A geometrically exact Cosserat shell-model including size effects, avoiding degeneracy in the thin shell limit. Existence of minimizers for zero Cosserat couple modulus.

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6th September 2004

Abstract

This paper establishes the existence of minimizers to a finite-strain, geometrically exact Cosserat plate model. The membrane energy of the investigated model is a quadratic, uniformly Legendre-Hadamard elliptic energy in contrast to classical approaches. The bending contribution is augmented by a curvature term representing an additional stiffness of the Cosserat theory and the corresponding nonlinear system of balance equations remains of second order. The lateral boundary conditions corresponding to simple support are non-standard. The model includes size effects, transverse shear resistance, drilling degrees of freedom and accounts implicitly for thickness extension and asymmetric shift of the midsurface. The formal thin shell "membrane" limit without classical h^3-bending term is non-degenerate due to the additional Cosserat curvature stiffness and control of drill rotations. In this formulation, the drill-rotations are strictly related to the size-effects of the Cosserat bulk model and not introduced artificially for numerical convenience. Upon linearization with zero Cosserat couple modulus $\mu_c = 0$ exclusively, we recover the well known infinitesimal-displacement Reissner-Mindlin model without size-effects and without drill-rotations.

It is shown that this new finite-strain Cosserat plate formulation is well-posed for $\mu_c = 0$ by means of the direct methods of variations. The midsurface deformation m is found in $H^1(\omega, \mathbb{R}^3)$. Decisive use is made of a dimensionally reduced version of an extended Korn's first inequality proved by the author.

Key words: shells, plates, membranes, thin films, polar materials, non-simple materials, solid mechanics, variational methods.

AMS 2000: 74K20, 74K25, 74B20, 74D10, 74A35, 74E05, 74G65, 74N15, 74K35

short-title: Existence for finite Cosserat plate with zero couple modulus.

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Contents

1 Introduction

1.1 Some aspects of shell theory

The dimensional reduction of a given continuum-mechanical model is already an old and mature subject and it has seen many "solutions". The different approaches toward elastic shell theory proposed in the literature and relevant references thereof are, therefore, too numerous to list here. The investigated model herein falls within the so called **derivation approach**, i.e., reducing a given three-dimensional model via physically reasonable constitutive assumptions on the kinematics to a two-dimensional model. This is opposed to either the **intrinsic approach** which views the shell from the onset as a two-dimensional surface and invokes concepts from differential geometry or the **asymptotic methods** which try to establish two-dimensional equations by formal expansion of the three-dimensional solution in power series in terms of a small parameter. The intrinsic approach is closely related to the **direct approach** which takes the shell to be a two-dimensional medium with additional **extrinsic directors** in the sense of a **restricted Cosserat surface** [CC09].[1] There, two-dimensional equilibrium in appropriate new resultant stress and strain variables is postulated ab-initio more or less independent of three-dimensional considerations, cf. [Ant95, GNW65, ET58, CD66b, CD66a, CW89, Rub00, PG89].

A detailed presentation of the different approaches in classical shell theories can be found in the monograph [Nag72]. A thorough mathematical analysis of linear, infinitesimal-displacement shell theory, based on asymptotic methods is to be found in [Cia98a] and the extensive references therein, see also [Cia97, Cia99, Ant95, DS96, Dik82, CSP95]. Excellent reviews and insightful discussions of the modelling and finite element implementation may be found in [SB92, San95, SB98, GSW89, GT92, WG93, BGS96, BR92] and in the series of papers [SF89, SFR89, SFR90, SRF90, SK92, SF92]. Properly invariant, geometrically exact, elastic plate theories are derived by formal asymptotic methods in [FRS93]. This formal derivation is extended to curvilinear shells in [Mia98, LM98]. Apart from the pure bending case [FJM02], which is rigourously justified as the Γ-limit [Mas92] of the three-dimensional model and which can be shown to be intrinsically well-posed, the obtained finite-strain models have not yet been shown to be well-posed. Indeed, the membrane energy contribution is notoriously not Legendre-Hadamard elliptic. The membrane model justified in [DR96] by Γ-convergence is geometrically exact and quasiconvex/elliptic but unfortunately does not coincide upon linearization with the otherwise well-established infinitesimal-displacement membrane model. Moreover, this model does not describe the detailed geometry of deformation in compression but reduces to a tension-field theory [Ste90].

There is no place here to comment further on the relative merits of each alternative approach. The "rational" of descend from three to two dimensions should in any case be complemented by an investigation of the intrinsic mathematical properties of the obtained reduced models. Today, the need to simulate the mechanical response of highly flexible thin structures allowing easily for finite rotations excludes the use of classical infinitesimal-displacement models, either of Reissner-Mindlin (5.11) or Kichhoff-Love type (5.14). Also, certain "intermediary" models allowing in principle for buckling like the "nonlinear" von Kármán plates [Cia97, p.403] and penalized "nonlinear" Reissner-Mindlin models [Dhi95][2] or "semilinear" Kirchhoff-Love plate models [Mon03] are not geometrically exact (not frame-indifferent). Nevertheless, the nonlinear von Kármán plate has been succesfully applied to the delamination problem of thin films [OG94, GO97, GDOC02].

Mielke [Mie95] established in the infinitesimal-displacement context that by using more than five ansatz-functions in a director model it is possible to obtain exponential decay estimates for the boundary layer and to establish therefore a St.Venant principle for linearized plates. While it is not clear how his methods can be transferred to the finite-strain case, they provide, independent of mechanical/physical considerations, a strong motivation to use a director ansatz also in the finite-strain case in order to better capture the boundary layer phenomena.

Indeed, so called shear-deformable theories with independent directors are usually preferred in the engineering community [AMZ02, CB03]. In view of an efficient finite element implementation one considers a hyperelastic, variationally based formulation with second-order Euler-

[1] Restricted, since no material length scale usually enters the direct approach, only the **relative thickness** h appears in the model. In terminology I distinguish between a "true" Cosserat model operating on $SO(3,\mathbb{R})$ and theories with any number of directors.

[2] Conceptually a von Kármán plate with one independent director $\vec{d}$ and addition of a penalisation term $\mu_c\left(\langle\vec{d},\partial_x m\rangle^2+\langle\vec{d},\partial_y m\rangle^2\right)$, $\mu_c\to\infty$ with m the sought midsurface deformation.

Lagrange equations and uses standard C^0-conforming elements. The prototype examples are models based on the Reissner-Mindlin kinematical assumption. There are numerous proposals in the engineering literature for a finite-strain, geometrically exact plate formulation, see e.g. [FS92, SB92, SB95, SB98, WG93, BGS96, BR92]. In many cases the need has been felt to devote specific attention to proper rotations $R \in \mathrm{SO}(3, \mathbb{R})$, since finite rotations are the dominant deformation mode of a flexible structure. This has led to the so called **drill-rotation formulation** which means that proper rotations either appear in the formulation as independent fields (leading to a restricted Cosserat surface) or they are an intermediary ingredient [HB89] in the numerical treatment (constraint Cosserat surface). While the computational merit of this approach is well documented, a mathematical analysis for such a family of finite-strain plate models is yet missing, both for the Cosserat surface and the constraint model. It may be speculated that those restricted Cosserat plates (obtained from classical non-polar bulk models or from direct modelling) though geometrically exact and allowing for transverse shear and the description of boundary layers, might not be well posed for certain membrane strain measures either, notably if Green-strains: $F^T F - 1\!\!1$ or Hencky-strains: $\ln F^T F$ are used. Another drawback from a modelling point of view is that the inclusion of drill-rotations is most often done in an ad-hoc fashion.

1.2 Limitations of existing shell models

The classical infinitesimal-displacement or finite-strain plate-models proposed in the literature lead to effective numerical schemes only if the relative thickness h of the structure is still appreciable, i.e. classical bending terms are present and regularize the computation. However, there is an abundance of new applications where very thin (absolutely thin) structures are used, e.g. very thin metal layers on a substrate (in computer hardware, for the characteristic relative thickness $h \leq 5 \cdot 10^{-4}$). In these cases, classical bending energy, which comes with a factor of h^2 compared with the membrane energy contribution, cannot play a stabilizing role for non-vanishing membrane energy. See [BJ99] for such a problem occurring in thin films. But, as we noted already, the **membrane terms** e.g. in a finite-strain, invariant Kirchhoff-Love plate [FRS93] or finite-strain Reissner-Mindlin model [FS92] are **non-elliptic** (degenerated) and the remaining minimization problem might not be well-posed even if classical bending is included.

It is also observed experimentally that **very thin structures** behave **comparably stiffer** than absolutely thicker structures while both have the **same relative thickness**. These **non-classical size effects** cannot be neglected for very thin structures [CCC+03]. Such effects are, however, not accounted for in classical theories.

In addition, classical infinitesimal-displacement or finite-strain shell models predict unrealistically high levels of smoothness, typically $m \in W^{1,4}(\omega, \mathbb{R}^3)$ for the midsurface m in both finite-strain Kirchhoff-Love and Reissner-Mindlin models and $m \in H^2(\omega, \mathbb{R}^3)$ in the finite-strain pure bending problem [FJM02] and the von Kármán model. This implies at least $C^{0,\alpha}(\omega)$ for the midsurface which rule out the description of boundary layer effects and possible failure along asymptotic lines of the surface.

1.3 Scope of study

I have therefore proposed a new shell model (described in (4.1)) for very thin almost rigid materials which should remedy some of the aforementioned shortcomings with a view towards a subsequent stringent mathematical analysis and possible stable finite element implementation. It is the goal to provide a model which is both theoretically and physically sound, such that its numerical implementation can concentrate on real convergence issues. Let me summarize what I require of a general, all purpose, consistent first approximation plate model. I require

1. A finite-strain formulation which is geometrically exact and allows for finite rotations.

2. The description of transverse shear, drill rotations, thickness stretch and asymmetric shift of the midsurface. This excludes normality assumptions for some director.

3. A qualitative resolution of the boundary layer and edge effect compared with the bulk model.

4. Well-posedness: existence, but not unqualified uniqueness in order to be able to describe buckling due to membrane forces, e.g. under lateral compression or lateral shear and avoiding unqualified smoothness for the midsurface, requiring only $m \in H^{1,2}(\omega, \mathbb{R}^3)$.

5. A hyperelastic, variational formulation with second-order Euler-Lagrange equations in view of an efficient finite element implementation with standard C^0-conforming elements.

6. A reduced energy density which is defined in terms of two-dimensional quantities with a clear physical meaning of these reduced two-dimensional quantities. Maximally h^3-bending contributions.

7. The incorporation of non-classical size effects without leading to trivial compactness arguments for the the midsurface m.[3] The model must also be "operative" without the classical h^3-bending contribution, i.e. in the formal "membrane" thin shell limit.

8. The consistency with classical plate models (infinitesimal displacement Reissner-Mindlin (5.11), infinitesimal-displacement Kirchhoff-Love (5.14)) upon linearization and consistency with rigourously justified finite-strain Kirchhoff-Love bending model [FJM02, FRS93] in pure bending for large samples (classical limit of vanishing internal length L_c).

1.4 Outline of this contribution

The basic idea to meet these requirements for a plate model is to descend from a three-dimensional Cosserat model. First, we introduce therefore in section (2) the underlying "parent" three-dimensional finite-strain frame-indifferent Cosserat model with **size effects** and already appearing **independent microrotations** $\overline{R}$, i.e. a **triad of rigid directors** $(\overline{R}_1|\overline{R}_2|\overline{R}_3) = \overline{R} \in \mathrm{SO}(3,\mathbb{R})$ and we recall the obtained existence results for this Cosserat bulk model. We then provide the restriction of the bulk model to a thin domain (3.1) on which the reduction is based. Applying our "rational" of dimensional descend we postulate in section (4.1) the full two-field minimization problem for the new Cosserat plate model [Nef03a, Nef04a]. It must be observed that the resulting Cosserat plate model cannot be obtained from a simple **energy projection**, such that the already obtained three-dimensional results do not apply.

The corresponding equilibrium problem defined over the two-dimensional referential domain $\omega \subset \mathbb{R}^2$ has six **d**egrees **o**f **f**reedom (three for the midsurface deformation $m : \omega \mapsto \mathbb{R}^3$ and three for the independent rotations $\overline{R} : \omega \mapsto \mathrm{SO}(3,\mathbb{R})$, 6 **dof**) and constitutes a nonlinear, partial differential elliptic system of six equations for basically six unknown functions. The derivation of these Euler-Lagrange equations is standard and therefore not presented. The model includes naturally one-drilling degree of freedom for in-plane rotations and accounts for thickness stretch and transverse shear. The drilling degree is strictly related to the size-effect of the bulk model and not specifically introduced in an ad hoc fashion by the dimensional reduction. The model features also a **non-standard boundary condition**, which is called **consistent coupling**.

In section (4.3), we derive a new Korn's first inequality for plates and elasto-plastic shells which is decisive for the mathematical treatment of models obtained in our variational context. Depending on material constants and boundary conditions, different mathematical existence theorems are proposed in section (4.4). Generically, we obtain for the midsurface deformation $m \in H^{1,2}(\omega, \mathbb{R}^3)$. For these results the direct methods of variations are used.

The **quasiconvexity** of the reduced energy functional $I(m, \overline{R})$ in the pair $(m, \overline{R})$ is rather easy to see, however, **unqualified coercivity** [PGC91][4] w.r.t. the midsurface deformation m depends crucially on the **uniform positivity** of the **Cosserat couple modulus** $\mu_c > 0$. The simpler existence of minimizers in this case is established elsewhere [Nef03a, Nef04a].

For zero Cosserat couple modulus $\mu_c = 0$, the lack of unqualified coercivity, however, can only be overcome by a certain control of the curvature in conjunction with the new Korn's inequality for plates.

In order to treat external loads for zero Cosserat couple modulus $\mu_c = 0$, the resultant load functional Π has to be adapted. This modification, which is already needed in the Cosserat bulk model, has been termed there **"principle of bounded external work"** [Nef04c] and expresses the observation that by simple translation of a solid in a force field only a finite amount of energy can be gained which is certainly true for any classical physical field. If we want to treat the non-standard boundary condition of very weak consistent coupling, we need to augment the energy functional with an additional curvature control on the lateral Dirichlet boundary γ_0. The mathematical analysis is also extended to a new Cosserat plate model appropriate for large stretch which has appealing physical features.

[3] Adding a second derivative $L_c^p \, \|D^2 m\|^p$ to the energy density would "resolve" all mathematical difficulties but lead to $m \in W^{2,p}(\omega, \mathbb{R}^3)$.

[4] In finite elasticity: $W(F) \geq c_1^+ \|F\|^p - c_2^+$, $p \geq 2$.

In order to relate the new finite-strain Cosserat plate model to classical approaches, we show then, that a **linearization** of the new plate model **with zero Cosserat couple modulus** $\mu_c = 0$ **results in** the classical infinitesimal-displacement **Reissner-Mindlin** model (without extra size effects and therefore without drill-rotations) and shear correction factor $\kappa = 1$. However, weaker boundary conditions for the increment of the director in the linearized infinitesimal-displacement Reissner-Mindlin model (5.11) are motivated. Nevertheless, this new boundary condition reduces to the classical condition on the increment of the normal in the linearized Kirchhoff-Love model (5.14). Finally, the treatment of external loads is detailed.

1.5 Notation

1.5.1 Notation for bulk material

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3\times3}$. The identity tensor on $\mathbb{M}^{3\times3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[X] = \langle X, 1\!\!1\rangle$ and $\operatorname{tr}[X]^2 = \langle X, 1\!\!1\rangle^2$. We let Sym and PSym denote the symmetric and positive definite symmetric tensors respectively. We adopt the usual abbreviations of Lie-group theory, i.e., $\mathrm{GL}(3,\mathbb{R}) := \{X \in \mathbb{M}^{3\times3} \;|\det[X] \neq 0\}$ the general linear group, $\mathrm{SL}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \;|\det[X] = 1\}$, $\mathrm{O}(3) := \{X \in \mathrm{GL}(3,\mathbb{R}) \mid X^T X = 1\!\!1\}$, $\mathrm{SO}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \;|X^T X = 1\!\!1,\ \det[X] = 1\}$ with corresponding Lie-algebras $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times3} \;|X^T = -X\}$ of skew symmetric tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{M}^{3\times3} \;|\operatorname{tr}[X] = 0\}$ of traceless tensors. With $\operatorname{Adj} X$ we denote the tensor of transposed cofactors $\mathrm{Cof(X)}$ such that $\operatorname{Adj} X = \det[X]\, X^{-1} = \mathrm{Cof(X)^T}$ if $X \in \mathrm{GL}(3,\mathbb{R})$. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$. For $X \in \mathbb{M}^{3\times3}$ we set for the deviatoric part $\operatorname{dev} X = X - \frac{1}{3}\operatorname{tr}[X]\, 1\!\!1 \in \mathfrak{sl}(3)$ and for vectors $\xi, \eta \in \mathbb{R}^n$ we have the tensor product $(\xi \otimes \eta)_{ij} = \xi_i\, \eta_j$.

We write the polar decomposition in the form $F = RU = \operatorname{polar}(F)U$ with $R = \operatorname{polar}(F)$ the orthogonal part of F. For a second order tensor X we define the third order tensor $\mathfrak{h} = \mathrm{D_x} X(x) = (\nabla(X(x).e_1), \nabla(X(x).e_2), \nabla(X(x).e_3)) = (\mathfrak{h}^1, \mathfrak{h}^2, \mathfrak{h}^3) \in \mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3} \cong \mathfrak{T}(3)$. For third order tensors $\mathfrak{h} \in \mathfrak{T}(3)$ we set $\|\mathfrak{h}\|^2 = \sum_{i=1}^3 \|\mathfrak{h}^i\|^2$ together with $\operatorname{sym}(\mathfrak{h}) := (\operatorname{sym}\mathfrak{h}^1, \operatorname{sym}\mathfrak{h}^2, \operatorname{sym}\mathfrak{h}^3)$ and $\operatorname{tr}[\mathfrak{h}] := (\operatorname{tr}\left[\mathfrak{h}^1\right], \operatorname{tr}\left[\mathfrak{h}^2\right], \operatorname{tr}\left[\mathfrak{h}^3\right]) \in \mathbb{R}^3$. Moreover, for any second order tensor X we define $X \cdot \mathfrak{h} := (X\mathfrak{h}^1, X\mathfrak{h}^2, X\mathfrak{h}^3)$ and $\mathfrak{h} \cdot X$, correspondingly. Quantities with a bar, e.g. the micropolar rotation $\overline{R}$, represent the micropolar replacement of the corresponding classical continuum rotation R. In general we work in the context of nonlinear, finite-strain elasticity. For the total deformation $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^3)$ we have the deformation gradient $F = \nabla\varphi \in C(\overline{\Omega}, \mathbb{M}^{3\times3})$. Furthermore, $S_1(F) = D_F W(F)$ and $S_2(F) = F^{-1} D_F W(F)$ denote the first and second Piola Kirchhoff stress tensors, respectively. Total time derivatives are written $\frac{\mathrm{d}}{\mathrm{dt}} X(t) = \dot{X}$. The first and second differential of a scalar valued function $W(F)$ are written $D_F W(F).H$ and $D_F^2 W(F).(H, H)$, respectively. We employ the standard notation of Sobolev spaces, i.e. $L^2(\Omega), H^{1,2}(\Omega), H^{1,2}_\circ(\Omega), W^{1,q}(\Omega)$, which we use indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions. The set $W^{1,q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$ denotes orthogonal tensors whose components are in $W^{1,q}(\Omega)$. Moreover, we set $\|X\|_\infty = \sup_{x\in\Omega} \|X(x)\|$. For $A \in C^1(\overline{\Omega}, \mathbb{M}^{3\times3})$ we define $\operatorname{Curl} A(x)$ as the operation curl applied row wise. We define $H^{1,2}_\circ(\Omega, \Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|_\Gamma} = 0\}$, where $\phi_{|_\Gamma} = 0$ is to be understood in the sense of traces and by $C_0^\infty(\Omega)$ we denote infinitely differentiable functions with compact support in Ω. We use capital letters to denote possibly large positive constants, e.g. C^+, K and lower case letters to denote possibly small positive constants, e.g. c^+, d^+. The smallest eigenvalue of a positive definite symmetric tensor P is abbreviated by $\lambda_{\min}(P)$.

1.5.2 Notation for plates and shells

Let $\omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\omega$ and let γ_0 be a smooth subset of $\partial\omega$ with non-vanishing 1-dimensional Hausdorff measure. The thickness of the plate is taken to be $h > 0$ with dimension length (contrary to Ciarlet's definition of the thickness to be 2ε, which difference leads only to various different constants in the resulting formulas). We denote by $\mathbb{M}^{n\times m}$ the set of matrices mapping $\mathbb{R}^n \mapsto \mathbb{R}^m$. For $H \in \mathbb{M}^{2\times3}$ and $\xi \in \mathbb{R}^3$ we employ also the notation $(H|\xi) \in \mathbb{M}^{3\times 3}$ to denote the matrix composed of H and the column ξ. Likewise $(v|\xi|\eta)$ is the matrix composed of the columns v, ξ, η. This allows us to write for $\varphi \in C^1\mathbb{R}^3, \mathbb{R}^3)$: $\nabla\varphi = (\varphi_x|\varphi_y|\varphi_z) = (\partial_x\varphi|\partial_y\varphi|\partial_z\varphi)$. The identity tensor on $\mathbb{M}^{2\times 2}$ will be denoted by $1\!\!1_2$. For $B \in \mathbb{M}^{2\times 2}$ we define $B^\flat = \begin{pmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{M}^{3\times 3}$. The mapping $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ is the deformation of the midsurface, ∇m is the corresponding deformation gradient and $\vec{n}_m$ is the outer unit normal on m. A matrix $X \in \mathbb{M}^{3\times 3}$ can now be written as $X = (X.e_2|X.e_2|X.e_3) = (X_1|X_2|X_3)$. We write $v : \mathbb{R}^2 \mapsto \mathbb{R}^3$ for the displacement of the midsurface, such that $m(x, y) = (x, y, 0)^T + v(x, y)$. The standard volume element is written $\mathrm{dx}\,\mathrm{dy}\,\mathrm{dz} = \mathrm{dV} = \mathrm{d}\omega\,\mathrm{dz}$.

2 The underlying finite-strain three-dimensional Cosserat model in variational form

In [Nef03b] a finite-strain, fully frame-indifferent, three-dimensional Cosserat micropolar model is introduced. The **two-field** problem has been posed in a variational setting. The task is to find a pair $(\varphi, \overline{R}) : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3 \times \mathrm{SO}(3, \mathbb{R})$ of deformation φ and **independent microrotation** $\overline{R} \in \mathrm{SO}(3, \mathbb{R})$ minimizing the energy functional I,

$$
\begin{aligned}
I(\varphi, \overline{R}) = \int_\Omega W_{\mathrm{mp}}(\overline{R}^T\nabla\varphi) + W_{\mathrm{curv}}(\overline{R}^T \mathrm{D_x}\overline{R}) - \Pi_f(\varphi) - \Pi_M(\overline{R})\,\mathrm{dV} \\
- \int_{\Gamma_S} \Pi_N(\varphi)\,\mathrm{dS} - \int_{\Gamma_C} \Pi_{M_c}(\overline{R})\,\mathrm{dS} \mapsto \min.\ \text{w.r.t. } (\varphi, \overline{R}), \qquad (2.1)
\end{aligned}
$$

together with the Dirichlet boundary condition of place for the deformation φ on Γ: $\varphi_{|_\Gamma} = g_{\mathrm{d}}$ and three possible **alternative** boundary conditions for the microrotations $\overline{R}$ on Γ,

$$
\overline{R}_{|_\Gamma} = \begin{cases} \overline{R}_{\mathrm{d}}\,, & \text{the case of } \textbf{rigid} \text{ prescription}\,, \\ \mathrm{polar}(\nabla\varphi)\,, & \text{the case of } \textbf{consistent coupling}\,, \\ \text{no condition for } \overline{R} \text{ on } \Gamma, & \textbf{induced Neumann-type} \text{ relations for } \overline{R} \text{ on } \Gamma\,. \end{cases} \qquad (2.2)
$$

The constitutive assumptions on the densities are

$$
\begin{aligned}
W_{\mathrm{mp}}(\overline{U}) &= \mu\,\|\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c\,\|\mathrm{skew}(\overline{U})\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2,\ \overline{U} = \overline{R}^T F\,,\ F = \nabla\varphi\,, \\
W_{\mathrm{curv}}(\mathfrak{K}) &= \mu\frac{L_c^{1+p}}{12}\,\left(1 + \alpha_4\,L_c^q\,\|\mathfrak{K}\|^q\right)\left(\alpha_5\,\|\mathrm{sym}\,\mathfrak{K}\|^2 + \alpha_6\,\|\mathrm{skew}\,\mathfrak{K}\|^2 + \alpha_7\,\mathrm{tr}\,[\mathfrak{K}]^2\right)^{\frac{1+p}{2}}\,, \qquad (2.3) \\
\mathfrak{K} &= \overline{R}^T\mathrm{D_x}\overline{R} := \left(\overline{R}^T\nabla(\overline{R}.e_1), \overline{R}^T\nabla(\overline{R}.e_2), \overline{R}^T\nabla(\overline{R}.e_3)\right),\ \text{the third order } \textbf{curvature tensor}\,.
\end{aligned}
$$

The total elastically stored energy $W = W_{\mathrm{mp}} + W_{\mathrm{curv}}$ is quadratic in the stretch $\overline{U}$ and possibly super-quadratic in the curvature $\mathfrak{K}$. The strain energy W_{mp} depends on the deformation gradient $F = \nabla\varphi$ and the microrotations $\overline{R} \in \mathrm{SO}(3, \mathbb{R})$, which do not necessarily coincide with the **continuum rotations** $R = \mathrm{polar}(F)$. The curvature energy W_{curv} depends moreover on the space derivatives $\mathrm{D_x}\overline{R}$ which describe the self-interaction of the microstructure.[5] In general, the **micropolar stretch tensor** $\overline{U}$ is **not symmetric** and does not coincide with the **symmetric continuum stretch** tensor $U = R^TF = \sqrt{F^TF}$. By abuse of notation we set $\|\mathrm{sym}\,\mathfrak{K}\|^2 := \sum_{i=1}^3 \|\mathrm{sym}\,\mathfrak{K}^i\|^2$ for third order tensors $\mathfrak{K}$, cf.(1.5.1).

Here $\Omega \subset \mathbb{R}^3$ is a domain with boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ is that part of the boundary, where Dirichlet conditions $g_{\mathrm{d}}, \overline{R}_{\mathrm{d}}$ for deformations and microrotations or coupling conditions for microrotations, are prescribed. $\Gamma_S \subset \partial\Omega$ is a part of the boundary, where traction boundary conditions in the form of the potential of applied surface forces Π_N are given with $\Gamma \cap \Gamma_S = \emptyset$. In addition, $\Gamma_C \subset \partial\Omega$ is the part of the boundary where the potential of external surface couples Π_{M_c} are applied with $\Gamma \cap \Gamma_C = \emptyset$. On the free boundary $\partial\Omega \setminus \{\Gamma \cup \Gamma_S \cup \Gamma_C\}$ corresponding natural boundary conditions for $(\varphi, \overline{R})$ apply. The potential of the external applied volume

[5] Observe that $\overline{R}^T\nabla(\overline{R}.e_i) \neq \overline{R}^T\partial_{x_i}\overline{R} \in \mathfrak{so}(3, \mathbb{R})$.

force is Π_f and Π_M takes on the role of the potential of applied external volume couples. For simplicity we assume

$$\Pi_f(\varphi) = \langle f, \varphi \rangle\,, \quad \Pi_M(\overline{R}) = \langle M, \overline{R} \rangle\,, \quad \Pi_N(\varphi) = \langle N, \varphi \rangle\,, \quad \Pi_{M_c}(\overline{R}) = \langle M_c, \overline{R} \rangle\,, \tag{2.4}$$

for the potentials of applied loads with given functions $f \in L^2(\Omega, \mathbb{R}^3)$, $M \in L^2(\Omega, \mathbb{M}^{3\times 3})$, $N \in L^2(\Gamma_S, \mathbb{R}^3)$, $M_c \in L^2(\Gamma_C, \mathbb{M}^{3\times 3})$.

The parameters μ, $\lambda > 0$ are the Lamé constants of classical isotropic elasticity, the additional parameter $\mu_c \geq 0$ is called the **Cosserat couple modulus**. For $\mu_c > 0$ the elastic strain energy density $W_{\mathrm{mp}}(\overline{U})$ is **uniformly convex** in $\overline{U}$. Moreover[6]

$$\begin{aligned} \forall\, F \in \mathrm{GL}^+(3, \mathbb{R}) :\; W_{\mathrm{mp}}(\overline{U}) &= W_{\mathrm{mp}}(\overline{R}^T F) \geq \mu_c \, \|\overline{R}^T F - 1\!\!1\|^2 = \mu_c \, \|F - \overline{R}\|^2 \\ &\geq \mu_c \inf_{R \in \mathrm{O}(3,\mathbb{R})} \|F - R\|^2 = \mu_c \, \mathrm{dist}^2(F, \mathrm{O}(3, \mathbb{R})) \\ &= \mu_c \, \mathrm{dist}^2(F, \mathrm{SO}(3, \mathbb{R})) = \mu_c \, \|F - \mathrm{polar}(F)\|^2 = \mu_c \, \|U - 1\!\!1\|^2 \,. \end{aligned} \tag{2.5}$$

In contrast, for $\mu_c = 0$ the strain energy density is **only convex** w.r.t. F and does not satisfy (2.5).

The parameter $L_c > 0$ (with dimension length) introduces an **internal length** which is **characteristic** for the material, e.g. related to the grain size in a polycrystal. The internal length $L_c > 0$ is responsible for **size effects** in the sense that smaller samples are relatively stiffer than larger samples. We assume throughout that $\alpha_5 > 0, \alpha_6 > 0, \alpha_7 \geq 0$. This implies the **coercivity of curvature**

$$\exists\, c^+ > 0 \quad \forall\, \mathfrak{K} \in \mathfrak{T}(3) : \quad W_{\mathrm{curv}}(\mathfrak{K}) \geq c^+ \, \|\mathfrak{K}\|^{1+p} \,, \tag{2.6}$$

which is a basic ingredient of the mathematical analysis.

The non-standard boundary condition of **consistent coupling** ensures that no unwanted non-classical, polar effects may occur at the Dirichlet boundary Γ. It implies for the micropolar stretch that $\overline{U}_{|_\Gamma} \in \mathrm{Sym}$ and for the second Piola-Kirchhoff stress tensor $S_2 := F^{-1} D_F W_{\mathrm{mp}}(\overline{U}) \in \mathrm{Sym}$ on Γ as in the classical, non-polar case.

We mention, that a linearization of this Cosserat bulk model with $\mu_c = 0$ for small displacement and small microrotations completely decouples the two fields of deformation and microrotations and leads to the classical linear elasticity problem for the deformation.[7] For more details on the modelling of the three-dimensional Cosserat model we refer the reader to [Nef03b].

2.1 Mathematical results for the three-dimensional Cosserat bulk problem

For conciseness we state only the obtained results for the case without external loads. It can be shown [Nef04a]:

Theorem 2.1 (Existence for 3D-finite-strain elastic Cosserat model with $\mu_c > 0$)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p}(\Omega, \mathrm{SO}(3, \mathbb{R}))$. Then (2.1) with $\mu_c > 0, \alpha_4 \geq 0, p \geq 1, q \geq 0$ and either free or rigid prescription for $\overline{R}$ on Γ admits at least one minimizing solution pair $(\varphi, \overline{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,1+p}(\Omega, \mathrm{SO}(3, \mathbb{R}))$. ■

Using the extended Korn's inequality Theorem 8.1, the following has been shown in [Nef03b, Nef04c]:

Theorem 2.2 (Existence for 3D-finite-strain elastic Cosserat model with $\mu_c = 0$)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p+q}(\Omega, \mathrm{SO}(3, \mathbb{R}))$. Then (2.1) with $\mu_c = 0, \alpha_4 > 0, p \geq 1, q > 1$ and either free or rigid prescription for $\overline{R}$ on Γ admits at least one minimizing solution pair $(\varphi, \overline{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega, \mathrm{SO}(3, \mathbb{R}))$. ■

[6] The condition $F \in \mathrm{GL}^+(3, \mathbb{R})$ is necessary, otherwise $\|F - \mathrm{polar}(F)\|^2 = \mathrm{dist}^2(F, \mathrm{O}(3, \mathbb{R})) < \mathrm{dist}^2(F, \mathrm{SO}(3, \mathbb{R}))$, as can be easily seen for the reflection $F = \mathrm{diag}(1, -1, 1)$.

[7] Thinking in the context of an infinitesimal-displacement Cosserat theory one might erroneously believe that $\mu_c > 0$ is strictly necessary also for a "true" finite-strain Cosserat theory.

3 Formal dimensional reduction of the Cosserat bulk model

3.1 The three-dimensional Cosserat problem on a thin domain

The basic task of any shell theory is a consistent reduction of some presumably "exact" 3D-theory to 2D. The general three-dimensional problem (2.1) will now be adapted to a shell-like theory. Let us assume that we are given a three-dimensional **absolutely thin domain**

$$\Omega_h := \omega \times [-\frac{h}{2}, \frac{h}{2}], \quad \omega \subset \mathbb{R}^2 \,, \tag{3.1}$$

with **transverse boundary** $\partial\Omega_h^{\text{trans}} = \omega \times \{-\frac{h}{2}, \frac{h}{2}\}$ and **lateral boundary** $\partial\Omega_h^{\text{lat}} = \partial\omega \times [-\frac{h}{2}, \frac{h}{2}]$, where ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\omega$ and $h > 0$ is the thickness. Moreover, assume we are given a deformation φ and microrotation $\overline{R}^{3d}$,

$$\varphi: \quad \Omega_h \subset \mathbb{R}^3 \mapsto \mathbb{R}^3 \,, \qquad \overline{R}^{3d} : \Omega_h \subset \mathbb{R}^3 \mapsto \mathrm{SO}(3,\mathbb{R}) \,, \tag{3.2}$$

solving the following two-field minimization problem on the thin domain Ω_h:

$$\begin{aligned}
I(\varphi, \nabla\varphi, \overline{R}, \mathrm{D_x}\overline{R}) &= \int_{\Omega_h} W_{\mathrm{mp}}(\overline{U}) + W_{\mathrm{curv}}(\mathfrak{K}) - \langle f, \varphi\rangle \,\mathrm{dV} - \int_{\partial\Omega_h^{\text{trans}} \cup \{\gamma_s \times [-\frac{h}{2}, \frac{h}{2}]\}} \langle N, \varphi\rangle \,\mathrm{dS} \mapsto \min . \text{ w.r.t. } (\varphi, \overline{R}) \,, \\
\overline{U} &= \overline{R}^T F, \quad \varphi_{|_{\Gamma_0^h}} = g_{\mathrm{d}}(x,y,z), \quad \Gamma_0^h = \gamma_0 \times [-\frac{h}{2}, \frac{h}{2}], \quad \gamma_0 \subset \partial\omega, \quad \gamma_s \cap \gamma_0 = \emptyset \,, \\
\overline{R}_{|_{\Gamma_0^h}} &= \mathrm{polar}(\nabla\varphi) \,, \quad \text{strong consistent coupling boundary condition} \,, \\
W_{\mathrm{mp}}(\overline{U}) &= \mu \,\| \mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c \,\| \mathrm{skew}(\overline{U})\|^2 + \frac{\lambda}{2} \,\mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2 , \\
W_{\mathrm{curv}}(\mathfrak{K}) &= \mu \frac{L_c^{1+p}}{12} \,(1 + \alpha_4 \,L_c^q \,\|\mathfrak{K}\|^q) \left(\alpha_5 \,\|\, \mathrm{sym}\, \mathfrak{K}\|^2 + \alpha_6 \,\|\, \mathrm{skew}\, \mathfrak{K}\|^2 + \alpha_7 \,\mathrm{tr}\,[\mathfrak{K}]^2\right)^{\frac{1+p}{2}} , \\
\mathfrak{K} &= \overline{R}^T \mathrm{D_x}\overline{R} = \left(\overline{R}^T\nabla(\overline{R}.e_1), \overline{R}^T\nabla(\overline{R}.e_2), \overline{R}^T\nabla(\overline{R}.e_3)\right) .
\end{aligned} \tag{3.3}$$

Without loss of mathematical generality we assume that $M, M_c \equiv 0$ in (2.4), i.e. that no external volume or surface couples are present in the bulk problem. We want to find a reasonable approximation $(\varphi_s, \overline{R}_s)$ of $(\varphi, \overline{R}^{3d})$ involving only two-dimensional quantities. For us, this dimensional reduction is based on a **formal dimensional reduction "rational"**, which is characterized as follows:

1. A **quadratic ansatz** through the thickness for the three-dimensional deformation: $\varphi_s(x,y,z) = m(x,y) + (z\varrho_m + \frac{z^2}{2}\varrho_b)\,\vec{d}$ with m the deformation of the midsurface, i.e. **normals** to the undeformed midsurface **remain straight**, but may be **elongated** and the **midsurface** may be **asymmetrically shifted**. The rotations are assumed to be constant over the thickness: $\overline{R}^{3d}(x,y,z) = \overline{R}_s(x,y) = \overline{R}(x,y)$. Restriction of the director $\vec{d}$ to the third column $\overline{R}_3 := \overline{R}.e_3$ of the already appearing microrotations.

2. **Exact analytical** determination of the two leading coefficients ϱ_m, ϱ_b from the three-dimensional transverse boundary condition on the upper and lower face of the plate in terms of the quadratic ansatz, **independent** of the Cosserat couple modulus μ_c. Simplification of the formulas for ϱ_m, ϱ_b in view of an assumed **almost rigid** ($\mu, \lambda \gg 1$) behaviour (4.8). Replaces "Condensation of the material law: $\sigma_{33} = 0$" in the classical infinitesimal-displacement theory.

3. **Analytical integration** of the bulk energy through the thickness with an approximated expression $\overline{F}_s = (\nabla m|\varrho_m \overline{R}_3) + z\,(\nabla\overline{R}_3|\varrho_b\overline{R}_3)$ for the reconstructed deformation gradient $\nabla\varphi_s$, consistent with a linear ansatz through the thickness to obtain a dimensionally reduced energy density $I(\varphi_s, \overline{F}_s, \overline{R}_s, \mathrm{D_x}\overline{R}_s)$. Amounts to "Typical inconsistency of derivation with naive energy projection."

4. **Non-standard Dirichlet boundary conditions for simple support**: no direct prescription of a director at the lateral Dirichlet boundary γ_0, instead requiring only a **weak coupling condition** to the extent that no polar effects may occur at the Dirichlet boundary, possibly weakening the boundary layer: "Avoiding the typical problem of Cosserat

theories as regards formulation of boundary conditions." Alternative Dirichlet boundary conditions are also possible: classical **rigid director prescription**: $\vec{d} = \overline{R}.e_3$ prescribed at γ_0 (**clamped**).

4 The new formal finite-strain Cosserat thin plate model with size effects

4.1 Statement of the formal Cosserat plate model

The proposed formal "rational" of dimensional descend leads us to **postulate** the following two-dimensional minimization problem for the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the microrotation of the plate (shell) $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3, \mathbb{R})$ on ω:

$$I(m, \overline{R}) = \int_\omega h\, W_{\mathrm{mp}}(\overline{U}) + h\, W_{\mathrm{curv}}(\mathfrak{K}_s) + \frac{h^3}{12} W_{\mathrm{bend}}(\mathfrak{K}_b)\, \mathrm{d}\omega \\ - \Pi(m, \overline{R}_3) \mapsto \min . \text{ w.r.t. } (m, \overline{R}) \,, \tag{4.1}$$

under the constraints

$$\overline{U} = \overline{R}^T \widehat{F}, \qquad \widehat{F} = (\nabla m | \overline{R}_3) \in \mathbb{M}^{3\times 3} \,, \tag{4.2}$$
$$\mathfrak{K}_s = \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0)\right) \in \mathfrak{T}(3)\,, \quad \mathfrak{K}_b = \mathfrak{K}_s^3 \,,$$

and the boundary conditions of place for the midsurface deformation m on the Dirichlet part of the lateral boundary γ_0,

$$m_{|_{\gamma_0}} = g_{\mathrm{d}}(x, y, 0)\,, \qquad \text{simply supported (fixed, welded)}\,. \tag{4.3}$$

The three possible **alternative** boundary conditions for the microrotations $\overline{R}$ on γ_0 are

$$\overline{R}_{|_{\gamma_0}} = \mathrm{polar}((\nabla m | \nabla g_{\mathrm{d}}(x,y,0).e_3))_{|_{\gamma_0}}\,, \quad \text{strong form of reduced consistent coupling}\,, \tag{4.4}$$
$$\forall\, A \in C_0^\infty(\gamma_0, \mathfrak{so}(3,\mathbb{R})) :$$
$$\int_{\gamma_0} \langle \overline{R}^T (\nabla m(x,y) | \nabla g_{\mathrm{d}}(x,y,0).e_3), A(x,y)\rangle \,\mathrm{ds} = 0\,, \qquad \text{very } \textbf{weak consistent coupling}\,,$$
$$\overline{R}_{3_{|_{\gamma_0}}} = \frac{\nabla g_{\mathrm{d}}(x,y,0).e_3}{\|\nabla g_{\mathrm{d}}(x,y,0).e_3\|}\,, \qquad \textbf{rigid director prescription}\,.$$

The constitutive assumptions on the reduced densities are[8]

$$W_{\mathrm{mp}}(\overline{U}) = \mu\, \|\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c\, \|\mathrm{skew}(\overline{U})\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2 \tag{4.5}$$
$$= \mu \underbrace{\|\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T\nabla m - 1\!\!1_2)\|^2}_{\text{shear-stretch energy}} + \mu_c \underbrace{\|\mathrm{skew}((\overline{R}_1|\overline{R}_2)^T\nabla m)\|^2}_{\text{first order drill energy}}$$
$$+ \underbrace{\frac{\kappa(\mu+\mu_c)}{2}\left(\langle \overline{R}_3, m_x\rangle^2 + \langle \overline{R}_3, m_y\rangle^2\right)}_{\text{classical transverse shear energy}} + \frac{\mu\lambda}{2\mu+\lambda} \underbrace{\mathrm{tr}\left[\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T\nabla m - 1\!\!1_2)\right]^2}_{\text{elongational stretch energy}},$$
$$W_{\mathrm{curv}}(\mathfrak{K}_s) = \mu \frac{L_c^{1+p}}{12}\,(1 + \alpha_4\, L_c^q\, \|\mathfrak{K}_s\|^q) \left(\alpha_5\, \|\mathrm{sym}\,\mathfrak{K}_s\|^2 + \alpha_6\, \|\mathrm{skew}\,\mathfrak{K}_s\|^2 + \alpha_7\, \mathrm{tr}\left[\mathfrak{K}_s\right]^2\right)^{\frac{1+p}{2}} \,,$$
$$\mathfrak{K}_s = \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0)\right)\,,$$
$$\mathfrak{K}_s = (\mathfrak{K}_s^1, \mathfrak{K}_s^2, \mathfrak{K}_s^3) \in \mathfrak{T}(3)\,, \quad \text{the reduced third order } \textbf{curvature tensor}\,,$$
$$W_{\mathrm{bend}}(\mathfrak{K}_b) = \mu\, \|\mathrm{sym}(\mathfrak{K}_b)\|^2 + \mu_c\, \|\mathrm{skew}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}(\mathfrak{K}_b)\right]^2\,,$$
$$\mathfrak{K}_b = \overline{R}^T(\nabla\overline{R}_3|0) = \mathfrak{K}_s^3\,, \quad \text{the second order } \textbf{non-symmetric bending tensor}\,.$$

[8] $\|\mathrm{skew}((\overline{R}_1|\overline{R}_2)^T\nabla m)\|^2 = \left(\langle \overline{R}_1, m_y\rangle - \langle \overline{R}_2, m_x\rangle\right)^2$.

The (relative) thickness of the plate (shell) is $h > 0$. The total elastically stored energy density due to **membrane-strain**, total **plate-curvature** and specific **plate-bending**

$$W = \underbrace{h\,W_{\mathrm{mp}}}_{\text{membrane}} + \underbrace{h\,W_{\mathrm{curv}}}_{\text{curvature}} + \underbrace{\frac{h^3}{12} W_{\mathrm{bend}}}_{\text{bending}}\,, \tag{4.6}$$

depends on the midsurface deformation gradient ∇m and microrotations $\overline{R}$ together with their space derivatives only through the frame-indifferent measures $\overline{U}$ and $\mathfrak{K}_s$. The **micropolar stretch tensor** $\overline{U}$ of the plate is in general **non-symmetric**, neither is the **micropolar reduced third order curvature tensor** $\mathfrak{K}_s$. The three-dimensional plate deformation is reconstructed as

$$\varphi_s(x,y,z) = m(x,y) + \left(z\,\varrho_m(x,y) + \frac{z^2}{2}\,\varrho_b(x,y)\right)\,\overline{R}(x,y).e_3\,, \tag{4.7}$$

where

$$\varrho_m = 1 - \frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right] + \frac{\langle N_{\mathrm{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)} = \underbrace{1 - \frac{\lambda}{2\mu+\lambda}\mathrm{tr}\left[\overline{U}-1\!\!1\right] + \frac{\langle N_{\mathrm{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)}}_{\text{first order thickness change due to elongational stretch}}\,,$$

$$\varrho_b = \underbrace{-\frac{\lambda}{2\mu+\lambda}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle + \frac{\langle N_{\mathrm{res}},\overline{R}_3\rangle}{(2\mu+\lambda)\,h}}_{\text{non-symmetric shift of the midsurface due to bending}} = -\frac{\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathfrak{K}_b\right] + \frac{\langle N_{\mathrm{res}},\overline{R}_3\rangle}{(2\mu+\lambda)\,h} \tag{4.8}$$

and N_{diff}, N_{res} as defined in (5.2). To first order, the reconstructed deformationgradient is given by $F_s = (\nabla m|\varrho_m\,\overline{R}_3)$. Here $\omega \subset \mathbb{R}^2$ is a domain with boundary $\partial\omega$ and $\gamma_0 \subset \partial\omega$ is that part of the boundary, where Dirichlet conditions g_{d} for deformations and microrotations and/or consistent coupling conditions for microrotations, respectively, are prescribed. The reduced external loading functional $\Pi(m,\overline{R}_3)$ is a linear form in $(m,\overline{R}_3)$ defined in (5.19) in terms of the underlying three-dimensional loads. The parameters μ, $\lambda > 0$ are the Lamé constants of classical elasticity, $\mu_c \geq 0$ is called the Cosserat couple modulus and $L_c > 0$ introduces the internal length. We assume throughout that $\alpha_5 > 0, \alpha_6 > 0, \alpha_7 \geq 0$. We have included the so called **shear correction factor** κ ($0 < \kappa \leq 1$) to keep in line with classical infinitesimal-displacement plate models (5.11). In our formal derivation, however, we obtain $\kappa = 1$. The reduced model (4.1) is fully **frame-indifferent**, meaning that

$$\forall\, Q \in \mathrm{SO}(3,\mathbb{R}):\quad W_{\mathrm{mp}}(Q\widehat{F},Q\overline{R}) = W_{\mathrm{mp}}(\widehat{F},\overline{R})\,,\quad \mathfrak{K}_s(Q\overline{R}) = \mathfrak{K}_s(\overline{R})\,. \tag{4.9}$$

The non-invariant term ϱ_m is only needed to reconstruct the 3D-deformation, which depends on the non-invariant loading.[9] **Strain** and **curvature** parts are **additively decoupled**, as in the underlying parent model (2.1). We note the appearance of the **harmonic mean** $\mathcal{H}$ and **arithmetic mean** $\mathcal{A}$

$$\frac{1}{2}\mathcal{H}(\mu,\frac{\lambda}{2}) = \frac{\mu\lambda}{2\mu+\lambda}\,,\qquad \kappa\,\mathcal{A}(\mu,\mu_c) = \kappa\,\frac{\mu+\mu_c}{2}\,. \tag{4.10}$$

4.2 The different cases for the Cosserat plate

As in the three-dimensional case [Nef03b], we may distinguish five different situations: (different values of p,q compared with the three-dimensional case)

I: $\boldsymbol{\mu_c > 0}$, $\boldsymbol{\alpha_4 \geq 0}$, $\mathbf{p \geq 1}$, $\mathbf{q \geq 0}$. Unconditional coercivity and unqualified existence, positive Cosserat couple modulus. **Fracture excluded**.

II: $\boldsymbol{\mu_c = 0}$, $\boldsymbol{\alpha_4 = 0}$, $\mathbf{p > 1}$, $\mathbf{q \geq 0}$. **Conditional coercivity**, zero Cosserat couple modulus. **Fracture excluded**.

III: $\boldsymbol{\mu_c = \infty}$, $\boldsymbol{\alpha_4 \geq 0}$, $\mathbf{p \geq 1}$, $\mathbf{q \geq 0}$. **Constrained gradient** Cosserat micropolar plate problem (indeterminate couple-stress plate model). Compatible Dirichlet boundary conditions: $m_{|\gamma_0} = g_{\mathrm{d}}$, $\mathrm{polar}((\nabla m|\varrho_m\vec{n}_m))_{|\gamma_0} = \mathrm{polar}(\nabla g_{\mathrm{d}})_{|\gamma_0}$. Similar to, but not identical with, a Kirchhoff-Love model.

[9] Of course, if the external tractions are rotated as well, we obtain invariance: $\langle Q.N_{\mathrm{diff}},Q.\overline{R}_3\rangle = \langle N_{\mathrm{diff}},\overline{R}_3\rangle$.

IV: $\mu_c = 0$, $\alpha_4 = 0$, $0 < p \le 1$, $q = 0$. Possibly $m \notin W^{1,1}(\omega, \mathbb{R}^3)$ due to lack of elastic coercivity, **including fracture** in multiaxial situations.

V: $\mu_c = 0, L_c = 0$. **Relaxation case**. Finite elasticity with free rotations and microstructure. Weak solutions of the nonlinear, non-elliptic problem based on the total elastic energy density

$$
\begin{aligned}
W(\nabla m, \vec{n}_m, \nabla \vec{n}_m) &= h \left(\mu \, \|U((\nabla m|\vec{n})) - 1\!\!1\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \mathrm{tr}\left[U((\nabla m|\vec{n})) - 1\!\!1\right]^2 \right) \\
&\quad + \frac{h^3}{12} \left(\mu \, \|U^{-1} \widehat{II}_m\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \mathrm{tr}\left[U^{-1}\widehat{II}_m\right]^2 \right), \\
U((\nabla m|\vec{n})) &= \sqrt{(\nabla m|\vec{n})^T (\nabla m|\vec{n})} = \sqrt{I_m} + e_3 \otimes e_3 \,,
\end{aligned}
$$

(I_m the first fundamental form, $\widehat{II}_m = II_m + e_3 \otimes e_3$, $II_m = -\nabla m^T \nabla \vec{n}_m \in \mathbb{M}^{2\times 2}$ the second fundamental form of the midsurface m), are stationary points of the minimization problem (4.1). Allowing in principle for **sharp interfaces**.

We refer to $0 < p < 1$, $q \ge 0$ as the **sub-critical case**, to $p = 1$, $q \ge 0$ as the **critical case** and to $p > 1$, $q \ge 0$ as the **super-critical case**. In this contribution we will treat mathematically exclusively the super-critical case II. The simpler case I and case III for positive Cosserat couple modulus $\mu_c > 0$ with rigid director prescription at the boundary are dealt with in [Nef04a]. The "fracture" case IV and the "relaxation" case V remain open at present.

It is easy to see that the membrane energy part W_{mp} in (4.1) is uniformly Legendre-Hadamard elliptic with ellipticity constant $\mu > 0$ independent of the value of the Cosserat couple modulus μ_c. As will be seen, a linearization of (4.1) with $\mu_c = 0$ and $p > 1$ (super-quadratic curvature energy W_{curv}) for small displacement and small microrotation does not decouple the fields, as in the three-dimensional situation, but leads formally to the infinitesimal-displacement, classical linear Reissner-Mindlin model (5.11).

4.3 The coercivity inequality in two-dimensions

In this section we show how to use the three-dimensional extended Korn's first inequality Theorem 8.1 in our reduced two-dimensional context of plates and shells in order to improve Legendre-Hadamard ellipticity to uniform positivity. In order to show that the elastic membrane energy is uniformly convex for zero Cosserat couple modulus $\mu_c = 0$ we look at the second differential of $W_{\mathrm{mp}}(\overline{R}^T \widehat{F})$ with respect to m

$$
D^2_{\nabla m} W_{\mathrm{mp}}(\overline{R}^T \widehat{F}).(\nabla\phi, \nabla\phi) \ge \frac{\mu}{2} \|(\nabla\phi|0)^T \overline{R} + \overline{R}^T (\nabla\phi|0)\|^2 \,. \tag{4.11}
$$

Set for simplicity $\mu = 2$ and consider the slightly more general quadratic form (appropriate for elastic shells: $F_p = \nabla\Theta$ with Θ a regular parametrization of the stress-free initial curvilinear shell surface and elasto-plastic shells: $F_p, \overline{R}_e$ arbitrary)

$$
\begin{aligned}
\|F_p^{-T} (\nabla\phi|0)^T \overline{R}_e + \overline{R}_e^T (\nabla\phi|0) F_p^{-1}\|^2 &= \|\overline{R}_e \left(F_p^{-T} (\nabla\phi|0)^T \overline{R}_e + \overline{R}_e^T (\nabla\phi|0) F_p^{-1} \right) \overline{R}_e^T \|^2 \\
&= \|(\overline{R}_e F_p)^{-T} (\nabla\phi|0)^T + (\nabla\phi|0)(\overline{R}_e F_p)^{-1}\|^2 \,,
\end{aligned} \tag{4.12}
$$

where $\phi : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and $\phi_{|_{\gamma_0}} = 0$ for $\gamma_0 \subset \partial\omega$. Extend now ϕ to $\bar{\phi} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ through

$$
\bar{\phi}(x,y,z) := \phi(x,y) \Rightarrow \bar{\phi}(x,y,z)_{|_{\gamma_0 \times [-\frac{h}{2}, \frac{h}{2}]}} = 0 \quad \text{and} \quad \nabla_{(x,y,z)} \bar{\phi}(x,y,z) = (\nabla_{(x,y)}\phi|0). \tag{4.13}
$$

For $\bar{\phi}$ it is possible to use the 3D-extended Korn's first inequality Theorem 8.1. To this end consider $\Omega_h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$ and the lateral Dirichlet boundary $\Gamma_0^h = \gamma_0 \times [-\frac{h}{2}, \frac{h}{2}] \subset \partial\Omega_h$. Then Γ_0^h has non-vanishing 2-dimensional Hausdorff measure. Set by abuse of notation $F_p = (\overline{R}_e F_p)$ for the moment. With smooth enough, invertible F_p it holds on applying Theorem 8.1 that

$$
\begin{aligned}
&\int_{\Omega_h} \|\nabla\bar{\phi}^T F_p^{-1} + F_p^{-T} \nabla\bar{\phi}\|^2 \, \mathrm{dV} \ge c_{3D}^+ \cdot \int_{\omega \times [-\frac{h}{2}, \frac{h}{2}]} \|\bar{\phi}\|^2 + \|\nabla\bar{\phi}\|^2 \, \mathrm{dV} \Rightarrow \\
&\int_\omega \int_{-\frac{h}{2}}^{\frac{h}{2}} \|\nabla\bar{\phi}^T F_p^{-1} + F_p^{-T} \nabla\bar{\phi}\|^2 \, \mathrm{d}\omega \, \mathrm{dz} \ge c_{3D}^+ \cdot \int_\omega \int_{-\frac{h}{2}}^{\frac{h}{2}} \|\bar{\phi}\|^2 + \|\nabla\bar{\phi}\|^2 \, \mathrm{d}\omega \, \mathrm{dz} \,.
\end{aligned} \tag{4.14}
$$

Since $\bar{\phi}$ is independent of z we may carry out the integration with respect to the transverse variable and get, however,

$$\int_\omega \|\nabla\bar{\phi}^T F_p^{-1} + F_p^{-T}\nabla\bar{\phi}\|^2 \,\mathrm{d}\omega \geq c_{3D}^+ \cdot \int_\omega \|\bar{\phi}\|^2 + \|\nabla\bar{\phi}\|^2 \,\mathrm{d}\omega\,, \tag{4.15}$$

or back in terms of ϕ

$$\int_\omega \|(\nabla\phi|0)^T F_p^{-1} + F_p^{-T}(\nabla\phi|0)\|^2 \,\mathrm{d}\omega \geq c_{3D}^+ \cdot \int_\omega \|\phi\|^2 + \|(\nabla\phi|0)\|^2 \;\mathrm{d}\omega. \tag{4.16}$$

Observe that the constant c_{3D}^+ is in fact independent of the thickness h (we could set $h = 1$) which might be surprising at first glance. This observation allows one to bound $m \in H_\circ^{1,2}(\omega,\mathbb{R}^3;\gamma_0)$, independent of the relative thickness h only in terms of the membrane energy $\int_\omega W(\nabla m, \overline{R})\;\mathrm{d}\omega$ if $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$ is smooth enough. Thus we have finally proved

Theorem 4.1 (Improved Korn's inequality for rigid shells)
Let $\omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and let $\gamma_0 \subset \partial\omega$ be a part of the boundary with non vanishing 1-dimensional Hausdorff measure. Define $H_\circ^{1,2}(\omega,\mathbb{R}^3;\gamma_0) := \{\phi \in H^{1,2}(\omega,\mathbb{R}^3)\mid \phi_{|_{\gamma_0}} = 0\}$ and let $F_p, F_p^{-1} \in W^{1,2+\delta}(\overline{\omega},\mathrm{GL}(3,\mathbb{R}))$. Then

$$\exists\, c^+ > 0 \quad \forall\,\phi \in H_\circ^{1,2}(\omega,\mathbb{R}^3;\gamma_0):\; \|(\nabla\phi|0)F_p^{-1}(x) + F_p^{-T}(x)(\nabla\phi|0)^T\|^2_{L^2(\omega)} \geq c^+\;\|\phi\|^2_{H^{1,2}(\omega)}\,,$$

and the constant is bounded away from zero for F_p, F_p^{-1} bounded in $W^{1,2+\delta}(\overline{\omega},\mathrm{GL}(3,\mathbb{R}))$.

Proof. The proof is based on the previous argument and on the strengthening of Theorem 8.1 proposed in [Pom03]. The Sobolev embedding shows that $F_p \in W^{1,2+\delta}(\overline{\omega},\mathrm{GL}(3,\mathbb{R}))$ may be identified with a continuous function. In order to show that the constant is uniformly bounded away from zero for bounded $F_p, F_p^{-1} \in W^{1,2+\delta}(\overline{\omega},\mathrm{GL}(3,\mathbb{R}))$ a contradiction argument as in [Nef04b] is employed which uses the fact that $W^{1,2+\delta}(\overline{\omega},\mathrm{GL}(3,\mathbb{R}))$ is compactly embedded in $C^0(\overline{\omega},\mathrm{GL}(3,\mathbb{R}))$. ■

4.4 Mathematical analysis for zero Cosserat couple modulus $\mu_c = 0$

The following results provide existence theorems for geometrically exact deduced elastic Cosserat plate models for the physically more realistic super-critical case.[10]

Theorem 4.2 (Existence for 2D-finite-strain elastic Cosserat model: case II)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\omega,\mathbb{R}^3)$ and polar$(\nabla g_{\mathrm{d}}) \in W^{1,1+p+q}(\omega,\mathrm{SO}(3,\mathbb{R}))$. *Moreover, let $\overline{f} \in L^1(\omega,\mathbb{R}^3)$ and suppose $\overline{N} \in L^1(\gamma_s,\mathbb{R}^3)$ together with $\overline{M} \in L^1(\omega,\mathbb{R}^3)$ and $\overline{M}_c \in L^1(\gamma_s,\mathbb{R}^3)$, see* (5.19). *Then* (4.1) *with material constants conforming to case II, boundary conditions for $\overline{R}$ of rigid director prescription on γ_0 and modified external potential $\Pi^\sharp$* (5.20) *admits at least one minimizing solution pair $(m,\overline{R}) \in H^1(\omega,\mathbb{R}^3)\times W^{1,1+p+q}(\omega,\mathrm{SO}(3,\mathbb{R}))$.*

Proof. We apply the direct methods of variations. First, the requirement on the data shows that

$$\forall\, m \in H^1(\omega,\mathbb{R}^3),\; \overline{R} \in W^{1,1+p+q}(\omega,\mathrm{SO}(3,\mathbb{R})):\quad \Pi^\sharp(m,\overline{R}_3) \leq C\,, \tag{4.17}$$

i.e. a uniform bound on the external loading functional. Let us define the admissible set

$$\begin{aligned}\mathcal{A} := \{m \in H^1(\omega,\mathbb{R}^3),\; \overline{R} \in W^{1,1+p+q}(\omega,\mathrm{SO}(3,\mathbb{R}))\,|\, m_{|_{\gamma_0}} = g_{\mathrm{d}}(x,y,0)\,,\\ \overline{R}_{3_{|_{\gamma_0}}} = \frac{\nabla g_{\mathrm{d}}(x,y,0).e_3}{\|\nabla g_{\mathrm{d}}(x,y,0).e_3\|}\,\}\,.\end{aligned} \tag{4.18}$$

[10]The proposed finite-strain results determine the macroscopic midsurface plate deformation $m \in H^1(\omega,\mathbb{R}^3)$ and not more. This means that discontinuous macroscopic deformations by cavities or the formation of holes are not excluded (possible mode I failure). If $\mu_c > 0$ fracture is effectively ruled out, which is, however, somewhat unrealistic. All results remain true for arbitrary shear correction factor $\kappa > 0$. For $\kappa = 0$, however, uniform Legendre-Hadamard ellipticity is lost.

With the prescription of the boundary data $g_{\rm d}$ it is clear that $I(g_{\rm d}(x,y,0),\mathrm{polar}(\nabla g_{\rm d}(x,y,0)))<\infty$, hence I is bounded above on $\mathcal{A}$. Consider a sequence of pairs of deformation m_k and rotations $\overline{R}^k$ in the admissible set $\mathcal{A}$ with bounded energy I. For such a sequence we have

$$\begin{aligned}\infty > I(m_k,\overline{R}^k) &= \int_\omega h\,W_{\rm mp}(\overline{U}_k)+h\,W_{\rm curv}(\mathfrak{K}_{s,k})+\frac{h^3}{12}\,W_{\rm bend}(\mathfrak{K}_{b,k})\,\mathrm{d}\omega-\Pi^\sharp(m_k,\overline{R}_3^k)\\ &\geq \int_\omega h\,W_{\rm mp}(\overline{U}_k)+h\,W_{\rm curv}(\mathfrak{K}_{s,k})+\frac{h^3}{12}\,W_{\rm bend}(\mathfrak{K}_{b,k})\,\mathrm{d}\omega-C\geq C_3\,,\end{aligned} \tag{4.19}$$

which implies that I is bounded below on $\mathcal{A}$ and the positive curvature energy $\int_\omega h\,W_{\rm curv}(\mathfrak{K}_{s,k})\,\mathrm{d}\omega$ can be bounded independent of $k\in\mathbb{N}$. Observe now that the curvature energy bounds the sequence of curvature tensors $\mathfrak{K}_{s,k}$ in $L^{1+p+q}(\omega,\mathfrak{T}(3))$ by the positivity assumption on the parameters $\alpha_5,\alpha_6>0$. Since $\|\mathfrak{K}_s\|=\|\overline{R}^T\mathrm{D_x}\overline{R}\|=\|\mathrm{D_x}\overline{R}\|$ pointwise, this implies that $\|\mathrm{D_x}\overline{R}^k\|_{L^{1+p+q}(\omega)}$ is bounded as well. Since $\|\overline{R}^k\|=\sqrt{3}$ pointwise, this shows the boundedness of $\overline{R}^k\subset W^{1,1+p+q}(\omega,\mathrm{SO}(3,\mathbb{R}))$, even without specific Dirichlet boundary conditions on the remaining "free" columns $\overline{R}.e_1,\overline{R}.e_2$.[11] This is a distinctive feature for exact rotations. A subsequence can be chosen such that $\mathfrak{K}_{s,k}\rightharpoonup\widehat{\mathfrak{K}}_s$ in $L^{1+p+q}(\omega,\mathfrak{T}(3))$, weakly. Since the boundedness of the rotations $\overline{R}^k$ holds true in the space $W^{1,1+p+q}(\omega,\mathrm{SO}(3,\mathbb{R}))$ with $1+p+q>N=3$, it is possible to extract a subsequence, not relabelled, such that $\overline{R}^k$ converges strongly to $\widehat{\overline{R}}\in C^0(\overline{\omega},\mathrm{SO}(3,\mathbb{R}))$ in the topology of $C^0(\overline{\omega},\mathrm{SO}(3,\mathbb{R}))$ on account of the Sobolev-embedding theorem.

Since I is bounded below on $\mathcal{A}$ we may consider from now on infimizing sequences of mid-surface deformations m_k and rotations $\overline{R}^k$ with

$$\lim_{k\to\infty} I(m_k,\overline{R}^k)=\inf_{(m,\overline{R})\in\mathcal{A}} I(m,\overline{R})\,. \tag{4.20}$$

Along the strongly convergent sequence of rotations, the corresponding sequence of mid-surface deformations m^k is also bounded in $H^1(\omega,\mathbb{R}^3)$. However, this is not due to a basically simple pointwise estimate as in case I ($\mu_c>0$) [Nef04a], but only true after integration over the domain ω: at face value we only control certain mixed symmetric expressions in the reconstructed deformation gradient. Let us therefore define $v_k\in H^{1,2}(\omega,\mathbb{R}^3)$ by $m^k=g_{\rm d}+(m^k-g_{\rm d})=g_{\rm d}+v_k$. Then we have

$$\begin{aligned}\infty > I(m_k,\overline{R}^k) &= \int_\omega h\,W_{\rm mp}(\overline{U}_k)+h\,W_{\rm curv}(\mathfrak{K}_{s,k})+\frac{h^3}{12}\,W_{\rm bend}(\mathfrak{K}_{b,k})\,\mathrm{d}\omega-\Pi^\sharp(m_k,\overline{R}_3^k)\\ &\geq \int_\omega h\,W_{\rm mp}(\overline{U}_k)-\Pi^\sharp(m_k,\overline{R}_3^k)\,\mathrm{d}\omega \geq \int_\omega h\,W_{\rm mp}(\overline{U}_k)\,\mathrm{d}\omega - C\\ &\geq \int_\omega h\,\frac{\mu}{4}\|\overline{R}^{k,T}(\nabla m_k|\overline{R}_3^k)+(\nabla m_k|\overline{R}_3^k)^T\overline{R}^k-2{1\!\!1}\|^2\,\mathrm{d}\omega-C\\ &= \int_\omega h\,\frac{\mu}{4}\|\overline{R}^T(\nabla m_k|\overline{R}_3)+(\nabla m_k|\overline{R}_3)^T\overline{R}\|^2\\ &\quad -4h\frac{\mu}{4}\mathrm{tr}\left[\overline{R}^T(\nabla m_k|\overline{R}_3)+(\nabla m_k|\overline{R}_3)^T\overline{R}\right]+4\,h\,\frac{\mu}{4}\|{1\!\!1}\|^2\,\mathrm{d}\omega-C\\ &\geq \int_\omega h\,\frac{\mu}{4}\,\|\overline{R}^{k,T}(\nabla m_k|0)+(\nabla m_k|0)^T\overline{R}^k\|^2\,\mathrm{d}\omega-C_1\,\|m_k\|_{H^{1,2}(\omega)}+C_2\\ &= \int_\omega h\,\frac{\mu}{4}\,\|(\overline{R}^k-\widehat{\overline{R}}+\widehat{\overline{R}})^T(\nabla v_k|0)+(\nabla v_k|0)^T(\overline{R}^k-\widehat{\overline{R}}+\widehat{\overline{R}})\|^2\,\mathrm{d}\omega-C_1\,\|v_k\|_{H^{1,2}(\omega)}+C_2\\ &\geq \int_\omega h\,\frac{\mu}{4}\underbrace{\|\widehat{\overline{R}}^T(\nabla v_k|0)+(\nabla v_k|0)^T\widehat{\overline{R}}\|^2}_{\text{combinations of derivatives}}\,\mathrm{d}\omega-C_3\,\|\widehat{\overline{R}}-\overline{R}^k\|_\infty\,\|v_k\|^2_{H^{1,2}(\omega)}\end{aligned} \tag{4.21}$$

[11] Without independent curvature control, nothing can be shown for $\mu_c=0$. This is the reason for the modification of the external loads.

$$
\begin{aligned}
&- (C_1 + 2\,\|\widehat{\overline{R}} - \overline{R}^k\|_\infty)\,\|v_k\|_{H^{1,2}(\omega)} + C_2 \\
\geq\ & (h\,\frac{\mu}{4}\,c_K^+ - C_3\,\|\widehat{\overline{R}} - \overline{R}^k\|_\infty)\,\|v_k\|^2_{H^{1,2}(\omega)} - (C_1 + 2\,\|\widehat{\overline{R}} - \overline{R}^k\|_\infty)\,\|v_k\|_{H^{1,2}(\omega)} + C_2\,,
\end{aligned}
$$

where we made use of the zero boundary conditions for v_k on γ_0 and applied the extended Korn's inequality Theorem 4.1 (note that $\overline{R}^{-T} = \overline{R}$ for exact rotations) yielding the positive constant c_K^+ for the continuous microrotation $\widehat{\overline{R}}$. Since $\|\widehat{\overline{R}} - \overline{R}^k\|_\infty \to 0$ we conclude the boundedness of v_k in $H^1(\omega, \mathbb{R}^3)$. Hence, m_k is bounded as well in $H^1(\omega, \mathbb{R}^3)$.

From the boundedness of m_k in $H^1(\omega, \mathbb{R}^3)$ we may extract a subsequence, not relabelled, such that $m_k \rightharpoonup \widehat{m} \in H^1(\omega, \mathbb{R}^3)$. Furthermore, we may always obtain a subsequence of $(m_k, \overline{R}^k)$ such that $\overline{U}_k = \overline{R}^{k,T}\hat{F}^k = \overline{R}^{k,T}(\nabla m_k|\overline{R}_3^k)$ converges weakly in $L^2(\omega)$ to $\widehat{\overline{U}} = \widehat{\overline{R}}^T(\nabla\widehat{m}|\widehat{\overline{R}}_3)$.

Weak convergence of $\mathrm{D_x}\overline{R}^k$ in $L^{1,1+p+q}(\omega, \mathfrak{T}(3))$ and strong convergence of $\overline{R}^k$ in $L^2(\omega)$ together show that the sequence of the third order curvature tensors $\mathfrak{K}_{s,k} = \overline{R}^{k,T}\mathrm{D_x}\overline{R}^k$ converges indeed weakly to the correct limit $\widehat{\overline{R}}^T\mathrm{D_x}\widehat{\overline{R}} = \widehat{\mathfrak{K}}_s$ in $L^1(\omega, \mathfrak{T}(3))$. But from above we know already that weak convergence for $\mathfrak{K}_{s,k}$ takes place in $L^2(\omega, \mathfrak{T}(3))$. Gathering the obtained statements we have

$$
\begin{aligned}
\overline{U}_k = \overline{R}^{k,T}\hat{F}^k \rightharpoonup \widehat{\overline{U}} &= \widehat{\overline{R}}^T(\nabla\widehat{m}|\widehat{\overline{R}}_3) \quad \text{in } L^2(\omega)\,, \\
\mathfrak{K}_{s,k} = \overline{R}^{k,T}\mathrm{D_x}\overline{R}^k \rightharpoonup \widehat{\mathfrak{K}}_s &= \widehat{\overline{R}}^T\mathrm{D_x}\widehat{\overline{R}} \quad \text{in } L^2(\omega, \mathfrak{T}(3))\,, \\
\mathfrak{K}_{b,k} &\rightharpoonup \widehat{\mathfrak{K}}_b \quad \text{in } L^2(\omega, \mathbb{M}^{3\times 3})\,, \\
m_k &\to \widehat{m} \quad \text{in } L^2(\omega, \mathbb{R}^3)\,, \\
\overline{R}^k &\to \widehat{\overline{R}} \quad \text{in } C(\omega, \mathrm{SO}(3,\mathbb{R}))\,.
\end{aligned}
\tag{4.22}
$$

Since the total energy is convex in the combined terms $(\overline{U}, \mathfrak{K}_s, \mathfrak{K}_b)$ we get

$$
\begin{aligned}
I(\widehat{m}, \widehat{\overline{R}}) &= \int_\omega h\,W_{\mathrm{mp}}(\widehat{\overline{U}}) + h\,W_{\mathrm{curv}}(\widehat{\mathfrak{K}}_s) + \frac{h^3}{12}\,W_{\mathrm{bend}}(\widehat{\mathfrak{K}}_b)\,\mathrm{d}\omega - \Pi^\sharp(\widehat{m}, \widehat{\overline{R}}_3) \\
&\leq \liminf_{k\to\infty} \int_\omega h\,W_{\mathrm{mp}}(\overline{U}_k) + h\,W_{\mathrm{curv}}(\mathfrak{K}_{s,k}) + \frac{h^3}{12}\,W_{\mathrm{bend}}(\mathfrak{K}_{b,k})\,\mathrm{d}\omega - \Pi^\sharp(m_k, \overline{R}_3^k) \\
&= \lim_{k\to\infty} I(m_k, \overline{R}^k) = \inf_{(m,\overline{R})\in\mathcal{A}} I(m, \overline{R})\,,
\end{aligned}
\tag{4.23}
$$

which implies that the limit pair $(\widehat{m}, \widehat{\overline{R}})$ is a minimizer and the Dirichlet boundary conditions for either midsurface deformation $\widehat{m}$ and "director" $\widehat{\overline{R}}_3$ are satisfied strongly by compact embedding in the sense of traces on γ_0. This finishes the argument. ■

Let us turn to a slightly modified energy functional for which it is possible to extend the previous existence result to very weak consistent coupling boundary conditions. This modification is not necessary for a properly linearized model together with a linearized weak consistent coupling condition (5.11). The augmented energy functional reads

$$
\begin{aligned}
I(m, \overline{R}) = \int_\omega h\,W_{\mathrm{mp}}(\overline{U}) + h\,W_{\mathrm{curv}}(\mathfrak{K}_s) + \frac{h^3}{12}\,W_{\mathrm{bend}}(\mathfrak{K}_b)\,\mathrm{d}\omega - \Pi(m, \overline{R}_3) \\
+ \underbrace{\int_{\gamma_0} h\,W_{\mathrm{curv}}(\mathfrak{K}_s)\,\mathrm{ds}}_{\text{augmented}} \mapsto \min.\ \text{w.r.t. } (m, \overline{R})\,.
\end{aligned}
\tag{4.24}
$$

The new curvature control on γ_0 imparts additional regularity for the change of the rotations from γ_0 to the interior of the domain ω. With this modification it is possible to show

Corollary 4.3 (Existence for very weak consistent coupling)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\omega, \mathbb{R}^3)$, $\mathrm{polar}(\nabla g_{\mathrm{d}}) \in W^{1,1+p+q}(\omega, \mathrm{SO}(3,\mathbb{R}))$, $\mathrm{polar}(\nabla g_{\mathrm{d}})_{|_{\gamma_0}} \in W^{1,1+p+q}(\gamma_0, \mathrm{SO}(3,\mathbb{R}))$ *and* $\partial_z g_{\mathrm{d}|_{\gamma_0}} \in$

$L^2(\gamma_0,\mathbb{R}^3)$. *Moreover, let* $\overline{f} \in L^1(\omega,\mathbb{R}^3)$ *and suppose* $\overline{N} \in L^1(\gamma_s,\mathbb{R}^3)$ *together with* $\overline{M} \in L^1(\omega,\mathbb{R}^3)$ *and* $\overline{M}_c \in L^1(\gamma_s,\mathbb{R}^3)$, *see* (5.19). *Then* (4.1) *with material constants conforming to case II, boundary conditions of very weak consistent coupling on* γ_0, *modified external potential* $\Pi^\sharp$ (5.20) *and augmented energy functional* (4.24) *admits at least one minimizing solution pair* $(m,\overline{R}) \in H^1(\omega,\mathbb{R}^3) \times W^{1,1+p+q}(\omega,\mathrm{SO}(3,\mathbb{R}))$.

Proof. We basically repeat the argument of Theorem 4.2. First, we define the modified admissible set

$$\begin{aligned}\mathcal{A} := \{m \in H^1(\omega,\mathbb{R}^3),\, \overline{R} \in W^{1,1+p+q}(\omega,\mathrm{SO}(3,\mathbb{R}))\mid m_{|_{\gamma_0}} = g_{\mathrm{d}}(x,y,0)\,,\\ \int_{\gamma_0} \langle \overline{R}^T(\nabla m(x,y)|\nabla g_{\mathrm{d}}(x,y,0).e_3), A(x,y)\rangle\,\mathrm{ds} = 0 \quad \forall\, A \in C_0^\infty(\gamma_0,\mathfrak{so}(3,\mathbb{R}))\ \}\,,\end{aligned} \tag{4.25}$$

which incorporates the consistent coupling condition in its weak, distributional form. In order to see that the set $\mathcal{A}$ is not empty take $\overline{R} = \mathrm{polar}(\nabla g_{\mathrm{d}})$ and $m = g_{\mathrm{d}}$. As in Theorem 4.2 one shows that I is bounded above and below on $\mathcal{A}$. We then choose minimizing sequences of midsurface deformations m_k and rotations $\overline{R}^k$ in $\mathcal{A}$. Thus, along the minimizing sequence $(m_k,\overline{R}^k)$

$$\forall\, k \in \mathbb{N}:\quad \int_{\gamma_0} \langle \overline{R}^{k,T}(\nabla m_k(x,y)|\nabla g_{\mathrm{d}}(x,y,0).e_3), A(x,y)\rangle\,\mathrm{ds} = 0\,, \tag{4.26}$$

for all testfunctions $A \in C_0^\infty(\gamma_0,\mathfrak{so}(3,\mathbb{R}))$. We need to investigate in which sense the weak/strong limits found in Theorem 4.2 satisfy this additional relation on γ_0. We observe that for smooth testfunctions $A \in C_0^\infty(\gamma_0,\mathfrak{so}(3,\mathbb{R}))$ and by partial integration

$$\begin{aligned}\int_{\gamma_0} \langle \overline{R}^{k,T}(\nabla m_k(x,y)|\nabla g_{\mathrm{d}}(x,y,0).e_3), A(x,y)\rangle\,\mathrm{ds} &= \int_{\gamma_0} \langle(\nabla m_k(x,y)|\nabla g_{\mathrm{d}}(x,y,0).e_3), \overline{R}^k A(x,y)\rangle\,\mathrm{ds}\\ &= \int_{\gamma_0} -\Big(\langle m_k, \partial_x\left[\overline{R}^k A.e_1\right] + \partial_y\left[\overline{R}^k A.e_2\right]\rangle\Big) + \langle \partial_z g_{\mathrm{d}}(x,y,0), \overline{R}^k A(x,y).e_3\rangle\,\mathrm{ds}\\ &= \int_{\gamma_0} -\Big(\langle m_k, [\partial_x\overline{R}^k]A.e_1 + \overline{R}^k\partial_x A.e_1 + [\partial_y\overline{R}^k]A.e_2 + \overline{R}^k\partial_y A.e_2\rangle\Big)\,\mathrm{ds}\\ &\quad + \int_{\gamma_0} \langle \partial_z g_{\mathrm{d}}(x,y,0), \overline{R}^k A(x,y).e_3\rangle\,\mathrm{ds}\,.\end{aligned} \tag{4.27}$$

The augmented curvature expression (4.24) on the lateral boundary γ_0 allows us to specify a subsequence of the rotations, such that $\mathrm{D_x}\overline{R}^k \rightharpoonup \mathrm{D_x}\widehat{\overline{R}} \in L^2(\gamma_0,\mathfrak{T}(3))$. Observe that the augmented boundary curvature term is also weakly lower semicontinuous under weak convergence at the boundary γ_0. Since

$$\begin{aligned} m_k &\to \widehat{m} \in L^2(\gamma_0,\mathbb{R}^3)\,, &&\text{due to compact embedding}\,,\\ \overline{R}^k &\to \widehat{\overline{R}}^k \in L^2(\gamma_0,\mathrm{SO}(3,\mathbb{R}))\,, &&\text{due to compact embedding}\,,\\ \partial_x\overline{R}^k &\rightharpoonup \partial_x\widehat{\overline{R}} \in L^2(\gamma_0,\mathbb{M}^{3\times 3})\,, &&\text{due to additional curvature control at } \gamma_0\,,\\ \partial_y\overline{R}^k &\rightharpoonup \partial_y\widehat{\overline{R}} \in L^2(\gamma_0,\mathbb{M}^{3\times 3})\,, &&\text{due to additional curvature control at } \gamma_0\,,\end{aligned} \tag{4.28}$$

we conclude that

$$\begin{aligned}\lim_{k\to\infty}\int_{\gamma_0} &\langle \overline{R}^{k,T}(\nabla m_k(x,y)|\nabla g_{\mathrm{d}}(x,y,0).e_3), A(x,y)\rangle\,\mathrm{ds}\\ &= \int_{\gamma_0} \langle \widehat{\overline{R}}^T(\nabla\widehat{m}(x,y)|\nabla g_{\mathrm{d}}(x,y,0).e_3), A(x,y)\rangle\,\mathrm{ds}\,.\end{aligned} \tag{4.29}$$

Hence the minimizing solution $(\widehat{m},\widehat{\overline{R}}) \in H^1(\omega,\mathbb{R}^3) \times W^{1,1+p+q}(\omega,\mathrm{SO}(3,\mathbb{R}))$ satisfies the very weak consistent coupling condition and the proof is finished. ■

Remark 4.4 (The thin shell "membrane" limit)
Observe that all stated results remain true if we skip the h^3-bending contribution since the decisive curvature control is afforded by W_{curv} in conjunction with the internal length $L_c > 0$. In this sense, the formal thin shell "membrane" limit is not degenerated.

5 A new finite-strain Cosserat plate for large stretch and local invertibility

While the preceding models have been motivated from a three-dimensional "parent" model which itself is appropriate only for small strain and finite rotations, let us present a modified model,[12] which in principle allows for arbitrary large stretch and which automatically preserves local invertibility if the reconstructed deformation is smooth. It is clear that such an extension is by no means unique. We propose the model

$$
\begin{aligned}
I(m,\overline{R}) &= \int_\omega h\, W_{\mathrm{mp}}(\overline{U}) + h\, W_{\mathrm{curv}}(\mathfrak{K}_s) + \frac{h^3}{12}\, W_{\mathrm{bend}}(\mathfrak{K}_b)\,\mathrm{d}\omega - \Pi(m,\overline{R}_3) \mapsto \min.\ \text{w.r.t. } (m,\overline{R}),\\
\overline{U} &= \overline{R}^T\widehat{F}, \quad \widehat{F} = (\nabla m|\overline{R}_3), \quad F_s = (\nabla m|\varrho_m\overline{R}_3)\,,\\
\varrho_m &= \frac{1}{1+\frac{\lambda}{2\mu+\lambda}(\det[\overline{U}]-1)} + \frac{\langle N_{\mathrm{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)}\,, \quad \text{modified thickness stretch}\,,\\
m_{|\gamma_0} &= g_{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported}\,, \qquad\qquad (5.1)\\
\overline{R}_{|\gamma_0} &= \mathrm{polar}((\nabla m|\nabla g_{\mathrm{d}}(x,y,0).e_3))_{|\gamma_0}\,, \quad \text{strong form of reduced consistent coupling}\,,\\
\overline{R}_{3_{|\gamma_0}} &= \frac{\nabla g_{\mathrm{d}}(x,y,0).e_3}{\|\nabla g_{\mathrm{d}}(x,y,0).e_3\|}\,, \quad \text{alternatively: rigid director prescription}\,,\\
W_{\mathrm{mp}}(\overline{U}) &= \mu\,\|\mathrm{sym}(\overline{U}-1\!\!1)\|^2 + \mu_c\,\|\mathrm{skew}(\overline{U})\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\frac{1}{2}\left((\det[\overline{U}]-1)^2 + (\frac{1}{\det[\overline{U}]}-1)^2\right)\,,\\
W_{\mathrm{curv}}(\mathfrak{K}_s) &= \mu\frac{L_c^{1+p}}{12}\,(1+\alpha_4\,L_c^q\,\|\mathfrak{K}_s\|^q)\left(\alpha_5\,\|\mathrm{sym}\,\mathfrak{K}_s\|^2 + \alpha_6\,\|\mathrm{skew}\,\mathfrak{K}_s\|^2 + \alpha_7\,\mathrm{tr}\,[\mathfrak{K}_s]^2\right)^{\frac{1+p}{2}}\,,\\
\mathfrak{K}_s &= \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0)\right)\,,\\
W_{\mathrm{bend}}(\mathfrak{K}_b) &= \mu\,\|\mathrm{sym}(\mathfrak{K}_b)\|^2 + \mu_c\,\|\mathrm{skew}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\,[\mathrm{sym}(\mathfrak{K}_b)]^2\,,\\
\mathfrak{K}_b &= \overline{R}^T(\nabla\overline{R}_3|0) = \mathfrak{K}_s^3\,, \quad \text{second order } \textbf{non-symmetric bending tensor}\,.
\end{aligned}
$$

Let us summarize the salient features of this model: First, $W_{\mathrm{mp}}(\overline{U}) \to \infty$ if $\det[\overline{U}] \to 0$. Thus, if minimizers exist, then $\det[\overline{U}] > 0$ a.e. and the minimizing surface is locally regular. The modified membrane energy contribution W_{mp} is polyconvex w.r.t. ∇m at given $\overline{R}$ and indeed uniformly Legendre-Hadamard elliptic, independent of $\mu_c \geq 0$. If $\overline{R}_3 = \vec{n}_m$, then

$$\det[\overline{U}]^2 = \|\,\mathrm{Cof}\,(\nabla m|0)\|^2 = \|m_x \times m_y\|^2 = \|m_x\|^2\|m_y\|^2 - \langle m_x, m_y\rangle^2 = \det[I_m]\,, \qquad (5.2)$$

with $\vec{n}_m$ the outer unit normal of the surface m and I_m the first fundamental form. This formula represents a pure, intrinsic measure of the surface stretch. If $W_{\mathrm{mp}}(\overline{U}) = 0$ then $\overline{U} = 1\!\!1$ even for $\mu_c = 0$ and without gradient constraint.[13] Moreover, it can be shown that for zero Cosserat couple modulus $\mu_c = 0$ and zero internal length $L_c = 0$, the pure bending problem coincides with the rigourously justified classical finite-strain bending problem given in [FJM02].

The modified thickness stretch ϱ_m, which is used only for the a posteriori reconstruction of the bulk deformation, has such an analytical form, that at finite energy one has $0 < \varrho_m < \infty$, in line with the underlying physical description without restriction on the kinematics and transverse fibers will always be monotonically elongated upon action of opposite tractions.

Moreover, $\varrho_m \equiv 1$ for $\lambda = 0$ (extreme compressibility, $\nu = 0$) and $\varrho_m = \frac{1}{\det[\overline{U}]}$ for $\lambda = \infty$ (exact incompressibility, $\nu = \frac{1}{2}$) such that $\det[F_s] = \det[(\nabla m|\varrho_m\overline{R}_3)] \equiv 1$, i.e. exact incompressibility for the reconstructed deformation.

[12]It is clear that a modification to large stretch does not concern the bending term since bending only plays a role for small stretch.

[13]It is easy to see, that $\mathrm{sym}(\overline{U}-1\!\!1) = 0$ implies $\overline{R}_3 = \vec{n}_m$. The remaining consideration leads to $X \in \mathbb{M}^{2\times 2}$: $\mathrm{sym}\,X = 1\!\!1_2$, $\det[X] = 1 \Rightarrow X = 1\!\!1_2$.

The modified formulation (5.1), however, still has the same linearized behaviour as the initial model (4.1) and reduces to the classical infinitesimal-displacement Reissner-Mindlin model (5.11) for the choice of parameters $\mu_c = 0$, $p > 1$, $\alpha_4 = 0$.[14] We can prove the following result:

Theorem 5.1 (Existence for Cosserat plate with large stretch)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_\mathrm{d} \in H^1(\omega, \mathbb{R}^3)$ and $\overline{R}_\mathrm{d} \in W^{1,1+p+q}(\omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $\overline{f} \in L^1(\omega, \mathbb{R}^3)$ and suppose $\overline{N} \in L^1(\gamma_s, \mathbb{R}^3)$ together with $\overline{M} \in L^1(\omega, \mathbb{R}^3)$ and $\overline{M}_c \in L^1(\gamma_s, \mathbb{R}^3)$, see (5.19). Then (5.1) with material constants conforming to case II and rigid director prescription for $\overline{R}$ on γ_0 admits at least one minimizing solution pair $(m, \overline{R}) \in H^1(\omega, \mathbb{R}^3) \times W^{1,1+p+q}(\omega, \mathrm{SO}(3,\mathbb{R}))$ with $\det[(\nabla m|\overline{R}_3)] > 0$ a.e. $(x, y) \in \omega$. ■

Proof. The proof mimics the arguments of the existence result Theorem 4.2 for case II. We only need to observe in addition, that the modified membrane energy W_mp is in fact polyconvex [Bal77b] at given $\overline{R}$ w.r.t. ∇m since $\left((\det[\overline{U}] - 1)^2 + (\frac{1}{\det[\overline{U}]} - 1)^2\right)$ is convex in $\det[\overline{U}]$. The modified membrane strain energy term provides us with the information that $\det[(\nabla m_k|\overline{R}_3^k)]$ is uniformly bounded in $L^2(\omega)$ for minimizing sequences. Hence we may always choose a minimizing sequence, such that $\det[(\nabla m_k|\overline{R}_3^k)] \rightharpoonup \zeta \in L^2(\omega)$, weakly. A further subsequence may be choosen, not relabelled, such that $\overline{R}^k \to \overline{R} \in C^0(\omega, \mathrm{SO}(3,\mathbb{R}))$, due to the compact embedding $W^{1,1+p}(\omega) \subset C^0(\omega)$ for $p > 1$. Moreover, $\nabla m_k \rightharpoonup \nabla\widehat{m} \in L^2(\omega, \mathbb{M}^{2\times 3})$, weakly, as in Theorem 4.2. For two space dimensions, this implies the strong convergence of $\mathrm{Cof}\,(\nabla m_k|0)$ in the sense of distributions [Bal77a, Th. 3.4]:

$$\forall\, \psi \in C_0^\infty(\omega): \qquad \int_\omega \mathrm{Cof}\,(\nabla m_k|0)\, \psi\, \mathrm{d}\omega \to \int_\omega \mathrm{Cof}\,(\nabla \widehat{m}|0)\, \psi\, \mathrm{d}\omega\,, \quad k \to \infty\,. \tag{5.3}$$

Let us analyze in more detail the term $\det[(\nabla m_k|\overline{R}_3^k)]$. One has upon expanding of the detrminant

$$\begin{aligned}
\det[(\nabla m_k|\overline{R}_3^k)] &= \sum_{i=1}^3 \overline{R}_{3,i}^k\ \mathrm{Cof}\,(\nabla m_k|0)_{3,i} = \sum_{i=1}^3 (\overline{R}_{3,i}^k - \widehat{\overline{R}}_{3,i} + \widehat{\overline{R}}_{3,i})\ \mathrm{Cof}\,(\nabla m_k|0)_{3,i} \\
&= \sum_{i=1}^3 (\overline{R}_{3,i}^k - \widehat{\overline{R}}_{3,i})\ \mathrm{Cof}\,(\nabla m_k|0)_{3,i} + \sum_{i=1}^3 \widehat{\overline{R}}_{3,i}\ \mathrm{Cof}\,(\nabla m_k|0)_{3,i} \\
&= \sum_{i=1}^3 (\overline{R}_{3,i}^k - \widehat{\overline{R}}_{3,i})\ \mathrm{Cof}\,(\nabla m_k|0)_{3,i} + \sum_{i=1}^3 (\widehat{\overline{R}}_{3,i} - \widehat{\overline{R}}^\varepsilon + \widehat{\overline{R}}^\varepsilon)\ \mathrm{Cof}\,(\nabla m_k|0)_{3,i} \\
&= \sum_{i=1}^3 (\overline{R}_{3,i}^k - \widehat{\overline{R}}_{3,i})\ \mathrm{Cof}\,(\nabla m_k|0)_{3,i} + \sum_{i=1}^3 (\widehat{\overline{R}}_{3,i} - \widehat{\overline{R}}_{3,i}^\varepsilon)\ \mathrm{Cof}\,(\nabla m_k|0)_{3,i} \\
&\qquad + \widehat{\overline{R}}_{3,i}^\varepsilon\ \mathrm{Cof}\,(\nabla m_k|0)_{3,i}\,,
\end{aligned} \tag{5.4}$$

where $\widehat{\overline{R}}^\varepsilon \in C^\infty$ is introduced as a mollification of $\widehat{\overline{R}}$. Now we integrate $\det[(\nabla m_k|\overline{R}_3^k)]$ over ω against an arbitrary function $\psi \in C_0^\infty(\omega)$:

$$\begin{aligned}
\int_\omega \det[(\nabla m_k|\overline{R}_3^k)]\, \psi\, \mathrm{d}\omega = \int_\omega & \sum_{i=1}^3 (\overline{R}_{3,i}^k - \widehat{\overline{R}}_{3,i})\ \mathrm{Cof}\,(\nabla m_k|0)_{3,i}\, \psi \\
& + \sum_{i=1}^3 (\widehat{\overline{R}}_{3,i} - \widehat{\overline{R}}_{3,i}^\varepsilon)\ \mathrm{Cof}\,(\nabla m_k|0)_{3,i}\, \psi \\
& + \widehat{\overline{R}}_{3,i}^\varepsilon\ \mathrm{Cof}\,(\nabla m_k|0)_{3,i}\, \psi\, \mathrm{d}\omega\,.
\end{aligned} \tag{5.5}$$

Since $\mathrm{Cof}\,(\nabla m_k|0)$ is bounded in $L^1(\omega)$ the first sum converges to zero because of strong convergence of $\overline{R}^k$. The second term can be made arbitrarily small for $\varepsilon \to 0$ and the third

[14] Because $\left((\det[\overline{U}] - 1)^2 + (\frac{1}{\det[\overline{U}]} - 1)^2\right) = 2\,\mathrm{tr}\,\left[\overline{U} - 1\!\!1\right]^2 + O(\|\overline{U} - 1\!\!1\|^3)$.

term converges because $\widehat{\overline{R}}_{3,i}^{\varepsilon}\,\psi \in C_0^\infty(\omega)$ is an admitted testfunction in (5.3). Altogether, the strong convergence of $\overline{R}_3^k$ in $C^0(\omega)$ and the strong convergence of $\mathrm{Cof}\,(\nabla m_k|0)$ in the sense of distributions [Bal77a, Th. 3.4] for two space-dimensions show that

$$\forall\, \psi \in C_0^\infty(\omega): \qquad \int_\omega \det[(\nabla m_k|\overline{R}_3^k)]\,\psi\,\mathrm{d}\omega \to \int_\omega \det[(\nabla \widehat{m}|\overline{R}_3)]\,\psi\,\mathrm{d}\omega\,, \quad k\to\infty\,. \tag{5.6}$$

Thus, $\det[(\nabla m_k|\overline{R}_3^k)] \to \det[(\nabla \widehat{m}|\widehat{\overline{R}}_3)]$, strongly in the sense of distributions as well. This implies for the weak limit ζ found above that $\zeta = \det[(\nabla \widehat{m}|\widehat{\overline{R}}_3)]$. The remainder proceeds as in Theorem 4.2. ■

Altogether, this shows that (5.1) represents a significant conceptual improvement of the initially proposed plate model (4.1), although (5.1) itself is not strictly obtained from a parent model in our framework of formal dimensional descend. The extension of Theorem 5.1 to very weak consistent coupling is straightforward along the lines of Corollary 4.3.

In order to bridge the gap to more standard approaches we investigate now the relations of the new model to classical Reissner-Mindlin formulations.

5.1 Linearized plate models

5.1.1 Relations to the classical infinitesimal-displacement Reissner-Mindlin model

Let us linearize a variant of the proposed new finite-strain Cosserat plate (4.1) for situations of small midsurface deformations and small curvature. We assume here $\alpha_4 = 0$, $q = 0$, $p > 1$.[15] We write $m(x,y) = (x,y,0)^T + v(x,y)$, with the displacement of the midsurface of the plate $v : \omega \mapsto \mathbb{R}^3$ and $\overline{R} = 1\!\!1 + \overline{A} + \ldots$, with $\overline{A} \in \mathfrak{so}(3,\mathbb{R})$ the infinitesimal-displacement microrotation. For the boundary deformation we write $g_{\mathrm{d}}(x,y,z) = (x,y,z)^T + u^{\mathrm{d}}(x,y,z)$, with the consequence, that $\nabla g_{\mathrm{d}}.e_3 = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 1 + u^{\mathrm{d}}_{3,z})$. The curvature tensors are expanded as

$$\begin{aligned}
\mathfrak{K}_b &= \overline{R}^T(\nabla \overline{R}_3|0) = (1\!\!1 + \overline{A} + \ldots)^T\,(\nabla[\overline{A}_3 + \overline{A}^2.e_3 + \ldots]|0) \approx (\nabla \overline{A}_3|0) + \ldots\,,\\
\mathfrak{K}_s &\approx \big((\nabla(\overline{A}.e_1)|0), (\nabla(\overline{A}.e_2)|0), (\nabla(\overline{A}.e_3)|0)\big) \in \mathfrak{T}(3)\,,
\end{aligned} \tag{5.7}$$

and the Cosserat micropolar plate stretch tensor expands like

$$\begin{aligned}
\overline{U} &= \overline{R}^T\hat{F} = \overline{R}^T(\nabla m|\overline{R}_3) = (1\!\!1 + \overline{A} + \ldots)^T\left(\left(\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right) + \nabla v|(1\!\!1 + \overline{A} + \ldots).e_3)\right)\\
&\approx 1\!\!1 + (\nabla v|\overline{A}_3) - \overline{A} + \ldots\,.
\end{aligned} \tag{5.8}$$

Since $p > 1$, the additional Cosserat curvature contribution has an exponent strictly bigger than two such that a linearization w.r.t. zero curvature $\mathfrak{K}_s$ does not yield any contribution of this term. The consistent coupling condition is also expanded:

$$\begin{aligned}
\overline{R}_{|\gamma_0} &= \mathrm{polar}(\nabla m|\nabla g_d.e_3)\,,\\
1\!\!1 + \overline{A} + \ldots &= \mathrm{polar}(1\!\!1 + (\nabla v|\partial_z u^{\mathrm{d}}) + \ldots) = 1\!\!1 + \mathrm{skew}((\nabla v|\partial_z u^{\mathrm{d}})) + \ldots \Rightarrow\\
\overline{A}_{|\gamma_0} &= \mathrm{skew}((\nabla v|\partial_z u^{\mathrm{d}}))_{|\gamma_0}\,.
\end{aligned} \tag{5.9}$$

We are formally left with the minimization problem for $v \in \mathbb{R}^3$ and $\overline{A} \in \mathfrak{so}(3,\mathbb{R})$:

$$\begin{aligned}
&\int_\omega h\left(\mu\,\|\mathrm{sym}((\nabla v|\overline{A}_3))\|^2 + \mu_c\,\|\mathrm{skew}((\nabla v|\overline{A}_3) - \overline{A})\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}((\nabla v|\overline{A}_3))\right]^2\right)\\
&\qquad + \frac{h^3}{12}\left(\mu\,\|\mathrm{sym}((\nabla \overline{A}_3|0))\|^2 + \mu_c\,\|\mathrm{skew}((\nabla \overline{A}_3|0))\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}((\nabla \overline{A}_3|0))\right]^2\right)\mathrm{d}\omega\\
&\qquad\quad - \Pi(v,\overline{A}_3) \mapsto \min.\ \text{w.r.t. } (v,\overline{A}),\\
&v_{|\gamma_0} = u^{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported (fixed, welded)}\,,\\
&\overline{A}_{|\gamma_0} = \mathrm{skew}((\nabla v|\partial_z u^{\mathrm{d}}))_{|\gamma_0}\,, \quad \text{lin. coupling} \Rightarrow \overline{A}_{3_{|\gamma_0}} = \left(\frac{u^{\mathrm{d}}_{1,z} - v_{3,x}}{2}, \frac{u^{\mathrm{d}}_{2,z} - v_{3,y}}{2}, 0\right)^T,\\
&\overline{A}_{3_{|\gamma_0}} = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T\,, \quad \text{alternatively: rigid director prescription}\,.
\end{aligned} \tag{5.10}$$

[15] The linearization for the case $\alpha_4 = 0$, $q = 0$, $p = 1, \mu_c > 0$ is similar to the static micropolar plate model derived by Eringen [Eri67, eq. 8.6].

Now consider the case of zero Cosserat couple modulus $\mu_c = 0$. In this case infinitesimal in-plane rotations (linearized drilling degrees of freedom: $\overline{A}_{12} = -\overline{A}_{21}$) do not "survive" the linearization process. Abbreviating now $\theta = (\theta_1, \theta_2, 0)^T = -\overline{A}_3$, we are left with the following set of equations for the displacement of the midsurface of the plate $v : [0,T] \times \overline{\omega} \mapsto \mathbb{R}^3$ and the infinitesimal increment of the director, the infinitesimal "director", $\theta : \omega \mapsto \mathbb{R}^3$:

$$
\begin{aligned}
&\int_\omega h \left(\mu \, \|\operatorname{sym} \nabla (v_1, v_2)\|^2 + \kappa \underbrace{\frac{\mu}{2} \|\nabla v_3 - \theta\|^2}_{\text{transverse shear energy}} + \frac{\mu\lambda}{2\mu + \lambda} \operatorname{tr} \left[\operatorname{sym} \nabla (v_1, v_2)\right]^2 \right) \\
&\qquad + \frac{h^3}{12} \left(\mu \, \|\operatorname{sym} \nabla \theta\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \operatorname{tr} \left[\operatorname{sym} \nabla \theta\right]^2 \right) \, \mathrm{d}\omega - \Pi(v, -\theta) \mapsto \min . \text{ w.r.t. } (v, \theta) \,, \\
&v_{|_{\gamma_0}} = u^{\mathrm{d}}(x, y, 0) \,, \qquad \text{simply supported} \,, \qquad\qquad (5.11) \\
&-\theta_{|_{\gamma_0}} = \left(\frac{u^{\mathrm{d}}_{1,z} - v_{3,x}}{2}, \frac{u^{\mathrm{d}}_{2,z} - v_{3,y}}{2}, 0 \right)^T \,, \quad \text{linearized consistent coupling} \,, \\
&-\theta_{|_{\gamma_0}} = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T \,, \qquad \text{alternatively: rigid director prescription} \,,
\end{aligned}
$$

with the so-called **shear correction factor** $\kappa = 1$.

A further reduction arises if we assume only normal displacements: $v_1 = v_2 = 0$. The resulting minimization problem for the deflection v_3 and the "director" θ is

$$
\begin{aligned}
&\int_\omega h \, \frac{\kappa\mu}{2} \|\nabla v_3 - \theta\|^2 + \frac{h^3}{12} \left(\mu \, \|\operatorname{sym} \nabla \theta\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \operatorname{tr} \left[\operatorname{sym} \nabla \theta\right]^2 \right) \, \mathrm{d}\omega \\
&\qquad - \Pi(v_3 \cdot e_3, -\theta) \mapsto \min . \text{ w.r.t. } (v_3, \theta), \qquad\qquad (5.12) \\
&v_{3|_{\gamma_0}} = u^{\mathrm{d}}_3 \,, \qquad \text{simply supported} \,, \\
&-\theta_{|_{\gamma_0}} = \left(\frac{u^{\mathrm{d}}_{1,z} - v_{3,x}}{2}, \frac{u^{\mathrm{d}}_{2,z} - v_{3,y}}{2}, 0 \right)^T \qquad \text{linearized consistent coupling} \,, \\
&-\theta_{|_{\gamma_0}} = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T \,, \quad \text{rigid director prescription} \,.
\end{aligned}
$$

In this last form with rigid boundary prescription, the Reissner-Mindlin plate-bending problem is classical and can be found in many textbooks, e.g. [Bra92, p.281] or [Ste95, AMZ02] with Reissner's value $\kappa = \frac{5}{6}$. It should be noted, however, that in our formal, variationally based finite-strain derivation with subsequent linearization there is no imminent reason to introduce $\kappa \neq 1$. In fact, the shear correction factor κ can be seen as a tuning parameter of the infinitesimal-displacement model which, for certain types of loading,[16] allows to **improve** the **order of convergence** of the infinitesimal-displacement Reissner-Mindlin solution to the three-dimensional linear elasticity solution [Rös99].[17]

Note the novel non-standard Dirichlet **boundary condition** of **linearized consistent coupling** for the remaining infinitesimal "director" θ, motivated from the consistency condition of the Cosserat bulk model. In contrast to the standard rigid director prescription, the new coupling condition seems to reduce the strength of the boundary layer. In a direct derivation of the Reissner-Mindlin plate equations (5.11) there is no reason to introduce this weakened condition. However, a mathematical analysis based on the consistent coupling condition shows that the new boundary condition can only be satisfied in the distributional sense on γ_0. Let us

[16] Hence the shear correction factor κ shows some similarity to the Cosserat couple modulus μ_c, whose influence on the solution of the three-dimensional problem is also strongly dependent on boundary conditions. For rather thick plates, it is known that the shear energy in (5.11) is overestimated, therefore, one is led to reduce the shear energy contribution a posteriori by taking $\kappa < 1$.

[17] It would be interesting to know the optimal shear correction factor $0 < \kappa \leq 1$ of the infinitesimal-displacement Reissner-Mindlin model with our reduced consistent coupling boundary condition. Such an optimized parameter should also be beneficial for the finite-strain Cosserat plate. However, it might turn out that the new boundary condition of weak consistent coupling makes the artificial introduction of $\kappa < 1$ superfluous. Note as well, that $\kappa = 0$ decouples the horizontal "membrane" displacement in (5.11) from the vertical component and the bending term. In this sense, κ acts similarly as the Cosserat couple modulus μ_c in the linear Cosserat bulk model.

define therefore the admissible set

$$\mathcal{A}^{\text{lin}} := \{v_3 \in H^1(\omega,\mathbb{R}),\ \theta \in H^1(\omega,\mathbb{R}^2) \mid v_{3|_{\gamma_0}} = u_3^{\text{d}}\,, \quad \int_\omega \|\theta\|^2 \,\mathrm{d}\omega \le |\omega|\,,$$

$$\forall\, \phi \in C_0^\infty(\gamma_0,\mathbb{R}^2):\ \int_{\gamma_0} \langle -2\theta - \begin{pmatrix} u^{\text{d}}_{1,z} \\ u^{\text{d}}_{2,z} \end{pmatrix}, \phi\rangle_{\mathbb{R}^2} - v_3 \cdot \operatorname{Div}\phi \,\mathrm{d}\omega = 0\,\}\,, \tag{5.13}$$

which incorporates the linearized consistent coupling condition in the distributional sense, the standard Dirichlet boundary condition at γ_0, as well as an additional consistency condition for the linearization.[18] One can easily show that (5.12) admits a minimizer in $\mathcal{A}^{\text{lin}}$. If $\|\theta\|_{L^2(\omega,\mathbb{R}^2)} < |\omega|$, the solution is unique.

5.1.2 The classical infinitesimal-displacement Kirchhoff-Love plate (Koiter model)

For the convenience of the reader we also supply the similar system of equations for the classical infinitesimal-displacement Kirchhoff-Love plate (also the Koiter model) which can be derived as linearization of the finite-strain Kirchhoff-Love plate. In terms of the midsurface displacement v we have to find a solution of the minimization problem for $v : \omega \subset \mathbb{R}^2 \mapsto \in \mathbb{R}^3$:

$$\int_\omega h\left(\mu\,\|\operatorname{sym}\nabla(v_1,v_2)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}\nabla(v_1,v_2)\right]^2\right)$$

$$+\frac{h^3}{12}\left(\mu\,\|D^2 v_3\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[D^2 v_3\right]^2\right)\mathrm{d}\omega - \Pi(v,-\nabla v_3) \mapsto \min.\ \text{w.r.t. } v\,,$$

$$v_{|_{\gamma_0}} = u^{\text{d}}(x,y,0)\,, \qquad \text{simply supported (fixed, welded)}\,, \tag{5.14}$$

$$-\nabla v_{3|_{\gamma_0}} = \left(\frac{u^{\text{d}}_{1,z} - v_{3,x}}{2}, \frac{u^{\text{d}}_{2,z} - v_{3,y}}{2}, 0\right)^T, \quad \text{lin. coupling} \Rightarrow -\nabla v_{3|_{\gamma_0}} = (u^{\text{d}}_{1,z}, u^{\text{d}}_{2,z}, 0)^T\,,$$

$$-\nabla v_{3|_{\gamma_0}} = (u^{\text{d}}_{1,z}, u^{\text{d}}_{2,z}, 0)^T\,, \quad \text{rigid prescription of the infinitesimal increment of the "normal"}\,.$$

This energy can also be obtained formally from (5.12) by constraining the linearized director to the linearized normal of the plate, i.e. setting $\theta = \nabla v_3$. If this is done, we observe that the new boundary condition of consistent coupling coincides in fact with the classical boundary condition of the Kirchhoff-Love plate.

5.2 The treatment of external loads

5.2.1 Dead load body forces for the thin plate

In the three-dimensional theory the dead load body forces $f(x,y,z) \in \mathbb{R}^3$ were simply included by appending the potential with the term $\int_{\Omega_h} f(x,y,z)\cdot\varphi(x,y,z)\,\mathrm{dV}$. We define

$$\hat{f}_0(x,y) := \int_{-h/2}^{h/2} f(x,y,z)\,\mathrm{dz}\,, \quad \hat{f}_1(x,y) := \int_{-h/2}^{h/2} z\, f(x,y,z)\,\mathrm{dz}\,, \tag{5.15}$$

such that $\hat{f}_0$, $\hat{f}_1$ are the zero and first moment of f in thickness direction.

5.2.2 Traction boundary conditions for the thin plate

In the three-dimensional theory the traction boundary forces $N(x,y,z) \in \mathbb{R}^3$ were simply included by appending the potential with the term $\int_{\partial\Omega_h^{\text{trans}} \cup \{\gamma_s \times [-\frac{h}{2},\frac{h}{2}]\}} N(x,y,z)\cdot\varphi(x,y,z)\,\mathrm{dS}$. We define

$$\hat{N}_{\text{lat},0}(x,y) := \int_{-h/2}^{h/2} N(x,y,z)\,\mathrm{dz}\,, \quad \hat{N}_{\text{lat},1}(x,y) := \int_{-h/2}^{h/2} z\, N(x,y,z)\,\mathrm{dz}\,, \tag{5.16}$$

such that $\hat{N}_{\text{lat},0}$, $\hat{N}_{\text{lat},1}$ are the zero and first moment of the tractions N at the lateral boundary γ_s in thickness direction. Moreover, we abbreviate

$$N_{\text{res}} := [N(x,y,\frac{h}{2}) + N(x,y,-\frac{h}{2})]\,, \quad N_{\text{diff}} := \frac{1}{2}[N(x,y,\frac{h}{2}) - N(x,y,-\frac{h}{2})]\,. \tag{5.17}$$

[18] The unit "director" $\overline{R}_3$ is expanded as $\overline{R}_3 = e_3 - \theta + \ldots$. Any θ with $\|\theta(x,y)\| > 1$ pointwise, is inconsistent with the minimal requirement $1 = \|\overline{R}_3.e_1\| \ge \|(e_3+\theta).e_1\|$. As a consequence, we impose $\int_\omega \|\theta\|^2\,\mathrm{d}\omega \le |\omega|$.

5.2.3 The external resultant loading functional Π

For a first approximation plate formulation we set to leading order:

$$\begin{aligned}
\overline{f} &= \hat{f}_0 + N_{\mathrm{res}}\,, && \textbf{resultant body force}\,,\\
\overline{M} &= \hat{f}_1 + h\,N_{\mathrm{diff}}\,, && \textbf{resultant body couple}\,,\\
\overline{N} &= \hat{N}_{\mathrm{lat},0}\,, && \textbf{resultant surface traction}\,,\\
\overline{M}_c &= \hat{N}_{\mathrm{lat},1}\,, && \textbf{resultant surface couple}\,.
\end{aligned} \tag{5.18}$$

The **resultant dead load loading functional** Π is then given by the **linear form**

$$\Pi(m,\overline{R}_3) = \int_\omega \langle \overline{f}, m\rangle + \langle \overline{M}, \overline{R}_3\rangle \,\mathrm{d}\omega + \int_{\gamma_s} \langle \overline{N}, m\rangle + \langle \overline{M}_c, \overline{R}_3\rangle \,\mathrm{ds}\,. \tag{5.19}$$

If we denote the dependence of Π on the loads of the underlying three-dimensional problem as $\Pi(f,N;\, m,\overline{R}_3)$, then it is easily seen that frame-indifference of the external loading functional is satisfied in the sense that $\Pi(Q.f,Q.N;\, Q.m,Q.\overline{R}_3) = \Pi(f,N;\, m,\overline{R}_3)$ for all rigid rotations $Q \in \mathrm{SO}(3,\mathbb{R})$. It is possible to use the **same functional form** of the loading functional **for all finite-strain and infinitesimal-displacement models**. We only need to replace $(m,\overline{R}_3)$ by $(m,\vec{n}_m)$, $(v,\overline{A}_3)$ for the different finite and linearized models, respectively.

5.2.4 The modified external resultant loading functional $\Pi^\sharp$

In view of a possible mathematical analysis of the case with zero Cosserat couple modulus $\mu_c = 0$ we need to modify (5.19) into a **live load resultant loading functional** $\Pi^\sharp$, which better reflects the observation that by arbitrary translation of a material in a conservative force field only a finite amount of work can be gained. This is certainly true for any real physical field. In the three-dimensional theory we have called this the **"principle of bounded external work"**. Therefore we define the **nonlinear form**

$$\Pi^\sharp(m,\overline{R}_3) = \int_\omega \langle \overline{f}, \frac{m}{1+[\|m\|-K]_+}\rangle + \langle \overline{M}, \overline{R}_3\rangle \,\mathrm{d}\omega + \int_{\gamma_s} \langle \overline{N}, \frac{m}{1+[\|m\|-K]_+}\rangle + \langle \overline{M}_c, \overline{R}_3\rangle \,\mathrm{ds}\,. \tag{5.20}$$

Here $K > 0$ is a possibly large constant and $[\cdot]_+$ denotes the positive part of its scalar argument. We note that (5.20) is automatically **bounded**, if $\overline{f},\overline{M} \in L^1(\omega,\mathbb{R}^3)$ and $\overline{M}_c,\overline{N} \in L^1(\gamma_s,\mathbb{R}^3)$. Moreover, the linearization of $\Pi^\sharp$ coincides with the linearization of Π.

6 Discussion and open problems

We have investigated a finite-strain, frame-indifferent, geometrically exact Cosserat plate model derived in [Nef03a, Nef04a]. For vanishing Cosserat couple modulus $\mu_c = 0$, the formulation is shown to be downwards compatible with traditional infinitesimal-displacement linear Reissner-Mindlin theories and shear-correction factor $\kappa = 1$. A detailed mathematical analysis for vanishing Cosserat couple modulus $\mu_c = 0$ of the finite-strain model is given. Existence of minimizers in appropriate Sobolev-spaces is shown despite the inherent nonlinearity of the problem and despite the lack of unqualified coercivity. The decisive tool is a novel two-dimensional version of an extendend Korn's first inequality.

From a mechanical and computational point of view, compared to more traditional, non-elliptic finite-strain Reissner-Mindlin and Kirchhoff-Love models, it seems to be the beneficial influence of the drill-rotations in conjunction with the internal length $L_c > 0$ which stabilizes the new Cosserat thin plate model. Comparing with other alternative plate models with constraint or independent rotations, the additional implementational burden for the new Cosserat plate models is small compared to the possible gain of having a well-posed model.

Certain limit cases related to Sobolev-embedding theorems must remain open for the moment, notably the case IV including possible fracture of the plate. They leave a wide field of challenging purely mathematical problems.

While we have large freedom of specifying boundary conditions for the microrotations at the lateral Dirichlet boundary γ_0, we prefer a generalization of the three-dimensional consistent coupling condition which provides maximal consistency with the classical "symmetric" situation. I expect that this new consistent coupling condition reduces the strength of the boundary layer. Further research should clarify, whether the inherently sound Cosserat plate model (4.1) can be obtained as a Γ-limit of the Cosserat bulk problem for vanishing thickness.

7 Acknowledgements

The first idea of this paper was conceived in the academic year 2001/2002 while the author held a visiting faculty position under the ASCI program in Michael Ortiz group at the California Institute of Technology, Graduate Aeronautical Laboratories, Pasadena. The author would like to thank Fehmi Cirak and Kerstin Weinberg for helpful discussions and Michael Ortiz for first directing his attention to thin structures. Helpful remarks of Alexander Mielke on a preliminary version are also gratefully acknowledged.

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8 Appendix

8.1 The coercivity inequality in three-dimensions

The decisive analytical tool for the treatment of the case $\mu_c = 0$, called case II (super-critical) in [Nef03b] is the following inequality establishing coercivity for the deformations:

Theorem 8.1 (Extended 3D-Korn's first inequality)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $\Gamma \subset \partial\Omega$ be a smooth part of the boundary with non vanishing 2-dimensional Hausdorff measure. Define $H^{1,2}_\circ(\Omega,\Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|_\Gamma} = 0\}$ and let $F_p, F_p^{-1} \in C^1(\overline{\Omega}, \mathrm{GL}(3,\mathbb{R}))$. Moreover suppose that $\operatorname{Curl} F_p \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$. Then

$$\exists\, c^+ > 0\ \forall\, \phi \in H^{1,2}_\circ(\Omega,\Gamma): \quad \|\nabla\phi F_p^{-1}(x) + F_p^{-T}(x)\nabla\phi^T\|^2_{L^2(\Omega)} \geq c^+\, \|\phi\|^2_{H^{1,2}(\Omega)}\,.$$

Proof. The proof can be found in [Nef02]. ■

Remark 8.2
Note that for $F_p = \nabla\Theta$ we would only have to deal with the classical Korn's inequality evaluated on the transformed domain $\Theta(\Omega)$. However, in general, F_p is **incompatible** *giving rise to a* **non-Riemannian manifold** *structure. Compare this to [CG01] for an interpretation and the physical relevance of the volume dislocation density tensor $\operatorname{Curl} F_p$. A* **Riemannian** *version of Korn's inequality has also been given in [CJ02].*

Motivated by the investigations in [Nef02] it has been shown recently by Pompe [Pom03] that the extended Korn's inequality can be viewed as a special case of a general class of coercivity inequalities for quadratic forms. He was able to show that indeed $F_p, F_p^{-1} \in C(\overline{\Omega}, \mathrm{GL}(3,\mathbb{R}))$ is sufficient for (8.1) to hold without any condition on the compatibility.

However, taking the special structure of the extended Korn's inequality again into account, work in progress suggests that continuity is not really necessary: instead $F_p, F_p^{-1} \in$

$L^\infty(\Omega, \mathrm{GL}(3,\mathbb{R}))$ *and* $\mathrm{Curl}\, F_p \in L^{3+\delta}(\Omega)$ *should suffice, whereas* $F_p, F_p^{-1} \in L^\infty(\Omega, \mathrm{GL}(3,\mathbb{R}))$ *alone is not sufficient, see the counterexample presented in [Pom03]. This last possible improvement has no consequences for the subsequent mathematical analysis, however.*