Runge-Kutta methods for Itô stochastic differential equations with scalar noise

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Abstract

A class of explicit stochastic Runge-Kutta methods for non-autonomous Itô stochastic differential equation systems w.r.t. a one-dimensional Wiener process is developed. General conditions for the coefficients of stochastic Runge-Kutta schemes ensuring convergence with order two in the weak sense are derived. As a solution of these conditions, coefficients for new stochastic Runge-Kutta schemes with three stages are given explicitly. Some results of a simulation study reveal their good performance in comparison with some other known schemes.

Key words: stochastic Runge-Kutta method, stochastic differential equation, weak approximation, numerical method
MSC: 65C30, 65L06, 60H35, 60H10

1 Introduction

In many disciplines like engineering, physical sciences or mathematical finance, dynamical systems disturbed by random effects are studied. Thus, in recent years there has been an increasing interest in modelling stochastic dynamical systems by stochastic differential equations (SDE). However, analytic solutions of SDEs are known only in a few cases. Therefore, the development of numerical methods for the approximation of the solutions of SDEs has become more and more important. Depending on the objective of investigation, two different criteria for the convergence of the approximations can be considered. For good pathwise approximation strong convergence of the numerical method is needed while for good probability distributional approximation weak convergence is the adequate criterion.

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Much work has been done in developing numerical methods converging in the strong or weak sense, as presented for example in Kloeden and Platen [4] and Milstein [9]. Derivative free Runge-Kutta type schemes similar to the deterministic setting have been proposed for strong approximation by Burrage and Burrage [1], Mauthner [8], Newton [11] and Rümelin [16], while Kloeden and Platen [4], Komori and Mitsui [5], Mackevicius and Navikas [7], Milstein [9,10], Rößler [14,15], Talay [17,18] and Tocino and Vigo-Aguiar [19] developed Runge-Kutta type schemes for weak approximation.

In the following a system of Itô stochastic differential equations w.r.t. a one-dimensional Wiener process is considered. A class of stochastic Runge-Kutta (SRK) methods for weak approximation of the solution of the Itô SDE is proposed in Section 3. This class of algorithms turns out to be a generalization of deterministic Runge-Kutta methods (see, e.g., [2]). General order conditions for the coefficients of a SRK method assuring weak convergence of order two are calculated in Section 4 by a comparison of Taylor expansions. Therefore, solving a system of 28 nonlinear equations leads to coefficients for an order two SRK scheme. Some coefficients defining order two SRK schemes are presented in Section 5 as an example. The results of a simulation study presented in Section 6 reveal the good performance of the the proposed SRK schemes in comparison with some other well known schemes.

2 Local and global weak convergence

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t \geq 0}$. A d-dimensional Itô stochastic differential equation with a one-dimensional Wiener process $(W_t)_{t \geq 0}$ can be written as the integral equation

$$X_t = X_{t_0} + \int_{t_0}^{t} a(s, X_s)\, ds + \int_{t_0}^{t} b(s, X_s)\, dW_s \tag{1}$$

w.r.t. a $\mathcal{F}_{t_0}$-measurable initial condition $X_{t_0} = x_0 \in \mathbb{R}^d$ with $E(\|x_0\|^{2l}) < \infty$ for some $l \in \mathbb{N}$. For $I = [t_0, T]$ with $0 \leq t_0 < T < \infty$ we suppose that the drift $a : I \times \mathbb{R}^d \to \mathbb{R}^d$ and the diffusion $b : I \times \mathbb{R}^d \to \mathbb{R}^d$ are measurable functions satisfying a global Lipschitz and a linear growth condition

$$\|a(t,x) - a(t,y)\| + \|b(t,x) - b(t,y)\| \leq C_1 \, \|x - y\|, \tag{2}$$
$$\|a(t,x)\|^2 + \|b(t,x)\|^2 \leq C_1^2 (1 + \|x\|^2) \tag{3}$$

for every $t \in I$, $x, y \in \mathbb{R}^d$ and a constant $C_1 > 0$. Then the existence and uniqueness theorem applies, i.e. SDE (1) admits a unique solution process X with $E(\|X_t\|^{2l}) \leq (1 + E(\|x_0\|^{2l})) \exp(C_2 \, (t - t_0))$ for $t \in I$ (see, e.g., [3,12]).

In the following, let $C_P^l(\mathbb{R}^d, \mathbb{R})$ denote the space of all l times continuously differentiable functions $g : \mathbb{R}^d \to \mathbb{R}$ with polynomial growth, i.e. there exists a constant $C_3 > 0$ and $r \in \mathbb{N}$, such that $|\partial_x^i g(x)| \leq C_3(1 + \|x\|^{2r})$ for all $x \in \mathbb{R}^d$ and any partial derivative of order $i \leq l$. If g depends not only on $x \in \mathbb{R}^d$ but also on a real parameter $t \in I$, then $g(t,x)$ belongs to $C_P^l(\mathbb{R}^d, \mathbb{R})$ w.r.t. x if $g(t,\cdot) \in C_P^l(\mathbb{R}^d, \mathbb{R})$ holds uniformly in $t \in I$.

For the sake of simplicity, let an equidistant discretization $I_h = \{t_0, t_1, \ldots, t_N\}$ of the time interval $I = [t_0, T]$ with step size $h = \frac{T-t_0}{N}$, $N \in \mathbb{N}$, and $t_n = t_0 + n\, h$ be given. Then, a one-step approximation with step size h is defined by

$$Y^{t,x}(t+h) = A(t, x, h, \xi) \tag{4}$$

where A is a $\mathbb{R}^d$-valued function and ξ is a vector of random variables. In the following we write Y_n instead of $Y_{t_n} = Y(t_n)$. With (4) we can construct the sequence

$$Y_0 = x_0, \qquad Y_{n+1} = A(t_n, Y_n, h, \xi_n), \qquad n = 0, 1, \ldots, N-1, \tag{5}$$

where ξ_0 is independent of Y_0, while ξ_n is independent of $Y_0, \ldots, Y_n$ and of $\xi_0, \ldots, \xi_{n-1}$ for $1 \leq n \leq N-1$.

Definition 1 *The time discrete approximation process $Y = (Y_t)_{t \in I_h}$ converges weakly with order p to X as $h \to 0$ at time T if for each $f \in C_P^{2(p+1)}(\mathbb{R}^d, \mathbb{R})$ exists a constant C_f, which does not depend on h, and a finite $h_0 > 0$ such that*

$$|E(f(X_T)) - E(f(Y_T))| \leq C_f\, h^p \tag{6}$$

holds for each $h \in]0, h_0[$.

In order to prove convergence of an approximation Y to the solution X with some order p in the weak sense, we make use of a result due to Milstein (see, e.g., [10,15]) giving the relation between local and global order of weak convergence. We suppose that the coefficients $a = (a^i)$ and $b = (b^i)$ of the SDE system (1) are continuous, satisfy a Lipschitz condition, and that $a^i, b^i \in C_P^{2(p+1)}(\mathbb{R}^d, \mathbb{R})$ for $i = 1, \ldots, d$. Let the expectations $E(\|Y_n\|^{2r})$ exist for sufficiently large r and are uniformly bounded with respect to N and $n = 0, 1, \ldots, N$. Now we assume that for all $f \in C_P^{2(p+1)}(\mathbb{R}^d, \mathbb{R})$ the following *local error estimation*

$$|E(f(X^{t,x}(t+h))) - E(f(Y^{t,x}(t+h)))| \leq K(x)\, h^{p+1} \tag{7}$$

is valid for $x \in \mathbb{R}^d$, $K \in C_P^0(\mathbb{R}^d, \mathbb{R})$ and $t, t+h \in I$. Then for all N and all $n = 0, 1, \ldots, N$ the following *global error estimation*

$$|E(f(X^{t_0, X_{t_0}}(t_n))) - E(f(Y^{t_0, X_{t_0}}(t_n)))| \leq C_4\, h^p \tag{8}$$

holds for all $f \in C_P^{2(p+1)}(\mathbb{R}^d, \mathbb{R})$, where C_4 is a constant, i.e. the method (5) has order of accuracy p in the sense of weak approximation.

3 Stochastic Runge-Kutta methods

In the well developed numerical analysis of deterministic ordinary differential equations, derivative free approximation methods are of particular interest. Especially Runge-Kutta methods are very much in demand. Therefore, we introduce a new class of generalized Runge-Kutta methods suitable for SDEs which don't involve derivatives of drift and diffusion coefficients.

Let $I_h = \{t_0, t_1, \ldots, t_N\}$ be a discretization of the time interval $I = [t_0, T]$ with $h_n = t_{n+1} - t_n > 0$. In the following, we consider a class of explicit stochastic Runge-Kutta methods of the type

$$Y_0 = x_0, \qquad Y_{n+1} = A_I(Y_n, h_n, \theta(h_n)), \qquad n = 0, 1, \ldots, N-1, \tag{9}$$

with a vector $\theta(h_n) = (\theta_1(h_n), \theta_2(h_n), \theta_3(h_n))$ of random variables defined by

$$\theta_1(h_n) = W_{t_{n+1}} - W_{t_n} = I_{(1),n}, \tag{10}$$

$$\theta_2(h_n) = \tfrac{1}{\sqrt{h_n}} \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} dW_s \, dW_{s_1} = \frac{I_{(1,1),n}}{\sqrt{h_n}}, \tag{11}$$

$$\theta_3(h_n) = \sqrt{h_n}. \tag{12}$$

Then an explicit stochastic Runge-Kutta method with s stages takes the form

$$\begin{aligned} Y_{n+1} = Y_n &+ \sum_{i=1}^{s} \alpha_i \, a(t_n + c_i^{(0)} h_n, H_i^{(0)}) \, h_n + \sum_{i=1}^{s} \gamma_i^{(1)} \, b(t_n + c_i^{(1)} h_n, H_i^{(1)}) \, \theta_1(h_n) \\ &+ \sum_{i=1}^{s} \gamma_i^{(2)} b(t_n + c_i^{(1)} h_n, H_i^{(1)}) \, \theta_2(h_n) \end{aligned} \tag{13}$$

for $n = 0, 1, \ldots, N-1$ with the support values

$$\begin{aligned} H_i^{(0)} &= Y_n + \sum_{j=1}^{i-1} A_{ij}^{(0)} \, a(t_n + c_j^{(0)} h_n, H_j^{(0)}) \, h_n + \sum_{j=1}^{i-1} {B_{ij}^{(1)}}^{(0)} \, b(t_n + c_j^{(1)} h_n, H_j^{(1)}) \, \theta_1(h_n) \\ H_i^{(1)} &= Y_n + \sum_{j=1}^{i-1} A_{ij}^{(1)} \, a(t_n + c_j^{(0)} h_n, H_j^{(0)}) \, h_n + \sum_{j=1}^{i-1} {B_{ij}^{(3)}}^{(1)} \, b(t_n + c_j^{(1)} h_n, H_j^{(1)}) \, \theta_3(h_n) \end{aligned} \tag{14}$$

for $i = 1, 2, \ldots, s$ with the weights $\alpha_i, \gamma_i^{(1)}, \gamma_i^{(2)} \in \mathbb{R}$ and the coefficients $A_{ij}^{(0)}, A_{ij}^{(1)}, {B_{ij}^{(1)}}^{(0)}, {B_{ij}^{(3)}}^{(1)} \in \mathbb{R}$, $j < i$. Here we have to remark, that for a deterministic ordinary differential equation, i.e. SDE (1) with $b \equiv 0$, the stochastic Runge-Kutta method reduces to the well known deterministic Runge-Kutta method, so the introduced class of SRK methods turns out to be a generalization of deterministic Runge-Kutta methods.

An explicit stochastic Runge-Kutta scheme is characterized by its coefficients and weights, which can be represented in the usual manner by a Butcher tableau as presented for a s-stage scheme in Figure 1.

$$
\begin{array}{c|cccc|cccc|cccc}
0 & & & & & & & & & & & & \\
c_2^{(0)} & A_{21}^{(0)} & & & & B_{21}^{(1)(0)} & & & & & & & \\
\vdots & \vdots & \ddots & & & \vdots & \ddots & & & & & & \\
c_s^{(0)} & A_{s1}^{(0)} & \cdots & A_{s,s-1}^{(0)} & & B_{s1}^{(1)(0)} & \cdots & B_{s,s-1}^{(1)(0)} & & & & & \\
\hline
0 & & & & & & & & & & & & \\
c_2^{(1)} & A_{21}^{(1)} & & & & B_{21}^{(3)(1)} & & & & & & & \\
\vdots & \vdots & \ddots & & & \vdots & \ddots & & & & & & \\
c_s^{(1)} & A_{s1}^{(1)} & \cdots & A_{s,s-1}^{(1)} & & B_{s1}^{(3)(1)} & \cdots & B_{s,s-1}^{(3)(1)} & & & & & \\
\hline
 & \alpha_1 & \cdots & \alpha_{s-1} & \alpha_s & \gamma_1^{(1)} & \cdots & \gamma_{s-1}^{(1)} & \gamma_s^{(1)} & \gamma_1^{(2)} & \cdots & \gamma_{s-1}^{(2)} & \gamma_s^{(2)}
\end{array}
$$

Fig. 1. Coefficients of a SRK method represented by a Butcher tableau.

4 Local expansion and order conditions

For the calculation of coefficients for a stochastic Runge-Kutta scheme (13) converging globally with order 2.0 in the weak sense, we make use of the already mentioned result due to Milstein on the relation between lokal and global order of convergence. Thus, we have to determine coefficients for a SRK scheme such that the approximation converges locally with at least third-order. Therefore, we give an expansion of the solution process $E^{t,x}(f(X(t+h)))$ on the one hand and of the approximation process $E^{t,x}(f(Y(t+h)))$ on the other hand. A comparison of the coefficients of the expansion up to order three together with an estimation of the truncation error yield conditions for the coefficients of a SRK scheme converging with order two.

To keep the calculations easily comprehensible, we consider in the following without loss of generality an autonomous d-dimensional Itô SDE

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)\, ds + \int_{t_0}^{t} b(X_s)\, dW_s, \qquad X_{t_0} = x_0 \tag{15}$$

w.r.t. a one-dimensional Wiener process and a deterministic initial value $x_0 \in \mathbb{R}^d$. Clearly, every non-autonomous system can be transformed into an autonomous system of the same type with one additional equation.

To begin with the expansion of $E^{t_0,x_0}(f(X(t_0+h)))$ with $h>0$, $t_0, t_0+h \in I$ and $x_0 \in \mathbb{R}^d$ for SDE system (15) we apply the diffusion operator L (see [4,12]):

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{d} a^i \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^{d} b^i b^j \frac{\partial^2}{\partial x^i \partial x^j} \tag{16}$$

Then Itô's formula yields for the solution of the autonomous SDE (15)

$$\begin{aligned} E^{t_0,x_0}(f(X(t_0+h))) &= f(x_0) + \int_{t_0}^{t_0+h} E^{t_0,x_0}(Lf(X_s))\,ds \\ &= f(x_0) + \int_{t_0}^{t_0+h} \Big(Lf(x_0) + \int_{t_0}^{s} E^{t_0,x_0}(L^2 f(X_u))\,du \Big)\,ds \\ &= f(x_0) + \int_{t_0}^{t_0+h} Lf(x_0)\,ds + \int_{t_0}^{t_0+h} \int_{t_0}^{s} L^2 f(x_0)\,du\,ds \\ &\quad + \int_{t_0}^{t_0+h} \int_{t_0}^{s} \int_{t_0}^{u} E^{t_0,x_0}(L^3 f(X_v))\,dv\,du\,ds \\ &= f(x_0) + Lf(x_0)\,h + L^2 f(x_0) \frac{1}{2} h^2 \\ &\quad + \int_{t_0}^{t_0+h} \int_{t_0}^{s} \int_{t_0}^{u} E^{t_0,x_0}(L^3 f(X_v))\,dv\,du\,ds \end{aligned} \tag{17}$$

with

$$Lf = \sum_{i=1}^{d} a^i \frac{\partial f}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^{d} b^i b^j \frac{\partial^2 f}{\partial x^i \partial x^j} \tag{18}$$

and

$$\begin{aligned} L^2 f = &\sum_{i=1}^{d} \frac{\partial f}{\partial x^i} \left(\sum_{j=1}^{d} a^j \frac{\partial a^i}{\partial x^j} + \frac{1}{2} \sum_{j,k=1}^{d} b^j\, b^k \frac{\partial^2 a^i}{\partial x^j \partial x^k} \right) \\ &+ \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x^i \partial x^j} \Bigg(a^i\, a^j + \sum_{k=1}^{d} a^k\, b^j \frac{\partial b^i}{\partial x^k} + \sum_{k=1}^{d} b^k\, b^j \frac{\partial a^i}{\partial x^k} \\ &+ \frac{1}{2} \sum_{k,l=1}^{d} b^k\, b^l \frac{\partial^2 b^i}{\partial x^k \partial x^l}\, b^j + \frac{1}{2} \sum_{k,l=1}^{d} b^k\, b^l \frac{\partial b^i}{\partial x^k} \frac{\partial b^j}{\partial x^l} \Bigg) \\ &+ \sum_{i,j,k=1}^{d} \frac{\partial^3 f}{\partial x^i \partial x^j \partial x^k} \left(a^k\, b^i\, b^j + \sum_{l=1}^{d} b^k\, b^l \frac{\partial b^i}{\partial x^l}\, b^j \right) \\ &+ \frac{1}{4} \sum_{i,j,k,l=1}^{d} \frac{\partial^4 f}{\partial x^i \partial x^j \partial x^k \partial x^l} \left(b^i\, b^j\, b^k\, b^l \right). \end{aligned} \tag{19}$$

For $f, a^i, b^i \in C_P^6(\mathbb{R}^d, \mathbb{R})$, $i = 1, \ldots, d$, we have $L^3 f \in C_P(\mathbb{R}^d, \mathbb{R})$. Therefore, a constant $C_5 > 0$ exists such that

$$|E^{t_0,x_0}(L^3 f(X_s))| \leq C_5 (1 + \|x_0\|^{2l}), \quad t_0 \leq s \in I. \tag{20}$$

holds for $x_0 \in \mathbb{R}^d$. Applying (20) to (17) yields

$$E^{t_0,x_0}(f(X(t_0+h))) = f(x_0) + Lf(x_0)h + L^2 f(x_0)\tfrac{1}{2}h^2 + O(h^3). \qquad (21)$$

Next, a Taylor expansion up to order three of the stochastic Runge-Kutta method (13) has to be calculated. Therefore, we write t_0 instead of t_n and $t = t_0 + h$ instead of t_{n+1}. Then the kth component of the approximation Y_{n+1} is calculated as $Y(t)$ with intermediate values $H_1^{(0)}(t), \ldots, H_s^{(0)}(t)$ and $H_1^{(1)}(t), \ldots, H_s^{(1)}(t)$ given by

$$\begin{aligned} Y^k(t) = Y^k(t_0) + \sum_{i=1}^{s} \alpha_i \, a^k(H_i^{(0)}(t))\, h + \sum_{i=1}^{s} \gamma_i^{(1)} b^k(H_i^{(1)}(t))\, \theta_1(h) \\ + \sum_{i=1}^{s} \gamma_i^{(2)} b^k(H_i^{(1)}(t))\, \theta_2(h) \end{aligned} \qquad (22)$$

with

$$\begin{aligned} H_i^{(0)^k}(t) &= Y^k(t_0) + \sum_{j=1}^{s} A_{ij}^{(0)} \, a^k(H_j^{(0)}(t))\, h + \sum_{j=1}^{s} B_{ij}^{(1)^{(0)}} \, b^k(H_j^{(1)}(t))\, \theta_1(h), \\ H_i^{(1)^k}(t) &= Y^k(t_0) + \sum_{j=1}^{s} A_{ij}^{(1)} \, a^k(H_j^{(0)}(t))\, h + \sum_{j=1}^{s} B_{ij}^{(3)^{(1)}} \, b^k(H_j^{(1)}(t))\, \theta_3(h) \end{aligned} \qquad (23)$$

for $k = 1, \ldots, d$ and $i = 1, \ldots, s$. Here, the most general case of implicit SRK methods is considered. Clearly, for explicit SRK methods we have to set $A_{ij}^{(0)} = A_{ij}^{(1)} = B_{ij}^{(1)^{(0)}} = B_{ij}^{(3)^{(1)}} = 0$ for $j \geq i$. For notational convenience, in this section we denote by $\theta_0(h) = h$ and by $\theta(h) = (\theta_0(h), \ldots, \theta_3(h))^T$ the corresponding 4-dimensional vector [1]. Here $\theta_l(0) = 0$ for $l = 0, 1, \ldots, 3$.

In order to give a Taylor expansion of the approximation process defined by the SRK method $Y(t) = A_I(Y(t_0), \theta(t-t_0))$ as a function of $\theta_0, \ldots, \theta_3$, the differential operator $\mathcal{D}^k$, $k \in \mathbb{N}$, is introduced as

$$\mathcal{D}^k = \sum_{\nu_1,\ldots,\nu_k=0}^{3} \Delta\theta_{\nu_1} \cdot \Delta\theta_{\nu_2} \cdot \ldots \cdot \Delta\theta_{\nu_k} \cdot \frac{\partial^k}{\partial\theta_{\nu_1}\partial\theta_{\nu_2}\ldots\partial\theta_{\nu_k}} \qquad (24)$$

with $\Delta\theta_\nu = \theta_\nu(h) - \theta_\nu(0)$ for $\nu = 0, \ldots, 3$ and we denote by $\mathcal{D}^0 \equiv \mathrm{Id}$. Under the assumption that f, a and b are sufficiently differentiable, we get for $m \in \mathbb{N}$

$$f(Y(t)) = \sum_{k=0}^{m} \frac{\mathcal{D}^k f(Y(t_0))}{k!} + \mathcal{R}_m(t_0, h) \qquad (25)$$

[1] Then $Y(t) = A_I(Y(t_0), \theta_0(t-t_0), \ldots, \theta_3(t-t_0))$ and $Y(t_0) = A_I(Y(t_0), 0, \ldots, 0)$.

with a remainder term $\mathcal{R}_m$ which can be written in Lagrange form as

$$\mathcal{R}_m(t_0, h) = \frac{\mathcal{D}^{m+1} f(Y(t_0 + \xi\, h))}{(m+1)!} \tag{26}$$

with $\xi \in]0,1[$ and $h = t - t_0$.

In the following we write briefly $A_I = A_I(x_0) = A_I(x_0, 0, \ldots, 0)$ and $f = f(x_0)$. Then applying the multi-dimensional Taylor formula (25) and taking the expectation gives

$$\begin{aligned}
E^{t_0,x_0}(f(Y(t))) &= f + \sum_{i=1}^{d} \frac{\partial f}{\partial x^i} \left(\frac{\partial A_I^i}{\partial \theta_0} \right) h + \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x^i \partial x^j} \left(\frac{1}{2} \frac{\partial A_I^i}{\partial \theta_1} \frac{\partial A_I^j}{\partial \theta_1} \right. \\
&+ \left. \frac{1}{4} \frac{\partial A_I^i}{\partial \theta_2} \frac{\partial A_I^j}{\partial \theta_2} \right) h + \sum_{i=1}^{d} \frac{\partial f}{\partial x^i} \left(\frac{1}{2} \frac{\partial^3 A_I^i}{\partial \theta_0 \partial {\theta_1}^2} + \frac{1}{2} \frac{\partial^2 A_I^i}{\partial {\theta_0}^2} \right) h^2 \\
&+ \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x^i \partial x^j} \left(\frac{\partial A_I^i}{\partial \theta_1} \frac{\partial^2 A_I^j}{\partial \theta_0 \partial \theta_1} + \frac{1}{2} \frac{\partial^2 A_I^i}{\partial \theta_1 \partial \theta_3} \frac{\partial^2 A_I^j}{\partial \theta_1 \theta_3} + \frac{1}{2} \frac{\partial A_I^i}{\partial \theta_1} \frac{\partial^3 A_I^j}{\partial \theta_1 \partial {\theta_3}^2} \right. \\
&+ \left. \frac{1}{2} \frac{\partial A_I^i}{\partial \theta_0} \frac{\partial A_I^j}{\partial \theta_0} + \frac{1}{4} \frac{\partial^2 A_I^i}{\partial \theta_2 \partial \theta_3} \frac{\partial^2 A_I^j}{\partial \theta_2 \partial \theta_3} + \frac{1}{4} \frac{\partial A_I^i}{\partial \theta_2} \frac{\partial^3 A_I^j}{\partial \theta_2 \partial {\theta_3}^2} + \frac{1}{2} \frac{\partial A_I^i}{\partial \theta_2} \frac{\partial^2 A_I^j}{\partial \theta_0 \partial \theta_2} \right) h^2 \\
&+ \sum_{i,j,k=1}^{d} \frac{\partial^3 f}{\partial x^i \partial x^j \partial x^k} \left(\frac{1}{2} \frac{\partial A_I^i}{\partial \theta_0} \frac{\partial A_I^j}{\partial \theta_1} \frac{\partial A_I^k}{\partial \theta_1} + \frac{1}{2} \frac{\partial^2 A_I^i}{\partial \theta_2 \partial \theta_3} \frac{\partial A_I^j}{\partial \theta_1} \frac{\partial A_I^k}{\partial \theta_1} \right. \\
&+ \left. \frac{\partial A_I^i}{\partial \theta_2} \frac{\partial A_I^j}{\partial \theta_1} \frac{\partial^2 A_I^k}{\partial \theta_1 \partial \theta_3} + \frac{1}{2} \frac{\partial A_I^i}{\partial \theta_2} \frac{\partial A_I^j}{\partial \theta_2} \frac{\partial^2 A_I^k}{\partial \theta_2 \partial \theta_3} + \frac{1}{4} \frac{\partial A_I^i}{\partial \theta_0} \frac{\partial A_I^j}{\partial \theta_2} \frac{\partial A_I^k}{\partial \theta_2} \right) h^2 \\
&+ \sum_{i,j,k,l=1}^{d} \frac{\partial^4 f}{\partial x^i \partial x^j \partial x^k \partial x^l} \left(\frac{1}{8} \frac{\partial A_I^i}{\partial \theta_1} \frac{\partial A_I^j}{\partial \theta_1} \frac{\partial A_I^k}{\partial \theta_1} \frac{\partial A_I^l}{\partial \theta_1} + \frac{5}{32} \frac{\partial A_I^i}{\partial \theta_2} \frac{\partial A_I^j}{\partial \theta_2} \frac{\partial A_I^k}{\partial \theta_2} \frac{\partial A_I^l}{\partial \theta_2} \right. \\
&+ \left. \frac{5}{8} \frac{\partial A_I^i}{\partial \theta_2} \frac{\partial A_I^j}{\partial \theta_2} \frac{\partial A_I^k}{\partial \theta_1} \frac{\partial A_I^l}{\partial \theta_1} \right) h^2 \\
&+ \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x^i \partial x^j} \left(\frac{1}{2} \frac{\partial A_I^i}{\partial \theta_2} \frac{\partial^2 A_I^j}{\partial \theta_2 \partial \theta_3} + \frac{\partial A_I^i}{\partial \theta_1} \frac{\partial^2 A_I^j}{\partial \theta_1 \partial \theta_3} \right) h^{\frac{3}{2}} \\
&+ \sum_{i,j,k=1}^{d} \frac{\partial^3 f}{\partial x^i \partial x^j \partial x^k} \left(\frac{1}{6} \frac{\partial A_I^i}{\partial \theta_3} \frac{\partial A_I^j}{\partial \theta_3} \frac{\partial A_I^k}{\partial \theta_3} + \frac{1}{2} \frac{\partial A_I^i}{\partial \theta_2} \frac{\partial A_I^j}{\partial \theta_1} \frac{\partial A_I^k}{\partial \theta_1} \right) h^{\frac{3}{2}} \\
&+ \sum_{i=1}^{d} \frac{\partial f}{\partial x^i} \left(\frac{1}{2} \frac{\partial^4 A_I^i}{\partial \theta_0 \partial {\theta_1}^2 \partial \theta_3} + \frac{1}{2} \frac{\partial^4 A_I^i}{\partial \theta_0 \partial {\theta_1}^2 \partial \theta_2} \right) h^{\frac{5}{2}} \\
&+ \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x^i \partial x^j} \left(\frac{1}{6} \frac{\partial A_I^i}{\partial \theta_1} \frac{\partial^4 A_I^j}{\partial \theta_1 \partial {\theta_3}^3} + \frac{\partial^2 A_I^i}{\partial \theta_1 \partial \theta_3} \frac{\partial^2 A_I^j}{\partial \theta_0 \partial \theta_1} + \frac{\partial A_I^i}{\partial \theta_1} \frac{\partial^3 A_I^j}{\partial \theta_0 \partial \theta_1 \partial \theta_3} \right. \\
&+ \frac{1}{2} \frac{\partial^2 A_I^i}{\partial \theta_2 \partial \theta_3} \frac{\partial^2 A_I^j}{\partial \theta_0 \partial \theta_2} + \frac{1}{2} \frac{\partial A_I^i}{\partial \theta_2} \frac{\partial^3 A_I^j}{\partial \theta_0 \partial \theta_2 \partial \theta_3} + \frac{\partial A_I^i}{\partial \theta_1} \frac{\partial^3 A_I^j}{\partial \theta_0 \partial \theta_1 \partial \theta_2} + \frac{1}{2} \frac{\partial A_I^i}{\partial \theta_2} \frac{\partial^3 A_I^j}{\partial \theta_0 \partial {\theta_1}^2} \\
&+ \left. \frac{1}{12} \frac{\partial A_I^i}{\partial \theta_2} \frac{\partial^4 A_I^j}{\partial \theta_2 \partial {\theta_3}^3} + \frac{1}{4} \frac{\partial^2 A_I^i}{\partial \theta_2 \partial \theta_3} \frac{\partial^3 A_I^j}{\partial \theta_2 \partial {\theta_3}^2} + \frac{1}{2} \frac{\partial^2 A_I^i}{\partial \theta_1 \partial \theta_3} \frac{\partial^3 A_I^j}{\partial \theta_1 \partial {\theta_3}^2} \right) h^{\frac{5}{2}}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i,j,k=1}^{d}\frac{\partial^3 f}{\partial x^i\partial x^j\partial x^k}\left(\frac{1}{2}\frac{\partial^2 A_I^i}{\partial\theta_0\partial\theta_2}\frac{\partial A_I^j}{\partial\theta_2}\frac{\partial A_I^k}{\partial\theta_2}+\frac{\partial A_I^i}{\partial\theta_0}\frac{\partial A_I^j}{\partial\theta_1}\frac{\partial^2 A_I^k}{\partial\theta_1\partial\theta_3}\right.\\
&+\frac{1}{2}\frac{\partial A_I^i}{\partial\theta_0}\frac{\partial A_I^j}{\partial\theta_2}\frac{\partial^2 A_I^k}{\partial\theta_2\partial\theta_3}+\frac{1}{2}\frac{\partial^2 A_I^i}{\partial\theta_0\partial\theta_2}\frac{\partial A_I^j}{\partial\theta_1}\frac{\partial A_I^k}{\partial\theta_1}+\frac{\partial A_I^i}{\partial\theta_2}\frac{\partial A_I^j}{\partial\theta_1}\frac{\partial^2 A_I^k}{\partial\theta_0\partial\theta_1}\\
&+\frac{\partial^2 A_I^i}{\partial\theta_2\partial\theta_3}\frac{\partial A_I^j}{\partial\theta_1}\frac{\partial^2 A_I^k}{\partial\theta_1\partial\theta_3}+\frac{1}{2}\frac{\partial A_I^i}{\partial\theta_2}\frac{\partial A_I^j}{\partial\theta_1}\frac{\partial^3 A_I^k}{\partial\theta_1\partial{\theta_3}^2}+\frac{1}{4}\frac{\partial^3 A_I^i}{\partial\theta_2\partial{\theta_3}^2}\frac{\partial A_I^j}{\partial\theta_1}\frac{\partial A_I^k}{\partial\theta_1}\\
&\left.+\frac{1}{2}\frac{\partial A_I^i}{\partial\theta_2}\frac{\partial^2 A_I^j}{\partial\theta_1\partial\theta_3}\frac{\partial^2 A_I^k}{\partial\theta_1\partial\theta_3}+\frac{1}{2}\frac{\partial A_I^i}{\partial\theta_2}\frac{\partial^2 A_I^j}{\partial\theta_2\partial\theta_3}\frac{\partial^2 A_I^k}{\partial\theta_2\partial\theta_3}+\frac{1}{4}\frac{\partial A_I^i}{\partial\theta_2}\frac{\partial A_I^j}{\partial\theta_2}\frac{\partial^3 A_I^k}{\partial\theta_2\partial{\theta_3}^2}\right)h^{\frac{5}{2}}\\
&+\sum_{i,j,k,l=1}^{d}\frac{\partial^4 f}{\partial x^i\partial x^j\partial x^k\partial x^l}\left(\frac{1}{2}\frac{\partial A_I^i}{\partial\theta_1}\frac{\partial A_I^j}{\partial\theta_1}\frac{\partial A_I^k}{\partial\theta_1}\frac{\partial^2 A_I^l}{\partial\theta_1\partial\theta_3}+\frac{1}{6}\frac{\partial A_I^i}{\partial\theta_2}\frac{\partial A_I^j}{\partial\theta_2}\frac{\partial A_I^k}{\partial\theta_2}\frac{\partial A_I^l}{\partial\theta_0}\right.\\
&+\frac{5}{4}\frac{\partial A_I^i}{\partial\theta_2}\frac{\partial A_I^j}{\partial\theta_1}\frac{\partial A_I^k}{\partial\theta_1}\frac{\partial^2 A_I^l}{\partial\theta_2\partial\theta_3}+\frac{5}{4}\frac{\partial A_I^i}{\partial\theta_2}\frac{\partial A_I^j}{\partial\theta_2}\frac{\partial A_I^k}{\partial\theta_1}\frac{\partial^2 A_I^l}{\partial\theta_1\partial\theta_3}+\frac{1}{2}\frac{\partial A_I^i}{\partial\theta_0}\frac{\partial A_I^j}{\partial\theta_2}\frac{\partial A_I^k}{\partial\theta_1}\frac{\partial A_I^l}{\partial\theta_1}\\
&\left.+\frac{5}{8}\frac{\partial A_I^i}{\partial\theta_2}\frac{\partial A_I^j}{\partial\theta_2}\frac{\partial A_I^k}{\partial\theta_2}\frac{\partial^2 A_I^l}{\partial\theta_2\partial\theta_3}\right)h^{\frac{5}{2}}\\
&+\sum_{i,j,k,l,r=1}^{d}\frac{\partial^5 f}{\partial x^i\partial x^j\partial x^k\partial x^l\partial x^r}\left(\frac{17}{120}\frac{\partial A_I^i}{\partial\theta_2}\frac{\partial A_I^j}{\partial\theta_2}\frac{\partial A_I^k}{\partial\theta_2}\frac{\partial A_I^l}{\partial\theta_2}\frac{\partial A_I^r}{\partial\theta_2}\right.\\
&\left.+\frac{17}{24}\frac{\partial A_I^i}{\partial\theta_2}\frac{\partial A_I^j}{\partial\theta_2}\frac{\partial A_I^k}{\partial\theta_2}\frac{\partial A_I^l}{\partial\theta_1}\frac{\partial A_I^r}{\partial\theta_1}+\frac{1}{4}\frac{\partial A_I^i}{\partial\theta_2}\frac{\partial A_I^j}{\partial\theta_1}\frac{\partial A_I^k}{\partial\theta_1}\frac{\partial A_I^l}{\partial\theta_1}\frac{\partial A_I^r}{\partial\theta_1}\right)h^{\frac{5}{2}}\\
&+E^{t_0,x_0}(\mathcal{R}_5(t_0,h)). \qquad (27)
\end{aligned}
$$

Due to the polynomial growth condition for f, a^i and b^i and to the fact that the random variables have an order of magnitude $O(\sqrt{h})$, it can be easily proved that $E^{t_0,x_0}(\mathcal{R}_5(t_0,h)) = O(h^3)$ (see [15] Proposition 2.6.1).

Having calculated the two expansions of the solution (17) and the approximation (27), a comparison of the coefficients of the expansions has to be carried out. First of all a comparison of the expressions of order h with $\frac{\partial f}{\partial x^i}$ and $\frac{\partial^2 f}{\partial x^i\partial x^j}$ yields the conditions

$$
\sum_{i=1}^{d}\frac{\partial A_I^i}{\partial\theta_0}(x)=\sum_{i=1}^{d}a^i(x), \qquad (28)
$$

$$
\sum_{i,j=1}^{d}\left(\frac{1}{2}\frac{\partial A_I^i}{\partial\theta_1}\frac{\partial A_I^j}{\partial\theta_1}+\frac{1}{4}\frac{\partial A_I^i}{\partial\theta_2}\frac{\partial A_I^j}{\partial\theta_2}\right)(x)=\sum_{i,j=1}^{d}\left(\frac{1}{2}b^i\,b^j\right)(x). \qquad (29)
$$

Considering order $h^{1.5}$ terms with $\frac{\partial^2 f}{\partial x^i\partial x^j}$ and $\frac{\partial^3 f}{\partial x^i\partial x^j\partial x^k}$ we get the conditions

$$
\sum_{i,j=1}^{d}\left(\frac{1}{2}\frac{\partial A_I^i}{\partial\theta_2}\frac{\partial^2 A_I^j}{\partial\theta_2\partial\theta_3}+\frac{\partial A_I^i}{\partial\theta_1}\frac{\partial^2 A_I^j}{\partial\theta_1\partial\theta_3}\right)(x)=0, \qquad (30)
$$

$$
\sum_{i,j,k=1}^{d}\left(\frac{1}{6}\frac{\partial A_I^i}{\partial\theta_2}\frac{\partial A_I^j}{\partial\theta_2}\frac{\partial A_I^k}{\partial\theta_2}+\frac{1}{2}\frac{\partial A_I^i}{\partial\theta_2}\frac{\partial A_I^j}{\partial\theta_1}\frac{\partial A_I^k}{\partial\theta_1}\right)(x)=0. \qquad (31)
$$

These equations have to hold for all $x \in \mathbb{R}^d$. Calculating the derivative of $A_I^i(x)$ w.r.t. θ_0, condition (28) is fulfilled if

$$\sum_{l=1}^{s} \alpha_l \, a^i(x) = a^i(x) \tag{32}$$

holds for $i = 1, \ldots, d$, that is if $\alpha^T e = 1$ with $e = (1, \ldots, 1)^T$. Considering condition (29) and calculating the derivative of $A_I^i(x)$ w.r.t. θ_1 and θ_2, we obtain

$$\frac{1}{2} \left(\sum_{l=1}^{s} \gamma_l^{(1)} b^i \right) \left(\sum_{l=1}^{s} \gamma_l^{(1)} b^j \right) (x) + \frac{1}{4} \left(\sum_{l=1}^{s} \gamma_l^{(2)} b^i \right) \left(\sum_{l=1}^{s} \gamma_l^{(2)} b^j \right) (x) = \frac{1}{2} \left(b^i \, b^j \right) (x) \tag{33}$$

for $i, j = 1, \ldots, d$. Therefore the equation $(\gamma^{(1)^T} e)^2 + \frac{1}{2} (\gamma^{(2)^T} e)^2 = 1$ has to be fulfilled. Next, equations (30) and (31) are analysed analogously. By calculating the derivatives of $A_I^i(x)$ w.r.t. θ_1, θ_2 and θ_3, we obtain the conditions

$$\begin{aligned} &\frac{1}{2} \left(\sum_{l=1}^{s} \gamma_l^{(2)} \, b^i \right) (x) \cdot \left(\sum_{k=1}^{d} \sum_{r,u=1}^{s} \gamma_r^{(2)} \, B_{r,u}^{(3)^{(1)}} \frac{\partial b^j}{\partial x^k} \, b^k \right) (x) \\ &+ \left(\sum_{k=1}^{s} \gamma_k^{(1)} \, b^i \right) (x) \cdot \left(\sum_{k=1}^{d} \sum_{r,u=1}^{s} \gamma_r^{(1)} B_{r,u}^{(3)^{(1)}} \frac{\partial b^j}{\partial x^k} \, b^k \right) (x) = 0 \end{aligned} \tag{34}$$

for $i, j = 1, \ldots, d$ and

$$\begin{aligned} &\frac{1}{6} \left(\sum_{l=1}^{s} \gamma_l^{(2)} \, b^i \right) (x) \cdot \left(\sum_{l=1}^{s} \gamma_l^{(2)} \, b^j \right) (x) \cdot \left(\sum_{l=1}^{s} \gamma_l^{(2)} \, b^k \right) (x) \\ &+ \frac{1}{2} \left(\sum_{l=1}^{s} \gamma_l^{(2)} \, b^i \right) (x) \cdot \left(\sum_{l=1}^{s} \gamma_l^{(1)} \, b^j \right) (x) \cdot \left(\sum_{l=1}^{s} \gamma_l^{(1)} \, b^k \right) (x) = 0 \end{aligned} \tag{35}$$

for $i, j, k = 1, \ldots, d$. Therefore, the two equations

$$\begin{aligned} &(\gamma^{(2)^T} e)(\gamma^{(2)^T} B^{(3)^{(1)}} e) + 2(\gamma^{(1)^T} e)(\gamma^{(1)^T} B^{(3)^{(1)}} e) = 0, \\ &\frac{1}{3} (\gamma^{(2)^T} e)^3 + (\gamma^{(2)^T} e)(\gamma^{(1)^T} e)^2 = 0 \end{aligned} \tag{36}$$

have to be fulfilled. As a result of these calculations, equations (32)–(35) lead to conditions for the coefficients of a stochastic Runge-Kutta scheme converging with order 1.0 in the weak sense.

Analogous calculations have to be performed for expressions of order h^2 and $h^{2.5}$ to obtain additional conditions for the coefficients of a stochastic Runge-Kutta scheme converging with order 2.0. By investigating the order h^2 terms with $\frac{\partial f}{\partial x^i}$ we get the condition

$$\sum_{i=1}^{d}\left(\frac{\partial^3 A_I^i}{\partial\theta_0\partial{\theta_1}^2}+\frac{\partial^2 A_I^i}{\partial{\theta_0}^2}\right)(x)=\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\frac{\partial a^i}{\partial x^j}\,a^j+\frac{1}{2}\sum_{j,k=1}^{d}\frac{\partial^2 a^i}{\partial x^j\partial x^k}\,b^j\,b^k\right)(x). \tag{37}$$

Again, the equation has to be fulfilled for all $x \in \mathbb{R}^d$. Calculating the necessary derivatives of A_I^i yields

$$\begin{aligned}&\sum_{r=1}^{s}\alpha_r\left(\sum_{u=1}^{s}B_{r,u}^{(1)(0)}\right)^2\sum_{j,k=1}^{d}\frac{\partial^2 a^i}{\partial x^j\partial x^k}\,b^j\,b^k\\&+2\sum_{r=1}^{s}\gamma_r^{(1)}\sum_{u=1}^{s}A_{r,u}^{(1)}\sum_{v=1}^{s}B_{u,v}^{(1)(0)}\sum_{j,k=1}^{d}\frac{\partial b^i}{\partial x^j}\frac{\partial a^j}{\partial x^k}\,b^k\\&+2\sum_{r=1}^{s}\alpha_r\sum_{u=1}^{s}A_{r,u}^{(0)}\sum_{j=1}^{d}\frac{\partial a^i}{\partial x^j}\,a^j=\sum_{j=1}^{d}\frac{\partial a^i}{\partial x^j}\,a^j+\frac{1}{2}\sum_{j,k=1}^{d}\frac{\partial^2 a^i}{\partial x^j\partial x^k}\,b^j\,b^k\end{aligned} \tag{38}$$

for $i=1,\ldots,d$. This equation holds if the coefficients fulfill $\alpha^T(B^{(1)(0)}e)^2=\frac{1}{2}$, ${\gamma^{(1)}}^T(A^{(1)}(B^{(1)(0)}e))=0$ and $\alpha^T A^{(0)}e=\frac{1}{2}$. Here, the product of two vectors $v=(v_1,\ldots,v_k)^T$ and $w=(w_1,\ldots,w_k)^T$ is defined by componentwise multiplication, i.e. $v\cdot w=(v_1w_1,\ldots,v_kw_k)^T$. Considering the terms of order h^2 with $\frac{\partial^2 f}{\partial x^i\partial x^j}$, we get the condition

$$\begin{aligned}&\sum_{i,j=1}^{d}\Bigg(2\frac{\partial A_I^i}{\partial\theta_1}\frac{\partial^2 A_I^j}{\partial\theta_0\partial\theta_1}+\frac{\partial^2 A_I^i}{\partial\theta_1\partial\theta_3}\frac{\partial^2 A_I^j}{\partial\theta_1\partial\theta_3}+\frac{\partial A_I^i}{\partial\theta_1}\frac{\partial^3 A_I^j}{\partial\theta_1\partial{\theta_3}^2}+\frac{\partial A_I^i}{\partial\theta_0}\frac{\partial A_I^j}{\partial\theta_0}\\&+\frac{\partial^2 A_I^i}{\partial\theta_2\partial\theta_3}\frac{\partial^2 A_I^j}{\partial\theta_2\partial\theta_3}+\frac{1}{2}\frac{\partial A_I^i}{\partial\theta_2}\frac{\partial^3 A_I^j}{\partial\theta_2\partial{\theta_3}^2}+\frac{\partial A_I^i}{\partial\theta_2}\frac{\partial^2 A_I^j}{\partial\theta_0\partial\theta_2}\Bigg)(x)\\&=\sum_{i,j=1}^{d}\Bigg(a^i\,a^j+\sum_{k=1}^{d}a^k\,b^j\,\frac{\partial b^i}{\partial x^k}+\sum_{k=1}^{d}b^k\,b^j\,\frac{\partial a^i}{\partial x^k}\\&+\frac{1}{2}\sum_{k,l=1}^{d}b^k\,b^l\,\frac{\partial^2 b^i}{\partial x^k\partial x^l}\,b^j+\frac{1}{2}\sum_{k,l=1}^{d}b^k\,b^l\,\frac{\partial b^i}{\partial x^k}\frac{\partial b^j}{\partial x^l}\Bigg)(x).\end{aligned} \tag{39}$$

By calculating the necessary derivatives of A_I^i, condition (39) can be written as the equations

$$\begin{aligned}&2\left(\sum_{r=1}^{s}\gamma_r^{(1)}\,b^j\right)\left(\sum_{r=1}^{s}\alpha_r\sum_{u=1}^{s}B_{r,u}^{(1)(0)}\sum_{k=1}^{d}\frac{a^i}{\partial x^k}\,b^k+\sum_{r=1}^{s}\gamma_r^{(1)}\sum_{u=1}^{s}A_{r,u}^{(1)}\sum_{k=1}^{d}\frac{\partial b^i}{\partial x^k}\,a^k\right)\\&+\left(\sum_{r=1}^{s}\gamma_r^{(1)}\sum_{u=1}^{s}B_{r,u}^{(3)(1)}\right)^2\sum_{k,l=1}^{d}b^k\,b^l\,\frac{\partial b^i}{\partial x^k}\frac{\partial b^j}{\partial x^l}+\left(\sum_{r=1}^{s}\gamma_r^{(1)}\right)\times\\&\times\left(\sum_{r=1}^{s}\gamma_r^{(1)}\left(\sum_{u=1}^{s}B_{r,u}^{(3)(1)}\right)^2\right)\sum_{k,l=1}^{d}\frac{\partial^2 b^i}{\partial x^k\partial x^l}\,b^k\,b^l\,b^j+2\left(\sum_{r=1}^{s}\gamma_r^{(1)}\right)\times\end{aligned}$$

$$
\begin{aligned}
&\times \left(\sum_{r=1}^{s} \gamma_r^{(1)} \sum_{u=1}^{s} B_{r,u}^{(3)^{(1)}} \sum_{v=1}^{s} B_{u,v}^{(3)^{(1)}}\right) \sum_{k,l=1}^{d} \frac{\partial b^i}{\partial x^k} \frac{\partial b^k}{\partial x^l} b^l\, b^j + \left(\sum_{r=1}^{s} \alpha_r\right)^2 a^i\, a^j \\
&+ \frac{1}{2}\left(\sum_{r=1}^{s} \gamma_r^{(2)} \sum_{u=1}^{s} B_{r,u}^{(3)^{(1)}}\right)^2 \sum_{k,l=1}^{d} \frac{\partial b^i}{\partial x^k} \frac{\partial b^j}{\partial x^l} b^k\, b^l + \frac{1}{2}\left(\sum_{r=1}^{s} \gamma_r^{(2)}\right) \times \\
&\times \left(\sum_{r=1}^{s} \gamma_r^{(2)} \left(\sum_{u=1}^{s} B_{r,u}^{(3)^{(1)}}\right)^2\right) \sum_{k,l=1}^{d} \frac{\partial b^i}{\partial x^k \partial x^l} b^k\, b^l\, b^j + \left(\sum_{r=1}^{s} \gamma_r^{(2)}\right) \times \\
&\times \left(\sum_{r=1}^{s} \gamma_r^{(2)} \sum_{u=1}^{s} B_{r,u}^{(3)^{(1)}} \sum_{v=1}^{s} B_{u,v}^{(3)^{(1)}}\right) \sum_{k,l=1}^{d} \frac{\partial b^i}{\partial x^k} \frac{\partial b^k}{\partial x^l} b^l\, b^j + \left(\sum_{r=1}^{s} \gamma_r^{(2)}\right) \times \\
&\times \left(\sum_{r=1}^{s} \gamma_r^{(2)} \sum_{u=1}^{s} A_{r,u}^{(1)}\right) \sum_{k=1}^{d} \frac{\partial b^i}{\partial x^k} a^k\, b^j = a^i\, a^j + \sum_{k=1}^{d} a^k\, b^j \frac{\partial b^i}{\partial x^k} \\
&+ \sum_{k=1}^{d} b^k\, b^j \frac{\partial a^i}{\partial x^k} + \frac{1}{2} \sum_{k,l=1}^{d} b^k\, b^l \frac{\partial^2 b^i}{\partial x^k \partial x^l} b^j + \frac{1}{2} \sum_{k,l=1}^{d} b^k\, b^l \frac{\partial b^i}{\partial x^k} \frac{\partial b^j}{\partial x^l} \qquad (40)
\end{aligned}
$$

for $i, j = 1, \ldots, d$. As a result of a comparison of coefficients, the equations are fulfilled if

$$
\begin{aligned}
&(\gamma^{(1)^T} e)(\alpha^T B^{(1)^{(0)}} e) = \frac{1}{2}, \\
&(\gamma^{(1)^T} e)(\gamma^{(1)^T} A^{(1)} e) + \frac{1}{2}(\gamma^{(2)^T} e)(\gamma^{(2)^T} A^{(1)} e) = \frac{1}{2}, \\
&\frac{1}{2}(\gamma^{(1)^T} B^{(3)^{(1)}} e)^2 + \frac{1}{4}(\gamma^{(2)^T} B^{(3)^{(1)}} e)^2 = \frac{1}{4}, \\
&(\gamma^{(1)^T} e)(\gamma^{(1)^T} (B^{(3)^{(1)}} (B^{(3)^{(1)}} e))) + \frac{1}{2}(\gamma^{(2)^T} e)(\gamma^{(2)^T} (B^{(3)^{(1)}} (B^{(3)^{(1)}} e))) = 0, \\
&\frac{1}{2}(\gamma^{(1)^T} e)(\gamma^{(1)^T} (B^{(3)^{(1)}} e)^2) + \frac{1}{4}(\gamma^{(2)^T} e)(\gamma^{(2)^T} (B^{(3)^{(1)}} e)^2) = \frac{1}{4}
\end{aligned}
\qquad (41)
$$

and $(\alpha^T e)^2 = 1$ hold. Next we consider the order h^2 terms with $\frac{\partial^3 f}{\partial x^i \partial x^j \partial x^k}$. Here we get the condition

$$
\begin{aligned}
&\sum_{i,j,k=1}^{d} \left(\frac{\partial A_I^i}{\partial \theta_0} \frac{\partial A_I^j}{\partial \theta_1} \frac{\partial A_I^k}{\partial \theta_1} + \frac{\partial^2 A_I^i}{\partial \theta_2 \partial \theta_3} \frac{\partial A_I^j}{\partial \theta_1} \frac{\partial A_I^k}{\partial \theta_1} \right. \\
&\left. +2 \frac{\partial A_I^i}{\partial \theta_2} \frac{\partial A_I^j}{\partial \theta_1} \frac{\partial^2 A_I^k}{\partial \theta_1 \partial \theta_3} + \frac{\partial A_I^i}{\partial \theta_2} \frac{\partial A_I^j}{\partial \theta_2} \frac{\partial^2 A_I^k}{\partial \theta_2 \partial \theta_3} + \frac{1}{2} \frac{\partial A_I^i}{\partial \theta_0} \frac{\partial A_I^j}{\partial \theta_2} \frac{\partial A_I^k}{\partial \theta_2} \right)(x) \qquad (42) \\
&= \sum_{i,j,k=1}^{d} \left(a^i\, b^j\, b^k + \sum_{l=1}^{d} \frac{\partial b^i}{\partial x^l} b^l\, b^j\, b^k \right)(x)
\end{aligned}
$$

The calculation of the necessary derivatives of A_I^i leads to the equations

$$\left(\sum_{r=1}^{s}\alpha_r\right)\left(\sum_{r=1}^{s}\gamma_r^{(1)}\right)^2 a^i\,b^j\,b^k + \left(\sum_{r=1}^{s}\gamma_r^{(2)}\sum_{u=1}^{s}B_{r,u}^{(3)(1)}\right)\left(\sum_{r=1}^{s}\gamma_r^{(1)}\right)^2\sum_{l=1}^{d}\frac{\partial b^i}{\partial x^l}\,b^l\,b^j\,b^k$$
$$+\,2\left(\sum_{r=1}^{s}\gamma_r^{(2)}\right)\left(\sum_{r=1}^{s}\gamma_r^{(1)}\right)\left(\sum_{r=1}^{s}\gamma_r^{(1)}\sum_{u=1}^{s}B_{r,u}^{(3)(1)}\right)\sum_{l=1}^{d}\frac{\partial b^i}{\partial x^l}\,b^l\,b^j\,b^k + \left(\sum_{r=1}^{s}\gamma_r^{(2)}\right)^2\times$$
$$\times\left(\sum_{r=1}^{s}\gamma_r^{(2)}\sum_{u=1}^{s}B_{r,u}^{(3)(1)}\right)\sum_{l=1}^{d}\frac{\partial b^i}{\partial x^l}\,b^l\,b^j\,b^k + \frac{1}{2}\left(\sum_{r=1}^{s}\alpha_r\right)\left(\sum_{r=1}^{s}\gamma_r^{(2)}\right)^2 a^i\,b^j\,b^k$$
$$= a^i\,b^j\,b^k + \sum_{l=1}^{d}\frac{\partial b^i}{\partial x^l}\,b^l\,b^j\,b^k \tag{43}$$

for $i,j,k = 1,\ldots,d$. These conditions are fulfilled if the two equations

$$\begin{aligned}&(\alpha^T e)({\gamma^{(1)}}^T e)^2 + \frac{1}{2}(\alpha^T e)({\gamma^{(2)}}^T e)^2 = 1,\\ &({\gamma^{(1)}}^T e)^2({\gamma^{(2)}}^T {B^{(3)}}^{(1)} e) + 2({\gamma^{(2)}}^T e)({\gamma^{(1)}}^T e)({\gamma^{(1)}}^T {B^{(3)}}^{(1)} e)\\ &+ ({\gamma^{(2)}}^T e)^2({\gamma^{(2)}}^T {B^{(3)}}^{(1)} e) = 1\end{aligned} \tag{44}$$

hold. Now, we have to analyse the order h^2 terms with $\frac{\partial^4 f}{\partial x^i \partial x^j \partial x^k \partial x^l}$. Here we get the condition

$$\begin{aligned}\sum_{i,j,k,l=1}^{d}&\left(\frac{1}{4}\frac{\partial A_I^i}{\partial\theta_1}\frac{\partial A_I^j}{\partial\theta_1}\frac{\partial A_I^k}{\partial\theta_1}\frac{\partial A_I^l}{\partial\theta_1} + \frac{5}{16}\frac{\partial A_I^i}{\partial\theta_2}\frac{\partial A_I^j}{\partial\theta_2}\frac{\partial A_I^k}{\partial\theta_2}\frac{\partial A_I^l}{\partial\theta_2}\right.\\ &\left.+\frac{5}{4}\frac{\partial A_I^i}{\partial\theta_1}\frac{\partial A_I^j}{\partial\theta_1}\frac{\partial A_I^k}{\partial\theta_2}\frac{\partial A_I^l}{\partial\theta_2}\right)(x) = \sum_{i,j,k,l=1}^{d}\left(\frac{1}{4}b^i\,b^j\,b^k\,b^l\right)(x)\end{aligned} \tag{45}$$

Again, the calculation of the necessary derivatives of A_I^i leads to the equations

$$\begin{aligned}&\frac{1}{4}\left(\sum_{r=1}^{s}\gamma_r^{(1)}\right)^4 b^i\,b^j\,b^k\,b^l + \frac{5}{16}\left(\sum_{r=1}^{s}\gamma_r^{(2)}\right)^4 b^i\,b^j\,b^k\,b^l\\ &+\frac{5}{4}\left(\sum_{r=1}^{s}\gamma_r^{(1)}\right)^2\left(\sum_{r=1}^{s}\gamma_r^{(2)}\right)^2 b^i\,b^j\,b^k\,b^l = \frac{1}{4}b^i\,b^j\,b^k\,b^l\end{aligned} \tag{46}$$

for $i,j,k,l = 1,\ldots,d$. By a comparison of coefficients we get the equation

$$\frac{1}{4}({\gamma^{(1)}}^T e)^4 + \frac{5}{16}({\gamma^{(2)}}^T e)^4 + \frac{5}{4}({\gamma^{(1)}}^T e)^2({\gamma^{(2)}}^T e)^2 = \frac{1}{4} \tag{47}$$

which has to be fulfilled by the coefficients of the scheme.

Finally, the same calculations have to be performed for the expressions of order $h^{2.5}$. Due to the fact that there exist only terms of integer orders in the expansion (17) of the solution, all Taylor terms of non-integer order $h^{2.5}$ have to vanish. These calculations are omitted since they are straightforward.

The approximation Y has to be uniformly bounded with respect to the number N of steps in order to assure convergence. It is possible to give sufficient conditions for the random variables and for some coefficients by making use of the structure of the analyzed explicit stochastic Runge-Kutta methods.

Proposition 2 *Let $a, b \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ satisfy the conditions (2) and (3) of the Existence and Uniqueness Theorem and let*

$$E\Big(\sum_{i=1}^{s} \sum_{l=1}^{q} \gamma_i^{(l)} \, \theta_l(h) \Big) = 0. \tag{48}$$

Additionally, suppose that each random variable $\theta_l(h)$ can be represented as $\sqrt{h} \cdot \vartheta_l$ with a bounded random variable ϑ not depending on h sucht that $\theta_l(h)$ and $\sqrt{h} \cdot \vartheta_l$ have the same distribution for $l = 1, \ldots, q$. Then the approximation Y by the explicit stochastic Runge-Kutta method has uniformly bounded moments, i.e. for $r \in \mathbb{N}$ the expectation $E(\|Y_n\|^{2r})$ is uniformly bounded w.r.t. the number of steps N for $n = 0, 1, \ldots, N$.

Proof. See [15] Proposition 2.6.4.

If we assume now that (17) and (27) coincide up to order 3, then the discrete time approximation process Y is of third order one-step accuracy and fulfills condition (7). As a result of this, the global error estimation (8) holds with $p = 2.0$ if the approximation process has uniformly bounded moments. Thus, the above considerations lead to the following corollary:

Corollary 3 *Let $f, a^i, b^i \in C_P^6(\mathbb{R}^d, \mathbb{R})$ for $i = 1, \ldots, d$ and let the approximation Y of X with $Y_{n+1} = A_I(Y_n, h_n, \theta(h_n))$ as defined in (22) satisfy the assumptions of Proposition 2. If the expansions (17) and (27) coincide up to order 3.0, then the explicit stochastic Runge-Kutta method defined by (22) converges with order 2.0 in the weak sense to the solution X of the stochastic differential equation system (15).*

Remark 4 *Analogous calculations as presented in this section can also be performed for a non-autonomous SDE of type (1) with the stochastic Runge-Kutta method (13). They yield conditions for the coefficients $c^{(0)}$ and $c^{(1)}$. It is easily seen that these additional conditions can be obtained if $A^{(0)}e$ is replaced by $c^{(0)}$ and $A^{(1)}e$ by $c^{(1)}$ in the previously calculated conditions for the autonomous case. Thus, the new conditions are fulfilled if we set*

$$c^{(0)} = A^{(0)} e \quad \text{and} \quad c^{(1)} = A^{(1)} e \tag{49}$$

for the coefficients $c_1^{(0)}, \ldots, c_s^{(0)}$ and $c_1^{(1)}, \ldots, c_s^{(1)}$ in the following.

5 Stochastic Runge-Kutta methods of order 2.0

In this section, general order conditions for a class of order two stochastic Runge-Kutta schemes with an arbitrary number of stages are presented. The approximation Y defined by the stochastic Runge-Kutta method (13) is a second-order global approximation of X if the conditions of Proposition 2 and Corollary 3 are fulfilled. For simplicity of notation, we make use of the common tensor notation, i.e. the product of vectors is defined by componentwise multiplication and we denote $e = (1, \ldots, 1)^T$. As a result of this, the following theorem giving general conditions for the coefficients of the stochastic Runge-Kutta method (13) applicable to non-autonomous stochastic differential equations (1) can be stated.

Theorem 5 *If the coefficients of the stochastic Runge-Kutta method (13) fulfill the equations*

1. $\alpha^T e = 1$
2. ${\gamma^{(2)}}^T e = 0$
3. $({\gamma^{(1)}}^T e)^2 = 1$
4. ${\gamma^{(1)}}^T {B^{(3)}}^{(1)} e = 0$

then the method has the order of convergence 1.0 in the weak sense. If in addition the equations

5. $\alpha^T ({B^{(1)}}^{(0)} e)^2 = \frac{1}{2}$
6. ${\gamma^{(1)}}^T ({B^{(3)}}^{(1)} ({B^{(3)}}^{(1)} ({B^{(3)}}^{(1)} e))) = 0$
7. $\alpha^T A^{(0)} e = \frac{1}{2}$
8. $({\gamma^{(1)}}^T e)(\alpha^T {B^{(1)}}^{(0)} e) = \frac{1}{2}$
9. $({\gamma^{(1)}}^T e)({\gamma^{(1)}}^T A^{(1)} e) = \frac{1}{2}$
10. ${\gamma^{(1)}}^T ({B^{(3)}}^{(1)} ({B^{(3)}}^{(1)} e)) = 0$
11. $({\gamma^{(1)}}^T e)({\gamma^{(1)}}^T ({B^{(3)}}^{(1)} e)^2) = \frac{1}{2}$
12. ${\gamma^{(1)}}^T ({B^{(3)}}^{(1)} (A^{(1)} ({B^{(1)}}^{(0)} e))) = 0$
13. $\alpha^T (({B^{(1)}}^{(0)} e)({B^{(1)}}^{(0)} ({B^{(3)}}^{(1)} e))) = 0$
14. ${\gamma^{(1)}}^T (({B^{(3)}}^{(1)} e)(A^{(1)} ({B^{(1)}}^{(0)} e))) = 0$
15. ${\gamma^{(1)}}^T (A^{(1)} ({B^{(1)}}^{(0)} ({B^{(3)}}^{(1)} e))) = 0$
16. ${\gamma^{(1)}}^T (({B^{(3)}}^{(1)} e)({B^{(3)}}^{(1)} ({B^{(3)}}^{(1)} e))) = 0$
17. ${\gamma^{(1)}}^T ({B^{(3)}}^{(1)} e)^3 = 0$
18. ${\gamma^{(1)}}^T (A^{(1)} ({B^{(1)}}^{(0)} e)) = 0$
19. $\alpha^T ({B^{(1)}}^{(0)} ({B^{(3)}}^{(1)} e)) = 0$
20. ${\gamma^{(1)}}^T ({B^{(3)}}^{(1)} (A^{(1)} e)) = 0$
21. ${\gamma^{(1)}}^T (({B^{(3)}}^{(1)} e)(A^{(1)} e)) = 0$
22. ${\gamma^{(2)}}^T A^{(1)} e = 0$
23. ${\gamma^{(2)}}^T ({B^{(3)}}^{(1)} ({B^{(3)}}^{(1)} e)) = 0$
24. ${\gamma^{(2)}}^T ({B^{(3)}}^{(1)} e)^2 = 0$
25. ${\gamma^{(2)}}^T (A^{(1)} ({B^{(1)}}^{(0)} e)^2) = 0$
26. ${\gamma^{(2)}}^T {B^{(3)}}^{(1)} e = 1$
27. ${\gamma^{(1)}}^T ({B^{(3)}}^{(1)} ({B^{(3)}}^{(1)} e)^2) = 0$
28. ${\gamma^{(2)}}^T (A^{(1)} ({B^{(1)}}^{(0)} e)) = 0$

are fulfilled and if $c^{(0)} = A^{(0)} e$ and $c^{(1)} = A^{(1)} e$ holds, then the stochastic Runge-Kutta method (13) has the order of convergence 2.0 in the weak sense.

As an example of an explicit stochastic Runge-Kutta scheme converging with order 1.0 in the weak sense, we mention the well known Euler-Maruyama scheme (see, e.g. [4]). Similar to the deterministic setting, the Euler-Maruyama scheme EM is a $s = 1$ stage stochastic Runge-Kutta scheme with the coefficients presented in Figure 2 and it can be easily proved that the conditions 1.–4. of Theorem 5 are fulfilled. For the determination of some coefficients for a

0			
0			
$(1,1)$	1	1	0

Fig. 2. SRK scheme EM with $p_D = 1.0$ and $p_S = 1.0$.

second order stochastic Runge-Kutta scheme of the introduced class, a system of 28 non-linear equations presented in Theorem 5 has to be solved. Here, the number of coefficients depends on the choice of the number of stages s of the considered scheme. In the following, different sets of coefficients defining order two stochastic Runge-Kutta schemes with $s = 3$ stages are presented. Due to some degrees of freedom in choosing the coefficients for the deterministic part, it is possible to calculate a SRK scheme converging with order three if it is applied to deterministic ordinary differential equations. Therefore, if the weights α_i and the coefficients $A_{ij}^{(0)}$ are chosen such that condition 1. and 7. of Theorem 5 and additionally the conditions (see, e.g. [2])

$$\alpha^T(A^{(0)}(A^{(0)}e)) = \frac{1}{6} \quad \text{and} \quad \alpha^T(A^{(0)}e)^2 = \frac{1}{3} \tag{50}$$

are fulfilled, then the SRK scheme is of order three in the case of $b \equiv 0$ in (1).

In the following let (p_D, p_S) with $p_D \geq p_S$ denote the order of convergence of the stochastic Runge-Kutta scheme if it is applied to a deterministic or stochastic differential equation, respectively. Thus, in any case the scheme converges in the weak sense with not less than order $p = p_S$ and we assume better convergence for schemes with $p_D > p_S$, at least for SDEs having small noise. The following schemes can be applied for systems of stochastic differen-

0									
$\frac{2}{3}$	$\frac{2}{3}$			1					
$\frac{2}{3}$	$-\frac{1}{3}$	1		0	0				
0									
1	1			1					
1	1	0		-1	0				
$(3,2)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$	$-\frac{1}{2}$

Fig. 3. SRK scheme RI1W1 with $p_D = 3.0$ and $p_S = 2.0$.

0									
$\frac{2}{3}$	$\frac{2}{3}$			$\frac{1}{3}$					
$\frac{2}{3}$	$-\frac{1}{3}$	1		$\frac{4}{3}$	0				
0									
1	1			1					
1	1	0		-1	0				
$(3,2)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$	$-\frac{1}{2}$

Fig. 4. SRK scheme RI2W1 with $p_D = 3.0$ and $p_S = 2.0$.

0									
1	1			$\frac{3-2\sqrt{6}}{5}$					
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$		$\frac{6+\sqrt{6}}{10}$	0				
0									
1	1			1					
1	1	0		-1	0				
$(3,2)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$	$-\frac{1}{2}$

Fig. 5. SRK scheme RI3W1 of order $p_D = 3.0$ and $p_S = 2.0$.

0									
1	1			$\frac{1}{3}$					
$\frac{5}{12}$	$\frac{25}{144}$	$\frac{35}{144}$		$\frac{-5}{6}$	0				
0									
$\frac{1}{4}$	$\frac{1}{4}$			$\frac{1}{2}$					
$\frac{1}{4}$	$\frac{1}{4}$	0		$-\frac{1}{2}$	0				
$(3,2)$	$\frac{1}{10}$	$\frac{3}{14}$	$\frac{24}{35}$	1	-1	-1	0	1	-1

Fig. 6. SRK scheme RI5W1 of order $p_D = 3.0$ and $p_S = 2.0$.

tial equations w.r.t. a one-dimensional Wiener process. The scheme RI1W1 presented in Figure 3 converges with order $(3,2)$, as well as the schemes RI2W1 in Figure 4, RI3W1 in Figure 5 and RI5W1 in Figure 6. The scheme PL1W1 presented in Figure 7 converges with order $(2,2)$ and represents the enhanced non-autonomous version of the explicit order 2.0 weak scheme for autonomous SDEs proposed by Platen [4]. Further sets of coefficients and also embedded versions of the presented schemes, which may be applied for step size control algorithms as described in [6], have been calculated by Rößler [13,15].

For weak convergence with order 2.0 it is possible to replace the increments of the Wiener process $\Delta W_n = W_{t_{n+1}} - W_{t_n}$, $n = 0, 1, \ldots, N-1$, in the SRK

0									
1	1			1					
0	0	0		0	0				
0									
1	1			1					
1	1	0		-1	0				
$(2,2)$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$	$-\frac{1}{2}$

Fig. 7. SRK scheme PL1W1 with $p_D = 2.0$ and $p_S = 2.0$.

method(13) by random variables $\Delta\hat{W}_n$ having similar moment properties. As in [4,10,15,18], the random variables $\Delta\hat{W}_n$ have to be independent, $\mathcal{F}_{t_{n+1}}$-measurable and the moment condition $E((\Delta\hat{W}_n)^i) = E((\Delta W_n)^i)$ has to be fulfilled for $i = 1, \ldots, 5$. So one can replace ΔW_n by an independent identically three-point distributed random variable $\Delta\hat{W}_n$ with $P(\Delta\hat{W}_n = \pm\sqrt{3h_n}) = \frac{1}{6}$ and $P(\Delta\hat{W}_n = 0) = \frac{2}{3}$, which can be easily simulated. Further, due to $I_{(1,1)} = \frac{1}{2}((\Delta W_n)^2 - h_n)$, all random variables of the stochastic Runge-Kutta method can be expressed by the bounded random variable $\Delta\hat{W}_n$ which fulfills the conditions of Proposition 2.

6 Simulation study

In this section, the proposed stochastic Runge-Kutta schemes are compared with some well known schemes like the Euler-Maruyama scheme (EM) having order 1.0, the scheme (PL1W1) based on a scheme proposed by Platen (see also [4]) of order 2.0, and with a scheme by Milstein (MI) (see [9], [10] or [4] p.487) of order 2.0 which is not derivative free. In order to get a fair competition, we allow the Euler-Maruyama scheme to work with smaller step size $h/3$. Then it has nearly the same computational effort as the 3-stages SRK schemes if the number of evaluations of the drift and diffusion are considered as a measure for the effort. However, we have to take into consideration that this comparison can't be completely fair since the number of simulated random variables is three times higher for the step size $h/3$ than for step size h.

Example 6 *First, a linear stochastic differential equation [4]*

$$dX_t = aX_t\,dt + bX_t\,dW_t, \qquad X_0 = x_0, \tag{51}$$

with constant coefficients a and b is considered, which is the Black-Scholes stochastic differential equation used for option pricing. We choose $a = 1.5$, $b = 0.1$, $x_0 = 0.1$ and $T = 1.0$. The functional f is taken as $f(x) = x$. Then *the expectation of the solution at time T is given by*

$$E(X_T) = x_0 \cdot \exp(a\,T). \tag{52}$$

Example 7 *As a second example, we consider a non-linear stochastic differential equation [4,7]*

$$dX_t = \left(\tfrac{1}{2}X_t + \sqrt{X_t^2+1}\right)\,dt + \sqrt{X_t^2+1}\,dW_t, \qquad X_0 = 0 \tag{53}$$

on the time interval $[0,T]$, having the solution

$$X_t = \sinh(t + W_t) \tag{54}$$

for $t \geq 0$. Here, we choose $f(x) = p(\operatorname{arcsinh}(x))$, where $p(z) = z^3 - 6z^2 + 8z$ is a polynomial. Then the expectation of the solution can be calculated as

$$E(f(X_t)) = t^3 - 3t^2 + 2t. \tag{55}$$

In the following simulations, we approximate the functional $u = E(f(X_t))$ by a Monte Carlo simulation using the sample average

$$u_{M,h} = \frac{1}{M}\sum_{k=1}^{M} f(Y_t(\omega_k)), \qquad \omega_k \in \Omega, \tag{56}$$

of M independent simulated realizations of the considered approximation Y_t. Then, the mean error of the weak approximation is given as

$$\hat{\mu} = u_{M,h} - E(f(X_t)) \tag{57}$$

and the empirical variance of the mean error is denoted by $\hat{\sigma}_\mu^2$.

Example 6 is used for the investigation of the empirical order of convergence. Therefore, $M = 80\,000\,000$ trajectories are simulated with different step sizes $2^{-2}, \ldots, 2^{-6}$ and the error $|\hat{\mu}|$ is considered at the end of the time period $T = 1.0$ for each considered step size. The results are presented in Table 1 and plotted on the left hand side of the Figures 8–11 with double logarithmic scale. Consequently one obtains the empirical order of convergence as the slope of the printed lines. The first two lines represent the results of the Euler-Maruyama scheme EM calculated with step size h and $h/3$, respectively. Both lines have a slope of ≈ 1.0. The third and fourth lines coincide in our investigation and represent the results of the schemes PL1W1 due to Platen and MI by Milstein. Both lines reveal a slope of ≈ 2.0 which corresponds to their analytical order of convergence. The fifth line represents the results of the considered stochastic Runge-Kutta scheme. The empirical order of convergence of the schemes RI1W1, RI2W1, RI3W1 and RI5W1 is significantly

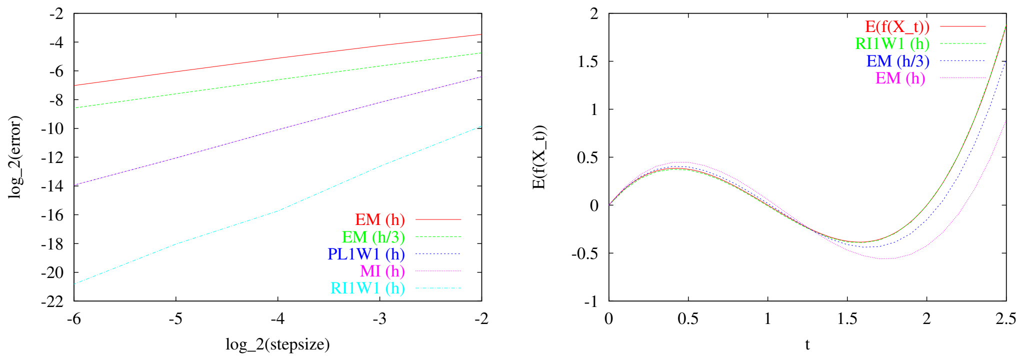

Fig. 8. Scheme RI1W1 with SDE (51) and (53).

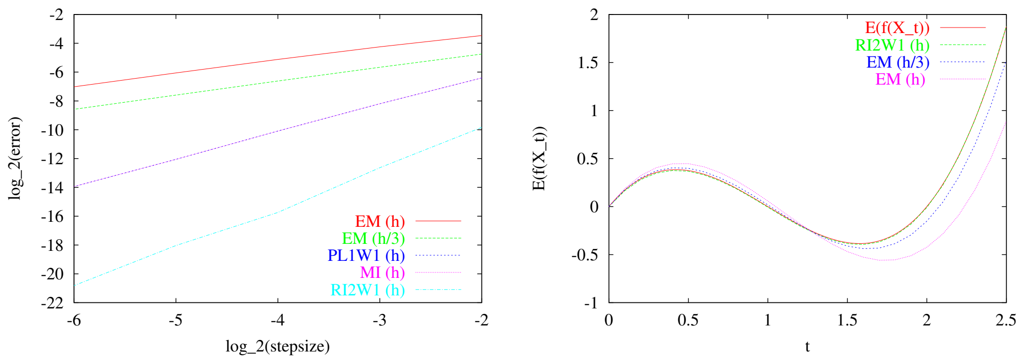

Fig. 9. Scheme RI2W1 with SDE (51) and (53).

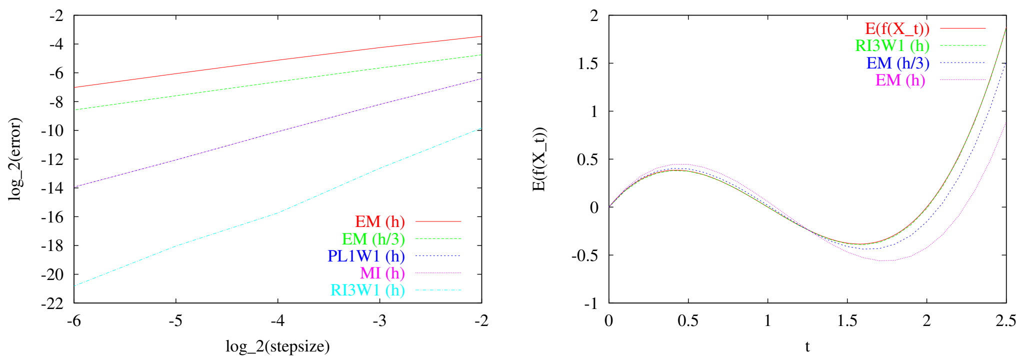

Fig. 10. Scheme RI3W1 with SDE (51) and (53).

higher than 2.0. This increase of the order of convergence goes back to $p_D = 3.0$ for the deterministic part of the schemes. It has to be pointed out that the better performance of the SRK schemes depends on the order of magnitude of the diffusion. Thus, the improved performance decreases as the coefficient b increases. However, also for higher values of b, the SRK schemes still perform better than the other schemes under consideration, as additional simulations revealed (see [15]).

For Example 7, the solution $E(f(X_t))$ is considered as a mapping from $[0,T]$ to $\mathbb{R}$ with $t \mapsto E(f(X_t))$. Therefore, a whole trajectory of the expectation has

Table 1
Simulations for the orders of convergence with SDE (51).

	h	$\lvert\hat{\mu}\rvert$	$\hat{\sigma}^2_\mu$		h	$\lvert\hat{\mu}\rvert$	$\hat{\sigma}^2_\mu$
RI1W1	2^{-2}	1.093351e-03	1.633037e-09	RI2W1	2^{-2}	1.093384e-03	1.578671e-09
	2^{-3}	1.570481e-04	1.602722e-09		2^{-3}	1.570562e-04	1.586961e-09
	2^{-4}	1.827831e-05	1.368048e-09		2^{-4}	1.828247e-05	1.364314e-09
	2^{-5}	3.720047e-06	1.677291e-09		2^{-5}	3.719851e-06	1.676111e-09
	2^{-6}	5.398320e-07	2.090135e-09		2^{-6}	5.397594e-07	2.089757e-09
RI3W1	2^{-2}	1.093414e-03	1.531615e-09	RI5W1	2^{-2}	1.097347e-03	1.524668e-09
	2^{-3}	1.570634e-04	1.573160e-09		2^{-3}	1.581176e-04	1.620628e-09
	2^{-4}	1.828613e-05	1.361033e-09		2^{-4}	2.235743e-05	1.683908e-09
	2^{-5}	3.719679e-06	1.675073e-09		2^{-5}	5.568290e-06	2.133419e-09
	2^{-6}	5.396956e-07	2.089425e-09		2^{-6}	1.857481e-06	2.756024e-09
EM (h)	2^{-2}	9.072138e-02	5.872681e-10	EM (h/3)	2^{-2}	3.717815e-02	1.154897e-09
	2^{-3}	5.274061e-02	8.745436e-10		2^{-3}	1.972579e-02	1.282961e-09
	2^{-4}	2.870995e-02	1.275816e-09		2^{-4}	1.017250e-02	1.566993e-09
	2^{-5}	1.502008e-02	1.789500e-09		2^{-5}	5.165140e-03	1.985171e-09
	2^{-6}	7.693292e-03	2.628478e-09		2^{-6}	2.607984e-03	2.774675e-09
MI	2^{-2}	1.180486e-02	1.496554e-09	PL1W1	2^{-2}	1.180486e-02	1.496554e-09
	2^{-3}	3.414500e-03	1.517499e-09		2^{-3}	3.414500e-03	1.517499e-09
	2^{-4}	9.167149e-04	1.723751e-09		2^{-4}	9.167149e-04	1.723751e-09
	2^{-5}	2.348662e-04	2.086635e-09		2^{-5}	2.348662e-04	2.086635e-09
	2^{-6}	6.369378e-05	2.850095e-09		2^{-6}	6.369378e-05	2.850095e-09

Table 2
SDE (53) simulated with step size $h = 0.1$ and $M = 8000000$ trajectories.

	t	$\lvert\hat{\mu}\rvert$	$\hat{\sigma}^2_\mu$		t	$\lvert\hat{\mu}\rvert$	$\hat{\sigma}^2_\mu$
$RI1W1$	0.5	9.199748e-03	1.829091e-04	$RI2W1$	0.5	7.107312e-03	1.830146e-04
	1.0	8.653046e-03	2.638963e-04		1.0	6.351854e-03	2.637400e-04
	1.5	8.681483e-03	2.503043e-04		1.5	1.069185e-02	2.497335e-04
	2.0	2.689141e-03	6.800987e-04		2.0	1.000730e-02	6.764457e-04
	2.5	1.010433e-02	1.695786e-03		2.5	2.494433e-03	1.686681e-03
$RI3W1$	0.5	9.531370e-04	1.821170e-04	$RI5W1$	0.5	7.648122e-05	2.685383e-04
	1.0	5.042670e-04	2.623512e-04		1.0	4.564679e-03	3.443047e-04
	1.5	7.368482e-03	2.483984e-04		1.5	1.411688e-02	2.947346e-04
	2.0	1.268896e-02	6.716069e-04		2.0	1.842642e-02	6.692763e-04
	2.5	6.030278e-03	1.673905e-03		2.5	9.733074e-03	1.607318e-03
$EM(h)$	0.5	7.216958e-02	1.733631e-04	$EM(h/3)$	0.5	2.398517e-02	1.835036e-04
	1.0	5.781451e-02	2.584442e-04		1.0	1.793959e-02	2.608021e-04
	1.5	9.356178e-02	2.256694e-04		1.5	3.630689e-02	2.402332e-04
	2.0	4.230208e-01	4.797263e-04		2.0	1.521508e-01	5.807490e-04
	2.5	9.859176e-01	9.776347e-04		2.5	3.501739e-01	1.352847e-03
MI	0.5	9.495915e-03	1.907192e-04	$PL1W1$	0.5	1.026393e-02	1.833368e-04
	1.0	1.365020e-02	2.709711e-04		1.0	7.923806e-03	2.643683e-04
	1.5	2.272141e-02	2.521346e-04		1.5	1.189430e-02	2.500735e-04
	2.0	3.326583e-02	6.752011e-04		2.0	2.052936e-02	6.716529e-04
	2.5	4.710692e-02	1.655876e-03		2.5	3.623635e-02	1.662453e-03

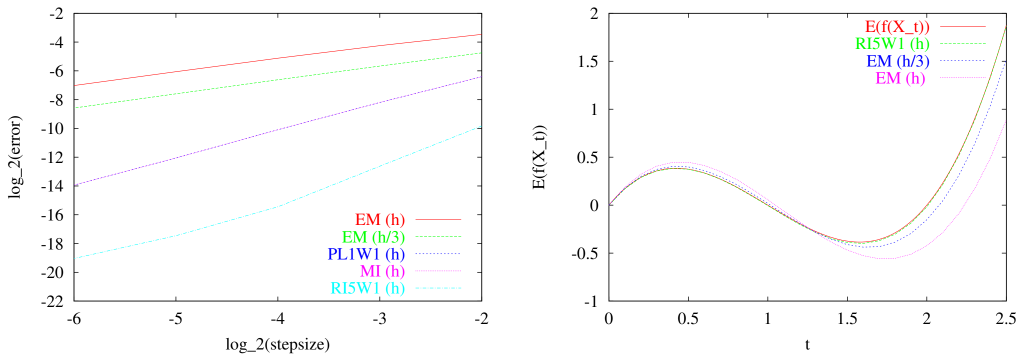

Fig. 11. Scheme RI5W1 with SDE (51) and (53).

to be simulated. Here, we choose the step size $h = 0.1$ and determine $u_{M,h}$ for all discretization times $t_n = n \cdot h$ for $n = 0, 1, \ldots, N$. To keep the results plotted on the right hand side of the Figures 8–11 clear, we restrict them to the approximations calculated by the considered SRK scheme with step size h and the Euler-Maruyama scheme with step size h and $h/3$. However, some calculated results for all considered schemes can be found in Table 2. A comparison of the presented results emphasizes the good performance of the introduced stochastic Runge-Kutta schemes in contrast to the well known schemes.

7 Conclusion

In this paper a new class of stochastic Runge-Kutta methods for systems of non-autonomous Itô stochastic differential equations with scalar noise is introduced. Firstly, general order conditions for the coefficients of the SRK methods with arbitrary number of stages are calculated by a comparison of the Taylor expansions of the solution and the approximation process. As a solution of a system of 28 nonlinear equations, coefficients for some different schemes of order $(2.0, 2.0)$ and $(3.0, 2.0)$ are calculated. The proposed stochastic Runge-Kutta methods turn out to be a generalization of the well known Runge-Kutta methods applied for deterministic ordinary differential equations. Finally, a simulation study reveals the good performance of the SRK schemes in comparison with the schemes EM, PL1W1 and MI proposed in literature. The main advantage of the presented SRK methods is that they are derivative free. Further, some higher oder of convergence than the assured order two can be observed for SRK schemes with $p_D > 2.0$ in case of stochastic differential equations with small noise. Thus the presented stochastic Runge-Kutta schemes offer significant benefits in comparison with the established schemes.

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