A Multicriterial Approach for Optimizing Bus Schedules and School Starting Times

Armin Fügenschuh and Alexander Martin

Darmstadt University of Technology, Schlossgartenstr. 7, D-64289 Darmstadt

Abstract. A successful optimization of public mass transit in rural areas concentrates on the traffic caused by pupils on their ways to school, for they are the largest group of customers. Besides a change in the schedules of the buses and the starting times of the trips, also the school starting time may become an integrated part of the planning process. We discuss the legal framework for this optimization problem in German states and counties, and present a multi-objective mixed-integer programming formulation for the simultaneous rectification of school and trip starting times. For its solution, we develop a two-stage decomposition heuristic, and apply it to real-world data sets from three different counties.

1 Introduction

The optimization of public mass transit is often based on a solution of a (multi-depot) vehicle scheduling problem, that is, the assignment of a set of vehicles to a set of bus trips with respect to several restrictions such that the number of deployed vehicles is minimized and the vehicles are used as efficiently as possible. Usually, the bus trips have fixed starting times, see [12], for instance.

The counties in Germany we focus on are rural, for their population density is rather low. The biggest city has no more than, say, 30,000 inhabitants, around 150,000 people live in an area of about 1,000 square kilometers. More than half of all pupils are coming to school by public transit, that is, about 10,000 pupils take the bus to 100 different schools. The average way to school has a length of around 10 km, a few pupils travel even more than 30 km twice a day. The optimization of public mass transit in rural areas is mostly an optimization of the traffic caused by pupils on their ways to school and back home, because they are the largest group of customers. Beside the morning and afternoon peaks, there is a much lower demand for public buses over the rest of the day.

It was noted by the consulting company *BPI-Consult*, a subsidiary of the Finnish *Jaakko Pöyry*, that a significant lower number of buses is needed, if the bus scheduling problem is solved together with the starting time problem, i.e., the simultaneous settlement of school and trip starting times [14]. Since then, BPI successfully consulted several counties, where they were able to find solutions which reduce the number of buses by 15 – 20%. Moreover, BPI does not only present a solution, instead they accompany the whole embedding process, including negotiations with all participating groups (bus companies, pupils, parents, teachers, schools, and the county government). Within this process it is sometimes necessary to re-optimize the problem, when new, previously unknown constraints emerge. Interestingly, their payment is proportional to the number of buses they saved in reality, not what they predicted in the beginning. However, their solutions are currently generated manually.

Generating solutions manually is a difficult task, even if we only concentrate on the morning peak, which gives a small planning horizon from 5:30 – 9:00 a.m.. Think of an average county consisting of about 2,000 bus stops, 100 schools, 250 bus trips, 10,000 pupils, and a bus company that deploys around 100 buses to serve all trips. Looking at this figures, it is not a surprise that a human problem solver is easily mired by the huge amount of variables and constraints, so that getting stuck in sub-optimal solutions is inevitable. Thus, the idea of an automatic planning tool was born.

The remainder of this article is organized as follows. In Section 2 we discuss the legal framework of the optimization. A survey of relevant articles is given in Section 3. The mathematical description of this multicriterial optimization problem is then developed in Section 4. For this model is not solvable with standard software, we come up with a heuristic in Section 5. The merits of this heuristic applied to some real-world problem instances can be found in Section 6. We finish with a discussion of the results and give an outlook on our future research activities in Section 7.

2 Legal Framework

In this section we give an overview on the legal restriction for the optimization of school buses and school starting times. Moreover, it will become clear, why a "county" somehow determines a natural border for the size of the problem.

Germany is divided into 16 states ("Länder") and sub-divided into 544 counties ("Landkreise und kreisfreie Städte"). Within Germany's federal governmental system, it is well-regulated which of their duties and tax-funded public services (p.e., waste collection, street preservation and rebuilding, or public mass transportation) the states delegate to county administrations. For our special interest, we consider, as a representative example, the state's school law of Mecklenburg-Vorpommern [2] and two administrative regulations [3], [4] from county Demmin and Mecklenburg-Vorpommern, respectively. The corresponding laws and regulations in other states and counties are similar to these.

Within the school law, the state entitles its counties to be responsible for the transport of all inhabitant pupils. The county has to carry out a public transport for all pupils attending public schools up to an age of 16, if they live too far away from school. The specification of what "too far away" means and how the public transport should be organized, is at the discretion of the counties, assessing the endurance of pupils and the traffic safety of their ways to school.

Based on this document, county Demmin in Mecklenburg-Vorpommern specifies all missing details in an administrative regulation [3]. There, the "way to school" is defined as the shortest path between the pupils' home and the school which is responsible for the area they live in. (This might differ from the school the pupil actually is attending.) A school way is considered as unacceptable for pupils up to age 10, if it is longer than 2 km, and longer than 4 km for all others. Exceptions to this general rule are dangerous ways along highways without pedestrian walks, for instance, where even shorter ways are unacceptable. In these cases, the county pays for the transportation cost of the pupils.

Moreover it is declared in [3] that the county is responsible for the organization of the transport by means of public transportation. Usually, this is done by public buses and trains, in some rare cases by special school buses or rented cars. The deployment of all vehicles has to be done in such a way that pupils arrive not more than 60 minutes before school starts and do not have to wait more than 90 minutes after the end of school. If this is assured, "paying for transportation" means that the county pays the fares for the pupils so that they can use one of the means of transportation mentioned above. The starting and the ending time of schools are settled in such a way that the vehicles (buses, trains, etc.) can be deployed in an economical way and traffic peaks are avoided. To achieve this, not all schools should start at the same time, rectified starting times are preferred. The transportation companies, in coordination with the school authorities, are responsible for a reconciliation of these times with public traffic.

By another administrative regulation of the state Mecklenburg-Vorpommern [4], schools are in general allowed to start between 7:30 and 8:30 a.m., exceptions regarding an earlier start are possible under certain circumstances. Here again, a rectification of school starting times is preferred. Thus, documents [3] and [4] constitute the legal foundation for the optimization problem we are aiming at.

Turning the laws and administrative regulations into an optimization model, we identified (after several discussions with BPI-Consult) the following variables, constraints and objectives. We focus on the following degrees of freedom (variables):

- The schedules of the buses,
- the starting times of the bus trips, and
- the starting times of the schools.

No other possible variables are issued, for example, planning the routes of the bus trips, or locating the bus stops. Moreover it is required that all pupils are using the same bus trips for their ways to school as they do it today. For

BPI-Consult, the decision on which variables the focus should lie, is mainly a political one. Changing the starting times of schools and bus trips causes already enough public opposition. For example, if some school doesn't start at, say, 7:40, but at 8:30, then the pupils leave home nearly one hour later, which might cause troubles for working parents. The same pupils then come home about one hour later, so they might therefore be no longer able to attend a sports club. Therefore, changing school starting times in a whole county at once is a very delicate issue, and political skills are needed to implement any given solution.

The decision variables are not independent from each other, they are coupled by the following constraints:

- The (legal) bounds on the school start (7:30 – 8:30 a.m.),
- lower and upper bounds on the waiting time for pupils at the school,
- bounds on the waiting time for pupils while transferring from one bus trip to another,
- bounds on the starting time of trips.

There are several conflicting goals that have to be addressed by the optimization. In particular, we want to minimize the following:

- The total number of deployed buses,
- the time for driving deadhead-trips,
- the standing times of buses between two trips,
- the absolute change of the schools' starting times,
- the absolute change of the starting times of the bus trips,
- the waiting times for pupils at their schools, and
- the waiting times at a transfer bus stop.

Thus it turns out that the integrated optimization of bus schedules and school starting times is a multicriterial discrete optimization problem.

3 Survey of the Literature

A wide range of transportation problems involving public bus transit, pupils and/or schools were already studied before. On some publications we will now take a closer look and summarize afterwards, why none of these exactly fit to the application we have in mind.

In [5] Bodin and Berman describe a procedure (heuristic algorithm) for a school bus routing and scheduling problem. The selection of the proper starting and ending times of the schools is pointed out as a key factor in transportation cost reduction. The implementation of their algorithm is tested on data from two school districts and results in about a 20% savings in cost. For the computation of school starting and ending times, they make use of a heuristic algorithm developed by Ferland and Fortin [11]. In their approach, however, all pupils are transported directly to the schools, and no transfer between trips is taken into account.

In [7] Braca et al. describe various issues related to the development of a computer software for the automatic solution of the school bus routing and scheduling problem in New York City. The transportation problem is divided into a morning and afternoon transportation problem, where the morning problem is considered as more difficult, because the time windows are tighter. They formulate the problem in terms of two different integer programs: By the first one, a set of feasible routes is generated, and by the other a subset is selected which is sufficient to serve all pupils (set covering).

In [6] Bowerman et al. present a multi-objective model for urban school bus routing problem arising in Ontario (Canada). An additional degree of freedom is the integration of the bus stop planning problem, that is locating bus stops and assigning pupils to them is a part of the whole. The output performance of the model is measured by multiple objectives in terms of efficiency (cost, number of buses), effectiveness (how well is the demand satisfied), and equity (length and load balance among all trips, walking distances to bus stops). For the solution of the model with its related sub-problems, they suggest a sequence of several heuristics, where the results of each heuristic are merged into the final output.

In [9] Desaulniers et al. consider the urban bus scheduling problem, a multi-depot vehicle scheduling problem with time windows. This problem consists of scheduling a fleet of vehicles (buses) to cover a set of tasks (trips) at minimum cost. Each task is restricted to begin within a prescribed time interval and vehicles are supplied by different depots. The problem is formulated as an integer nonlinear multi-commodity network flow model with time variables and is solved using a column generation approach embedded in a branch-and-bound framework.

In [8] Corberan et al. address the problem of routing school buses in rural areas. Their approach is based on a node routing model with multiple objectives that arise from two conflicting goals, costs and quality of service. Their solution algorithm uses construction and improvement heuristics and combines different solutions within an evolutionary framework.

So far, none of the presented models completely fits to our problem, mainly for some or all of the following reasons. With exception of [11], the school starting times are fixed and cannot be changed to save buses. In all modeling approaches, pupils are always transported directly to school, and changing the bus is not allowed. Locating bus stops, designing routes (trips) and assigning pupils to routes is sometimes part of the optimization, but for us these are input figures. Finally, scheduling drivers is not an issue for us: Since our time horizon is small (from 5:30 to 9:00 a.m.), planning of breaks is not needed.

4 An Integrated Multi-Objective Mixed-Integer Programming Model

In the sequel we describe a model for the integrated planning problem of bus and school starting times and bus scheduling in the language of mixed-integer programming. This model is based on the so-called two-indexed formulation of the multiple traveling salesman problem with time windows (mTSP-TW), see [15] for instance.

4.1 Sets and Parameters

We start with a detailed description of the necessary input sets and parameters of the model, see Table 1 for an overview.

Let $\mathcal{V}$ be the set of all bus trips in the given county. A trip $t \in \mathcal{V}$ is a sequence of bus stops, each having an arrival and a departure time assigned to. The time difference between the departure at the first and the arrival at the last bus stop, i.e. the *service duration*, is denoted by $\delta_t^{\text{trip}} \in \mathbb{Z}_+$. The current starting time for trip t, i.e., the departure time of a bus at the first bus stop in t, is given by $\hat{\alpha}_t \in \mathbb{Z}_+$. We assume a time window $\underline{\alpha}_t, \overline{\alpha}_t \in \mathbb{Z}_+, \underline{\alpha}_t \leq \overline{\alpha}_t$ is given, in which the planned trip starting time is supposed to be. The trips in $\mathcal{V}$ play different roles in the transportation of pupils to schools. The following types have to be distinguished: school trips, feeder and collector trips, and free trips, definitions of which are given in the subsequel.

Let the set $\mathcal{A} \subset \mathcal{V} \times \mathcal{V}$ contain all pairs of trips (t_1, t_2) that can be connected in some schedule: In principle, every trip may be served by a new bus from the depot. When this bus arrives at the last bus stop it is either sent back to the depot or it is re-used to serve another trip. The connection of trips which are then served by the same bus is a *block* or *schedule.* The intermediate trip from the last bus stop of trip t_1 to the first bus stop of trip t_2, where no passengers are transported, is called a *shift* or a *deadhead trip* (see Figure 1). The duration of the deadhead trip is given by $\delta_{t_1 t_2}^{\text{shift}} \in \mathbb{Z}_+$.

Let $\mathcal{S}$ be the set of all schools in the county under consideration. The current starting time for school $s \in \mathcal{S}$ is given by $\hat{\tau}_s \in \mathbb{Z}_+$. It is allowed to shift this starting time within some time window $\underline{\tau}_s, \overline{\tau}_s \in \mathbb{Z}_+, \underline{\tau}_s \leq \overline{\tau}_s$. Usually, this time window reflects the legal bounds on the school starting time (7:30 – 8:30 a.m.). Each school has some bus stops (usually, this is exactly one bus stop), where pupils get off the bus and walk the rest of the way to school.

The set $\mathcal{P} \subset \mathcal{S} \times \mathcal{V}$ consists of pairs (s, t), where trip t transports pupils to a bus stop of school s. In this case we say, t is a *school trip* for s (see Figure 2). The number of transported pupils by this trip is $p_{st} \in \mathbb{Z}_+$. The time difference between the departure at the first bus stop of t and the arrival at the bus stop of s is denoted by $\delta_{st}^{\text{school}} \in \mathbb{Z}_+$. There is another time window for the pupils $\underline{\omega}_{st}^{\text{school}}, \overline{\omega}_{st}^{\text{school}} \in \mathbb{Z}_+, \underline{\omega}_{st}^{\text{school}} \leq \overline{\omega}_{st}^{\text{school}}$, specifying the minimal and maximal waiting time relative to the school starting time. The lower bound $\underline{\omega}_{st}^{\text{school}}$ is

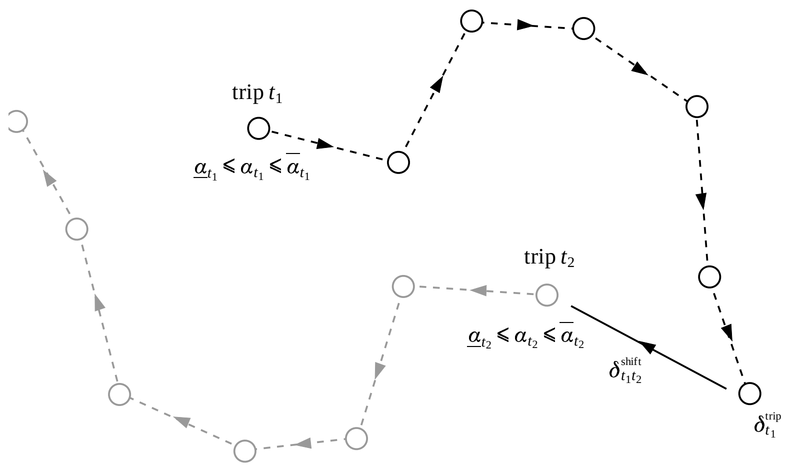

Fig. 1. Two trips t_1, t_2 connected by a deadhead trip.

chosen according to the walking time from the bus stop where the pupils are dropped off, whereas the upper bound $\overline{\omega}^{\text{school}}_{st}$ is due to law restrictions. A typical time window is 5 – 45 minutes.

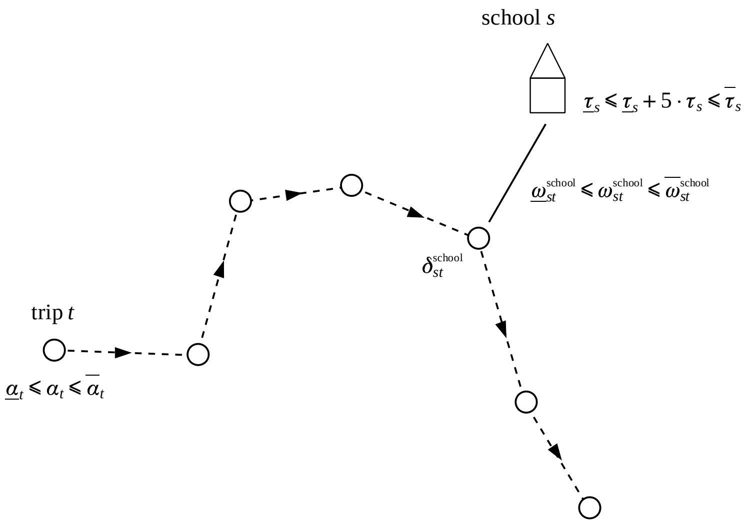

Fig. 2. School trip t for school s.

Let $\mathcal{C} \subset \mathcal{V} \times \mathcal{V}$ be the set of pairs (t_1, t_2), where trip t_1 transports pupils to a so-called *changing bus stop*, where they leave the bus and transfer to trip t_2. We say, t_1 is a *feeder trip* for t_2 and, vice versa, t_2 is a *collector trip* for t_1 (see Figure 3). The number of transferring pupils between t_1 and t_2 is $p_{t_1t_2} \in \mathbb{Z}_+$. The driving time from the first bus stop of feeder trip t_1 to the changing bus stop is denoted by $\delta^{\text{feeder}}_{t_1t_2} \in \mathbb{Z}_+$. For the collector trip, the corresponding parameter is $\delta^{\text{collector}}_{t_1t_2} \in \mathbb{Z}_+$. At the changing bus stop, a time window $\underline{\omega}^{\text{change}}_{t_1t_2}, \overline{\omega}^{\text{change}}_{t_1t_2} \in \mathbb{Z}_+, \underline{\omega}^{\text{change}}_{t_1t_2} \le \overline{\omega}^{\text{change}}_{t_1t_2}$ for the minimal and maximal waiting time is given. Typically, this time window is 0 – 10 minutes.

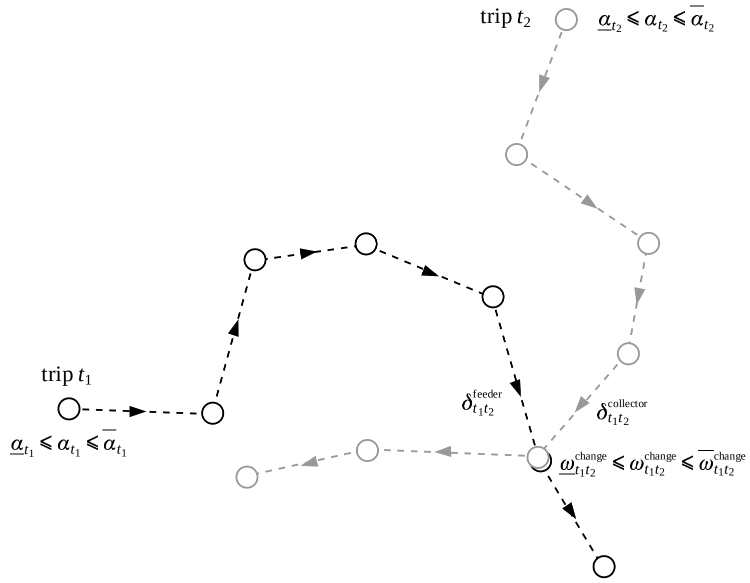

Fig. 3. Two bus trips t_1, t_2 with $(t_1, t_2) \in \mathcal{C}$.

Note that a trip can have more than one type, p.e., it can be both a school and feeder trip.

All trips $t \in \mathcal{V}$ that are not school, feeder or collector trips are called *free trips*. Free trips are obviously not important for the transport of pupils. However, they also have to be served by some bus. The time windows for these trips are usually quite narrow, for example plus/minus 15 minutes relative to the current starting time of the respective trip.

4.2 Variables and Bounds

The model contains binary, integer and continuous variables, see Table 1 for an overview.

For every trip $t \in \mathcal{V}$ the decision variables $v_t, w_t \in \{0, 1\}$ indicate if trip t is the first or the last trip in some block, respectively. For every pair of trips

$(t_1, t_2) \in \mathcal{A}$ the variable $x_{t_1 t_2} \in \{0, 1\}$ indicates if t_1 and t_2 are in sequence in some block, that is, the same bus serves trip t_2 directly after finishing trip t_1.

The school starting time is required to be in discrete time slots of 5 minutes (7:30, 7:35, 7:40, etc.). For every school $s \in \mathcal{S}$ we introduce an integer variable $\tau_s \in \mathbb{Z}_+$ with

$$0 \leq \tau_s \leq \frac{\overline{\tau}_s - \underline{\tau}_s}{5}. \tag{1}$$

Thus the school starting time can be computed as $\underline{\tau}_s + 5 \cdot \tau_s$.

For every trip $t \in \mathcal{V}$ we introduce a continuous variable $\alpha_t \in \mathbb{R}_+$ representing its starting time, i.e., the departure of a bus at the first bus stop. Bounds on these variables are given by the corresponding time windows,

$$\underline{\alpha}_t \leq \alpha_t \leq \overline{\alpha}_t. \tag{2}$$

For every $(s, t) \in \mathcal{P}$ the variable $\omega_{st}^{\text{school}} \in \mathbb{R}_+$ keeps track of the waiting time for the pupils at the school bus stop relative to the school starting time. These variables are also bounded by time windows,

$$\underline{\omega}_{st}^{\text{school}} \leq \omega_{st}^{\text{school}} \leq \overline{\omega}_{st}^{\text{school}}. \tag{3}$$

The waiting time for pupils at the changing bus stop is settled by the variable $\omega_{t_1 t_2}^{\text{change}} \in \mathbb{R}_+$ for every $(t_1, t_2) \in \mathcal{C}$. These variables are bounded from below and above,

$$\underline{\omega}_{t_1 t_2}^{\text{change}} \leq \omega_{t_1 t_2}^{\text{change}} \leq \overline{\omega}_{t_1 t_2}^{\text{change}}. \tag{4}$$

If two trips $(t_1, t_2) \in \mathcal{A}$ are connected by a deadhead trip, then there might be some waiting time for the bus driver before the start of trip t_2. For this, we introduce variables $\omega_{t_1 t_2}^{\text{shift}} \in \mathbb{R}_+$ for all $(t_1, t_2) \in \mathcal{A}$. Finally, the variables Δ_s^{school} for all $s \in \mathcal{S}$ and Δ_t^{trip} for all $t \in \mathcal{V}$ measure the absolute value of the time shift between current and planned school and bus starting time, respectively.

4.3 Constraints

Each trip is served by exactly one bus. It has either a unique predecessor or it is the first one in some block:

$$\sum_{(t_1, t_2) \in \mathcal{A}} x_{t_1 t_2} + v_{t_2} = 1, \quad \forall\, t_2 \in \mathcal{V}. \tag{5}$$

Moreover, it either has a unique successor or it is the last one in some block:

$$\sum_{(t_1, t_2) \in \mathcal{A}} x_{t_1 t_2} + w_{t_1} = 1, \quad \forall\, t_1 \in \mathcal{V}. \tag{6}$$

Table 1. Sets, parameters and variables

$\mathcal{V}$	$\ni t$	bus trips (nodes)
$\mathcal{A}$	$\ni (t_1, t_2)$	connectable trips (arcs)
$\mathcal{S}$	$\ni s$	schools
$\mathcal{P}$	$\ni (s, t)$	school-trip pairings
$\mathcal{C}$	$\ni (t_1, t_2)$	feeder and collector trip pairings
$\hat{\tau}_s$		current school starting time
$\hat{\alpha}_t$		current trip starting time
$\delta_{t_1}^{\text{trip}}$		time for serving entire trip
$\delta_{st}^{\text{school}}$		time for serving trip from start to school
$\delta_{t_1 t_2}^{\text{feeder}}$		time for feeder trip from start to change b.s.
$\delta_{t_1 t_2}^{\text{collector}}$		time for collector trip from start to change b.s.
$\delta_{t_1 t_2}^{\text{shift}}$		time for deadhead trip
$\underline{\tau}_s, \overline{\tau}_s$		bounds on school starting time (lower, upper)
$\underline{\alpha}_t, \overline{\alpha}_t$		bounds on trip starting time
$\underline{\omega}_{st}^{\text{school}}, \overline{\omega}_{st}^{\text{school}}$		bounds on waiting time for pupils at school
$\underline{\omega}_{t_1 t_2}^{\text{change}}, \overline{\omega}_{t_1 t_2}^{\text{change}}$		bounds on waiting time at change bus stop
p_{st}		number of pupils on school bus
$p_{t_1 t_2}$		number of transferring pupils between trips
v_t	$\in \{0, 1\}$	first trip in block
w_t	$\in \{0, 1\}$	last trip in block
$x_{t_1 t_2}$	$\in \{0, 1\}$	connected trips in block
τ_s	$\in \mathbb{Z}_+$	time-slot of school starting time
α_t	$\in \mathbb{R}_+$	trip starting time
$\omega_{st}^{\text{school}}$	$\in \mathbb{R}_+$	waiting time for pupils at school
$\omega_{t_1 t_2}^{\text{change}}$	$\in \mathbb{R}_+$	waiting time for pupils at change bus stop
$\omega_{t_1 t_2}^{\text{shift}}$	$\in \mathbb{R}_+$	waiting time for bus after deadhead trip
Δ_s^{school}	$\in \mathbb{R}_+$	absolute value of school starting time shift
Δ_t^{trip}	$\in \mathbb{R}_+$	absolute value of trip starting time shift

If trips t_1 and t_2 are connected, then trip t_2 can only start after the bus has finished trip t_1 and shifted from the end of t_1 to the start of t_2. Waiting is permitted, if the bus arrives there before the start of t_2. Using a sufficient big value for M, these constraints can be formulated in terms of linear equations:

$$\begin{aligned} &\alpha_{t_1} + \delta_{t_1}^{\text{trip}} + \delta_{t_1 t_2}^{\text{shift}} + \omega_{t_1 t_2}^{\text{shift}} - M \cdot (1 - x_{t_1 t_2}) \le \alpha_{t_2}, \\ &\alpha_{t_1} + \delta_{t_1}^{\text{trip}} + \delta_{t_1 t_2}^{\text{shift}} + \omega_{t_1 t_2}^{\text{shift}} + M \cdot (1 - x_{t_1 t_2}) \ge \alpha_{t_2}, \quad \forall\ (t_1, t_2) \in \mathcal{A}. \end{aligned} \tag{7}$$

For every $(t_1, t_2) \in \mathcal{C}$, the starting times of both bus trips must be synchronized in such way that t_2 arrives at the changing bus stop after trip t_1 within a small time window. This is assured by the following equations:

$$\alpha_{t_1} + \delta_{t_1 t_2}^{\text{feeder}} + \omega_{t_1 t_2}^{\text{change}} = \alpha_{t_2} + \delta_{t_1 t_2}^{\text{collector}}, \quad \forall\ (t_1, t_2) \in \mathcal{C}. \tag{8}$$

For every $(s, t) \in \mathcal{P}$, the starting times of trip t and school s have to be chosen such that the waiting time restrictions for the pupils at school s are met. Thus, we add the following equations to the model in order to synchronize the start of bus trips and schools:

$$\alpha_t + \delta_{st}^{\text{school}} + \omega_{st}^{\text{school}} = \underline{\tau}_s + 5 \cdot \tau_s, \quad \forall\ (s,t) \in \mathcal{P}. \tag{9}$$

The relative time shifts of school and trip starting times are transformed into absolute values by the following inequalities: For the schools

$$\begin{aligned} \hat{\tau}_s - (\underline{\tau}_s + 5 \cdot \tau_s) &\leq \Delta_s^{\text{school}}, \\ (\underline{\tau}_s + 5 \cdot \tau_s) - \hat{\tau}_s &\leq \Delta_s^{\text{school}}, \quad \forall\ s \in \mathcal{S}, \end{aligned} \tag{10}$$

and for the bus trips

$$\begin{aligned} \hat{\alpha}_t - \alpha_t &\leq \Delta_t^{\text{trip}}, \\ \alpha_t - \hat{\alpha}_t &\leq \Delta_t^{\text{trip}}, \quad \forall\ t \in \mathcal{V}. \end{aligned} \tag{11}$$

For abbreviation we write $\mathcal{X}$ for the solution space, that is the set of points

$$\begin{aligned} \mathcal{X} := \{(&v, w, x, \tau, \alpha, \omega^{\text{shift}}, \omega^{\text{school}}, \omega^{\text{change}}, \Delta^{\text{school}}, \Delta^{\text{trip}}) : \\ &(1) - (4), (5) - (11) \text{ are fulfilled}, \\ &v, w, x \text{ binary}, \tau \text{ integer}, \alpha \text{ continuous}\}. \end{aligned}$$

4.4 Objective Functions

As indicated in Section 2, there are seven targets for the optimization:
The total number of deployed vehicles:

$$f_1(v) := \sum_{t \in \mathcal{V}} v_t,$$

the driving times of all deadhead trips:

$$f_2(x) := \sum_{(t_1,t_2) \in \mathcal{A}} \delta_{t_1 t_2}^{\text{shift}} \cdot x_{t_1 t_2},$$

the waiting times of buses between two trips, after the deadhead trip:

$$f_3(\omega^{\text{shift}}) := \sum_{(t_1,t_2) \in \mathcal{A}} \omega_{t_1 t_2}^{\text{shift}},$$

the waiting times for pupils at their schools:

$$f_4(\omega^{\text{school}}) := \sum_{(s,t) \in \mathcal{P}} p_{st} \cdot \omega_{st}^{\text{school}},$$

the waiting times at the transfer bus stops:

$$f_5(\omega^{\text{change}}) := \sum_{(t_1,t_2) \in \mathcal{C}} p_{t_1 t_2} \cdot \omega_{t_1 t_2}^{\text{change}},$$

the absolute change of the school starting times:

$$f_6(\Delta^{\text{school}}) := \sum_{s \in \mathcal{S}} \Delta_s^{\text{school}},$$

and the absolute change of the starting times of the bus trips:

$$f_7(\Delta^{\text{trip}}) := \sum_{t \in \mathcal{V}} \Delta_t^{\text{trip}}.$$

Following the notation of [10], our optimization problem can be stated as follows:

$$\begin{array}{rl} \min & (f_1, f_2, f_3, f_4, f_5, f_6, f_7) \\ \text{subject to} & (v, w, x, \tau, \alpha, \omega^{\text{shift}}, \omega^{\text{school}}, \omega^{\text{change}}, \Delta^{\text{school}}, \Delta^{\text{trip}}) \in \mathcal{X}. \end{array} \tag{12}$$

5 Two-Stage Decomposition of the Model

From a theoretical point of view, we remark that (12) is NP-hard. Checking feasibility with a fixed number of vehicles is NP-complete, for it is a special case of the mTSP-TW with static time windows, which is known to be NP-complete, see [13]. Thus we cannot expect a polynomial algorithm for its solution, unless $P = NP$. In order to obtain good feasible solutions, we now describe a (heuristic) decomposition into two parts. For this, it is necessary to take a closer look at the objective functions.

According to our project partner BPI, the most important goal is the reduction of the number of buses f_1, and second, their efficient deployment f_2. These two goals are lexicographically ordered. All other goals $f_3, \ldots, f_7$ can be treated on a subordinate level. Thus, we split (12) into a two-stage optimization problem, where in stage one a schedule of buses is computed such that preferably few buses are in use and the total length of all deadhead-trips is minimal. At this stage, it is not necessary to compute starting times for the schools and the trips. It only has to be assured that the schedules of the buses are feasible in such way that the starting times can be computed. Their actual computation according to the subordinate goals is then done in stage two, where the schedules from stage one are taken as fixed input values.

5.1 Stage One: Bus Scheduling

In stage one, the bus scheduling problem is addressed. The crucial point is that a schedule (v, w, x) is only feasible, if it can be extended to a solution vector $(v, w, x, \tau, \alpha, \omega^{\text{shift}}, \omega^{\text{school}}, \omega^{\text{change}}, \Delta^{\text{school}}, \Delta^{\text{trip}}) \in \mathcal{X}$. That is, we seek for a solution of

$$\begin{array}{rl} \text{lexmin} & (f_1, f_2) \\ \text{subject to} & (5), (6) \\ & (v, w, x) \in \{0,1\}^{\mathcal{V} \times \mathcal{V} \times \mathcal{A}} \\ & \exists\ (v, w, x, \tau, \alpha, \omega^{\text{shift}}, \omega^{\text{school}}, \omega^{\text{change}}, \Delta^{\text{school}}, \Delta^{\text{trip}}) \in \mathcal{X}. \end{array} \tag{13}$$

This lexmin-problem can be turned into a single-objective mixed-integer programming problem by scaling f_1 by a large value such that f_1 dominates f_2 for all feasible solutions. Among the currently most successful methods for solving mixed-integer programming problems are linear programming based branch-and-bound algorithms, where the underlying linear programming relaxations are possibly strengthened by cutting planes. Unfortunately, todays state-of-the-art commercial MIP-solvers (such as CPLEX [1]) are not able to handle (13) as mixed-integer program for problem instances of realistic size. Computing a solution together with a proof of optimality in reasonable time is out of reach, see Table 4. Thus, we solve problem (13) by the following greedy-type heuristic, see Table 2 for an overview.

Consider the graph $(\mathcal{V}, \mathcal{A})$. In each step of this heuristic, a local-best deadhead-trip from $\mathcal{A}$ is selected. In the case of the classical multiple traveling salesman problem with fixed time windows (mTSP-TW), a greedy strategy might, for example, select an arc $(t_1^*, t_2^*) \in \mathcal{A}$ with minimal distance $\delta^{\text{shift}}_{t_1 t_2}$, i.e.,

$$(t_1^*, t_2^*) = \operatorname{argmin}\{\delta^{\text{shift}}_{t_1 t_2} : (t_1, t_2) \in \mathcal{A}\}, \tag{14}$$

provided that the time windows are not violated, meaning $\underline{\alpha}_{t_1} + \delta^{\text{trip}}_{t_1} + \delta^{\text{shift}}_{t_1 t_2} \leq \overline{\alpha}_{t_2}$. In case of (13), this strategy leads to poor solutions (with a high number of deployed buses, see Table 5). Here, each possible connection of trips $(t_1, t_2) \in \mathcal{A}$ needs a score that does not only take into account the time for the deadhead trip, but also takes care of the necessary changes in the corresponding time windows. If the time windows are narrowed too early in the course of the algorithms, the number of deployed buses quickly increases, for no "flexibility" remains. Thus, we introduce a scoring function that prefers those connections that do not (or at least not too much) change time windows of other trips or schools. For this, we define

$$\begin{aligned} s_{t_1 t_2} := \quad & \delta^{\text{shift}}_{t_1 t_2} & (15)\\ & + |\underline{\alpha}_{t_1} + \delta^{\text{trip}}_{t_1} + \delta^{\text{shift}}_{t_1 t_2} - \underline{\alpha}_{t_2}| & (16)\\ & + |\overline{\alpha}_{t_1} + \delta^{\text{trip}}_{t_1} + \delta^{\text{shift}}_{t_1 t_2} - \overline{\alpha}_{t_2}|, \quad \forall\ (t_1, t_2) \in \mathcal{A}. & (17) \end{aligned}$$

There is no contribution of (16) and (17) to the score $s_{t_1 t_2}$ if and only if the time windows of trips t_1 and t_2 perfectly coincide in the sense that the time window for trip t_1 at the first bus stop of trip t_2 equals the starting time window of trip t_2, i.e., $\underline{\alpha}_{t_1} + \delta^{\text{trip}}_{t_1} + \delta^{\text{shift}}_{t_1 t_2} = \underline{\alpha}_{t_2}$, and $\overline{\alpha}_{t_1} + \delta^{\text{trip}}_{t_1} + \delta^{\text{shift}}_{t_1 t_2} = \overline{\alpha}_{t_2}$. After selecting

$$(t_1^*, t_2^*) = \operatorname{argmin}\{s_{t_1 t_2} : (t_1, t_2) \in \mathcal{A}\}, \tag{18}$$

setting $x_{t_1^* t_2^*} := 1$, and removing all arcs from $\mathcal{A}$ that would lead to cycles, the time windows for the school and trip starting times are updated. School and trip starting times are coupled by the minimum and maximum waiting time restriction of inequalities (9). The time window for the starting time of school s can be propagated onto the starting time window of trip t for all $(s,t) \in \mathcal{P}$. From (9) we obtain

$$\alpha_t \le \tau_s - \delta_{st}^{\text{school}} - \underline{\omega}_{st}^{\text{school}} \le \overline{\tau}_s - \delta_{st}^{\text{school}} - \underline{\omega}_{st}^{\text{school}}. \tag{19}$$

We now compare the right-hand side of (19) with the previous upper bound $\overline{\alpha}_t$ on α_t. If it is less, a new upper bound is found. In general, we set:

$$\overline{\alpha}_t := \min \{\overline{\alpha}_t, \overline{\tau}_s - \delta_{st}^{\text{school}} - \underline{\omega}_{st}^{\text{school}}\}, \quad \forall\ (s,t) \in \mathcal{P}. \tag{20}$$

In the same way, an improved lower bound can be derived from (9):

$$\underline{\alpha}_t := \max \{\underline{\alpha}_t, \underline{\tau}_s - \delta_{st}^{\text{school}} - \overline{\omega}_{st}^{\text{school}}\}, \quad \forall\ (s,t) \in \mathcal{P}. \tag{21}$$

Vice versa, the trip time window can be propagated onto the school time window by (9). Since school starts are required to be in discrete 5-minute time slots, rounding has an additional influence on the time windows. With $\lceil a \rceil_b := \lceil \frac{a}{b} \rceil \cdot b$, $\lfloor a \rfloor_b := \lfloor \frac{a}{b} \rfloor \cdot b$ for $a, b \in \mathbb{R}, b \neq 0$, we obtain:

$$\begin{aligned} \underline{\tau}_s &:= \max \{\underline{\tau}_s, \lceil \underline{\alpha}_t + \delta_{st}^{\text{school}} + \underline{\omega}_{st}^{\text{school}} \rceil_5\}, \\ \overline{\tau}_s &:= \min \{\overline{\tau}_s, \lfloor \overline{\alpha}_t + \delta_{st}^{\text{school}} + \overline{\omega}_{st}^{\text{school}} \rfloor_5\}, \quad \forall\ (s,t) \in \mathcal{P}. \end{aligned} \tag{22}$$

The same idea can be applied for the time windows of feeder and collector trips, which are coupled by (8). Then the improved bounds are given by:

$$\begin{aligned} \overline{\alpha}_{t_1} &:= \min \{\overline{\alpha}_{t_1}, \overline{\alpha}_{t_2} + \delta_{t_1t_2}^{\text{collector}} - \delta_{t_1t_2}^{\text{feeder}} - \underline{\omega}_{t_1t_2}^{\text{change}}\}, \\ \underline{\alpha}_{t_1} &:= \max \{\underline{\alpha}_{t_1}, \underline{\alpha}_{t_2} + \delta_{t_1t_2}^{\text{collector}} - \delta_{t_1t_2}^{\text{feeder}} - \overline{\omega}_{t_1t_2}^{\text{change}}\}, \\ \overline{\alpha}_{t_2} &:= \min \{\overline{\alpha}_{t_2}, \overline{\alpha}_{t_1} + \delta_{t_1t_2}^{\text{feeder}} + \overline{\omega}_{t_1t_2}^{\text{change}} - \delta_{t_1t_2}^{\text{collector}}\}, \\ \underline{\alpha}_{t_2} &:= \max \{\underline{\alpha}_{t_2}, \underline{\alpha}_{t_1} + \delta_{t_1t_2}^{\text{feeder}} + \underline{\omega}_{t_1t_2}^{\text{change}} - \delta_{t_1t_2}^{\text{collector}}\}, \quad \forall\ (t_1,t_2) \in \mathcal{C}. \end{aligned} \tag{23}$$

If some trips are connected in a schedule by the heuristic, then we obtain further constraints on the trip time windows:

$$\begin{aligned} \overline{\alpha}_{t_1} &:= \min \{\overline{\alpha}_{t_1}, \overline{\alpha}_{t_2} - \delta_{t_1}^{\text{trip}} - \delta_{t_1t_2}^{\text{shift}}\}, \\ \underline{\alpha}_{t_2} &:= \max \{\underline{\alpha}_{t_2}, \underline{\alpha}_{t_1} + \delta_{t_1}^{\text{trip}} + \delta_{t_1t_2}^{\text{shift}}\}, \quad \forall\ (t_1,t_2) \in \mathcal{A}, x_{t_1t_2} = 1. \end{aligned} \tag{24}$$

Preprocessing steps (20) – (24) are iteratively repeated for all $(s,t) \in \mathcal{P}$, all $(t_1,t_2) \in \mathcal{C}$, and all $(t_1,t_2) \in \mathcal{A}$ with $x_{t_1t_2} = 1$, respectively, until no bound is improved any more.

Moreover, it is possible to fix some decision variables to their respective bounds. Consider some deadhead trip $(t_1,t_2) \in \mathcal{A}$. If the upper bound $\overline{\alpha}_{t_2}$ is tightened such that trip t_2 cannot be connected with trip t_1, even if starting t_1 as early as possible, then the corresponding arc can be removed from the graph, and the corresponding decision variable is fixed to zero, applying inequality (7):

$$\forall\ (t_1,t_2) \in \mathcal{A} : \underline{\alpha}_{t_1} + \delta_{t_1}^{\text{trip}} + \delta_{t_1t_2}^{\text{shift}} > \overline{\alpha}_{t_2} \ \Rightarrow\ x_{t_1t_2} = 0. \tag{25}$$

Consider a trip $t_2 \in \mathcal{V}$. If we can fix $x_{t_1 t_2} = 0$ for all preceding trips $t_1 \in \mathcal{V}$ with $(t_1, t_2) \in \mathcal{A}$ by (25), then the upper bound on the starting time of trip t_2 was too low for a bus to serve trip t_2 after having already served any other trip. Hence, a new bus from the depot is required to serve trip t_2, i.e., t_2 is the first trip in some block, and v_{t_2} can be fixed to 1 by (5). Vice versa, consider a trip $t_1 \in \mathcal{V}$ where we fixed $x_{t_1 t_2} = 0$ by (25) for all succeeding trips t_2 with $(t_1, t_2) \in \mathcal{A}$, then the lower bound $\underline{\alpha}_{t_1}$ on the starting time of trip t_1 was too high so that a bus cannot serve any other trip after serving this trip. Hence, the bus is sent back to the depot, i.e., trip t_1 is the last trip in some block. Thus, w_{t_1} can be fixed to 1 by (6).

The above steps are repeated iteratively, until all binary variables (v, w, x) are either fixed to zero or one. Then, we end up with a schedule for all buses and strengthened bounds on the time variables, which is now input for the second stage of the optimization.

Table 2. Greedy algorithm for stage one

(1) Input: An instance $\mathcal{I}$
(2) Update time windows, fix variables, remove infeasible arcs in $\mathcal{A}$
(3) **Repeat** the following steps
(4) Compute best shift $(t_1^*, t_2^*) = \text{argmin}\{s_{t_1 t_2} : (t_1, t_2) \in \mathcal{A}\}$
(5) Set $x_{t_1^* t_2^*} = 1$
(6) Update time windows, fix variables, remove infeasible arcs in $\mathcal{A}$
(7) **Until** no more free arcs in $\mathcal{A}$
(8) Output: Feasible solution $(\mathbf{v}, \mathbf{w}, \mathbf{x})$, new time windows for α, τ

5.2 Stage Two: Starting Time Computation

The goal of stage two is the computation of school and trip starting times, given the bus schedules from stage one. During the computations in stage one, the time windows have been significantly narrowed. If a time window shrinks to a single point, then the corresponding variable, α_t or τ_s, can be fixed to this value. However, for the majority of all time windows, there is still some flexibility, see Table 6. In contrast to stage one, where the objectives f_1, f_2 are lexicographically ordered, there is no obvious ordering among the objectives $f_3, \ldots, f_7$ to solve the remaining problem, i.e.,

$$\begin{aligned} &\min\ (f_3, f_4, f_5, f_6, f_7) \\ &\text{subject to } (\mathbf{v}, \mathbf{w}, \mathbf{x}, \tau, \alpha, \omega^{\text{shift}}, \omega^{\text{school}}, \omega^{\text{change}}, \Delta^{\text{school}}, \Delta^{\text{trip}}) \in \mathcal{X}', \end{aligned} \tag{26}$$

where $\mathbf{v}, \mathbf{w}, \mathbf{x}$ are the fixed variables corresponding to the bus schedule computed in stage one, and $\mathcal{X}' \subseteq \mathcal{X}$ is the remaining set of solutions, due to the narrowed time windows. We can transform (26) into the a single-objective, parametric mixed-integer program by the well-known weighted sum scalarization approach (see [10], for instance). A scalar vector $(\lambda_3, \dots, \lambda_7) \geq 0$, $\sum_{i=3}^{7} \lambda_i = 1$ is used to replace the multiple objectives by the single objective function

$$\begin{aligned} &\min \textstyle\sum_{i=3}^{7} \lambda_i \cdot f_i \\ &\text{subject to } (\mathbf{v}, \mathbf{w}, \mathbf{x}, \tau, \alpha, \omega^{\text{shift}}, \omega^{\text{school}}, \omega^{\text{change}}, \Delta^{\text{school}}, \Delta^{\text{trip}}) \in \mathcal{X}', \end{aligned} \tag{27}$$

which can be solved to optimality by a MIP solver within a few seconds. However, it is necessary to select appropriate weights $(\lambda_3, \dots, \lambda_7)$ that lead to good solutions. To support this selection process, we first examine the solution space of stage two by computing lower and upper bounds $\underline{\varepsilon}_i, \overline{\varepsilon}_i$ on each objective f_i for $i = 3, \dots, 7$. For this, we solve a sequence of problems of type (27), where the negative resp. positive unit vectors are taken as weights in the objective function. That means, the minimization resp. maximization of only one goal is kept as objective function, the other goals are neglected, see Table 7. Thereafter, it is possible to compare each single objective value in the solution of (27) for a given set of weights $(\lambda_3, \dots, \lambda_7)$ with its respective lower and upper bounds. The weights are adjusted until an appropriate solution is found, which then has to be implemented in real-world, see Table 8.

6 Computational Results

Real-world data for the model was provided by BPI. The three test instances are from counties where manually generated solutions were implemented in the last years. The data of instance Example 1 is taken from a county situated in Mecklenburg-Vorpommern, in the north-east of Germany. The counties of Example 2 and Example 3 are both located in Nordrhein-Westfalen, in the mid-west of Germany. From each county, only the trips and the schools (i.e., sets $\mathcal{V}$ and $\mathcal{S}$) have to be given explicitly. All other data is generated automatically, see Table 3 for the sizes of the instances. If (13) is transformed

Table 3. Size of sets for the three test instances

Set	$\mathcal{V}$	$\mathcal{A}$	$\mathcal{S}$	$\mathcal{P}$	$\mathcal{C}$
Ex. 1	247	60,762	43	195	165
Ex. 2	490	239,610	102	574	406
Ex. 3	192	36,672	76	263	133

into a single-objective mixed integer program (where we take $10000 \cdot f_1 + f_2$ as objective function), optimal solutions can be computed using CPLEX [1]. We use CPLEX8.1 with default parameter settings on a 2.6GHz Intel Pentium-IV with a given time limit of 3,600 sec. (1 hour). Table 4 shows the best solutions found and the gap between lower and upper bound. The large gaps in each of the three cases are due to the poor quality of the linear programming relaxation of the mixed-integer model. For instance, in Example 1 the lower bound equals the deployment of only 16 buses, whereas the upper bound (best feasible solution found by CPLEX within the time limit) uses 74 of them. The

Table 4. Computational results with CPLEX8.1

	upper bound			lower bound	gap
	buses	deadhead trips	objective	objective	
Ex. 1	74	10:17	740,617	161,561	78.19%
Ex. 2	191	21:38	1,911,298	183,232	90.41%
Ex. 3	87	10:33	870,633	99,167	88.61%

results of the heuristic used in stage one are depicted in Table 5. Different results are obtained for the two presented scoring functions; "greedy_1" uses selection rule (14), whereas "greedy_2" uses strategy (18). The running time for the heuristics is very low for all examples. The solutions found in stage one reduce the number of deployed buses by 11–15% (compared the the current schedules used by the respective bus companies). Moreover, in all three cases the heuristic solutions are better than the solutions found by the MIP solver. Also note that the reduced number of buses leads to an increase in the length of the deadhead trips, for the buses are now deployed in a more efficient way. Table 6 summarizes the size of the time windows before the optimization (in

Table 5. Computational results from stage one

	current schedules		greedy_1		greedy_2		time
	buses	deadhead trips	buses	deadhead trips	buses	deadhead trips	
Ex. 1	82	7:03	82	5:45	73	11:08	2 sec.
Ex. 2	226	7:21	199	15:00	190	19:30	4 sec.
Ex. 3	96	7:21	86	9:07	82	11:55	2 sec.

the input data), and after stage one of the optimization. If a time window shrinks to a single value, then the corresponding variable can be fixed. The number of free variables yields some information about the degrees freedom left for stage two; compared with Table 3, the majority of all variables is not fixed. The solution space $\mathcal{X}'$ is smaller than the solution space $\mathcal{X}$ due to the decreased time windows. For the weighted sum approach for stage two, it is helpful to know between which bounds the objectives concerning the starting times (i.e., $f_3, \dots, f_7$) can be. The upper and lower bounds $\underline{\varepsilon}_i, \overline{\varepsilon}_i$ on f_i for each $i \in \{3, 4, 5, 6, 7\}$ are shown in Table 7. Our finally chosen solutions within these bounds uses $\lambda_i = 0.2$ for all $i = 3, \dots, 7$, i.e., all goals are equally weighted. The objective values for each goal are listed in the column "after" in Table 8. Compared to the current waiting times (in the columns "before"

Table 6. Size of time windows before and after stage one

	τ			α			ω^{school}			ω^{change}		
	before	after	free	before	after	free	before	after	free	before	after	free
Ex. 1	36:45	15:50	43	199:41	112:44	247	51:40	47:11	195	27:30	27:30	165
Ex. 2	79:50	14:30	68	356:14	143:49	465	141:14	69:27	526	67:40	40:45	379
Ex. 3	67:25	25:00	70	171:30	57:19	192	64:15	51:55	263	22:10	20:52	133

Table 7. Lower and upper bounds on the goals in stage two

	$\underline{\varepsilon}_3$	$\overline{\varepsilon}_3$	$\underline{\varepsilon}_4$	$\overline{\varepsilon}_4$	$\underline{\varepsilon}_5$	$\overline{\varepsilon}_5$	$\underline{\varepsilon}_6$	$\overline{\varepsilon}_6$	$\underline{\varepsilon}_7$	$\overline{\varepsilon}_7$
Ex. 1	0:13	38:35	35:37	63:40	3:18	18:37	1:40	15:55	10:19	78:26
Ex. 2	1:59	55:37	115:13	165:17	12:36	33:35	27:50	40:25	86:37	177:46
Ex. 3	2:11	24:41	38:47	80:26	1:41	17:39	8:50	26:20	19:03	59:50

in Table 8), it is interesting to see, that the quality of service is improved significantly. For example, the sum of all waiting times for buses after deadheading is reduced by 42–72%, and the sum of all waiting times for pupils at their schools (goal f_4) is reduced by 25–40%. Moreover, comparing these solutions with the bounds given in Table 7, one can see that we are closer to the lower than to the respective upper bounds.

Table 8. Results of stage two

	f_3		f_4		f_5		f_6		f_7	
	before	after	before	after	before	after	before	after	before	after
Ex. 1	32:45	10:57	59:10	41:45	5:01	5:00	–	2:50	–	14:47
Ex. 2	29:21	8:37	159:10	121:19	13:17	16:12	–	28:05	–	99:19
Ex. 3	9:27	5:32	72:53	43:03	5:04	4:03	–	11:45	–	25:48

7 Conclusions and Future Research

In this article, we presented a multi-objective linear mixed-integer model for the integrated optimization of bus schedules and school starting times. For its solution, we proposed a two-stage heuristic. This heuristic was designed to become part of a software tool that supports the planning part of BPI's work. We applied it to data sets from three German counties, where for each of them a significant reduction in the number of deployed buses could be achieved. Moreover, it was possible to increase the quality-of-service, in terms of a reduced waiting time for the pupils at their schools.

Our focus for future research lies on a reduction of the solution time for the mixed-integer programming problem. The inclusion of high-dimensional valid inequalities in a branch-and-cut framework or a column-generation approach are promising directions.

References

1. ILOG CPLEX Division, 889 Alder Avenue, Suite 200, Incline Village, NV 89451, USA. Information available at URL http://www.cplex.com.
2. Schulgesetz Mecklenburg-Vorpommern vom 15.05.1996. Stand Feb. 2000, 68. (In German).
3. Satzung über die Anerkennung der notwendigen Kosten für die Schülerbeförderung des Landkreises Demmin. Aus: Kreisanzeiger des Landkreises Demmin, Nr. 6/97. Amtliche Mitteilungen und Informationen, 7 – 8. (In German).
4. Hinweise zur Schulorganisation für allgemein bildende Schulen. Verwaltungsvorschrift des Ministeriums für Bildung, Wissenschaft und Kultur vom 21. Juli 2000. Aus: Mitteilungsblatt des Ministeriums für Bildung, Wissenschaft und Kultur Mecklenburg-Vorpommern, Nr. 8/2000, 362 – 365. (In German).
5. Bodin L., Berman L. (1979), Routing and Scheduling of School Buses by Computer. Transportation Science **13** (2), 113 – 129.
6. Bowerman R.L., Hall G.B., Calamai P.H. (1995), A Multi-Objective Optimisation Approach to School Bus Routing Problems. Transp. Research A **28** (5), 107 – 123.
7. Braca J., Bramel J., Posner B., Simchi-Levi D. (1997), A Computerized Approach to the New York City School Bus Routing Problem. IIE Transactions **29**, 693 – 702.
8. Corberan A., Fernandez E., Laguna M., Marti R. (2000), Heuristic Solutions to the Problem of Routing School Buses with Multiple Objectives. Technical Report TR08-2000, Dep. of Statistics and OR, University of Valencia, Spain.
9. Desaulniers G., Lavigne J., Soumis F. (1998), Multi-Depot Vehicle Scheduling with Time Windows and Waiting Costs. Europ. J. of Op. Res. **111**, 479 – 494.
10. Ehrgott M. (2000), Multicriteria Optimization. Springer Verlag, Heidelberg.
11. Ferland J.A., Fortin L. (1989), Vehicles Scheduling with Sliding Time Windows. European Journal of Operational Research **38** (2), 213 – 226.
12. Löbel A. (1997), Optimal Vehicle Scheduling in Public Transit. Shaker Verlag, Aachen.
13. Savelsbergh M. (1986), Local Search for Routing Problems with Time Windows. Annals of Operations Research **4**, 285 – 305.
14. Stöveken P. (2000), Wirtschaftlicherer Schulverkehr: ÖPNV-Optimierung mit erfolgsabhängiger Honorierung. Der Nahverkehr **3**, 65 – 68. (In German).
15. Toth P., Vigo D. (2002), The Vehicle Routing Problem. SIAM Monographs on Discrete Mathematics and Applications. SIAM, Philadelphia.