Local existence and uniqueness for a geometrically exact membrane-plate with viscoleastic transverse shear resistance.

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6th October 2004

Abstract

We prove the local existence and uniqueness to a geometrically exact, observer-invariant membrane-plate model introduced by the author. The model consists of an elliptic partial differential system of equations describing the equilibrium response of the membrane which is nonlinearly coupled with a viscoelastic evolution equation for exact rotations, taking on the role of an orthonormal triad of directors. This coupling introduces a viscoelastic transverse shear resistance.

Refined elliptic regularity results together with a new extended Korn's first inequality for plates and shells allow to proceed by a fixed point argument in appropriately chosen Sobolev-spaces in order to prove existence and uniqueness.

Key words: membranes, plates, thin films, energy minimization, viscoelasticity, transverse shear, elliptic systems.
AMS 2000: 74K15, 74K20, 74G65

short-title: Well-posedness for a viscoleastic membrane-plate.

final version for: Mathematical Methods in the Applied Sciences (MMAS), accepted for publication

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1 Introduction

1.1 A finite viscoelastic membrane-plate model

We study a geometrically exact, **observer-invariant** membrane-plate model that has been derived in [34] which incorporates viscoelastic transverse shear resistance due to an additional field of independently evolving rotations $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$.[1] The model in a variational formulation reads: find the deformation of the midsurface of the membrane-plate $m : [0,T] \times \overline{\omega} \mapsto \mathbb{R}^3$ and the independent local **viscoelastic rotation** $\overline{R} : [0,T] \times \overline{\omega} \mapsto \mathrm{SO}(3,\mathbb{R})$ such that m minimizes on $\overline{\omega}$

$$\int_\omega h\, W(F,\overline{R}) - \langle \overline{f}, m\rangle \,\mathrm{d}\omega \mapsto \min . \text{w.r.t. } m \text{ at given } \overline{R}\,, \tag{1.1}$$

with prescribed Dirichlet boundary conditions for simple support $m_{|_{\gamma_0}}(t,x,y) = g_\mathrm{d}(t,x,y)$, $(x,y) \in \gamma_0 \subset \partial\omega$. The constitutive assumptions on the densities are

$$\begin{aligned} W(F,\overline{R}) &:= \frac{\mu}{4}\, \|F^T\overline{R} + \overline{R}^T F - 2 1\!\!1\|^2 + \frac{\lambda}{8}\,\mathrm{tr}\left[F^T\overline{R} + \overline{R}^T F - 2 1\!\!1\right]^2, \\ F &= (\nabla m|\varrho_m\,\overline{R}_3), \quad \varrho_m = 1 - \frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right] + \frac{\langle N_\mathrm{diff}, \overline{R}_3\rangle}{2\mu+\lambda}\,. \end{aligned} \tag{1.2}$$

The local viscoelastic evolution for the "moving three-frame" $\overline{R}(t,x,y) \in \mathrm{SO}(3,\mathbb{R})$ is given by

$$\begin{aligned} &\frac{\mathrm{d}_{\hat\omega}}{\mathrm{dt}}\overline{R}(t) = \nu^+ \cdot \mathrm{skew}\,(B^\mathrm{res}) \cdot \overline{R}(t)\,, \quad B^\mathrm{res} = B^{\mathrm{res},0}_\mathrm{mech} \quad \text{or} \quad B^{\mathrm{res},0}_\mathrm{tc}, \quad \nu^+ = \nu^+(F,\overline{R}) \in \mathbb{R}^+\,, \\ &B^{\mathrm{res},0}_\mathrm{mech} = \mu\, F\overline{R}^T, \quad B^{\mathrm{res},0}_\mathrm{tc} = \left[\mu(2\,1\!\!1 - F\overline{R}^T) + \lambda\,[3 - \langle F\overline{R}^T, 1\!\!1\rangle 1\!\!1]\right] F\overline{R}^T, \quad \overline{R}(0) \in \mathrm{SO}(3,\mathbb{R})\,. \end{aligned} \tag{1.3}$$

This evolution equation guarantees that indeed exact rotations are determined whatever form the resultant (res) generator of the group $B^\mathrm{res} \in \mathbb{M}^{3\times3}$ has. By $\frac{\mathrm{d}_{\hat\omega}}{\mathrm{dt}}$ we mean the **observer-invariant (corotated) time derivative** on $\mathrm{SO}(3,\mathbb{R})$

$$\frac{\mathrm{d}_{\hat\omega}}{\mathrm{dt}}[R(t)] := \frac{\mathrm{d}}{\mathrm{dt}}[R(t)] - \widehat{\omega}(t)\cdot R(t)\,, \quad \widehat{\omega} := \frac{\mathrm{d}}{\mathrm{dt}}[Q(t)]\cdot Q(t)^T\,, \tag{1.4}$$

where $Q(t) \in \mathrm{SO}(3,\mathbb{R})$ is the rotation of the current frame with respect to the inertial frame and $\widehat{\omega}$ is the corresponding angular velocity. Without loss of generality, we confine attention to the inertial frame, i.e. $\widehat{\omega} \equiv 0$ and $\frac{\mathrm{d}_{\hat\omega}}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}$. The term $\nu^+ \in \mathbb{R}^+$ represents a scalar valued function introducing viscoelasticity and specified subsequently. $\overline{R}^0$ is the initial condition for the viscoelastic rotation part. Transverse shear ($\overline{R}_3 \neq \vec{n}_m$, where $\vec{n}_m$ is the unit normal to the surface given by m) occurs viscoelastically. $B^{\mathrm{res},0}_\mathrm{mech}$ or $B^{\mathrm{res},0}_\mathrm{tc}$ are alternative constitutive choices for B^res in (1.3). $B^{\mathrm{res},0}_\mathrm{mech}$ is mechanically motivated (mech) while $B^{\mathrm{res},0}_\mathrm{tc}$ is in addition thermodynamically consistent (tc). This notation derives from the underlying modelling paper [34].

Here, $\omega \subset \mathbb{R}^2$ denotes the **flat** referential domain of the membrane-plate with smooth boundary $\partial\omega$ and $\gamma_0 \subset \partial\omega$ is a part of the boundary supposed to have full one-dimensional Hausdorff measure. The **relative thickness** of the plate is $h > 0$, $\overline{f}$ denotes the applied resultant body loading while N_diff denotes a resultant surface couple (see (6.7)). The function ϱ_m accounts for **thickness stretch** of the membrane which is linearly coupled to the **membrane stretch** $\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]$, such that locally stretching the membrane decreases the thickness.

The three-dimensional deformation $\varphi_s : \overline{\omega} \times [-\frac{h}{2},\frac{h}{2}] \mapsto \mathbb{R}^3$ of the underlying thin structure is supposed to be reconstructed by

$$\varphi_s(x,y,z) = m(x,y) + z\,\varrho_m(x,y)\,\overline{R}_3(x,y)\,, \quad z \in [-\frac{h}{2},\frac{h}{2}], \tag{1.5}$$

where $\overline{R}_3 := \overline{R}.e_3$ and corresponding **reconstructed deformation gradient** $\nabla_{(x,y,z)}\varphi_s(x,y,0) := F = (\nabla m|\varrho_m\,\overline{R}_3)$, evaluated at the midsurface $z = 0$. Viewing (1.5) as an ansatz for the three-dimensional deformation with yet indetermined ϱ_m and inserting this ansatz into the underlying three-dimensional problem the form of the factor ϱ_m turns out to be an **exact analytical consequence** of the thickness-averaged three-dimensional stress conditions at the upper and lower face of the plate. The other notation is found in the appendix.

[1]The rotations $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$ can be thought of as a viscoelastically adjusted orthonormal triad of directors.

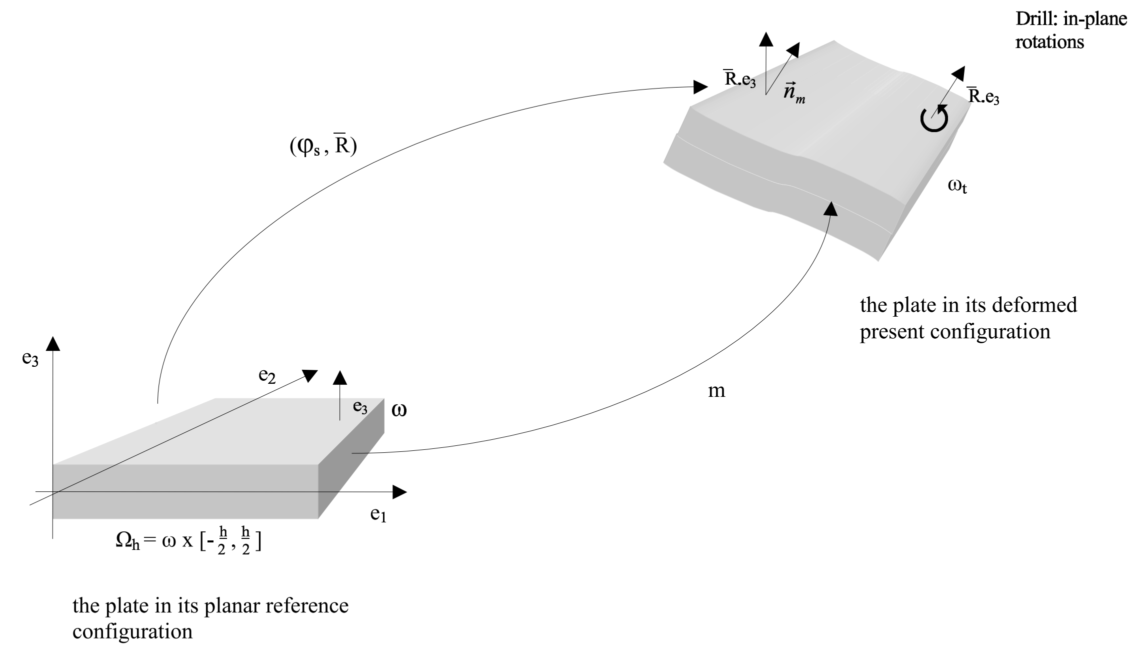

Figure 1: The assumed membrane-plate kinematics incorporating viscoelastic transverse shear ($\overline{R}_3 \neq \vec{n}_m$), instantaneous (elastic) thickness stretch ($\varrho_m \neq 1$) and viscoelastic drill-rotations. Reconstructed three-dimensional deformation $\varphi_s(x,y,z) = m(x,y) + z\,\varrho_m(x,y)\,\overline{R}_3$, midsurface deformation m, independent viscoelastic rotation $\overline{R}$.

The introduced problem (1.1,1.2,1.3) is **observer-invariant** (geometrically exact) in the sense that if the pair $(m, \overline{R})$ is a solution then for arbitrary $Q(t) \in \mathrm{SO}(3,\mathbb{R})$ the rigidly rotated pair $(Q(t).m, Q(t)\overline{R})$ is also a solution to rotated data. This requirement is crucial for a consistent description in continuum mechanics but violated by whatever infinitesimal-displacement models. This necessary requirement introduces automatically a certain type of nonlinearity which we aim to analyze.

It is also important to note that after all $W(F,\overline{R})$ depends at most quadratically on ∇m, the membrane deformation gradient, at given $\overline{R}$, despite appearance in (1.2). This can be seen by a lengthy but straightforward calculation given in (6.3). It shows that in terms of what will be called the **reduced reconstructed deformation gradient** $\widehat{F} = (\nabla m|\overline{R}_3)$ and $N_{\mathrm{diff}} = 0$ in fact

$$\begin{aligned} W(F,\overline{R}) &= \mu\,\|\,\mathrm{sym}\left(F^T\overline{R} - 1\!\!1\right)\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[\mathrm{sym}\left(F^T\overline{R} - 1\!\!1\right)\right]^2 \\ &= \mu\,\|\,\mathrm{sym}\left(\widehat{F}^T\overline{R} - 1\!\!1\right)\|^2 + \frac{\mu\lambda}{(2\mu+\lambda)}\,\mathrm{tr}\left[\mathrm{sym}\left(\widehat{F}^T\overline{R} - 1\!\!1\right)\right]^2, \end{aligned} \tag{1.6}$$

showing the apparent change of the Lamé moduli for the three-dimensional structure (μ,λ) to the reduced (homogenized) moduli of the two-dimensional structure $(\mu, \frac{\mu\lambda}{(2\mu+\lambda)})$. Note that $\frac{\mu\lambda}{(2\mu+\lambda)} = \frac{1}{2}\mathcal{H}(\mu,\frac{\lambda}{2})$ with $\mathcal{H}$ the **harmonic mean**. This is a characteristic feature of lower-dimensional theories which otherwise would not be asymptotically correct.

The goal of this contribution is to prove the well-posedness of (1.1,1.2,1.3). More precisely, we show the following result, for which we choose the positive function ν^+ in the viscoelastic flow part 1.3 formally similar to a conventional **Norton-Hoff** formulation of viscoplasticity theory

$$\nu^+ = \frac{1}{\eta}\left(1 + \left[\frac{\|\,\mathrm{skew}(\mu\,F\overline{R}^T)\|-0}{\bar\sigma_0}\right]_+^{r_0+1}\right)^{k_0}\cdot\left[\frac{\|\,\mathrm{skew}\,(B^{\mathrm{res}})\,\|-0}{\bar\sigma_0}\right]_+^{r_0}\cdot\frac{1}{\|\,\mathrm{skew}\,(B^{\mathrm{res}})\,\|} \tag{1.7}$$

with $\bar\sigma_0 = 1[\mathrm{MPa}]$, non-dimensional parameters $r_0, k_0 \geq 1$ and η plays the role of a relaxation time with units $[\eta] = \mathrm{sec}$. Within this setting we show

Theorem 1.1 (Local existence and uniqueness for problem (1.1,1.2,1.3))
Let $h > 0$ and $\omega \subset \mathbb{R}^2$ be a bounded smooth domain and suppose for the displacement boundary data $g_{\mathrm{d}} \in C^1(\mathbb{R}, H^{3,2}(\omega, \mathbb{R}^3))$ and for the resultant body force $\overline{f} \in C^1(\mathbb{R}, H^{1,2}(\omega, \mathbb{R}^3))$. Assume for the initial condition $\overline{R}^0 \in H^{2,2}(\omega, \mathrm{SO}(3)))$. Then there exists a time $t_1 > 0$ such that the initial boundary value problem (1.1,1.2,1.3) with ν^+ in the form (1.7)*, pure displacement boundary data and $N_{\mathrm{diff}} = 0$ admits a unique solution*

$$(m, \overline{R}) \in C([0, t_1], H^{3,2}(\omega, \mathbb{R}^3)) \times C^1([0, t_1], H^{2,2}(\omega, \mathrm{SO}(3))). \qquad \blacksquare$$

1.2 Relation to existing work

The dimensional reduction of a given model is already an old and mature subject and it has seen many "solutions". The different approaches toward elastic shell theory proposed in the literature and relevant references thereof are, therefore, too numerous to list here. In any case our proposal falls within the so called **derivation approach**, i.e., reducing a given three-dimensional model via (physically) reasonable constitutive assumptions to a two-dimensional model as opposed to either the **intrinsic** approach which views the shell from the onset as a two-dimensional surface and invokes concepts from differential geometry or the **asymptotic methods** which try to establish two-dimensional equations by formal expansion of the three-dimensional solution in power series in terms of a small parameter. The intrinsic approach is closely related to the **direct** approach which takes the shell to be a directed medium in the sense of a **restricted Cosserat-theory** [12].[2] A detailed presentation of the classical shell theories can be found in [28]. A thorough mathematical analysis of linear, infinitesimal shell theory, based on asymptotic methods is to be found in [8] and the extensive references therein, see also [7, 10, 1, 13, 14]. Reviews and insightful discussions of the modelling and finite element implementation may be found in [41, 39, 40, 23, 24, 2, 4] and in the series of papers [42, 44, 45, 47, 46, 43, 11]. Properly invariant elastic plate theories for membrane and bending are derived by formal asymptotic methods in [21] and extended to the case of curvilinear coordinates in [27, 26].

The mathematical analysis establishing the wellposedness of all the infinitesimal linearized models is fairly well established and will not be our concern.

In the finite-strain, geometrically exact elastic case, mostly based on the Saint Venant-Kirchhoff free energy density $\mu \|E\|^2 + \frac{\lambda}{2}\mathrm{tr}\,[E]^2$ where $E = \frac{1}{2}(F^T F - 1\!\!1)$, the formal asymptotic methods are still successful in that they identify again leading membrane and bending terms. As far as the occurring membrane contribution is concerned, it is the form (3.50) which is given e.g. in [22, 21, 27]. However, variational methods based on scaling assumptions and Γ-convergence [15] suggest a fundamentally different membrane term which leads to a non-resistance of the membrane plate/shell in compression.[3] The non-resistance to compression in this analysis is related to the use of the quasiconvex hull[4] QW_0 of a dimensionally reduced St.Venant Kirchhoff energy, see (3.52). This quasiconvex hull, surprisingly enough, can be given in closed form [17, 25] and shows to be in general positive but zero in the compression range.

The classical linear models proposed in the literature lead to effective numerical schemes only if the thickness h of the structure is still appreciable, i.e. classical bending terms are present and regularize the computation. However, there is an abundance of new applications where very thin structures are used, e.g. very thin metal layers on a substrate (in computer hardware, for the characteristic non dimensional relative thickness $h \leq 5 \cdot 10^{-4}$). See [3] for an application to thin films.

Since locally rotating the thin structure is energetically "cheap" compared to stretching, we are forced to consider models including finite rotations in an objective manner. But the proposed finite-strain membrane terms found in the literature are either **non-elliptic** and the remaining (minimization) problem is not well-posed or they lead to the aforementioned non-resistance in compression. We view the model (1.1,1.2,1.3) as a partial answer to these problems. A different approach to the same problem has been taken in [35], where balance equations for rotations are prescribed instead of evolution equations as in (1.3).

[2] Restricted, since no material length scale enters the direct approach, only the thickness h appears.

[3] They remark [16, p.550]: "...then the corresponding nonlinear membranes offer no resistance to crumpling. This is an empirical fact, witnessed by anyone who ever played with a deflated balloon."

[4] "... the fact that this function is not quasiconvex already implied that it had to be relaxed in order to give rise to a well posed problem." [16, p.575].

1.3 Preliminaries and general mathematical framework

Let us outline how we show that the nonlinear problem (1.1,1.2,1.3) admits a unique local solution. Since we will heavily use elliptic regularity, we confine attention to the case without external surface tractions.[5] At "frozen" rotations $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$ the corresponding system of elastic balance of linear momentum proves to be a linear, second order, strictly Legendre-Hadamard elliptic boundary value problem with **non-constant coefficients** set by $\overline{R}$. This system has variational structure in the sense that the equilibrium part of (1.1,1.2,1.3) is equivalent to the elastic minimization problem

$$\forall\, t \in [0,T]: \quad I(m(t), \overline{R}(t)) \mapsto \min . \text{ w.r.t. } m, \quad m(t) \in g_d(t) + H_\circ^{1,2}(\omega, \mathbb{R}^3; \gamma_0)\,, \tag{1.8}$$

where

$$I(m,\overline{R}) = \int_\omega h\, W(F,\overline{R}) - \langle \overline{f}, m\rangle \,\mathrm{d}\omega\,, \quad F = (\nabla m|\overline{R}_3)\,, \tag{1.9}$$
$$W(F,\overline{R}) := \frac{\mu}{4}\, \|F^T\overline{R} + \overline{R}^T F - 2\mathbb{1}\|^2 + \frac{2\mu\lambda}{8(2\mu+\lambda)}\, \mathrm{tr}\left[F^T\overline{R} + \overline{R}^T F - 2\mathbb{1}\right]^2 .$$

The weak form of the corresponding equilibrium equation is given by

Lemma 1.2 (Weak form of static elastic problem)
A minimizer $m \in H^{1,2}(\omega,\mathbb{R}^3)$ of (1.8) is a weak solution to the equilibrium problem

$$0 = \int_\omega h\,\langle D_F W(F,\overline{R}), (\nabla\phi|0)\rangle - \langle \overline{f}, \phi\rangle \,\mathrm{d}\omega \qquad \forall\, \phi \in H_0^{1,2}(\omega,\mathbb{R}^3)\,. \tag{1.10}$$

If the appearing quantities are smooth enough, this is equivalent to the strong form

$$0 = h \text{ Div } \overline{R}\left[\mu(F^T\overline{R} + \overline{R}^T F - 2\mathbb{1}) + \frac{2\mu\lambda}{2\mu+\lambda}\, \mathrm{tr}\left[F^T\overline{R} - \mathbb{1}\right]\mathbb{1}\right] + \overline{f}\,. \tag{1.11}$$

For the reduced reconstructed deformation gradient $F = (\nabla m|\overline{R}_3)$ it holds that

$$F^T\overline{R} = (\nabla m|\overline{R}_3)^T\overline{R} = \left((\nabla m|0) + (0|0|\overline{R}_3)\right)^T\overline{R} = (\nabla m|0)^T\overline{R} + (0|0|e_3)\,, \tag{1.12}$$

and we have also the alternative representation

$$\begin{aligned} h \text{ Div } \overline{R}&\left[\mu((\nabla m|0)^T\overline{R} + \overline{R}^T(\nabla m|0)) + \frac{2\mu\lambda}{2\mu+\lambda}\, \mathrm{tr}\left[(\nabla m|0)^T\overline{R}\right]\mathbb{1}\right] = \\ &-\overline{f} + h \text{ Div } \left[2\left(\mu + 3\frac{\mu\lambda}{2\mu+\lambda}\right)\overline{R}\right] . \end{aligned} \tag{1.13}$$

■

Note the appearance of a **"virtual" body force** contribution on the right hand side in (1.13) due to the inhomogeneities inherent in $\overline{R}$ which can be seen as a permanent source of internal stresses. This weak form (1.13) can be written in the shortcut form

$$h \text{ Div } \mathbb{D}(\overline{R}(x,y)).(\nabla m|0) = -\overline{f} + h \text{ Div}\, V(\overline{R}(x,y)), \quad m_{|_{\partial\omega}} = g_{\mathrm{d}}\,, \tag{1.14}$$

where we introduced the corresponding elasticity tensor $\mathbb{D}$ and the additional right-hand side contribution V according to the next definition in line with (1.13):

Definition 1.3 (Homogenized two-dimensional elasticity tensor)
We define the two dimensional elasticity tensor $\mathbb{D} : \mathbb{M}^{3\times3} \mapsto \mathrm{Lin}(\mathbb{M}^{3\times3}, \mathbb{M}^{3\times3})$ and the right hand side $V : \mathbb{M}^{3\times3} \mapsto \mathbb{M}^{3\times3}$ by

$$\forall\, H \in \mathbb{M}^{3\times3}: \quad \mathbb{D}(\overline{R}).H := \overline{R}\left[\mu(H^T\overline{R} + \overline{R}^T H) + \frac{2\mu\lambda}{2\mu+\lambda}\, \mathrm{tr}\left[H^T\overline{R}\right]\mathbb{1}\right],$$
$$V(\overline{R}) := 2\left(\mu + 3\frac{\mu\lambda}{2\mu+\lambda}\right)\overline{R}\,, \tag{1.15}$$

[5]The case with non-vanishing transverse surface tractions N_{diff} can be easily included since it involves only a modification of the resultant body force.

respectively. Note that $\mathbb{D}$ is a **nonlinear** *mapping with respect to $\overline{R}$, while V remains* **linear** *and*

$$\mathbb{D}(1\!\!1).H := \left[\mu(H^T + H) + \frac{2\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\,[H]1\!\!1\right] \tag{1.16}$$

is the two-dimensional homogenized elasticity tensor of linear elasticity. ■

A startling difficulty which we encounter in the treatment of (1.13) is that the elasticity tensor $\mathbb{D} = \mathbb{D}(\overline{R})$, although turning out to be uniformly Legendre-Hadamard elliptic, does not induce a pointwise uniformly positive bilinear form on the symmetrized strains as in (3.49) for $\overline{R} = 1\!\!1$, ($\overline{A} = 0$). To see nevertheless the uniform Legendre-Hadamard ellipticity, we prove

Lemma 1.4 (Uniform Legendre-Hadamard ellipticity)
Assume that $\overline{R} : \omega \mapsto \mathrm{SO}(3,\mathbb{R})$. Then the system (1.13) with elasticity tensor $\mathbb{D}$ given by Definition 1.3 is uniformly Legendre-Hadamard elliptic in the sense that

$$\exists\, c^+ > 0\, \forall\, \xi \in \mathbb{R}^3,\ \eta \in \mathbb{R}^2 : \quad \langle \mathbb{D}(\overline{R}(x,y)).(\xi\otimes\eta|0), (\xi\otimes\eta|0)\rangle \geq c^+\, \|\xi\|^2_{\mathbb{R}^3}\, \|\eta\|^2_{\mathbb{R}^2}\,, \tag{1.17}$$

and the **ellipticity constant** *is* **independent** *of $\overline{R}(x,y)$.*

Proof. Set $\hat\eta = (\eta_1,\eta_2,0)^T$ with $\eta \in \mathbb{R}^2$ implying $\xi\otimes\hat\eta = (\xi\otimes\eta|0)$. For $\mathbb{D}$ given by Definition 1.3 we have

$$\begin{aligned}
&\langle \mathbb{D}(\overline{R}(x,y)).(\xi\otimes\eta|0), (\xi\otimes\eta|0)\rangle = D^2_{\nabla m} W((\nabla m|\overline{R}_3), \overline{R}).((\xi\otimes\eta|0), (\xi\otimes\eta|0))\\
&= \frac{\mu}{2}\,\|\overline{R}^T(\xi\otimes\eta|0) + (\xi\otimes\eta|0)^T\overline{R}\|^2 + \frac{\mu\lambda}{2(2\mu+\lambda)}\mathrm{tr}\left[\overline{R}^T(\xi\otimes\eta|0) + (\xi\otimes\eta|0)^T\overline{R}\right]^2\\
&\geq \frac{\mu}{2}\,\|\overline{R}^T(\xi\otimes\eta|0) + (\xi\otimes\eta|0)^T\overline{R}\|^2\\
&= \mu\,\|\overline{R}^T(\xi\otimes\eta|0)\|^2 + \mu\,\langle \overline{R}^T(\xi\otimes\eta|0), (\xi\otimes\eta|0)^T\overline{R}\rangle\\
&= \mu\,\|(\xi\otimes\eta|0)\|^2 + \mu\,\langle \overline{R}^T.\xi\otimes\hat\eta, (\hat\eta\otimes\xi)\overline{R}\rangle\\
&= \mu\,\|(\xi\otimes\hat\eta)\|^2 + \mu\,\langle \overline{R}^T.\xi\otimes\hat\eta, \hat\eta\otimes\overline{R}^T.\xi\rangle\\
&\geq \mu\,\|\xi\otimes\hat\eta\|^2 + \mu\,\langle \overline{R}^T.\xi, \hat\eta\rangle^2 \geq \mu\,\|\xi\|^2_{\mathbb{R}^3}\,\|\hat\eta\|^2_{\mathbb{R}^3} = \mu\,\|\xi\|^2_{\mathbb{R}^3}\,\|\eta\|^2_{\mathbb{R}^2}\,.
\end{aligned} \tag{1.18}$$

The uniformity of the estimate is only true since rotations $\overline{R}(x,y) \in \mathrm{SO}(3,\mathbb{R})$ leave length constant: $\|\overline{R}.\xi\| = \|\xi\|$. ■

Despite the missing pointwise uniform positivity, we prove the existence, uniqueness and regularity of solutions to the boundary value problem (1.13). The existence part for (1.13) relies heavily on the following Theorem recently proved by the author extending Korn's first inequality to non-constant coefficients and overcoming the lack of uniform positivity of (1.8). This theorem has been proved in the context of multiplicative plasticity, from which the notation F_p originates.

Theorem 1.5 (Extended 3D-Korn's first inequality)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $\Gamma \subset \partial\Omega$ be a smooth part of the boundary with non vanishing 2-dimensional Hausdorff measure. Define $H^{1,2}_\circ(\Omega,\Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|_\Gamma} = 0\}$ and let $F_p, F_p^{-1} \in C^1(\overline{\Omega}, \mathrm{GL}(3,\mathbb{R}))$. Moreover suppose that $\mathrm{Curl}\,F_p \in C^1(\overline{\Omega}, \mathbb{M}^{3\times3})$. Then

$$\begin{aligned}
&\exists\, c^+ > 0\ \forall\ \phi \in H^{1,2}_\circ(\Omega,\Gamma) :\\
&\qquad \|(\nabla\phi)F_p^{-1}(x) + F_p^{-T}(x)(\nabla\phi)^T\|^2_{L^2(\Omega)} \geq c^+\ \|\phi\|^2_{H^{1,2}(\Omega)}\,.
\end{aligned} \tag{1.19}$$

Proof. The proof has been presented in [31]. ■

Remark 1.6
Note that for $F_p = \nabla\Theta$ we would only have to deal with the classical Korn's inequality evaluated on the transformed domain $\Theta(\Omega)$. This is the **compatible** *case. However, in general, F_p is* **incompatible** *such that the problem can be viewed as posed on a* **non-Riemannian manifold** *. Compare to [5] for an interpretation and the physical relevance of the quantity $\mathrm{Curl}\,F_p$. It comes as no surprise that in finite plasticity the incompatibility of F_p should play an important role.*

Motivated by the investigations in [31], it has been shown recently by Pompe [38] that the extended Korn's inequality can be viewed as a special case of a general class of coercive inequalities for quadratic forms. He was able to show that indeed $F_p, F_p^{-1} \in C(\overline{\Omega}, \mathrm{GL}(3,\mathbb{R}))$ is sufficient for Theorem 1.5 to hold without any condition on the compatibility.

However, taking the special structure of the extended Korn's inequality again into account, work in progress suggests that continuity is not really necessary: instead $F_p, F_p^{-1} \in L^\infty(\Omega, \mathrm{GL}(3,\mathbb{R}))$ and $\mathrm{Curl}\, F_p \in L^{3+\delta}(\Omega)$ should suffice, whereas $F_p, F_p^{-1} \in L^\infty(\Omega, \mathrm{GL}(3,\mathbb{R}))$ alone is not sufficient, see the counterexample presented in [38]. The possible improvement has no bearing on our further development. ■

As a consequence of the three-dimensional coercivity inequality it is possible to prove

Theorem 1.7 (Extended Korn's inequality for rigid shells)
Let $\omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and let $\gamma_0 \subset \partial\omega$ be a part of the boundary with non vanishing 1-dimensional Hausdorff measure. Define $H_\circ^{1,2}(\omega, \mathbb{R}^3; \gamma_0) := \{\phi \in H^{1,2}(\omega, \mathbb{R}^3),\ |\ \phi_{|\gamma_0} = 0\}$ and let $F_p, F_p^{-1} \in W^{1,2+\delta}(\overline{\omega}, \mathrm{GL}(3,\mathbb{R}))$. Then

$$\begin{aligned} &\exists\, c^+ > 0 \quad \forall\, \phi \in H_\circ^{1,2}(\omega, \mathbb{R}^3; \gamma_0) : \\ &\qquad \|(\nabla\phi|0)F_p^{-1}(x) + F_p^{-T}(x)(\nabla\phi|0)^T\|^2_{L^2(\omega)} \geq c^+ \, \|\phi\|^2_{H^{1,2}(\omega)}, \end{aligned} \tag{1.20}$$

and the constant is bounded away from zero for F_p, F_p^{-1} bounded in $W^{1,2+\delta}(\overline{\omega}, \mathrm{GL}(3,\mathbb{R}))$.

Proof. The idea is to extend the function ϕ in a suitable manner to three dimensions and to use Theorem 1.5 in the strengthened form proposed in [38]. The Sobolev embedding shows that $F_p \in W^{1,2+\delta}(\overline{\omega}, \mathrm{GL}(3,\mathbb{R}))$ may be identified with a continuous function. A contradiction argument as in [32] shows that the constant is bounded away from zero since $W^{1,2+\delta}(\overline{\omega}, \mathrm{GL}(3,\mathbb{R}))$ is compactly embedded in $C(\overline{\omega}, \mathrm{GL}(3,\mathbb{R}))$. For details consult [29, 33]. ■

Continuing with our general development we observe that the solution m of (1.13) depends nonlinearly on $\overline{R}$. Despite this nonlinearity, we establish Lipschitz-continuous-dependence of the solution to (1.8) with respect to the data and coefficients $\overline{R}$, by looking at the weak problem (1.13) in the form (1.14) and using sharp elliptic estimates.

The conceptual idea to treat the nonlinear **coupled** viscoelastic evolution problem is straightforward: the ordinary differential equation may be written in the following form

$$\frac{\mathrm{d}}{\mathrm{dt}}\overline{R}(t) = \mathfrak{f}(F(\overline{R}), \overline{R}) \cdot \overline{R}, \tag{1.21}$$

with $\mathfrak{f} : \mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3} \mapsto \mathrm{Lin}(\mathbb{M}^{3\times3}, \mathbb{M}^{3\times3})$ where $F(\overline{R}) = (\nabla m(\overline{R})|\overline{R}_3)$. Here $m(\overline{R})$ is the solution of the elliptic boundary value problem (1.13) at given $\overline{R}$. It remains to show that the right hand side of (1.21) as a function of $\overline{R}$ is locally Lipschitz-continuous in appropriate spaces allowing to apply the local existence and uniqueness theorem for nonlinear evolution equations in Banach spaces based on Banach's fixed point theorem, cf. (6.2).

2 Local existence and uniqueness proof

2.1 First step: the static elastic subproblem

We have already indicated that in the static case for frozen variables $\overline{R}$ the elastic equilibrium system in (1.13) is a linear, strictly Legendre-Hadamard elliptic second order boundary value problem with non-constant coefficients and variational structure.[6] We exploit this structure and apply the direct methods of the calculus of variations to show that there exists a unique weak solution to (1.13) at frozen variables $\overline{R}$ which satisfies an additional uniform estimate.

Theorem 2.1 (Existence of minimizers)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\omega, \mathbb{R}^3)$ and for the rotations $\overline{R} \in W^{1,p}(\omega, \mathrm{SO}(3,\mathbb{R}))$, $p > 2$. Moreover, assume for the resultant body force

[6]This corresponds essentially to the **elastic trial step** in current algorithmic formulations of viscoplasticity.

$\overline{f} \in L^2(\omega, \mathbb{R}^3)$. *Then the variational problem*

$$
\begin{aligned}
& I(m, \overline{R}) \mapsto \min.\ \text{w.r.t. } m, \quad m \in g_{\mathrm{d}} + H_\circ^{1,2}(\omega, \mathbb{R}^3; \gamma_0)\,, \\
& I(m, \overline{R}) := \int_\omega h\, W(F, \overline{R}) - \langle \overline{f}, m \rangle \,\mathrm{d}\omega\,, \quad F = (\nabla m | \overline{R}_3), \quad \overline{U} = \overline{R}^T F\,, \qquad (2.1) \\
& W(F, \overline{R}) := \frac{\mu}{4}\, \|F^T \overline{R} + \overline{R}^T F - 2 1\!\!1\|^2 + \frac{\lambda^*}{8}\, \mathrm{tr}\left[F^T \overline{R} + \overline{R}^T F - 2 1\!\!1\right]^2 \\
& \qquad = \mu\, \|\,\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \frac{\lambda^*}{2} \mathrm{tr}\left[\overline{U} - 1\!\!1\right]^2, \quad \lambda^* = \frac{2\mu\lambda}{2\mu + \lambda}\,,
\end{aligned}
$$

admits at least one minimizing midsurface deformation $m \in H^1(\omega, \mathbb{R}^3)$.

Proof. With the prescription of g_{d} it is clear that $I(g_{\mathrm{d}}, \overline{R}) < \infty$. Consider any sequence of functions $m^k \in H^{1,2}(\omega, \mathbb{R}^2)$ for which the energy remains bounded. At face value, along the sequence, we only control certain mixed symmetric expressions in the reconstructed deformation gradient $(\nabla m_k | \overline{R}_3)$. Let us define $v_k \in H^{1,2}(\omega, \mathbb{R}^3)$ by $m^k = g_{\mathrm{d}} + (m^k - g_{\mathrm{d}}) = g_{\mathrm{d}} + v_k$. Then we have

$$
\begin{aligned}
\infty > I(m_k, \overline{R}) &= \int_\omega h\, W(\overline{U}_k) - \langle \overline{f}, m_k \rangle \,\mathrm{d}\omega \geq \int_\omega h\, W_{\mathrm{mp}}(\overline{U}_k) \,\mathrm{d}\omega - C \|m_k\|_{L^2(\omega)} \\
&\geq \int_\omega h\, \frac{\mu}{4} \|\overline{R}^T (\nabla m_k | \overline{R}_3) + (\nabla m_k | \overline{R}_3)^T \overline{R} - 2 1\!\!1\|^2 \,\mathrm{d}\omega - C\, \|m_k\|_{H^{1,2}(\omega)} \\
&= \int_\omega h\, \frac{\mu}{4} \|\overline{R}^T (\nabla m_k | \overline{R}_3) + (\nabla m_k | \overline{R}_3)^T \overline{R}\|^2 \\
&\quad - 4h\, \frac{\mu}{4} \mathrm{tr}\left[\overline{R}^T (\nabla m_k | \overline{R}_3) + (\nabla m_k | \overline{R}_3)^T \overline{R}\right] + 4\, h\, \frac{\mu}{4} \|1\!\!1\|^2 \,\mathrm{d}\omega - C\, \|m_k\|_{H^{1,2}(\omega)} \\
&\geq \int_\omega h\, \frac{\mu}{4}\, \|\overline{R}^T (\nabla m_k | 0) + (\nabla m_k | 0)^T \overline{R}\|^2 \,\mathrm{d}\omega - C_1\, \|m_k\|_{H^{1,2}(\omega)} + C_2 \qquad (2.2) \\
&\geq \int_\omega h\, \frac{\mu}{4}\, \underbrace{\|\overline{R}^T (\nabla v_k | 0) + (\nabla v_k | 0)^T \overline{R}\|^2}_{\text{combinations of derivatives}} \,\mathrm{d}\omega - C_1\, \|v_k\|_{H^{1,2}(\omega)} + C_2 \\
&\geq \frac{h\,\mu}{4}\, c_K^+\, \|v_k\|^2_{H^{1,2}(\omega)} - C_1\, \|v_k\|_{H^{1,2}(\omega)} + C_2\,,
\end{aligned}
$$

where we made use of the zero boundary conditions for v_k on γ_0 and applied the extended Korn's inequality Theorem 1.7 (note again that $\overline{R}^{-T} = \overline{R}$) yielding the positive constant c_K^+ for the continuous microrotation $\overline{R}$. We conclude that I is bounded below and that the sequence v_k is bounded in $H^1(\omega)$. Hence, m_k is bounded as well in $H^1(\omega)$.

Since I is bounded below, we can consider an infimizing sequence $m_k \in H^{1,2}(\omega, \mathbb{R}^3)$ with

$$
\lim_{k \to \infty} I(m_k, \overline{R}) = \inf_{m \in H^{1,2}(\omega, \mathbb{R}^3)} I(m, \overline{R})\,. \qquad (2.3)
$$

Due to the boundedness of m_k we may extract a subsequence, not relabelled, such that $m_k \rightharpoonup \tilde{m} \in H^1(\omega, \mathbb{R}^3)$.

Now we obtain that $\overline{U_k} = \overline{R}^T (\nabla m_k | \overline{R}_3) \rightharpoonup \tilde{\overline{U}} = \overline{R}^T (\nabla \tilde{m} | \overline{R}_3)$ by construction. Since the total energy is convex in $\overline{U}$ (indeed quadratic in the non-symmetric $\overline{U}$) we get

$$
\begin{aligned}
I(\tilde{m}, \overline{R}) &= \int_\omega h\, W(\tilde{\overline{U}}) - \langle \overline{f}, \tilde{m} \rangle \,\mathrm{d}\omega \leq \liminf_{k \to \infty} \int_\omega h\, W(\overline{U}_k) - \langle \overline{f}, m_k \rangle \,\mathrm{d}\omega \\
&= \lim_{k \to \infty} I(m_k, \overline{R})\,, \qquad (2.4)
\end{aligned}
$$

which implies that the weak limit $\tilde{m}$ is a minimizer. ■

Corollary 2.2 (Uniqueness of minimizers)
Let $\omega \subset \mathbb{R}^2$ *be a bounded Lipschitz domain and assume for the boundary data* $g_{\mathrm{d}} \in H^1(\omega, \mathbb{R}^3)$ *and* $\overline{R} \in W^{1,p}(\omega, \mathrm{SO}(3, \mathbb{R}))$, $p > 2$. *Moreover, let* $\overline{f} \in L^2(\omega, \mathbb{R}^3)$. *Then the variational problem (2.1) has a unique minimizing midsurface deformation* $m \in H^1(\omega, \mathbb{R}^3)$.

Proof. We show that the functional $I(m,\overline{R})$ is strictly convex w.r.t. $m \in H^{1,2}(\omega,\mathbb{R}^3)$. This can be seen by computing the second derivative of I. Since I is quadratic w.r.t. m the bilinear form induced by the second derivative is given for $\phi \in H^{1,2}(\omega,\mathbb{R}^3)$ by

$$\begin{aligned} D_m^2 I(m,\overline{R}).(\phi,\phi) &= \int_\omega h\,\frac{\mu}{2}\|(\nabla\phi|0)^T\overline{R} + \overline{R}^T(\nabla\phi|0)\|^2 + \frac{\lambda^*}{4}\operatorname{tr}\left[(\nabla\phi|0)^T\overline{R} + \overline{R}^T(\nabla\phi|0)\right]^2 \mathrm{d}\omega \\ &\geq \int_\omega h\,\frac{\mu}{2}\|(\nabla\phi|0)^T\overline{R} + \overline{R}^T(\nabla\phi|0)\|^2\,\mathrm{d}\omega\,. \end{aligned} \tag{2.5}$$

For the displacement problem we have zero boundary conditions for ϕ on γ_0. Hence, applying Theorem 1.7 yields uniform positivity. ■

Lemma 2.3 (Uniform Gårding-type estimate for the minimizer)
Let $\omega \subset \mathbb{R}^2$ be a bounded smooth domain and assume for the boundary data now $g_{\mathrm{d}} \in H^{3,2}(\omega,\mathbb{R}^3)$ and $\overline{R} \in \mathcal{M}$ with $\mathcal{M}$ defined in GA.3 below and order of elliptic regularity $k = 1$. Moreover, let $\overline{f} \in L^2(\omega,\mathbb{R}^3)$. Then the unique minimizing solution $m \in H^{1,2}(\omega)$ to (2.1) satisfies the (rough) estimate

$$\begin{aligned} &\exists\, C^+_{\mathcal{M}}(\|g_{\mathrm{d}}\|_{3,2,\omega}, \|\overline{f}\|_{2,\omega}) > 0 \quad \forall\,\overline{R} \in \mathcal{M} \\ &\|m\|_{1,2,\omega} \leq C^+_{\mathcal{M}}(\|g_{\mathrm{d}}\|_{3,2,\omega}, \|\overline{f}\|_{2,\omega}) \cdot \left(1 + \|g_{\mathrm{d}}\|_{3,2,\omega} + \|\overline{f}\|_{2,\omega}\right) \end{aligned} \tag{2.6}$$

and $C^+_{\mathcal{M}}(\|g_{\mathrm{d}}\|_{3,2,\omega}, \|\overline{f}\|_{2,\omega})$ is a continuous function of $\|g_{\mathrm{d}}\|_{3,2,\omega}$ and $\|\overline{f}\|_{2,\omega}$.

Proof. Idea: recall the estimates (2.2) of Theorem 2.1 which bounds m from above. With the assumptions on the coefficients $\overline{R}$ we have by Theorem 1.7 that the appearing constants in Theorem 2.1 are bounded independent of the coefficients for $\overline{R}$ bounded in $H^{2,2}(\omega)$; notably the constant c_K^+ is bounded away from zero in this case. The bound from above can be made explicit by taking as comparison function g_{d}.

Since we have to keep track of the appearing constants, however, we must proceed in more detail: Set $m = v + g_{\mathrm{d}}$ with $v \in H^{1,2}(\omega,\mathbb{R}^3)$ and let $F = (\nabla m|\overline{R}_3)$. To simplify notation we write ∇v for $(\nabla_{(x,y)}v|0)$. We have algebraically

$$\begin{aligned} W(F,\overline{R}) &= \frac{\mu}{4}\,\|\overline{R}^T F + F^T\overline{R} - 2\,1\!\!1\|^2 + \frac{\lambda^*}{8}\,\operatorname{tr}\left[\overline{R}^T F + F^T\overline{R} - 2\,1\!\!1\right]^2 \\ &\geq \frac{\mu}{4}\,\|\overline{R}^T\nabla v + \nabla v^T\overline{R}\|^2 - 2\,\mu\,\|\overline{R}^T\|^2\,\|\nabla v\|\,\|\nabla g_{\mathrm{d}}\| - 2\,\mu\sqrt{3}\,\|\overline{R}^T\|\,\|\nabla v\| + \\ &\quad \frac{\mu}{4}\,\|\overline{R}^T\nabla g_{\mathrm{d}} + \nabla g_{\mathrm{d}}^T\overline{R} - 21\!\!1\|^2. \end{aligned} \tag{2.7}$$

Integrating over ω and making use of Theorem 1.7 with $\overline{R},\overline{R}^T \in H^{2,2}(\omega,\mathrm{SO}(3,\mathbb{R})) \subset C^{0,\frac{1}{2}}(\overline{\omega})$ we get for all $m \in H^{1,2}(\omega,\mathbb{R}^3)$

$$\begin{aligned} \int_\omega h\;W(F,\overline{R}) - \langle\overline{f},m\rangle\,\mathrm{d}\omega &\geq \mu\;\underbrace{h\,c_K^+(\overline{R})\,\|v\|^2_{H^{1,2}(\omega)}}_{\text{extended 2D-Korn}} - 2\,\mu\,h\,\|\overline{R}^{-1}\|^2_\infty\,\|\nabla g_{\mathrm{d}}\|_\infty\|v\|_{H^{1,2}(\omega)} \\ &\quad - 2\,\mu\,h\,\sqrt{3}\|\overline{R}^T\|_\infty\,\|v\|_{H^{1,2}(\omega)} + \int_\omega h\;\frac{\mu}{4}\,\|\overline{R}^{-T}\nabla g_{\mathrm{d}}^T + \nabla g_{\mathrm{d}}\overline{R}^{-1} - 21\!\!1\|^2\;\mathrm{d}\omega \\ &\quad - \|\overline{f}\|_{L^2(\omega)}\,\left(\|v\|_{L^2(\omega)} + \|g_{\mathrm{d}}\|_{L^2(\omega)}\right)\,. \end{aligned} \tag{2.8}$$

Since m is a minimizer, we have by estimating from above and using $\langle X, 1\!\!1\rangle^2 \leq 3\|X\|^2$

$$\begin{aligned} &\int_\omega h\;W(F,\overline{R}) - \langle\overline{f},m\rangle\;\mathrm{d}\omega \leq \\ &\left(\frac{\mu}{4} + \frac{3\lambda^*}{8}\right)\int_\omega h\;\|\overline{R}^T\nabla g_{\mathrm{d}} + \nabla g_{\mathrm{d}}^T\overline{R} - 21\!\!1\|^2\;\mathrm{d}\omega + \|\overline{f}\|_{L^2(\omega)}\,\|g_{\mathrm{d}}\|_{L^2(\omega)} \\ &\leq \frac{\mu}{4}\int_\omega h\;\|\overline{R}^T\nabla g_{\mathrm{d}} + \nabla g_{\mathrm{d}}^T\overline{R} - 21\!\!1\|^2\;\mathrm{d}\omega + \\ &\quad \frac{3\lambda^*}{2}\,h\,|\omega|\left(\|\overline{R}^T\|^2_\infty\,\|\nabla g_{\mathrm{d}}\|^2_\infty + 2\sqrt{3}\|\overline{R}^T\|_\infty\,\|\nabla g_{\mathrm{d}}\|_\infty + 3\right) + \|\overline{f}\|_{L^2(\omega)}\,\|g_{\mathrm{d}}\|_{L^2(\omega)}. \end{aligned} \tag{2.9}$$

This implies together with estimate (2.8) (the term with $\frac{\mu}{4}h$ cancels and $h < 1$ without loss of generality) the inequality

$$
\begin{aligned}
&\frac{3\lambda^* h}{2}\,|\omega|\,\Big(\|\overline{R}^{-1}\|_\infty^2\,\|\nabla g_{\rm d}\|_\infty^2 + 2\sqrt{3}\|\overline{R}^T\|_\infty\,\|\nabla g_{\rm d}\|_\infty + 3\Big) + 2\,\|\overline{f}\|_{L^2(\omega)}\,\|g_{\rm d}\|_{L^2(\omega)} \geq \qquad (2.10)\\
&\mu\, h\, c_k^+(\overline{R})\,\|v\|^2_{H^{1,2}(\omega)} - 2\,\mu\, h\,\|\overline{R}^T\|_\infty^2\,\|\nabla g_{\rm d}\|_\infty \|v\|_{H^{1,2}(\omega)}\\
&\quad - 2\,\mu\, h\,\sqrt{3}\|\overline{R}^{-1}\|_\infty\,\|v\|_{H^{1,2}(\omega)} - \|\overline{f}\|_{L^2(\omega)}\,\|v\|_{2,\Omega}\\
&\geq \mu\, h\, c_k^+(\overline{R})\,\|v\|^2_{H^{1,2}(\omega)} - 2\mu\sqrt{3}\left(1 + \|\overline{R}^{-1}\|_\infty^2\right)\left[\|\overline{R}^{-1}\|_\infty + \|\nabla g_{\rm d}\|_\infty + \|\overline{f}\|_{L^2(\omega)}\right]\cdot \|v\|_{H^{1,2}(\omega)}\,.
\end{aligned}
$$

Hence a rough estimate yields

$$
\begin{aligned}
&5\,\lambda^*\, h\,|\omega|\,\left(1 + \|\overline{R}^T\|_\infty\,\|\nabla g_{\rm d}\|_\infty\right)^2 + 2\,\|\overline{f}\|_{L^2(\omega)}\,\|g_{\rm d}\|_{L^2(\omega)} \geq \qquad (2.11)\\
&\mu\, h\, c_K^+(\overline{R})\,\|v\|^2_{H^{1,2}(\omega)} - 5\,\mu\,\left(1 + \|\overline{R}^T\|_\infty^2\right)\left[\|\overline{R}^T\|_\infty + \|\nabla g_{\rm d}\|_\infty + \|\overline{f}\|_{L^2(\omega)}\right]\cdot\|v\|_{H^{1,2}(\omega)}\,.
\end{aligned}
$$

After further rearranging we get a quadratic inequality in $\|v\|_{H^{1,2}(\omega)}$

$$
\begin{aligned}
0 \geq\ &\|v\|^2_{H^{1,2}(\omega)} - \frac{5}{c_K^+(\overline{R})}\left(1 + \|\overline{R}^{-1}\|_\infty^2\right)\left[\|\overline{R}^{-1}\|_\infty + \|\nabla g_{\rm d}\|_\infty + \|\overline{f}\|_{L^2(\omega)}\right]\cdot\|v\|_{H^{1,2}(\omega)}\\
&- \frac{5\,\lambda^*\,|\omega|}{\mu\, c_K^+(\overline{R})}\left(1 + \|\overline{R}^{-1}\|_\infty\,\|\nabla g_{\rm d}\|_\infty\right)^2 - \frac{2}{\mu\, h\, c_K^+(\overline{R})}\,\|\overline{f}\|_{L^2(\omega)}\,\|g_{\rm d}\|_{L^2(\omega)}. \qquad (2.12)
\end{aligned}
$$

Since $0 \geq x^2 - bx - c \Rightarrow x \leq b + \sqrt{c}$, the former yields (with Young's inequality on f, g and $\sqrt{c_1^2 + c_2^2} \leq (c_1 + c_2)$ for positive constants c_1, c_2)

$$
\begin{aligned}
\|v\|_{H^{1,2}(\omega)} \leq\ &\left[\frac{5}{c_K^+(\overline{R})}\left(1 + \|\overline{R}^T\|_\infty^2\right) + \sqrt{\frac{5\,\lambda^*\,|\omega|}{\mu\, c_K^+(\overline{R})}}\,\frac{\left(1 + \|\overline{R}^T\|_\infty\,\|\nabla g_{\rm d}\|_\infty\right)}{\|\overline{R}^T\|_\infty + \|\nabla g_{\rm d}\|_\infty + \|\overline{f}\|_{L^2(\omega)}}\right. \qquad (2.13)\\
&\left.+ \frac{1}{\mu\, h\, c_K^+(\overline{R})}\,\frac{\|\overline{f}\|_{L^2(\omega)} + \|g_{\rm d}\|_{L^2(\omega)}}{\|\overline{R}^T\|_\infty + \|\nabla g_{\rm d}\|_\infty + \|\overline{f}\|_{L^2(\omega)}}\right]\cdot\left[\|\overline{R}^T\|_\infty + \|\nabla g_{\rm d}\|_\infty + \|\overline{f}\|_{L^2(\omega)}\right].
\end{aligned}
$$

Since $\|\overline{R}\| = \|\overline{R}^T\| = \sqrt{3}$ we obtain

$$
\begin{aligned}
\|v\|_{H^{1,2}(\omega)} \leq\ &\left[\frac{5\cdot 4}{c_K^+(\overline{R})} + \sqrt{\frac{5\,\lambda^*\,|\omega|}{\mu\, c_K^+(\overline{R})}}\,\frac{(1 + \sqrt{3}\,\|\nabla g_{\rm d}\|_\infty)}{\sqrt{3}} + \right.\\
&\left.\frac{1}{\sqrt{3}\,\mu\, h\, c_K^+(\overline{R})}\left(\|\overline{f}\|_{L^2(\omega)} + \|g_{\rm d}\|_{L^2(\omega)}\right)\right]\cdot\left[\sqrt{3} + \|\nabla g_{\rm d}\|_\infty + \|\overline{f}\|_{L^2(\omega)}\right]. \qquad (2.14)
\end{aligned}
$$

With the embedding $H^{m,2}(\omega) \hookrightarrow C^{m-\frac{n}{2}}(\overline{\omega})$ we get the estimate for v from which we obtain easily the desired estimate in terms of m. ■

2.2 Second step: higher regularity and continuous dependence

2.2.1 Definitions and assumptions

In order to simplify the investigation of the elliptic system (1.13) with respect to regularity and continuous dependence and to place it in a more general context we introduce the

Definition 2.4 (General assumption, GA)

GA.1 $\Omega \subset \mathbb{R}^n$ *is a bounded domain with smooth boundary and space dimension* n.

GA.2 We call $k \in \mathbb{N}$ *the order of elliptic regularity, and assume throughout that* $2\cdot(k+1) > n$.

GA.3 (Local boundedness of the elasticity tensor and part of the right hand side) There exists $K_1 > 0$

$$\mathbb{D}:\ \mathbb{M}^{3\times 3} \mapsto \mathrm{Lin}(\mathbb{M}^{3\times 3}, \mathbb{M}^{3\times 3}), \qquad V:\ \mathbb{M}^{3\times 3} \mapsto \mathbb{M}^{3\times 3},$$

$$\mathcal{M} := \{A : \Omega \mapsto \mathbb{M}^{3\times 3} \mid \|A\|_{k+1,2,\Omega} \le K_1\},$$
$$\exists\, C_{\mathcal{M}} : \quad \forall\, A \in \mathcal{M} : \quad \|\mathbb{D}(A)\|_{k+1,2,\Omega},\ \|V(A)\|_{k+1,2,\Omega} \le C_{\mathcal{M}}.$$

GA.4 (Uniform Legendre-Hadamard ellipticity on $\mathcal{M}$) For all $\xi \in \mathbb{R}^3$, $\eta \in \mathbb{R}^2$ it holds

$$\exists\, c^+_{e,\mathcal{M}} > 0 : \forall\, x \in \Omega : \forall\, A \in \mathcal{M} : \quad \langle \mathbb{D}(A(x)).(\xi \otimes \eta|0), (\xi \otimes \eta|0)\rangle \ge c^+_{e,\mathcal{M}} \cdot \|\xi\|^2_{\mathbb{R}^3}\, \|\eta\|^2_{\mathbb{R}^2}.$$

GA.5 (Local Lipschitz continuity)

$$\exists\, L_{\mathcal{M}} : \forall\, A, B \in \mathcal{M} : \quad \|\mathbb{D}(A) - \mathbb{D}(B)\|_{k+1,2,\Omega} \le L_{\mathcal{M}} \cdot \|A - B\|_{k+1,2,\Omega},$$
$$\exists\, L_{\mathcal{M}} : \forall\, A, B \in \mathcal{M} : \quad \|V(A) - V(B)\|_{k+1,2,\Omega} \le L_{\mathcal{M}} \cdot \|A - B\|_{k+1,2,\Omega}.$$

If (GA.1,GA.2,GA.3,GA.4,GA.5) holds we say that **GA** *holds. Note that condition GA.5 already implies GA.3 but for convenience GA.3 is stated separately.* ■

2.2.2 The difference of two solutions

The difference of two solutions m_A, m_B of (1.13) for different data (forces f_A, f_B, boundary displacement g_A, g_B and rotations A, B), is governed by the system

$$\begin{aligned} h \text{ Div } \mathbb{D}(A(x)).(\nabla(m_A - m_B)|0) &= h \text{ Div } (\mathbb{D}(B(x)) - \mathbb{D}(A(x)).(\nabla m_B|0)) \\ &\quad + f_A - f_B + h \text{ Div}(V(A) - V(B)), \\ (m_A - m_B)_{|\partial\omega} &= g_A - g_B. \end{aligned} \tag{2.15}$$

Therefore we investigate now the following general elliptic problem, where the data f, g do in general not coincide with the actual resultant body force $\overline{f}$ and the actual Dirichlet data g_{d}. We have

Lemma 2.5 (General linear system)
Let $\overline{R} \in H^{2,2}(\omega, \mathrm{SO}(3))$ be given and set $A = \overline{R}$. Suppose that $\mathbb{D}$ has the form postulated in Definition 1.3 and assume for the generalized Dirichlet boundary data $g \in H^{3,2}(\omega)$ and for some generalized body force $f \in L^2(\omega)$. Then the linear problem

$$\text{Div } \mathbb{D}(A).\nabla u = f, \quad u_{|\partial\omega} = g, \tag{2.16}$$

has a unique weak solution $u \in H^{1,2}(\omega)$.

Proof. The same ideas as in Theorem 2.1 and Corollary 2.2 carry over. As corresponding energy expression we have only to take

$$W_{\mathbb{D}}(F, \overline{R}) = \frac{\mu}{4}\, \|F^T\overline{R} + \overline{R}^T F\|^2 + \frac{\lambda^*}{8} \text{ tr } \left[F^T\overline{R} + \overline{R}^T F\right]^2. \qquad ■$$

Now we provide the specialization of the elliptic regularity result to the situation treated in Lemma 2.5.

Theorem 2.6 (Improved Hilbert space elliptic regularity with L^2-part)
Assume **GA** *and $A \in \mathcal{M}$. Consider the linear divergence form elliptic system*

$$\text{Div}\, \mathbb{D}(A).\nabla u = f(x), \quad u_{|\partial\Omega} = g(x). \tag{2.17}$$

Assume that (2.17) admits at least one weak solution $u \in H^{1,2}(\Omega)$ for all $g \in H^{k+2,2}(\Omega)$ and all $f \in H^{k,2}(\Omega)$. Then the following estimate is valid:

$$\|u\|_{k+2,2,\Omega} \le C^+(\Omega, \|\mathbb{D}(A)\|_{k+1,2,\Omega}) \cdot \left(\|g\|_{k+2,2,\Omega} + \|f\|_{k,2,\Omega} + \|u\|_{2,\Omega}\right), \tag{2.18}$$

and the appearing constant $C^+(\Omega, \|\mathbb{D}(A)\|_{k+1,2,\Omega})$ is uniform on $\mathcal{M}$.

Proof. The transformation $v = u - g$ allows us to consider

$$\operatorname{Div} \mathbb{D}(A).\nabla v = f(x) + \operatorname{Div} \mathbb{D}(A).\nabla g \;, \quad v_{|\partial\Omega} = 0. \tag{2.19}$$

If we apply Theorem 6.1 to (2.19) we get the estimate

$$\begin{aligned}
\|v\|_{k+2,2,\Omega} \leq & C^+(\Omega, c_e^+) P(\|\mathbb{D}(A)\|_{k+1,2,\Omega}) \left(\|\operatorname{Div} \mathbb{D}(A).\nabla g\|_{k,2,\Omega} + \|f\|_{k,2,\Omega} + \|v\|_{2,\Omega}\right) \\
\leq & C^+(\Omega, c_e^+) P(\|\mathbb{D}(A)\|_{k+1,2,\Omega}) \left(\|\mathbb{D}(A)\|_{k+1,2,\Omega} \|g\|_{k+2,2,\Omega} + \|f\|_{k,2,\Omega} + \|v\|_{2,\Omega}\right) \\
\leq & C^+(\Omega, c_e^+) P(\|\mathbb{D}(A)\|_{k+1,2,\Omega})[1 + \|\mathbb{D}(A)\|_{k+1,2,\Omega}] \\
& \left(\|g\|_{k+2,2,\Omega} + \|f\|_{k,2,\Omega} + \|u\|_{2,\Omega} + \|g\|_{2,\Omega}\right).
\end{aligned} \tag{2.20}$$

This yields for $u = v + g$

$$\begin{aligned}
\|u\|_{k+2,2,\Omega} \leq 2 & \left(1 + C^+(\Omega, c_e^+) P(\|\mathbb{D}(A)\|_{k+1,2,\Omega}) \, [1 + \|\mathbb{D}(A)\|_{k+1,2,\Omega}]\right) \\
& \left(\|g\|_{k+2,2,\Omega} + \|f\|_{k,2,\Omega} + \|u\|_{2,\Omega}\right).
\end{aligned} \tag{2.21}$$

Now take

$$C^+(\Omega, \|\mathbb{D}(A)\|_{k+1,2,\Omega}) = 2 \left(1 + C^+(\Omega, c_e^+) \, P(\|\mathbb{D}(A)\|_{k+1,2,\Omega}) \, [1 + \|\mathbb{D}(A)\|_{k+1,2,\Omega}]\right).$$

This ends the proof since $C^+(\Omega, c_e^+)$ is uniformly bounded above on $\mathcal{M}$ by (GA.4) and Theorem 6.1. ∎

Theorem 2.7 (Uniform estimates for bounded coefficients)
Assume **GA** *and $A \in \mathcal{M}$. Consider the linear divergence form elliptic system*

$$\operatorname{Div} \mathbb{D}(A).\nabla u = f(x) \;, \quad u_{|\partial\Omega} = g(x). \tag{2.22}$$

Assume that (2.22) has a unique weak solution $u \in H^{1,2}(\Omega)$ for all $g \in H^{k+2,2}(\Omega)$ and all $f \in H^{k,2}(\Omega)$. In addition assume that a **uniform** *Gårding type $L^2(\Omega)$-estimate on $\mathcal{M}$ is available, i.e.,*

$$\exists\, C_{\mathcal{M}} > 0 : \quad \forall\, A \in \mathcal{M} : \quad \|u\|_{2,\Omega} \leq C_{\mathcal{M}} \cdot \left(\|g\|_{k_1+2,2,\Omega} + \|f\|_{k_2,2,\Omega}\right), \tag{2.23}$$

with $\max(k_1, k_2) \leq k$. Then the following uniform estimate is true:

$$\|u\|_{k+2,2,\Omega} \leq C^+(\Omega, \mathcal{M}) \cdot \left(\|g\|_{k+2,2,\Omega} + \|f\|_{k,2,\Omega}\right), \tag{2.24}$$

and the appearing constant $C^+(\Omega, \mathcal{M})$ is uniform on $\mathcal{M}$.

Proof. An application of Theorem 2.6 will give the result. ∎

Theorem 2.8 (Lipschitz-continuous dependence of solutions)
Assume **GA** *and let $A, B \in \mathcal{M}$. Assume for the boundary data $g_A, g_B \in H^{k+2,2}(\Omega)$ and for the body forces $f_A, f_B \in H^{k,2}(\Omega)$. Consider the two systems*

$$\begin{aligned}
\operatorname{Div}\; \mathbb{D}(A(x)).\nabla u &= f_A(x) + \operatorname{Div} V(A) & \qquad \operatorname{Div}\; \mathbb{D}(B(x)).\nabla u &= f_B(x) + \operatorname{Div} V(B) \\
u_{|\partial\Omega} &= g_A(x) & u_{|\partial\Omega} &= g_B(x).
\end{aligned} \tag{2.25}$$

Assume that both systems verify the assumptions made in Theorem 2.7. Denote the (unique) solutions $u_A, u_B \in H^{1,2}(\Omega)$, respectively. Then the following estimate holds:

$$\begin{aligned}
\|u_A - u_B\|_{k+2,2,\Omega} \leq & C^+(\Omega, \mathcal{M}) \cdot \left(1 + \|B\|_{k+1,2,\Omega} + \|g_B\|_{k+2,2,\Omega} + \|f_B\|_{k,2,\Omega}\right) \\
& \left(\|A - B\|_{k+1,2,\Omega} + \|g_A - g_B\|_{k+2,2,\Omega} + \|f_A - f_B\|_{k,2,\Omega}\right),
\end{aligned} \tag{2.26}$$

with $C^+(\Omega, \mathcal{M})$ uniformly bounded on $\mathcal{M}$.

Proof. Consider

$$\begin{aligned}\text{Div }\mathbb{D}(A(x)).\nabla u_A &= f_A(x) + \text{Div }V(A) \qquad & \text{Div }\mathbb{D}(B(x)).\nabla u_B &= f_B(x) + \text{Div }V(B)\\ u_{A|\partial\Omega} &= g_A(x) & u_{B|\partial\Omega} &= g_B(x). \end{aligned} \tag{2.27}$$

Taking the difference of the two equations leads us to consider

$$\begin{aligned}\text{Div }\mathbb{D}(A(x)).\nabla(u_A - u_B) &= \text{Div }(\mathbb{D}(B(x)) - \mathbb{D}(A(x)).\nabla u_B) + f_A - f_B + \text{Div}(V(A) - V(B))\\ (u_A - u_B)_{|\partial\Omega} &= g_A - g_B. \end{aligned} \tag{2.28}$$

By the assumption on A and the elasticity tensor $\mathbb{D}(A)$ we know that the system (2.28) has a unique solution $(u_A - u_B)$. Together with the regularity assumption made for A and $\mathbb{D}(A)$ in GA we can apply Theorem 2.7 to (2.28) and get the estimate

$$\begin{aligned}\|u_A - u_B\|_{k+2,2,\Omega} &\le C^+(\Omega,\mathcal{M})\cdot\Big(\|\text{ Div}(\mathbb{D}(B) - \mathbb{D}(A)).\nabla u_B\|_{k,2,\Omega} + \|\text{ Div}(V(B) - V(A))\|_{k,2,\Omega} +\\ &\qquad \|g_A - g_B\|_{k+2,2,\Omega} + \|f_A - f_B\|_{k,2,\Omega}\Big)\\ &\le C^+(\Omega,\mathcal{M})\cdot\Big(\|\mathbb{D}(A) - \mathbb{D}(B)\|_{k+1,2,\Omega}\cdot\|u_B\|_{k+2,2,\Omega} + \|V(B) - V(A)\|_{k+1,2,\Omega} +\\ &\qquad \|g_A - g_B\|_{k+2,2,\Omega} + \|f_A - f_B\|_{k,2,\Omega}\Big). \end{aligned} \tag{2.29}$$

Again with Theorem 2.7 applied to the solution u_B we have

$$\|u_B\|_{k+2,2,\Omega} \le C^+(\Omega,\mathcal{M})\cdot\Big(\|g_B\|_{k+2,2,\Omega} + \|f_B\|_{k,2,\Omega} + \|V(B)\|_{k+1,2,\Omega}\Big). \tag{2.30}$$

Combining these two estimates and using (GA.5) for $\mathbb{D}, V$ ends the argument. ■

Corollary 2.9 (Lipschitz-continuous solution operator; time dependent coefficients) *Assume that for a given family of coefficients $\mathfrak{M} := \{A_t \in \mathcal{M}|\ t > 0\}$, the family of related elasticity tensors $\mathbb{D}(A_t)$ verifies all conditions of Theorem 2.7. For given constants $K_1, K_2, K_3 > 0$ define the set of admissible boundary data $\mathfrak{G} := \{g \in H^{k+2,2}(\Omega)|\ \|g\|_{k+2,2,\Omega} \le K_2\}$ and the set of admissible body loads $\mathfrak{F} := \{f \in H^{k,2}(\Omega)|\ \|f\|_{k,2,\Omega} \le K_3\}$. Let the boundary data $g_t \in \mathfrak{G}$ and the body forces $f_t \in \mathfrak{F}$ be given. Then the family of corresponding linear elliptic systems (parametrized by $t \in \mathbb{R}$)*

$$\text{Div }\mathbb{D}(A_t).\nabla\varphi_t = f_t(x) + \text{Div }V(A_t)\ , \quad \varphi_{t|\partial\Omega} = g_t(x). \tag{2.31}$$

allows for a Lipschitz-continuous solution operator T on $\mathfrak{M}\times\mathfrak{G}\times\mathfrak{F}$ such that $\varphi_t = T(A_t, g_t, f_t)$ and

$$\begin{aligned}\|T(A, g_A, f_A) - T(B, g_B, f_B)\|_{k+2,2,\Omega} \le &C^+(\Omega,\mathcal{M})\cdot\Big(1 + \|B\|_{k+1,2,\Omega} + \|g_B\|_{k+2,2,\Omega} + \|f_B\|_{k,2,\Omega}\Big)\\ &\Big(\|A - B\|_{k+1,2,\Omega} + \|g_A - g_B\|_{k+2,2,\Omega} + \|f_A - f_B\|_{k,2,\Omega}\Big)\ , \end{aligned} \tag{2.32}$$

for $A, B \in \mathfrak{M},\ g_A, g_B \in \mathfrak{G},\ f_A, f_B \in \mathfrak{F}$. The corresponding Lipschitz constant L^+ on $\mathfrak{M}\times\mathfrak{G}\times\mathfrak{F}$ has the form

$$L^+ = C^+(\Omega,\mathcal{M})\cdot\Big(1 + \|B\|_{k+1,2,\Omega} + \|g_B\|_{k+2,2,\Omega} + \|f_B\|_{k,2,\Omega}\Big)\ . \tag{2.33}$$

On $\mathfrak{M}\times\mathfrak{G}\times\mathfrak{F}$ the Lipschitz-constant is uniformly bounded by

$$L^+ = C^+(\Omega,\mathcal{M})\,(1 + K_1 + K_2 + K_3)\,. \tag{2.34}$$

Hence a family of elliptic systems of the above type has corresponding solution operators with uniform Lipschitz-constant whenever $\|A\|_{k+1,2,\Omega}, \|g_A\|_{k+2,2,\Omega}, \|f_A\|_{k,2,\Omega}$ are bounded due to Theorem 2.7. ■

Remark 2.10 (Nonlinear solution operator)
Let $A_t \in \mathcal{M}$ and f_t, g_t as before. Then the mapping $(f_t, g_t) \mapsto T(A_0, g_t, f_t)$ is linear while the mapping $A_t \mapsto T(A_t, g_0, f_0)$ is nonlinear. Hence the solution depends nonlinearly on the (time dependent) elasticity tensor although the problem is linear for frozen (fixed at time t_0) elasticity tensor $\mathbb{D}(A_{t_0})$. ■

The previous development has been fairly general. Therefore, in the final part of the proof we specialize Ω to $\omega \subset \mathbb{R}^2$ in Definition 2.4 and set $n = 2$.

2.3 Third step: the coupled nonlinear viscoelastic evolution problem

In this final part of the proof we consider the coupled viscoelastic evolution problem. The coupled problem (1.1,1.2,1.3) is formally equivalent to

$$\frac{\mathrm{d}}{\mathrm{dt}}\overline{R}(t) = \mathfrak{f}\big(\nabla_x T(\overline{R}(t), g_{\mathrm{d}}(t), \overline{f}(t)), \overline{R}(t)\big) \cdot \overline{R}(t)\,, \tag{2.35}$$

with

$$\mathfrak{f} : \mathbb{M}^{3\times 3} \times \mathbb{M}^{3\times 3} \mapsto \mathrm{Lin}(\mathbb{M}^{3\times 3}, \mathbb{M}^{3\times 3})\,, \qquad \mathfrak{f}\big(F, \overline{R}(t)\big) = \nu^+ \cdot \mathrm{skew}(B^{\mathrm{res}}(t)) \in \mathfrak{so}(3, \mathbb{R})\,, \tag{2.36}$$

where B^{res}, defined in (1.3), is an expression depending on $\overline{R}$ and on the (reduced) reconstructed deformation gradient $F = (\nabla m|\overline{R}_3) = (\nabla_x T(\overline{R}, g_{\mathrm{d}}, \overline{f})|\overline{R}_3)$. Here $T(\overline{R}, g_{\mathrm{d}}, \overline{f})$ is, at this stage, formally defined to be the solution operator of the static equilibrium part (1.13) in (1.1,1.2,1.3).

The choice for ν^+ in (1.7) implies that $\mathfrak{f} \in C^3(\mathbb{M}^{3\times 3} \times \mathbb{M}^{3\times 3}, \mathrm{Lin}(\mathbb{M}^{3\times 3}, \mathbb{M}^{3\times 3}))$, considered pointwise.

Remark 2.11 (Flow rule on Sobolev space)
Set $M := \{v \in H^{k+1,2}(\omega) |\ \|v\|_{k+1,2,\omega} \leq K\}$. Then due to Sobolev's embedding theorem it is easy to see that for $\mathfrak{f} \in C^{k+2}(\mathbb{M}^{3\times 3} \times \mathbb{M}^{3\times 3}, \mathbb{M}^{9\times 9})$ and forall $v_1, v_2 \in M$ the estimate

$$\|\mathfrak{f}(v_1) - \mathfrak{f}(v_2)\|_{k+1,2,\omega} \leq \sup_{\|\xi\|\leq \tilde{K}} \|\mathfrak{f}(\xi)\|_{C^{k+2}(\mathbb{R}^{27}, \mathbb{M}^{9\times 9})} \cdot C^+(\omega, M) \cdot \|v_1 - v_2\|_{k+1,2,\omega}\,, \tag{2.37}$$

holds. ■

It remains to identify the precise spaces on which to consider this evolution problem in the framework of a local existence and uniqueness result for ordinary differential equations in Banach-spaces, cf. Theorem 6.2. We let

$$\widehat{U} := H^{2,2}(\omega, \mathrm{GL}(3, \mathbb{R})), \quad X := H^{2,2}(\omega, \mathrm{SO}(3, \mathbb{R})) \tag{2.38}$$

and set $Y := H^{3,2}(\omega, \mathbb{R}^3)$ and $Z := H^{1,2}(\omega, \mathbb{R}^3)$. Assume that $A^0 = \overline{R}^0 \in X$ is given and for positive constants K_1, K_2, K_3 let

$$\begin{aligned} \mathcal{M} &:= \{A \in X |\ \|A - A^0\|_{2,2,\omega} \leq K_1\}\,, \quad \mathcal{Y} := \{y \in Y |\ \|y\|_{3,2,\omega} \leq K_2\}\,, \\ \mathcal{Z} &:= \{z \in Z |\ \|z\|_{1,2,\omega} \leq K_3\}. \end{aligned} \tag{2.39}$$

Observe that by construction of the flow rule $\frac{\mathrm{d}}{\mathrm{dt}}\overline{R}(t) = \mathfrak{X}_2 \cdot \overline{R}(t)$ with $\mathfrak{X}_2 \in \mathfrak{so}(3)$ we know a priori that $\overline{R}(x,t) \in \mathrm{SO}(3, \mathbb{R})$. Assume for the Dirichlet boundary data $g_{\mathrm{d}} \in C^1([0,T], \mathcal{Y})$ and for the resultant body forces $\overline{f} \in C^1([0,T], \mathcal{Z})$. In view of the specifications of spaces and data we show presently that the nonlinear, infinite-dimensional evolution problem

$$\frac{\mathrm{d}}{\mathrm{dt}}\overline{R}(t) = \mathfrak{f}\big(\nabla_x T(\overline{R}(t), g_{\mathrm{d}}(t), \overline{f}(t)), \overline{R}(t)\big) \cdot \overline{R}(t)\,, \tag{2.40}$$

fits into the formal framework set forth in Theorem 6.2.

First we proceed to show that it is possible to define a solution operator $m = T(\overline{R}, g_{\mathrm{d}}, \overline{f})$ to the static equilibrium part (1.13) of (1.1,1.2,1.3) and that this operator is indeed Lipschitz-continuous on the bounded set $\mathcal{M} \times \mathcal{Y} \times \mathcal{Z}$. We have

Lemma 2.12 (Existence of solution operator T)
For given local rotation $\overline{R} \in H^{2,2}(\omega, \mathrm{SO}(3, \mathbb{R}))$, Dirichlet boundary data $g_{\mathrm{d}} \in H^{3,2}(\omega, \mathbb{R}^3)$ and resultant body force $\overline{f} \in H^{1,2}(\omega, \mathbb{R}^3)$ the elliptic problem (1.13) admits an operator T with

$$T : H^{2,2}(\omega, \mathrm{SO}(3, \mathbb{R})) \times H^{3,2}(\omega, \mathbb{R}^3) \times H^{1,2}(\omega, \mathbb{R}^3) \mapsto H^{3,2}(\omega, \mathbb{R}^3) \tag{2.41}$$

such that $m = T(\overline{R}, g_{\mathrm{d}}, \overline{f})$ is the unique solution to (1.13).

Proof. Due to Theorem 2.1 and Corollary 1.2 we know that solutions $m = m(\overline{R}, g_{\rm d}, \overline{f})$ of (1.13) exist. With Definition 1.3 it is obvious that $\mathbb{D}, V \in C^\infty$. Remark 2.11 shows that (GA.3) and (GA.5) are satisfied for $\mathbb{D}, V$ on $\mathcal{M}$. Moreover, by Corollary 1.4 we see that (GA.4) is true. If we choose the order of elliptic regularity $k = 1$ for the space dimension $n = 2$ then (GA.2) holds as well. The domain $\omega \subset \mathbb{R}^2$ has smooth boundary, therefore (GA.1) holds. Theorem 2.2 may therefore be applied and shows that the solutions of the boundary value problem (1.13) are unique, which establishes existence of the solution operator.

Now the asserted regularity part: Lemma 2.3 proves a uniform $H^{1,2}(\omega)$ estimate for the solution m if $g_{\rm d} \in \mathcal{Y}$, $\overline{f} \in \mathcal{Z}$ on $\mathcal{M}$. With Lemma 2.5 we make sure that the assumptions needed for elliptic regularity in Theorem 2.7 are verified. Hence Theorem 2.7 establishes higher regularity if the data are smooth; for $k = 1$ we obtain $H^{3,2}(\omega, \mathbb{R}^3)$. ■

Lemma 2.13 (Lipschitz continuity of solution operator T)
Under the same assumptions as in Lemma 2.12 the solution operator T is uniformly Lipschitz-continuous on the bounded set

$$\mathcal{M} \times \mathcal{Y} \times \mathcal{Z} \subset H^{2,2}(\omega, \mathrm{SO}(3, \mathbb{R})) \times H^{3,2}(\omega, \mathbb{R}^3) \times H^{1,2}(\omega, \mathbb{R}^3)\,. \tag{2.42}$$

Proof. Taking into account Lemma 2.12 we are entitled to apply Theorem 2.8 and Corollary 2.9. This shows

$$\begin{aligned} &\|T(A, g_A, f_A) - T(B, g_B, f_B)\|_{k+2,2,\omega} \le \\ &\quad C^+(\omega, \mathcal{M}) \cdot \Big(1 + \|B\|_{k+1,2,\omega} + \|g_B\|_{k+2,2,\omega} + \|f_B\|_{k+2,2,\omega}\Big) \\ &\qquad \Big(\|A - B\|_{k+1,2,\omega} + \|g_A - g_B\|_{k+2,2,\omega} + \|f_A - f_B\|_{k,2,\omega}\Big)\,. \end{aligned} \tag{2.43}$$

Hence, $T(\overline{R}, g_{\rm d}, \overline{f})$ is a Lipschitz continuous operator with uniform Lipschitz constant L^+ on $\mathcal{M} \times \mathcal{Y} \times \mathcal{Z}$. ■

By restricting the former estimate on T to the first gradient of T we obtain

Corollary 2.14 (Lipschitz continuity for the gradient of T)
The gradient $\nabla_x T(\overline{R}, g_{\rm d}, \overline{f})$ satisfies a similar uniform Lipschitz estimate as T does, namely

$$\begin{aligned} &\|\nabla_x T(A, g_A, f_A) - \nabla_x T(B, g_B, f_B)\|_{k+1,2,\omega} \le \\ &\quad C^+(\omega, \mathcal{M}) \cdot \Big(1 + \|B\|_{k+1,2,\omega} + \|g_B\|_{k+2,2,\omega} + \|f_B\|_{k+2,2,\omega}\Big) \\ &\qquad \Big(\|A - B\|_{k+1,2,\omega} + \|g_A - g_B\|_{k+2,2,\omega} + \|f_A - f_B\|_{k,2,\omega}\Big)\,. \end{aligned} \tag{2.44}$$ ■

Hence on $\mathcal{M} \times \mathcal{Y} \times \mathcal{Z}$ we get

$$\begin{aligned} &\|\nabla_x T(A, g_A, f_A) - \nabla_x T(B, g_B, f_B)\|_{k+1,2,\omega} \le \\ &\quad C^+(\omega, \mathcal{M}) \cdot (1 + K_1 + K_3)\, \Big(\|A - B\|_{k+1,2,\omega} + \|g_A - g_B\|_{k+2,2,\omega} + \|f_A - f_B\|_{k,2,\omega}\Big)\,. \end{aligned} \tag{2.45}$$

This is enough to see that the operator $G(\overline{R}, g_{\rm d}, \overline{f}) := \nabla_x T(\overline{R}, g_{\rm d}, \overline{f})$ satisfies the assumptions of Theorem 6.2.

Moreover, Remark 2.11 applied to $\mathfrak{f} \in C^3(\mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3}, \mathbb{M}^{6\times6})$ shows that $\mathfrak{f}$, viewed as a function $\mathfrak{f} : U \times X \mapsto \mathrm{Lin}(X, X)$ is locally Lipschitz-continuous on $\mathcal{M}$. Therefore, we may finally apply Theorem 6.2 giving us a unique local in time solution $\overline{R} \in C^1([0, t_1], \mathcal{M})$ to the ordinary differential system of equations (2.35). Since $m(t) = T(\overline{R}(t), g_{\rm d}(t), \overline{f}(t))$, the pair

$$(m, \overline{R}) \in C([0, t_1], H^{3,2}(\omega, \mathbb{R}^3)) \times C^1([0, t_1], H^{2,2}(\omega, \mathrm{SO}(3)))\,, \tag{2.46}$$

is the unique local in time solution of (1.1,1.2,1.3). Thus we have finally proved Theorem 1.1. ■

3 A glimpse on the modelling

3.1 The non-elliptic relaxation limit

In [34] it is shown that due to the underlying isotropy the resulting nonlinear membrane-plate model (1.1,1.2,1.3) with $B = B^{\mathrm{res},0}_{\mathrm{mech}}$ approaches in the equilibrium limit $\nu^+ \to \infty$ (vanishing elastic viscosity = zero relaxation limit $\eta \to 0$ viz. for arbitrary slow processes) formally the **intrinsic, purely elastic**[7] membrane-plate problem

$$\int_\omega h\, W_\infty(U((\nabla m|\vec{n}))) - \langle \overline{f}, m\rangle \, \mathrm{d}\omega \mapsto \text{ stat . w.r.t. } m \in g_{\mathrm{d}} + H^{1,2}_{\circ}(\omega, \mathbb{R}^3; \gamma_0)\,, \tag{3.47}$$

where

$$W_\infty(U) := \mu\, \|U - 1\!\!1\|^2 + \frac{\mu\lambda}{(2\mu+\lambda)}\, \mathrm{tr}\, [U - 1\!\!1]^2\,, \quad \widehat{F} = (\nabla m|\vec{n}_m)\,, \tag{3.48}$$

with $U = (\widehat{F}^T\widehat{F})^{\frac{1}{2}} = R^T F$ the **symmetric** elastic stretch, $U - 1\!\!1$ the elastic Biot strain tensor and $\vec{n}_m$ the unit normal on the parametrized surface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$. The system (3.47) is a geometrically exact equilibrium membrane-plate model for small elastic strains and finite deformations in the classical sense with no extra internal dissipation. The transition from (1.1,1.2,1.3) to (3.47) however, is not entirely trivial since it is not just the replacement of the independent viscoelastic rotation $\overline{R}$ in (1.1,1.2,1.3) by the continuum rotation $R = \mathrm{polar}(\widehat{F})$ in (3.47). Moreover we must note the subtle change from **global minimization** in (1.1,1.2,1.3) to a **stationarity** requirement only in (3.47).

Note as well that the equilibrium energy $W_\infty(U)$ is a **non-quasiconvex, non-elliptic** elastic energy w.r.t. ∇m but convex in the symmetric continuum stretch U, **satisfying** in fact the **Baker-Ericksen inequalities**. Currently there are no mathematical theorems available establishing the existence of minimizers based directly on W_∞. In this sense, the viscoelastic formulation (1.1,1.2,1.3) provides a **physical regularization** of the occurring **loss of ellipticity** in (3.47). Up to a different strain measure ($U = \sqrt{\widehat{F}^T\widehat{F}}$ instead of $\overline{C} = \widehat{F}^T\widehat{F}$), the model (3.47) coincides with (3.50).

In order to put the new model into some perspective, let us consider a formal linearization.

3.2 Partial linearization for the thin viscoelastic membrane plate

To put our modelling development into perspective, we simplify (1.1,1.2,1.3) further by writing $m(x, y) = (x, y, 0)^T + v(x, y)^T$, where v is the displacement of the midsurface and assume for the viscoelastic rotations $\overline{R}(x, y) = \exp(\overline{A}(x, y))$ with $\overline{A} \in \mathfrak{so}(3, \mathbb{R})$ small.

Expanding (1.1,1.2,1.3) yields to leading order in $\overline{A}$ the following set of equations for the displacement of the midsurface of the plate $v : [0, T] \times \overline{\omega} \mapsto \mathbb{R}^3$ and the skew part $\overline{A} : [0, T] \times \overline{\omega} \mapsto \mathfrak{so}(3, \mathbb{R})$:

$$\int_\omega h\, W_{\mathrm{lin}}(\nabla v, \overline{A}) - \langle \overline{f}, v\rangle \, \mathrm{d}\omega \mapsto \min . \text{ w.r.t. } v \text{ at fixed } \overline{A}\,, \tag{3.49}$$

$$W_{\mathrm{lin}}(\nabla v, \overline{A}) = \mu\, \|\, \mathrm{sym}((\nabla v|\overline{A}_3) + \overline{A}^T(\nabla v|0))\|^2 + \frac{\mu\lambda}{(2\mu+\lambda)}\, \mathrm{tr}\left[\mathrm{sym}((\nabla v|\overline{A}_3) + \overline{A}^T(\nabla v|0))\right]^2,$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\overline{A}(t) = -\nu^+\overline{A} + \nu^+\, \mathrm{skew}\left((\nabla v|\overline{A}_3) + \overline{A}^T(\nabla v|0)\right)\,,$$

where the evolution equation is linear in $\overline{A}$ but the coupled model is nonlinear due to the presence of the multiplicative term $\overline{A}^T(\nabla v|0)$. Note that we have not assumed that ∇v is small since an expansion to first order in ∇v leaves v indetermined in general, due to possible infinitesimal bending modes, in which case the classical infinitesimal bending plate (Kirchhoff plate) equations can be used.

We observe that for $\overline{A} = 0$ and in the absence of external forces the elasticity part alone decouples into pure in-plane deformation (to which μ and λ contribute) and pure transverse displacement. The transverse displacement $v_3(x, y)$ is then simply determined through $\Delta v_3 = 0$,

[7]intrinsic: only depending on the first fundamental form of the surface m.

i.e. like the static elastic membrane of the classical theory. For $\overline{A} = 0$ the elastic problem has **constant coefficients** and is coercive on account of the standard Korn's inequality [6]. In the case that $\overline{A} = 0$ and only vertical body forces $\hat{f} = (0, 0, f_3)$ are present, the problem reduces to $(v_1, v_2) = (0, 0)$ and for the vertical deflection $\mu\,\Delta v_3 = f_3$. We wish to emphasize that getting a membrane problem for the thin plate is a classical fact [7, p.356]: "...a thin nonlinearly elastic body submitted to its own weight does not behave like a (bending) plate, but indeed like a membrane."

In order to relate our development to existing geometrically exact membrane formulations we present two alternative propositions from the literature adapted to our notation.

3.3 The finite-strain membrane model of Fox/Simo

In [21] the following geometrically exact, frame-indifferent membrane model has been derived by formal asymptotic analysis based on the St. Venant-Kirchhoff energy. In a variational form the model can be written in our notation in the form of a minimization problem for the deformation of the midsurface of the membrane $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ on ω:

$$
\begin{aligned}
&\int_\omega h\, W_{\mathrm{mp}}(\overline{C})\, \mathrm{d}\omega - \Pi(m, \vec{n}_m) \mapsto \min . \text{ w.r.t. } m\,, \quad m_{|_{\gamma_0}} = g_{\mathrm{d}}(x, y, 0)\,,\\
&\overline{C} = \widehat{F}^T \widehat{F}, \quad \widehat{F} = (\nabla m | \vec{n}_m), \quad F_s = (\nabla m | \varrho_m\, \vec{n}_m)\,, \qquad (3.50)\\
&\varrho_m = \frac{\langle N_{\mathrm{diff}}, \vec{n}_m\rangle}{(2\mu + \lambda)} + \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)} \mathrm{tr}\left[\overline{C} - 1\!\!1\right] + \frac{\langle N_{\mathrm{diff}}, \vec{n}_m\rangle^2}{(2\mu+\lambda)^2}}\,, \quad \text{first order thickness stretch}\,,\\
&W_{\mathrm{mp}}(\overline{C}) = \frac{\mu}{4}\, \|\overline{C} - 1\!\!1\|^2 + \frac{2\mu\lambda}{8(2\mu+\lambda)}\, \mathrm{tr}\left[\overline{C} - 1\!\!1\right]^2\\
&\qquad = \frac{\mu}{4}\, \|\nabla m^T \nabla m - 1\!\!1_2\|^2 + \frac{2\mu\lambda}{8(2\mu+\lambda)}\, \mathrm{tr}\left[\nabla m^T \nabla m - 1\!\!1_2\right]^2\\
&\qquad = \frac{\mu}{4}\, \|I_m - 1\!\!1_2\|^2 + \frac{2\mu\lambda}{8(2\mu+\lambda)}\, \mathrm{tr}\left[I_m - 1\!\!1_2\right]^2, \quad I_m = \nabla m^T \nabla m\text{: } \textbf{first fundamental form}\,.
\end{aligned}
$$

The reconstructed membrane deformation $\varphi_s(x, y, z) = m(x, y) + z\varrho_m\, \vec{n}_m$ yields the plane stress condition $S_1(\nabla\varphi_s(x, y, 0).e_3 = 0$, which is only consistent with three-dimensional equilibrium if there are no normal tractions at the transverse boundary and indeed, in [21, p.176] it is assumed that $N_{\mathrm{diff}} \equiv 0$, for otherwise, formal asymptotic expansion is impossible. In this case we have the identity

$$
W_{\mathrm{mp}}(\overline{C}) = \frac{\mu}{4}\, \|F_s^T F_s - 1\!\!1\|^2 + \frac{\lambda}{8}\, \mathrm{tr}\left[F_s^T F_s - 1\!\!1\right]^2, \quad \overline{C} = \widehat{F}^T \widehat{F}\,. \qquad (3.51)
$$

It is easily seen that the resultant membrane strain energy $W_{\mathrm{mp}}(\overline{C})$ is neither quasiconvex nor Legendre-Hadamard elliptic. Moreover, the resultant membrane strain energy density **does not satisfy the Baker-Ericksen inequalities** in contrast to the equilibrium model (3.47).

3.4 The finite-strain, quasiconvex membrane model of Le Dret/Raoult

By means of Γ-convergence arguments based on the St. Venant-Kirchhoff energy and a natural scaling assumptions LeDret and Raoult [16] derive the following quasiconvex geometrically exact, frame-indifferent minimization problem which is, however, degenerate in compression. The membrane deformation $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ satisfies on ω:

$$
\begin{aligned}
&\int_\omega h\, QW_0(\nabla m)\, \mathrm{d}\omega - \Pi(m, \vec{n}_m) \mapsto \min . \text{ w.r.t. } m\,, \quad m_{|_{\gamma_0}} = g_{\mathrm{d}}(x, y, 0)\,, \qquad (3.52)\\
&W_0(\nabla m) := \inf_{\eta \in \mathbb{R}^3} W((\nabla m|\eta)^T (\nabla m|\eta))\,, \quad W(C) = \frac{\mu}{4}\, \|C - 1\!\!1\|^2 + \frac{\lambda}{8}\, \mathrm{tr}\left[C - 1\!\!1\right]^2,\\
&\widehat{\varrho}_m := \begin{cases} \varrho_m & 1 - \frac{\lambda}{(2\mu+\lambda)}\left[\|\nabla m\|^2 - 2\right] \geq 0\,, \quad (\nabla m | \widehat{\varrho}_m \vec{n}) \in \mathrm{GL}^+(3, \mathbb{R}) \\ 0 & 1 - \frac{\lambda}{(2\mu+\lambda)}\left[\|\nabla m\|^2 - 2\right] < 0\,, \quad (\nabla m | \widehat{\varrho}_m \vec{n}) \notin \mathrm{GL}^+(3, \mathbb{R}) \end{cases} \quad \text{implies}\\
&W_0(\nabla m) = W((\nabla m | \widehat{\varrho}_m \vec{n}_m)^T (\nabla m | \widehat{\varrho}_m \vec{n}_m)) = W_{\mathrm{mp}}(\overline{C}) \quad \text{if} \quad \widehat{\varrho}_m = \varrho_m\,,
\end{aligned}
$$

with the definition of $\overline{C}$, ϱ_m and W_{mp} given in (3.50). QW_0 denotes the quasiconvex hull of W_0 which can be determined analytically showing the degenerate feature that $QW_0 = 0$ in uniform compression. In compression, this model can only predict the stresses in the membrane appropriately while the geometry of deformation cannot be accounted for.

4 Discussion and concluding remarks

Having proved a local existence theorem for the nonlinear viscoelastic membrane model (1.1,1.2,1.3) we observe that the existence time in general will depend crucially on the smoothness of the values of the local rotations $\overline{R}$, i.e., the smoothness of the elasticity tensor $\mathbb{D}$. If bifurcations occur they must then be attributed to a severe loss of smoothness of these elastic moduli. It is still an open problem whether the viscoelastic system (1.1,1.2,1.3) admits global in time solutions for small data. This may not be true.

In closing, a number of possible extensions of the theory are worth mentioning. The general mathematical methodology of (1.1,1.2,1.3) is not confined to a viscoelastic membrane plate. Indeed, an extension to viscoelastic membrane-shells and viscoelastic-viscoplastic membrane-shells is possible.

First numerical computations [49] with the relaxation time η of the order 0.01 and $B^{\mathrm{res}} = B^{\mathrm{res},0}_{\mathrm{mech}}$ confirm the general applicability of the viscoelastic membrane-plate model (1.1,1.2,1.3) for structural applications of thin components compared with standard models and corroborate the excellent properties of (1.1,1.2,1.3) with this choice in the evolution of the "viscoelastic" rotations.

5 Acknowledgements

The main idea of this paper was conceived in the academic year 2001/2002 while the author held a visiting faculty position under the ASCI program in Michael Ortiz group at the California Institute of Technology, Graduate Aeronautical Laboratories. The author would like to thank Fehmi Cirak and Kerstin Weinberg for helpful discussions and Michael Ortiz for first directing his attention to thin structures. Thanks are due to Hans Dieter Alber for continued support.

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6 Appendix

6.1 Notation

6.1.1 Notation for bulk material

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times 3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times 3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3\times 3}$. The identity tensor on $\mathbb{M}^{3\times 3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[X] = \langle X, 1\!\!1\rangle$. We let Sym and PSym denote the symmetric and positive definite symmetric tensors respectively. We adopt the usual abbreviations of Lie-group theory, i.e., $\mathrm{GL}(3,\mathbb{R}) := \{X \in \mathbb{M}^{3\times 3}\ |\det[X] \neq 0\}$ the general linear group, $\mathrm{SL}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R})\ |\det[X] = 1\}$, $\mathrm{O}(3) := \{X \in \mathrm{GL}(3,\mathbb{R})\ |\ X^TX = 1\!\!1\}$, $\mathrm{SO}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R})\ |X^TX = 1\!\!1,\ \det[X] = 1\}$ with corresponding Lie-algebras $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times 3}\ |X^T = -X\}$ of skew symmetric tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{M}^{3\times 3}\ |\operatorname{tr}[X] = 0\}$ of traceless tensors. With $\operatorname{Adj} X$ we denote the tensor of transposed cofactors $\mathrm{Cof(X)}$ such that $\operatorname{Adj} X = \det[X]\,X^{-1} = \mathrm{Cof(X)}^{\mathrm{T}}$ if $X \in \mathrm{GL}(3,\mathbb{R})$. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$. For $X \in \mathbb{M}^{3\times 3}$ we set for the deviatoric part $\operatorname{dev} X = X - \frac{1}{3}\operatorname{tr}[X]\,1\!\!1 \in \mathfrak{sl}(3)$ and for vectors $\xi, \eta \in \mathbb{R}^n$ we have the tensor product $(\xi \otimes \eta)_{ij} = \xi_i\,\eta_j$.

We write the polar decomposition in the form $F = RU = \operatorname{polar}(F)\,U$ with $R = \operatorname{polar}(F)$ the orthogonal part of F. In general we work in the context of nonlinear, finite elasticity. For the total deformation $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^3)$ we have the deformation gradient $F = \nabla\varphi \in C(\overline{\Omega}, \mathbb{M}^{3\times 3})$. Furthermore, $S_1(F)$ and $S_2(F)$ denote the first and second Piola Kirchhoff stress tensors, respectively. Total time derivatives are written $\frac{\mathrm{d}}{\mathrm{dt}}X(t) = \dot{X}$. The first and second differential of a scalar valued function $W(F)$ are written $D_F W(F).H$ and $D_F^2 W(F).(H, H)$, respectively. We employ the standard notation of Sobolev spaces, i.e. $L^2(\Omega), H^{1,2}(\Omega), H^{1,2}_\circ(\Omega)$, which we use indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions. Moreover, we set $\|X\|_\infty = \sup_{x\in\Omega}\|X(x)\|$. For $A \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$ we define $\operatorname{Curl} A(x)$ as the operation curl applied row wise. We define $H^{1,2}_\circ(\Omega, \Gamma) := \{\phi \in H^{1,2}(\Omega)\ |\ \phi_{|_\Gamma} = 0\}$, where $\phi_{|_\Gamma} = 0$ is to be understood in the sense of traces and by $C_0^\infty(\Omega)$ we denote infinitely differentiable functions with compact support in Ω. We use capital letters to denote possibly large positive constants, e.g. C^+, K and lower case letters to denote possibly small positive constants, e.g. c^+, d^+. The smallest eigenvalue of a positive definite symmetric tensor P is abbreviated by $\lambda_{\min}(P)$.

6.1.2 Notation for membrane shells

Let $\omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\omega$ and let γ_0 be a smooth subset of $\partial\omega$ with non-vanishing 1-dimensional Hausdorff measure. The relative thickness of the plate is taken to be $h > 0$ with dimension length (contrary to Ciarlet's definition of the thickness to be 2ε, which difference leads only to various different constants in the resulting formulas). We denote by $\mathbb{M}^{n\times m}$ the set of matrices mapping $\mathbb{R}^n \mapsto \mathbb{R}^m$. For $H \in \mathbb{M}^{2\times 3}$ and $\xi \in \mathbb{R}^3$ we employ also the notation $(H|\xi) \in \mathbb{M}^{3\times 3}$ to denote the matrix composed of H and the column ξ. Likewise $(v|\xi|\eta)$ is the matrix composed of the columns v, ξ, η. The identity tensor on $\mathbb{M}^{2\times 2}$ will be denoted by $1\!\!1_2$. The mapping $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ is the deformation of the midsurface, ∇m is the corresponding deformation gradient and $m_x = (m_{1,x}, m_{2,x}, m_{3,x})^T$, $m_y = (m_{1,y}, m_{2,y}, m_{3,y})^T$. We write $v : \mathbb{R}^2 \mapsto \mathbb{R}^3$ for the displacement of the midsurface, such that $m(x, y) = (x, y, 0)^T + v(x, y)$. The standard volume element is written $\mathrm{dx\,dy\,dz} = \mathrm{dV} = \mathrm{d}\omega\,\mathrm{dz}$.

6.2 The treatment of external loads

In this subsection we supply the reader with the consistent definition of resultant loads for the two-dimensional structure, starting from given three-dimensional loads.

6.2.1 Dead load body forces for the thin plate

Let $\Omega_h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$ be the underlying thin, flat three-dimensional domain. In the three-dimensional theory the dead load body forces $f(x, y, z) \in \mathbb{R}^3$ were simply included in the variational formulation by appending the potential with the term

$$\int_{\Omega_h} f(x, y, z) \cdot \varphi(x, y, z)\,\mathrm{dV}\,. \tag{6.1}$$

We define

$$\hat{f}_0(x, y) := \int_{-h/2}^{h/2} f(x, y, z)\,\mathrm{dz}\,, \quad \hat{f}_1(x, y) := \int_{-h/2}^{h/2} z\,f(x, y, z)\,\mathrm{dz}\,, \tag{6.2}$$

such that $\hat{f}_0$, $\hat{f}_1$ are the zero and first moment of f in thickness direction.

6.2.2 Traction boundary conditions for the thin plate

In the three-dimensional theory the traction boundary forces $N(x,y,z) \in \mathbb{R}^3$ were simply included by appending the potential with the term

$$\int_{\partial\Omega_h^{\mathrm{trans}} \cup \{\gamma_s \times [-\frac{h}{2},\frac{h}{2}]\}} N(x,y,z) \cdot \varphi(x,y,z) \, \mathrm{dS}\,, \tag{6.3}$$

where $\partial\Omega_h^{\mathrm{trans}} = \omega \times \{-\frac{h}{2}, \frac{h}{2}\}$ is the transverse boundary. We define

$$\hat{N}_{\mathrm{lat},0}(x,y) := \int_{-h/2}^{h/2} N(x,y,z)\,\mathrm{dz}\,, \quad \hat{N}_{\mathrm{lat},1}(x,y) := \int_{-h/2}^{h/2} z\, N(x,y,z)\,\mathrm{dz}\,, \tag{6.4}$$

such that $\hat{N}_{\mathrm{lat},0}$, $\hat{N}_{\mathrm{lat},1}$ are the zero and first moment of the tractions N at the lateral boundary in thickness direction. Moreover, we define

$$N_{\mathrm{res}} := [N(x,y,\frac{h}{2}) + N(x,y,-\frac{h}{2})]\,, \quad N_{\mathrm{diff}} := \frac{1}{2}[N(x,y,\frac{h}{2}) - N(x,y,-\frac{h}{2})]\,. \tag{6.5}$$

6.2.3 The external loading functional

Let us gather the influences of the external loading terms. To leading order we have

$$\begin{aligned}
\overline{f} &= \hat{f}_0 + N_{\mathrm{res}}\,, && \textbf{resultant body force}\\
\overline{M} &= \hat{f}_1 + h\,N_{\mathrm{diff}}\,, && \textbf{resultant body couple}\\
\overline{N} &= \hat{N}_{\mathrm{lat},0}\,, && \textbf{resultant lateral surface traction}\\
\overline{M}_c &= \hat{N}_{\mathrm{lat},1}\,, && \textbf{resultant lateral surface couple}\,.
\end{aligned} \tag{6.6}$$

The **resultant loading functional** Π is given by

$$\Pi(m,\overline{R}_3) = \int_\omega \langle \overline{f}, m\rangle + \langle \overline{M}, \overline{R}_3\rangle \,\mathrm{d}\omega + \int_{\gamma_s} \langle \overline{N}, m\rangle + \langle \overline{M}_c, \overline{R}_3\rangle \,\mathrm{ds}\,. \tag{6.7}$$

If we denote the dependence of Π on the loads of the underlying three-dimensional problem as $\Pi(f,N;\, m,\overline{R}_3)$, then it is easily seen that frame-indifference of the external loading functional is satisfied in the sense that $\Pi(Q.f, Q.N;\, Q.m, Q.\overline{R}_3) = \Pi(f,N;\, m,\overline{R}_3)$ for all rigid rotations $Q \in \mathrm{SO}(3,\mathbb{R})$. Since in the viscoelastic membrane-plate model (1.1,1.2,1.3), $\overline{R}$ is only a parameter in the static variational problem, the dependence of the resultant loading functional Π on the rotations $\overline{R}$ can be dropped.

6.3 Thickness stretch and homogenized moduli

Here we show, how the formulation with thickness stretch ϱ_m can be reduced to a formulation without thickness stretch to the effect that ϱ_m leaves a trace in the homogenized moduli of the two-dimensional structure. Recall that

$$\begin{aligned}
W(F,\overline{R}) &:= \frac{\mu}{4}\,\|F^T\overline{R} + \overline{R}^T F - 2\mathbb{1}\|^2 + \frac{\lambda}{8}\,\mathrm{tr}\left[F^T\overline{R} + \overline{R}^T F - 2\mathbb{1}\right]^2, \\
F &= (\nabla m|\varrho_m\,\overline{R}_3), \quad \varrho_m = 1 - \frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]\,.
\end{aligned} \tag{6.8}$$

We define $\varrho := \frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]$. In a first step, we note

$$\begin{aligned}
\overline{R}^T(\nabla m|\varrho_m\,\overline{R}_3) &= \overline{R}^T(\nabla m|0) + (0|0|\varrho_m\,e_3) = \overline{R}^T(\nabla m|0) + (0|0|e_3) + (0|0|\varrho\,e_3)\\
&= \overline{R}^T(\nabla m|\overline{R}_3) + (0|0|\varrho_m\,e_3)\,.
\end{aligned} \tag{6.9}$$

In a second step we obtain that

$$\frac{\mu}{4}\,\|(\nabla m|\varrho_m\,\overline{R}_3)^T\overline{R} + \overline{R}^T(\nabla m|\varrho_m\,\overline{R}_3) - 2\mathbb{1}\|^2 = \frac{\mu}{4}\,\|(\nabla m|\overline{R}_3)^T\overline{R} + \overline{R}^T(\nabla m|\overline{R}_3) - 2\mathbb{1}\|^2 + \mu\,\varrho(\nabla m,\overline{R})^2\,, \tag{6.10}$$

where we have used the orthogonality $\langle \mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3) - \mathbb{1}), (0|0|\varrho\,e_3)\rangle = 0$. Similarly, we get

$$\begin{aligned}
&\frac{\lambda}{8}\,\mathrm{tr}\left[(\nabla m|\varrho_m\,\overline{R}_3)^T\overline{R} + \overline{R}^T(\nabla m|\varrho_m\,\overline{R}_3) - 2\mathbb{1}\right]^2 = \frac{\lambda}{8}\left(\mathrm{tr}\left[(\nabla m|\overline{R}_3)^T\overline{R} + \overline{R}^T(\nabla m|\overline{R}_3) - 2\mathbb{1}\right] - 2\,\varrho(\nabla m,\overline{R})\right)^2\\
&= \frac{\lambda}{8}\left(2\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right] - 2\frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]\right)^2\\
&= \frac{\lambda}{2}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]^2\left(1 - \frac{\lambda}{2\mu+\lambda}\right)^2 = \frac{\lambda}{2}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]^2\frac{(2\mu)^2}{(2\mu+\lambda)^2}\,.
\end{aligned} \tag{6.11}$$

In addition

$$\begin{aligned}
&\mu\,\varrho^2 + \frac{\lambda}{2}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]^2\frac{(2\mu)^2}{(2\mu+\lambda)^2} = \mu\,\frac{\lambda^2}{(2\mu+\lambda)^2}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]^2 + \frac{\lambda}{2}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]^2\frac{(2\mu)^2}{(2\mu+\lambda)^2}\\
&= \left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]^2\frac{\mu\,\lambda^2 + 2\mu^2\lambda}{(2\mu+\lambda)^2} = \frac{\mu\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]^2 = \frac{\mu\lambda}{2\mu+\lambda}\,\frac{\mathrm{tr}\left[(\nabla m|\overline{R}_3)^T\overline{R} + \overline{R}^T(\nabla m|\overline{R}_3) - 2\mathbb{1}\right]^2}{4}\\
&= \frac{2\mu\lambda}{8(2\mu+\lambda)}\mathrm{tr}\left[(\nabla m|\overline{R}_3)^T\overline{R} + \overline{R}^T(\nabla m|\overline{R}_3) - 2\mathbb{1}\right]^2\,.
\end{aligned} \tag{6.12}$$

Combining (6.10) and (6.12) shows (1.6).

6.4 Sharp ellipticity type estimates

For the exposition of the static case we need sharp a priori estimates for elliptic systems of second order with non-constant coefficients in divergence form. Ebenfeld [20] has recently proved the following new sharpened a priori estimate which we give adapted to our situation and our notation.

Theorem 6.1 (General improved sharp Hilbert space elliptic regularity)
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Consider the divergence-form linear system

$$\mathrm{Div}\ \mathbb{C}(x).\nabla u = f(x)\ , \quad u_{|\partial\Omega} = 0\,, \tag{6.1}$$

with $f \in H^{k,2}(\Omega)$ and homogeneous boundary data. Let $\mathbb{C} : \Omega \subset \mathbb{R}^3 \mapsto \mathrm{Lin}(\mathbb{M}^{3\times3}, \mathbb{M}^{3\times3})$ be the fourth order elasticity tensor. Suppose $\mathbb{C} \in H^{k+1,2}(\Omega)$ with $2 \cdot (k+1) > n$ and assume that for arbitrary $\xi, \eta \in \mathbb{R}^n$ it holds

$$\exists\, c_e^+ > 0 \quad \forall\, x \in \Omega : \quad \langle \mathbb{C}(x).(\xi \otimes \eta), \xi \otimes \eta \rangle \geq c_e^+ \cdot \|\xi\|^2\, \|\eta\|^2\,, \tag{6.2}$$

i.e., that the system is uniformly Legendre-Hadamard elliptic with ellipticity constant c_e^+. Assume that the system admits at least one weak solution $u \in H^{1,2}(\Omega)$. Then the following estimate is valid

$$\|u\|_{k+2,2,\Omega} \leq C^+(\Omega, c_e^+) P(\|\mathbb{C}\|_{k+1,2,\Omega}) \left(\|f\|_{k,2,\Omega} + \|u\|_{2,\Omega} \right)\,, \tag{6.3}$$

where $P : \mathbb{R} \mapsto \mathbb{R}$ is a polynomial of finite order and the appearing constant is independent of $u, f, \mathbb{C}$ and in addition $C^+(\Omega, c_e^+)$ is bounded above for $c_e^+ > 0$.

Proof. See [18, 19] and compare with [48, p.75] for comparable results on elliptic regularity for linear second order elliptic systems on other scales. The main advantage of the new theorem is to precisely track how the regularity of the coefficients enter the elliptic estimate. Precise estimates of this form had not been available previously. ■

6.5 Local existence for ordinary differential equations in Banach-spaces

Theorem 6.2 (Unique local existence)
Let $\widehat{U}, X, Y, Z$ be arbitrary Banach-spaces with norms $\|\cdot\|_{\widehat{U}}, \|\cdot\|_X, \|\cdot\|_Y, \|\cdot\|_Z$ respectively. Assume that $\mathfrak{f} : \widehat{U} \times X \mapsto \mathrm{Lin}(X, X)$ is locally Lipschitz-continuous and let the initial value $y^0 \in X$ be given. Let $G : X \times Y \times Z \mapsto \widehat{U}$ be an operator which is Lipschitz continuous on the set $\mathcal{M} \times \mathcal{Y} \times \mathcal{Z}$ with $\mathcal{M} := \{y \in X \mid \|y - y^0\|_X \leq K\}$ and $\mathcal{Y} \subset Y$, $\mathcal{Z} \subset Z$ bounded in Y, Z, respectively, i.e., there is a positive constant L^+ such that

$$\begin{aligned}
&\exists\, L^+ > 0 : \quad \forall (x_1, a_1, b_1), (x_2, a_2, b_2) \in \mathcal{M} \times \mathcal{Y} \times \mathcal{Z} : \\
&\|G(x_1, a_1, b_1) - G(x_2, , a_2, b_2)\|_{\widehat{U}} \leq L^+ \cdot (\|x_1 - x_2\|_X + \|a_1 - a_2\|_Y + \|b_1 - b_2\|_Z)\,.
\end{aligned}$$

Moreover, assume that $\alpha \in C^1([0,T], \mathcal{Y})$, $\beta \in C^1([0,T], \mathcal{Z})$ are given functions. Then there is some $0 < t_1 \in \mathbb{R}$ such that the initial value problem

$$\frac{\mathrm{d}}{\mathrm{dt}} y(t) = \mathfrak{f}\big(G(y(t), \alpha(t), \beta(t)), y(t)\big) \cdot y(t)\,, \quad y(0) = y^0\,, \tag{6.4}$$

has a unique solution $y \in C^1([0, t_1], \mathcal{M})$. ■