1|X_2|X_3)$. We write $v : \mathbb{R}^2 \mapsto \mathbb{R}^3$ for the displacement of the midsurface, such that $m(x, y) = (x, y, 0)^T + v(x, y)$. The standard volume element is written $\mathrm{dx}\,\mathrm{dy}\,\mathrm{dz} = \mathrm{dV} = \mathrm{d}\omega\,\mathrm{dz}$.

2 The underlying finite-strain three-dimensional Cosserat model in variational form

In [54] a finite-strain, fully frame-indifferent, three-dimensional Cosserat micropolar model is introduced. The **two-field** problem has been posed in a variational setting. The task is to find a pair $(\varphi, \overline{R}) : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3 \times \mathrm{SO}(3, \mathbb{R})$ of deformation φ and **independent microrotation** $\overline{R} \in \mathrm{SO}(3, \mathbb{R})$ minimizing the energy functional I,

$$\begin{aligned} I(\varphi, \overline{R}) = \int_\Omega W_{\mathrm{mp}}(\overline{R}^T\nabla\varphi) + W_{\mathrm{curv}}(\overline{R}^T \mathrm{D_x}\overline{R}) - \Pi_f(\varphi) - \Pi_M(\overline{R})\,\mathrm{dV} \\ - \int_{\Gamma_S} \Pi_N(\varphi)\,\mathrm{dS} - \int_{\Gamma_C} \Pi_{M_c}(\overline{R})\,\mathrm{dS} \mapsto \min .\ \text{w.r.t. } (\varphi, \overline{R})\,, \end{aligned} \tag{2.1}$$

together with the Dirichlet boundary condition of place for the deformation φ on Γ: $\varphi_{|\Gamma} = g_{\mathrm{d}}$ and three possible **alternative** boundary conditions for the microrotations $\overline{R}$ on Γ,

$$\overline{R}_{|\Gamma} = \begin{cases} \overline{R}_{\mathrm{d}}\,, & \text{the case of } \textbf{rigid} \text{ prescription}\,, \\ \mathrm{polar}(\nabla\varphi)\,, & \text{the case of } \textbf{strong consistent coupling}\,, \\ \text{no condition for } \overline{R} \text{ on } \Gamma, & \textbf{induced Neumann-type} \text{ relations for } \overline{R} \text{ on } \Gamma\,. \end{cases} \tag{2.2}$$

The constitutive assumptions on the densities are

$$\begin{aligned} W_{\mathrm{mp}}(\overline{U}) &= \mu\,\|\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c\,\|\mathrm{skew}(\overline{U})\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2, \ \overline{U} = \overline{R}^T F\,,\ F = \nabla\varphi\,, \\ W_{\mathrm{curv}}(\mathfrak{K}) &= \mu\frac{L_c^{1+p}}{12}\,(1 + \alpha_4\,L_c^q\,\|\mathfrak{K}\|^q)\,\left(\alpha_5\,\|\mathrm{sym}\,\mathfrak{K}\|^2 + \alpha_6\,\|\mathrm{skew}\,\mathfrak{K}\|^2 + \alpha_7\,\mathrm{tr}\,[\mathfrak{K}]^2\right)^{\frac{1+p}{2}}\,, \\ \mathfrak{K} &= \overline{R}^T \mathrm{D_x}\overline{R} := \left(\overline{R}^T\nabla(\overline{R}.e_1), \overline{R}^T\nabla(\overline{R}.e_2), \overline{R}^T\nabla(\overline{R}.e_3)\right)\,, \text{ the third order } \textbf{curvature tensor}\,, \end{aligned} \tag{2.3}$$

under the minimal requirement $p \geq 1$, $q \geq 0$. The total elastically stored energy $W = W_{\mathrm{mp}} + W_{\mathrm{curv}}$ is quadratic in the stretch $\overline{U}$ and possibly super-quadratic in the curvature $\mathfrak{K}$. The strain energy W_{mp} depends on the deformation gradient $F = \nabla\varphi$ and the microrotations $\overline{R} \in \mathrm{SO}(3, \mathbb{R})$, which do not necessarily coincide with the **continuum rotations** $R = \mathrm{polar}(F)$. The curvature energy W_{curv} depends moreover on the space derivatives $\mathrm{D_x}\overline{R}$ which describe the self-interaction of the microstructure.[3] In general, the **micropolar stretch tensor** $\overline{U}$ is **not symmetric** and does not coincide with the **symmetric continuum stretch** tensor $U = R^T F = \sqrt{F^T F}$. By abuse of notation we set $\|\mathrm{sym}\,\mathfrak{K}\|^2 := \sum_{i=1}^3 \|\mathrm{sym}\,\mathfrak{K}^i\|^2$ for third order tensors $\mathfrak{K}$, cf.(1.3.1).

Here $\Omega \subset \mathbb{R}^3$ is an open domain with boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ is that part of the boundary, where Dirichlet conditions $g_{\mathrm{d}}, \overline{R}_{\mathrm{d}}$ for deformations and microrotations or coupling conditions for microrotations, are prescribed. $\Gamma_S \subset \partial\Omega$ is a part of the boundary, where traction boundary conditions in the form of the potential of applied surface forces Π_N are given with $\Gamma \cap \Gamma_S = \emptyset$. In addition, $\Gamma_C \subset \partial\Omega$ is the part of the boundary where the potential of external surface couples

[3] Observe that $\overline{R}^T\nabla(\overline{R}.e_i) \neq \overline{R}^T\partial_{x_i}\overline{R} \in \mathfrak{so}(3, \mathbb{R})$.

Π_{M_c} are applied with $\Gamma \cap \Gamma_C = \emptyset$. On the free boundary $\partial\Omega \setminus \{\Gamma \cup \Gamma_S \cup \Gamma_C\}$ corresponding natural boundary conditions for $(\varphi, \overline{R})$ apply. The potential of the external applied volume force is Π_f and Π_M takes on the role of the potential of applied external volume couples. For simplicity we assume

$$\Pi_f(\varphi) = \langle f, \varphi \rangle\,, \quad \Pi_M(\overline{R}) = \langle M, \overline{R} \rangle\,, \quad \Pi_N(\varphi) = \langle N, \varphi \rangle\,, \quad \Pi_{M_c}(\overline{R}) = \langle M_c, \overline{R} \rangle\,, \tag{2.4}$$

for the potentials of applied loads with given functions $f \in L^2(\Omega, \mathbb{R}^3)$, $M \in L^2(\Omega, \mathbb{M}^{3\times 3})$, $N \in L^2(\Gamma_S, \mathbb{R}^3)$, $M_c \in L^2(\Gamma_C, \mathbb{M}^{3\times 3})$.

The parameters μ, $\lambda > 0$ are the Lamé constants of classical isotropic elasticity, the additional parameter $\mu_c \geq 0$ is called the **Cosserat couple modulus**. For $\mu_c > 0$ the elastic strain energy density $W_{\mathrm{mp}}(\overline{U})$ is **uniformly convex** in $\overline{U}$. Moreover

$$\begin{aligned}
\forall F \in \mathrm{GL}^+(3,\mathbb{R}):\ W_{\mathrm{mp}}(\overline{U}) &= W_{\mathrm{mp}}(\overline{R}^T F) \geq \min(\mu, \mu_c)\, \|\overline{R}^T F - 1\!\!1\|^2 = \min(\mu, \mu_c)\, \|F - \overline{R}\|^2 \\
&\geq \min(\mu, \mu_c) \inf_{R \in \mathrm{O}(3,\mathbb{R})} \|F - R\|^2 = \min(\mu, \mu_c)\, \mathrm{dist}^2(F, \mathrm{O}(3,\mathbb{R})) \\
&= \min(\mu, \mu_c)\, \mathrm{dist}^2(F, \mathrm{SO}(3,\mathbb{R})) = \min(\mu, \mu_c)\, \|F - \mathrm{polar}(F)\|^2 \\
&= \min(\mu, \mu_c)\, \|U - 1\!\!1\|^2\,.
\end{aligned} \tag{2.5}$$

In contrast, for $\mu_c = 0$ the strain energy density is **only convex** w.r.t. F and does not satisfy (2.5).[4]

The parameter $L_c > 0$ (with dimension length) introduces an **internal length** which is **characteristic** for the material, e.g. related to the grain size in a polycrystal. The internal length $L_c > 0$ is responsible for **size effects** in the sense that smaller samples are relatively stiffer than larger samples. We assume throughout that $\alpha_4, \alpha_5, \alpha_6 > 0, \alpha_7 \geq 0$. This implies the **coercivity of curvature**

$$\exists\, c^+ > 0 \quad \forall\, \mathfrak{K} \in \mathfrak{T}(3): \quad W_{\mathrm{curv}}(\mathfrak{K}) \geq c^+ \, \|\mathfrak{K}\|^{1+p+q}\,, \tag{2.6}$$

which is a basic ingredient of the mathematical analysis.

The non-standard boundary condition of **strong consistent coupling** ensures that no unwanted non-classical, polar effects may occur at the Dirichlet boundary Γ. It implies for the micropolar stretch that $\overline{U}_{|_\Gamma} \in \mathrm{Sym}$ and for the second Piola-Kirchhoff stress tensor $S_2 := F^{-1} D_F W_{\mathrm{mp}}(\overline{U}) \in \mathrm{Sym}$ on Γ as in the classical, non-polar case. We refer to the weaker boundary condition $\overline{U}_{|_\Gamma} \in \mathrm{Sym}$ as **weak consistent coupling**.

We mention, that a linearization of this Cosserat bulk model with $\mu_c = 0$ for small displacement and small microrotations completely decouples the two fields of deformation and microrotations and leads to the classical linear elasticity problem for the deformation.[5] For more details on the modelling of the three-dimensional Cosserat model we refer the reader to [54].

2.1 Mathematical results for the three-dimensional Cosserat bulk problem

For conciseness we state only the obtained results for the case without external loads. It can be shown:

Theorem 2.1 (Existence for 3D-finite-strain elastic Cosserat model with $\mu_c > 0$) *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p}(\Omega, \mathrm{SO}(3,\mathbb{R}))$. Then (2.1) with $\mu_c > 0$, $\alpha_4 \geq 0$, $p \geq 1$, $q \geq 0$ and either*

[4] The condition $F \in \mathrm{GL}^+(3,\mathbb{R})$ is necessary, otherwise $\|F - \mathrm{polar}(F)\|^2 = \mathrm{dist}^2(F, \mathrm{O}(3,\mathbb{R})) < \mathrm{dist}^2(F, \mathrm{SO}(3,\mathbb{R}))$, as can be easily seen for the reflection $F = \mathrm{diag}(1, -1, 1)$.

[5] Thinking in the context of an infinitesimal-displacement Cosserat theory one might erroneously believe that $\mu_c > 0$ is strictly necessary also for a "true" finite-strain Cosserat theory.

free or rigid prescription for $\overline{R}$ on Γ admits at least one minimizing solution pair $(\varphi, \overline{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,1+p}(\Omega, \mathrm{SO}(3, \mathbb{R}))$. ■

Using the extended Korn's inequality [48, 56], the following has been shown in [54, 50]:

Theorem 2.2 (Existence for 3D-finite-strain elastic Cosserat model with $\mu_c = 0$) *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p+q}(\Omega, \mathrm{SO}(3, \mathbb{R}))$. Then (2.1) with $\mu_c = 0, \alpha_4 > 0, p \geq 1, q > 1$ and either free or rigid prescription for $\overline{R}$ on Γ admits at least one minimizing solution pair* $(\varphi, \overline{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega, \mathrm{SO}(3, \mathbb{R}))$. ■

3 Formal dimensional reduction of the Cosserat bulk model

3.1 The three-dimensional Cosserat problem on a thin domain

The basic task of any shell theory is a consistent reduction of some presumably "exact" 3D-theory to 2D. The general three-dimensional problem (2.1) will now be adapted to a shell-like theory. Let us assume that we are given a three-dimensional **absolutely thin domain**

$$\Omega_h := \omega \times [-\frac{h}{2}, \frac{h}{2}], \quad \omega \subset \mathbb{R}^2 \,, \tag{3.1}$$

with **transverse boundary** $\partial\Omega_h^{\mathrm{trans}} = \omega \times \{-\frac{h}{2}, \frac{h}{2}\}$ and **lateral boundary** $\partial\Omega_h^{\mathrm{lat}} = \partial\omega \times [-\frac{h}{2}, \frac{h}{2}]$, where ω is a bounded open domain in $\mathbb{R}^2$ with smooth boundary $\partial\omega$ and $h > 0$ is the thickness. Moreover, assume we are given a deformation φ and microrotation $\overline{R}^{3d}$,

$$\varphi : \quad \Omega_h \subset \mathbb{R}^3 \mapsto \mathbb{R}^3 \,, \qquad \overline{R}^{3d} : \Omega_h \subset \mathbb{R}^3 \mapsto \mathrm{SO}(3, \mathbb{R}) \,, \tag{3.2}$$

solving the following two-field minimization problem on the thin domain Ω_h:

$$\begin{aligned}
I(\varphi, \nabla\varphi, \overline{R}^{3d}, \mathrm{D_x}\overline{R}^{3d}) &= \int_{\Omega_h} W_{\mathrm{mp}}(\overline{U}) + W_{\mathrm{curv}}(\mathfrak{K}) - \langle f, \varphi \rangle \, \mathrm{dV} - \int_{\partial\Omega_h^{\mathrm{trans}} \cup \{\gamma_s \times [-\frac{h}{2}, \frac{h}{2}]\}} \langle N, \varphi \rangle \, \mathrm{dS} \mapsto \min . \text{ w.r.t. } (\varphi, \overline{R}) \,, \\
\overline{U} &= \overline{R}^{3d,T} F, \quad \varphi_{|_{\Gamma_0^h}} = g_{\mathrm{d}}(x, y, z), \quad \Gamma_0^h = \gamma_0 \times [-\frac{h}{2}, \frac{h}{2}], \quad \gamma_0 \subset \partial\omega, \quad \gamma_s \cap \gamma_0 = \emptyset \,, \\
\overline{U}_{|_{\Gamma_0^h}} &= \overline{R}^{3d,T} \nabla\varphi_{|_{\Gamma_0^h}} \in \mathrm{Sym}(3) \,, \quad \text{weak consistent coupling boundary condition } \textbf{or} \\
\overline{R}^{3d} &: \quad \text{free on } \Gamma_0^h, \text{ alternative Neumann-type boundary condition} \,, \\
W_{\mathrm{mp}}(\overline{U}) &= \mu \, \| \mathrm{sym}(\overline{U} - 1\!\!1) \|^2 + \mu_c \, \| \mathrm{skew}(\overline{U}) \|^2 + \frac{\lambda}{2} \, \mathrm{tr} \left[\mathrm{sym}(\overline{U} - 1\!\!1) \right]^2 \,, \\
W_{\mathrm{curv}}(\mathfrak{K}) &= \mu \frac{L_c^{1+p}}{12} \, (1 + \alpha_4 \, L_c^q \, \|\mathfrak{K}\|^q) \left(\alpha_5 \, \| \mathrm{sym} \, \mathfrak{K} \|^2 + \alpha_6 \, \| \mathrm{skew} \, \mathfrak{K} \|^2 + \alpha_7 \, \mathrm{tr} \, [\mathfrak{K}]^2 \right)^{\frac{1+p}{2}} \,, \\
\mathfrak{K} &= \overline{R}^{3d,T} \mathrm{D_x} \overline{R}^{3d} = \left(\overline{R}^{3d,T} \nabla(\overline{R}^{3d}.e_1), \overline{R}^{3d,T} \nabla(\overline{R}^{3d}.e_2), \overline{R}^{3d,T} \nabla(\overline{R}^{3d}.e_3) \right) .
\end{aligned}$$

Without loss of mathematical generality we assume that $M, M_c \equiv 0$ in (2.4), i.e. that no external volume or surface couples are present in the bulk problem. We want to find a reasonable approximation $(\varphi_s, \overline{R}_s)$ of $(\varphi, \overline{R}^{3d})$ involving only two-dimensional quantities.

3.2 Transformation on a fixed domain

In order to apply standard techniques of Γ-convergence, we transform the problem onto a **fixed domain** Ω_1, independent of the thickness $h > 0$. Define therefore

$$\Omega_1 = \omega \times [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}^3\,, \quad \omega \subset \mathbb{R}^2\,. \tag{3.3}$$

The scaling transformation

$$\begin{aligned}
&\zeta : \eta \in \Omega_1 \subset \mathbb{R}^3 \mapsto \mathbb{R}^3, \quad \zeta(\eta_1,\eta_2,\eta_3) := (\eta_1,\eta_2,h\cdot\eta_3)\,,\\
&\nabla_\eta\zeta(\eta) = \begin{pmatrix}1&0&0\\0&1&0\\0&0&h\end{pmatrix}, \quad \mathrm{Cof}\,\nabla_\eta\zeta(\eta) = \begin{pmatrix}h&0&0\\0&h&0\\0&0&1\end{pmatrix}, \quad \det[\nabla_\eta\zeta(\eta)] = h\,,\\
&\zeta^{-1} : \xi \in \Omega_h \subset \mathbb{R}^3 \mapsto \mathbb{R}^3, \quad \zeta^{-1}(\xi_1,\xi_2,\xi_3) := (\xi_1,\xi_2,\frac{\xi_3}{h}), \quad \nabla_\xi[\zeta^{-1}(\xi)] = \begin{pmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{h}\end{pmatrix},
\end{aligned} \tag{3.4}$$

is such that ζ maps Ω_1 into Ω_h and $\zeta(\Omega_1) = \Omega_h$. We consider the correspondingly scaled function (subsequently, scaled functions defined on Ω_1 will be indicated with a superscript $\sharp$) $\varphi^\sharp : \Omega_1 \to \mathbb{R}^3$, defined by

$$\begin{aligned}
\varphi(\xi_1,\xi_2,\xi_3) &= \varphi^\sharp(\zeta^{-1}(\xi_1,\xi_2,\xi_3)) \quad \forall\,\xi \in \Omega_h\,; \quad \varphi^\sharp(\eta) = \varphi(\zeta(\eta)) \quad \forall\,\eta \in \Omega_1\,,\\
F^\sharp(\eta) &= \nabla_\eta\varphi^\sharp(\eta) = \nabla_\xi\varphi(\zeta(\eta))\cdot\nabla_\xi\zeta(\eta)\,,\\
\nabla\varphi(\xi_1,\xi_2,\xi_3) &= \nabla\varphi^\sharp(\zeta^{-1}(\xi_1,\xi_2,\xi_3))\cdot\begin{pmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{h}\end{pmatrix},\\
&= \Big(\partial_{\eta_1}\varphi^\sharp(\eta_1,\eta_2,\eta_3)|\partial_{\eta_2}\varphi^\sharp(\eta_1,\eta_2,\eta_3)|\frac{1}{h}\partial_{\eta_3}\varphi^\sharp(\eta_1,\eta_2,\eta_3)\Big) =: \nabla_\eta^h\varphi^\sharp = F_h^\sharp\,.
\end{aligned} \tag{3.5}$$

Similarly, we define a scaled rotation tensor $\overline{R}^{3d,\sharp} : \Omega_1 \subset \mathbb{R}^3 \mapsto \mathrm{SO}(3,\mathbb{R})$ by

$$\begin{aligned}
\overline{R}^{3d}(\xi_1,\xi_2,\xi_3) &= \overline{R}^{3d,\sharp}(\zeta^{-1}(\xi_1,\xi_2,\xi_3)) \quad \forall\xi \in \Omega_h\,; \quad \overline{R}^{3d,\sharp}(\eta) = \overline{R}^{3d}(\zeta(\eta)) \quad \forall\,\eta \in \Omega_1\,,\\
\nabla_\xi[\overline{R}^{3d}(\xi_1,\xi_2,\xi_3).e_i] &= \nabla_\eta[\overline{R}^{3d,\sharp}(\eta).e_i]\cdot(\nabla_\eta\zeta(\eta))^{-1}\\
&= \Big(\partial_{\eta_1}[\overline{R}^{3d,\sharp}(\eta).e_i]|\partial_{\eta_2}[\overline{R}^{3d,\sharp}(\eta).e_i]|\frac{1}{h}\partial_{\eta_3}[\overline{R}^{3d,\sharp}(\eta).e_i]\Big)\\
&=: \nabla_\eta^h[\overline{R}^{3d,\sharp}(\eta).e_i] \in \mathbb{M}^{3\times 3}\,,\\
\mathrm{D}_\eta^{\mathrm{h}}\overline{R}_h^{3d,\sharp}(\eta) &:= \Big(\nabla_\eta^h[\overline{R}^{3d,\sharp}(\eta).e_1],\, \nabla_\eta^h[\overline{R}^{3d,\sharp}(\eta).e_2],\, \nabla_\eta^h[\overline{R}^{3d,\sharp}(\eta).e_3]\Big) \in \mathfrak{T}(3)\,.
\end{aligned} \tag{3.6}$$

This defines the scaled third order curvature tensor $\mathfrak{K}_h^\sharp : \Omega_1 \mapsto \mathfrak{T}(3)$

$$\begin{aligned}
\mathfrak{K}_h^\sharp(\eta) = \Big(&\overline{R}^{3d,\sharp,T}(\eta)\Big(\partial_{\eta_1}[\overline{R}^{3d,\sharp}(\eta).e_1]|\partial_{\eta_2}[\overline{R}^{3d,\sharp}(\eta).e_1]|\frac{1}{h}\partial_{\eta_3}[\overline{R}^{3d,\sharp}(\eta).e_1]\Big)\,,\\
&\overline{R}^{3d,\sharp,T}(\eta)\Big(\partial_{\eta_1}[\overline{R}^{3d,\sharp}(\eta).e_2]|\partial_{\eta_2}[\overline{R}^{3d,\sharp}(\eta).e_2]|\frac{1}{h}\partial_{\eta_3}[\overline{R}^{3d,\sharp}(\eta).e_2]\Big)\,,\\
&\overline{R}^{3d,\sharp,T}(\eta)\Big(\partial_{\eta_1}[\overline{R}^{3d,\sharp}(\eta).e_3]|\partial_{\eta_2}[\overline{R}^{3d,\sharp}(\eta).e_3]|\frac{1}{h}\partial_{\eta_3}[\overline{R}^{3d,\sharp}(\eta).e_3]\Big)\Big)\\
= \Big(&\overline{R}^{3d,\sharp,T}(\eta)\nabla_\eta^h[\overline{R}^{3d,\sharp}(\eta).e_1],\, \overline{R}^{3d,\sharp,T}(\eta)\nabla_\eta^h[\overline{R}^{3d,\sharp}(\eta).e_2],\, \overline{R}^{3d,\sharp,T}(\eta)\nabla_\eta^h[\overline{R}^{3d,\sharp}(\eta).e_3]\Big)\\
= \,&\overline{R}^{3d,\sharp,T}\,\mathrm{D}_\eta^{\mathrm{h}}\overline{R}^{3d,\sharp}(\eta)\,.
\end{aligned} \tag{3.7}$$

Moreover, we define similarly scaled functions by setting

$$f^\sharp(\eta) := f(\zeta(\eta)), \qquad g_{\rm d}^\sharp(\eta) = g_{\rm d}(\zeta(\eta))\,, \qquad N^\sharp(\eta) := N(\zeta(\eta))\,. \tag{3.8}$$

In terms of the introduced scaled deformations and rotations

$$\varphi^\sharp: \quad \Omega_1 \subset \mathbb{R}^3 \mapsto \mathbb{R}^3\,, \qquad \overline{R}^{3d,\sharp}: \Omega_1 \subset \mathbb{R}^3 \mapsto \mathrm{SO}(3,\mathbb{R})\,, \tag{3.9}$$

the scaled problem solves the following two-field minimization problem on the fixed domain Ω_1:

$$\begin{aligned}
I^\sharp(\varphi^\sharp, \nabla\varphi^\sharp, \overline{R}^{3d,\sharp}, \mathrm{D}^{\rm h}_\eta \overline{R}^{3d,\sharp}) &= \int_{\eta\in\Omega_1} \Big[W_{\rm mp}(\overline{U}^\sharp_h) + W_{\rm curv}(\mathfrak{K}^\sharp_h) - \langle f^\sharp, \varphi^\sharp\rangle\Big]\ \det[\nabla\zeta(\eta)]\,\mathrm{dV}_\eta \\
&\quad - \int_{\partial\Omega_1^{\rm trans}\cup\{\gamma_s\times[-\frac12,\frac12]\}} \langle N^\sharp, \varphi^\sharp\rangle\,\|\operatorname{Cof}\nabla\zeta(\eta).\vec{n}\|\,\mathrm{dS}_\eta\,, \\
&= h \int_{\eta\in\Omega_1} W_{\rm mp}(\overline{U}^\sharp_h) + W_{\rm curv}(\mathfrak{K}^\sharp_h) - \langle f^\sharp, \varphi^\sharp\rangle\,\mathrm{dV}_\eta \\
&\quad - \int_{\partial\Omega_1^{\rm trans}} \langle N^\sharp, \varphi^\sharp\rangle\,1\,\mathrm{dS}_\eta - \int_{\gamma_s\times[-\frac12,\frac12]} \langle N^\sharp, \varphi^\sharp\rangle\,h\,\mathrm{dS}_\eta \mapsto \min.\ \text{w.r.t.}\ (\varphi^\sharp, \overline{R}^\sharp)\,, \\
\overline{U}^\sharp_h &= \overline{R}^{3d,\sharp,T} F^\sharp_h, \quad \varphi^\sharp_{|_{\Gamma^1_0}}(\eta) = g^\sharp_{\rm d}(\eta) = g_{\rm d}(\zeta(\eta)) = g_{\rm d}(\eta_1,\eta_2,h\cdot\eta_3)\,, \qquad (3.10)\\
\Gamma^1_0 &= \gamma_0\times[-\frac12,\frac12], \quad \gamma_0\subset\partial\omega, \quad \gamma_s\cap\gamma_0=\emptyset\,, \\
\overline{U}^\sharp_{h|_{\Gamma^1_0}} &= \overline{R}^{3d,\sharp,T}\nabla^\sharp_\eta\varphi^\sharp_{|_{\Gamma^1_0}} \in \mathrm{Sym}(3)\,, \quad \text{weak consistent coupling boundary condition } \textbf{or} \\
\overline{R}^{3d,\sharp}: &\quad \text{free on } \Gamma^1_0, \text{ alternative Neumann-type boundary condition}\,, \\
W_{\rm mp}(\overline{U}^\sharp_h) &= \mu\,\|\mathrm{sym}(\overline{U}^\sharp_h - 1\!\!1)\|^2 + \mu_c\,\|\mathrm{skew}(\overline{U}^\sharp_h)\|^2 + \frac{\lambda}{2}\operatorname{tr}\Big[\mathrm{sym}(\overline{U}^\sharp_h - 1\!\!1)\Big]^2, \\
W_{\rm curv}(\mathfrak{K}^\sharp_h) &= \mu\frac{L_c^{1+p}}{12}\,\Big(1+\alpha_4\,L_c^q\,\|\mathfrak{K}^\sharp_h\|^q\Big)\,\Big(\alpha_5\,\|\mathrm{sym}\,\mathfrak{K}^\sharp_h\|^2 + \alpha_6\,\|\mathrm{skew}\,\mathfrak{K}^\sharp_h\|^2 + \alpha_7\operatorname{tr}\Big[\mathfrak{K}^\sharp_h\Big]^2\Big)^{\frac{1+p}{2}}, \\
\mathfrak{K}^\sharp_h &= \overline{R}^{3d,\sharp,T}\,\mathrm{D}^{\rm h}_\eta\overline{R}^{3d,\sharp}_h(\eta)\,.
\end{aligned}$$

3.3 The rescaled variational Cosserat bulk problem

Since the energy $\frac{1}{h}I^\sharp$ would not be finite for $h\to 0$ if tractions $N^\sharp$ on the transverse boundary were present, the investigations are in principle restricted to the case of $N^\sharp = 0$ on $\partial\Omega_1^{\rm trans}$.[6] For conciseness we therefore investigate finally the following simplified and rescaled ($N^\sharp, f^\sharp = 0$, $g_{\rm d}(\xi_1,\xi_2,\xi_3) := g_{\rm d}(\xi_1,\xi_2)$) two-field minimization problem on Ω_1 with respect to Γ-convergence (without the factor $h>0$ now):

$$\begin{aligned}
I^\sharp_h(\varphi^\sharp, \nabla\varphi^\sharp, \overline{R}^{3d,\sharp}, \mathrm{D}^{\rm h}_\eta\overline{R}^{3d,\sharp}) &= \int_{\eta\in\Omega_1} W_{\rm mp}(\overline{U}^\sharp_h) + W_{\rm curv}(\mathfrak{K}^\sharp_h)\,\mathrm{dV}_\eta \mapsto \min.\ \text{w.r.t.}\ (\varphi^\sharp, \overline{R}^\sharp)\,, \\
\overline{U}^\sharp_h &= \overline{R}^{3d,\sharp,T} F^\sharp_h, \quad \varphi^\sharp_{|_{\Gamma^1_0}}(\eta) = g^\sharp_{\rm d}(\eta) = g_{\rm d}(\zeta(\eta)) = g_{\rm d}(\eta_1,\eta_2,h\cdot\eta_3) = g_{\rm d}(\eta_1,\eta_2,0)\,, \\
\Gamma^1_0 &= \gamma_0\times[-\frac12,\frac12], \quad \gamma_0\subset\partial\omega, \\
\overline{R}^{3d,\sharp}: &\quad \text{free on } \Gamma^1_0, \text{ Neumann-type boundary condition}\,, \qquad (3.11)
\end{aligned}$$

[6]The thin plate limit $h\to 0$ obviously cannot support non-vanishing transverse surface loads.

$$
\begin{aligned}
W_{\mathrm{mp}}(\overline{U}^\sharp_h) &= \mu \,\| \operatorname{sym}(\overline{U}^\sharp_h - 1\!\!1)\|^2 + \mu_c \,\| \operatorname{skew}(\overline{U}^\sharp_h)\|^2 + \frac{\lambda}{2} \operatorname{tr}\left[\operatorname{sym}(\overline{U}^\sharp_h - 1\!\!1)\right]^2 ,\\
W_{\mathrm{curv}}(\mathfrak{K}^\sharp_h) &= \mu \frac{L_c^{1+p}}{12} \left(1 + \alpha_4 \, L_c^q \, \|\mathfrak{K}^\sharp_h\|^q\right) \left(\alpha_5 \,\| \operatorname{sym} \mathfrak{K}^\sharp_h\|^2 + \alpha_6 \,\| \operatorname{skew} \mathfrak{K}^\sharp_h\|^2 + \alpha_7 \operatorname{tr}\left[\mathfrak{K}^\sharp_h\right]^2\right)^{\frac{1+p}{2}} ,\\
\mathfrak{K}^\sharp_h &= \overline{R}^{3d,\sharp,T} \, \mathrm{D}^{\mathrm{h}}_\eta \overline{R}^{3d,\sharp}(\eta) \,.
\end{aligned}
$$

Here we assume that the boundary condition g_{d} is already independent of the transverse variable. For simplicity, we restrict furthermore attention to the weakest possible response, namely the **Neumann boundary conditions** on the microrotations $\overline{R}^\sharp$.[7] Moreover, for simplicity, we assume

$$p \geq 1 \,, \quad q > 1 \,, \tag{3.12}$$

from now on, such that both cases $\mu_c > 0$ and $\mu_c = 0$ can be considered simultanuously. External loads of various sort can be treated by Remark 4.5.

Within the rescaled formulation (3.11) we want to investigate the possible limit behaviour for $h \to 0$ and **fixed internal length** $L_c > 0$. While it does not make much sense to let $h \to 0$ at fixed in-plane elongation $L > 0$, since from a physical consideration, there is an absolute lower bound on the thickness in terms of the internal length L_c, we may consider a **sequence of plates**, with small **relative thickness** h kept **constant** in a first place, but whose **in-plane elongation** L is **increased** together with a **simultaneous increase of the dimensions of the microstructure**, to the effect that the internal length L_c, transformed to a unit domain ω remains constant.[8] In a second step, the relative thickness h is decreased.

3.4 On the choice of the scaling

As will be seen later, the Γ-limit, if it exists, is unique. The only choice, which influences then the final form of the Γ-limit is given by the **initial scaling assumptions** made on the unknowns, in order to relate them to the fixed domain Ω_1 and the **assumption on the scaling of the energies**, here $\frac{1}{h} I^\sharp < \infty$. Our scaling ansatz is consistent with the one proposed in [23, 29], but not consistent with the one taken in [12], which scales transverse components of the displacement different in order to extract more information from the Γ-limit. Since we deal with a "two-field" model there is no imminent possibility to scale the fields differently.

The justification for our choice is given by the apparent consistency of the results with formal developments and its linearization stability. Here we see that the **scaling assumptions** also **introduce** a certain arbitrariness in the development. For example, starting from classical nonlinear elasticity, considering the present scaling for the unknowns and assuming $\frac{1}{h^5} I^\sharp < \infty$, a nonlinear von Kármán plate can be rigourously justified by Γ-convergence [29].

4 Recapitulation of facts from Γ-convergence

Let us briefly recapitulate the notions involved by using Γ-convergence. For a detailed treatment we refer e.g. to [42, 10]. We start by defining the lower and upper Γ-limit. In the following, X will always denote a metric space such that sequential compactness and compactness coincide. Moreover, we set $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. We consider now a sequence of energy functionals $I_{h_j} : X \mapsto \overline{\mathbb{R}}\,, h_j \to 0$.

[7] We could as well treat the rigid case, i.e. $\overline{R}^\sharp_{|_{\Gamma_0^1}} = \overline{R}_{\mathrm{d}}$. The case of weak consistent coupling would need additional provisions, the three-dimensional existence result already needs additional control in order to define the then necessary boundary terms.

[8] This is tantamount to assuming that the building blocks of the larger plates are themselves enlarged with the same ratio.

Definition 4.1 (Lower and upper Γ-limit)
Let X be a metric space and let $I_{h_j} : X \mapsto \overline{\mathbb{R}}\,, h_j \to 0$ be a sequence of functionals. For $x \in X$ we define

$$\Gamma - \liminf_{h_j} I_{h_j} : X \mapsto \overline{\mathbb{R}}\,, \quad \Gamma - \liminf_{h_j} I_{h_j}(x) := \inf\{\liminf_{h_j} I_{h_j}(x_{h_j})\,, \quad x_{h_j} \to x\}\,,$$
$$\Gamma - \limsup_{h_j} I_{h_j} : X \mapsto \overline{\mathbb{R}}\,, \quad \Gamma - \limsup_{h_j} I_{h_j}(x) := \inf\{\limsup_{h_j} I_{h_j}(x_{h_j})\,, \quad x_{h_j} \to x\}\,. \quad \blacksquare$$

It is clear that $\Gamma - \liminf_{h_j} I_{h_j}$ and $\Gamma - \limsup_{h_j} I_{h_j} : X \to \mathbb{R}$ always exist and are uniquely determined.

Definition 4.2 (Γ-convergence)
Let X be a metric space. We say that a sequence of functionals $I_{h_j} : X \mapsto \overline{\mathbb{R}}$ Γ-converges in X to the limit functional $I_0 : X \mapsto \overline{\mathbb{R}}$, if for all $x \in X$ we have

$$\forall\, x \in X : \forall\, x_{h_j} \to x : \quad I_0(x) \le \liminf_{h_j \to 0} I_{h_j}(x_{h_j})\,, \qquad (\liminf\textbf{-inequality})$$
$$\forall\, x \in X : \exists\, x_{h_i} \to x : \quad I_0(x) \ge \limsup_{h_i \to 0} I_{h_i}(x_{h_i})\,, \qquad \textbf{(recovery sequence)}\,. \quad \blacksquare$$

Corollary 4.3
Let X be a metric space. The sequence of functionals $I_{h_j} : X \mapsto \overline{\mathbb{R}}$ Γ-converges in X to $I_0 : X \mapsto \mathbb{R}$ if and only if

$$\Gamma - \liminf_{h_j} I_{h_j} = \Gamma - \limsup_{h_j} I_{h_j} = I_0\,. \quad \blacksquare$$

Remark 4.4 (Lower semicontinuity of the Γ-limit)
The lower and upper Γ-limits are always lower semicontinuous, hence the Γ-limit is a lower semicontinuous functional. Moreover, if the Γ-limit exists, it is unique.

Remark 4.5 (Stability under continuous perturbations)
Assume that $I_{h_j} : X \mapsto \overline{\mathbb{R}}$ Γ-converges in X to $I_0 : X \mapsto \overline{\mathbb{R}}$ and let $\Pi : X \mapsto \overline{\mathbb{R}}$, independent of h_j, be continuous. If $I_{h_j} + \Pi$ is Γ-convergent, then it holds

$$(\Gamma - \lim_{h_j}[I_{h_j} + \Pi])(x) = (\Gamma - \lim_{h_j} I_{h_j})(x) + \Pi(x) = I_0(x) + \Pi(x)\,, \tag{4.13}$$

see [10, p.23]or [42, Prop. 6.21]. Note that in the general case, the constant functional Π can influence whether or not Γ-convergence takes place, which necessitates the additional prior assumption on existence of the Γ-limit, compared to [10, p.23], cf. [42, Prop. 6.17]. $\blacksquare$

Let us also recapitulate the important **equi-coerciveness** property. First we recall **coerciveness** of an integral functional.[9]

Definition 4.6 (Coerciveness)
The integral functional $I : X \mapsto \overline{\mathbb{R}}$ is **coercive** *w.r.t. X, if for each fixed $C > 0$ the closure of the set $\{x \in X \,|\, I(x) \le C\}$ is compact in X, i.e. I has compact sub-levels.* $\blacksquare$

[9] A typical instant of coerciveness is given for $X = L^p(\Omega, \mathbb{R}^3)\,, 1 < p < \infty$ with Ω a bounded domain with smooth boundary and

$$I(\varphi) = \begin{cases} \int_\Omega W(\nabla\varphi)\,\mathrm{dV} & \text{if } \varphi \in W^{1,p}(\Omega, \mathbb{R}^3)\,, \quad \varphi_{|\partial\Omega} = 0\,, \\ +\infty & \text{else}\,, \end{cases} \tag{4.14}$$

with the **local coercivity assumption** $W(F) \ge c_1^+ \|\nabla\varphi\|^p - c_2^+$. Coerciveness follows by Poincaré's inequality and Rellich's compact embedding $W^{1,p}(\Omega, \mathbb{R}^3) \subset L^p(\Omega, \mathbb{R}^3)$. Recall that linear elasticity does not satisfy a local coercivity condition. This is the cause for some technical problems of the theory.

Following [42, p.70] we introduce

Definition 4.7 (Equi-coerciveness)
The sequence of integral functionals $I_{h_j} : X \mapsto \overline{\mathbb{R}}$ is **equi-coercive***, if for each fixed $C > 0$ there exists a compact set $K_C \subset X$ such that $\{x \in X \,|\, I_{h_j}(x) \leq C\} \subset K_C$,* **independent** *of $h_j > 0$.* ■

Hence, if we know that I_{h_j} is equi-coercive over X and that along a sequence $\varphi_j \in X$ it holds that $I_{h_j}(\varphi_j) \leq C$, then we can extract a subsequence, φ_{j_k} converging in the topology of X to some limit element $\varphi \in X$.

Theorem 4.8 (Characterization of equi-coerciveness)
The sequence of integral functionals $I_{h_j} : X \mapsto \mathbb{R}$ is equi-coercive if and only if there exists a lower semicontinuous coercive function $\Psi : X \mapsto \overline{\mathbb{R}}$ such that $I_{h_j} \geq \Psi$ on X for every $h_j > 0$.

Proof. [42, Prop. 7.7]. ■
The following theorem concerns the convergence of the minimum values of an equi-coercive sequence of functions.

Theorem 4.9 (Coerciveness of the Γ-limit)
Suppose that the sequence of integral functionals $I_{h_j} : X \mapsto \mathbb{R}$ is equi-coercive. Then the upper and lower Γ-limit are both coercive and

$$\min_{x \in X} \left(\Gamma - \liminf_{h_j} I_{h_j} \right)(x) = \liminf_{h_j} \inf_{x \in X} I_{h_j}(x) \,. \tag{4.15}$$

If, in addition, the sequence of integral functionals $I_{h_j} : X \mapsto \overline{\mathbb{R}}$ Γ-converges to a functional $I_0 : X \mapsto \overline{\mathbb{R}}$, then I_0 itself is **coercive** *and*

$$\min_{x \in X} I_0(x) = \lim_{h_j} \inf_{x \in X} I_{h_j}(x) \,. \tag{4.16}$$

Proof. [42, Theo. 7.8]. ■
Note that equi-coercivity is an additional feature in the development of Γ-convergence arguments, which allows to simplify proofs considerably through compactness arguments. As far as Γ-convergence is concerned, it may be useful to recall [10, p.19] that **minimizers of the Γ-limit variational problem may not be a limit of minimizers, so that Γ-convergence must be interpreted as a choice criterion.** In addition, the Γ-limit of a constant sequence of functionals J, which is not lower semicontinuous, does not coincide with the constant functional J, instead one has $(\Gamma - \lim J)(x) < J(x)$. In this case, $(\Gamma - \lim J)(x) = QJ(x)$, where QJ is the quasiconvex hull of J. In the case of non lower semicontinuous functionals, the Γ-limit is therefore introducing a different physical setting. Fortunately, in our application, we are always dealing with lower-semicontinuous functions.

5 The "two-field" Cosserat Γ-limit

5.1 The spaces and admissible sets

Now let us proceed to the investigation of the Γ-limit for the rescaled problem (3.11). We do not use $I^{\sharp}_{h_j}$ directly in our investigation of Γ-convergence, since this would imply working with the weak topology of $H^{1,2}(\Omega_1, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega_1, \mathrm{SO}(3,\mathbb{R}))$, which does not give rise to a metric space. Instead, we define the "bulk" spaces X, X' and the "two-dimensional" spaces X_ω, X'_ω. First, for $p \geq 1$, $q > 1$ we define the number $r > 1$ by

$$\frac{1}{1+p+q} + \frac{1}{r} = \frac{1}{2} \quad \Rightarrow \quad r = \frac{2(1+p+q)}{(1+p+q) - 2} \,, \tag{5.17}$$

such that $L^{1+p+q} \cdot L^r \subset L^2$. Note that for $1+p+q>3$ it holds that $r<6$ which implies the compact embedding $H^{1,2}(\Omega_1,\mathbb{R}^3) \subset L^r(\Omega_1,\mathbb{R}^3)$. Now define the spaces

$$
\begin{aligned}
X &:= \{(\varphi,\overline{R}) \in L^r(\Omega_1,\mathbb{R}^3) \times L^{1+p+q}(\Omega_1,\mathrm{SO}(3,\mathbb{R}))\}\,,\\
X' &:= \{(\varphi,\overline{R}) \in H^{1,2}(\Omega_1,\mathbb{R}^3) \times W^{1,1+p+q}(\Omega_1,\mathrm{SO}(3,\mathbb{R}))\,,\\
X_\omega &:= \{(\varphi,\overline{R}) \in L^r(\omega,\mathbb{R}^3) \times L^{1+p+q}(\omega,\mathrm{SO}(3,\mathbb{R}))\}\,,\\
X'_\omega &:= \{(\varphi,\overline{R}) \in H^{1,2}(\omega,\mathbb{R}^3) \times W^{1,1+p+q}(\omega,\mathrm{SO}(3,\mathbb{R}))\}\,,
\end{aligned}
\tag{5.18}
$$

and the admissible sets

$$
\begin{aligned}
\mathcal{A}' &:= \{(\varphi,\overline{R}) \in H^{1,2}(\Omega_1,\mathbb{R}^3) \times W^{1,1+p+q}(\Omega_1,\mathrm{SO}(3,\mathbb{R}))\,, \quad \varphi_{|_{\Gamma_0^1}}(\eta) = g_{\mathrm{d}}^\sharp(\eta) \quad \}\,, &(5.19)\\
\mathcal{A}'_\omega &:= \{(\varphi,\overline{R}) \in H^{1,2}(\omega,\mathbb{R}^3) \times W^{1,1+p+q}(\omega,\mathrm{SO}(3,\mathbb{R}))\,, \quad \varphi_{|_{\gamma_0}}(\eta_1,\eta_2) = g_{\mathrm{d}}^\sharp(\eta_1,\eta_2,0) \quad \}\,,\\
\mathcal{A}'_{\Omega_1,\omega} &:= \{(\varphi,\overline{R}) \in H^{1,2}(\Omega_1,\mathbb{R}^3) \times W^{1,1+p+q}(\omega,\mathrm{SO}(3,\mathbb{R}))\,, \quad \varphi_{|_{\Gamma_0^1}}(\eta) = g_{\mathrm{d}}^\sharp(\eta) \quad \}\,,
\end{aligned}
$$

We note the compact embedding $X' \subset X$ and the natural inclusions $X_\omega \subset X$ and $X'_\omega \subset X'$. Now we extend the rescaled energies to the space X through redefining

$$
I_h^\sharp(\varphi^\sharp,\nabla\varphi^\sharp,\overline{R}^\sharp,\mathrm{D}_\eta^{\mathrm{h}}\overline{R}^\sharp) = \begin{cases} I_h^\sharp(\varphi^\sharp,\nabla\varphi^\sharp,\overline{R}^\sharp,\mathrm{D}_\eta^{\mathrm{h}}\overline{R}^\sharp) & \text{if } (\varphi^\sharp,\overline{R}^\sharp) \in \mathcal{A}' \\ +\infty & \text{else in } X\,, \end{cases}
\tag{5.20}
$$

by abuse of notation. This is a classical trick used in applications of Γ-convergence. It has the additional virtue of incorporating the boundary conditions already in the energy functional. In the following, Γ-convergence results will be shown with respect to the encompassing metric space X.[10]

5.2 The transverse averaging operator

For $\varphi \in L^2(\Omega_1,\mathbb{R}^3)$ let us define the averaging operator over the transverse variable η_3

$$
\mathrm{Av}: L^2(\Omega_1,\mathbb{R}^3) \mapsto L^2(\omega,\mathbb{R}^3)\,, \quad \mathrm{Av}\,.\varphi(\eta_1,\eta_2) := \int_{-1/2}^{1/2} \varphi(\eta_1,\eta_2,\eta_3)\,\mathrm{d}\eta_3\,.
\tag{5.21}
$$

It is clear that averaging with respect to the transverse variable η_3 commutes with differentiation w.r.t. the planar variables η_1,η_2, i.e.

$$
[\mathrm{Av}\,.\nabla_{(\eta_1,\eta_2)}\varphi(\eta_1,\eta_2,\eta_3)](\eta_1,\eta_2) = \nabla_{(\eta_1,\eta_2)}[\mathrm{Av}\,.\varphi(\eta_1\eta_1\eta_1)](\eta_1,\eta_2)\,,
\tag{5.22}
$$

for suitable regular functions φ. For a convex function $f: \mathbb{M}^{3\times 2} \mapsto \mathbb{R}$ Jensen's inequality implies

$$
\begin{aligned}
\int_\omega f(\nabla_{(\eta_1,\eta_2)}[\mathrm{Av}\,.\varphi](\eta_1,\eta_2))\,\mathrm{d}\omega &= \int_\omega f([\mathrm{Av}\,.\nabla_{(\eta_1,\eta_2)}\varphi](\eta_1,\eta_2))\,\mathrm{d}\omega\\
&\le \int_\omega \int_{-1/2}^{1/2} f(\nabla_{(\eta_1,\eta_2)}\varphi(\eta_1,\eta_2,\eta_3))\,\mathrm{d}\eta_3\,\mathrm{d}\omega\\
&= \int_{\Omega_1} f(\nabla_{(\eta_1,\eta_2)}\varphi(\eta_1,\eta_2,\eta_3))\,\mathrm{dV}_\eta\,.
\end{aligned}
\tag{5.23}
$$

[10] Of course, X, X' as such are not vectorspaces, since we cannot add two rotations. Nevertheless, $L^r(\Omega_1,\mathrm{SO}(3,\mathbb{R})) \subset L^r(\Omega_1,\mathbb{M}^{3\times 3})$ and this space is a Banach space.

5.3 The Γ-limit variational "membrane" problem

We claim that for strictly positive Cosserat couple modulus $\mu_c > 0$ the Γ-limit for problem (3.11) is given by the following limit energy functional $I_0^\sharp : X \mapsto \overline{\mathbb{R}}$,

$$I_0^\sharp(\varphi, \overline{R}) := \begin{cases} \int_{\Omega_1} W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla \operatorname{Av}.\varphi, \overline{R}) + W_{\mathrm{curv}}^{\mathrm{hom}}(\mathfrak{K}_s)\, \mathrm{d}\omega - \Pi(\operatorname{Av}.\varphi, \overline{R}_3) & (\varphi, \overline{R}) \in \mathcal{A}'_{\Omega_1,\omega} \\ +\infty & \text{else in } X\,. \end{cases} \tag{5.24}$$

The proof will be given in Section 6. If we identify the averaged deformation $\operatorname{Av}.\varphi$ with the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$, this problem determines in fact a purely two-dimensional minimization problem for the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the microrotation of the plate (shell) $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3, \mathbb{R})$ on ω:

$$I_0^\sharp(m, \overline{R}) = \int_\omega W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla m, \overline{R}) + W_{\mathrm{curv}}^{\mathrm{hom}}(\mathfrak{K}_s)\, \mathrm{d}\omega - \Pi(m, \overline{R}_3) \mapsto \min\,.\ \text{w.r.t. } (m, \overline{R})\,, \tag{5.25}$$

and the boundary conditions of place for the midsurface deformation m on the Dirichlet part of the lateral boundary $\gamma_0 \subset \partial\omega$,

$$m_{|_{\gamma_0}} = g_{\mathrm{d}}(x, y, 0) = \operatorname{Av}.g_{\mathrm{d}}(x, y, 0)\,, \qquad \text{simply supported (fixed, welded)}\,. \tag{5.26}$$

The boundary conditions for the microrotations $\overline{R}$ are automatically determined in the variational process. The dimensionally homogenized local density is[11] [12]

$$\begin{aligned} W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla m, \overline{R}) &:= \mu \underbrace{\|\operatorname{sym}((\overline{R}_1|\overline{R}_2)^T \nabla m - 1\!\!1_2)\|^2}_{\text{"intrinsic" shear-stretch energy}} + \mu_c \underbrace{\|\operatorname{skew}((\overline{R}_1|\overline{R}_2)^T \nabla m)\|^2}_{\text{"intrinsic" first order drill energy}} \\ &+ \underbrace{2\mu \frac{\mu_c}{\mu + \mu_c} \left(\langle \overline{R}_3, m_x\rangle^2 + \langle \overline{R}_3, m_y\rangle^2\right)}_{\text{homogenized transverse shear energy}} + \frac{\mu\lambda}{2\mu + \lambda} \underbrace{\operatorname{tr}\left[\operatorname{sym}((\overline{R}_1|\overline{R}_2)^T \nabla m - 1\!\!1_2)\right]^2}_{\text{homogenized elongational stretch energy}}\,. \end{aligned} \tag{5.27}$$

The dimensionally homogenized curvature density is given by

$$\begin{aligned} W_{\mathrm{curv}}^{\mathrm{hom}}(\mathfrak{K}_s) &:= \inf_{A \in \mathfrak{so}(3,\mathbb{R})} W_{\mathrm{curv}}^*(\overline{R}^T \partial_{\eta_1} \overline{R}, \overline{R}^T \partial_{\eta_2} \overline{R}, A)\,, \\ \mathfrak{K}_s &= \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0)\right) = \overline{R}^T(x, y) \mathrm{D_x} \overline{R}(x, y)\,, \\ \mathfrak{K}_s &= (\mathfrak{K}_s^1, \mathfrak{K}_s^2, \mathfrak{K}_s^3) \in \mathfrak{T}(3)\,, \quad \text{the reduced third order } \textbf{curvature tensor}\,, \end{aligned} \tag{5.28}$$

where W_{curv}^* is an equivalent representation of the bulk curvature energy in terms of skew-symmetric arguments

$$\begin{aligned} W_{\mathrm{curv}}(\mathfrak{K}) &= W_{\mathrm{curv}}^*(\overline{R}^T \partial_{\eta_1} \overline{R}, \overline{R}^T \partial_{\eta_2} \overline{R}, \overline{R}^T \partial_{\eta_3} \overline{R})\,, \\ W_{\mathrm{curv}}^* &: \mathfrak{so}(3, \mathbb{R}) \times \mathfrak{so}(3, \mathbb{R}) \times \mathfrak{so}(3, \mathbb{R}) \mapsto \mathbb{R}^+\,, \end{aligned} \tag{5.29}$$

with $\overline{R}^T \partial_{\eta_i} \overline{R} \in \mathfrak{so}(3, \mathbb{R})$ since $\partial_{\eta_i}[\overline{R}^T \overline{R}] = \partial_{\eta_i} 1\!\!1 = 0$. We note that W_{curv}^* remains a convex function in its argument and so is $W_{\mathrm{curv}}^{\mathrm{hom}}(\mathfrak{K}_s)$. Moreover, $W_{\mathrm{curv}}^{\mathrm{hom}}(\mathfrak{K}_s) = W_{\mathrm{curv}}(\mathfrak{K}_s)$ for $W_{\mathrm{curv}}(\mathfrak{K}) = \widehat{W}(\|\mathfrak{K}\|)$.

[11] $\|\operatorname{skew}((\overline{R}_1|\overline{R}_2)^T \nabla m)\|^2 = \left(\langle \overline{R}_1, m_y\rangle - \langle \overline{R}_2, m_x\rangle\right)^2$. Note that $\|\operatorname{skew}((\overline{R}_1|\overline{R}_2)^T \nabla m)\| = 0$ does not imply that $\overline{R}_3 = \vec{n}_m$.

[12] In the following, "intrinsic" refers to classical surface geometry, where intrinsic quantities are those which depend only on the first fundamental form $I_m = \nabla m^T \nabla m \in \mathbb{M}^{2\times 2}$ of the surface. Then "intrinsic" in our terminology are terms, which reduce to such a dependence in the continuum limit $\overline{R} = \operatorname{polar}(\nabla m|\vec{n})$. For example $(\overline{R}_1|\overline{R}_2)^T \nabla m = \sqrt{\nabla m^T \nabla m}$, in this case.

In (5.25) Π denotes a general external loading functional, continuous in the topology of X, cf. Remark 4.5 and (14.19). It is clear that the limit functional $I_0^\sharp$ is weakly lower semicontinuous in the topology of $X' = H^{1,2}(\Omega, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$ by simple convexity arguments. We note the twofold appearance of the **harmonic mean** $\mathcal{H}$,[13]

$$\frac{1}{2}\mathcal{H}(\mu, \frac{\lambda}{2}) = \frac{\mu\lambda}{2\mu+\lambda}\,, \qquad \mathcal{H}(\mu,\mu_c) = 2\mu\,\frac{\mu_c}{\mu+\mu_c}\,. \tag{5.30}$$

A major advantage of this formulation is that the dimensionally homogenized formulation remains fully **frame-indifferent**. Note that the limit functional $I_0^\sharp$ is consistent with the following **plane stress** requirement (c.f. (6.48))

$$\forall\, \eta_3 \in [-\frac{1}{2}, \frac{1}{2}]: \quad S_1(\eta_1,\eta_2,\eta_3).e_3 = 0\,, \tag{5.31}$$

i.e. a vanishing normal stress over the entire thickness of the plate, while for any given thickness $h > 0$ from 3D-equilibrium one can only infer **zero normal stresss at the upper and lower faces**

$$\langle \overline{R}^T(\eta_1,\eta_2,\pm 1/2) S_1(\eta_1,\eta_2,\pm 1/2).e_3, e_3\rangle = 0\,. \tag{5.32}$$

In this sense, **the Cosserat "membrane" Γ-limit underestimates the real stresses, notably the transverse shear stresses**.

5.4 The borderline case $\mu_c = 0$

Since it is not possible to establish equi-coercivity for $\mu_c = 0$, we are not in a position to state a rigourous Γ-limit result based directly on the proof of the result for $\mu_c > 0$ in this case. However, since the energy functional $I_{h_j}^\sharp$ for $\mu_c > 0$ is strictly bigger than the same functional for $\mu_c = 0$, independent of $h_j > 0$, it is easy to see [42, Prop. 6.7] that on X we have the inequalities

$$\Gamma - \liminf I^\sharp_{h_j}{}_{|_{\mu_c=0}} \le \Gamma - \limsup I^\sharp_{h_j}{}_{|_{\mu_c=0}} \le \lim_{\mu_c\to 0}\left(\Gamma - \lim I^\sharp_{h_j}{}_{|_{\mu_c>0}}\right) =: I_0^{\sharp,0}\,, \tag{5.33}$$

where

$$I_0^{\sharp,0}(\varphi,\overline{R}) = \begin{cases} \int_\omega W_{\mathrm{mp}}^{\mathrm{hom},0}(\nabla \operatorname{Av}.\varphi, \overline{R}) + W_{\mathrm{curv}}^{\mathrm{hom}}(\mathfrak{K}_s)\,\mathrm{d}\omega - \Pi(\operatorname{Av}.\varphi, \overline{R}_3) & (\varphi,\overline{R}) \in \mathcal{A}_0^{\mathrm{mem}} \\ +\infty & \text{else in } X\,, \end{cases} \tag{5.34}$$

with $\mathcal{A}_0^{\mathrm{mem}}$ defined in (7.90) and the corresponding local energy density in terms of $m = \operatorname{Av}.\varphi$ is

$$W_{\mathrm{mp}}^{\mathrm{hom},0}(\nabla m, \overline{R}) := \mu\,\underbrace{\|\mathrm{sym}(\overline{R}_1|\overline{R}_2)^T\nabla m - 1\!\!1_2\|^2}_{\text{"intrinsic" shear-stretch energy}} + \frac{\mu\lambda}{2\mu+\lambda}\,\underbrace{\mathrm{tr}\left[\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T\nabla m - 1\!\!1_2)\right]^2}_{\text{homogenized elongational stretch energy}}\,. \tag{5.35}$$

Observe that the upper bound $I_0^{\sharp,0}$ for the $\Gamma - \limsup$ energy functional is **not coercive** w.r.t. $H^{1,2}(\omega,\mathbb{R}^3)$ due to the now missing transverse shear contribution, while it retains lower-semicontinuity. This degeneration remains true for whatever form the Γ-limit for $\mu_c = 0$ has, should it exist. We complement the investigation of the geometrically exact case $\mu_c = 0$ with an estimate for the $\Gamma - \liminf$ in Section 7, which shows altogether, that $I_0^{\sharp,0}$ is indeed the Γ-limit for zero Cosserat couple modulus $\mu_c = 0$.

[13] For $a, b \ge 0$ the harmonic, arithmetic and geometric mean are defined as $\mathcal{H}(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}$, $\mathcal{A}(a,b) = \frac{a+b}{2}$, $\mathcal{G}(a,b) = \sqrt{a\,b}$, respectively and one has the chain of inequalities $\mathcal{H}(a,b) \le \mathcal{G}(a,b) \le \mathcal{A}(a,b)$.

For $\mu = \mu_c$, however, the limit energy $W_{\rm mp}^{\rm hom}$ coincides with the respective plate energy $W_{\rm mp}$ derived in terms of the formal ansatz given in (8.1). If $\mu_c > 0$, then coercivity and well-posedness of the limit problem can be established by a local coercivity argument and Poincaré's inequality or can be inferred from equi-coerciveness and Theorem 4.9.

The loss of coercivity for $\mu_c = 0$ is primarily a loss of control for the "transverse" components $\langle m_x, \overline{R}_3\rangle, \langle m_y, \overline{R}_3\rangle$, while w.r.t. the remaining "in-plane" components compactness for minimizing sequences, whose midsurface deformations are supposed to be already bounded in $L^r(\omega)$, can be established (appropriate use of an extended Korn's second inquality, c.f. (7.102)).

As far as linearization consistency is concerned, it is an easy matter to show (see (14.11)) that the linearization for $\mu_c = 0$ of the frame-indifferent Γ-limit $I_0^{\sharp,0}$ w.r.t. small midsurface displacement $v : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ **and** small curvature decouples the fields of infinitesimal midsurface displacement and infinitesimal microrotations: after descaling we are left with the classical infinitesimal "membrane" plate problem for $v : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$

$$\begin{aligned}
&\int_\omega h \left(\mu \,\| \operatorname{sym} \nabla(v_1, v_2)\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \operatorname{tr}\left[\operatorname{sym} \nabla(v_1, v_2)\right]^2\right) \mathrm{d}\omega \\
&\qquad\qquad - \langle f, \langle v, e_1\rangle \cdot e_1 + \langle v, e_2\rangle \cdot e_2\rangle \mapsto \min . \text{ w.r.t. } v\,, \qquad (5.36)\\
&\langle v, e_i\rangle_{|_{\gamma_0}} = \langle u^{\rm d}(x,y,0), e_i\rangle\,, \quad i = 1,2 \qquad \text{simply supported (horizontal components only)}\,,
\end{aligned}$$

which leaves the vertical midsurface displacement v_3 indetermined due to the non-resistance of a linear "membrane" plate to vertical deflections. This problem coincides with a linearization[14] of the nonlinear membrane plate problem proposed in [27, par.4.3], based on purely formal asymptotic methods applied to the St.Venant-Kirchhoff energy. The variational problem (5.36) is as well the Γ-limit of the classical linear elasticity bulk problem (if corresponding scaling asumptions are made, compare with [3, Th.4.2], [8] or [12, Th.1.11.2] and (14.2)). The classical linear bulk model in turn can be obtained as linearization for $\mu_c = 0$ of the Cosserat bulk problem. Hence, **for $\mu_c = 0$ exclusively, linearization and taking the Γ-limit commute with the Γ-limit of classical linear elasticity.**[15]

5.5 The borderline case $\mu_c = \infty$

This case is interesting, because the rigourous Γ-limit for $\mu_c = \infty$ still gives rise to an independent field of microrotations $\overline{R}$, while the Cosserat bulk problem for $\mu_c = \infty$ degenerates into a constraint theory (a so called interdeterminate couple-stress model), where $\overline{R}$ coincides necessarily with the continuum rotations $\operatorname{polar}(F)$ from the polar decomposition.

The Γ-limit variational problem reads: find the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the microrotation of the plate (shell) $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3,\mathbb{R})$ on ω such that for $I_0^{\sharp,\infty} : X \mapsto \overline{\mathbb{R}}$ in terms of the averaged deformation $m = \mathrm{Av}\,.\varphi$,

$$I_0^{\sharp,\infty}(m, \overline{R}) \mapsto \min . \text{ w.r.t. } (m, \overline{R})\,, \tag{5.37}$$

with

$$I_0^{\sharp,\infty}(m,\overline{R}) = \begin{cases} \int_\omega W_{\rm mp}^{\rm hom,\infty}(\nabla m, \overline{R}) + W_{\rm curv}^{\rm hom}(\mathfrak{K}_s)\,\mathrm{d}\omega - \Pi(m, \overline{R}_3) & (m,\overline{R}) \in \mathcal{A}_\omega^{\prime,\infty} \\ +\infty & \text{else in } X\,, \end{cases} \tag{5.38}$$

the admissible set

$$\begin{aligned}
\mathcal{A}_\omega^{\prime,\infty} := \{(m,\overline{R}) \in H^{1,2}(\omega,\mathbb{R}^3) \times W^{1,1+p+q}(\omega, \mathrm{SO}(3,\mathbb{R}))\,, \quad & m_{|_{\gamma_0}}(\eta_1,\eta_2) = g_{\rm d}^\sharp(\eta_1,\eta_2,0)\,, \\
\langle \overline{R}_1, m_y\rangle = \langle \overline{R}_2, m_x\rangle \quad & \}\,, \qquad (5.39)
\end{aligned}$$

[14] Expansion of the first fundamental form I_m of the midsurface m w.r.t. planar initial configuration yields $I_m - 1\!\!1_2 = \nabla m^T \nabla m - 1\!\!1_2 \approx \operatorname{sym} \nabla_{(x,y)}(v_1, v_2) + O(\|\nabla v\|^2)$. Hence control on vertical deflections v_3 is lost during linearization.

[15] As is well known [14, p.464] this is not the case with the membrane Γ-limit considered in [23], based on the non-elliptic St.Venant-Kirchhoff energy.

and the corresponding dimensionally homogenized local energy density is

$$W_{\mathrm{mp}}^{\mathrm{hom},\infty}(\nabla m,\overline{R}) := \mu \underbrace{\|(\overline{R}_1|\overline{R}_2)^T\nabla m - 1\!\!1_2\|^2}_{\text{"intrinsic" shear-stretch energy}} + \underbrace{2\mu\left(\langle\overline{R}_3,m_x\rangle^2+\langle\overline{R}_3,m_y\rangle^2\right)}_{\text{homogenized transverse shear energy}}$$
$$+\frac{\mu\lambda}{2\mu+\lambda}\underbrace{\mathrm{tr}\left[\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T\nabla m - 1\!\!1_2)\right]^2}_{\text{homogenized elongational stretch energy}}\ . \tag{5.40}$$

Note that $\mu_c=\infty$ **effectively rules out in-plane drill rotations** [40, 28]. Moreover, the transverse shear energy is doubled, but transverse shear is still possible. In this sense, the resulting homogenized transverse shear modulus excludes what could be called "transverse shear locking" in accordance with the "Poisson thickness locking" which occurs, if the correct homogenized volumetric modulus is not taken.[16]

6 Proof of Γ-convergence for positive Cosserat couple modulus $\mu_c > 0$

Let us continue by proving the claim on the form of the Γ-limit for strictly positive Cosserat couple modulus $\mu_c > 0$.

6.1 Equi-coercivity of $I^\sharp_{h_j}$, compactness and dimensional reduction

Theorem 6.1 (Equi-coercivity of $I^\sharp_{h_j}$)
For positive Cosserat couple modulus $\mu_c>0$ the sequence of rescaled energy functionals $I^\sharp_{h_j}$ defined in (3.11) *is equi-coercive on the space X.*

Proof. It is clear that for given $h>0$ the problem (3.11) admits a minimizing pair $(\varphi^\sharp_h,\overline{R}^\sharp_h)\in H^{1,2}(\Omega_1,\mathbb{R}^3)\times W^{1,1+p+q}(\Omega_1,\mathrm{SO}(3,\mathbb{R}))$ by the obvious scaling transformation of the minimizing solution of the bulk problem for values of $p\geq 1, q>1$ and for both $\mu_c>0$ and $\mu_c=0$.[17] This is especially true for Neumann boundary conditions on the microrotations, since for exact rotations, $\|\overline{R}\|=\sqrt{3}$. This leads to a control of microrotations in $W^{1,1+p+q}(\Omega_1,\mathrm{SO}(3,\mathbb{R}))$ already without specification of Dirichlet boundary data on the microrotations.

Consider now a sequence $h_j\to 0$ for $j\to\infty$. By inspection of the existence proof for the Cosserat bulk problem, it will become clear that for corresponding sequences $(\varphi^\sharp_{h_j},\overline{R}^\sharp_{h_j})\in H^{1,2}(\Omega_1,\mathbb{R}^3)\times W^{1,1+p+q}(\Omega_1,\mathrm{SO}(3,\mathbb{R}))=X'$ with $I^\sharp_{h_j}(\varphi^\sharp_{h_j},\overline{R}^\sharp_{h_j})<\infty$ bounded independent of h_j (not necessarily minimizers) we obtain a bound on the sequence $(\varphi^\sharp_{h_j},\overline{R}^\sharp_{h_j})$ in X', independent of h_j. To see this, note that for $\mu_c>0$, it is immediate that $\nabla^h_\eta\varphi^\sharp = F^\sharp_h$ is **bounded** in $L^2(\Omega_1,\mathbb{M}^{3\times 3})$, **independent** of $\overline{R}^\sharp_{h_j}$ on account of the decisive **local coercivity** condition

$$\begin{aligned} W_{\mathrm{mp}}(\overline{R}^{\sharp,T}_{h_j}F^\sharp_{h_j}) &\geq \min(\mu_c,\mu)\,\|\overline{R}^{\sharp,T}_{h_j}F^\sharp_{h_j}-1\!\!1\|^2=\min(\mu_c,\mu)\,\left(\|F^\sharp_{h_j}\|^2-2\langle\overline{R}^{\sharp,T}_{h_j}F^\sharp_{h_j},1\!\!1\rangle+3\right)\\ &\geq \min(\mu_c,\mu)\left(\|F^\sharp_{h_j}\|^2-2\sqrt{3}\|F^\sharp_{h_j}\|+3\right)\ , \end{aligned}\tag{6.41}$$

[16] $\lim_{\lambda\to\infty}\frac{1}{2}\mathcal{H}(\mu,\frac{\lambda}{2})=\mu<\infty$ but $\lim_{\lambda\to\infty}\frac{1}{2}\mathcal{A}(\mu,\frac{\lambda}{2})=\infty$.

[17] In contrast to Γ-convergence arguments based on the St.Venant-Kirchhoff energy [23] which might not admit minimizers for any given $h>0$.

and after integration

$$
\begin{aligned}
\infty > I^\sharp_{h_j}(\varphi^\sharp_{h_j},\overline{R}^\sharp_{h_j}) &> \int_{\Omega_1} W_{\mathrm{mp}}(\overline{U}^\sharp_{h_j}) + W_{\mathrm{curv}}(\mathfrak{K}^\sharp_{h_j})\,\mathrm{dV}_\eta \geq \int_{\Omega_1} W_{\mathrm{mp}}(\overline{U}^\sharp_{h_j})\,\mathrm{dV}_\eta \\
&\geq \int_{\Omega_1} \min(\mu_c,\mu)\left(\|F^\sharp_{h_j}\|^2 - 2\sqrt{3}\|F^\sharp_{h_j}\| + 3\right)\,\mathrm{dV}_\eta \qquad (6.42)\\
&\geq \min(\mu_c,\mu)\int_{\Omega_1}\left(\left[\|\partial_{\eta_1}\varphi^\sharp\|^2 + \|\partial_{\eta_2}\varphi^\sharp\|^2 + \frac{1}{{h_j}^2}\|\partial_{\eta_3}\varphi^\sharp\|^2\right]\right.\\
&\qquad\qquad \left. -2\sqrt{3}\left[\|\partial_{\eta_1}\varphi^\sharp\| + \|\partial_{\eta_2}\varphi^\sharp\| + \frac{1}{h_j}\|\partial_{\eta_3}\varphi^\sharp\|\right] + 3\right)\,\mathrm{dV}_\eta\,.
\end{aligned}
$$

This implies a bound, independent of h_j, for the gradient $\nabla\varphi^\sharp_{h_j}$ in $L^2(\Omega_1,\mathbb{R}^3)$. The Dirichlet boundary conditions for $\varphi^\sharp_{h_j}$ together with Poincaré's inequality yield the boundedness of $\varphi^\sharp_{h_j}$ in $H^{1,2}(\Omega_1,\mathbb{R}^3)$.[18] With a similar argument, based on the local coercivity of curvature, the bound on $\overline{R}^\sharp_{h_j}$ can be obtained: we need only to observe that for a constant $c^+ > 0$, depending on the positivity of $\alpha_4,\alpha_5,\alpha_6,\alpha_7$, but independent of h_j,

$$
\begin{aligned}
\infty > I^\sharp_{h_j}(\varphi^\sharp_{h_j},\overline{R}^\sharp_{h_j}) &> \int_{\Omega_1} W_{\mathrm{mp}}(\overline{U}^\sharp_{h_j}) + W_{\mathrm{curv}}(\mathfrak{K}^\sharp_{h_j})\,\mathrm{dV}_\eta \geq \int_{\Omega_1} W_{\mathrm{curv}}(\mathfrak{K}^\sharp_{h_j})\,\mathrm{dV}_\eta \qquad (6.43)\\
\geq \int_{\Omega_1} c^+\,\|\mathfrak{K}^\sharp_{h_j}\|^{1+p+q}\,\mathrm{dV}_\eta &= c^+\int_{\Omega_1}\|\overline{R}^{\sharp,T}_{h_j}\mathrm{D}^{\mathrm{h_j}}_\eta\overline{R}^\sharp_{h_j}\|^{1+p+q}\,\mathrm{dV}_\eta = c^+\int_{\Omega_1}\|\mathrm{D}^{\mathrm{h_j}}_\eta\overline{R}^\sharp_{h_j}\|^{1+p+q}\,\mathrm{dV}_\eta\,,
\end{aligned}
$$

which establishes a bound on the gradient of rotations $\nabla^{h_j}_\eta[\overline{R}^\sharp_{h_j}(\eta).e_i]$, $i=1,2,3$, independent of h_j. Moreover, $\|\overline{R}^\sharp_{h_j}\| = \sqrt{3}$, establishing the $W^{1,1+p+q}(\Omega_1,\mathrm{SO}(3,\mathbb{R}))$ bound on $\overline{R}^\sharp_{h_j}$. Thus we may obtain a subsequence, not relabelled, such that

$$
\varphi^\sharp_{h_j} \rightharpoonup \varphi^\sharp_0 \quad \text{in } H^{1,2}(\Omega_1,\mathbb{R}^3)\,,\quad \overline{R}^\sharp_{h_j} \rightharpoonup \overline{R}^\sharp_0 \quad \text{in } W^{1,1+p+q}(\Omega_1,\mathrm{SO}(3,\mathbb{R}))\,. \qquad (6.44)
$$

Both weak limits $(\varphi^\sharp_0,\overline{R}^\sharp_0)$ must be independent of the transverse coordinate η_3, otherwise the energy $I^\sharp_{h_j}$ could not remain finite for $h_j\to 0$, see (6.42) and compare with the definition of $\mathrm{D}^{\mathrm{h_j}}_\eta$ in (3.6). Hence the solution must be found in terms of functions defined on the two-dimensional domain ω. In this sense the domain of the limit problem is two-dimensional and the corresponding space is X_ω. Since the embedding $X' \subset X$ is compact, it is shown that the sequence of energy functionals $I^\sharp_{h_j}$ is equi-coercive w.r.t. X. ■

6.2 Lower bound-the $\liminf$-condition

If $I^\sharp_0$ is the Γ-limit of the sequence of energy functionals $I^\sharp_{h_j}$ then we must have ($\liminf$-inequality) that

$$
I^\sharp_0(\varphi_0,\overline{R}_0) \leq \liminf_{h_j} I^\sharp_{h_j}(\varphi^\sharp_{h_j},\overline{R}^\sharp_{h_j})\,, \qquad (6.45)
$$

whenever

$$
\varphi^\sharp_{h_j} \to \varphi^\sharp_0 \quad \text{in } L^r(\Omega_1,\mathbb{R}^3)\,,\quad \overline{R}^\sharp_{h_j} \to \overline{R}^\sharp_0 \quad \text{in } L^{1+p+q}(\Omega_1,\mathrm{SO}(3,\mathbb{R}))\,, \qquad (6.46)
$$

for arbitrary $(\varphi^\sharp_0,\overline{R}^\sharp_0)\in X$. Observe that we can restrict attention to sequences $(\varphi^\sharp_{h_j},\overline{R}^\sharp_{h_j})\in X$ such that $I^\sharp_{h_j}(\varphi^\sharp_{h_j},\overline{R}^\sharp_{h_j}) < \infty$ since otherwise the statement is true anyway. Sequences with

[18] This argument fails for the limit case $\mu_c = 0$ since local coercivity does not hold.

$I^\sharp_{h_j}(\varphi^\sharp_{h_j}, \overline{R}^\sharp_{h_j}) < \infty$ are uniformly bounded in the space X', as seen previously. This implies weak convergence of a subsequence in X'. But we know already that the original sequences converge strongly in X to the limit $(\varphi^\sharp_0, \overline{R}^\sharp_0) \in X$. Hence we must have as well weak convergence to $\varphi^\sharp_0 \in H^{1,2}(\omega, \mathbb{R}^3)$ and $\overline{R}^\sharp_0 \in W^{1,1+p+q}(\omega, \mathrm{SO}(3,\mathbb{R}))$, independent of the transverse variable η_3.

In a first step we consider now the **local energy contribution**: along sequences $(\varphi^\sharp_{h_j}, \overline{R}^\sharp_{h_j}) \in X$ with finite energy $I^\sharp_{h_j}$, the third column of the deformation gradient $\nabla^{h_j}_\eta \varphi^\sharp_{h_j}$ remains bounded but otherwise indetermined. Therefore, a trivial lower bound is obtained by minimizing the effect of the derivative in this direction in the local energy W_{mp}. To continue our development, we need some calculations: For smooth $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$, $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3,\mathbb{R})$ define the "director"-vector $b^* \in \mathbb{R}^3$ formally through

$$W^{\mathrm{hom}}_{\mathrm{mp}}(\nabla m, \overline{R}) = W_{\mathrm{mp}}(\overline{R}^T(\nabla m|b^*)) := \inf_{b\in\mathbb{R}^3} W_{\mathrm{mp}}(\overline{R}^T(\nabla m|b))\,. \tag{6.47}$$

The vector b^*, which realizes this infimum, can be explicitly determined. Set $\tilde{F} := (\nabla m|b^*)$. The corresponding local optimality condition reads

$$\begin{aligned}
\forall\, \delta b^* \in \mathbb{R}^3 :\quad & \langle DW_{\mathrm{mp}}(\overline{R}^T(\nabla m|b^*)), \overline{R}^T(0|0|\delta b^*)\rangle = 0 \;\Rightarrow \\
& \langle \overline{R}\, DW_{\mathrm{mp}}(\overline{R}^T(\nabla m|b^*)), (0|0|\delta b^*)\rangle = 0 \;\Rightarrow \\
& \overline{R}\, DW_{\mathrm{mp}}(\overline{R}^T(\nabla m|b^*)).e_3 = 0 \;\Rightarrow D_{\tilde{F}} W_{\mathrm{mp}}(\overline{R}^T(\nabla m|b^*)).e_3 = 0 \Rightarrow \\
& \quad S_1((\nabla m|b^*), \overline{R}).e_3 = 0\,.
\end{aligned} \tag{6.48}$$

Since

$$S_1(F, \overline{R}) = \overline{R}\left(\mu\left(F^T\overline{R} + \overline{R}^T F - 2\,1\!\!1\right) + 2\mu_c\ \mathrm{skew}(\overline{R}^T F) + \lambda\,\mathrm{tr}\left[\overline{R}^T F - 1\!\!1\right] 1\!\!1\right) \tag{6.49}$$

and

$$\begin{aligned}
\overline{R}^T\tilde{F} &= \begin{pmatrix} \langle \overline{R}_1, m_x\rangle & \langle \overline{R}_1, m_y\rangle & \langle \overline{R}_1, b^*\rangle \\ \langle \overline{R}_2, m_x\rangle & \langle \overline{R}_2, m_y\rangle & \langle \overline{R}_2, b^*\rangle \\ \langle \overline{R}_3, m_x\rangle & \langle \overline{R}_3, m_y\rangle & \langle \overline{R}_3, b^*\rangle \end{pmatrix}, \\
\tilde{F}^T\overline{R} + \overline{R}^T\tilde{F} - 2\,1\!\!1 &= \begin{pmatrix} 2[\langle \overline{R}_1, m_x\rangle - 1] & \langle \overline{R}_1, m_y\rangle + \langle \overline{R}_2, m_x\rangle & \langle \overline{R}_1, b^*\rangle + \langle \overline{R}_3, m_x\rangle \\ \langle \overline{R}_2, m_x\rangle + \langle \overline{R}_1, m_y\rangle & 2[\langle \overline{R}_2, m_y\rangle - 1] & \langle \overline{R}_2, b^*\rangle + \langle \overline{R}_3, m_y\rangle \\ \langle \overline{R}_3, m_x\rangle + \langle \overline{R}_1, b^*\rangle & \langle \overline{R}_3, m_y\rangle + \langle \overline{R}_2, b^*\rangle & 2[\langle \overline{R}_3, b^*\rangle - 1] \end{pmatrix}, \\
\mathrm{skew}(\overline{R}^T\tilde{F}) &= \begin{pmatrix} 0 & \frac{1}{2}\left(\langle \overline{R}_1, m_y\rangle - \langle \overline{R}_2, m_x\rangle\right) & \frac{1}{2}\left(\langle \overline{R}_1, b^*\rangle - \langle \overline{R}_3, m_x\rangle\right) \\ * & 0 & \frac{1}{2}\left(\langle \overline{R}_2, b^*\rangle - \langle \overline{R}_3, m_y\rangle\right) \\ * & * & 0 \end{pmatrix},
\end{aligned} \tag{6.50}$$

the (plane-stress) requirement $S_1.e_3 = 0$ (6.48) implies

$$\begin{aligned}
&\mu \begin{pmatrix} \langle \overline{R}_1, b^*\rangle + \langle \overline{R}_3, m_x\rangle \\ \langle \overline{R}_2, b^*\rangle + \langle \overline{R}_3, m_y\rangle \\ 2[\langle \overline{R}_3, b^*\rangle - 1] \end{pmatrix} + \mu_c \begin{pmatrix} \langle \overline{R}_1, b^*\rangle - \langle \overline{R}_3, m_x\rangle \\ \langle \overline{R}_2, b^*\rangle - \langle \overline{R}_3, m_y\rangle \\ 0 \end{pmatrix} \\
&\qquad + \lambda\left(\langle \overline{R}_1, m_x\rangle + \langle \overline{R}_2, m_y\rangle + \langle \overline{R}_3, b^*\rangle - 3\right)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned} \tag{6.51}$$

The solution of the last system can conveniently be expressed in the orthonormal triad $(\overline{R}_1, \overline{R}_2, \overline{R}_3)$ as

$$\begin{aligned}
b^* &= \frac{\mu_c - \mu}{\mu + \mu_c}\langle \overline{R}_3, m_x\rangle\, \overline{R}_1 + \frac{\mu_c - \mu}{\mu + \mu_c}\langle \overline{R}_3, m_y\rangle\, \overline{R}_2 + \varrho^*_m\, \overline{R}_3\,, \\
\varrho^*_m &= 1 - \frac{\lambda}{2\mu + \lambda}\left[\langle (\nabla m|0), \overline{R}\rangle - 2\right].
\end{aligned} \tag{6.52}$$

Note that for $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$ and $\nabla m \in L^2(\Omega_1,\mathbb{R}^3)$ it follows that $b^* \in L^2(\Omega_1,\mathbb{R}^3)$. Reinserting the solution b^* we have

$$\overline{R}^T\tilde{F} = \begin{pmatrix} \langle\overline{R}_1,m_x\rangle & \langle\overline{R}_1,m_y\rangle & \frac{\mu_c-\mu}{\mu+\mu_c}\langle\overline{R}_3,m_x\rangle \\ \langle\overline{R}_2,m_x\rangle & \langle\overline{R}_2,m_y\rangle & \frac{\mu_c-\mu}{\mu+\mu_c}\langle\overline{R}_3,m_y\rangle \\ \langle\overline{R}_3,m_x\rangle & \langle\overline{R}_3,m_y\rangle & \varrho_m^* \end{pmatrix},$$

$$\tilde{F}^T\overline{R} + \overline{R}^T\tilde{F} - 2\,1\!\!1 = \begin{pmatrix} 2[\langle\overline{R}_1,m_x\rangle - 1] & \langle\overline{R}_1,m_y\rangle + \langle\overline{R}_2,m_x\rangle & \left(1+\frac{\mu_c-\mu}{\mu+\mu_c}\right)\langle\overline{R}_3,m_x\rangle \\ \langle\overline{R}_2,m_x\rangle + \langle\overline{R}_1,m_y\rangle & 2[\langle\overline{R}_2,m_y\rangle - 1] & \left(1+\frac{\mu_c-\mu}{\mu+\mu_c}\right)\langle\overline{R}_3,m_y\rangle \\ \left(1+\frac{\mu_c-\mu}{\mu+\mu_c}\right)\langle\overline{R}_3,m_x\rangle & \left(1+\frac{\mu_c-\mu}{\mu+\mu_c}\right)\langle\overline{R}_3,m_y\rangle & 2[\varrho_m^*-1] \end{pmatrix},$$

$$\mathrm{skew}(\overline{R}^T\tilde{F}) = \begin{pmatrix} 0 & \frac{1}{2}\left(\langle\overline{R}_1,m_y\rangle - \langle\overline{R}_2,m_x\rangle\right) & \frac{1}{2}\left(\left(\frac{\mu_c-\mu}{\mu+\mu_c}-1\right)\langle\overline{R}_3,m_x\rangle\right) \\ * & 0 & \frac{1}{2}\left(\left(\frac{\mu_c-\mu}{\mu+\mu_c}-1\right)\langle\overline{R}_3,m_y\rangle\right) \\ * & * & 0 \end{pmatrix},$$

$$1+\frac{\mu_c-\mu}{\mu+\mu_c} = \frac{2\mu_c}{\mu+\mu_c}, \qquad \frac{\mu_c-\mu}{\mu+\mu_c} - 1 = \frac{-2\mu}{\mu+\mu_c}. \tag{6.53}$$

We obtain finally for $W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla m,\overline{R}) := W_{\mathrm{mp}}(\overline{R}^T(\nabla m|b^*))$ with $\tilde{\overline{U}} = \overline{R}^T(\nabla m|b^*) = \overline{R}^T\tilde{F}$ after a lengthy but otherwise straightforward computation

$$\begin{aligned} W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla m,\overline{R}) &:= W_{\mathrm{mp}}(\tilde{\overline{U}}) = \mu\,\|\mathrm{sym}(\tilde{\overline{U}} - 1\!\!1)\|^2 + \mu_c\,\|\mathrm{skew}(\tilde{\overline{U}})\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[\mathrm{sym}(\tilde{\overline{U}} - 1\!\!1)\right]^2 \\ &= \mu\,\|\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T\nabla m - 1\!\!1_2)\|^2 + \mu_c\,\|\mathrm{skew}((\overline{R}_1|\overline{R}_2)^T\nabla m)\|^2 \\ &+ 2\mu\,\frac{\mu_c}{\mu+\mu_c}\left(\langle\overline{R}_3,m_x\rangle^2 + \langle\overline{R}_3,m_y\rangle^2\right) + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T\nabla m - 1\!\!1_2)\right]^2. \end{aligned} \tag{6.54}$$

Along the sequence $(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp)$ we have by construction,

$$W_{\mathrm{mp}}(\overline{R}_{h_j}^{\sharp,T}\nabla_\eta^{h_j}\varphi_{h_j}^\sharp) = W_{\mathrm{mp}}(\overline{R}_{h_j}^{\sharp,T}(\nabla_{(\eta_1,\eta_2)}\varphi_{h_j}^\sharp|\frac{1}{h_j}\partial_{\eta_3}\varphi_{h_j}^\sharp)) \geq W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla_{(\eta_1,\eta_2)}\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp). \tag{6.55}$$

Hence, integrating and taking the $\liminf$ also

$$\liminf_{h_j}\int_{\Omega_1} W_{\mathrm{mp}}(\overline{R}_{h_j}^{\sharp,T}\nabla_\eta^{h_j}\varphi_{h_j}^\sharp)\,\mathrm{dV}_\eta \geq \liminf_{h_j}\int_{\Omega_1} W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla_{(\eta_1,\eta_2)}\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp)\,\mathrm{dV}_\eta. \tag{6.56}$$

Now we use weak convergence of $\varphi_{h_j}^\sharp$ and strong convergence of $\overline{R}_{h_j}^\sharp$, together with the convexity w.r.t. ∇m and continuity w.r.t. $\overline{R}$ of $\int_{\Omega_1} W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla m,\overline{R})\,\mathrm{dV}_\eta$ to get lower semi-continuity of the right hand side in (6.56) and to obtain altogether

$$\liminf_{h_j}\int_{\Omega_1} W_{\mathrm{mp}}(\overline{R}_{h_j}^{\sharp,T}\nabla_\eta^{h_j}\varphi_{h_j}^\sharp)\,\mathrm{dV}_\eta \geq \int_{\Omega_1} W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla_{(\eta_1,\eta_2)}\varphi_0^\sharp, \overline{R}_0^\sharp)\,\mathrm{dV}_\eta. \tag{6.57}$$

Next we are concerned with the **curvature contribution**: it is always possible to uniquely rewrite the curvature energy expression in terms of skew-symmetric quantities

$$\begin{aligned} &W_{\mathrm{curv}}^* : \mathfrak{so}(3,\mathbb{R}) \times \mathfrak{so}(3,\mathbb{R}) \times \mathfrak{so}(3,\mathbb{R}) \mapsto \mathbb{R}^+, \\ &W_{\mathrm{curv}}^*(\overline{R}^T\partial_{\eta_1}\overline{R}, \overline{R}^T\partial_{\eta_2}\overline{R}, \overline{R}^T\partial_{\eta_3}\overline{R}) := W_{\mathrm{curv}}(\mathfrak{K}), \end{aligned} \tag{6.58}$$

where $\overline{R}^T\partial_{\eta_i}\overline{R} \in \mathfrak{so}(3,\mathbb{R})$ since $\partial_{\eta_i}[\overline{R}^T\overline{R}] = 0$. We note that W_{curv}^* remains a convex function in its argument since $\mathfrak{K} \in \mathfrak{T}(3)$ can be obtained by a linear mapping from $(\overline{R}^T\partial_{\eta_1}\overline{R}, \overline{R}^T\partial_{\eta_2}\overline{R}, \overline{R}^T\partial_{\eta_3}\overline{R}) \in$

$\mathfrak{so}(3,\mathbb{R})\times\mathfrak{so}(3,\mathbb{R})\times\mathfrak{so}(3,\mathbb{R})$. We define the "homogenized" (relaxed) curvature energy through

$$\begin{aligned} W^{*,\text{hom}}_{\text{curv}}(\overline{R}^T\partial_{\eta_1}\overline{R},\overline{R}^T\partial_{\eta_2}\overline{R}) :&= W^{*}_{\text{curv}}(\overline{R}^T\partial_{\eta_1}\overline{R},\overline{R}^T\partial_{\eta_2}\overline{R},A^*) \\ &= \inf_{A\in\mathfrak{so}(3,\mathbb{R})} W^{*}_{\text{curv}}(\overline{R}^T\partial_{\eta_1}\overline{R},\overline{R}^T\partial_{\eta_2}\overline{R},A)\,, \end{aligned} \tag{6.59}$$

and set accordingly

$$\begin{aligned} W^{\text{hom}}_{\text{curv}}(\mathfrak{K}_s) &:= W^{*,\text{hom}}_{\text{curv}}(\overline{R}^T\partial_{\eta_1}\overline{R},\overline{R}^T\partial_{\eta_2}\overline{R})\,, \\ \mathfrak{K}_s &= \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0),\overline{R}^T(\nabla(\overline{R}.e_2)|0),\overline{R}^T(\nabla(\overline{R}.e_3)|0)\right)\,, \end{aligned} \tag{6.60}$$

in terms of the reduced curvature tensor $\mathfrak{K}_s\in\mathfrak{T}(3)$.

Similarly to (6.48) the infinitesimal rotation $A^*\in\mathfrak{so}(3,\mathbb{R})$, which realizes the infimum in (6.59), can be explicitely determined. For the moment we refrain from giving the final result. Suffice it to note that $W^{\text{hom}}_{\text{curv}}$ is uniquely defined, remains convex in its argument and has the same growth as W_{curv}. Then

$$\begin{aligned} W_{\text{curv}}(\overline{R}^{\sharp,T}_{h_j}\mathrm{D}^{\mathrm{h_j}}_{\eta}\overline{R}^{\sharp}_{h_j}) &= W^{*}_{\text{curv}}(\overline{R}^{\sharp,T}_{h_j}\partial_{\eta_1}\overline{R}^{\sharp}_{h_j},\overline{R}^{\sharp,T}_{h_j}\partial_{\eta_2}\overline{R}^{\sharp}_{h_j},\frac{1}{h_j}\overline{R}^{\sharp,T}_{h_j}\partial_{\eta_3}\overline{R}^{\sharp}_{h_j}) \\ &\geq W^{*,\text{hom}}_{\text{curv}}(\overline{R}^{\sharp,T}_{h_j}\partial_{\eta_1}\overline{R}^{\sharp}_{h_j},\overline{R}^{\sharp,T}_{h_j}\partial_{\eta_2}\overline{R}^{\sharp}_{h_j})\,. \end{aligned} \tag{6.61}$$

Integrating the last inequality, taking the lim inf on both sides and using that $W^{*,\text{hom}}_{\text{curv}}$ is convex in its argument, together with weak convergence of the two in-plane components of the curvature tensor, i.e.

$$(\overline{R}^{\sharp,T}_{h_j}\partial_{\eta_1}\overline{R}^{\sharp}_{h_j},\overline{R}^{\sharp,T}_{h_j}\partial_{\eta_2}\overline{R}^{\sharp}_{h_j},0)\rightharpoonup(\overline{R}^{\sharp,T}_{0}\partial_{\eta_1}\overline{R}^{\sharp}_{0},\overline{R}^{\sharp,T}_{0}\partial_{\eta_2}\overline{R}^{\sharp}_{0},0)\text{ in }L^{1+p+q}(\Omega_1,\mathfrak{T}(3))\,, \tag{6.62}$$

we obtain

$$\begin{aligned} \liminf_{h_j}\int_{\Omega_1} W_{\text{curv}}(\overline{R}^{\sharp,T}_{h_j}\mathrm{D}^{\mathrm{h_j}}_{\eta}\overline{R}^{\sharp}_{h_j})\,\mathrm{dV}_\eta &\geq \int_{\Omega_1} W^{*,\text{hom}}_{\text{curv}}(\overline{R}^{\sharp,T}_{h_j}\partial_{\eta_1}\overline{R}^{\sharp}_{h_j},\overline{R}^{\sharp,T}_{h_j}\partial_{\eta_2}\overline{R}^{\sharp}_{h_j})\,\mathrm{dV}_\eta \\ &= \int_{\Omega_1} W^{\text{hom}}_{\text{curv}}(\overline{R}^{\sharp,T}_{0}D\overline{R}^{\sharp}_{0})\,\mathrm{dV}_\eta\,. \end{aligned} \tag{6.63}$$

Then, because $W_{\text{curv}},W_{\text{mp}}\geq 0$,

$$\begin{aligned} &\liminf_{h_j}\int_{\Omega_1} W_{\text{mp}}(\overline{R}^{\sharp,T}_{h_j}\nabla^{h_j}_{\eta}\varphi^{\sharp}_{h_j})+W_{\text{curv}}(\overline{R}^{\sharp,T}_{h_j}\mathrm{D}^{\mathrm{h_j}}_{\eta}\overline{R}^{\sharp}_{h_j})\,\mathrm{dV}_\eta \\ &\geq \liminf_{h_j}\int_{\Omega_1} W_{\text{mp}}(\overline{R}^{\sharp,T}_{h_j}\nabla^{h_j}_{\eta}\varphi^{\sharp}_{h_j})\,\mathrm{dV}_\eta+\liminf_{h_j}\int_{\Omega_1} W_{\text{curv}}(\overline{R}^{\sharp,T}_{h_j}\mathrm{D}^{\mathrm{h_j}}_{\eta}\overline{R}^{\sharp}_{h_j})\,\mathrm{dV}_\eta \\ &\geq \int_{\Omega_1} W^{\text{hom}}_{\text{mp}}(\nabla_{(\eta_1,\eta_2)}\varphi^{\sharp}_0,\overline{R}^{\sharp}_0)\,\mathrm{dV}_\eta+\int_{\Omega_1} W^{\text{hom}}_{\text{curv}}(\overline{R}^{\sharp,T}_{0}D\overline{R}^{\sharp}_{0})\,\mathrm{dV}_\eta \\ &= \int_{\Omega_1} W^{\text{hom}}_{\text{mp}}(\nabla_{(\eta_1,\eta_2)}\varphi^{\sharp}_0,\overline{R}^{\sharp}_0)+W^{\text{hom}}_{\text{curv}}(\overline{R}^{\sharp,T}_{0}D\overline{R}^{\sharp}_{0})\,\mathrm{dV}_\eta\,, \end{aligned} \tag{6.64}$$

where we used (6.57) and (6.63). Now we use that $\varphi^\sharp_0$ is independent of the transverse variable η_3, which allows us to insert the averaging operator without any change to see that

$$\int_{\Omega_1} W^{\text{hom}}_{\text{mp}}(\nabla_{(\eta_1,\eta_2)}\varphi^{\sharp}_0,\overline{R}^{\sharp}_0)=\int_{\Omega_1} W^{\text{hom}}_{\text{mp}}(\nabla_{(\eta_1,\eta_2)}\operatorname{Av}.\varphi^{\sharp}_0,\overline{R}^{\sharp}_0)\,\mathrm{dV}_\eta=\int_{\omega} W^{\text{hom}}_{\text{mp}}(\nabla_{(\eta_1,\eta_2)}\operatorname{Av}.\varphi^{\sharp}_0,\overline{R}^{\sharp}_0)\,\mathrm{d}\omega\,, \tag{6.65}$$

since $\overline{R}_0^\sharp$ is also independent of the transverse variable. Hence we obtain altogether the desired lim inf-inequality

$$I_0^\sharp(\varphi_0^\sharp, \overline{R}_0^\sharp) \le \liminf_{h_j} I_{h_j}^\sharp(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp) \tag{6.66}$$

for

$$\begin{aligned} I_0^\sharp(\varphi_0, \overline{R}_0) &:= \int_{\Omega_1} W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla_{(\eta_1,\eta_2)} \operatorname{Av}.\varphi_0, \overline{R}_0) + W_{\mathrm{curv}}^{\mathrm{hom}}(\overline{R}_0^T D\overline{R}_0)\, \mathrm{dV}_\eta \\ &= \int_\omega W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla_{(\eta_1,\eta_2)} \operatorname{Av}.\varphi_0, \overline{R}_0) + W_{\mathrm{curv}}^{\mathrm{hom}}(\overline{R}_0^T D\overline{R}_0)\, \mathrm{d}\omega\,. \end{aligned}$$

■

6.3 Upper bound-the recovery sequence

Now we show that the lower bound will actually be reached. A sufficient requirement for the recovery sequence is that

$$\begin{aligned} &\forall\, (\varphi_0, \overline{R}_0) \in X = L^r(\Omega_1, \mathbb{R}^3) \times L^{1+p+q}(\Omega_1, \mathrm{SO}(3,\mathbb{R})) \\ &\quad \exists\, \varphi_{h_j}^\sharp \to \varphi_0 \quad \text{in } L^r(\Omega_1, \mathbb{R}^3)\,, \quad \overline{R}_{h_j}^\sharp \to \overline{R}_0 \quad \text{in } L^{1+p+q}(\Omega_1, \mathrm{SO}(3,\mathbb{R}))\,: \\ &\qquad \limsup I_{h_j}^\sharp(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp) \le I_0^\sharp(\varphi_0, \overline{R}_0)\,. \end{aligned} \tag{6.67}$$

Observe that this is now only a condition if $I_0^\sharp(\varphi_0, \overline{R}_0) < \infty$. In this case the uniform coercivity of $I_{h_j}^\sharp(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp)$ over $X' = H^{1,2}(\Omega_1, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega_1, \mathrm{SO}(3,\mathbb{R}))$ implies that we can restrict attention to sequences $(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp)$ converging weakly to some $(\varphi_0, \overline{R}_0) \in H^{1,2}(\omega, \mathbb{R}^3) \times W^{1,1+p+q}(\omega, \mathrm{SO}(3,\mathbb{R})) = X'_\omega$, defined over the two-dimensional domain ω only. Note, however, that finally it is strong convergence in X which matters.

The natural candidate for the recovery sequence for the bulk deformation is given by the "reconstruction"

$$\varphi_{h_j}^\sharp(\eta_1, \eta_2, \eta_3) := m(\eta_1, \eta_2) + h_j\, \eta_3\, b^*(\eta_1, \eta_2) = \varphi_0(\eta_1, \eta_2) + h_j\, \eta_3\, b^*(\eta_1, \eta_2)\,, \tag{6.68}$$

where, with the abbreviation $m = \varphi_0 = \operatorname{Av}.\varphi_0$ at places,

$$\begin{aligned} b^*(\eta_1, \eta_2) &:= \frac{\mu_c - \mu}{\mu + \mu_c} \langle \overline{R}_{0,3}, m_x \rangle\, \overline{R}_{0,1} + \frac{\mu_c - \mu}{\mu + \mu_c} \langle \overline{R}_{0,3}, m_y \rangle\, \overline{R}_{0,2} + \varrho_m^*\, \overline{R}_{0,3}\,, \\ \varrho_m^* &= 1 - \frac{\lambda}{2\mu + \lambda} \left[\langle (\nabla m | 0), \overline{R}_0 \rangle - 2 \right]\,. \end{aligned} \tag{6.69}$$

Observe that $b^* \in L^2(\omega, \mathbb{R}^3)$. Convergence of $\varphi_{h_j}^\sharp$ in $L^r(\Omega_1, \mathbb{R}^3)$ to the limit φ_0 as $h_j \to 0$ is obvious.

The reconstruction for the rotation $\overline{R}_0$ is, however, not obvious since on the one hand we have to maintain the rotation constraint along the sequence and on the other hand we must approach the lower bound, which excludes the simple reconstruction $\overline{R}_{h_j}^\sharp(\eta_1, \eta_2, \eta_3) = \overline{R}_0(\eta_1, \eta_2)$. In order to meet both requirements we consider therefore

$$\overline{R}_{h_j}^\sharp(\eta_1, \eta_2, \eta_3) := \overline{R}_0(\eta_1, \eta_2) \cdot \exp\left(h_j\, \eta_3\, A^*(\eta_1, \eta_2) \right)\,, \tag{6.70}$$

where $A^* \in \mathfrak{so}(3,\mathbb{R})$ is the term obtained in (6.59), depending on the given $\overline{R}_0$ and we note that $A^* \in L^{1+p+q}(\omega, \mathfrak{so}(3,\mathbb{R}))$ by the coercivity of W_{curv}^*. It is clear that $\overline{R}_{h_j}^\sharp \in \mathrm{SO}(3,\mathbb{R})$, since $\exp : \mathfrak{so}(3,\mathbb{R}) \mapsto \mathrm{SO}(3,\mathbb{R})$ and we have the convergence $\overline{R}_{h_j}^\sharp \to \overline{R}_0$ in $L^{1+p+q}(\Omega_1, \mathrm{SO}(3,\mathbb{R}))$ for $h_j \to 0$.

Since neither b^* nor A^* need be differentiable, we have to consider slightly modified recovery sequences, however. With fixed $\varepsilon > 0$ choose $b_\varepsilon \in W^{1,2}(\omega,\mathbb{R}^3)$ such that $\|b_\varepsilon - b^*\|_{L^2(\omega,\mathbb{R}^3)} < \varepsilon$ and similarly for A^* choose $A_\varepsilon \in W^{1,1+p+q}(\omega,\mathfrak{so}(3,\mathbb{R}))$ such that $\|A_\varepsilon - A^*\|_{L^{1+p+q}(\omega,\mathfrak{so}(3,\mathbb{R}))} < \varepsilon$. This allows us to present finally our **recovery sequence**

$$\begin{aligned}
\varphi^\sharp_{h_j,\varepsilon}(\eta_1,\eta_2,\eta_3) &:= \varphi_0(\eta_1,\eta_2) + h_j\,\eta_3\, b_\varepsilon(\eta_1,\eta_2)\,,\\
\overline{R}^\sharp_{h_j,\varepsilon}(\eta_1,\eta_2,\eta_3) &:= \overline{R}_0(\eta_1,\eta_2)\cdot \exp\left(h_j\,\eta_3\,A_\varepsilon(\eta_1,\eta_2)\right)\,.
\end{aligned} \tag{6.71}$$

This definition implies

$$\begin{aligned}
\nabla\varphi^\sharp_{h_j,\varepsilon}(\eta_1,\eta_2,\eta_3) &= (\nabla\varphi_0(\eta_1,\eta_2)|h_j\,b_\varepsilon(\eta_1,\eta_2)) + h_j\,\eta_3\,(\nabla b_\varepsilon(\eta_1,\eta_2)|0)\,,\\
\overline{R}^{\sharp,T}_{h_j,\varepsilon}\partial_{\eta_1}\overline{R}^\sharp_{h_j,\varepsilon} &= \exp\left(h_j\,\eta_3\,A_\varepsilon\right)^T\overline{R}_0^T\left[\partial_{\eta_1}\overline{R}_0\,\exp\left(h_j\,\eta_3\,A_\varepsilon\right) + \overline{R}_0\,D\exp\left(h_j\,\eta_3\,A_\varepsilon\right).[h_j\,\eta_3\partial_{\eta_1}A_\varepsilon]\right]\,,\\
\overline{R}^{\sharp,T}_{h_j,\varepsilon}\partial_{\eta_2}\overline{R}^\sharp_{h_j,\varepsilon} &= \exp\left(h_j\,\eta_3\,A_\varepsilon\right)^T\overline{R}_0^T\left[\partial_{\eta_2}\overline{R}_0\,\exp\left(h_j\,\eta_3\,A_\varepsilon\right) + \overline{R}_0\,D\exp\left(h_j\,\eta_3\,A_\varepsilon\right).[h_j\,\eta_3\partial_{\eta_2}A_\varepsilon]\right]\,,\\
\overline{R}^{\sharp,T}_{h_j,\varepsilon}\partial_{\eta_3}\overline{R}^\sharp_{h_j,\varepsilon} &= \exp\left(h_j\,\eta_3\,A_\varepsilon\right)^T\overline{R}_0^T\left[\partial_{\eta_3}\overline{R}_0\,\exp\left(h_j\,\eta_3\,A_\varepsilon\right) + \overline{R}_0\,D\exp\left(h_j\,\eta_3\,A_\varepsilon\right).[h_j\,A_\varepsilon]\right]\\
&= h_j\,\exp\left(h_j\,\eta_3\,A_\varepsilon\right)^T D\exp\left(h_j\,\eta_3\,A_\varepsilon\right).[A_\varepsilon]\,,
\end{aligned} \tag{6.72}$$

with $\partial_{\eta_i}A_\varepsilon \in \mathfrak{so}(3,\mathbb{R})$. In view of the prominent appearance of the exponential in these expressions it is useful to briefly recapitulate the basic features of the matrix exponential exp acting on $\mathfrak{so}(3,\mathbb{R})$. We note

$$\begin{aligned}
&\exp:\ \mathfrak{so}(3,\mathbb{R}) \mapsto \mathrm{SO}(3,\mathbb{R}) \quad \text{is infinitely differentiable}\,,\\
&\forall\, A \in \mathfrak{so}(3,\mathbb{R}):\quad \|\exp(A)\| = \sqrt{3}\ \Rightarrow\\
&\exp: L^{1+p+q}(\Omega_1,\mathfrak{so}(3,\mathbb{R})) \mapsto L^{1+p+q}(\Omega_1,\mathrm{SO}(3,\mathbb{R})) \quad \text{is continuous}\,,\\
&D\exp: \mathfrak{so}(3,\mathbb{R}) \mapsto \mathrm{Lin}(\mathfrak{so}(3,\mathbb{R}),\mathbb{M}^{3\times 3}) \quad \text{is locally continuous}\,,\\
&\forall\, H \in \mathfrak{so}(3,\mathbb{R}):\quad D\exp(0).H = H\,,\\
&\forall\, A, H \in \mathfrak{so}(3,\mathbb{R}): \exp(A)^T\cdot D\exp(A).H \in \mathfrak{so}(3,\mathbb{R})\,.
\end{aligned} \tag{6.73}$$

Note that by appropriately choosing $h_j, \varepsilon > 0$ we can arrange that strong convergence of (6.72) to the correct limit still obtains by using $(6.73)_3$. Now abbreviate

$$\begin{aligned}
\tilde{\overline{U}} &:= \overline{R}_0^T(\nabla\varphi_0(\eta_1,\eta_2)|b^*) \in \mathbb{M}^{3\times 3}\,,\\
\tilde{\overline{V}}^\varepsilon_{h_j} &:= \overline{R}^{\sharp,T}_{h_j,\varepsilon}\left[(\nabla\varphi_0(\eta_1,\eta_2)|b_\varepsilon(\eta_1,\eta_2)) + h_j\,\eta_3\,(\nabla b_\varepsilon(\eta_1,\eta_2)|0)\right] \in \mathbb{M}^{3\times 3}\,,\\
\tilde{\overline{V}}^\varepsilon_0 &:= \overline{R}_0^T(\nabla\varphi_0(\eta_1,\eta_2)|b_\varepsilon(\eta_1,\eta_2)) \in \mathbb{M}^{3\times 3}\,,\\
\mathfrak{k}^{\sharp,i}_{h_j,\varepsilon} &:= \overline{R}^{\sharp,T}_{h_j,\varepsilon}\partial_{\eta_i}\overline{R}^\sharp_{h_j,\varepsilon} \in \mathfrak{so}(3,\mathbb{R}),\qquad i = 1,2,3\,,\\
\mathfrak{k}^i_0 &:= \overline{R}_0^T\partial_{\eta_i}\overline{R}_0 \in \mathfrak{so}(3,\mathbb{R}),\qquad i = 1,2\,,\\
\tilde{A}_{h_j,\varepsilon} &:= \exp\left(h_j\,\eta_3\,A_\varepsilon(\eta_1,\eta_2)\right)^T D\exp\left(h_j\,\eta_3\,A_\varepsilon(\eta_1,\eta_2)\right).[A_\varepsilon] \in \mathfrak{so}(3,\mathbb{R})\,,\\
\mathfrak{K}^\sharp_{h_j,\varepsilon} &:= \overline{R}^{\sharp,T}_{h_j,\varepsilon}\mathrm{D}^{\mathrm{h_j}}_\eta\overline{R}^\sharp_{h_j,\varepsilon}(\eta_1,\eta_2,\eta_3) \in \mathfrak{T}(3)\\
\mathfrak{K}_0(\eta_1,\eta_2) &= \overline{R}_0^T D\overline{R}_0(\eta_1,\eta_2) \in \mathfrak{T}(3)\,.
\end{aligned} \tag{6.74}$$

We note that by the smoothness of $A_\varepsilon \in W^{1,1+p+q}(\omega_1,\mathfrak{so}(3,\mathbb{R}))$

$$\begin{aligned}
&\|\tilde{A}_{h_j,\varepsilon} - A_\varepsilon\|_{L^{1+p+q}(\Omega_1,\mathfrak{so}(3,\mathbb{R}))} \to 0 \quad \text{if} \quad h_j \to 0\,,\\
&\|\mathfrak{k}^{\sharp,i}_{h_j,\varepsilon} - \mathfrak{k}^i_0\|_{L^{1+p+q}(\Omega_1,\mathfrak{so}(3,\mathbb{R}))} \to 0 \quad \text{if} \quad h_j \to 0\,,\\
&\|\tilde{\overline{V}}^\varepsilon_{h_j} - \tilde{\overline{V}}^\varepsilon_0\|_{L^2(\Omega_1,\mathbb{M}^{3\times 3})} \to 0 \quad \text{if} \quad h_j \to 0\,,\\
&\|\tilde{\overline{V}}^\varepsilon_{h_j} - \tilde{\overline{U}}\|_{L^2(\Omega_1,\mathbb{M}^{3\times 3})} \to 0 \quad \text{if} \quad h_j,\varepsilon \to 0\,.
\end{aligned} \tag{6.75}$$

The abbreviations in (6.74) imply

$$I^\sharp_{h_j}(\varphi^\sharp_{h_j,\varepsilon}, \overline{R}^\sharp_{h_j,\varepsilon}) = \int_{\Omega_1} W_{\rm mp}(\tilde{\overline{V}}^\varepsilon_{h_j}) + W^*_{\rm curv}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \frac{1}{h_j}\overline{R}^{\sharp,T}_{h_j,\varepsilon}\partial_{\eta_3}\overline{R}^\sharp_{h_j,\varepsilon})\,{\rm dV}_\eta$$

$$= \int_{\Omega_1} W_{\rm mp}(\tilde{\overline{V}}^\varepsilon_{h_j}) + W^*_{\rm curv}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon})\,{\rm dV}_\eta\,, \tag{6.76}$$

where we used that $h_j \cdot b_\varepsilon$ in the definition of the recovery deformation gradient $(6.72)_1$ is cancelled by the factor $\frac{1}{h_j}$ in the definition of $I^\sharp_{h_j}$. Whence, adding and subtracting $W_{\rm mp}(\tilde{\overline{U}})$

$$I^\sharp_{h_j}(\varphi^\sharp_{h_j,\varepsilon}, \overline{R}^\sharp_{h_j,\varepsilon}) = \int_{\Omega_1} W_{\rm mp}(\tilde{\overline{U}}) + W_{\rm mp}(\tilde{\overline{V}}^\varepsilon_{h_j}) - W_{\rm mp}(\tilde{\overline{U}}) + W^*_{\rm curv}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon})\,{\rm dV}_\eta$$

$$= \int_{\Omega_1} W_{\rm mp}(\tilde{\overline{U}}) + W_{\rm mp}(\tilde{\overline{U}} + \tilde{\overline{V}}^\varepsilon_{h_j} - \tilde{\overline{U}}) - W_{\rm mp}(\tilde{\overline{U}}) + W_{\rm curv}(\mathfrak{K}_{h_j})\,{\rm dV}_\eta$$

since $W_{\rm mp}$ and $W_{\rm curv}$ are both positive, we get from the triangle inequality

$$\leq \int_{\Omega_1} W_{\rm mp}(\tilde{\overline{U}}) + |W_{\rm mp}(\tilde{\overline{U}} + \tilde{\overline{V}}^\varepsilon_{h_j} - \tilde{\overline{U}}) - W_{\rm mp}(\tilde{\overline{U}})| + W^*_{\rm curv}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon})\,{\rm dV}_\eta$$

expanding the quadratic energy $W_{\rm mp}$ we obtain

$$= \int_{\Omega_1} W_{\rm mp}(\tilde{\overline{U}}) + |W_{\rm mp}(\tilde{\overline{U}}) + \langle DW_{\rm mp}(\tilde{\overline{U}}), \tilde{\overline{V}}^\varepsilon_{h_j} - \tilde{\overline{U}}\rangle$$

$$+ D^2W_{\rm mp}(\tilde{\overline{U}}).(\tilde{\overline{V}}^\varepsilon_{h_j} - \tilde{\overline{U}}, \tilde{\overline{V}}^\varepsilon_{h_j} - \tilde{\overline{U}})| + W^*_{\rm curv}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon})\,{\rm dV}_\eta \tag{6.77}$$

$$\leq \int_{\Omega_1} W_{\rm mp}(\tilde{\overline{U}}) + \|DW_{\rm mp}(\tilde{\overline{U}})\|\,\|\tilde{\overline{V}}^\varepsilon_{h_j} - \tilde{\overline{U}}\| + C\,\|\tilde{\overline{V}}^\varepsilon_{h_j} - \tilde{\overline{U}}\|^2 + W^*_{\rm curv}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon})\,{\rm dV}_\eta$$

for $\|\tilde{\overline{V}}^\varepsilon_{h_j} - \tilde{\overline{U}}\| \leq 1$ we have

$$\leq \int_{\Omega_1} W_{\rm mp}(\tilde{\overline{U}}) + \Big(C + \|DW_{\rm mp}(\tilde{\overline{U}})\|\Big)\,\|\tilde{\overline{V}}^\varepsilon_{h_j} - \tilde{\overline{U}}\| + W^*_{\rm curv}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon})\,{\rm dV}_\eta$$

since $\|DW_{\rm mp}(\tilde{\overline{U}})\| \leq C_2\,\|\tilde{\overline{U}}\|$ we obtain

$$\leq \int_{\Omega_1} W_{\rm mp}(\tilde{\overline{U}}) + \Big(C + \|\tilde{\overline{U}}\|\Big)\,\|\tilde{\overline{V}}^\varepsilon_{h_j} - \tilde{\overline{U}}\| + W^*_{\rm curv}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon})\,{\rm dV}_\eta$$

and by Hölder's inequality we get

$$\leq \int_{\Omega_1} W_{\rm mp}(\tilde{\overline{U}}) + W^*_{\rm curv}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon})\,{\rm dV}_\eta + \Big(C + \|\tilde{\overline{U}}\|_{L^2(\Omega_1)}\Big)\,\|\tilde{\overline{V}}^\varepsilon_{h_j} - \tilde{\overline{U}}\|_{L^2(\Omega_1)}\,.$$

Continuing the estimate with regard to $W^*_{\rm curv}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon})$ and adding and subtracting $\tilde{\overline{V}}^\varepsilon_0$ we may obtain

$$I^\sharp_{h_j}(\varphi^\sharp_{h_j,\varepsilon}, \overline{R}^\sharp_{h_j,\varepsilon}) \leq \int_{\Omega_1} W_{\rm mp}(\tilde{\overline{U}}) + W^*_{\rm curv}(\mathfrak{k}^1_0, \mathfrak{k}^2_0, A^*) + W^*_{\rm curv}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon})$$

$$- W^*_{\rm curv}(\mathfrak{k}^1_0, \mathfrak{k}^2_0, A^*)\,{\rm dV}_\eta$$

$$+ \Big(C + \|\tilde{\overline{U}}\|_{L^2(\Omega_1)}\Big)\,\|\tilde{\overline{V}}^\varepsilon_{h_j} - \tilde{\overline{V}}^\varepsilon_0 + \tilde{\overline{V}}^\varepsilon_0 - \tilde{\overline{U}}\|_{L^2(\Omega_1)}$$

$$\leq \int_{\Omega_1} W_{\rm mp}(\tilde{\overline{U}}) + W^*_{\rm curv}(\mathfrak{k}^1_0, \mathfrak{k}^2_0, A^*)\,{\rm dV}_\eta$$

$$
\begin{aligned}
&+\|W^*_{\text{curv}}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon},\mathfrak{k}^{\sharp,2}_{h_j,\varepsilon},\tilde{A}_{h_j,\varepsilon})-W^*_{\text{curv}}(\mathfrak{k}^1_0,\mathfrak{k}^2_0,A_\varepsilon)\|_{L^1(\Omega_1)} \\
&+\|W^*_{\text{curv}}(\mathfrak{k}^1_0,\mathfrak{k}^2_0,A_\varepsilon)-W^*_{\text{curv}}(\mathfrak{k}^1_0,\mathfrak{k}^2_0,A^*)\|_{L^1(\Omega_1)} \\
&\quad+\left(C+\|\tilde{\overline{U}}\|_{L^2(\Omega_1)}\right)\left(\|\tilde{\overline{V}}^\varepsilon_{h_j}-\tilde{\overline{V}}^\varepsilon_0\|_{L^2(\Omega_1)}+\|\tilde{\overline{V}}^\varepsilon_0-\tilde{\overline{U}}\|_{L^2(\Omega_1)}\right).
\end{aligned} \tag{6.78}
$$

Now take $h_j \to 0$ to obtain by the continuity of W^*_{curv} in its first two arguments and $(6.75)_3$

$$
\begin{aligned}
\limsup_{h_j\to 0} I^\sharp_{h_j}(\varphi^\sharp_{h_j,\varepsilon},\overline{R}^\sharp_{h_j,\varepsilon}) \le &\int_{\Omega_1} W_{\text{mp}}(\tilde{\overline{U}})+W^*_{\text{curv}}(\mathfrak{k}^1_0,\mathfrak{k}^2_0,A^*)\,\mathrm{dV}_\eta \\
&+\|W^*_{\text{curv}}(\mathfrak{k}^1_0,\mathfrak{k}^2_0,A_\varepsilon)-W^*_{\text{curv}}(\mathfrak{k}^1_0,\mathfrak{k}^2_0,A^*)\|_{L^1(\Omega_1)} \\
&\quad+\left(C+\|\tilde{\overline{U}}\|_{L^2(\Omega_1)}\right)\|\tilde{\overline{V}}^\varepsilon_0-\tilde{\overline{U}}\|_{L^2(\Omega_1)}.
\end{aligned} \tag{6.79}
$$

Since

$$
\begin{aligned}
\|\tilde{\overline{V}}^\varepsilon_0-\tilde{\overline{U}}\|^2 &= \|\overline{R}^T_0(\nabla\varphi_0(\eta_1,\eta_2)|b_\varepsilon)-\overline{R}^T_0(\nabla\varphi_0(\eta_1,\eta_2)|b^*)\|^2 \\
&= \|\overline{R}^T_0((\nabla\varphi_0(\eta_1,\eta_2)|b_\varepsilon)-(\nabla\varphi_0(\eta_1,\eta_2)|b^*))\|^2 \\
&= \|(\nabla\varphi_0(\eta_1,\eta_2)|b_\varepsilon)-(\nabla\varphi_0(\eta_1,\eta_2)|b^*)\|^2=\|b_\varepsilon-b^*\|^2,
\end{aligned} \tag{6.80}
$$

we get, by letting $\varepsilon \to 0$ and using now the continuity of W^*_{curv} in its last argument together with $\|A_\varepsilon - A^*\|_{L^{1+p+q}(\omega,\mathfrak{so}(3,\mathbb{R}))} < \varepsilon$, the bound

$$
\begin{aligned}
\limsup_{h_j\to 0} I^\sharp_{h_j}(\varphi^\sharp_{h_j,\varepsilon},\overline{R}^\sharp_{h_j,\varepsilon}) &\le \int_{\Omega_1} W_{\text{mp}}(\tilde{\overline{U}})+W^*_{\text{curv}}(\mathfrak{k}^1_0,\mathfrak{k}^2_0,A^*)\,\mathrm{dV}_\eta = \int_{\Omega_1} W_{\text{mp}}(\tilde{\overline{U}})+W^{*,\text{hom}}_{\text{curv}}(\mathfrak{k}^1_0,\mathfrak{k}^2_0)\,\mathrm{dV}_\eta \\
&= \int_{\Omega_1} W^{\text{hom}}_{\text{mp}}(\nabla\varphi_0,\overline{R}_0)+W^{\text{hom}}_{\text{curv}}(\mathfrak{K}_0)\,\mathrm{dV}_\eta.
\end{aligned} \tag{6.81}
$$

Since $\varphi_0, \overline{R}_0$ are two-dimensional (independent of the transverse variable), we may write as well

$$
\begin{aligned}
\limsup_{h_j\to 0} I^\sharp_{h_j}(\varphi^\sharp_{h_j,\varepsilon},\overline{R}^\sharp_{h_j,\varepsilon}) &\le \int_{\Omega_1} W^{\text{hom}}_{\text{mp}}(\nabla_{(\eta_1,\eta_2)}\operatorname{Av}.\varphi_0,\overline{R}_0)+W^{\text{hom}}_{\text{curv}}(\mathfrak{K}_0)\,\mathrm{dV}_\eta \\
&= \int_{\omega} W^{\text{hom}}_{\text{mp}}(\nabla_{(\eta_1,\eta_2)}\operatorname{Av}.\varphi_0,\overline{R}_0)+W^{\text{hom}}_{\text{curv}}(\mathfrak{K}_0)\,\mathrm{d}\omega = I^\sharp_0(\varphi_0,\overline{R}_0),
\end{aligned} \tag{6.82}
$$

which shows the desired upper bound. Note that the appearance of the averaging operator Av is not strictly necessary since the limit problem for $\mu_c > 0$ is independent of the transverse variable anyhow. ■

7 Proof of Γ-convergence for zero Cosserat couple modulus $\mu_c = 0$ without equi-coercivity

In this part we show that the the formal limit of $\mu_c \to 0$ of the Γ-limit for $\mu_c > 0$ is in fact the Γ-limit for $\mu_c = 0$. First, we investigate a lower bound of the rescaled three-dimensional formulation for the limit case $\mu_c = 0$. We consider a family of functionals $I^{\sharp,\text{mem}}_h : X' \mapsto \overline{\mathbb{R}}$, where all transverse shear terms have been omitted, more precisely

$$
\begin{aligned}
I_h^{\sharp,\mathrm{mem}}(\varphi^\sharp,\nabla\varphi^\sharp,\overline{R}^{3d,\sharp},\mathrm{D}_\eta^{\mathrm{h}}\overline{R}^{3d,\sharp}) &= \int_{\eta\in\Omega_1} W_{\mathrm{mp}}(\overline{U}_h^{\sharp,\mathrm{mem}}) + W_{\mathrm{curv}}(\mathfrak{K}_h^\sharp)\,\mathrm{dV}_\eta \mapsto \min.\ \text{w.r.t. } (\varphi^\sharp,\overline{R}^\sharp)\,,\\
\overline{U}_h^\sharp &= \overline{R}^{3d,\sharp,T}F_h^\sharp, \quad \varphi^\sharp_{|_{\Gamma_0^1}}(\eta) = g_{\mathrm{d}}^\sharp(\eta) = g_{\mathrm{d}}(\zeta(\eta)) = g_{\mathrm{d}}(\eta_1,\eta_2,h\cdot\eta_3) = g_{\mathrm{d}}(\eta_1,\eta_2,0)\,,\\
\overline{U}_h^{\sharp,\mathrm{mem}} &= \begin{pmatrix} \overline{U}^\sharp_{h,11} & \overline{U}^\sharp_{h,12} & 0\\ \overline{U}^\sharp_{h,21} & \overline{U}^\sharp_{h,22} & 0\\ 0 & 0 & \overline{U}^\sharp_{h,33}\end{pmatrix} = \begin{pmatrix} \langle \overline{R}_1^{3d,\sharp},\partial_{\eta_1}\varphi^\sharp\rangle & \langle \overline{R}_1^{3d,\sharp},\partial_{\eta_2}\varphi^\sharp\rangle & 0\\ \langle \overline{R}_2^{3d,\sharp},\partial_{\eta_1}\varphi^\sharp\rangle & \langle \overline{R}_2^{3d,\sharp},\partial_{\eta_2}\varphi^\sharp\rangle & 0\\ 0 & 0 & \frac{1}{h}\langle \overline{R}_3^{3d,\sharp},\partial_{\eta_3}\varphi^\sharp\rangle\end{pmatrix},\\
\Gamma_0^1 &= \gamma_0\times[-\frac{1}{2},\frac{1}{2}], \quad \gamma_0\subset\partial\omega,\\
\overline{R}^{3d,\sharp}: &\quad \text{free on } \Gamma_0^1, \text{ Neumann-type boundary condition}\,, \qquad (7.83)\\
W_{\mathrm{mp}}(\overline{U}_h^{\sharp,\mathrm{mem}}) &= \mu\,\|\operatorname{sym}(\overline{U}_h^{\sharp,\mathrm{mem}}-1\!\!1)\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}(\overline{U}_h^{\sharp,\mathrm{mem}}-1\!\!1)\right]^2 =: W_{\mathrm{mp}}^{\mathrm{mem}}(\nabla\varphi_h^\sharp,\overline{R}^{3d,\sharp})\,,\\
W_{\mathrm{curv}}(\mathfrak{K}_h^\sharp) &= \mu\frac{L_c^{1+p}}{12}\left(1+\alpha_4 L_c^q\,\|\mathfrak{K}_h^\sharp\|^q\right)\left(\alpha_5\,\|\operatorname{sym}\mathfrak{K}_h^\sharp\|^2+\alpha_6\,\|\operatorname{skew}\mathfrak{K}_h^\sharp\|^2+\alpha_7\operatorname{tr}\left[\mathfrak{K}_h^\sharp\right]^2\right)^{\frac{1+p}{2}},\\
\mathfrak{K}_h^\sharp &= \overline{R}^{3d,\sharp,T}\,\mathrm{D}_\eta^{\mathrm{h}}\overline{R}^{3d,\sharp}(\eta)\,.
\end{aligned}
$$

Accordingly, we define the admissible set

$$
\begin{aligned}
\mathcal{A}_h^{\mathrm{mem}} := \{(\varphi,\overline{R})\in X \mid \operatorname{sym}\overline{U}_h^{\sharp,\mathrm{mem}} &\in L^2(\Omega_1,\mathbb{M}^{3\times3}),\ \overline{R}\in W^{1,1+p+q}(\Omega_1,\mathrm{SO}(3,\mathbb{R}))\,,\\
\varphi^\sharp_{|_{\Gamma_0^1}}(\eta) &= g_{\mathrm{d}}^\sharp(\eta) = g_{\mathrm{d}}(\eta_1,\eta_2,0)\,\}\,. \qquad (7.84)
\end{aligned}
$$

As in (5.20) we extend the rescaled energies to the larger space X through redefining

$$
I_h^{\sharp,\mathrm{mem}}(\varphi^\sharp,\nabla\varphi^\sharp,\overline{R}^\sharp,\mathrm{D}_\eta^{\mathrm{h}}\overline{R}^\sharp) = \begin{cases} I_h^{\sharp,\mathrm{mem}}(\varphi^\sharp,\nabla\varphi^\sharp,\overline{R}^\sharp,\mathrm{D}_\eta^{\mathrm{h}}\overline{R}^\sharp) & \text{if } (\varphi^\sharp,\overline{R}^\sharp)\in\mathcal{A}_h^{\mathrm{mem}}\\ +\infty & \text{else in } X\,. \end{cases} \qquad (7.85)
$$

Observe that

$$
I^\sharp_{h|_{\mu_c=0}}(\varphi^\sharp,\nabla\varphi^\sharp,\overline{R}^{3d,\sharp},\mathrm{D}_\eta^{\mathrm{h}}\overline{R}^{3d,\sharp}) \geq I_h^{\sharp,\mathrm{mem}}(\varphi^\sharp,\nabla\varphi^\sharp,\overline{R}^{3d,\sharp},\mathrm{D}_\eta^{\mathrm{h}}\overline{R}^{3d,\sharp})\,, \qquad (7.86)
$$

which implies [42, Prop. 6.7] that

$$
\Gamma-\liminf_h I^\sharp_{h|_{\mu_c=0}} \geq \Gamma-\liminf_h I_h^{\sharp,\mathrm{mem}}\,. \qquad (7.87)
$$

Hence $\Gamma-\liminf I_h^{\sharp,\mathrm{mem}}$ provides a lower bound for $\Gamma-\liminf I^\sharp_{h|_{\mu_c=0}}$. Putting inequalities (5.33) and (7.87) together, we obtain the natural chain of inequalities on X

$$
\begin{aligned}
\Gamma-\liminf I_h^{\sharp,\mathrm{mem}} &\leq \Gamma-\liminf I^\sharp_{h|_{\mu_c=0}}\\
&\leq \Gamma-\limsup I^\sharp_{h|_{\mu_c=0}} \leq \lim_{\mu_c\to0}\left(\Gamma-\lim I^\sharp_{h|_{\mu_c>0}}\right) =: I_0^{\sharp,0}\,. \qquad (7.88)
\end{aligned}
$$

7.1 Conjecture on the corresponding form of $\Gamma-\liminf$

We conjecture that the $\Gamma-\liminf$ for (7.83) is given by the following energy functional $I_0^{\sharp,\mathrm{mem}}: X\mapsto\overline{\mathbb{R}}$,

$$
I_0^{\sharp,\mathrm{mem}}(\varphi,\overline{R}) := \begin{cases} \int_\omega W_{\mathrm{mp}}^{\mathrm{hom},0}(\nabla_{(\eta_1,\eta_2)}\operatorname{Av}.\varphi(\eta_1,\eta_2,\eta_3),\overline{R}) + W_{\mathrm{curv}}^{\mathrm{hom}}(\mathfrak{K}_s)\,\mathrm{d}\omega & (\varphi,\overline{R})\in\mathcal{A}_0^{\mathrm{mem}}\\ +\infty & \text{else in } X\,, \end{cases} \qquad (7.89)
$$

where $W_{\rm mp}^{\rm hom,0}$ is defined in (5.35) and the admissible set is now

$$\mathcal{A}_0^{\rm mem} := \{(\varphi,\overline{R}) \in X \mid \mathrm{sym}\,(\overline{R_1}|\overline{R}_2)^T\nabla_{(\eta_1,\eta_2)}\,\mathrm{Av}\,.\varphi \in L^2(\Omega_1,\mathbb{M}^{2\times 2}),\ \overline{R}\in W^{1,1+p+q}(\omega,\mathrm{SO}(3,\mathbb{R}))\,,$$
$$\varphi^\sharp_{|_{\Gamma_0^1}}(\eta) = g_{\rm d}^\sharp(\eta) = g_{\rm d}(\eta_1,\eta_2,0)\,\}\,. \tag{7.90}$$

Note that this $\Gamma-\liminf$ conjecture makes no statement about the behaviour through the thickness of the in-plane components of the deformation: the limit problem $I_0^{\sharp,\rm mem}$ as such would not be entirely two-dimensional.

7.2 A lower bound for the "membrane" lower bound

We show presently that

$$I_0^{\sharp,\rm mem}(\varphi_0,\overline{R}_0) \le \liminf_{h_j} I_{h_j}^{\sharp,\rm mem}(\varphi^\sharp_{h_j},\overline{R}^\sharp_{h_j})\,, \tag{7.91}$$

whenever

$$\varphi^\sharp_{h_j}\to\varphi^\sharp_0 \quad \text{in } L^r(\Omega_1,\mathbb{R}^3)\,,\quad \overline{R}^\sharp_{h_j}\to\overline{R}^\sharp_0 \quad \text{in } L^{1+p+q}(\Omega_1,\mathrm{SO}(3,\mathbb{R}))\,, \tag{7.92}$$

for arbitrary $(\varphi_0^\sharp,\overline{R}_0^\sharp)\in X$. Observe that we can restrict attention to sequences $(\varphi^\sharp_{h_j},\overline{R}^\sharp_{h_j})\in X$ such that $I_{h_j}^{\sharp,\rm mem}(\varphi^\sharp_{h_j},\overline{R}^\sharp_{h_j})<\infty$ since otherwise the statement is true anyway. The statement (7.91) implies that

$$I_0^{\sharp,\rm mem} \le \Gamma - \liminf_{h_j} I_{h_j}^{\sharp,\rm mem}\,, \tag{7.93}$$

which is "almost" the conjecture (7.89) since $I_0^{\sharp,\rm mem}$ could be strictly smaller. If $I_{h_j}^{\sharp,\rm mem}(\varphi^\sharp_{h_j},\overline{R}^\sharp_{h_j})<\infty$, then equicoercivity w.r.t. rotations remains untouched by a change from $W_{\rm mp}$ to $W_{\rm mp}^{\rm mem}$ in the local energy. Hence, as usual by now, we can restrict attention to sequences of rotations $\overline{R}^\sharp_{h_j}$ converging weakly to some $\overline{R}_0\in W^{1,1+p+q}(\omega,\mathrm{SO}(3,\mathbb{R}))$, defined over the two-dimensional domain ω only. **However, we cannot conclude that φ_0 is independent of the transverse variable, contrary to the case with $\mu_c>0$.**

Along sequences $(\varphi^\sharp_{h_j},\overline{R}^\sharp_{h_j})\in X$ with finite energy the product $\frac{1}{h_j}\langle\overline{R}_{h_j,3},\partial_{\eta_3}\varphi^\sharp_{h_j}\rangle$ remains bounded but otherwise indeterminate. Therefore, a trivial lower bound is obtained by minimizing the effect in the 33-component in the local energy $W_{\rm mp}^{\rm mem}$. To do this, we need some calculations: for smooth $\varphi:\Omega_1\mapsto\mathbb{R}^3$, $\overline{R}:\omega\subset\mathbb{R}^2\mapsto\mathrm{SO}(3,\mathbb{R})$ define the "director"-vector $b^*=(0,0,\varrho^*)^T\in\mathbb{R}^3$ with $b(\varrho)=(0,0,\varrho)^T\in\mathbb{R}^3$ formally through

$$W_{\rm mp}^{\rm hom,0}(\nabla_{(\eta_1,\eta_2)}\varphi,\overline{R}) = W_{\rm mp}^{\rm mem}(\overline{R}^T(\nabla_{(\eta_1,\eta_2)}\varphi|b^*)) := \inf_{\varrho\in\mathbb{R}} W_{\rm mp}^{\rm mem}(\overline{R}^T(\nabla_{(\eta_1,\eta_2)}\varphi|b(\varrho)))\,. \tag{7.94}$$

The real number ϱ^*, which realizes this infimum, can be explicitly determined. Without giving the calculation, which follows as in (6.48) we obtain

$$\varrho^* = 1-\frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla_{(\eta_1,\eta_2)}\varphi|0),\overline{R}\rangle-2\right] = 1-\frac{\lambda}{2\mu+\lambda}\mathrm{tr}\left[\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T\nabla_{(\eta_1,\eta_2)}\varphi-1\!\!1_2)\right]. \tag{7.95}$$

Note that if $\overline{R}\in\mathrm{SO}(3,\mathbb{R})$ and $\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T\nabla_{(\eta_1,\eta_2)}\varphi-1\!\!1_2)\in L^2(\Omega_1,\mathbb{R}^3)$ one has $\varrho^*\in L^2(\Omega_1,\mathbb{R}^3)$. We obtain for $W_{\rm mp}^{\rm hom,0}(\nabla_{(\eta_1,\eta_2)}\varphi,\overline{R}) := W_{\rm mp}(\overline{R}^T(\nabla_{(\eta_1,\eta_2)}\varphi|b^*))$ after a lengthy but straightforward computation

$$\begin{aligned} W_{\rm mp}^{\rm hom,0}(\nabla_{(\eta_1,\eta_2)}\varphi,\overline{R}) &:= \mu\,\|\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T\nabla_{(\eta_1,\eta_2)}\varphi-1\!\!1_2)\|^2 \\ &\quad+\frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T\nabla_{(\eta_1,\eta_2)}\varphi-1\!\!1_2)\right]^2. \end{aligned} \tag{7.96}$$

Along the sequence $(\varphi^\sharp_{h_j}, \overline{R}^\sharp_{h_j})$ we have therefore by construction,

$$W^{\mathrm{mem}}_{\mathrm{mp}}(\overline{R}^{\sharp,T}_{h_j}\nabla^{h_j}_\eta\varphi^\sharp_{h_j}) = W^{\mathrm{mem}}_{\mathrm{mp}}(\overline{R}^{\sharp,T}_{h_j}(\nabla_{(\eta_1,\eta_2)}\varphi^\sharp_{h_j}|\frac{1}{h_j}\partial_{\eta_3}\varphi^\sharp_{h_j})) \geq W^{\mathrm{hom},0}_{\mathrm{mp}}(\nabla_{(\eta_1,\eta_2)}\varphi^\sharp_{h_j}, \overline{R}^\sharp_{h_j}) \,. \tag{7.97}$$

Hence, integrating and taking the lim inf also

$$\liminf_{h_j}\int_{\Omega_1} W_{\mathrm{mp}}(\overline{R}^{\sharp,T}_{h_j}\nabla^{h_j}_\eta\varphi^\sharp_{h_j})\,\mathrm{dV}_\eta \geq \liminf_{h_j}\int_{\Omega_1} W^{\mathrm{hom},0}_{\mathrm{mp}}(\nabla_{(\eta_1,\eta_2)}\varphi^\sharp_{h_j}, \overline{R}^\sharp_{h_j})\,\mathrm{dV}_\eta \,. \tag{7.98}$$

As in (6.57) (and subsequently) the proof of statement (7.91) would be finished, if we could show weak convergence of $\nabla_{(\eta_1,\eta_2)}\varphi^\sharp_{h_j}$ in $L^2(\Omega_1, \mathbb{M}^{3\times 3})$ whenever $\varphi^\sharp_{h_j} \to \varphi^\sharp_0$ strong in $L^r(\Omega_1, \mathbb{R}^3)$ and $I^{\sharp,\mathrm{mem}}_{h_j}(\varphi^\sharp_{h_j}, \overline{R}^\sharp_{h_j}) < \infty$. Boundedness and weak convergence of $\nabla_{(\eta_1,\eta_2)}\varphi^\sharp_{h_j}$ in $L^2(\Omega_1, \mathbb{M}^{3\times 3})$ is, however, not clear at all, since we now basically control only the "intrinsic" term $\|\operatorname{sym}((\overline{R}_1|\overline{R}_2)^T\nabla_{(\eta_1,\eta_2)}\varphi - 1\!\!1_2)\|^2$ in the integrand. Instead, we will prove a weaker statement, namely that

$$(\overline{R}^\sharp_{1,h_j}|\overline{R}^\sharp_{2,h_j})^T\nabla_{(\eta_1,\eta_2)}\varphi^\sharp_{h_j} \rightharpoonup (\overline{R}^\sharp_{1,0}|\overline{R}^\sharp_{2,0})^T\nabla_{(\eta_1,\eta_2)}\varphi^\sharp_0 \in L^2(\Omega_1, \mathbb{M}^{2\times 2}) \,, \tag{7.99}$$

after showing, that the above expression makes sense along the sequence, since $\nabla_{(\eta_1,\eta_2)}\varphi^\sharp_{h_j}$ is not yet explained if we know only that $\varphi^\sharp_{h_j} \in L^r(\Omega_1, \mathbb{R}^3)$.

In order to give a precise meaning to the expression in (7.99) along the sequence we define first for smooth $\phi \in C^\infty(\overline{\Omega}_1, \mathbb{R}^3)$ and $\overline{R} \in W^{1,1+p+q}(\Omega_1, \mathrm{SO}(3,\mathbb{R}))$ an intermediate function Ψ,

$$\Psi : \Omega_1 \mapsto \mathbb{R}^2 \,, \quad \Psi(\eta_1,\eta_2,\eta_3) := \begin{pmatrix} \langle \overline{R}_1, \phi\rangle \\ \langle \overline{R}_2, \phi\rangle \end{pmatrix} . \tag{7.100}$$

This implies that $\Psi \in W^{1,1+p+q}(\Omega_1, \mathbb{R}^2)$. It holds

$$\begin{aligned} (\overline{R}_1|\overline{R}_2)^T\nabla_{(\eta_1,\eta_2)}\phi &= \begin{pmatrix} \langle \overline{R}_1, \partial_{\eta_1}\phi\rangle & \langle \overline{R}_1, \partial_{\eta_2}\phi\rangle \\ \langle \overline{R}_2, \partial_{\eta_1}\phi\rangle & \langle \overline{R}_2, \partial_{\eta_2}\phi\rangle \end{pmatrix} ,\ D(\overline{R}_1|\overline{R}_2).\phi := \begin{pmatrix} \langle \partial_{\eta_1}\overline{R}_1, \phi\rangle & \langle \partial_{\eta_2}\overline{R}_1, \phi\rangle \\ \langle \partial_{\eta_1}\overline{R}_2, \phi\rangle & \langle \partial_{\eta_2}\overline{R}_2, \phi\rangle \end{pmatrix} , \\ \nabla\Psi &= \begin{pmatrix} \partial_{\eta_1}\langle \overline{R}_1, \phi\rangle & \partial_{\eta_2}\langle \overline{R}_1, \phi\rangle \\ \partial_{\eta_1}\langle \overline{R}_2, \phi\rangle & \partial_{\eta_2}\langle \overline{R}_2, \phi\rangle \end{pmatrix} = (\overline{R}_1|\overline{R}_2)^T\nabla_{(\eta_1,\eta_2)}\phi + D(\overline{R}_1|\overline{R}_2).\phi \,. \end{aligned} \tag{7.101}$$

The last equality shows

$$(\overline{R}_1|\overline{R}_2)^T\nabla_{(\eta_1,\eta_2)}\phi := \nabla_{(\eta_1,\eta_2)}\Psi - D(\overline{R}_1|\overline{R}_2).\phi \,. \tag{7.102}$$

We note the local estimate

$$\begin{aligned} \|\operatorname{sym}\nabla_{(\eta_1,\eta_2)}\Psi\|^2 &= \|\operatorname{sym}((\overline{R}_1|\overline{R}_2)^T\nabla_{(\eta_1,\eta_2)}\phi) + \operatorname{sym}(D(\overline{R}_1|\overline{R}_2).\phi)\|^2 \\ &\leq 2\|\operatorname{sym}((\overline{R}_1|\overline{R}_2)^T\nabla_{(\eta_1,\eta_2)}\phi)\|^2 + 2\|\operatorname{sym}(D(\overline{R}_1|\overline{R}_2).\phi)\|^2 \\ &\leq 2\|\operatorname{sym}((\overline{R}_1|\overline{R}_2)^T\nabla_{(\eta_1,\eta_2)}\phi)\|^2 + 2\|D(\overline{R}_1|\overline{R}_2).\phi\|^2 \\ &\leq 2\|\operatorname{sym}((\overline{R}_1|\overline{R}_2)^T\nabla_{(\eta_1,\eta_2)}\phi)\|^2 + 2\|D(\overline{R}_1|\overline{R}_2)\|^2\cdot\|\phi\|^2 \,. \end{aligned} \tag{7.103}$$

The last inequality implies after integration and Hölder's inequality (reminder $r = \frac{2(1+p+q)}{(1+p+q)-2}$, c.f. (5.17))

$$\begin{aligned} &\int_{\Omega_1} \|\operatorname{sym}\nabla_{(\eta_1,\eta_2)}\Psi\|^2\,\mathrm{dV}_\eta \\ &\quad\leq 2\int_{\Omega_1} \|\operatorname{sym}((\overline{R}_1|\overline{R}_2)^T\nabla_{(\eta_1,\eta_2)}\phi)\|^2\,\mathrm{dV}_\eta + 2\|\overline{R}\|^2_{W^{1,1+p+q}(\Omega_1)}\,\|\phi\|^2_{L^r(\Omega_1,\mathbb{R}^3)} \,. \end{aligned} \tag{7.104}$$

Moreover,

$$\int_{\Omega_1} \|\operatorname{sym} \nabla_{(\eta_1,\eta_2)} \Psi\|^2 + \|\Psi\|^2 \, \mathrm{dV}_\eta \leq 2 \int_{\Omega_1} \|\operatorname{sym}((\overline{R}_1|\overline{R}_2)^T \nabla_{(\eta_1,\eta_2)} \phi)\|^2 \, \mathrm{dV}_\eta$$
$$+ 2\|\overline{R}\|^2_{W^{1,1+p+q}(\Omega_1)} \, \|\phi\|^2_{L^r(\Omega_1,\mathbb{R}^3)} + 2\,\|\phi\|^2_{L^2(\Omega_1,\mathbb{R}^3)} \,, \quad (7.105)$$

since $\|\Psi\|^2 = \langle \overline{R}_1, \phi \rangle^2 + \langle \overline{R}_2, \phi \rangle^2 \leq \|\overline{R}_1\|^2 \|\phi\|^2 + \|\overline{R}_2\|^2 \|\phi\|^2 = 2\,\|\phi\|^2$. Furthermore, adding and subtracting $1\!\!1_2$

$$\begin{aligned}
&\int_{\Omega_1} \|\operatorname{sym} \nabla_{(\eta_1,\eta_2)} \Psi\|^2 + \|\Psi\|^2 \, \mathrm{dV}_\eta \\
&\leq 2 \int_{\Omega_1} \|\operatorname{sym}((\overline{R}_1|\overline{R}_2)^T \nabla_{(\eta_1,\eta_2)} \phi)\|^2 \, \mathrm{dV}_\eta + 2\|\overline{R}\|^2_{W^{1,1+p+q}(\Omega_1)} \, \|\phi\|^2_{L^r(\Omega_1,\mathbb{R}^3)} + 2\,\|\phi\|^2_{L^2(\Omega_1,\mathbb{R}^3)} \\
&= 2 \int_{\Omega_1} \|\operatorname{sym}((\overline{R}_1|\overline{R}_2)^T \nabla_{(\eta_1,\eta_2)} \phi - 1\!\!1_2 + 1\!\!1_2)\|^2 \, \mathrm{dV}_\eta \\
&\qquad + 2\|\overline{R}\|^2_{W^{1,1+p+q}(\Omega_1)} \, \|\phi\|^2_{L^r(\Omega_1,\mathbb{R}^3)} + 2\,\|\phi\|^2_{L^2(\Omega_1,\mathbb{R}^3)} \\
&\leq \int_{\Omega_1} 4\|\operatorname{sym}((\overline{R}_1|\overline{R}_2)^T \nabla_{(\eta_1,\eta_2)} \phi - 1\!\!1_2)\|^2 + 4\|1\!\!1_2\|^2 \, \mathrm{dV}_\eta \qquad (7.106) \\
&\qquad + 2\|\overline{R}\|^2_{W^{1,1+p+q}(\Omega_1)} \, \|\phi\|^2_{L^r(\Omega_1,\mathbb{R}^3)} + 2\,\|\phi\|^2_{L^2(\Omega_1,\mathbb{R}^3)} \,.
\end{aligned}$$

Hence, considering $\varphi^\sharp_{h_j}$ instead of ϕ we obtain along the sequence $(\varphi^\sharp_{h_j}, \overline{R}^\sharp_{h_j}) \in X$ with

$$I^{\sharp,\mathrm{mem}}_{h_j}(\varphi^\sharp_{h_j}, \overline{R}^\sharp_{h_j}) < \infty \,, \qquad (7.107)$$

the additional uniform bound

$$\begin{aligned}
&\int_{\Omega_1} \|\operatorname{sym} \nabla_{(\eta_1,\eta_2)} \Psi\|^2 + \|\Psi\|^2 \, \mathrm{dV}_\eta \\
&\quad \leq \frac{4}{\mu} I^{\sharp,\mathrm{mem}}_{h_j}(\varphi^\sharp_{h_j}, \overline{R}^\sharp_{h_j}) + \int_{\Omega_1} 4\|1\!\!1_2\|^2 \, \mathrm{dV}_\eta \qquad (7.108) \\
&\qquad + 2\|\overline{R}^\sharp_{h_j}\|^2_{W^{1,1+p+q}(\Omega_1)} \, \|\varphi^\sharp_{h_j}\|^2_{L^r(\Omega_1,\mathbb{R}^3)} + 2\,\|\varphi^\sharp_{h_j}\|^2_{L^2(\Omega_1,\mathbb{R}^3)} < \infty \,.
\end{aligned}$$

The classical Korn's second inequality without boundary conditions implies therefore that

$$\begin{aligned}
\infty &> \int_{\Omega_1} \|\operatorname{sym} \nabla_{(\eta_1,\eta_2)} \Psi_{h_j}\|^2 + \|\Psi_{h_j}\|^2 \, \mathrm{dV}_\eta \\
&= \int_{-1/2}^{1/2} \left[\int_\omega \|\operatorname{sym} \nabla_{(\eta_1,\eta_2)} \Psi_{h_j}(\eta_1,\eta_2,\eta_3)\|^2 + \|\Psi_{h_j}(\eta_1,\eta_2,\eta_3)\|^2 \, \mathrm{d}\omega \right] \mathrm{d}\eta_3 \\
&\geq \int_{-1/2}^{1/2} \left[c^+_K \int_\omega \|\nabla_{(\eta_1,\eta_2)} \Psi_{h_j}(\eta_1,\eta_2,\eta_3)\|^2 + \|\Psi_{h_j}(\eta_1,\eta_2,\eta_3)\|^2 \, \mathrm{d}\omega \right] \mathrm{d}\eta_3 \,, \qquad (7.109)
\end{aligned}$$

which allows us to conclude the boundedness of $\nabla_{(\eta_1,\eta_2)} \Psi_{h_j}$ in $L^2(\Omega_1, \mathbb{R}^2)$ and weak convergence of this sequence of gradients to a limit. By construction we know already that $\Psi_{h_j} \to \Psi_0 \in L^2(\Omega_1, \mathbb{R}^2)$ (assumed strong convergence of $\overline{R}_{h_j}$ and $\varphi^\sharp_{h_j}$). Hence $\nabla_{(\eta_1,\eta_2)} \Psi_{h_j}$ converges weakly to $\nabla_{(\eta_1,\eta_2)}\Psi_0$. Since we know as well that $\partial_{\eta_i}\overline{R}^\sharp_{h_j} \rightharpoonup \partial_{\eta_i}\overline{R}^\sharp_0$ in $L^{1+p+q}(\Omega_1,\mathbb{M}^{3\times 3})$, $i=1,2$ and $\varphi^\sharp_{h_j}\to\varphi^\sharp_0$ in $L^r(\Omega_1,\mathbb{R}^3)$ we obtain

$$D(\overline{R}^\sharp_{1,h_j}|\overline{R}^\sharp_{2,h_j}).\phi^\sharp_{h_j} \rightharpoonup D(\overline{R}^\sharp_{1,0}|\overline{R}^\sharp_{2,0}).\phi^\sharp_0 \in L^2(\Omega_1,\mathbb{M}^{2\times 2})\,. \tag{7.110}$$

Looking now back at (7.102) shows that

$$(\overline{R}^\sharp_{1,h_j}|\overline{R}^\sharp_{2,h_j})^T\nabla_{(\eta_1,\eta_2)}\varphi^\sharp_{h_j} \in L^2(\Omega_1,\mathbb{M}^{2\times 2})\,, \tag{7.111}$$

is a well defined expression for which (7.99) holds. Due to the convexity of $W^{\text{hom},0}_{\text{mp}}$ in the argument $\text{sym}((\overline{R}_1|\overline{R}_2)^T\nabla_{(\eta_1,\eta_2)}\varphi$, we may pass to the limit in (7.98) to obtain

$$\liminf_{h_j}\int_{\Omega_1} W_{\text{mp}}(\overline{R}^{\sharp,T}_{h_j}\nabla^{h_j}_\eta\varphi^\sharp_{h_j})\,\mathrm{dV}_\eta \geq \int_{\Omega_1} W^{\text{hom},0}_{\text{mp}}(\nabla_{(\eta_1,\eta_2)}\varphi^\sharp_0,\overline{R}^\sharp_0)\,\mathrm{dV}_\eta\,. \tag{7.112}$$

The convexity of $W^{\text{hom},0}_{\text{mp}}$ and Jensen's inequality (5.23) show then easily

$$\begin{aligned}\int_\omega W^{\text{hom},0}_{\text{mp}}(\nabla_{(\eta_1,\eta_2)}\,\text{Av}\,.\varphi(\eta_1,\eta_2),\overline{R})\,\mathrm{d}\omega &\leq \int_\omega\int_{-1/2}^{1/2} W^{\text{hom},0}_{\text{mp}}(\nabla_{(\eta_1,\eta_2)}\varphi(\eta_1,\eta_2,\eta_3),\overline{R})\,\mathrm{d}\eta_3\,\mathrm{d}\omega\\ &= \int_{\Omega_1} W^{\text{hom},0}_{\text{mp}}(\nabla_{(\eta_1,\eta_2)}\varphi(\eta_1,\eta_2,\eta_3),\overline{R})\,\mathrm{dV}_\eta \end{aligned}\tag{7.113}$$

Combining (7.113) with (7.112) shows

$$\liminf_{h_j}\int_{\Omega_1} W_{\text{mp}}(\overline{R}^{\sharp,T}_{h_j}\nabla^{h_j}_\eta\varphi^\sharp_{h_j})\,\mathrm{dV}_\eta \geq \int_\omega W^{\text{hom},0}_{\text{mp}}(\nabla_{(\eta_1,\eta_2)}\,\text{Av}\,.\varphi(\eta_1,\eta_2),\overline{R})\,\mathrm{d}\omega\,. \tag{7.114}$$

The proof of (7.91) is finished along the lines of (6.57). Note that (7.111) does definitely not yield control of $\nabla_{(\eta_1,\eta_2)}\varphi^\sharp_{h_j}$ in $L^2(\Omega_1,\mathbb{M}^{3\times 2})$. ■
To finish the proof of Γ-convergence for zero Cosserat couple modulus we observe that we have shown (c.f.(7.93)) in this section that on $X = L^r(\Omega_1,\mathbb{R}^3)\times L^{1+p+q}(\Omega_1,\text{SO}(3,\mathbb{R}))$

$$\begin{aligned} I^{\sharp,\text{mem}}_0 &\leq \Gamma-\liminf I^{\sharp,\text{mem}}_h \leq \Gamma-\liminf I^\sharp_{h|_{\mu_c=0}}\\ &\leq \Gamma-\limsup I^\sharp_{h|_{\mu_c=0}} \leq \lim_{\mu_c\to 0}\left(\Gamma-\lim I^\sharp_{h|_{\mu_c>0}}\right) =: I^{\sharp,0}_0\,. \end{aligned}\tag{7.115}$$

Since, however, $I^{\sharp,\text{mem}}_0 = I^{\sharp,0}_0$, the last inequality is in fact an equality, which shows that

$$\Gamma-\lim I^\sharp_{h_j|_{\mu_c=0}} = I^{\sharp,0}_0\,. \tag{7.116}$$

This gives us complete information on the behaviour of sequences of minimizing problems for $\mu_c=0$, should such sequences exist and converge to a limit in the encompassing space X. ■

8 The new formal finite-strain Cosserat thin plate model with size effects

8.1 Statement of the formal Cosserat plate model

The proposed formal "rational" of dimensional descend leads us to **postulate** the following two-dimensional minimization problem for the deformation of the midsurface $m:\omega\subset\mathbb{R}^2\mapsto\mathbb{R}^3$

and the microrotation of the plate (shell) $\overline{R}:\omega\subset\mathbb{R}^2\mapsto \mathrm{SO}(3,\mathbb{R})$ on ω:

$$I(m,\overline{R})=\int_\omega h\,W_{\mathrm{mp}}(\overline{U})+h\,W_{\mathrm{curv}}(\mathfrak{K}_s)+\frac{h^3}{12}\,W_{\mathrm{bend}}(\mathfrak{K}_b)\,\mathrm{d}\omega$$
$$-\,\Pi(m,\overline{R}_3)\mapsto\min\,.\ \text{w.r.t.}\ (m,\overline{R})\,, \tag{8.1}$$

under the constraints

$$\overline{U}=\overline{R}^T\widehat{F},\qquad \widehat{F}=(\nabla m|\overline{R}_3)\in\mathbb{M}^{3\times3}\,, \tag{8.2}$$
$$\mathfrak{K}_s=\Big(\overline{R}^T(\nabla(\overline{R}.e_1)|0),\overline{R}^T(\nabla(\overline{R}.e_2)|0),\overline{R}^T(\nabla(\overline{R}.e_3)|0)\Big)\in\mathfrak{T}(3)\,,\quad \mathfrak{K}_b=\mathfrak{K}_s^3\,,$$

and the boundary conditions of place for the midsurface deformation m on the Dirichlet part of the lateral boundary γ_0,

$$m_{|_{\gamma_0}}=g_{\mathrm{d}}(x,y,0)\,,\qquad \text{simply supported (fixed, welded)}\,. \tag{8.3}$$

The three possible **alternative** boundary conditions for the microrotations $\overline{R}$ on γ_0 are

$$\overline{R}_{|_{\gamma_0}}=\mathrm{polar}((\nabla m|\nabla g_{\mathrm{d}}(x,y,0).e_3))_{|_{\gamma_0}}\,,\qquad \text{strong form of reduced consistent coupling}\,, \tag{8.4}$$
$$\forall\,A\in C_0^\infty(\gamma_0,\mathfrak{so}(3,\mathbb{R}))\,:$$
$$\int_{\gamma_0}\langle\overline{R}^T\,(\nabla m(x,y)|\nabla g_{\mathrm{d}}(x,y,0).e_3),A(x,y)\rangle\,\mathrm{ds}=0\,,\qquad \text{very } \textbf{weak consistent coupling}\,,$$
$$\overline{R}_{3_{|_{\gamma_0}}}=\frac{\nabla g_{\mathrm{d}}(x,y,0).e_3}{\|\nabla g_{\mathrm{d}}(x,y,0).e_3\|}\,,\qquad \textbf{rigid director prescription}\,.$$

The constitutive assumptions on the reduced densities are[19]

$$W_{\mathrm{mp}}(\overline{U})=\mu\,\|\mathrm{sym}(\overline{U}-1\!\!1)\|^2+\mu_c\,\|\mathrm{skew}(\overline{U})\|^2+\frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U}-1\!\!1)\right]^2 \tag{8.5}$$
$$=\mu\,\underbrace{\|\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T\nabla m-1\!\!1_2)\|^2}_{\text{shear-stretch energy}}+\mu_c\,\underbrace{\|\mathrm{skew}((\overline{R}_1|\overline{R}_2)^T\nabla m)\|^2}_{\text{first order drill energy}}$$
$$+\underbrace{\frac{\kappa(\mu+\mu_c)}{2}\left(\langle\overline{R}_3,m_x\rangle^2+\langle\overline{R}_3,m_y\rangle^2\right)}_{\text{classical transverse shear energy}}+\frac{\mu\lambda}{2\mu+\lambda}\,\underbrace{\mathrm{tr}\left[\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T\nabla m-1\!\!1_2)\right]^2}_{\text{elongational stretch energy}}\,,$$
$$W_{\mathrm{curv}}(\mathfrak{K}_s)=\mu\frac{L_c^{1+p}}{12}\,(1+\alpha_4\,L_c^q\,\|\mathfrak{K}_s\|^q)\,\left(\alpha_5\,\|\mathrm{sym}\,\mathfrak{K}_s\|^2+\alpha_6\,\|\mathrm{skew}\,\mathfrak{K}_s\|^2+\alpha_7\,\mathrm{tr}\,[\mathfrak{K}_s]^2\right)^{\frac{1+p}{2}}\,,$$
$$\mathfrak{K}_s=\Big(\overline{R}^T(\nabla(\overline{R}.e_1)|0),\overline{R}^T(\nabla(\overline{R}.e_2)|0),\overline{R}^T(\nabla(\overline{R}.e_3)|0)\Big)\,,$$
$$\mathfrak{K}_s=(\mathfrak{K}_s^1,\mathfrak{K}_s^2,\mathfrak{K}_s^3)\in\mathfrak{T}(3)\,,\quad \text{the reduced third order } \textbf{curvature tensor}\,,$$
$$W_{\mathrm{bend}}(\mathfrak{K}_b)=\mu\,\|\mathrm{sym}(\mathfrak{K}_b)\|^2+\mu_c\,\|\mathrm{skew}(\mathfrak{K}_b)\|^2+\frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\,[\mathrm{sym}(\mathfrak{K}_b)]^2\,,$$
$$\mathfrak{K}_b=\overline{R}^T(\nabla\overline{R}_3|0)=\mathfrak{K}_s^3\,,\quad \text{the second order } \textbf{non-symmetric bending tensor}\,.$$

The (relative) thickness of the plate (shell) is $h>0$. The total elastically stored energy density due to **membrane-strain**, total **plate-curvature** and specific **plate-bending**

$$W=\underbrace{h\,W_{\mathrm{mp}}}_{\text{membrane}}+\underbrace{h\,W_{\mathrm{curv}}}_{\text{curvature}}+\underbrace{\frac{h^3}{12}W_{\mathrm{bend}}}_{\text{bending}}\,, \tag{8.6}$$

[19] $\|\mathrm{skew}((\overline{R}_1|\overline{R}_2)^T\nabla m)\|^2=\left(\langle\overline{R}_1,m_y\rangle-\langle\overline{R}_2,m_x\rangle\right)^2$.

depends on the midsurface deformation gradient ∇m and microrotations $\overline{R}$ together with their space derivatives only through the frame-indifferent measures $\overline{U}$ and $\mathfrak{K}_s$. The **micropolar stretch tensor** $\overline{U}$ of the plate is in general **non-symmetric**, neither is the **micropolar reduced third order curvature tensor** $\mathfrak{K}_s$. The three-dimensional plate deformation is reconstructed as

$$\varphi_s(x,y,z) = m(x,y) + \left(z\,\varrho_m(x,y) + \frac{z^2}{2}\,\varrho_b(x,y)\right)\,\overline{R}(x,y).e_3\,, \tag{8.7}$$

where

$$\begin{aligned}
\varrho_m &= 1 - \frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right] + \frac{\langle N_{\text{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)} = \underbrace{1 - \frac{\lambda}{2\mu+\lambda}\text{tr}\left[\overline{U} - 1\!\!1\right] + \frac{\langle N_{\text{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)}}_{\text{first order thickness change due to elongational stretch}}\,,\\
\varrho_b &= \underbrace{-\frac{\lambda}{2\mu+\lambda}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle + \frac{\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)\,h}}_{\text{non-symmetric shift of the midsurface due to bending}} = -\frac{\lambda}{2\mu+\lambda}\,\text{tr}\left[\mathfrak{K}_b\right] + \frac{\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)\,h}
\end{aligned} \tag{8.8}$$

and N_{diff}, N_{res} as defined in (14.3). To first order, the reconstructed deformationgradient is given by $F_s = (\nabla m|\varrho_m\,\overline{R}_3)$. Here $\omega \subset \mathbb{R}^2$ is a domain with boundary $\partial\omega$ and $\gamma_0 \subset \partial\omega$ is that part of the boundary, where Dirichlet conditions g_{d} for deformations and microrotations and/or consistent coupling conditions for microrotations, respectively, are prescribed. The reduced external loading functional $\Pi(m,\overline{R}_3)$ is a linear form in $(m,\overline{R}_3)$ defined in (14.19) in terms of the underlying three-dimensional loads. The parameters μ, $\lambda > 0$ are the Lamé constants of classical elasticity, $\mu_c \geq 0$ is called the Cosserat couple modulus and $L_c > 0$ introduces the internal length. We assume throughout that $\alpha_5 > 0, \alpha_6 > 0, \alpha_7 \geq 0$. We have included the so called **shear correction factor** κ $(0 < \kappa \leq 1)$ to keep in line with classical infinitesimal-displacement plate models (14.11). In our formal derivation, however, we obtain $\kappa = 1$. The reduced model (8.1) is fully **frame-indifferent**, meaning that

$$\forall\, Q \in \text{SO}(3,\mathbb{R}):\quad W_{\text{mp}}(Q\widehat{F}, Q\overline{R}) = W_{\text{mp}}(\widehat{F},\overline{R})\,,\quad \mathfrak{K}_s(Q\overline{R}) = \mathfrak{K}_s(\overline{R})\,. \tag{8.9}$$

The non-invariant term ϱ_m is only needed to reconstruct the 3D-deformation, which depends on the non-invariant loading.[20] **Strain** and **curvature** parts are **additively decoupled**, as in the underlying parent Cosserat bulk model (2.1). We note the appearance of the **harmonic mean** $\mathcal{H}$ and **arithmetic mean** $\mathcal{A}$

$$\frac{1}{2}\mathcal{H}(\mu,\frac{\lambda}{2}) = \frac{\mu\lambda}{2\mu+\lambda}\,,\qquad \kappa\,\mathcal{A}(\mu,\mu_c) = \kappa\,\frac{\mu+\mu_c}{2}\,. \tag{8.10}$$

8.2 Mathematical results for the formal Cosserat thin plate model

For conciseness we state only the obtained results for the case without external loads. It can be shown directly, without recourse to three-dimensional considerations [49]:

Theorem 8.1 (Existence for 2D-Cosserat thin plate with $\mu_c > 0$ and $\kappa > 0$)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\text{d}} \in H^1(\omega,\mathbb{R}^3)$ and $\overline{R}_{\text{d}} \in W^{1,1+p}(\omega,\text{SO}(3,\mathbb{R}))$. Then (8.1) with $\mu_c > 0$, $\kappa > 0$, $\alpha_4 \geq 0$, $p \geq 1$, $q \geq 0$ and either free or rigid prescription for $\overline{R}$ on γ_0 admits at least one minimizing solution pair $(m,\overline{R}) \in H^1(\omega,\mathbb{R}^3) \times W^{1,1+p}(\omega,\text{SO}(3,\mathbb{R}))$. ∎

Using the extended Korn's inequality [48, 56], the following has been shown in [52]:

[20] Of course, if the external tractions are rotated as well, we obtain invariance: $\langle Q.N_{\text{diff}}, Q.\overline{R}_3\rangle = \langle N_{\text{diff}},\overline{R}_3\rangle$.

Theorem 8.2 (Existence for 2D-Cosserat thin plate with $\mu_c = 0$ and $\kappa > 0$)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p+q}(\omega, \mathrm{SO}(3,\mathbb{R}))$. Then (8.1) with $\mu_c = 0$, $\kappa > 0$, $\alpha_4 > 0$, $p \geq 1$, $q > 0$ and either free or rigid prescription for $\overline{R}$ on γ_0 admits at least one minimizing solution pair $(m, \overline{R}) \in H^1(\omega, \mathbb{R}^3) \times W^{1,1+p+q}(\omega, \mathrm{SO}(3,\mathbb{R}))$. ■

9 On the form of the transverse shear energy for non-vanishing thickness and the shear correction factor κ

Γ-convergence describes the thin shell limit, but misses of course the fact that in actual computations of thin structures one wants to describe a material with finite thickness, which can sustain some amount of transverse shear.

If we compare the two different limit models (5.25),(8.1) described herein, we see that $\lim_{h_j \to 0} \frac{1}{h_j} I(m, \overline{R})$ in (8.1) coincides with the Γ-limit $I_0^\sharp$ in (5.25) as far as the local energy contribution W_{mp} is concerned, apart from the coefficient of the transverse shear energy. How then should the transverse shear contribution a priori look like, starting from a three-dimensional view-point?[21]

There is a large number of papers concerned with the effective (homogenized) coefficient of the transverse shear energy for isotropic linear elastic bulk material. The transverse shear deformation in the finite-strain Cosserat approach is proportional to $\left(\langle \overline{R}_3, m_x \rangle, \langle \overline{R}_3, m_y \rangle\right)$. The corresponding transverse shear energy is proportional to $\langle \overline{R}_3, m_x \rangle^2 + \langle \overline{R}_3, m_y \rangle^2$. If we assume **no warping** (transverse sections remain straight), i.e. an ansatz of the form $\varphi(x, y, z) = m(x, y) + \varrho^+(z)\,\overline{R}(x, y).e_3$ with $\varrho^+ : \mathbb{R} \mapsto \mathbb{R}^+$ and a constant director $\overline{R}.e_3$ over the thickness, the transverse shear energy is generally over-estimated. This ansatz leads to a linear distribution of the transverse shear-stresses in the plate.

From direct equilibrium considerations for the bulk it follows, however, that the director should be S-shaped over the thickness. Including this effect amounts to introduce warping. This corresponds to a "weaker" kinematical ansatz $\varphi(x, y, z) = m(x, y) + \varrho^+(z)\, Q(z)\, \overline{R}(x, y).e_3$ with an additional independent rotation field $Q \in \mathrm{SO}(3,\mathbb{R})$, depending only on the transverse variable z [73, 74]. It leads to a quadratic distribution of the transverse shear stresses in thickness direction. In order to relieve the effect of not including warping in the simpler ansatz, the introduction of the **shear correction factor** κ can be motivated.

For both presented models, the transverse shear energy in our notation can be written in the form

$$\mathrm{G}' \left(\langle \overline{R}_3, m_x \rangle^2 + \langle \overline{R}_3, m_y \rangle^2 \right) , \tag{9.11}$$

with a constitutive coefficient G', the **transverse shear modulus** $[\mathrm{G}'] = [\mathrm{N/m^2}]$.[22] Summarizing, we have

$$\begin{aligned}
\mathrm{G}' &= \kappa\, \mathcal{A}(\mu, \mu_c) = \kappa\, \frac{\mu + \mu_c}{2} && \text{formal reduction (8.1)}\,, \\
\mathrm{G}' &= \mathcal{H}(\mu, \mu_c) = 2\mu\, \frac{\mu_c}{\mu + \mu_c} && \Gamma\text{-limit (5.25)}\ , \\
\mathrm{G}' &= \kappa\, \mathcal{A}(\mu, 0) = \kappa\, \frac{\mu}{2} && \text{classical linear Reissner-Mindlin (14.10)}\ ,
\end{aligned} \tag{9.12}$$

[21]The possible difference between W_{curv} and $W_{\mathrm{curv}}^{\mathrm{hom}}$ is not our concern, since the constitutive coefficients of W_{curv} are rather a matter of convenience at present, as long as coercivity of curvature is guaranteed.
[22]Mindlin's notation [45, eq.7].

with $\kappa \geq 0$, the so called **shear correction factor**.[23] There are various values for the shear correction factor κ proposed in the engineering literature, among them prominently

$$\begin{aligned}
\kappa &= \frac{\pi^2}{12} \approx 0.8225\,, && \text{Mindlin's value [45]}\ ,\\
\kappa &= \frac{87}{100} = 0.8700\,, && \text{Babuska's value for } \nu = 0.3\,,\\
\kappa &= \frac{10}{12-2\nu} \approx 0.8772\,, && \text{Zhilin's value for } \nu = 0.3 \text{ [1]}\ ,\\
\kappa &= \frac{10}{12} \approx 0.8333\,, && \text{Reissner's value [57, 58]}\,,\\
\kappa &= \frac{10}{12-7\nu} \approx 1.01 && \text{Rössle's value for } \nu = 0.3\,,\\
\frac{\pi^2}{12} &\leq \kappa < 1\,, && \text{Altenbach's estimate [1]}\,.
\end{aligned} \tag{9.13}$$

These values for κ are proposed in terms of best fitting of certain simple infinitesimal three-dimensional quasistatic or dynamic test cases. Mindlin's value $\kappa = \frac{\pi^2}{12}$ is obtained from a best fit of the first eigenfrequency of the linearized plate model as compared to the three-dimensional linear elasticity solution. Reissner's value appears through an additional assumption regarding the stress distribution through the thickness [57, eq.10]. Babuska's value [5] is based on numerical "experiments". By dimensional analysis it can be shown [1] that κ should depend on the Lamé constants only through the Poisson ratio $0 < \nu < \frac{1}{2}$. Another motivation for the introduction of κ is obtained by trying to optimize the rate of convergence of the linear Reissner-Mindlin model to the solution of the linear elasticity model as $h \to 0$. This is the argument for Rössle's value [59]. The fact that there κ might be bigger than one cannot easily be accepted from a purely engineering point of view.

For $0 \leq \kappa = \frac{4\mu\mu_c}{(\mu+\mu_c)^2} \leq 1$ it holds that $\kappa\,\mathcal{A}(\mu,\mu_c) = \mathcal{H}(\mu,\mu_c)$. Hence, in view of our deduction of the Γ-limit as compared to the formal reduction and the general inequality $\mathcal{H}(\mu,\mu_c) \leq \mathcal{A}(\mu,\mu_c)$ together with the linearization consistency of the Γ-limit (5.34) if $\mu_c = 0$ **it is strongly suggested that $\kappa < 1$, in accordance with engineering practice**, also in the finite strain case.

The question of the form of the homogenized transverse shear energy is as well related to the observation, that the Γ-limit energy functional for $\mu_c = 0$, should it exist, will necessarily loose coercivity, which can directly be traced to the missing transverse shear contribution but this loss of coercivity is not due to the missing drill-energy. In this respect, note that $W_{\mathrm{mp}}(\overline{U})$ in (8.5) leads to a coercive formulation w.r.t. the midsurface deformation m also for $\mu_c = 0$. Moreover, in a linearized context, this energy is asymptotically correct for $\mu_c = 0$ and $\kappa = 1$, cf. (14.11).

For numerical calculations, the "homogenized" energy $I_0^{\sharp,0}$, which is indeed the Γ-limit energy functional for $\mu_c = 0$, can hardly be regarded as suitable in this case. From a more practical, computational viewpoint then, the introduction of a strictly positive shear correction factor $0 < \kappa < 1$ is fully justified and provides exactly that necessary minimal change of the local energy used in $I_0^{\sharp,0}$, in order to re-establish first strict Legendre-Hadamard ellipticity w.r.t. m (but not local strict convexity) and second coercivity for the midsurface in $H^{1,2}(\omega,\mathbb{R}^3)$. This underlines the salient features of the formal derivation together with $\mu_c = 0$ and $0 < \kappa \leq 1$.

[23]"In the classical Reissner-Mindlin model, the shear stresses $\sigma_{13}, \sigma_{23} (= \langle \overline{R}_3, m_x\rangle, \langle \overline{R}_3, m_y\rangle)$ are constant through the thickness of the plate. However, three-dimensional traction free boundary conditions at the upper and lower face of the shell imply that at these faces, the stresses have to be zero, hence also the shear stresses have to be zero. An analysis of equilibrium for an elastic beam shows that the shear stress should be quadratic through the thickness and vanish at the faces. A constant shear stress distribution over the thickness overestimates therefore the shear energy. A correction factor, known as the shear correction factor is often used to reduce the energy associated with transverse shear and accurate estimates of this factor can be made for elastic beams and shells. For nonlinear materials, however, it is difficult to estimate a shear correction factor." [6, p.554].

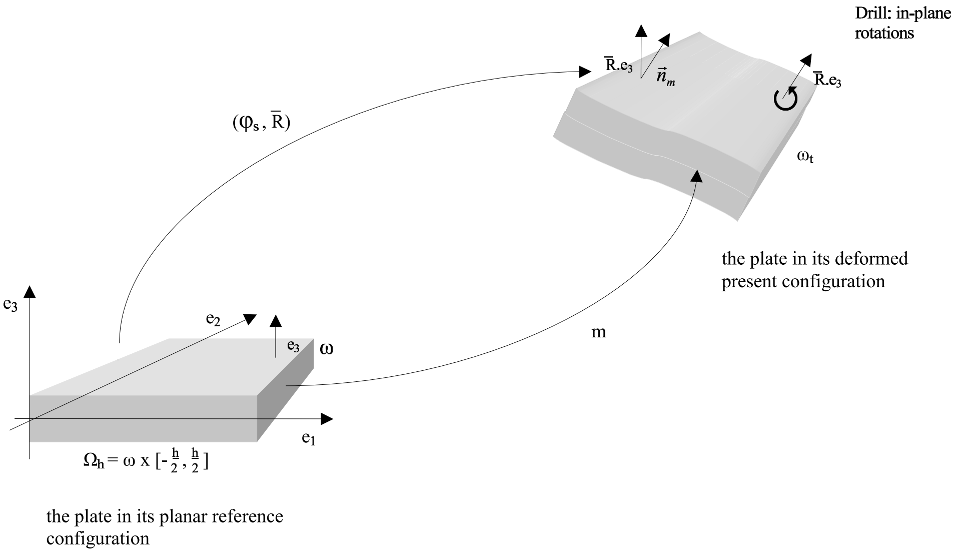


Figure 1: The assumed Cosserat plate kinematics incorporating transverse shear ($\overline{R}_3 \neq \vec{n}_m$), thickness stretch ($\varrho_m \neq 1$) and drill-rotations. Reconstructed three-dimensional deformation $\varphi_s : \Omega_h \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$, reconstructed microrotation $\overline{R}^{3d} : \Omega_h \subset \mathbb{R}^3 \mapsto \mathrm{SO}(3,\mathbb{R})$, $\overline{R}^{3d}(x,y,z) = \overline{R}(x,y)$, midsurface deformation $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and microrotation of the plate $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3,\mathbb{R})$.

10 Consequences for the Cosserat couple modulus μ_c

It is generally accepted in the engineering literature that really thin structures cannot support a non-vanishing transverse shear contribution. We introduce therefore the **postulate**

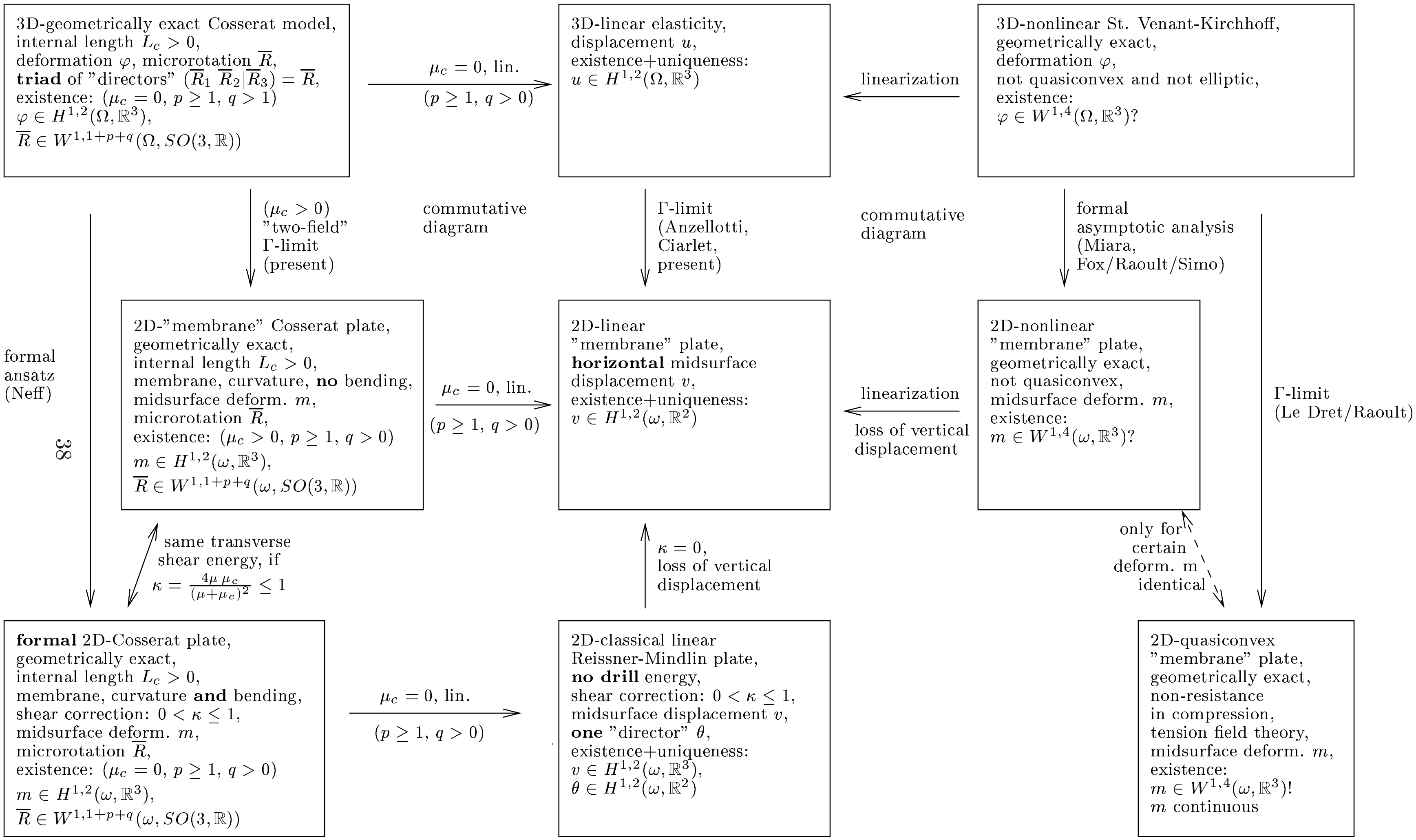
Postulate 10.1 (Vanishing transverse shear)
Regardless of material constants, in the limit of arbitrarily thin, homogeneous isotropic structures, i.e. for $h \to 0$, transverse shear effects are altogether absent. ■

Since the Γ-limit faithfully describes the leading order term for vanishing thickness, this postulate implies that the Cosserat couple modulus μ_c must vanish as well, since otherwise one would have to deal with a remaining homogenized transverse shear contribution in the thin plate limit.

This statement has far reaching consequences: it has never been possible to unequivocally identify specific values for the Cosserat couple modulus $\mu_c > 0$ in the experimentally oriented literature. In light of our development the problem can be resolved in the following way: $\mu_c > 0$ **in the Cosserat bulk model is a numerical tuning or penalty parameter but not a material constant**. That μ_c should be zero as a material constant has been conjectured by the first author already in [49, 52]. The unexpected formal proof of this statement has been reached now by our Γ-convergence result.

A striking consequence of this reasoning is that **a linear Cosserat bulk model describing faithfully the behaviour of a material body, does not exist**, since for $\mu_c = 0$ the linearized fields of infinitesimal displacement and infinitesimal microrotation decouple, see [54]. In summary Postulate 10.1 implies that the **infinitesimal Cauchy stress tensor σ must always be symmetric**.

11 Schematic overview of relations between lower dimensional models and Γ-convergence



12 Open problems and discussion

We have rigourously justified the dimensional homogenization of a geometrically exact Cosserat bulk model to its two-dimensional counterpart by use of Γ-convergence arguments. In starting from a "true" Cosserat bulk model, the appearance of an independent director field $\overline{R}_3$ is most natural. The argument is given for plates (flat reference configuration) only, but it is straightforward to extend the result to genuine shells with curvilinear reference configuration and it should be noted that the extension to shells is independent of geometrical features of the curvilinear reference configuration. The inclusion of transverse shear effects makes the distinction between elliptic, parabolic and hyperbolic surfaces in a certain sense obsolete. A welcome feature of the obtained Γ-limit is its linearization consistency.

Perhaps not so clear is an extension to the weak consistent coupling boundary condition in the Cosserat bulk problem, which might have an influence on the form of the homogenized transverse shear energy.

As a by-product of our development, we have obtained information on the numerical value of the Cosserat couple modulus μ_c in the bulk model: it should be set to zero which implies the symmetry of the infinitesimal Cauchy stresses σ. Moreover, for $\mu_c = 0$, a value $0 < \kappa < 1$ for the shear correction factor in the formal model is physically consistent, amounts to the inclusion of transverse shear and computationally stabilizes the model. In this sense, the classical linear Reissner-Mindlin model, which is not a Γ-limit of classical linear elasticity can now be seen as linearization of the geometrically exact Cosserat $\Gamma - \limsup$ for $\mu_c = 0$ with additional transverse shear stabilization.

The proposed two-dimensional Cosserat "membrane" plate (shell) model may as well have applications in those cases, where classical surface theory is not sufficient. This can be the case, if the surface to be investigated is not smooth enough, i.e. $m \notin H^{2,2}(\omega, \mathbb{R}^3)$ in the presence of failure along asymptotic lines of the surface. Our Γ-limit formulation is in principle well-posed for midsurface parametrizations $m \in H^{1,2}(\omega, \mathbb{R}^3)$.

Future work should investigate the numerical virtues of the formulation with non-vanishing transverse shear energy.

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14 Appendix

14.1 The Γ-limit for the rescaled linear bulk problem

Γ-limit investigations for the classical linear bulk problem are already well-known [12, 3]. However, while giving generically consistent results, they are based on different scaling assumptions. In order to establish linearization consistency of our formulation, it is therefore necessary to use the same scaling for the linear problem as for the finite-strain problem.

While we want to draw finally conclusions as regards classical linear elasticity, we study in a first step a quadratic functional which is strictly bigger than that of linear elasticity if we put $\mu_c > 0$. Let us investigate therefore the Γ-limit of the sequence of quadratic energy functionals defined for $\varphi^\sharp \in H^1(\Omega_1, \mathbb{R}^3)$

$$\begin{aligned}
J_h^\sharp(\varphi^\sharp) &= \int_{\eta\in\Omega_1} W_{\mathrm{lin}}(F_h^\sharp) - \langle f^\sharp, \varphi^\sharp\rangle \, \mathrm{dV}_\eta \mapsto \min . \text{ w.r.t. } \varphi^\sharp \,, \\
F_h^\sharp &= \left(\partial_{\eta_1}\varphi^\sharp(\eta_1,\eta_2,\eta_3)|\partial_{\eta_2}\varphi^\sharp(\eta_1,\eta_2,\eta_3)|\frac{1}{h}\partial_{\eta_3}\varphi^\sharp(\eta_1,\eta_2,\eta_3)\right) \,, \\
\varphi^\sharp_{|_{\Gamma_0^1}}(\eta) &= g_{\mathrm{d}}^\sharp(\eta) = g_{\mathrm{d}}(\zeta(\eta)) = g_{\mathrm{d}}(\eta_1,\eta_2,h\cdot\eta_3) = g_{\mathrm{d}}(\eta_1,\eta_2,0) \,, \\
\Gamma_0^1 &= \gamma_0 \times [-\frac{1}{2},\frac{1}{2}], \quad \gamma_0 \subset \partial\omega \,, \\
W_{\mathrm{lin}}(F_h^\sharp) &= \mu \,\|\operatorname{sym}(F_h^\sharp - 1\!\!1)\|^2 + \mu_c \,\|\operatorname{skew}(F_h^\sharp - 1\!\!1)\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}(F_h^\sharp - 1\!\!1)\right]^2 .
\end{aligned} \tag{14.1}$$

This rescaled formulation can be easily obtained from the finite strain formulation (3.11) by setting $\overline{R}^{3d,\sharp}(\eta) = 1\!\!1$ and neglecting curvature contributions altogether. Note that this is not the rescaled formulation of a linear Cosserat bulk model, since infinitesimal rotations are absent.

The major advantage of this definition for $J_h^\sharp$ is that the Γ-limit $J_0^\sharp$ can be immediately read of based on the finite-strain development. The Γ-limit for problem (14.1) is given by the following two-dimensional minimization problem for the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ on ω:

$$J_0^\sharp : L^r(\Omega_1, \mathbb{R}^3) \mapsto \overline{\mathbb{R}}\,,$$
$$J_0^\sharp(m) = \int_\omega W_{\rm lin}^{\rm hom}(\nabla m) - \langle f^\sharp, m\rangle \, {\rm d}\omega \mapsto \min .\ \text{w.r.t. } m\,, \quad m_{|_{\gamma_0}} = g_{\rm d}(\eta_1, \eta_2, 0)\,. \tag{14.2}$$

The dimensionally homogenized quadratic density is

$$\begin{aligned} W_{\rm lin}^{\rm hom}(\nabla m) &= \mu \, \| {\rm sym}((e_1|e_2)^T \nabla m - 1\!\!1_2)\|^2 + \mu_c \, \|{\rm skew}((e_1|e_2)^T \nabla m - 1\!\!1_2)\|^2 \\ &\quad + 2\mu \, \frac{\mu_c}{\mu + \mu_c} \left(\langle e_3, m_x\rangle^2 + \langle e_3, m_y\rangle^2 \right) + \frac{\mu\lambda}{2\mu+\lambda} \, {\rm tr}\left[{\rm sym}((e_1|e_2)^T \nabla m - 1\!\!1_2)\right]^2 . \end{aligned} \tag{14.3}$$

Proof. The proof follows with minor changes from the nonlinear proof of (5.25). One only has to replace $\overline{R}$ by $1\!\!1$ and skip the curvature part. Equi-coerciveness follows from local coercivity for $\mu_c > 0$. ■

In terms of the midsurface displacement $v \in \mathbb{R}^3$ we obtain equivalently the formulation

$$J_0^\sharp(v) = \int_\omega \widehat{W}_{\rm lin}^{\rm hom}(\nabla v) - \langle f^\sharp, v\rangle \, {\rm d}\omega \mapsto \min .\ \text{w.r.t. } v\,, \quad v_{|_{\gamma_0}} = g_{\rm d}(\eta_1, \eta_2, 0) - (\eta_1, \eta_2, 0)^T\,. \tag{14.4}$$

The dimensionally homogenized quadratic density reads then

$$\begin{aligned} \widehat{W}_{\rm lin}^{\rm hom}(\nabla v) &= \mu \, \| {\rm sym}\, \nabla_{(\eta_1,\eta_2)}(v_1, v_2)\|^2 + \mu_c \, \|{\rm skew}\, \nabla_{(\eta_1,\eta_2)}(v_1, v_2)\|^2 \\ &\quad + 2\mu \, \frac{\mu_c}{\mu + \mu_c} \left(v_{3,\eta_1}^2 + v_{3,\eta_2}^2 \right) + \frac{\mu\lambda}{2\mu+\lambda} \, {\rm tr}\left[{\rm sym}\, \nabla_{(\eta_1,\eta_2)}(v_1, v_2)\right]^2 . \end{aligned} \tag{14.5}$$

Since $J_h^\sharp$ for $\mu_c > 0$ is strictly bigger than the same functional for $\mu_c = 0$, independent of $h > 0$, it is easy to see [42, Prop. 6.7] that

$$\Gamma - \liminf J^\sharp_{h_{|\mu_c=0}} \le \Gamma - \limsup J^\sharp_{h_{|\mu_c=0}} \le \lim_{\mu_c \to 0} \left(\Gamma - \lim J^\sharp_{h_{|\mu_c>0}} \right) =: J_0^{\sharp,0}\,, \tag{14.6}$$

and we obtain an upper bound for the $\Gamma - \limsup$ of classical linear elasticity by taking $\mu_c = 0$ in (14.4). Setting $\mu_c = 0$ in (14.4) decouples the horizontal from the vertical components in which case one has to assume that body forces have no vertical component and boundary data are purely horizontal in order for the remaining classical linear "membrane" problem to be well-posed. This is a degeneration of the classical linearized formulation: **a linear "membrane" plate cannot sustain its own weight without being pre-stressed**, which is well known.

14.2 Linearized plate models

14.2.1 Relations to the classical infinitesimal-displacement Reissner-Mindlin model

Let us linearize a variant of the proposed new finite-strain Cosserat plate (8.1) for situations of small midsurface deformations and small curvature. We assume here $\alpha_4 = 0$, $q = 0$, $p > 1$.[24] We write $m(x,y) = (x,y,0)^T + v(x,y)$, with the displacement of the midsurface of the plate $v : \omega \mapsto \mathbb{R}^3$ and $\overline{R} = 1\!\!1 + \overline{A} + \ldots$, with $\overline{A} \in \mathfrak{so}(3,\mathbb{R})$ the infinitesimal-displacement microrotation. For the boundary deformation we write $g_{\rm d}(x,y,z) = (x,y,z)^T + u^{\rm d}(x,y,z)$, with the consequence, that $\nabla g_{\rm d}.e_3 = (u^{\rm d}_{1,z}, u^{\rm d}_{2,z}, 1 + u^{\rm d}_{3,z})$. The curvature tensors are expanded as

$$\begin{aligned} \mathfrak{K}_b &= \overline{R}^T (\nabla \overline{R}_3|0) = (1\!\!1 + \overline{A} + \ldots)^T \left(\nabla[\overline{A}_3 + \overline{A}^2.e_3 + \ldots]|0\right) \approx (\nabla \overline{A}_3|0) + \ldots\,, \\ \mathfrak{K}_s &\approx \left((\nabla(\overline{A}.e_1)|0), (\nabla(\overline{A}.e_2)|0), (\nabla(\overline{A}.e_3)|0)\right) \in \mathfrak{T}(3)\,, \end{aligned} \tag{14.7}$$

[24] The linearization for the case $\alpha_4 = 0$, $q = 0$, $p = 1, \mu_c > 0$ is similar to the static micropolar plate model derived by Eringen [26, eq. 8.6].

and the Cosserat micropolar plate stretch tensor expands like

$$\begin{aligned}\overline{U} &= \overline{R}^T \hat{F} = \overline{R}^T (\nabla m|\overline{R}_3) = (1\!\!1 + \overline{A} + \ldots)^T \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix} \right) + \nabla v | (1\!\!1 + \overline{A} + \ldots).e_3) \right) \\ &\approx 1\!\!1 + (\nabla v|\overline{A}_3) - \overline{A} + \ldots . \end{aligned} \tag{14.8}$$

Since $p > 1$, the additional Cosserat curvature contribution has an exponent strictly bigger than two such that a linearization w.r.t. zero curvature $\mathfrak{K}_s$ does not yield any contribution of this term. The consistent coupling condition is also expanded:

$$\begin{aligned} \overline{R}_{|\gamma_0} &= \mathrm{polar}(\nabla m|\nabla g_d.e_3)\,, \\ 1\!\!1 + \overline{A} + \ldots &= \mathrm{polar}(1\!\!1 + (\nabla v|\partial_z u^{\mathrm{d}}) + \ldots) = 1\!\!1 + \mathrm{skew}((\nabla v|\partial_z u^{\mathrm{d}})) + \ldots \Rightarrow \\ \overline{A}_{|\gamma_0} &= \mathrm{skew}((\nabla v|\partial_z u^{\mathrm{d}}))_{|\gamma_0}\,. \end{aligned} \tag{14.9}$$

We are formally left with the minimization problem for $v \in \mathbb{R}^3$ and $\overline{A} \in \mathfrak{so}(3,\mathbb{R})$:

$$\begin{aligned} &\int_\omega h \left(\mu \, \|\mathrm{sym}((\nabla v|\overline{A}_3))\|^2 + \mu_c \, \|\mathrm{skew}((\nabla v|\overline{A}_3) - \overline{A})\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \mathrm{tr}\left[\mathrm{sym}((\nabla v|\overline{A}_3))\right]^2 \right) \\ &\qquad + \frac{h^3}{12} \left(\mu \, \|\mathrm{sym}((\nabla \overline{A}_3|0))\|^2 + \mu_c \, \|\mathrm{skew}((\nabla \overline{A}_3|0))\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \mathrm{tr}\left[\mathrm{sym}((\nabla \overline{A}_3|0))\right]^2 \right) \mathrm{d}\omega \\ &\qquad\quad - \Pi(v, \overline{A}_3) \mapsto \min . \text{ w.r.t. } (v, \overline{A}), \\ &v_{|\gamma_0} = u^{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported (fixed, welded)}\,, \\ &\overline{A}_{|\gamma_0} = \mathrm{skew}((\nabla v|\partial_z u^{\mathrm{d}}))_{|\gamma_0}\,, \quad \text{lin. coupling} \Rightarrow \overline{A}_{3_{|\gamma_0}} = \left(\frac{u^{\mathrm{d}}_{1,z} - v_{3,x}}{2}, \frac{u^{\mathrm{d}}_{2,z} - v_{3,y}}{2}, 0 \right)^T, \\ &\overline{A}_{3_{|\gamma_0}} = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T\,, \quad \text{alternatively: rigid director prescription}\,. \end{aligned} \tag{14.10}$$

Now consider the case of zero Cosserat couple modulus $\mu_c = 0$. In this case infinitesimal in-plane rotations (linearized drilling degrees of freedom: $\overline{A}_{12} = -\overline{A}_{21}$) do not "survive" the linearization process. Abbreviating now $\theta = (\theta_1, \theta_2, 0)^T = -\overline{A}_3$, we are left with the following set of equations for the displacement of the midsurface of the plate $v : [0,T] \times \overline{\omega} \mapsto \mathbb{R}^3$ and the infinitesimal increment of the director, the infinitesimal "director", $\theta : \omega \mapsto \mathbb{R}^3$:

$$\begin{aligned} &\int_\omega h \left(\mu \, \|\mathrm{sym}\, \nabla(v_1,v_2)\|^2 + \kappa \underbrace{\frac{\mu}{2} \|\nabla v_3 - \theta\|^2}_{\text{transverse shear energy}} + \frac{\mu\lambda}{2\mu+\lambda} \mathrm{tr}\left[\mathrm{sym}\, \nabla(v_1,v_2)\right]^2 \right) \\ &\qquad + \frac{h^3}{12} \left(\mu \, \|\mathrm{sym}\, \nabla\theta\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \mathrm{tr}\left[\mathrm{sym}\, \nabla\theta\right]^2 \right) \mathrm{d}\omega - \Pi(v, -\theta) \mapsto \min . \text{ w.r.t. } (v,\theta)\,, \\ &v_{|\gamma_0} = u^{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported}\,, \\ &-\theta_{|\gamma_0} = \left(\frac{u^{\mathrm{d}}_{1,z} - v_{3,x}}{2}, \frac{u^{\mathrm{d}}_{2,z} - v_{3,y}}{2}, 0 \right)^T, \quad \text{linearized consistent coupling}\,, \\ &-\theta_{|\gamma_0} = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T\,, \qquad \text{alternatively: rigid director prescription}\,, \end{aligned} \tag{14.11}$$

with the so-called **shear correction factor** $\kappa = 1$.

A further reduction arises if we assume only normal displacements: $v_1 = v_2 = 0$. The resulting minimization problem for the deflection v_3 and the "director" θ is

$$\int_\omega h\,\frac{\kappa\mu}{2}\,\|\nabla v_3-\theta\|^2+\frac{h^3}{12}\left(\mu\,\|\operatorname{sym}\nabla\theta\|^2+\frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}\nabla\theta\right]^2\right)\,\mathrm{d}\omega$$
$$-\,\Pi(v_3\cdot e_3,-\theta)\mapsto\min.\ \text{w.r.t.}\ (v_3,\theta),\tag{14.12}$$
$$v_{3|_{\gamma_0}}=u_3^{\mathrm{d}}\,,\qquad\text{simply supported}\,,$$
$$-\theta_{|_{\gamma_0}}=\left(\frac{u^{\mathrm{d}}_{1,z}-v_{3,x}}{2},\frac{u^{\mathrm{d}}_{2,z}-v_{3,y}}{2},0\right)^T\qquad\text{linearized consistent coupling}\,,$$
$$-\theta_{|_{\gamma_0}}=(u^{\mathrm{d}}_{1,z},u^{\mathrm{d}}_{2,z},0)^T\,,\quad\text{rigid director prescription}\,.$$

In this last form with rigid boundary prescription, the Reissner-Mindlin plate-bending problem is classical and can be found in many textbooks, e.g. [9, p.281] or [72, 4] with Reissner's value $\kappa=\frac{5}{6}$. It should be noted, however, that in our formal, variationally based finite-strain derivation with subsequent linearization there is no imminent reason to introduce $\kappa\neq 1$. In fact, the shear correction factor κ can be seen as a tuning parameter of the infinitesimal-displacement model which, for certain types of loading,[25] allows to **improve** the **order of convergence** of the infinitesimal-displacement Reissner-Mindlin solution to the three-dimensional linear elasticity solution [59].[26]

Note the novel non-standard Dirichlet **boundary condition** of **linearized consistent coupling** for the remaining infinitesimal "director" θ, motivated from the consistency condition of the Cosserat bulk model. In contrast to the standard rigid director prescription, the new coupling condition seems to reduce the strength of the boundary layer. In a direct derivation of the Reissner-Mindlin plate equations (14.11) there is no reason to introduce this weakened condition. However, a mathematical analysis based on the consistent coupling condition shows that the new boundary condition can only be satisfied in the distributional sense on γ_0. Let us define therefore the admissible set

$$\mathcal{A}^{\mathrm{lin}}:=\{v_3\in H^1(\omega,\mathbb{R}),\,\theta\in H^1(\omega,\mathbb{R}^2)\,|\,v_{3|_{\gamma_0}}=u_3^{\mathrm{d}}\,,\quad\int_\omega\|\theta\|^2\,\mathrm{d}\omega\leq|\omega|\,,$$
$$\forall\,\phi\in C_0^\infty(\gamma_0,\mathbb{R}^2):\ \int_{\gamma_0}\langle-2\theta-\begin{pmatrix}u^{\mathrm{d}}_{1,z}\\u^{\mathrm{d}}_{2,z}\end{pmatrix},\phi\rangle_{\mathbb{R}^2}-v_3\cdot\operatorname{Div}\phi\,\mathrm{d}\omega=0\,\}\,,\tag{14.13}$$

which incorporates the linearized consistent coupling condition in the distributional sense, the standard Dirichlet boundary condition at γ_0, as well as an additional consistency condition for the linearization.[27] One can easily show that (14.12) admits a minimizer in $\mathcal{A}^{\mathrm{lin}}$. If $\|\theta\|_{L^2(\omega,\mathbb{R}^2)}<|\omega|$, the solution is unique.

14.2.2 The classical infinitesimal-displacement Kirchhoff-Love plate (Koiter model)

For the convenience of the reader we also supply the similar system of equations for the classical infinitesimal-displacement Kirchhoff-Love plate (also the Koiter model) which can be derived as

[25] Hence the shear correction factor κ shows some similarity to the Cosserat couple modulus μ_c, whose influence on the solution of the three-dimensional problem is also strongly dependent on boundary conditions. For rather thick plates, it is known that the shear energy in (14.11) is overestimated, therefore, one is led to reduce the shear energy contribution a posteriori by taking $\kappa<1$.

[26] It would be interesting to know the optimal shear correction factor $0<\kappa\leq 1$ of the infinitesimal-displacement Reissner-Mindlin model with our reduced consistent coupling boundary condition. Such an optimized parameter should also be beneficial for the finite-strain Cosserat plate. However, it might turn out that the new boundary condition of weak consistent coupling makes the artificial introduction of $\kappa<1$ superfluous. Note as well, that $\kappa=0$ decouples the horizontal "membrane" displacement in (14.11) from the vertical component and the bending term. In this sense, κ acts similarly as the Cosserat couple modulus μ_c in the linear Cosserat bulk model.

[27] The unit "director" $\overline{R}_3$ is expanded as $\overline{R}_3=e_3-\theta+\ldots$. Any θ with $\|\theta(x,y)\|>1$ pointwise, is inconsistent with the minimal requirement $1=\|\overline{R}_3.e_1\|\geq\|(e_3+\theta).e_1\|$. As a consequence, we impose $\int_\omega\|\theta\|^2\,\mathrm{d}\omega\leq|\omega|$.

linearization of the finite-strain Kirchhoff-Love plate. In terms of the midsurface displacement v we have to find a solution of the minimization problem for $v : \omega \subset \mathbb{R}^2 \mapsto \in \mathbb{R}^3$:

$$\begin{aligned}
&\int_\omega h \left(\mu \,\|\operatorname{sym} \nabla(v_1, v_2)\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \operatorname{tr}\left[\operatorname{sym} \nabla(v_1, v_2)\right]^2\right) \\
&\qquad + \frac{h^3}{12}\left(\mu \,\|D^2 v_3\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \operatorname{tr}\left[D^2 v_3\right]^2\right) \mathrm{d}\omega - \Pi(v, -\nabla v_3) \mapsto \min . \text{ w.r.t. } v\,, \\
&v_{|\gamma_0} = u^{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported (fixed, welded)}\,, \qquad (14.14) \\
&-\nabla v_{3|\gamma_0} = \left(\frac{u^{\mathrm{d}}_{1,z} - v_{3,x}}{2}, \frac{u^{\mathrm{d}}_{2,z} - v_{3,y}}{2}, 0\right)^T, \quad \text{lin. coupling} \Rightarrow -\nabla v_{3|\gamma_0} = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T\,, \\
&-\nabla v_{3|\gamma_0} = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T\,, \quad \text{rigid prescription of the infinitesimal increment of the "normal"}\,.
\end{aligned}$$

This energy can also be obtained formally from (14.12) by constraining the linearized director to the linearized normal of the plate, i.e. setting $\theta = \nabla v_3$. If this is done, we observe that the new boundary condition of consistent coupling coincides in fact with the classical boundary condition of the Kirchhoff-Love plate.

14.3 The treatment of external loads

14.3.1 Dead load body forces for the thin plate

In the three-dimensional theory the dead load body forces $f(x,y,z) \in \mathbb{R}^3$ were simply included by appending the potential with the term $\int_{\Omega_h} f(x,y,z) \cdot \varphi(x,y,z)\,\mathrm{dV}$. We define

$$\hat{f}_0(x,y) := \int_{-h/2}^{h/2} f(x,y,z)\,\mathrm{dz}\,, \quad \hat{f}_1(x,y) := \int_{-h/2}^{h/2} z\, f(x,y,z)\,\mathrm{dz}\,, \tag{14.15}$$

such that $\hat{f}_0$, $\hat{f}_1$ are the zero and first moment of f in thickness direction.

14.3.2 Traction boundary conditions for the thin plate

In the three-dimensional theory the traction boundary forces $N(x,y,z) \in \mathbb{R}^3$ were simply included by appending the potential with the term $\int_{\partial\Omega_h^{\mathrm{trans}} \cup \{\gamma_s \times [-\frac{h}{2}, \frac{h}{2}]\}} N(x,y,z) \cdot \varphi(x,y,z)\,\mathrm{dS}$. We define

$$\hat{N}_{\mathrm{lat},0}(x,y) := \int_{-h/2}^{h/2} N(x,y,z)\,\mathrm{dz}\,, \quad \hat{N}_{\mathrm{lat},1}(x,y) := \int_{-h/2}^{h/2} z\, N(x,y,z)\,\mathrm{dz}\,, \tag{14.16}$$

such that $\hat{N}_{\mathrm{lat},0}$, $\hat{N}_{\mathrm{lat},1}$ are the zero and first moment of the tractions N at the lateral boundary γ_s in thickness direction. Moreover, we abbreviate

$$N_{\mathrm{res}} := [N(x,y,\frac{h}{2}) + N(x,y,-\frac{h}{2})]\,, \quad N_{\mathrm{diff}} := \frac{1}{2}[N(x,y,\frac{h}{2}) - N(x,y,-\frac{h}{2})]\,. \tag{14.17}$$

14.3.3 The external resultant loading functional Π

For a first approximation plate formulation we set to leading order:

$$\begin{aligned}
\overline{f} &= \hat{f}_0 + N_{\mathrm{res}}\,, & &\textbf{resultant body force}\,, \\
\overline{M} &= \hat{f}_1 + h\, N_{\mathrm{diff}}\,, & &\textbf{resultant body couple}\,, \\
\overline{N} &= \hat{N}_{\mathrm{lat},0}\,, & &\textbf{resultant surface traction}\,, \\
\overline{M}_c &= \hat{N}_{\mathrm{lat},1}\,, & &\textbf{resultant surface couple}\,.
\end{aligned} \tag{14.18}$$

The **resultant dead load loading functional** Π is then given by the **linear form**

$$\Pi(m, \overline{R}_3) = \int_\omega \langle \overline{f}, m\rangle + \langle \overline{M}, \overline{R}_3\rangle \,\mathrm{d}\omega + \int_{\gamma_s} \langle \overline{N}, m\rangle + \langle \overline{M}_c, \overline{R}_3\rangle \,\mathrm{ds}\,. \tag{14.19}$$

If we denote the dependence of Π on the loads of the underlying three-dimensional problem as $\Pi(f, N;\, m, \overline{R}_3)$, then it is easily seen that frame-indifference of the external loading functional is satisfied in the sense that $\Pi(Q.f, Q.N;\, Q.m, Q.\overline{R}_3) = \Pi(f, N;\, m, \overline{R}_3)$ for all rigid rotations $Q \in \mathrm{SO}(3, \mathbb{R})$. It is possible to use the **same functional form** of the loading functional **for all finite-strain and infinitesimal-displacement models**. We only need to replace $(m, \overline{R}_3)$ by $(m, \vec{n}_m)$, $(v, \overline{A}_3)$ for the different finite and linearized models, respectively.

14.3.4 The modified external resultant loading functional $\Pi^\sharp$

In view of a possible mathematical analysis of the case with zero Cosserat couple modulus $\mu_c = 0$ we need to modify (14.19) into a **live load resultant loading functional** $\Pi^\sharp$, which better reflects the observation that by arbitrary translation of a material in a conservative force field only a finite amount of work can be gained. This is certainly true for any real physical field. In the three-dimensional theory we have called this the **"principle of bounded external work"**. Therefore we define the **nonlinear form**

$$\Pi^\sharp(m, \overline{R}_3) = \int_\omega \langle \overline{f}, \frac{m}{1 + [\|m\| - K]_+}\rangle + \langle \overline{M}, \overline{R}_3\rangle \,\mathrm{d}\omega + \int_{\gamma_s} \langle \overline{N}, \frac{m}{1 + [\|m\| - K]_+}\rangle + \langle \overline{M}_c, \overline{R}_3\rangle \,\mathrm{ds}\,. \tag{14.20}$$

Here $K > 0$ is a possibly large constant and $[\cdot]_+$ denotes the positive part of its scalar argument. We note that (14.20) is automatically **bounded**, if $\overline{f}, \overline{M} \in L^1(\omega, \mathbb{R}^3)$ and $\overline{M}_c, \overline{N} \in L^1(\gamma_s, \mathbb{R}^3)$. Moreover, the linearization of $\Pi^\sharp$ coincides with the linearization of Π