Very weak solutions of stationary and instationary Navier-Stokes equations with nonhomogeneous data

Dedicated to Professor Dr. H. Amann on the Occasion of his 65th Birthday

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Abstract

We investigate several aspects of *very weak solutions* u to stationary and nonstationary Navier-Stokes equations in a bounded domain $\Omega \subseteq \mathbb{R}^3$. This notion was introduced by Amann [3], [4] for the nonstationary case with nonhomogeneous boundary data $u|_{\partial\Omega} = g$ leading to a new and very large solution class. Here we are mainly interested to investigate the "largest possible" class for the more general problem with arbitrary divergence $k = \operatorname{div} u$, boundary data $g = u|_{\partial\Omega}$ and an external force f, as weak as possible. In principle, we will follow Amann's approach.

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1 Introduction

Throughout this paper $\Omega \subset \mathbb{R}^3$ is a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$ and $N = N(x) \in \mathbb{R}^3$ denotes the unit outer normal at $x = (x_1, x_2, x_3) \in \partial\Omega$. In $\Omega \times [0, T)$, where $0 < T \leq \infty$, we consider the system of Navier-Stokes equations in the very general form

$$u_t - \Delta u + u \cdot \nabla u + \nabla p = f, \quad \operatorname{div} u = k, \quad u|_{\partial\Omega} = g, \quad u|_{t=0} = u_0 \tag{1.1}$$

with nonhomogeneous data $f = \operatorname{div} F$ and k, g, u_0 satisfying

$$\begin{aligned} F &= (F_{ij})_{i,j=1,2,3} \in L^s\big(0,T; L^r(\Omega)\big), \qquad k \in L^s\big(0,T, L^r(\Omega)\big), \\ g &= (g_1, g_2, g_3) \in L^s\big(0,T; W^{-\frac{1}{q},q}(\partial\Omega)\big), \quad u_0 \in \mathcal{J}^{q,s}(\Omega), \end{aligned} \tag{1.2}$$

where $3 < q < \infty$, $2 < s < \infty$, $1 < r < q$ such that $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$, $\frac{2}{s} + \frac{3}{q} = 1$. For simplicity, we assume that the coefficient of viscosity equals 1. See (2.19) concerning the space of initial values $\mathcal{J}^{q,s}(\Omega)$ as a space of functionals. Further we suppose the compatibility condition

$$\int_\Omega k(t)dx = \int_{\partial\Omega} N \cdot g(t) dS \quad \text{for a.a. } t \in [0,T). \tag{1.3}$$

The largest possible class in the context of very weak solutions seems to be Serrin's uniqueness class $L^s\big(0,T; L^q(\Omega)\big)$ for u defined by $\frac{2}{s} + \frac{3}{q} = 1$. Indeed, we cannot expect, up to now, to obtain the desired uniqueness and regularity properties for any larger class.

To obtain the relation which defines a very weak solution u of (1.1), we follow Amann [3], [4] in principle and apply formally to (1.1) the test function $w \in C_0^1\big([0,T); C^2_{0,\sigma}(\overline{\Omega})\big)$, where $C^2_{0,\sigma}(\overline{\Omega}) = \{v \in C^2(\overline{\Omega})^3 : \operatorname{div} v = 0 \text{ in } \Omega, v|_{\partial\Omega} = 0\}$. Then an integration by parts yields the relation

$$\begin{aligned}\int_0^T \big(- \langle u, w_t\rangle_\Omega - \langle u, \Delta w\rangle_\Omega + \langle g, N\cdot\nabla w\rangle_{\partial\Omega} - \langle uu, \nabla w\rangle_\Omega - \langle ku, w\rangle_\Omega\big) dt \quad (1.4)\\ = \langle u_0, w(0)\rangle_\Omega - \int_0^T \langle F, \nabla w\rangle_\Omega dt, \quad w \in C_0^1\big([0,T); C^2_{0,\sigma}(\overline{\Omega})\big).\end{aligned}$$

Here $\langle\cdot,\cdot\rangle_\Omega$ is the usual $L^q - L^{q'}$-pairing in Ω and $\langle g(t), N\cdot\nabla w(t)\rangle_{\partial\Omega}$ denotes the value of the boundary distribution $g(t) \in W^{-\frac{1}{q},q}(\partial\Omega)$ at the normal derivative $w_N(t) = N\cdot\nabla w(t)|_{\partial\Omega}$; furthermore, $\langle u_0, w(0)\rangle_\Omega$ means the value of the functional $u_0 \in \mathcal{J}^{q,s}(\Omega)$ at $w(0) = w|_{t=0} \in C^2_{0,\sigma}(\overline{\Omega})$, see (2.19), and $uu = u\otimes u = (u_iu_j)_{i,j=1,2,3}$ for $u = (u_1,u_2,u_3)$. We also use the relation $u\cdot\nabla u = (u\cdot\nabla)u = \operatorname{div}(uu) - ku$. Note that $\operatorname{div} F = \big(\sum_{i=1}^3 (\partial/\partial x_i)F_{ij}\big)_{j=1,2,3}$.

Using in (1.4) in particular the test function $w \in C_0^1\big([0,T); C^2_{0,\sigma}(\Omega)\big)$ with $C^2_{0,\sigma}(\Omega) = \{v \in C_0^2(\Omega)^3;\ \operatorname{div} v = 0\}$, we obtain, together with some appropriate distribution p, the validity of the first equation in (1.1) in the sense of distributions. The second equation $\operatorname{div} u = k$ in (1.1) must be supposed additionally to (1.4). To explain the boundary condition $u|_{\partial\Omega} = g$ in (1.1) we observe that the normal derivative $w_N(t)$ of $w(t)$ at $\partial\Omega$ has the form

$$w_N(t) = N\cdot\nabla w(t) = \big(\operatorname{curl} w(t)\big)\times N, \tag{1.5}$$

and therefore, the relation (1.4) contains only the tangential component $N\times g$ of u at $\partial\Omega$. Indeed, we will show that the tangential component of u is well defined by (1.4) as a distribution on $\partial\Omega$, and we will derive an explicit formula, see (4.5). The (well defined) condition $N\cdot u(t)|_{\partial\Omega} = N\cdot g(t)$ for the normal component of u at $\partial\Omega$ must be supposed additionally. This leads to a precise formulation of the boundary condition $u_{|_{\partial\Omega}} = g$, see Remarks 3(2) in Section 4 below. Thus we suppose, additionally to (1.4), the conditions

$$\operatorname{div} u(t) = k(t), \quad N \cdot u(t)_{|_{\partial\Omega}} = N \cdot g(t) \quad \text{for a.a. } t \in [0,T). \tag{1.6}$$

This leads to the following

Definition 1 Assume that the data F, k, g and u_0 satisfy (1.2) and (1.3). Then $u \in L^s\big(0,T; L^q(\Omega)\big)$ is called a *very weak solution* of the Navier-Stokes system (1.1) if the conditions (1.4) and (1.6) are satisfied.

Note that a very weak solution u need not have any differentiability property in space and time. In particular, u need not satisfy any energy inequality with finite energy $\|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u\|^2_{L^2(0,T;L^2(\Omega))} < \infty$ like weak solutions in the sense of Hopf. This justifies the notation "very weak solution". On the other hand, a very weak solution is unique – a fact which is not known in general for weak solutions in the sense of Hopf.

The notion of very weak solutions is not new for homogeneous data $k = 0$ and $g = 0$, see [3], [4], [14], and the literature therein. However, Amann's notion of very weak solutions in [3], [4] for $k = 0$ and boundary values $g \neq 0$ introduces a completely new aspect. It leads to new solution classes of very low smoothness in space such that the boundary condition $u_{|_{\partial\Omega}} = g$ is not defined by usual trace theorems but more generally by the conditions (1.3), (1.4). This will have interesting applications.

Following Amann's approach our aim is to extend the solution classes for (1.1) to $\operatorname{div} u = k \neq 0$ and to the "weakest" possible case $u_{|_{\partial\Omega}} = g \in W^{-\frac{1}{q},q}(\partial\Omega)$ for a.a. $t \in [0,T]$, leading to the solution class in Definition 1 without any smoothness in space and only satisfying Serrin's condition, see Theorem 1. We will develop such a theory also for the linear nonstationary Stokes system, see Theorem 4. Further we improve the results on very weak solutions of stationary Stokes and Navier-Stokes systems developed in [14] to the more general case that $k \in L^r(\Omega)$, $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$, see Theorems 2 and 3.

Concerning the initial condition $u_{|_{t=0}} = u_0$ in (1.1) we note, see Theorem 1, that $A_q^{-1} P_q u$ is well defined as a continuous function on $[0,T)$ with values in $L^q_\sigma(\Omega)$; here, A_q means the Stokes operator and P_q the Helmholtz projection. We obtain the well defined condition

$$A_q^{-1} P_q u_{|_{t=0}} = A_q^{-1} P_q u_0, \tag{1.7}$$

which can be understood as the precise meaning of $u_{|_{t=0}} = u_0$, see (2.19) for $A_q^{-1} P_q u_0$ and the proof of Theorem 1.

Thus each condition in the system (1.1) has a well defined direct meaning for a very weak solution u. The first two equations hold in the sense of distributions on

$\Omega\times(0,T)$, $u|_{\partial\Omega} = g$ holds in the sense of distributions in $\partial\Omega$, and $u|_{t=0} = u_0$ holds in the sense of (1.7). Moreover, if the data are sufficiently smooth, u coincides with the usual strong solution.

Our main theorem on the system (1.1) reads as follows.

Theorem 1 *Suppose the data F, k, g and u_0 satisfy* (1.2) *and* (1.3) *with* $3 < q < \infty$, $2 < s < \infty$, $1 < r < q$, $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$, $\frac{2}{s} + \frac{3}{q} = 1$. *Then there exists some* $T' = T'(F, k, g, u_0) > 0$, $0 < T' \le T$, *and a uniquely determined very weak solution* $u \in L^s\big(0, T'; L^q(\Omega)\big)$ *of the system* (1.1) *satisfying* $A_q^{-1} P_q u_t \in L^{s/2}\big(0, T''; L^q_\sigma(\Omega)\big)$ *for all* $0 < T'' < T'$, *and* $A_q^{-1} P_q u \in C\big([0, T'); L^q_\sigma(\Omega)\big)$. *The existence interval* $[0, T')$ *is determined by the condition* (4.23)*, depending on the data, and includes the case* $T' = T = \infty$ *if the data are sufficiently small.*

Up to now we cannot prove that there exists an open maximal existence interval as in [3], [4] for the case $k = 0$. The reasons are the very weak assumptions on g and k in (1.2).

In the linearized case $u \cdot \nabla u \equiv 0$ we have to omit the term $\langle u \cdot \nabla u, w\rangle_\Omega = -\langle uu, \nabla w\rangle_\Omega - \langle ku, w\rangle_\Omega$ in (1.4), and we may omit the restriction $\frac{2}{s} + \frac{3}{r} = 1$ in (1.2), which is caused by the nonlinear term. Then we can show the existence and uniqueness of a very weak solution $u \in L^s\big(0, T; L^q(\Omega)\big)$ of the linearized system (1.1) satisfying the estimate

$$\begin{aligned} &\|A_q^{-1} P_q u_t\|_{L^s(0,T;L^q(\Omega))} + \|u\|_{L^s(0,T;L^q(\Omega))} \\ &\quad \le C\big(\|u_0\|_{\mathcal{J}^{q,s}(\Omega)} + \|F\|_{L^s(0,T;L^r(\Omega))} + \|k\|_{L^s(0,T;L^r(\Omega))} + \|g\|_{L^s(0,T;W^{-\frac{1}{q},q}(\partial\Omega))}\big) \end{aligned} \tag{1.8}$$

with $C = C(\Omega, q, s) > 0$, see Theorem 4 in Section 4 below.

In the stationary case we consider the system

$$-\Delta u + u \cdot \nabla u + \nabla p = f, \quad \operatorname{div} u = k, \quad u|_{\partial\Omega} = g \tag{1.9}$$

with data $f = \operatorname{div} F$ and k, g satisfying

$$F = (F_{ij}) \in L^r(\Omega),\ k \in L^r(\Omega),\ g \in W^{-\frac{1}{q},q}(\partial\Omega),\ \int_\Omega k\,dx = \int_{\partial\Omega} N \cdot g\,dS, \tag{1.10}$$

with $3 < q < \infty$, $1 < r < q$, $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$.

An obvious modification of the nonstationary case yields the following

Definition 2 Assume that the data F, k and g satisfy (1.10). Then $u \in L^q(\Omega)$ is called a *very weak solution* of the stationary Navier-Stokes system (1.9) if the relation

$$-\langle u, \Delta w\rangle_\Omega + \langle g, N \cdot \nabla w\rangle_{\partial\Omega} - \langle uu, \nabla w\rangle_\Omega - \langle ku, w\rangle_\Omega = -\langle F, \nabla w\rangle_\Omega, \tag{1.11}$$

holds for all $w \in C^2_{0,\sigma}(\overline{\Omega})$, and the conditions

$$\operatorname{div} u = k, \quad N \cdot u_{|_{\partial\Omega}} = N \cdot g \tag{1.12}$$

are satisfied.

In this case we obtain the following result.

Theorem 2 *Suppose the data F, k and g satisfy (1.10) with $3 < q < \infty$, $1 < r < q$, $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$. There exists a constant $K = K(\Omega, q) > 0$ such that if*

$$\|F\|_{L^r(\Omega)} + \|k\|_{L^r(\Omega)} + \|g\|_{W^{-\frac{1}{q},q}(\partial\Omega)} \leq K, \tag{1.13}$$

then we obtain a uniquely determined very weak solution $u \in L^q(\Omega)$ to the stationary Navier-Stokes system (1.9). This solution satisfies the estimate

$$\|u\|_{L^q(\Omega)} \leq C\big(\|F\|_{L^r(\Omega)} + \|k\|_{L^r(\Omega)} + \|g\|_{W^{-\frac{1}{q},q}(\partial\Omega)}\big) \tag{1.14}$$

with $C = C(\Omega, q) > 0$.

Similarly as in (1.5) we obtain for $x \in \partial\Omega$ the identity

$$w_N = N \cdot \nabla w = (\operatorname{curl} w) \times N, \quad w \in C^2_{0,\sigma}(\overline{\Omega}). \tag{1.15}$$

Setting $w \in C^2_{0,\sigma}(\Omega)$ in (1.11) we obtain that $-\Delta u + u \cdot \nabla u + \nabla p = f$ holds in the sense of distributions with some distribution p. In the stationary case we can also prove that each very weak solution u has a well defined trace $u_{|_{\partial\Omega}}$ even in the space $W^{-\frac{1}{q},q}(\partial\Omega)$, and there is an explicit representation formula for $u_{|_{\partial\Omega}}$, see (3.6).

Note that Theorem 2 improves the result in [14] where $\|k\|_{L^r(\Omega)}$ is replaced by the stronger norm $\|k\|_{L^q(\Omega)}$.

In the linearized case $u \cdot \nabla u \equiv 0$, we omit $\langle uu, \nabla w\rangle_\Omega$, $\langle ku, w\rangle_\Omega$ in relation (1.11), and the existence result together with estimate (1.14) holds without any smallness condition for every $1 < q < \infty$, $1 < r < q$, $\frac{1}{3} + \frac{1}{q} \geq \frac{1}{r}$, see Theorem 3 in Section 3 below.

The improvement concerning $\|k\|_{L^r(\Omega)}$ leads to a certain scaling invariance in the following sense. Let $\lambda > 0$, consider some ball $B_a(x_0) \subseteq \mathbb{R}^3$ with radius $a > 0$ and center $x_0 \in \mathbb{R}^3$, and let F, k, g be data as in (1.10) with $\Omega = B_a(x_0)$, $3 < q < \infty$, $1 < r < q$, $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$. Then it is easy to show that $u \in L^q\big(B_a(x_0)\big)$ is a very weak solution to the system (1.9) on Ω with data F, k, g iff $\tilde{u} \in L^q(\tilde{\Omega})$, $\tilde{\Omega} = B_{a/\lambda}(x_0)$, is a very weak solution of the system

$$-\Delta\tilde{u} + \tilde{u} \cdot \nabla\tilde{u} + \nabla\tilde{p} = \operatorname{div}\tilde{F}, \quad \operatorname{div}\tilde{u} = \tilde{k}, \quad \tilde{u}_{|_{\partial\tilde{\Omega}}} = \tilde{g} \tag{1.16}$$

on $\tilde{\Omega}$, where $\tilde{u}, \tilde{p}, \tilde{F}, \tilde{k}, \tilde{g}$ are defined by

$$\begin{array}{rclrcl}
\tilde{u}(x) & = & \lambda u\big(\lambda(x-x_0)+x_0\big), & \tilde{p}(x) & = & \lambda^2 p\big(\lambda(x-x_0)+x_0\big),\\
\tilde{F}(x) & = & \lambda^2 F\big(\lambda(x-x_0)+x_0\big), & \tilde{k}(x) & = & \lambda^2 k\big(\lambda(x-x_0)+x_0\big),\\
\tilde{g}(x) & = & \lambda g\big(\lambda(x-x_0)+x_0\big). & & &
\end{array}$$

Then we conclude that if $K_{a,q} = K\big(B_a(x_0), q\big) > 0$ is the constant in (1.13) for $B_a(x_0)$, then $\lambda^{1-\frac{3}{q}} K_{a,q}$ can be chosen as the corresponding constant for $B_{a/\lambda}(x_0)$. We also conclude that (1.14) holds for u, F, k, g in $\Omega = B_a(x_0)$ with $C = C\big(B_a(x_0), q\big)$ iff (1.14) holds for $\tilde{u}, \tilde{F}, \tilde{k}, \tilde{g}$ in $\tilde{\Omega} = B_{a/\lambda}(x_0)$ with the same constant C. These properties have several applications in the local regularity theory. Similar results hold in the nonstationary theory.

The proofs of these theorems are organized as follows. First we consider the linearized stationary and nonstationary system and prove explicit representation formulas. Applying these formulas to the nonlinear system leads to a fixed point problem which can be solved by Banach's fixed point theorem for sufficiently small data. In Section 2 we prepare several preliminaries.

2 Notations and Preliminaries

Classical Function Spaces. Let $1 < q < \infty$ and $q' = \frac{q}{q-1}$ such that $\frac{1}{q} + \frac{1}{q'} = 1$. We need the usual spaces $L^q(\Omega)$ and $W^{\alpha,q}(\Omega)$, $W_0^{\alpha,q}(\Omega)$, $\alpha \geq 0$, with norms $\|\cdot\|_{L^q(\Omega)} = \|\cdot\|_{q,\Omega}$ and $\|\cdot\|_{W^{\alpha,q}(\Omega)} = \|\cdot\|_{\alpha;q,\Omega}$, resp. The space $W^{-\alpha,q}(\Omega) = W_0^{\alpha,q'}(\Omega)'$ is the dual space of $W_0^{\alpha,q'}(\Omega)$ with the natural duality pairing $\langle\cdot,\cdot\rangle_\Omega$ and the norm $\|\cdot\|_{W^{-\alpha,q}(\Omega)} = \|\cdot\|_{-\alpha;q,\Omega}$. Thus, e.g., $\langle f, h\rangle_\Omega$ means the value of the functional $f \in W^{-\alpha,q}(\Omega)$ at $h \in W_0^{\alpha,q'}(\Omega)$. Similarly, for functions on the boundary, $L^q(\partial\Omega)$ and $W^{\alpha,q}(\partial\Omega), W^{-\alpha,q'}(\partial\Omega)$, $\alpha \geq 0$, with norms $\|\cdot\|_{L^q(\partial\Omega)} = \|\cdot\|_{q,\partial\Omega}$ and $\|\cdot\|_{W^{\alpha,q}(\partial\Omega)} = \|\cdot\|_{\alpha;q,\partial\Omega}$, $\|\cdot\|_{W^{-\alpha,q}(\partial\Omega)} = \|\cdot\|_{-\alpha;q,\partial\Omega}$, resp., and the duality pairing $\langle\cdot,\cdot\rangle_{\partial\Omega}$ are the corresponding notions for $\partial\Omega$. In particular, the pairing between $L^q(\partial\Omega)$ and its dual space $L^{q'}(\partial\Omega) = L^q(\partial\Omega)'$ is given by

$$\langle f, g\rangle_{\partial\Omega} = \int_{\partial\Omega} f \cdot g \, dS$$

where $\int_{\partial\Omega} \ldots dS$ means the surface integral on $\partial\Omega$, see [24] and [26], p. 33, p. 40. For more details on these spaces cf. [1], [11], [12], [26], [28]. In general, we use the same symbol for scalar, vector, and tensor valued spaces.

By $C_0^\nu(\Omega)$, $C^\nu(\Omega)$, $C^\nu(\overline{\Omega})$, $C^\nu(\partial\Omega)$, $\nu = 0, 1, \ldots$ and $\nu = \infty$, we denote the usual spaces of smooth functions. We set $C_0^\nu(\overline{\Omega}) = \{v \in C^\nu(\overline{\Omega});\, v|_{\partial\Omega} = 0\}$. The space of distributions $C_0^\infty(\Omega)'$ is the dual space of the test space $C_0^\infty(\Omega)$ with the usual topology, the duality pairing is again denoted by $\langle\cdot,\cdot\rangle_\Omega$. Similarly, the space

$C^\infty(\partial\Omega)'$ of boundary distributions is the dual space of the test space $C^\infty(\partial\Omega)$ with the duality pairing $\langle\cdot,\cdot\rangle_{\partial\Omega}$. This test space has the form $C^\infty(\partial\Omega)$ since $\partial\Omega$ has no boundary.

Spaces of solenoidal vector valued functions are denoted by appending "σ". Thus we have $C^\nu_{0,\sigma}(\Omega) = \{v \in C^\nu_0(\Omega);\ \operatorname{div} v = 0\}$ and $C^\nu_{0,\sigma}(\overline{\Omega}) = \{v \in C^\nu_0(\overline{\Omega});\ \operatorname{div} v = 0\}$. The corresponding functional space for solenoidal test functions $C^\infty_{0,\sigma}(\Omega)$ is the dual space $C^\infty_{0,\sigma}(\Omega)'$, again with pairing $\langle\cdot,\cdot\rangle_\Omega$. By a theorem of de Rham, [27], Chapter I, Proposition 1.1, a distribution $d = (d_1, d_2, d_3) \in C^\infty_0(\Omega)'$ with $\langle d, v\rangle_\Omega = 0$ for all $v \in C^\infty_{0,\sigma}(\Omega)$ has the form $d = \nabla h$ with some scalar distribution h.

Let $L^q_\sigma(\Omega)$ be the closure of $C^\infty_{0,\sigma}(\Omega)$ in the norm $\|\cdot\|_{q,\Omega}$. Then $L^{q'}_\sigma(\Omega) = L^q_\sigma(\Omega)'$ is the dual space of $L^q_\sigma(\Omega)$ with pairing $\langle\cdot,\cdot\rangle_\Omega$.

Traces and Extensions. Let $\alpha > \frac{1}{q}$, α an integer. Then the *trace map* $f \mapsto f|_{\partial\Omega}$ is a well defined bounded operator from $W^{\alpha,q}(\Omega)$ onto $W^{\alpha-\frac{1}{q},q}(\partial\Omega)$. Conversely, there exists a linear and bounded *extension operator* E: $h \mapsto E_h$ from $W^{\alpha-\frac{1}{q},q}(\partial\Omega)$ into $W^{\alpha,q}(\Omega)$ satisfying $E_h|_{\partial\Omega} = h$. Thus it holds $\|E_h\|_{\alpha;q,\Omega} \le C\|h\|_{\alpha-\frac{1}{q};q,\partial\Omega}$ with $C = C(\Omega,\alpha,q) > 0$.

Let $1 < r \le q$, $\frac{1}{3} + \frac{1}{q} \ge \frac{1}{r}$, and let $f = (f_1, f_2, f_3) \in L^q(\Omega)$ with $\operatorname{div} f \in L^r(\Omega)$. Then we use Green's identity

$$\langle \operatorname{div} f, E_h\rangle_\Omega = \langle N\cdot f, h\rangle_{\partial\Omega} - \langle f, \nabla E_h\rangle_\Omega$$

for $h \in W^{1-\frac{1}{q'},q'}(\partial\Omega) = W^{\frac{1}{q},q'}(\partial\Omega)$ and with the extension operator $E : W^{\frac{1}{q},q'}(\partial\Omega) \to W^{1,q'}(\Omega)$. This leads, using the embedding property $\|E_h\|_{r',\Omega} \le C(\|E_h\|_{q',\Omega} + \|\nabla E_h\|_{q',\Omega})$, $C = C(\Omega,q,r) > 0$, to the estimate

$$|\langle N\cdot f, h\rangle_{\partial\Omega}| \le C(\|f\|_{q,\Omega} + \|\operatorname{div} f\|_{r,\Omega})\|h\|_{\frac{1}{q};q',\partial\Omega} \tag{2.1}$$

with $C = C(\Omega,q,r) > 0$. Hence the trace $N\cdot f|_{\partial\Omega} \in W^{-\frac{1}{q},q}(\partial\Omega)$ of the normal component of f at $\partial\Omega$ is well defined and it holds the estimate

$$\|N\cdot f\|_{-\frac{1}{q};q,\partial\Omega} \le C(\|f\|_{q,\Omega} + \|\operatorname{div} f\|_{r,\Omega}). \tag{2.2}$$

Using the corresponding identity

$$\langle \operatorname{curl} f, E_h\rangle_\Omega = \langle N\times f, E_h\rangle_{\partial\Omega} + \langle f, \operatorname{curl} E_h\rangle_\Omega$$

now for $h = (h_1, h_2, h_3) \in W^{1-\frac{1}{q'},q'}(\partial\Omega)$, we obtain the following trace property: If $f = (f_1, f_2, f_3) \in L^q(\Omega)$, $\operatorname{curl} f \in L^r(\Omega)$, then the trace $N\times f|_{\partial\Omega} \in W^{-\frac{1}{q},q}(\partial\Omega)$ of the tangential component of f at $\partial\Omega$ is well defined and it holds the estimate

$$\|N\times f\|_{-\frac{1}{q};q,\partial\Omega} \le C(\|f\|_{q,\Omega} + \|\operatorname{curl} f\|_{r,\Omega}) \tag{2.3}$$

with $C = C(\Omega, q, r) > 0$. The identity $f = (N \cdot f)N + (N \times f) \times N$ at $\partial\Omega$ shows that it is justified to call $N \times f|_{\partial\Omega}$ the tangential component of f at $\partial\Omega$.

Let $f \in L^q(\Omega)$ with $\int_\Omega f\,dx = 0$. Then there exists some $b = b^f \in W_0^{1,q}(\Omega)$ with $\operatorname{div} b^f = f$ such that $f \mapsto b^f$ is a linear mapping satisfying

$$\|b^f\|_{1;q,\Omega} \le C\|f\|_{q,\Omega}, \quad C = C(\Omega, q) > 0. \tag{2.4}$$

If moreover $f \in W_0^{1,q}(\Omega)$, then $b^f \in W_0^{2,q}(\Omega)$ and

$$\|b^f\|_{2;q,\Omega} \le C\|\nabla f\|_{q,\Omega}, \quad C = C(\Omega, q) > 0, \tag{2.5}$$

see [5], [11], Theorem III.3.2 and [26], p. 68.

Using properties of the weak Neumann problem in $L^q(\Omega)$, see [23], we find for each $h \in W^{-\frac{1}{q},q}(\partial\Omega)$ some $E^h = (E_1^h, E_2^h, E_3^h) \in L^q(\Omega)$ with $\operatorname{div} E^h \in L^r(\Omega)$, $N \cdot E^h|_{\partial\Omega} = h$, such that $h \mapsto E^h$ is a linear map satisfying

$$\|E^h\|_{q,\Omega} + \|\operatorname{div} E^h\|_{r,\Omega} \le C\|h\|_{-\frac{1}{q};q,\partial\Omega} \tag{2.6}$$

with $C = C(\Omega, q, r) > 0$.

Let $h = (h_1, h_2, h_3) \in W^{1-\frac{1}{q},q}(\partial\Omega)$. Then we find an extension $w^h \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ such that $N \cdot \nabla w^h|_{\partial\Omega} = h$ depending linearly on h; moreover,

$$\|w^h\|_{2;q,\Omega} \le C\|h\|_{1-1/q;q,\partial\Omega} \tag{2.7}$$

with $C = C(\Omega, q) > 0$, see [22], Theorem 5.8, p. 104, or [28], 5.4.4, p. 385.

If additionally $N \cdot h|_{\partial\Omega} = 0$, then we can show that $\operatorname{div} w^h|_{\partial\Omega} = 0$ and $N \cdot \nabla w^h|_{\partial\Omega} = -N \times \operatorname{curl} w^h|_{\partial\Omega} = h$, see [14]. This yields $\int_\Omega \operatorname{div} w^h dx = 0$, $\operatorname{div} w^h \in W_0^{1,q}(\Omega)$, and we find $b = b(w^h) \in W_0^{2,q}(\Omega)$ satisfying $\operatorname{div} b = \operatorname{div} w^h$ and (2.4), (2.5). Setting $\hat{w}^h = w^h - b(w^h)$ we see that $\hat{w}^h \in W^{2,q}(\Omega)$ satisfies

$$\hat{w}^h|_{\partial\Omega} = 0, \quad N \cdot \nabla \hat{w}^h|_{\partial\Omega} = -N \times \operatorname{curl} \hat{w}^h|_{\partial\Omega} = h, \quad \operatorname{div} \hat{w}^h = 0. \tag{2.8}$$

Moreover, the mapping $h \mapsto \hat{w}^h$ is linear and

$$\|\hat{w}^h\|_{2;q,\Omega} \le C\|h\|_{1-1/q;q,\partial\Omega}, \quad C = C(\Omega, q) > 0, \tag{2.9}$$

see [14], (2.14).

The Stokes Operator. Let $f = (f_1, f_2, f_3) \in L^q(\Omega)$. Then, the weak Neumann problem $\Delta H = \operatorname{div} f$, $N \cdot (\nabla H - f)|_{\partial\Omega} = 0$, has a unique solution $\nabla H \in L^q(\Omega)$ satisfying the estimate

$$\|\nabla H\|_{q,\Omega} \le C\|f\|_{q,\Omega} \tag{2.10}$$

with $C = C(\Omega, q) > 0$, see [23]. Setting $P_q f = f - \nabla H$ we define the *Helmholtz projection operator* P_q as a bounded operator from $L^q(\Omega)$ onto $L^q_\sigma(\Omega)$, satisfying $P_q^2 = P_q$ and

$$\langle P_q f, g\rangle_\Omega = \langle f, P_{q'} g\rangle_\Omega, \quad \text{for all } f \in L^q(\Omega), g \in L^{q'}(\Omega)\,.$$

Hence $P_q' = P_{q'}$ holds for the dual operator P_q' of P_q.

The *Stokes operator* A_q, with domain

$$D(A_q) = L^q_\sigma(\Omega) \cap W^{1,q}_0(\Omega) \cap W^{2,q}(\Omega) \subseteq L^q_\sigma(\Omega)$$

and range $R(A_q) = L^q_\sigma(\Omega)$, is defined by $A_q u = -P_q \Delta u$, $u \in D(A_q)$. The fractional power $A_q^\alpha : D(A_q^\alpha) \to L^q_\sigma(\Omega) = R(A_q^\alpha)$ with $D(A_q) \subseteq D(A_q^\alpha) \subseteq L^q_\sigma(\Omega)$, $0 \le \alpha \le 1$, is well defined, bijective and its inverse $(A_q^\alpha)^{-1} = A_q^{-\alpha}$ is a bounded operator from $L^q_\sigma(\Omega)$ into $L^q_\sigma(\Omega)$ with range $R(A_q^{-\alpha}) = D(A_q^\alpha)$; furthermore, the operator $(A_q^\alpha)' = A_{q'}^\alpha$ is the dual operator of A_q^α. The norms $\|u\|_{2;q,\Omega}$ and $\|A_q u\|_{q,\Omega}$ are equivalent for $u \in D(A_q)$; analogously, the norms $\|u\|_{1;q,\Omega}$ and $\|A_q^{1/2} u\|_{q,\Omega}$ are equivalent for $u \in D(A_q^{\frac{1}{2}}) = L^q_\sigma(\Omega) \cap W^{1,q}_0(\Omega)$. Note that the space $D(A_q^\alpha)$ endowed with the graph norm $\|A_q^\alpha u\|_{q,\Omega}$, $u \in D(A_q^\alpha)$, is a Banach space. Furthermore, we mention the important embedding property

$$\|u\|_{q,\Omega} \le C\|A_\gamma^\alpha u\|_{\gamma,\Omega}, \quad u \in D(A_\gamma^\alpha), \quad 1 < \gamma \le q, \quad 2\alpha + \frac{3}{q} = \frac{3}{\gamma}, \tag{2.11}$$

with $C = C(\Omega, q) > 0$. See [2], [9], [13], [17], [18], [21], [25], [26], [28] concerning proofs and further properties of the Stokes operator. Finally we observe that $A_q u = A_\rho u$ holds if $u \in D(A_q) \cap D(A_\rho)$, $1 < q < \infty$, $1 < \rho < \infty$.

It is well-known that $-A_q$ generates a bounded analytic semigroup $\{e^{-tA_q} : t \ge 0\}$, see [2], [16], [19], [25], [26], and that

$$\|A_q^\alpha e^{-tA_q} v\|_{q,\Omega} \le C e^{-\delta t} t^{-\alpha} \|v\|_{q,\Omega}, \quad v \in L^q_\sigma(\Omega),\ t > 0, \tag{2.12}$$

with constants $C = C(\Omega, q) > 0$, $\delta = \delta(\Omega, q) > 0$.

Let $0 < \alpha \le 1$, $1 < q < \infty$, and let $d = (d_1, d_2, d_3) \in C_0^\infty(\Omega)'$ be a distribution. Assume that $\langle d, v\rangle_\Omega$ is well defined for all $v \in D(A_{q'}^\alpha)$ and is continuous in the norm $\|A_{q'}^\alpha v\|_{q',\Omega}$, i.e., there exists a constant $C > 0$ such that $|\langle d, v\rangle_\Omega| \le C\|A_{q'}^\alpha v\|_{q',\Omega}$. In other words, the functional $\langle d, v\rangle_\Omega$, $v \in D(A_{q'}^\alpha)$, is a well-defined element of the dual space $D(A_{q'}^\alpha)'$ of $D(A_{q'}^\alpha)$. Writing formally $\langle d, v\rangle_\Omega = \langle d, P_{q'} v\rangle_\Omega = \langle P_q d, v\rangle_\Omega$, we call $P_q d = \langle P_q d, \cdot\rangle_\Omega$ the restriction of the functional d to test functions $v \in D(A_{q'}^\alpha)$, giving P_q a generalized meaning; in short, we write $P_q d \in D(A_{q'}^\alpha)'$.

Let $d \in C_0^\infty(\Omega)'$ with $P_q d \in D(A_{q'}^\alpha)'$. Since $R(A_{q'}^\alpha) = L^{q'}_\sigma(\Omega)$, there exists a uniquely determined element $d^* \in L^q_\sigma(\Omega)$ satisfying the relation $\langle d, v\rangle_\Omega =$

$\langle d^*, A_{q'}^{\alpha} v\rangle_\Omega$. We set $d^* = A_q^{-\alpha} P_q d$, giving the operator $A_q^{-\alpha}$ a generalized meaning. Thus $A_q^{-\alpha} P_q d \in L^q_\sigma(\Omega)$ is well defined by the relation

$$\langle d, v\rangle_\Omega = \langle P_q d, v\rangle_\Omega = \langle P_q d, A_{q'}^{-\alpha} A_{q'}^{\alpha} v\rangle_\Omega = \langle A_q^{-\alpha} P_q d, A_{q'}^{\alpha} v\rangle_\Omega\,, \tag{2.13}$$

$v \in D(A_{q'}^{\alpha})$, similarly as in the theory of distributions. We conclude that the operation $A_q^{-\alpha} P_q d \in L^q_\sigma(\Omega)$ is well defined by (2.13) if $d \in C_0^\infty(\Omega)'$ and $P_q d \in D(A_{q'}^{\alpha})'$.

To obtain examples, we consider a functional f in the (vector valued) Bessel potential space $H_q^{-2\alpha}(\Omega)$ for $0 < \alpha \le \frac{1}{2}$, see [1], [2], [28]. Then $P_q f \in D(A_{q'}^{\alpha})'$, since $C_{0,\sigma}^\infty(\Omega)$ is dense in the Banach space $D(A_{q'}^{\alpha})$ because of $\alpha \le \frac{1}{2}$. Therefore, $A_q^{-\alpha} P_q f \in L^q_\sigma(\Omega)$ is well defined in this case.

Let $1 < r \le q$, $\frac{1}{3} + \frac{1}{q} \ge \frac{1}{r}$, and let $u = (u_1, u_2, u_3) \in L^q(\Omega)$. Assume that the distribution $d = \Delta u$ is continuous in the norm $\|A_{r'}^{\frac{1}{2}} \cdot \|_{r',\Omega}$. Then the element $A_r^{-1/2} P_r \Delta u \in L^r_\sigma(\Omega)$ is well defined by the relation

$$\langle A_r^{-\frac{1}{2}} P_r \Delta u,\, A_{r'}^{\frac{1}{2}}\, v\rangle_\Omega = \langle u, \Delta v\rangle_\Omega, \quad v \in C_{0,\sigma}^\infty(\Omega), \tag{2.14}$$

according to (2.13); see also [14], Section 2. Here we use that $C_{0,\sigma}^\infty(\Omega)$ is dense in the Banach space $D(A_{r'}^{\frac{1}{2}})$.

Let $F = (F_{ij}) \in L^r(\Omega)$ where $1 < r \le q$, $\frac{1}{3} + \frac{1}{q} \ge \frac{1}{r}$, and set $d = \operatorname{div} F$. Then using (2.11) and the estimate

$$\|\nabla v\|_{r',\Omega} \le C_1 \|A_{r'}^{\frac{1}{2}} v\|_{r',\Omega} \le C_2 \|A_{q'} v\|_{q',\Omega}\,, \quad v \in D(A_{q'})\,,$$

$C_i = C_i(\Omega, q, r) > 0$, $i = 1, 2$, we see that the distribution $v \mapsto -\langle F, \nabla v\rangle_\Omega$ is contained in $D(A_{q'})'$. Therefore, the element $\hat{F} = -A_q^{-1} P_q \operatorname{div} F \in L^q_\sigma(\Omega)$ is well defined by the relation

$$\langle \hat{F}, A_{q'} v\rangle_\Omega = -\langle A_q^{-1} P_q \operatorname{div} F, A_{q'} v\rangle_\Omega = -\langle \operatorname{div} F, v\rangle_\Omega \tag{2.15}$$

and it holds

$$\|\hat{F}\|_{q,\Omega} = \|A_q^{-1} P_q \operatorname{div} F\|_{q,\Omega} \le C \|F\|_{r,\Omega} \tag{2.16}$$

with $C = C(\Omega, q, r) > 0$; see [26], III, Lemma 2.6.1 concerning similar operations.

Spaces $L^q(0,T;X)$. Let $1 < q, s < \infty$. Then we introduce the usual Bochner space $L^s(0,T;X)$ with norm $\|\cdot\|_{L^s(0,T;X)} = \big(\int_0^T \|\cdot\|_X^s\, dt\big)^{1/s}$ where X is any Banach space with norm $\|\cdot\|_X$.

In the case $X = W^{\alpha,q}(\Omega)$, $-1 \le \alpha \le 1$, we set $\|\cdot\|_{L^s(0,T;W^{\alpha,q}(\Omega))} = \|\cdot\|_{\alpha;q,s,\Omega}$, and for $X = W^{\alpha,q}(\partial\Omega)$ let $\|\cdot\|_{L^s(0,T;W^{\alpha,q}(\partial\Omega))} = \|\cdot\|_{\alpha;q,s,\partial\Omega}$. Finally, if $X = L^q(\Omega)$ or $X = L^q(\partial\Omega)$, we set $\|\cdot\|_{L^s(0,T;L^q(\Omega))} = \|\cdot\|_{q,s,\Omega}$, and $\|\cdot\|_{L^s(0,T;L^q(\partial\Omega))} = \|\cdot\|_{q,s,\partial\Omega}$, resp. As duality pairing we define

$$\langle f, g\rangle_{\Omega,T} = \int_0^T \langle f, g\rangle_\Omega\, dt \quad \text{with} \quad \langle f, g\rangle_\Omega = \int_\Omega f \cdot g\, dx \tag{2.17}$$

for $f = (f_1, f_2, f_3) \in L^s(0,T; L^q(\Omega))$, $g = (g_1, g_2, g_3) \in L^{s'}(0,T; L^{q'}(\Omega))$. Similarly, we use the notation

$$\langle f, g\rangle_{\partial\Omega,T} = \int_0^T \langle f, g\rangle_{\partial\Omega}\, dt \quad \text{with} \quad \langle f, g\rangle_{\partial\Omega} = \int_{\partial\Omega} f \cdot g\, dS.$$

We also need the spaces $C^\nu\big([0,T); X\big)$, $\nu = 0, 1, 2, \ldots$, of X-valued functions $v(t)$, such that $v, (d/dt)v, \ldots, (d/dt)^\nu v$ are continuous on $[0,T)$. We set $C^0\big([0,T); X\big) = C\big([0,T); X\big)$. The space $C^1_0\big([0,T); X\big)$ is the subspace of $C^1\big([0,T); X\big)$ consisting of functions v with compact support contained in $[0,T)$, whereas $C^1_0\big((0,T); X\big)$ is the subspace of $C^1\big([0,T); X\big)$-functions v with compact support contained in $(0,T)$.

Let $f \in L^s\big(0,T; L^q_\sigma(\Omega)\big)$. Then there exists a unique function $v \in L^s\big(0,T; D(A_q)\big)$ with $v_t \in L^s\big(0,T; L^q_\sigma(\Omega)\big)$ and $v \in C\big([0,T); L^q_\sigma(\Omega)\big)$, satisfying the evolution system

$$v_t + A_q v = f, \quad 0 \le t < T, \quad v(0) = 0.$$

To be more precise,

$$v(t) = \int_0^t e^{-(t-\tau)A_q} f(\tau) d\tau, \quad 0 \le t < T,$$

and it holds the 'maximal regularity' estimate

$$\|v_t\|_{q,s,\Omega} + \|A_q v\|_{q,s,\Omega} \le C\|f\|_{q,s,\Omega}, \quad C = C(\Omega, q, s) > 0, \tag{2.18}$$

see [19], [25].

The Space of Initial Values. Let $1 < q, s < \infty$. The space of initial values $\mathcal{J}^{q,s}(\Omega)$ consists of distributions u_0 satisfying $A_q^{-1} P_q u_0 \in L^q_\sigma(\Omega)$, see (2.13), and an additional integrability condition in time. To be more precise, let

$$\mathcal{J}^{q,s}(\Omega) = \{u_0 \in C_0^\infty(\Omega)' :\ A_q^{-1} P_q u_0 \in L^q_\sigma(\Omega),\ \int_0^\infty \|A_q e^{-tA_q} A_q^{-1} P_q u_0\|_q^s\, dt < \infty\} \tag{2.19}$$

and

$$\|u_0\|_{\mathcal{J}^{q,s}(\Omega)} = \|A_q^{-1} P_q u_0\|_{q,\Omega} + (\int_0^\infty \|A_q e^{-tA_q} A_q^{-1} P_q u_0\|_q^s\, dt)^{\frac{1}{s}}.$$

Obviously, $\|\cdot\|_{\mathcal{J}^{q,s}(\Omega)}$ defines a seminorm in $\mathcal{J}^{q,s}(\Omega)$ which becomes a norm if we identify two elements $u_0, v_0 \in \mathcal{J}^{q,s}(\Omega)$ satisfying $\|A_q^{-1} P_q(u_0 - v_0)\|_{\mathcal{J}^{q,s}(\Omega)} = 0$, i.e., $u_0 - v_0$ is a gradient. Of course, since $w(0) \in C^2_{0,\sigma}(\overline{\Omega})$ in (1.4) is solenoidal, initial values can be prescribed only modulo gradients.

As an example, let $u_0 \in C_0^\infty(\Omega)'$ and assume that $A_q^{-\frac{1}{s}+\varepsilon} P_q u_0 \in L^q_\sigma(\Omega)$ where $0 < \varepsilon < \frac{1}{s}$. Then by (2.12) $u_0 \in \mathcal{J}^{q,s}(\Omega)$. Using similar calculations as in [4] we can show that $u_0 \in B^{-2/s}_{q,s}(\Omega)$ is sufficient for $u_0 \in \mathcal{J}^{q,s}(\Omega)$ in the case $s \geq 2$; here $B^{-2/s}_{q,s}(\Omega)$ denotes a Besov space, see [2], [3], [4], [28].

Consider $u \in L^s\big(0,T;L^q(\Omega)\big)$ such that its time derivative $(A_q^{-1}P_q u)_t = A_q^{-1}P_q u_t \in L^s\big(0,T;L^q_\sigma(\Omega)\big)$ in the sense of distributions. Then, after redefining on a null set of $[0,T)$, we get $A_q^{-1}P_q u \in C\big([0,T);L^q_\sigma(\Omega)\big)$. Thus $A_q^{-1}P_q u(t) \in L^q_\sigma(\Omega)$ is well defined for each $t \in [0,T)$, and therefore $A_q^{-1}P_q u(0) = A_q^{-1}P_q u_0$ in (1.7) is well defined.

Let $v_0 \in L^q_\sigma(\Omega)$ such that $\int_0^\infty \|A_q e^{-tA_q} v_0\|_q^s \, dt < \infty$, let $f \in L^s\big(0,T;L^q_\sigma(\Omega)\big)$ and consider the general system $v_t + A_q v = f, v(0) = v_0$. Then we apply (2.18) to $\hat{v}(t) = v(t) - e^{-tA_q}v_0$, obtain the estimate

$$\|v_t\|_{q,s,\Omega} + \|A_q v\|_{q,s,\Omega} \leq C\left(\left(\int_0^T \|A_q e^{-tA_q}v_0\|_q^s \, dt\right)^{\frac{1}{s}} + \|f\|_{q,s,\Omega}\right) \tag{2.20}$$

with $C = C(\Omega,q,s) > 0$, and the representation formula

$$v(t) = e^{-tA_q}v_0 + \int_0^t e^{-(t-\tau)A_q} f(\tau)\, d\tau, \quad 0 \leq t < T. \tag{2.21}$$

3 Stationary very weak solutions

First we consider the linearized stationary system

$$-\Delta u + \nabla p = f, \quad \operatorname{div} u = k, \quad u_{|\partial\Omega} = g \tag{3.1}$$

with data $f = \operatorname{div} F$ and k, g satisfying

$$F \in L^r(\Omega),\ k \in L^r(\Omega),\ g \in W^{-\frac{1}{q},q}(\partial\Omega),\ \int_\Omega k\, dx = \int_{\partial\Omega} N \cdot g\, dS \tag{3.2}$$

where $1 < q < \infty$, $1 < r \leq q$, $\frac{1}{3} + \frac{1}{q} \geq \frac{1}{r}$. Here we follow [14] in principle, but in [14] the stronger condition $k \in L^q(\Omega)$ is supposed.

Modifying Definition 2 in an obvious way for the linearized case, $u \in L^q(\Omega)$ is called a *very weak solution* of the system (3.1) with data (3.2) if

$$-\langle u, \Delta w\rangle_\Omega + \langle g, N\cdot\nabla w\rangle_{\partial\Omega} = -\langle F, \nabla w\rangle_\Omega, \quad w \in C^2_{0,\sigma}(\overline{\Omega}), \tag{3.3}$$

and additionally the conditions (1.12) are satisfied, i.e., $\operatorname{div} u = k$ and $N\cdot u = N\cdot g$ on $\partial\Omega$. Our main result on (3.1), improving [14], Theorem 3, reads as follows.

Theorem 3 *Suppose the data F, k, g satisfy (3.2) with $1 < q < \infty$, $1 < r \leq q$, $\frac{1}{3} + \frac{1}{q} \geq \frac{1}{r}$. Then there exists a unique very weak solution $u \in L^q(\Omega)$ of the system (3.1) satisfying the estimate*

$$\|u\|_{q,\Omega} \leq C(\|F\|_{r,\Omega} + \|k\|_{r,\Omega} + \|g\|_{-1/q;q,\partial\Omega}) \tag{3.4}$$

with $C = C(\Omega,q,r) > 0$.

Remarks 1 **(1)** Setting $w \in C^\infty_{0,\sigma}(\Omega)$ in (3.3) and using de Rham's argument we find some pressure $p \in W^{-1,q}(\Omega)$ such that $-\Delta u + \nabla p = f$ holds in the sense of distributions, and that $\|p\|_{-1;q,\Omega}$ satisfies the same estimate as $\|u\|_{q,\Omega}$ in (3.4). Moreover we conclude from (3.3) that

$$A_r^{-\frac{1}{2}} P_r \Delta u = -A_r^{-\frac{1}{2}} P_r \operatorname{div} F, \tag{3.5}$$

where $A_r^{-\frac{1}{2}} P_r \operatorname{div} F$ is defined by the relation $\langle A_r^{-\frac{1}{2}} P_r \operatorname{div} F, A_{r'}^{\frac{1}{2}} v\rangle_\Omega = -\langle F, \nabla v\rangle_\Omega$, $v \in C^\infty_{0,\sigma}(\Omega)$, see (2.14) and (2.15).

(2) Assume for a moment that u is sufficiently smooth. Then, inserting $w = \hat{w}^h$ from (2.8), (2.9) in the expression $\langle u, \Delta w\rangle_\Omega$, and using integration by parts, we obtain an explicit trace formula for $u_{|_{\partial\Omega}}$ as a functional in $W^{-\frac{1}{q},q}(\partial\Omega)$. To be more precise, the map $h \mapsto \langle u_{|_{\partial\Omega}}, h\rangle_{\partial\Omega}$ for $h \in W^{1-1/q',q'}(\partial\Omega) = W^{\frac{1}{q},q'}(\partial\Omega)$, $N \cdot h_{|_{\partial\Omega}} = 0$ is defined by

$$\langle u_{|_{\partial\Omega}}, h\rangle_{\partial\Omega} = \langle u, \Delta\hat{w}^h\rangle_\Omega - \langle A_r^{-\frac{1}{2}} P_r \Delta u, A_{r'}^{\frac{1}{2}} \hat{w}^h\rangle_\Omega, \quad h \in W^{-\frac{1}{q},q'}(\partial\Omega). \tag{3.6}$$

Using (2.9) with q replaced by q' we then obtain the estimate

$$|\langle u_{|_{\partial\Omega}}, h\rangle_{\partial\Omega}| \le C(\|u\|_{q,\Omega} + \|A_r^{-\frac{1}{2}} P_r \Delta u\|_{r,\Omega})\|h\|_{1/q;q';\partial\Omega} \tag{3.7}$$

with $C = C(\Omega, q, r) > 0$. Formula (3.6) is well defined for each very weak solution $u \in L^q(\Omega)$ and yields an explicit formula for the tangential component of $u_{|_{\partial\Omega}}$. The normal component $N \cdot u_{|_{\partial\Omega}}$ is well defined by (1.12), (2.1). This shows that the trace $u_{|_{\partial\Omega}} \in W^{-\frac{1}{q},q}(\partial\Omega)$ is well defined for a very weak solution u, and (2.1), (3.7) yield the estimate

$$\|u_{|_{\partial\Omega}}\|_{-\frac{1}{q};q,\partial\Omega} \le C(\|u\|_{q,\Omega} + \|A_r^{-\frac{1}{2}} P_r \Delta u\|_{r,\Omega} + \|\operatorname{div} u\|_{r,\Omega}) \tag{3.8}$$

with $C = C(\Omega, q, r) > 0$.

(3) We conclude that a very weak solution $u \in L^q(\Omega)$ of (3.1) satisfies the conditions (1.12), (3.5) and the condition $u_{|_{\partial\Omega}} = g$ as elements of $W^{-\frac{1}{q},q}(\partial\Omega)$. Conversely, if $u \in L^q(\Omega)$ satisfies (1.12), (3.5), and the (well defined) trace $u_{|_{\partial\Omega}}$ is equal to g, then u is a very weak solution of (3.1).

Proof of Theorem 3. Following in principle [14], we first assume that $u \in L^q(\Omega)$ is a given very weak solution u of (3.1), and prepare some estimates.

Using the trace map $W^{1,q'}(\Omega) \to W^{1-1/q',q'}(\partial\Omega)$ and the embedding estimate (2.11) we obtain that

$$|\langle g, N \cdot \nabla w\rangle_{\partial\Omega}| \le C\|g\|_{-\frac{1}{q};q,\partial\Omega}\, \|v\|_{q',\Omega}, \quad w \in C^2_{0,\sigma}(\overline{\Omega}),\ v = A_{q'} w \tag{3.9}$$

with $C = C(\Omega, q) > 0$. Therefore, the functional $v \mapsto \langle g, N \cdot \nabla A_{q'}^{-1} v\rangle_{\partial\Omega}$ is continuous in $\|v\|_{q',\Omega}$, and it holds $\langle g, N \cdot \nabla A_{q'}^{-1} v\rangle_{\partial\Omega} = \langle G, v\rangle_\Omega$ with some unique $G \in L^q_\sigma(\Omega)$ satisfying $\|G\|_{q,\Omega} \le C\|g\|_{-\frac{1}{q};q,\partial\Omega}$.

Similarly, $v \mapsto \langle F, \nabla A_{q'}^{-1} v\rangle_\Omega$, $v \in A_{q'}w$, is continuous in $\|v\|_{q',\Omega}$, and with $\hat{F} = -A_q^{-1} P_q \operatorname{div} F$, see (2.15), we get that $\langle F, \nabla A_{q'}^{-1} v\rangle_\Omega = \langle \hat{F}, v\rangle_\Omega$ and $\|\hat{F}\|_{q,\Omega} \le C\|F\|_{r,\Omega}$ with $C = C(\Omega, q, r) > 0$ by (2.16).

Using E^h for $h = N \cdot g$, cf. (2.6), the compatibility condition in (3.2) yields $\int_\Omega (\operatorname{div} E^h - k)dx = 0$. Hence there exists $b \in W_0^{1,r}(\Omega)$ satisfying $\operatorname{div} b = \operatorname{div} E^h - k$ and, due to (2.6), (2.4),

$$\|b\|_{q,\Omega} \le C_1\|\nabla b\|_{r,\Omega} \le C_2(\|\operatorname{div} E^h\|_{r,\Omega} + \|k\|_{r,\Omega}) \tag{3.10}$$

where $C_i = C_i(\Omega, q, r) > 0,\ i = 1, 2$. Then we use the solution $\nabla H \in L^q(\Omega)$ of the weak Neumann problem

$$\Delta H = \operatorname{div}(E^h - b) = k = \operatorname{div} u, \quad N \cdot \nabla H\big|_{\partial\Omega} = N \cdot (E^h - b)\big|_{\partial\Omega},$$

and applying (2.10), (2.6), (3.10) leads to the estimate

$$\|\nabla H\|_{q,\Omega} \le C_1\|E^h - b\|_{q,\Omega} \le C_2(\|g\|_{-\frac{1}{q};q,\partial\Omega} + \|k\|_{r,\Omega}) \tag{3.11}$$

with $C_i = C_i(\Omega, q, r) > 0$. Further we get from (2.10) that $\|\nabla H\|_{q,\Omega} \le C\|u\|_{q,\Omega}$ with $C = C(\Omega, q) > 0$. Obviously, ∇H only depends on the data k, g.

Using (2.2) and (2.3) with $\operatorname{curl} \nabla H = 0$ we conclude that $\nabla H\big|_{\partial\Omega} \in W^{-\frac{1}{q},q}(\partial\Omega)$ is well defined, and that

$$\|\nabla H\|_{-\frac{1}{q};q,\partial\Omega} \le C(\|g\|_{-\frac{1}{q};q,\partial\Omega} + \|k\|_{r,\Omega}) \tag{3.12}$$

with $C = C(\Omega, q, r) > 0$.

Set $\hat{u} = P_q u = u - \nabla H \in L^q_\sigma(\Omega)$ and $\hat{g} = g - \nabla H\big|_{\partial\Omega} \in W^{-\frac{1}{q},q}(\partial\Omega)$. Then

$$|\langle \hat{g}, N \cdot \nabla w\rangle_{\partial\Omega}| \le C(\|g\|_{-\frac{1}{q};q,\partial\Omega} + \|k\|_{r,\Omega})\|v\|_{q',\Omega}, \quad w \in C^2_{0,\sigma}(\overline{\Omega}),\ v = A_{q'}w, \tag{3.13}$$

cf. (3.9). As above, we construct $\hat{G} \in L^q_\sigma(\Omega)$ satisfying $\langle \hat{g}, N \cdot \nabla A_{q'}^{-1} v\rangle_{\partial\Omega} = \langle \hat{G}, v\rangle_\Omega$, and the estimate

$$\|\hat{G}\|_{q,\Omega} \le C(\|g\|_{-\frac{1}{q};q,\partial\Omega} + \|k\|_{r,\Omega}) \tag{3.14}$$

holds with $C = C(\Omega, q, r) > 0$.

In the next step we use the relation $\langle \nabla H, \Delta w\rangle_\Omega = \langle \nabla H, N \cdot \nabla w\rangle_{\partial\Omega}, w$ as in (3.13), which follows from using an approximation of H by smooth functions and an integration by parts. Then by (3.3) a calculation leads to $\langle \hat{g}, N \cdot \nabla w\rangle_{\partial\Omega} + \langle F, \nabla w\rangle_\Omega = -\langle \hat{u}, v\rangle_\Omega$, and inserting $\hat{G}, \hat{F}$ yields $\langle \hat{G}, v\rangle_\Omega + \langle \hat{F}, v\rangle_\Omega = -\langle \hat{u}, v\rangle_\Omega$.

From the regularity properties of the Stokes operator $A_{q'}$, see [25], we know that the set of all $v = A_{q'}w$ with $w \in C^2_{0,\sigma}(\overline{\Omega})$ is dense in $L^{q'}_\sigma(\Omega)$. Thus we may use the last relation for all $v \in L^{q'}_\sigma(\Omega)$, and we get the representation formula

$$u = \nabla H - \hat{G} - \hat{F} \tag{3.15}$$

for the given very weak solution u.

Since the right hand side of (3.15) depends only on the data F, k, g, we can use (3.15) to construct $u \in L^q(\Omega)$. Then the same arguments as above show that u satisfies (3.3) and (1.12). Thus u defined by (3.15) is a very weak solution of (3.1). Since each given very weak solution of (3.1) has the form (3.15), we obtain the uniqueness assertion. The estimate (3.4) follows from (3.12), (3.14), and (2.16). This proves Theorem 3. ∎

Remarks 2 **(1)** Suppose that the data F, k, g in (3.1), (3.2) satisfy the stronger conditions $F \in L^q(\Omega)$, $k \in L^q(\Omega)$, $g \in W^{1-1/q,q}(\partial\Omega)$, $1 < q < \infty$. Then the very weak solution u in Theorem 3 satisfies $u \in W^{1,q}(\Omega)$ and estimate

$$\|u\|_{1;q,\Omega} \le C(\|F\|_{q,\Omega} + \|k\|_{q,\Omega} + \|g\|_{1-1/q;q,\partial\Omega}) \tag{3.16}$$

with $C = C(\Omega, q) > 0$. The existence of such a solution u of (3.1) is well known, see [9], [11], [13]. Since u is obviously also a very weak solution which is unique, we conclude this regularity property; see also [14], Lemma 4.

(2) In the same way we conclude that if the data in (3.1), (3.2) satisfy the conditions $f = \operatorname{div} F \in L^q(\Omega)$, $k \in W^{1,q}(\Omega)$, $g \in W^{2-1/q,q}(\partial\Omega)$, then this solution satisfies

$$\|u\|_{2;q,\Omega} \le C(\|f\|_{q,\Omega} + \|k\|_{1;q,\Omega} + \|g\|_{2-1/q;q,\partial\Omega}) \tag{3.17}$$

with $C = C(\Omega, q) > 0$. Thus (3.16) and (3.17) are regularity properties of the very weak solution u if the data are sufficiently smooth.

Proof of Theorem 2. Following [14] we first consider a given very weak solution $u \in L^q(\Omega)$ of the system (1.9). Using similar arguments as in the previous proof we obtain, since $u \cdot \nabla u = \operatorname{div}(uu) - ku$ and $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$, that

$$|\langle u \cdot \nabla u, w \rangle_\Omega| \le C(\|u\|^2_{q,\Omega} + \|k\|_{r,\Omega}\, \|u\|_{q,\Omega}) \|\nabla w\|_{r',\Omega}, \quad w \in C^2_{0,\sigma}(\overline{\Omega}). \tag{3.18}$$

Hence we find $W(u) \in L^r(\Omega)$ satisfying

$$\langle u \cdot \nabla u, w \rangle_\Omega = \langle \operatorname{div}(uu) - ku, w \rangle_\Omega = \langle \operatorname{div} W(u), w \rangle_\Omega = -\langle W(u), \nabla w \rangle_\Omega, \tag{3.19}$$

and

$$\|W(u)\|_{r,\Omega} \le C(\|u\|^2_{q,\Omega} + \|k\|_{r,\Omega}\, \|u\|_{q,\Omega}) \tag{3.20}$$

with $C = C(\Omega, q, r) > 0$, similarly as in (2.15), (2.16). We see that u is a very weak solution of the linear system

$$-\Delta u + \nabla p = \operatorname{div}\big(F - W(u)\big), \quad \operatorname{div} u = k, \quad u_{|\partial\Omega} = g. \tag{3.21}$$

Then (3.15) leads to the formula

$$u = \nabla H - \hat{G} - \hat{F} - \hat{W}(u) \tag{3.22}$$

where $\nabla H, \hat{G}, \hat{F}, \hat{W}(u)$ are determined by

$$\begin{aligned}
\Delta H &= k, \quad N \cdot (\nabla H - g)\big|_{\partial\Omega} = 0, \\
\langle \hat{G}, v\rangle_\Omega &= \langle g - \nabla H\big|_{\partial\Omega}, N \cdot \nabla A_{q'}^{-1} v\rangle_{\partial\Omega}, \\
\langle \hat{F}, v\rangle_\Omega &= \langle F, \nabla A_{q'}^{-1} v\rangle_\Omega, \\
\langle \hat{W}(u), v\rangle_\Omega &= -\langle uu, \nabla A_{q'}^{-1} v\rangle_\Omega - \langle ku, A_{q'}^{-1} v\rangle_\Omega \quad \text{for all } w \in C^2_{0,\sigma}(\overline{\Omega}),\ v = A_{q'} w.
\end{aligned}$$

Setting $\hat{u} = u - \nabla H$, $\mathcal{F}(\hat{u}) = -\hat{F} - \hat{G} - \hat{W}(\hat{u} + \nabla H)$, we obtain from (3.22) the equation $\hat{u} = \mathcal{F}(\hat{u})$, which can be solved by Banach's fixed point theorem. This leads to the desired solution $u = \hat{u} + \nabla H$.

For this purpose we use similar estimates as in the previous proof, and obtain

$$\begin{aligned}
\|\mathcal{F}(\hat{u})\|_{q,\Omega} \leq\ & C(\|\hat{u}\|_{q,\Omega} + \|g\|_{-\frac{1}{q};q,\partial\Omega} + \|k\|_{r,\Omega})^2 \\
& + C\|k\|_{r,\Omega}\big(\|\hat{u}\|_{q,\Omega} + \|g\|_{-\frac{1}{q};q,\partial\Omega} + \|k\|_{r,\Omega}\big) \\
& + C\big(\|F\|_{r,\Omega} + \|g\|_{-\frac{1}{q};q,\partial\Omega} + \|k\|_{r,\Omega}\big)
\end{aligned} \tag{3.23}$$

with $C = C(\Omega, q) > 0$. Setting $a = C$, $\beta = \|g\|_{-\frac{1}{q};q,\partial\Omega} + \|k\|_{r,\Omega}$, $\gamma = C(\|F\|_{r,\Omega} + \beta) + \beta$, $\delta = C\|k\|_{r,\Omega}$, we obtain the estimate

$$\|\mathcal{F}(\hat{u})\|_{q,\Omega} + \beta \leq a(\|\hat{u}\|_{q,\Omega} + \beta)^2 + \delta(\|\hat{u}\|_{q,\Omega} + \beta) + \gamma. \tag{3.24}$$

Then we consider the closed ball $\mathcal{B} = \{\hat{u} \in L^q_\sigma(\Omega);\ \|\hat{u}\|_{q,\Omega} + \beta \leq y_1\}$ where $y_1 > 0$ means the smallest root of the equation $y = ay^2 + \delta y + \gamma$. Supposing the smallness condition $4a\gamma + 2\delta < 1$ we get $y_1 > \beta$ and $\|\mathcal{F}(\hat{u}) - \mathcal{F}(\hat{v})\|_{q,\Omega} \leq \hat{a}\|\hat{u} - \hat{v}\|_{q,\Omega}$ with some $0 < \hat{a} < 1$. Now Banach's fixed point theorem yields a unique solution $\hat{u} \in \mathcal{B}$ with $\hat{u} = \mathcal{F}(\hat{u})$, see [26], V.4.2, for details. Then $u = \hat{u} + \nabla H$ solves (3.22) and is a very weak solution of (1.9). The smallness condition $4a\gamma + 2\delta < 1$ can be written in the form (1.13).

To prove uniqueness we follow [14] and assume that there exists another very weak solution $v \in L^q(\Omega)$ of the system (1.9) with the same data F, k, g as for u. Setting $U = u - v$ we can show that the equation

$$\langle U, \Delta w + v \cdot \nabla w + u \cdot (\nabla w)^T + kw\rangle_\Omega = 0 \tag{3.25}$$

is satisfied for all $w \in C^2_{0,\sigma}(\overline{\Omega})$, where "$T$" denotes the transpose. Note that v need not satisfy any smallness condition, but that $\|u\|_q$ is small. Then, by standard arguments, we solve for each $G \in C_0^\infty(\Omega)$ the modified Stokes system

$$-\Delta w - v \cdot \nabla w - u \cdot (\nabla w)^T - kw + \nabla\pi = G, \quad \operatorname{div} w = 0,\ w\big|_{\partial\Omega} = 0$$

to get a solution $(w, \nabla\pi) \in \big(W^{2,3/2}(\Omega) \cap W_0^{1,3/2}(\Omega)\big) \times L^{3/2}(\Omega)$. Since $C^2_{0,\sigma}(\overline{\Omega})$ is dense in $\big(W^{2,3/2}(\Omega) \cap W_0^{1,3/2}(\Omega)\big) \cap L^{3/2}_\sigma(\Omega)$, we may insert this w in (3.25) which finally shows that $U = 0$, $u = v$. This proves Theorem 2. The estimate (1.14) is an easy consequence. ■

4 Nonstationary very weak solutions

Consider the linearized nonstationary system

$$u_t - \Delta u + \nabla p = f, \quad \operatorname{div} u = k, \quad u_{|_{\partial\Omega}} = g, \quad u_{|_{t=0}} = u_0 \tag{4.1}$$

with data $f = \operatorname{div} F$ and k, g, u_0 satisfying

$$\begin{aligned} &F \in L^s\big(0,T; L^r(\Omega)\big), \quad k \in L^s\big(0,T; L^r(\Omega)\big), \quad g \in L^s\big(0,T; W^{-\frac{1}{q},q}(\partial\Omega)\big), \\ &u_0 \in \mathcal{J}^{q,s}(\Omega), \quad \int_\Omega k\,dx = \int_{\partial\Omega} N\cdot g\,dS \quad \text{for a.a. } t \in [0,T), \end{aligned} \tag{4.2}$$

where $1 < r \le q < \infty$, $\frac{1}{3} + \frac{1}{q} \ge \frac{1}{r}$, $1 < s < \infty$.

Modifying Definition 1 in an obvious way for the linearized case, a vector field $u \in L^s\big(0,T; L^q(\Omega)\big)$ is called a *very weak solution* of the system (4.1) with data (4.2) if

$$-\langle u, w_t\rangle_{\Omega,T} - \langle u, \Delta w\rangle_{\Omega,T} + \langle g, N\cdot\nabla w\rangle_{\partial\Omega,T} = \langle u_0, w(0)\rangle_\Omega - \langle F, \nabla w\rangle_{\Omega,T}, \tag{4.3}$$

$w \in C^1_0\big([0,T); C^2_{0,\sigma}(\overline{\Omega})\big)$, and additionally the conditions (1.6) are satisfied, i.e., $\operatorname{div} u = k$ and $N\cdot u_{|_{\partial\Omega}} = N\cdot g$ a.e. in $(0,T)$.

Our main result on this system reads as follows.

Theorem 4 *Suppose that the data F, k, g and u_0 satisfy (4.2) with $1 < r \le q < \infty$, $\frac{1}{3} + \frac{1}{q} \ge \frac{1}{r}$, $1 < s < \infty$. Then there exists a unique very weak solution $u \in L^s\big(0,T; L^q(\Omega)\big)$ of the system (4.1), satisfying*

$$A_q^{-1}P_q u_t \in L^s\big(0,T; L^q_\sigma(\Omega)\big), A_q^{-1}P_q u \in C\big([0,T); L^q_\sigma(\Omega)\big), A_q^{-1}P_q u_{|_{t=0}} = A_q^{-1}P_q u_0$$

and

$$\|A_q^{-1}P_q u_t\|_{q,s,\Omega} + \|u\|_{q,s,\Omega} \le C\big(\|u_0\|_{\mathcal{J}^{q,s}(\Omega)} + \|F\|_{r,s,\Omega} + \|k\|_{r,s,\Omega} + \|g\|_{-\frac{1}{q};q,s,\partial\Omega}\big) \tag{4.4}$$

with $C = C(\Omega, q, r, s) > 0$.

Remarks 3 **(1)** Setting in particular $w \in C_0^\infty\big((0,T); C^\infty_{0,\sigma}(\Omega)\big)$ in (4.3), we obtain, see [26], p. 248, p. 202, [27], the existence of a distribution p such that $u_t - \Delta u + \nabla p = f$ holds in $\Omega \times (0,T)$ in the sense of distributions.

(2) Let $h = (h_1, h_2, h_3) \in C_0^1\big((0,T); W^{1-1/q',q'}(\partial\Omega)\big)$ with $N \cdot h_{|_{\partial\Omega}} = 0$. Then $h(t) \mapsto \hat{w}^{h(t)}$, see (2.8), (2.9), is a linear mapping satisfying $(\hat{w}^h)_t = \hat{w}^{h_t}$. We may insert $w = \hat{w}^h$ in (4.3) and obtain the formula

$$\langle g, h\rangle_{\partial\Omega,T} = \langle u, \hat{w}^{h_t}\rangle_{\Omega,T} + \langle u, \Delta\hat{w}^h\rangle_{\Omega,T} - \langle F, \nabla\hat{w}^h\rangle_{\Omega,T}. \tag{4.5}$$

Since the normal component $N \cdot h$ of the test function h is zero, this formula yields a well defined expression for the tangential component $N \times g$ of the boundary values. It is easy to see using integration by parts that $N \times g$ coincides with the usual trace $N \times u_{|_{\partial\Omega}}$ if u is sufficiently smooth. Therefore, we may call the right hand side of (4.5) the trace $N \times u_{|_{\partial\Omega}}$ of the tangential component of u at $\partial\Omega$ in the sense of distributions. Since the normal component $N \cdot u_{|_{\partial\Omega}}$ of u at $\partial\Omega$ is well defined by (1.6), we get an explicit trace formula for $u_{|_{\partial\Omega}}$ in the sense of distributions at $\partial\Omega$ which coincides with the usual trace of u at $\partial\Omega$ if u is smooth. This yields a precise meaning of the general boundary condition $u_{|_{\partial\Omega}} = g$ in the sense of boundary distributions.

(3) Since $w(0)$ in (4.3) is solenoidal we expect that the initial condition $u_{|_{t=0}} = u_0$ only makes sense "modulo gradients". Therefore, the condition $A_q^{-1}P_q u_{|_{t=0}} = A_q^{-1}P_q u_0$, see (1.7), seems to be the adequate precise formulation of the initial condition $u_{|_{t=0}} = u_0$. If u is sufficiently smooth, we need additional (necessary) compatibility conditions in order to reach that $u(0) = u_0$, see (4.14).

Proof of Theorem 4. Let $E(t) = E^{k(t),g(t)} \in L^q(\Omega)$ be the very weak solution of the stationary system

$$-\Delta E(t) + \nabla p(t) = 0, \quad \operatorname{div} E(t) = k(t), \;\; E(t)_{|_{\partial\Omega}} = g(t) \text{ for a.a. } t \in [0,T]. \tag{4.6}$$

Then from (3.3) we obtain the relation $\langle g, N \cdot \nabla w\rangle_{\partial\Omega,T} = \langle E, \Delta w\rangle_{\Omega,T}$ for every $w \in C_0^1\big([0,T); C^2_{0,\sigma}(\overline{\Omega})\big)$. Therefore, given a very weak solution u, (4.3) can be written in the form

$$-\langle A_q^{-1}P_q u, v_t\rangle_{\Omega,T} - \langle u - E, \Delta A_{q'}^{-1}v\rangle_{\Omega,T} = \langle A_q^{-1}P_q u_0, v(0)\rangle_\Omega - \langle F, \nabla A_{q'}^{-1}v\rangle_{\Omega,T} \tag{4.7}$$

where $w \in C_0^1\big([0,T); C^2_{0,\sigma}(\overline{\Omega})\big)$, $v = A_{q'}w$. Since $A_q^{-1}P_q u_0 \in L^q_\sigma(\Omega)$, see (2.19), and since $\operatorname{div}(u - E) = 0$, $N \cdot (u - E)_{|_{\partial\Omega}} = 0$ yielding $u - E = P_q(u - E)$ we obtain that

$$\begin{aligned} \langle u - E, \Delta A_{q'}^{-1}v\rangle_{\Omega,T} &= \langle P_q(u - E), \Delta A_{q'}^{-1}v\rangle_{\Omega,T} \\ &= \langle u - E, P_{q'}\Delta A_{q'}^{-1}v\rangle_{\Omega,T} = -\langle u - E, v\rangle_{\Omega,T}. \end{aligned}$$

Further we use (2.15) for a.a. $t \in [0,T)$ and get a unique $\hat{F} = -A_q^{-1}P_q \operatorname{div} F \in L^s\big(0,T; L^q_\sigma(\Omega)\big)$ satisfying the relation

$$\langle F, \nabla A_{q'}^{-1}v\rangle_{\Omega,T} = \langle \hat{F}, v\rangle_{\Omega,T} \quad \text{for all } v = A_{q'}w, \; w \in C_0^1\big([0,T); C^2_{0,\sigma}(\overline{\Omega})\big). \tag{4.8}$$

This leads to the relation

$$-\langle A_q^{-1}P_q u, v_t\rangle_{\Omega,T} = \langle A_q^{-1}P_q u_0, v(0)\rangle_\Omega - \langle P_q u, v\rangle_{\Omega,T} + \langle P_q E, v\rangle_{\Omega,T} + \langle A_q^{-1}P_q \operatorname{div} F, v\rangle_{\Omega,T}. \tag{4.9}$$

Then a standard argument shows, see [27], III, 1.1 or [26], IV, 1.3, that $A_q^{-1}P_q u_t \in L^s\big(0,T;L^q_\sigma(\Omega)\big)$ is well defined, that

$$A_q^{-1}P_q u \in C\big([0,T);L^q_\sigma(\Omega)\big), \quad A_q^{-1}P_q u(0) = A_q^{-1}P_q u_0,$$

and that the evolution system

$$(A_q^{-1}P_q u)_t + A_q(A_q^{-1}u) = A_q^{-1}P_q \operatorname{div} F + P_q E^{k,g}, \quad (A_q^{-1}P_q u)(0) = A_q^{-1}P_q u_0 \tag{4.10}$$

is satisfied. From (2.21) we now obtain the representation formula

$$\begin{aligned} \hat{u}(t) &\equiv P_q u(t) \\ &= A_q e^{-tA_q} A_q^{-1}P_q u_0 + \int_0^t A_q e^{-(t-\tau)A_q}\big(A_q^{-1}P_q \operatorname{div} F + P_q E^{k,g}\big)d\tau \end{aligned} \tag{4.11}$$

for the very weak solution u. As in (3.22) we get $\hat{u}(t) = P_q u(t) = u(t) - \nabla H(t)$ where $\nabla H(t)$ is determined by $\Delta H(t) = k(t)$, $N \cdot \big(\nabla H(t) - g(t)\big)\big|_{\partial\Omega} = 0$ for a.a. $t \in [0,T)$. Since $\nabla H(t)$ only depends on $g(t), k(t)$, see (3.11), we obtain by (4.11) a formula for $u = \hat{u} + \nabla H$ which determines u uniquely by the data F, k, g and u_0.

Now use (4.11) to construct a very weak solution u. Using the same calculation as above we obtain the existence assertion of u, the uniqueness of which follows from the representation (4.11). The estimate (4.4) is based on (2.20) and the estimates of $E, \hat{F}$, see (3.4) and (2.16).

Using (2.20) we conclude that the term $\|u_0\|_{\mathcal{J}^{q,s}(\Omega)}$ in (4.4) can be replaced by the weaker norm

$$\|u_0\|_{\mathcal{J}^{q,s}_T(\Omega)} \equiv \Big(\int_0^T \|A_q e^{-tA_q} A_q^{-1}P_q u_0\|^s_{q,\Omega}\, dt\Big)^{\frac{1}{s}}. \tag{4.12}$$

Now the proof of Theorem 4 is complete. ∎

Next we consider some regularity properties. Suppose the data $f = \operatorname{div} F$ and k, g, u_0 of the system (4.1) satisfy the stronger conditions

$$\begin{aligned} &F \in L^s\big(0,T;W^{1,q}(\Omega)\big),\ k \in L^s\big(0,T;W^{1,q}(\Omega)\big),\ k_t \in L^s\big(0,T;L^r(\Omega)\big), \\ &g \in L^s\big(0,T;W^{2-1/q,q}(\partial\Omega)\big),\ g_t \in L^s\big(0,T;W^{-\frac{1}{q},q}(\partial\Omega)\big),\ u_0 \in W^{2,q}(\Omega) \end{aligned} \tag{4.13}$$

with $1 < r \le q < \infty$, $\frac{1}{3} + \frac{1}{q} \ge \frac{1}{r}$, $1 < s < \infty$, and the compatibility conditions

$$\int_\Omega k\, dx = \int_{\partial\Omega} N\cdot g\, dS \ \text{ for } t \in [0,T), \quad u_0\big|_{\partial\Omega} = g(0), \quad \operatorname{div} u_0 = k(0). \tag{4.14}$$

Observe that $g(0)$ and $k(0)$ are well defined because of the assumptions on k_t, g_t. For simplicity the assumption on u_0 is not optimally chosen.

Then we will show the existence of a unique solution $u \in L^s\big(0,T;W^{2,q}(\Omega)\big)$ with $u_t \in L^s\big(0,T;L^q(\Omega)\big)$ of the system (4.1), together with a pressure function p such that $\nabla p \in L^s\big(0,T;L^q(\Omega)\big)$, satisfying the estimate

$$\begin{aligned} &\|u_t\|_{q,s,\Omega} + \|u\|_{2;q,s,\Omega} + \|\nabla p\|_{q,s,\Omega} \\ &\quad \le C\Big(\|u_0\|_{2;q,\Omega} + \|f\|_{q,s,\Omega} + \|k\|_{1;q,s,\Omega} + \|k_t\|_{r,s,\Omega} \\ &\qquad + \|g\|_{2-1/q;q,s,\partial\Omega} + \|g_t\|_{-1/q;q,s,\partial\Omega}\Big) \end{aligned} \tag{4.15}$$

with $C = C(\Omega,q,s) > 0$. The equations $u_t - \Delta u + \nabla p = f$, $\operatorname{div} u = k$ and $u|_{\partial\Omega} = g$ in (4.1) are satisfied in the strong sense for a.a. $t \in [0,T)$; hence $u \in C\big([0,T);L^q(\Omega)\big)$ and $u(0) = u_0$ is well defined. The initial value $u_0 \in W^{2,q}(\Omega)$ can be treated as a functional from $\mathcal{J}^{q,s}(\Omega)$, see (2.19). Thus we see that (4.13), (4.14) are stronger than the conditions (4.2).

Therefore, Theorem 4 yields a unique very weak solution $u \in L^s\big(0,T;L^q(\Omega)\big)$ to (4.1) which coincides with each more regular solution by the uniqueness property.

To show the existence of a regular solution u satisfying (4.15) we first suppose that such a solution is given. Let $E = E^{k,g}$ be chosen as in (4.6). Then $E(0) = E^{k(0),g(0)}$ satisfies the system

$$-\Delta E(0) + \nabla p(0) = 0, \quad \operatorname{div} E(0) = k(0), \quad E(0)|_{\partial\Omega} = g(0)$$

and (3.17) shows that $E(0) \in W^{2,q}(\Omega)$. Using (4.14) we see that $u_0 - E(0)|_{\partial\Omega} = 0$, $\operatorname{div}\big(u_0 - E(0)\big) = 0$ which leads to $u_0 - E(0) \in D(A_q)$. Further, using $E_t = (E^{k,g})_t = E^{k_t,g_t}$, the assumptions on k, k_t, g, g_t, estimate (3.4) with u, F, k, g replaced by $E_t, 0, k_t, g_t$, and the estimate (3.17) with u, f, k, g replaced by $E, 0, k, g$, we obtain the estimate

$$\begin{aligned} &\|E_t\|_{q,s,\Omega} + \|E\|_{2;q,s,\Omega} \\ &\quad \le C\big(\|k\|_{1;q,s,\Omega} + \|k_t\|_{r,s,\Omega} + \|g\|_{2-1/q;q,s,\Omega} + \|g_t\|_{-\frac{1}{q};q,s,\partial\Omega}\big) \end{aligned} \tag{4.16}$$

with $C = C(\Omega,q,s) > 0$.

Setting $\tilde{u}(t) = u(t) - E(t)$ we obtain the evolution system

$$\tilde{u}_t + A_q\tilde{u} = P_q f - P_q E_t,\ \operatorname{div}\tilde{u} = 0,\ \tilde{u}|_{\partial\Omega} = 0,\ \tilde{u}|_{t=0} = u_0 - E(0).$$

Then (2.21) yields the representation formula

$$u(t) = E(t) + e^{-tA_q}\big(u_0 - E(0)\big) + \int_0^t e^{-(t-\tau)A_q}(P_q f - P_q E_\tau)\,d\tau, \quad 0 \le t < T, \tag{4.17}$$

for the given regular solution u.

In the next step we use (4.17) to construct the desired solution u, and we use (2.20), (4.16). Further we apply (3.17) to $E(0)$, and use that $k(0) = \operatorname{div} u_0$, $g(0) = u_0|_{\partial\Omega}$. This yields the regularity properties of u, the estimate (4.15) for u, and its uniqueness. The pressure term ∇p, constructed by de Rham's argument, can be written in the form $\nabla p = f - u_t + \Delta u$ proving (4.15) for p.

Proof of Theorem 1. First let u be a given solution of (1.1) for some $0 < T' \le T$ with the properties of this theorem. Further we consider the solution $E = E^{F,k,g,u_0}$ of the corresponding linearized system

$$E_t - \Delta E + \nabla \hat{p} = \operatorname{div} F, \quad \operatorname{div} E = k, \quad E|_{\partial\Omega} = g, \quad E|_{t=0} = u_0$$

according to Theorem 4. Setting $\tilde{u} = u - E$, the calculation as in (3.19) shows that $\tilde{u}$ is a very weak solution of the linear system

$$\tilde{u}_t - \Delta\tilde{u} + \nabla\tilde{p} = -\operatorname{div} W(u), \quad \operatorname{div}\tilde{u} = 0, \quad \tilde{u}|_{\partial\Omega} = 0, \quad \tilde{u}|_{t=0} = 0, \tag{4.18}$$

where $W(u)$ is defined as in (3.19); in particular, $\operatorname{div} W(u) = \operatorname{div}(uu) - ku$. Using (3.20) we conclude that $W(u) \in L^{s/2}\big(0,T''; L^r(\Omega)\big)$ for $0 < T'' < T'$. If $T' < \infty$ we set $T'' = T'$. Thus we may use the representation formula (4.11) with $k = 0$, $g = 0$, $u_0 = 0$, $F = -W(u) = -W(\tilde{u} + E)$ and with s replaced by $s/2$. Hence

$$\tilde{u}(t) = (\mathcal{F}(\tilde{u})(t) := -\int_0^t A_q e^{-(t-\tau)A_q} A_q^{-1} P_q \operatorname{div} W(u)\, d\tau, \quad 0 \le t < T'. \tag{4.19}$$

To solve (4.19) by Banach's fixed point theorem we have to estimate $\|\mathcal{F}(\tilde{u})\|_{q,s,\Omega}$ where $\|\cdot\|_{q,s,\Omega} = \|\cdot\|_{q,s,\Omega,T'} = \big(\int_0^{T'} \|\cdot\|_{q,\Omega}^s\, dt\big)^{\frac{1}{s}}$. Let $\alpha' = \frac{3}{2q}$ yielding $\frac{1}{2} - \alpha' + \frac{1}{s} = \frac{1}{s/2}$ and $2\alpha' + \frac{3}{q} = \frac{3}{q/2}$. Using (2.12) and (2.11) with $\alpha = \alpha'$, $\gamma = \frac{q}{2}$, we get that

$$\|\mathcal{F}(\tilde{u})(t)\|_{q,\Omega} \le C \int_0^t \frac{1}{(t-\tau)^{1/2+\alpha'}} \|A_{q/2}^{-1/2} P_{q/2} \operatorname{div} W(u)\|_{q/2}\, d\tau\, .$$

Looking at the integrand, we apply the estimate

$$\|A_{q/2}^{-1/2} P_{q/2}\,(ku)\|_{q/2} \le C \|P_\gamma(ku)\|_\gamma \le C\|k\|_r \|u\|_q$$

which is based on (2.11) with $\alpha = \frac{1}{2}$ and $\frac{1}{\gamma} = \frac{1}{3} + \frac{2}{q} = \frac{1}{r} + \frac{1}{q}$ and on Hölder's inequality. Furthermore, $\|A_{q/2}^{-1/2} P_{q/2} \operatorname{div}(uu)\|_{q/2} \le C\|uu\|_{q/2} \le C\|u\|_q^2$ since

$$|\langle A_{q/2}^{-1/2} P_{q/2} \operatorname{div}(uu), \varphi\rangle_\Omega| = |-\langle uu, \nabla A_{(q/2)'}^{-1/2} P_{(q/2)'}\varphi\rangle_\Omega| \le C\|uu\|_{q/2}\|\varphi\|_{(q/2)'}$$

for all $\varphi \in L^{(q/2)'}(\Omega)$. Summarizing we conclude that

$$\|\mathcal{F}(\tilde{u})(t)\|_{q,\Omega} \le C \int_0^t \frac{1}{(t-\tau)^{1/2+\alpha'}} \big(\|u\|_q^2 + \|k\|_r \|u\|_q\big)\, d\tau\, .$$

Then the Hardy-Littlewood inequality, see [26], p. 103, [28],

$$\Big(\int_0^T |\int_0^t (t-\tau)^{\alpha-1} h(\tau)\,d\tau|^s\,dt\Big)^{\frac{1}{s}} \le C\,\Big(\int_0^T |h(t)|^{\tilde s}\,dt\Big)^{\frac{1}{\tilde s}}$$

with $\alpha = \frac{1}{2} - \alpha'$ and $\tilde s = \frac{s}{2}$ yields the estimate

$$\|\mathcal{F}(\tilde u)\|_{q,s,\Omega} \le C\Big(\big(\|\tilde u\|_{q,s,\Omega} + \|E\|_{q,s,\Omega}\big)^2 + \|k\|_{r,s,\Omega}\big(\|\tilde u\|_{q,s,\Omega} + \|E\|_{q,s,\Omega}\big)\Big) \quad (4.20)$$

with $C = C(\Omega, q, s) > 0$.

Setting $a = C$, $\beta = \|E\|_{q,s,\Omega}$, $\delta = C\|k\|_{r,s,\Omega}$ and $\gamma = \beta$, (4.20) is equivalent to the estimate

$$\|\mathcal{F}(\tilde u)\|_{q,s,\Omega} + \beta \le a(\|\tilde u\|_{q,s,\Omega} + \beta)^2 + \delta(\|\tilde u\|_{q,s,\Omega} + \beta) + \gamma; \quad (4.21)$$

cf. (3.24) in the proof of Theorem 2 for the stationary case. Thus, in the same way as in that proof, we obtain a solution $\tilde u \in L^s\big(0,T'; L^q(\Omega)\big)$ of the fixed point equation $\tilde u = \mathcal{F}(\tilde u)$ if the condition $4a\gamma + 2\delta < 1$, i.e.,

$$4C\Big(\int_0^{T'} \|E\|^s_{q,\Omega} dt\Big)^{\frac{1}{s}} + 2C\Big(\int_0^{T'} \|k\|^s_{r,\Omega} dt\Big)^{\frac{1}{s}} < 1\,. \quad (4.22)$$

is satisfied. Using (4.4) and (4.12) we may use also the (weaker) smallness condition

$$\begin{aligned} &\Big(\int_0^{T'} \|A_q e^{-tA_q} A_q^{-1} P_q u_0\|^s_{q,\Omega} dt\Big)^{\frac{1}{s}} + \Big(\int_0^{T'} \|F\|^s_{r,\Omega} dt\Big)^{\frac{1}{s}} \\ &\quad + \Big(\int_0^{T'} \|k\|^s_{r,\Omega} dt\Big)^{\frac{1}{s}} + \Big(\int_0^{T'} \|g\|^s_{-\frac{1}{q};q,\partial\Omega} dt\Big)^{\frac{1}{s}} < \frac{1}{C} \end{aligned} \quad (4.23)$$

with $C = C(\Omega, q, s) > 0$. This condition is always satisfied if $T' > 0$ is sufficiently small; note that the case $T' = \infty$ is possible.

Writing (4.19) in the form

$$\begin{aligned} A_q^{-1}\tilde u(t) &= A_q^{-1} P_q u(t) - A_q^{-1} P_q E(t) \\ &= -\int_0^t e^{-(t-\tau)A_q} A_q^{-1} P_q \operatorname{div} W(u) d\tau, \quad 0 \le t < T', \end{aligned} \quad (4.24)$$

we conclude using (2.20), (2.16), (3.20), together with Hölder's inequality, that

$$\begin{aligned} \|(A_q^{-1}\tilde u)_t\|_{q,s/2,\Omega} &\le C_1\|A_q^{-1} P_q \operatorname{div} W(u)\|_{q,s/2,\Omega} \le C_2\|W(u)\|_{r,s/2,\Omega} \\ &\le C_3\big(\|u\|^2_{q,s,\Omega} + \|k\|_{r,s,\Omega}\,\|u\|_{q,s,\Omega}\big) < \infty \end{aligned}$$

with C_i, $i = 1,2,3$, depending on Ω, q, s.

Furthermore, we obtain from (4.4) that

$$A_q^{-1} P_q E_t \in L^s\big(0,T''; L^q_\sigma(\Omega)\big) \subseteq L^{s/2}\big(0,T''; L^q_\sigma(\Omega)\big), \quad 0 < T'' < T'.$$

This proves that $A_q^{-1} P_q u_t \in L^{s/2}\big(0,T''; L^q_\sigma(\Omega)\big)$ for all T'' with $0 < T'' < T'$, and all $0 < T'' < \infty$ if $T' = \infty$.

A calculation shows that u defined by $u = \tilde{u} + E$ is a very weak solution of (1.1). To prove the uniqueness of u we assume that $v \in L^s\big(0,T'; L^q(\Omega)\big)$ is another very weak solution of (1.1). Setting $U = u - v$ we obtain in the same way as in (4.18) that U is a very weak solution of the system

$$\begin{aligned} &U_t - \Delta U + \nabla P = -\operatorname{div}(Uu) - \operatorname{div}(vU) + kU, \\ &\operatorname{div} U = 0, \quad U_{|\partial\Omega} = 0, \quad U_{|t=0} = 0. \end{aligned} \tag{4.25}$$

The same method as used for (4.18) and (4.19) then leads to the estimate

$$\|U\|_{q,s,\Omega} \leq C(\|u\|_{q,s,\Omega} + \|v\|_{q,s,\Omega} + \|k\|_{r,s,\Omega})\, \|U\|_{q,s,\Omega} \tag{4.26}$$

with $C = C(\Omega, q, s) > 0$. Since $\|\cdot\|_{q,s,\Omega} = \|\cdot\|_{q,s,\Omega,T'}$, we observe that C does not depend on T'. Thus we can choose $T'' \in (0, T')$ such that

$$\|u\|_{q,s,\Omega,T''} + \|v\|_{q,s,\Omega,T''} + \|k\|_{r,s,\Omega,T''} \leq \frac{1}{2C}\,.$$

This leads to $\|U\|_{q,s,\Omega,T''} \leq 0$, hence $U = 0$ and $u = v$ on the interval $[0, T'']$. If $T'' < T$ we can continue this procedure and get $u = v$ on $[0, T')$ in finitely many steps. The proof of Theorem 1 is complete. ■

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