Characteristics method for the problem of viscoelastic fluid flow of PTT model

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Abstract.— *We formulate and analyze a characteristics finite element approximation of a class of flows in viscoelastic fluids described by the Phan-Thien-Tanner model. Compared to the classical Oldroyd model, the considered model presents further difficulties due to the presence of nonlinear terms of exponential type in the constitutive equation. In this paper, we propose a characteristics-based method to treat the transport part of the equations. The stress, velocity and pressure approximations are P_1 discontinuous, P_2 continuous and P_1 continuous finite element, respectively. By assuming that the continuous problem admits a sufficiently smooth and sufficiently small solution, and using a fixed point method, we show existence of solution to the approximate problem. We also give an error bound for the numerical solution.*

Keywords: Numerical Analysis; Finite element method; Viscoelastic fluids; Characteristics method.

1 Introduction

Viscoelastic fluid flows are a subject of very intensive research activities since they include a wide variety of difficulties that typically arise in the numerical approximation of partial differential equations describing and consequently, determining their dynamics. The set of governing equations differ from one to another by the constitutive equation that closes the system. In this paper, a Phan-Thien-Tanner (PTT) model [1] is selected for being more physically realistic than the Oldroyd-B model extensively studied in literature, see for example [5, 3] and further references are cited therein. The PTT model inherits the main difficulties from the Oldroyd-B equations due to the convection part. Moreover, the major numerical difficulty in PTT model lies in the presence of nonlinear

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stress function in the constitutive equation. In our work, we consider a PTT model with nonlinear term of exponential form.

During the last decade, many numerical methods have been introduced and experimented for finite element approximation of viscoelastic fluid flows, compare among others, [15, 14, 17, 18, 12, 16]. Simulation of viscoelastic fluid flows is still a considerable task in case of convection dominated problems; particularly when the Weissenberg number goes close to the limit, see [11] for more details. Therefore, one important aspect which must be considered when approximating numerical solution to viscoelastic flows is the treatment of convection part in the governing equations. In many cases, convection terms are source of instabilities and order deterioration for instance, in upwind discontinuous finite element, streamline upwinding, or Petrov-Galerkin methods, we refer the reader to [12, 15] for more discussions.

The characteristics methods, on the other hand, make use of the hyperbolic nature of the governing equations. They combine the fixed Eulerian mesh with a particle tracking along the characteristic curves. These methods have been successfully applied to approximate convection-dominated flows [9]. The method characteristics has been also used for stationary problems as those studied in [6]. In order to avoid the above mentioned difficulties in viscoelastic flows, authors in [3] analyzed the characteristics method. However, the analysis they used can only be applied to the special Oldroyd-B model. In this paper we adapt the same ideas to the PTT model by incorporation of a pseudo-time derivative of stress variable and discretization of the constitutive equation along the characteristic trajectories using a backward time integration. It is a promising technique when PTT model is to be solved in conjunction with the time-dependent equations.

Although there is much work on the error analysis for finite element approximations to both steady and evolutionary viscoelastic problems, error estimates of numerical methods for the PTT model have not been investigated as much. Our aim in this work is to provide a numerical analysis of finite element approximation of a viscoelastic fluid flow obeying the PTT model. The central assumptions for carrying this analysis out are: (i) the continuous problem admits a sufficiently smooth and sufficiently small solution (ii) the approximate stress, velocity and pressure are P_1 discontinuous, P_2 continuous and P_1 continuous finite element, respectively. Using Brouwer fixed point theorem we shall prove that the error is $\mathcal{O}\left(\dfrac{h^2}{\sqrt{k}} + k\right)$.

2 The PTT problem and its finite element approximation

Modeling viscoelastic incompressible flows requires a constitutive equation for stress, conservation of mass and transport of momentum. The PTT model we consider in this paper for steady-state and creeping flows consists of the following non-dimensional set of governing equations,

$$(O)\quad\left\{\begin{array}{rcll} F(\sigma)\sigma+\lambda(u\cdot\nabla)\sigma+\lambda g_a(\sigma,\nabla u)-2\alpha d(u) &=& 0, & \text{in}\quad\Omega,\\ -\nabla\cdot\sigma-2(1-\alpha)\nabla\cdot d(u)+\nabla p &=& f, & \text{in}\quad\Omega,\\ \nabla\cdot u &=& 0, & \text{in}\quad\Omega,\\ u &=& 0, & \text{on}\quad\Gamma,\end{array}\right.$$

where

$$\begin{aligned} F(\sigma) &= \exp\left(\epsilon\frac{\lambda}{\alpha}\mathrm{tr}(\sigma)\right), \qquad \mathrm{tr}(\sigma)=\sum_i \sigma_{ii},\\ g_a(\sigma,\nabla u) &= \frac{1-a}{2}\left(\sigma\nabla u+\nabla u^T\sigma\right)-\frac{1+a}{2}\left(\nabla u\sigma+\sigma\nabla u^T\right), \qquad -1\le a\le 1,\\ d(u) &= \frac{1}{2}\left(\nabla u+(\nabla u)^T\right),\end{aligned}$$

with σ is the extra-stress tensor, u is the velocity field, p is the pressure, λ ($\lambda>0$) is the Weissenberg number, α ($0<\alpha<1$) is the viscosity coefficient, $d(u)$ is the rate of deformation tensor, $g_a(\cdot,\cdot)$ is a bilinear application and ϵ (ϵ small) is a dimensionless material parameter. Here ∇ is the gradient operator and superscript ^{T} denotes matrix transpose. The set of governing equations (O) is defined in a bounded domain $\Omega\subset I\!R^2$ with Lipschitz continuous boundary Γ. Note that for $a=-1$ (or $a=1$) in g_a, the above equations lead to the well known upper convected (or lower convected) Maxwell models. Furthermore, by setting $\epsilon=0$ in $F(\sigma)$ we recover the standard Oldroyd-B model.

Throughout this paper we shall use the following notations: $(\cdot,\cdot)$ and $|\cdot|$ denote, respectively, the scalar product and the norm of $L^2(\Omega)-$space; $|\cdot|_{s,p}$ and $\|\cdot\|_{s,p}$ denote, respectively, the usual semi-norm and norm on the Sobolev space $(W^{s,p}(\Omega))^n$, $s\in[0,\infty[$, $p\in[1,\infty[$. We define the following spaces,

$$\begin{aligned} T &= \left\{\tau=(\tau_{ij});\tau_{ij}=\tau_{ji};\quad \tau_{ij}\in L^2(\Omega);1\le i,j\le 2\right\},\\ X &= H_0^1(\Omega)=\left\{v\in H^1(\Omega)^2;\quad v\mid_\Gamma=0\right\},\\ Q &= L_0^2(\Omega)=\left\{q\in L^2(\Omega);\quad \int_\Omega q=0\right\}.\end{aligned}$$

The spatial domain Ω is supposed to be polygonal equipped with a regular family of triangulation $\mathcal{T}_h$ made of triangles, $\bar\Omega=\bigcup_{K\in\mathcal{T}_h}K$, and there exists ν_0 and ν_1 such that

$$h\nu_0\le h_K\le\nu_1\rho_K,$$

where h_K is the diameter of K, ρ_K is the diameter of the greatest ball included in K, $h = \max_{K \in \mathcal{T}_h} h_K$ and h_{max} denotes the diameter of Ω. Next, we define the associated norm

$$| \cdot |_{m,p,h}^p = \begin{cases} \sum_{K \in \mathcal{T}_h} | \cdot |_{m,p,K}^p, & \text{if} \quad p < +\infty, \\ \max_{K \in \mathcal{T}_h} | \cdot |_{m,\infty,K}^p, & \text{else.} \end{cases}$$

Let $P_r(K)$ denote the space of polynomials of degree less or equal to r on $K \in \mathcal{T}_h$. For the approximation of (u,p) we use the Hood-Taylor finite element spaces given by

$$\begin{aligned} X_h &= \left\{ v \in X; \quad v\,|_K \in P_2(K)^2, \quad \forall K \in \mathcal{T}_h \right\}, \\ Q_h &= \left\{ q \in Q \cap C^0(\bar{\Omega}); \quad q\,|_K \in P_1(K), \quad \forall K \in \mathcal{T}_h \right\}, \\ V_h &= \{ v \in X_h; \quad (q, \nabla \cdot v) = 0, \quad \forall q \in Q_h \}. \end{aligned}$$

It is known that the pair (X_h, Q_h) satisfies the following inf sup condition from [7], there exists $\beta > 0$ independent of h such that

$$\inf_{q \in Q_h} \sup_{v \in X_h} \frac{(q, \nabla \cdot v)}{| q || d(v) |} \geq \beta > 0. \tag{2.1}$$

The stress tensor σ is approximated by P_1 discontinuous finite elements space

$$T_h = \{ \tau \in T; \quad \tau\,|_K \in P_1(K)^4, \quad \forall K \in \mathcal{T}_h \}.$$

In the sequel, we shall use the following inverse inequalities (see for example [8, 7]):

Lemma **2.1** *Let $m = 0, 1$ and $1 \leq p, q \leq +\infty$, there exist positive constants C_1 and C_2 independent of h such that for all $v \in P_r(K)$ we have*

$$\begin{aligned} | v |_{m,q,K} &\leq C_1 h_K^{2/q-2/p} | v |_{m,p,K}, & (2.2) \\ | v |_{1,q,K} &\leq C_2 h_K^{2/q-2/p-1} | v |_{0,p,K}\,. \quad \blacksquare & (2.3) \end{aligned}$$

If we set $W_h = \{ v \in H^1(\Omega),\ v\,|_K \in P_r(K),\ \forall K \in \mathcal{T}_h \}$, then we have the next result:

Lemma **2.2** *Let $m = 0, 1$ and $1 \leq p, q \leq +\infty$. We assume that $\mathcal{T}_h$ is uniformly regular. Then, there exist positive constants C_1 and C_2 independent of $\mathcal{T}_h$ such that for all $v \in W_h$ we have*

$$\begin{aligned} | v |_{m,q,h} &\leq C_1 h^{\min\{0, 2/q-2/p\}} | v |_{m,p,h}, & (2.4) \\ | v |_{1,q,h} &\leq C h^{-1+\min\{0, 2/q-2/p\}} | v |_{0,p,h}\,. \quad \blacksquare & (2.5) \end{aligned}$$

Let P_h be the $L^2(\Omega)-$orthogonal projection onto W_h, then we have the following error estimate whose proof can be found in [7]:

LEMMA **2.3** *Let $m = 0, 1,\ 1 \leq p \leq +\infty$. There exists a positive constant C_3 independent of h such that*

$$\mid v - P_h v \mid_{m,p,K} \leq C_3 h_K^{r+1-m} \mid v \mid_{r+1,p,K}, \quad \forall v \in W^{r+1,p}(K). \quad \blacksquare \tag{2.6}$$

We shall use also the following Sobolev's embedding theorems, the proof of which is given in [8]:

LEMMA **2.4** *Let $m \geq 0$ be an integer. The following embedding holds algebraically and topologically*

$$W^{1,4}(\Omega) \subset L^\infty(\Omega), \qquad H^2(\Omega) \subset L^\infty(\Omega),$$
$$W^{m+1,2}(\Omega) \subset W^{m,q}(\Omega), \qquad \forall q \in [1, +\infty[,$$
$$W^{m,p}(\Omega) \subset C^0(\bar{\Omega}), \qquad \forall 1 \leq p \leq +\infty \quad \text{such} \quad \text{that} \quad mp > 2. \quad \blacksquare$$

3 Characteristics method for the steady convection problem

In this section, we recall some preliminary results given in [4, 2]. We also express a thick green formula (compare reference [2] for details) which will be used in the proof of the error estimate in the next section.
Let consider the following scalar problem

$$(P) \quad \begin{cases} u \cdot \nabla\sigma + c\sigma &= f, \qquad \text{in} \quad \Omega, \\ \sigma &= g_1, \qquad \text{on} \quad \Gamma^-, \end{cases}$$

where $\Gamma^- = \{x \in \Gamma,\ u \cdot n < 0\}$ with n is the unit outward normal to Γ. We shall assume that $u \in (W^{1,\infty}(\Omega))^2$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ and we make the assumption, $c \geq c_0 > 0$. We suppose that there exists $g \in H^1(\Omega)$ such that $g = g_1$ on Γ^-.
To discretize the convection term $(u \cdot \nabla)\sigma$ we adapt the well established transport-diffusion algorithm in [9] to the stationary case. The idea behind this approach is based on combining the method of characteristics with finite elements discretization. Similar technique has been used by the authors in [6] for steady-state problems. Thus, we replace the problem (P) by the following modified equations

$$(P^k) \quad \begin{cases} \dfrac{\sigma^k(x) - \sigma^k\left(X^k(x)\right)}{k} + c\sigma^k(x) &= f(x), \quad \text{in} \quad \Omega, \\ \sigma^k\left(X^k(x)\right) &= g, \qquad \text{if} \quad X^k(x) \notin \Omega, \end{cases}$$

where $X^k(x) = X(x, k, 0)$, $X(x, t, \tau)$ being the trajectory of a particle which will reach the point x at time t. Hence, a space-discrete approximation of the problem under study can be carried out by using finite elements in a classical Galerkin formulation. As previously mentioned in [6], the problem (P^k) can be viewed as a backward time integration of the non-stationary problem which involves incorporation of a pseudo-time step k. Indeed, if σ is the solution of (P), then $\tilde{\sigma}(x,t) = \sigma(x)$ is the solution of the evolution problem

$$(\tilde{P}) \quad \begin{cases} \dfrac{D\tilde{\sigma}}{Dt} + c\tilde{\sigma} &= f, \qquad \text{in} \quad \Omega\times]0,T[, \\ \qquad\quad \tilde{\sigma} &= g, \qquad \text{on} \quad \Gamma^-, \\ \tilde{\sigma}(x,0) &= \sigma(x), \quad \text{in} \quad \Omega, \end{cases}$$

where $\dfrac{D\tilde{\sigma}}{Dt}$ denotes the total derivative of $\tilde{\sigma}$ in the direction of the stationary flow u defined by

$$\frac{D\tilde{\sigma}}{Dt} = \frac{\partial\tilde{\sigma}}{\partial t} + (u\cdot\nabla)\tilde{\sigma}.$$

Discretization of $(\tilde{P})$ by the method of characteristics leads to the following scheme

$$\frac{\tilde{\sigma}^{n+1}(x) - \tilde{\sigma}^n\left(X^n(x)\right)}{k} + c\tilde{\sigma}^{n+1}(x) = f, \quad \text{in} \quad \Omega, \tag{3.1}$$

where $X^n(x) = X(x, t^{n+1}, t^n)$, k is the time step, $t^n = nk$, and X is the solution of the ordinary differential equation of the trajectories

$$\begin{cases} \dfrac{dX}{d\tau}(x,t,\tau) &= u\left(X(x,t,\tau)\right), \\ X(x,t,\tau) &= x. \end{cases}$$

Now, (3.1) leads to (P^k) by observing that both $\tilde{\sigma}^{n+1}$ and $\tilde{\sigma}^n$ are approximation of σ. The second step consists in using a classical finite element method to discretize (P^k). We consider the following variational formulation:
Find $\sigma^k \in L^2(\Omega)$ such that

$$\frac{1}{k}\left(\sigma^k, \tau\right) - \frac{1}{k}\left(\sigma^k(X^k(\cdot)), \tau\right)_{0,\Omega_1} + \left(c\sigma^k, \tau\right) = (f,\tau) + \frac{1}{k}\left(g,\tau\right)_{0,\Omega_2}, \quad \forall\tau \in L^2(\Omega), \tag{3.2}$$

where $\Omega_1 = \left\{x \in \Omega; X^k(x) \in \Omega\right\}$, $\Omega_2 = \left\{x \in \Omega; X^k(x) \notin \Omega\right\}$ and $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$. In what follows, we introduce a natural norm well adapted to the study of characteristics method such that when k goes to zero it reduces to the discontinuous Galerkin norm.

Let $V_h = \{v \in L^2(\Omega), \quad v\mid_K \in P_r(K), \quad \forall K \in \mathcal{T}_h\}$. The discrete variational problem associated to (3.2) is then:
Find $\sigma_h^k \in V_h$ such that

$$\frac{1}{k}\left(\sigma_h^k, \tau\right) - \frac{1}{k}\left(\sigma_h^k\left(X^k(\cdot)\right), \tau\right)_{0,\Omega_1} + \left(c\sigma_h^k, \tau\right) = (f, \tau) + \frac{1}{k}(g, \tau)_{0,\Omega_2}, \quad \forall \tau \in V_h. \quad (3.3)$$

More precisely we have

$$\frac{1}{k}\int_\Omega \sigma_h^k \tau_h dx - \frac{1}{k}\int_{\Omega_1} \sigma_h^k(X^k(\cdot))\tau_h(x)dx + \int_\Omega c\sigma_h^k\tau_h dx = \int_\Omega f\tau_h dx + \frac{1}{k}\int_{\Omega_2} g\tau_h dx.$$

The variational formulation (3.3) is then equivalent to

$$(P_h^k) \qquad B_c(\sigma_h^k, \tau_h) = L_c(\tau_h), \qquad \forall \tau_h \in V_h. \quad (3.4)$$

with

$$B_c(\sigma_h^k, \tau_h) = \frac{1}{k}\int_\Omega \sigma_h^k\tau_h dx - \frac{1}{k}\int_{\Omega_1} \sigma_h^k\left(X^k(\cdot)\right)\tau_h(x)dx + \int_\Omega c\sigma_h^k\tau_h dx, \quad (3.5)$$

$$L_c(\tau_h) = \int_\Omega f\tau_h dx + \frac{1}{k}\int_{\Omega_2} g\tau_h dx. \quad (3.6)$$

When the entry field is Ω filing and under a technical hypothesis, we have shown in [4] the existence and the uniqueness for both, the continuous and the discrete variational problems (P^k) and (P_h^k). We have also shown an error estimate of order $\mathcal{O}\left(\frac{h^{r+1}}{k} + h^{r+1} + k\right)$ with $L^2(\Omega)-$norm.
In [2], we have introduced the natural norm $\|\cdot\|_{h,k}$ as

$$\begin{aligned}\| \sigma_h^k \|_{h,k}^2 &= \int_\Omega (\sigma_h^k)^2 dx + \frac{1}{k}\int_{\Omega_2} (\sigma_h^k)^2(x)dx + \\ &\quad \frac{1}{k}\int_{\Omega_1} \left(\sigma_h^k(x) - \sigma_h^k\left(X^k(x)\right)\right)^2 dx + \int_{\Omega_\theta} \frac{(\sigma_h^k)^2\left(X^k(x)\right)}{k} dx, \end{aligned} \quad (3.7)$$

where $\Omega_\theta = \theta^k(\Omega)\backslash\Omega$ and $\theta^k(x) = X(x, t, t+k)$. We show that B_c is elliptic in norm $\|\cdot\|_{h,k}$ under a suitable hypothesis. This result implies that problem (P_h^k) has a unique solution and we give an error estimate of order $\mathcal{O}\left(\frac{h^{r+1}}{\sqrt{k}} + h^{r+1} + k\right)$ with the norm $\|\cdot\|_{h,k}$.
We define

$$D_k(y) = \frac{\left|\frac{dx}{dy}\right| - 1}{k}, \quad (3.8)$$

where $\left|\frac{dx}{dy}\right| = \left|\frac{d\theta^k(y)}{dy}\right|$ is the determinant of the Jacobian matrix and $\theta^k(y) = X(y, t, t+k)$.

Lemma **3.1** *(thick Green formula)*

$$\frac{1}{k}\int_{\Omega_2}\sigma(x)\tau(x)dx+\int_{\Omega_1}\left(\frac{\sigma(x)-\sigma(X^k(x))}{k}\tau(x)-\frac{\tau(X^k(x))-\tau(x)}{k}\sigma\left(X^k(x)\right)\right)dx+$$
$$\int_{\Omega}D_k(y)\sigma(y)\tau(y)dy=\int_{\Omega_\theta}\frac{\sigma\left(X^k(x)\right)\tau\left(X^k(x)\right)}{k}dx. \quad \blacksquare \tag{3.9}$$

For the proof of this Lemma, we refer to [2].

Lemma **3.2** *Suppose that $u\in W^{2,\infty}(\Omega)$, then there exists C independent of k such that*

$$\mid D_k(y)-\nabla\cdot u(y)\mid\leq CkM^2\left(1+M+M^3+M^4\right),$$

where $M=\mid u\mid_{2,\infty}$. $\blacksquare$

The proof of this Lemma can be found in [3].

Remark **3.1** *In the following section we suppose that $u=0$ on the boundary Γ that simplifies $\Omega_2=\emptyset$ and $\Omega_\theta=\emptyset$.*

4 Approximation of PTT model and error bound

The first equation of (O), which is hyperbolic in σ when u is fixed, is approximated by characteristics method introduced in [2]. The idea is to replace problem (O) by the following set of equations

$$(O^k)\begin{cases} F(\sigma^k)\sigma^k+\lambda\dfrac{\sigma^k(x)-\sigma^k\left(X^k(x)\right)}{k}+\lambda g_a\left(\sigma^k,\nabla u^k\right)-2\alpha d(u^k) = 0, & \text{in } \Omega,\\ -\nabla\cdot\sigma^k-2(1-\alpha)\nabla\cdot d(u^k)+\nabla p^k = f, & \text{in } \Omega,\\ \nabla\cdot u^k = 0, & \text{in } \Omega,\\ u^k = 0, & \text{on } \Gamma,\end{cases}$$

where $X^k(x)=S(x,t,t-k)$, $S(x,t,\tau)$ being the approximation of trajectory of particle defined by

$$\begin{cases}\dfrac{dS}{d\tau} = u^k\left(S(x,t,\tau)\right),\\ S(x,t,t) = x.\end{cases}$$

An operator B on $X_h\times T_h\times T_h$ is defined by

$$B(u,\sigma,\tau)=\left(\frac{\sigma(x)-\sigma(X^k(x))}{k},\tau\right)+\frac{1}{2}(D_k^u\sigma,\tau), \tag{4.1}$$

where $D_k^u(y) = \dfrac{\left|\dfrac{dx}{dy}\right| - 1}{k}$, $\left|\dfrac{dx}{dy}\right| = \left|\dfrac{d\theta^k(y)}{dy}\right|$ is the determinant of the Jacobian matrix and $\theta^k(y) = S(y, t, t+k)$.
The problem (O) is approximated by problem (O_h^k) as follows:
Find $(\sigma_h^k, \tau_h) \in T_h \times V_h$ such that

$$\left(F(\sigma_h^k)\sigma_h^k, \tau_h\right) + \lambda B\left(u_h^k, \sigma_h^k, \tau_h\right) + \lambda\left(g_a(\sigma_h^k, \nabla u_h^k), \tau_h\right) - 2\alpha\left(d(u_h^k), \tau_h\right) = 0, \quad \forall \tau_h \in T_h, \tag{4.2}$$

$$\left(\sigma_h^k, d(v_h)\right) + 2(1-\alpha)\left(d(u_h^k), d(v_h)\right) = (f, v_h), \quad \forall v_h \in V_h. \tag{4.3}$$

If we define the bilinear form A on $T \times X$ by

$$A\left((\sigma, u), (\tau, v)\right) = 2\alpha\left(\sigma, d(v)\right) - 2\alpha\left(d(u), \tau\right) + 4\alpha(1-\alpha)\left(d(u), d(v)\right), \tag{4.4}$$

then problem (O_h^k) becomes
Find $(\sigma_h^k, \tau_h) \in T_h \times V_h$ such that

$$\left(F(\sigma_h^k)\sigma_h^k, \tau_h\right) + A\left((\sigma_h^k, u_h^k), (\tau_h, v_h)\right) + \lambda B\left(u_h^k, \sigma_h^k, \tau_h\right) + \lambda\left(g_a(\sigma_h^k, \nabla u_h^k), \tau_h\right) = 2\alpha(f, v_h), \qquad \forall (\tau_h, v_h) \in T_h \times V_h. \tag{4.5}$$

THEOREM **4.1** *There exist positive constants M_0, h_0 and k_0 such that if problem (O) admits a solution $(\sigma, u, p) \in W^{2,\infty}(\Omega) \times (W^{3,2}(\Omega) \cap W^{2,\infty}(\Omega)) \times (H^2(\Omega) \cap L_0^2(\Omega))$ satisfying*

$$\max\Big\{\| \sigma \|_{2,\infty}, \| u \|_{3,2}, \| u \|_{2,\infty}, \| p \|_{2,2}\Big\} \le M_0,$$

then for all $h \le h_0$, $k \le k_0$ such that $\exists\ a_1, a_2$: $a_1 k \le h \le a_2\sqrt{k}$, problem (O_h^k) admits a solution $(\sigma_h^k, u_h^k, p_h^k) \in T_h \times X_h \times Q_h$ and there exists a positive constant C independent of h and k such that

$$| \sigma - \sigma_h^k | + | d(u - u_h^k) | \ \le\ C\left(\frac{h^2}{\sqrt{k}} + k\right), \tag{4.6}$$

$$| p - p_h^k | \ \le\ C\left(\frac{h^2}{\sqrt{k}} + k\right). \quad \square \tag{4.7}$$

Proof. We start by considering the problem (O_h^k). For this purpose we define a mapping $\phi : T_h \times V_h \longrightarrow T_h \times V_h$ which maps (σ_1, u_1) to $(\sigma_2, u_2) = \phi(\sigma_1, u_1)$, where $(\sigma_2, u_2) \in T_h \times V_h$ satisfies

$$\left(F(\sigma_1)\sigma_2, \tau\right) + A\left((\sigma_2, u_2), (\tau, v)\right) + \lambda B\left(u_1, \sigma_2, \tau\right) + \lambda\left(g_a(\sigma_1, \nabla u_1), \tau\right) = 2\alpha(f, v), \qquad \forall (\tau, v) \in T_h \times V_h. \tag{4.8}$$

The proof is split in four parts listed as follows:

(1) ϕ is well defined and bounded on bounded sets.

(2) ϕ is continuous on $T_h \times V_h$.

(3) There exists a ball B_h^k in $T_h \times V_h$ with center (σ, u) solution of problem (O) such that B_h^k is non empty and $\phi(B_h^k) \subset B_h^k$ for $\| \sigma \|_{2,\infty}, \| u \|_{3,2}, \| u \|_{2,\infty}, \| p \|_{2,2}$ sufficiently small.

(4) Brouwer's theorem gives then the existence of a fixed point (σ_h^k, u_h^k) of ϕ solution of problem (O_h^k) and satisfying (4.6). Existence of p_h^k and error bound (4.7) comes from the inf-sup condition (2.1) on (X_h, Q_h).

(1) ϕ is well defined and bounded on bounded sets:
Integration by "thick" parts (see Lemma 3.1) of (4.1) gives

$$B\left(u, \sigma, \tau\right) = -\left(\frac{\tau(x) - \tau(X^k(x))}{k}, \sigma(X^k(x))\right) - \frac{1}{2}\left(D_k \tau, \sigma\right), \tag{4.9}$$

which implies some "coercivity" of B

$$B\left(u, \sigma, \sigma\right) = \frac{1}{2k}\left(\sigma(x) - \sigma(X^k(x)), \sigma(x) - \sigma(X^k(x))\right). \tag{4.10}$$

From this property and the coercivity of $\mathcal{A}_1$ given by

$$\mathcal{A}_1\left((\sigma, u), (\sigma, u)\right) = \left(F(\sigma_1)\sigma, \tau\right) + A\left((\sigma, u), (\tau, v)\right). \tag{4.11}$$

we get

$$\mathcal{A}_1\left((\sigma, u), (\sigma, u)\right) \geq e^{(-2\epsilon\frac{\lambda}{\alpha}|\sigma_1|_{0,\infty})} \mid \sigma \mid^2 + 4\alpha(1-\alpha) \mid d(u) \mid^2, \tag{4.12}$$

it follows that the finite dimensional problem (4.8) has a unique solution.
To prove that ϕ is bounded on bounded sets we remark that $g_a(\sigma, \nabla u)$ is a linear combination of terms like $\sigma \nabla u$. We have,

$$\mid (\sigma_1 \nabla u_1, \tau) \mid \leq \mid \sigma_1 \mid_{0,\infty} \mid u_1 \mid_{1,2} \mid \tau \mid .$$

Taking $(\tau, v) = (\sigma_2, u_2)$ in (4.8) and using coercivity this gives

$$\mid \sigma_2 \mid + \mid d(u_2) \mid \leq C\left(\mid \sigma_1 \mid_{0,\infty} \mid u_1 \mid_{1,2} + \mid f \mid_{-1}\right),$$

which proves that ϕ is bounded on bounded sets.
(2) ϕ is continuous on $T_h \times V_h$:
Let $(\sigma_2, u_2) = \phi(\sigma_1, u_1)$ and $(\tau_2, v_2) = \phi(\tau_1, v_1)$ we have

$$\begin{aligned}(F(\tau_1)\tau_2, \tau) + A\left((\tau_2, v_2), (\tau, v)\right) + \lambda B\left(v_1, \tau_2, \tau\right) = \\ -\lambda\left(g_a(\tau_1, \nabla v_1), \tau\right) + 2\alpha\left(f, v\right), \quad \forall (\tau, v) \in T_h \times V_h.\end{aligned} \tag{4.13}$$

Subtracting (4.13) from (4.8) we have

$$(F(\sigma_1)(\sigma_2-\tau_2),\tau)+A\left((\sigma_2-\tau_2,u_2-v_2),(\tau,v)\right)+\lambda B\left(u_1,\sigma_2-\tau_2,\tau\right)=$$
$$((F(\sigma_1)-F(\tau_1))\tau_2,\tau)-\lambda B\left(u_1,\tau_2,\tau\right)+\lambda B\left(v_1,\tau_2,\tau\right)-$$
$$\lambda(g_a(\sigma_1,\nabla u_1)-g_a(\tau_1,\nabla v_1),\tau),\qquad \forall(\tau,v)\in T_h\times V_h.$$

Setting $\bar\sigma=\sigma_2-\tau_2$, $\bar u=u_2-v_2$, $\tau=\bar\sigma$ and $v=\bar u$ we obtain

$$(F(\sigma_1)\bar\sigma,\bar\sigma)+A((\bar\sigma,\bar u),(\bar\sigma,\bar u))+\lambda B(u_1,\bar\sigma,\bar\sigma)=((F(\sigma_1)-F(\tau_1))\tau_2,\bar\sigma)-$$
$$\lambda B(u_1,\tau_2,\bar\sigma)+\lambda B(v_1,\tau_2,\bar\sigma)-\lambda(g_a(\sigma_1,\nabla u_1)-g_a(\tau_1,\nabla v_1),\bar\sigma).$$

For the terms g_a and B, the following estimations hold (see [3] for details),

$$\mid\lambda(g_a(\sigma_1,\nabla u_1)-g_a(\tau_1,\nabla v_1),\bar\sigma)\mid\leq C\lambda\mid\bar\sigma\mid(\mid\sigma_1\mid_{0,\infty}\mid u_1-v_1\mid_{1,2}+\mid v_1\mid_{1,\infty}\mid\sigma_1-\tau_1\mid).$$

$$\begin{aligned}B(v_1,\tau_2,\bar\sigma)-B(u_1,\tau_2,\bar\sigma)\ \leq\ & C\sum_{K\in\mathcal{T}_h}\mid\tau_2\mid_{1,\infty,K}\mid u_1-v_1\mid_{0,\infty}\mid\bar\sigma\mid+\\ & CB(u_1,\bar\sigma,\bar\sigma)\sum_{K\in\mathcal{T}_h}\mid\tau_2\mid_{1,\infty,K}\mid u_1-v_1\mid_{0,\infty}+\\ & C\mid\tau_2\mid\mid\bar\sigma\mid\mid u_1-v_1\mid_{0,\infty}.\end{aligned}$$

On the other hand we have,

$$((F(\sigma_1)-F(\tau_1))\tau_2,\bar\sigma)=((F(\sigma_1-\tau_1)-1)F(\tau_1)\tau_2,\bar\sigma)\leq$$
$$e^{(\frac{2\epsilon\lambda}{\alpha}|\tau_1|_{0,\infty})}\mid F(\sigma_1-\tau_1)-1\mid\mid\tau_2\mid\mid\bar\sigma\mid,$$

then using a second order Taylor expansion of the exponential function we get for $\epsilon_1\in]0,\epsilon[$

$$((F(\sigma_1)-F(\tau_1))\tau_2,\bar\sigma)\leq$$
$$Ce^{(\frac{2\epsilon\lambda}{\alpha}|\tau_1|_{0,\infty})}\mid\frac{\epsilon\lambda}{\alpha}tr(\sigma_1-\tau_1)\mid\mid(1+\frac{\epsilon\lambda}{2\alpha}tr(\sigma_1-\tau_1)e^{\epsilon_1\frac{\lambda}{\alpha}tr(\sigma_1-\tau_1)})\mid\tau_2\mid\mid\bar\sigma\mid\leq$$
$$Ce^{(\frac{2\epsilon\lambda}{\alpha}|\tau_1|_{0,\infty})}\frac{\epsilon\lambda}{\alpha}\mid(1+\frac{\epsilon\lambda}{2\alpha})(\mid\sigma_1\mid_{0,\infty}+\mid\tau_1\mid_{0,\infty})e^{\epsilon_1\frac{\lambda}{\alpha}(|\sigma_1|_{0,\infty}+|\tau_1|_{0,\infty})}\mid\sigma_1-\tau_1\mid_{0,\infty}\mid\tau_2\mid\mid\bar\sigma\mid.$$

Thus, ϕ is continuous on $T_h\times V_h$.
(3) Existence of invariant ball for ϕ:
Let C^* be given, we define a ball

$$B_h^k=\left\{(\tau,v)\in T_h\times V_h;\quad\mid\tau-\sigma\mid\leq C^*\left(\frac{h^2}{\sqrt{k}}+k\right),\ \ \mid d(v-u)\mid\leq C^*\left(\frac{h^2}{\sqrt{k}}+k\right)\right\}.$$

Since $(u,p) \in (W^{3,2}(\Omega) \cap W^{2,\infty}(\Omega))) \times H^2(\Omega)$ and from Lemma 2.3, there exists $(\tilde{u}, \tilde{p}) \in V_h \times Q_h$ such that

$$\begin{aligned} \| u - \tilde{u} \|_{1,2} &\leq \bar{C}_1 h^2 \| u \|_{3,2}, \\ | p - \tilde{p} | &\leq \bar{C}_1 h^2 \| p \|_{2,2} . \end{aligned}$$

Let $\tilde{\sigma}$ be the orthogonal projection of σ on T_h in $L^2(\Omega)^4$. We have,

$$| \sigma - \tilde{\sigma} | \leq \bar{C} h^2 \| \sigma \|_{2,2} \leq \bar{C}_1 h^2 \| \sigma \|_{2,\infty} .$$

Let $M = max\,(\| \sigma \|_{2,\infty}, \| u \|_{2,\infty}, \| u \|_{3,2}, \| p \|_{2,2})$. In order to ensure that $(\tilde{\sigma}, \tilde{u}) \in B_h^k$ it suffices to impose that $\bar{C}_1 M \leq C^*$.
The exact solution (σ, u, p) of problem (O) satisfies the following consistency relation

$$\begin{aligned} &(F(\sigma)\sigma, \tau) + A((\sigma, u), (\tau, v)) + \lambda B(u, \sigma, \tau) = \\ &\qquad 2\alpha(p, \nabla \cdot v) - \lambda(g_a(\sigma, \nabla u), \tau) + 2\alpha(f, v) + \lambda(E, \tau) + \frac{\lambda}{2}(D_k^u \sigma, \tau), \end{aligned} \tag{4.14}$$

with $E = \dfrac{\sigma(x) - \sigma(X^k(x))}{k} - u \cdot \nabla \sigma$.
Subtracting (4.8) from the equality (4.14), we obtain

$$\begin{aligned} &(F(\sigma)\sigma - F(\sigma_1)\sigma_2, \tau) + A((\sigma - \sigma_2, u - u_2), (\tau, v)) + \lambda B(u_1, \sigma - \sigma_2, \tau) = \\ &\qquad \lambda B(u_1, \sigma, \tau) - \lambda B(u, \sigma, \tau) + \lambda(g_a(\sigma_1, \nabla u_1) - g_a(\sigma, \nabla u), \tau) + \\ &\qquad\qquad 2\alpha(p, \nabla \cdot v) + \lambda(E, \tau) + \frac{\lambda}{2}(D_k^u \sigma, \tau). \end{aligned} \tag{4.15}$$

As usual, to obtain an error bound it is preferable to use $(\sigma_2 - \tilde{\sigma}, u_2 - \tilde{u})$. Inserting $(\tilde{\sigma}, \tilde{u})$ in (4.15) gives

$$\begin{aligned} &(F(\sigma_1)(\sigma_2 - \tilde{\sigma}), \tau) + A((\sigma_2 - \tilde{\sigma}, u_2 - \tilde{u}), (\tau, v)) + \lambda B(u_1, \sigma_2 - \tilde{\sigma}, \tau) = \\ &\quad (F(\sigma)\sigma, \tau) - (F(\sigma_1)\tilde{\sigma}, \tau) - \lambda B(u_1, \sigma, \tau) + \lambda B(u, \sigma, \tau) - \lambda(g_a(\sigma_1, \nabla u_1) - \\ &\qquad g_a(\sigma, \nabla u), \tau) - 2\alpha(p - \tilde{p}, \nabla \cdot v) + A((\sigma - \tilde{\sigma}, u - \tilde{u}), (\tau, v)) + \\ &\qquad \lambda B(u_1, \sigma - \tilde{\sigma}, \tau) + \lambda(E, \tau) + \frac{\lambda}{2}(D_k^u \sigma, \tau), \qquad \forall (\tau, v) \in T_h \times V_h. \end{aligned}$$

Setting $\hat{\sigma} = \sigma_2 - \tilde{\sigma}$, $\hat{u} = u_2 - \tilde{u}$, $\tau = \hat{\sigma}$ and $v = \hat{u}$ we obtain

$$\begin{aligned} &(F(\sigma_1)\hat{\sigma}, \hat{\sigma}) + A((\hat{\sigma}, \hat{u}), (\hat{\sigma}, \hat{u})) + \lambda B(u_1, \hat{\sigma}, \hat{\sigma}) = -2\alpha(p - \tilde{p}, \nabla \cdot \tilde{u}) + \\ &A((\sigma - \tilde{\sigma}, u - \tilde{u}), (\hat{\sigma}, \hat{u})) + (F(\sigma)\sigma, \hat{\sigma}) - (F(\sigma_1)\tilde{\sigma}, \hat{\sigma}) + \lambda B(u, \sigma, \hat{\sigma}) - \lambda B(u_1, \sigma, \hat{\sigma}) + \\ &\lambda(g_a(\sigma, \nabla u) - g_a(\sigma_1, \nabla u_1), \hat{\sigma}) + \lambda B(u_1, \sigma - \tilde{\sigma}, \hat{\sigma}) + \lambda(E, \hat{\sigma}) + \frac{\lambda}{2}(D_k^u \sigma, \hat{\sigma}). \end{aligned} \tag{4.16}$$

Now, we take $(\sigma_1, u_1) \in B_h^k$. In order to prove that M and h_0 can be chosen such that $\phi(B_h^k) \subset B_h^k$, $\forall h \leq h_0$, we use (4.16), first a coercivity relation can be derived directly from (4.12) and (4.10) as

$$\begin{aligned}
&(F(\sigma_1)\hat{\sigma}, \hat{\sigma}) + A((\hat{\sigma}, \hat{u}), (\hat{\sigma}, \hat{u})) + \lambda B(u_1, \hat{\sigma}, \hat{\sigma}) \geq \\
&\qquad e^{(-2\epsilon\frac{\lambda}{\alpha}|\sigma_1|_{0,\infty})} \mid \hat{\sigma} \mid^2 + 4\alpha(1-\alpha) \mid d(\hat{u}) \mid^2 + \\
&\qquad \frac{\lambda}{2k}(\hat{\sigma}(x) - \hat{\sigma}(X_{u_1}^k(x)), \hat{\sigma}(x) - \hat{\sigma}(X_{u_1}^k(x))). \qquad (4.17)
\end{aligned}$$

Next, we will give an estimate of (4.16). More precisely, the following results are shown

$$\begin{aligned}
\mid (p - \tilde{p}, \nabla \cdot \hat{u}) &\leq Ch^2 M \mid d(\hat{u}) \mid, & (4.18)\\
\mid A((\sigma - \tilde{\sigma}, u - \tilde{u}), (\hat{\sigma}, \hat{u})) \mid &\leq CMh^2(\mid \hat{\sigma} \mid^2 + \mid d(\hat{u}) \mid^2)^{\frac{1}{2}}, & (4.19)\\
\mid \lambda(g_a(\sigma, \nabla u) - g_a(\sigma_1, \nabla u_1), \hat{\sigma}) \mid &\leq C\lambda\Big[MC^*(\frac{h^2}{\sqrt{k}} + k) + \\
&\quad C^{*2}(\frac{h^2}{\sqrt{k}} + k)(\frac{h}{\sqrt{k}} + \frac{k}{h})\Big] \mid \hat{\sigma} \mid, & (4.20)\\
\mid \lambda B(u, \sigma, \hat{\sigma}) - \lambda B(u_1, \sigma, \hat{\sigma}) \mid &\leq \Big[C\lambda MC^*(\frac{h^2}{\sqrt{k}} + k) + \\
&\quad C\lambda kM(C^* + M)^2(1 + F)\Big] \mid \hat{\sigma} \mid, & (4.21)\\
\lambda B(u_1, \sigma - \tilde{\sigma}, \hat{\sigma}) &\leq C\lambda M\frac{h^2}{\sqrt{k}}B(u_1, \hat{\sigma}, \hat{\sigma}) + \\
&\quad C\lambda kM(C^* + M)^2(1 + F)h^2 \mid \hat{\sigma} \mid + \\
&\quad CM\lambda h\left(\frac{h^2}{\sqrt{k}} + k\right) \mid \hat{\sigma} \mid, & (4.22)\\
\lambda(E, \hat{\sigma}) &\leq C\lambda kM^3 \mid \hat{\sigma} \mid. & (4.23)\\
\frac{\lambda}{2}(D_k^u\sigma, \hat{\sigma}) &\leq C\lambda kM^2\left(1 + M + M^3 + M^4\right) \mid \hat{\sigma} \mid, & (4.24)\\
(F(\sigma)\sigma, \hat{\sigma}) - (F(\sigma_1)\tilde{\sigma}, \hat{\sigma}) &\leq CMe^{\epsilon\frac{\lambda}{\alpha}M_1}h^2 \mid \hat{\sigma} \mid + \\
&\quad C\frac{\epsilon\lambda}{\alpha}C^*\frac{h^2}{\sqrt{k}}M \mid \hat{\sigma} \mid, & (4.25)
\end{aligned}$$

where,

$$\begin{aligned}
F &= C^*(1 + M^2 + M^3) + C^{*2}(M + M^2) + C^{*3}(1 + M) + C^{*4} + M + M^3 + M^4.\\
M_1 &= M + Mh^{\frac{1}{2}} + Mh + C^*\frac{h}{\sqrt{k}}.
\end{aligned}$$

We recall that the estimations (4.18)-(4.24) are collected from [3]. For the estimation of the term F, we proceed as follows

$$\begin{aligned}(F(\sigma)\sigma,\hat{\sigma})-(F(\sigma_1)\tilde{\sigma},\hat{\sigma}) &= (F(\sigma_1)(\sigma-\tilde{\sigma}),\hat{\sigma})-(F(\sigma_1)\sigma,\hat{\sigma})+(F(\sigma)\sigma,\hat{\sigma})\\ &= (F(\sigma_1)(\sigma-\tilde{\sigma}),\hat{\sigma})+(\sigma F(\sigma)(1-F(\sigma_1-\sigma)),\hat{\sigma}). \quad (4.26)\end{aligned}$$

The first term in the right-hand side of (4.26) can be bounded as

$$\begin{aligned}(F(\sigma_1)(\sigma-\tilde{\sigma}),\hat{\sigma}) &\leq \mid F(\sigma_1)\mid\mid \sigma-\tilde{\sigma}\mid\mid\hat{\sigma}\mid,\\ &\leq e^{2\epsilon\frac{\lambda}{\alpha}|\sigma_1|_{0,\infty}}\mid\sigma-\tilde{\sigma}\mid\mid\hat{\sigma}\mid.\end{aligned}$$

If (σ_1,u_1) belongs to ball B_h^k, then

$$\begin{aligned}\mid\sigma_1\mid_{0,\infty} &\leq \mid\sigma\mid_{0,\infty}+\mid\sigma-\tilde{\sigma}\mid_{0,\infty}+\mid\tilde{\sigma}-\sigma_1\mid_{0,\infty},\\ &\leq \mid\sigma\mid_{0,\infty}+\mid\sigma-\tilde{\sigma}\mid_{1,4}+Ch^{-1}\mid\tilde{\sigma}-\sigma_1\mid,\\ &\leq C(M+Mh^{\frac{1}{2}}+h^{-1}\mid\sigma-\tilde{\sigma}\mid+h^{-1}\mid\tilde{\sigma}-\sigma_1\mid),\\ &\leq C\left(M+Mh^{\frac{1}{2}}+Mh+C^*\frac{h}{\sqrt{k}}\right)=CM_1,\end{aligned}$$

then consequently we have

$$(F(\sigma_1)(\sigma-\tilde{\sigma}),\hat{\sigma})\leq CMe^{\epsilon\frac{\lambda}{\alpha}M_1}h^2\mid\hat{\sigma}\mid.$$

For the second term in the right-hand side of (4.26) we have

$$\begin{aligned}(\sigma F(\sigma)(1-F(\sigma_1-\sigma)),\hat{\sigma}) &\leq C\mid\sigma\mid_{0,\infty}e^{2\epsilon\frac{\lambda}{\alpha}|\sigma|_{0,\infty}}\mid 1-F(\sigma_1-\sigma)\mid\mid\hat{\sigma}\mid,\\ &\leq C\mid\sigma\mid_{0,\infty}e^{2\epsilon\frac{\lambda}{\alpha}|\sigma|_{0,\infty}}\mid\frac{\epsilon\lambda}{\alpha}tr(\sigma_1-\sigma)\mid\mid(1+\\ &\quad\frac{\epsilon\lambda}{2\alpha}tr(\sigma_1-\sigma)e^{\epsilon_1\frac{\lambda}{\alpha}tr(\sigma_1-\sigma)})\mid\mid\hat{\sigma}\mid,\\ &\leq C\mid\sigma\mid_{0,\infty}e^{2\epsilon\frac{\lambda}{\alpha}|\sigma|_{0,\infty}}\frac{\epsilon\lambda}{\alpha}\mid\sigma_1-\sigma\mid\mid\hat{\sigma}\mid\mid(1+\\ &\quad\frac{\epsilon\lambda}{2\alpha})(\mid\sigma_1\mid_{0,\infty}+\mid\sigma\mid_{0,\infty})e^{\epsilon_1\frac{\lambda}{\alpha}(|\sigma_1|_{0,\infty}+|\sigma|_{0,\infty})}\mid,\\ &\leq CMe^{2\epsilon\frac{\lambda}{\alpha}|\sigma|_{0,\infty}}\frac{\epsilon\lambda}{\alpha}C^*\frac{h^2}{k}\mid\hat{\sigma}\mid(1+\\ &\quad\frac{\epsilon\lambda}{2\alpha})(M+M_1)e^{\epsilon_1\frac{\lambda}{\alpha}(M+M_1)},\\ &\leq C\frac{\epsilon\lambda}{\alpha}C^*\frac{h^2}{\sqrt{k}}M\mid\hat{\sigma}\mid,\end{aligned}$$

then we get

$$(-F(\sigma_1)\tilde{\sigma}, \hat{\sigma}) + (F(\sigma)\sigma, \hat{\sigma}) \leq CMe^{\epsilon\frac{\lambda}{\alpha}M_1}h^2 \mid \hat{\sigma} \mid + C\frac{\epsilon\lambda}{\alpha}C^* \frac{h^2}{\sqrt{k}} M \mid \hat{\sigma} \mid .$$

Using (4.18) to (4.25) we obtain with $\delta = \sqrt{\alpha(1-\alpha)}$

$$\begin{aligned}
\mid \hat{\sigma} \mid + 2\delta \mid d(\hat{u}) \mid + \lambda^{\frac{1}{2}}(B(u_1, \hat{\sigma}, \hat{\sigma}))^{\frac{1}{2}} \leq\ & C_1\delta^{-1}Mh^2 + C_2MC^*\left(\frac{h^2}{\sqrt{k}} + k\right) + \\
& C_3\lambda kM(C^* + M)^2(1+F)(1+h^2) + \\
& C_4M(1+h)\left(\frac{h^2}{\sqrt{k}} + k\right) + \\
& C_5\lambda kM^3 + CMe^{\epsilon\frac{\lambda}{\alpha}M_1}h^2. \qquad (4.27)
\end{aligned}$$

Hence, for C^* sufficiently small we can choose M_0 small enough to ensure that for $M \leq M_0$ the right-hand side in (4.27) is smaller than $\frac{C^*}{2}\left(\frac{h^2}{\sqrt{k}} + k\right)$. Then

$$\begin{aligned}
\mid \sigma_2 - \sigma \mid + 2\delta \mid d(u_2 - u) \mid \ &\leq \ \mid \hat{\sigma} \mid + 2\delta \mid d(\hat{u}) \mid + \mid \tilde{\sigma} - \sigma \mid + 2\delta \mid d(\tilde{u} - u) \mid, \\
&\leq \ \frac{1}{2}C^*\left(\frac{h^2}{\sqrt{k}} + k\right) + C(1+2\delta)Mh^2, \\
&\leq \ C^*\left(\frac{h^2}{\sqrt{k}} + k\right),
\end{aligned}$$

by decreasing k_0 and M_0 if necessary. This proves that $\phi(B_h^k) \subset B_h^k$ and from Brouwer's fixed point theorem follows the existence of $(\sigma_h^k, u_h^k) \in T_h \times V_h$ satisfying (4.6).

(4) End of the proof of the theorem:

The application ϕ satisfies on B_h^k the hypothesis of Brouwer's fixed point theorem and consequently admits a fixed point $(\sigma_h^k, \sigma_h^k) \in B_h^k$. Then

$$\begin{aligned}
\mid \sigma - \sigma_h^k \mid \ &\leq \ C\left(\frac{h^2}{\sqrt{k}} + k\right), \\
\mid d(u - u_h^k) \mid \ &\leq \ C\left(\frac{h^2}{\sqrt{k}} + k\right),
\end{aligned}$$

It remains to bound $\mid p - p_h^k \mid$. To do so, we have

$$(\sigma_h^k, d(v)) - 2(1-\alpha)(d(u_h^k), d(v)) = (f, v), \qquad \forall v \in V_h.$$

From the infsup condition (2.1) on (X_h, Q_h) there exists $p_h \in Q_h$ such that

$$(\sigma_h^k, d(v)) - 2(1-\alpha)(d(u_h^k), d(v)) - (p_h^k, \nabla \cdot v) = (f, v), \qquad \forall v \in X_h.$$

On the other hand, and since p is regular

$$(\sigma, d(v)) - 2(1-\alpha)(d(u), d(v)) - (p, \nabla \cdot v) = (f, v), \qquad \forall v \in X_h.$$

Thus,

$$(\sigma - \sigma_h^k, d(v)) - 2(1-\alpha)(d(u-u_h^k), d(v)) = (p - p_h^k, \nabla \cdot v), \qquad \forall v \in X_h.$$

This implies,

$$(\tilde{p} - p_h^k, \nabla \cdot v) = (\tilde{p} - p, \nabla \cdot v) + (\sigma - \sigma_h^k, d(v)) - 2(1-\alpha)(d(u-u_h^k), d(v)), \qquad \forall v \in X_h.$$

Using again the infsup condition (2.1) we get

$$\begin{aligned} |\, \tilde{p} - p_h^k \,| &\leq C \left(|\, \tilde{p} - p \,| + |\, \sigma_h^k - \sigma \,| + |\, d(u - u_h^k) \,| \right), \\ &\leq C \left(\frac{h^2}{\sqrt{k}} + k \right), \end{aligned}$$

which concludes the proof of (4.7).

■

REMARK **4.1** *We would like to mention that the results in Theorem 4.1 can be extended to a discontinuous upwinding approximation of σ leading to an error bound of order* $\mathcal{O}\left(\frac{h^{r+\frac{1}{2}}}{\sqrt{k}} + k \right)$.

References

[1] N. PHAN-THIEN, RI. TANNER. *A new constitutive equation derived from network theory*, J. Non-Newtonian Fluid Mech. 2 (1977) 353-365.

[2] J. BARANGER, A. MACHMOUM. *A "natural" norm for the method of characteristics*, to appear in M^2AN.

[3] A. MACHMOUM, D. ESSELAOUI. *Finite element approximation of viscoelastic fluid flow using characteristics method*, Comput. Methods Appl. Mech. Engrg. 190 (2001) 5603-5618.

[4] J. BARANGER, D. ESSELAOUI, A. MACHMOUM. *Error estimate for convection problem with characteristics method* Numer. Algorithms 21 (1999) 49-56.

[5] J. BARANGER, D. SANDRI *Finite element approximation of viscoelastic fluid flow Existence of solutions and error bounds. I-Discontinuous constraints*, Numer. Math. 63 (1992) 13-27.

[6] A. Bermudez, J. Durany. *La méthode des caractéristiques pour les problèmes de convection-diffusion stationnaires* M^2AN, vol, 21 n^o 1, 7-26, 1987.

[7] V. Girault, P. A. Raviart. *Finite element method for Navier Stokes equations, Theory and Algorithms*. Berlin Heidelberg New York : Springer 1986.

[8] P.G. Ciarlet. *The finite element method for elliptic problems* (North Holland, Amsterdam, 1978).

[9] O. Pironneau. *On the transport-diffusion algorithm and its application to the Navier-Stokes equations*, Numer. Math. 38 (1982) 309.

[10] R. Keunings, *On the high Weissenberg Number problem*, J. Non-Newtonian Fluid Mech. 20 (1986) 209-226.

[11] R.G. Larson. *Constitutive equations for polymer melts and solutions*, Butterworths Series in Chemical Engineering, 1988.

[12] J.M. Marchal and M.J. Crochet, *A new finite element for calculating viscoelastic flow*, J. Non-Newtonian Fluid Mech. 26 (1987) 449-451.

[13] M. Fortin, A. Fortin. *Une note sur les méthodes de caractéristiques et de* Lesaint-Raviart *pour les problèmes hyperboliques stationnaires*. M^2AN vol, 23, n^o 4, 593-596, 1989.

[14] M. Fortin, A. Fortin. *A preconditioned GMRES algorithm for the numerical solution of viscoelastic fluid flows*, J. Non-Newtonian Fluid Mech. 36 (1990) 277-288.

[15] M. Fortin, A. Fortin. *A new approach for the finite element simulation of viscoelastic fluid flows*, J. Non-Newtonian Fluid Mech. 32(1989) 295-310.

[16] M. Fortin, A. Zine, J.F. Agassant. *Computing viscoelastic fluid flow problem at low cost*, J. Non-Newtonian fluid Mech. 45(1992) 209-229.

[17] M. Fortin, A. Zine. *An improved GMRES method for solving viscoelastic fluid flow problems*, J. Non-Newtonian fluid Mech. 42(1992) 1-18.

[18] M. Fortin, D. Esselaoui, *A finite element procedure for viscoelastic flows*, Int. J. Numer. Method 7 (1987).

[19] F.G. Basombrio, *Flows in viscoelastic fluids treated by the method of characteristics*, J. Non-Newtonian Fluid Mech. 39 (1991) 17-34.

[20] D. Sandri, *Remarques sur une formulation à trois champs du problème de Stokes equations*, Numer. Math. 27 (1993) 817-841.