Integral representations of positive functionals

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Abstract

For positive linear functionals on complex commutative *-algebras, we prove abstract Bochner and Plancherel Theorems without any hypothesis of non-degeneracy. A central positive functional on a *-algebra is decomposed as the sum of a non-degenerate and a totally degenerate positive linear functional by relating the non-degenerate part to the natural trace of an associated Hilbert algebra.[1] [2]

Let ω be a positive linear functional on a complex commutative Banach *-algebra with unit. The Abstract Bochner Theorem yields a unique regular Borel measure μ on the involutive part $\hat{A}$ of the Gelfand spectrum of A such that $\omega(a) = \int \hat{a}\, d\mu$ holds for each $a \in A$. The Abstract Plancherel Theorem states that the natural representation of A on $L^2(\mu)$ by multiplication of functions is unitarily equivalent to the representation associated with ω by the Gelfand–Naĭmark–Segal construction.

One would like to have such theorems under more general hypotheses. For example, let A be the convolution algebra of continuous functions on a non-discrete locally compact abelian group, and let ω be evaluation at the unit element. Sufficiently general theorems for this situation are proved by Fell and Doran [6, VI.21.4 and VI.21.6]. They assume that A is a dense *-subalgebra of a commutative hermitian Banach *-algebra (which need not be unital), that the positive linear functional ω on A satisfies a certain boundedness condition, and that the Gelfand–Naĭmark–Segal representation associated with ω is non-degenerate. Their results are special cases of the content of the first part of this article. It turns out that the topology of A is irrelevant. In fact, it seems more transparent to develop the theory for a positive linear functional ω on an abstract algebra A. If A is unital then one can still factor ω over an associated C*-algebra, so that one can

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apply the above results for commutative A, and Choquet Theory for non-commutative A. In the case of a non-unital but commutative algebra, we use a weaker boundedness condition than Fell and Doran which is not only sufficient but also necessary, and we drop the hypothesis of non-degeneracy. The treatment of possibly degenerate positive functionals has the important consequence that such a functional can be written as the sum of a non-degenerate and a totally degenerate part. By a different technique, we also establish this decomposition for central positive linear functionals on non-commutative algebras. The original motivation for an integral representation of a possibly degenerate positive linear functional on a non-unital algebra is related to the unitary representation theory of infinite-dimensional Lie groups of Harish–Chandra type [1].

Here is an overview of the content of this article. Let ω be a positive linear functional on a complex commutative *-algebra A, and let μ be a compactly supported Borel measure on the involutive spectrum $\hat{A}$. We say that μ represents ω if all $a, b, c \in A$ satisfy $\omega(abc) = \int \hat{a}\hat{b}\hat{c}\, d\mu$. Examples show that this is the best kind of representation which one can achieve for general ω. If a representing measure exists then ω is exponentially bounded, which means that A acts by bounded operators on the Gelfand–Naĭmark–Segal Hilbert space associated with ω. This condition is assumed throughout. Among the general properties of a representing measure μ which we prove, the most important is that the closed support of μ is a locally compact subset $\hat{A}(\sigma_\omega) \subseteq \hat{A}$ which is uniquely determined by ω and can easily be described in terms of ω. Conversely, the existence of a representing measure μ_ω on $\hat{A}(\sigma_\omega)$ can be deduced from the Riesz Representation Theorem for positive linear functionals $C_c(\hat{A}(\sigma_\omega))$, and under the uniqueness conditions appearing in that theorem, the representing measure μ_ω is unique. The positive linear functional ω admits an extension to the unitization of A if and only if μ_ω is finite and satisfies $\omega(a) = \int \hat{a}\, d\mu_\omega$ for all $a \in A$. Similarly, the equation $\omega(ab) = \int \hat{a}\hat{b}\, d\mu_\omega$ holds for all $a, b \in A$ if and only if the Gelfand–Naĭmark–Segal representation associated with ω is non-degenerate. In this case, we say that ω is non-degenerate. For general ω, the non-degenerate part of this representation is isomorphic to the representation of A on $L^2(\mu_\omega)$ by $a.f := \hat{a} \cdot f$. This is our general version of the Abstract Plancherel Theorem.

In the second part of this article, we replace the hypothesis that the *-algebra A is commutative by the weaker assumption that the exponentially bounded positive linear functional ω on A is central, which means that $\omega(ab) = \omega(ba)$ holds for all $a, b \in A$. This assumption is sufficient for the construction of a Hilbert algebra structure on the image of A under the Gelfand–Naĭmark–Segal representation. The natural trace on that Hilbert algebra leads to a non-degenerate positive linear functional ω_1 on A such that $\omega_0 := \omega - \omega_1$ is a totally degenerate positive linear functional. The decomposition $\omega = \omega_0 + \omega_1$ into a totally degenerate and a non-degenerate part is essentially unique (i.e. it is unique on the linear span of AA). This decomposition of ω corresponds to a direct sum decomposition of the Gelfand–Naĭmark–Segal module. The construction of a Hilbert algebra from ω also leads to a factorization of ω through a homomorphism from A into a hermitian Banach *-algebra.

If a generalization of the integral representation developed in the first part to central positive linear functionals ω is possible, it will probably be based on the Hilbert algebra associated with ω. We conclude this paper with a first step in this direction. Assume that the *-algebra A has countable dimension or is a separable Banach *-algebra. Then the natural trace mentioned above is defined on the positive cone of a separable C*-algebra. For such traces, Dixmier [5] has constructed a decomposition as an integral over the quasi-spectrum of the C*-algebra. We describe his construction and show that it specializes to the results in the first part if A is commutative. However, it must be emphasized that Dixmier's theory is intimately connected with the assumption of separability.

1 Hilbert spaces with reproducing kernel

Let X be a set. A *positive definite (complex-valued) kernel on* X is a function $K\colon X \times X \to \mathbb{C}$ such that for all finite sequences $x_1, \ldots, x_n \in X$, the matrix $\big(K(x_k, x_j)\big)_{j,k}$ is positive semi-definite. Since positive semi-definite matrices are hermitian, the relation $K(y,x) = \overline{K(x,y)}$ holds for all $x, y \in X$.

Similarly, one can define positive definite kernels on X with values in $B(V)$ for a Hilbert space V. For this generalization, the reader is referred to Neeb [7]. The following results on positive definite kernels have been specialized from Section I.1 of that monograph, which ends with historical comments.

We write $\mathbb{C}^X$ for the complex vector space of all complex-valued functions on X, and $\mathbb{C}^{(X)}$ for the subspace of all elements of $\mathbb{C}^X$ with finite support. Recall that a subset of a topological vector space is called *total* if its linear span is dense.

1.1 Lemma (Associated kernels). *Let X be a set, and let $\mathcal{H} \subseteq \mathbb{C}^X$ be a vector subspace with a Hilbert space structure such that the point evaluation $f \mapsto f(x)\colon \mathcal{H} \to \mathbb{C}$ is continuous for every $x \in X$. Since this is a linear functional on $\mathcal{H}$, every x gives rise to a unique vector $K^{\mathcal{H}}_x \in \mathcal{H}$ such that all $f \in \mathcal{H}$ satisfy $f(x) = \langle f, K^{\mathcal{H}}_x \rangle$. The map*

$$K^{\mathcal{H}}\colon X \times X \longrightarrow \mathbb{C}, \quad (y,x) \longmapsto \langle K^{\mathcal{H}}_x, K^{\mathcal{H}}_y \rangle$$

is a positive definite kernel, and it satisfies $K^{\mathcal{H}}(\cdot, x) = K^{\mathcal{H}}_x$ for every $x \in X$. It is called the reproducing kernel of the Hilbert space $\mathcal{H}$.

Proof. The equation $K^{\mathcal{H}}(y,x) = \langle K_x^{\mathcal{H}}, K_y^{\mathcal{H}}\rangle = K_x^{\mathcal{H}}(y)$ for $x, y \in X$ follows from the definitions. If $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ then

$$\sum_{j,k=1}^{n} \lambda_j\, K^{\mathcal{H}}(x_k, x_j)\, \overline{\lambda_k} = \sum_{j,k=1}^{n} \lambda_j\, \langle K_{x_j}^{\mathcal{H}}, K_{x_k}^{\mathcal{H}}\rangle\, \overline{\lambda_k}$$
$$= \left\langle \sum_{j=1}^{n} \lambda_j K_{x_j}^{\mathcal{H}}, \sum_{k=1}^{n} \lambda_k K_{x_k}^{\mathcal{H}} \right\rangle \geq 0.$$

Hence the matrix $\big(K^{\mathcal{H}}(x_k, x_j)\big)_{j,k}$ is positive semi-definite. $\square$

1.2 Theorem (Hilbert spaces with reproducing kernel). *Let X be a set.*

(a) *Let $K\colon X \times X \to \mathbb{C}$ be a positive definite kernel. Then there is a unique vector subspace $\mathcal{H}_K \subseteq \mathbb{C}^X$ and a unique scalar product $\langle \cdot, \cdot \rangle_K$ on $\mathcal{H}_K$ such that $\mathcal{H}_K$ is a Hilbert space with continuous point evaluations, and $K^{\mathcal{H}_K} = K$. The space $\mathcal{H}_K$ is called* the Hilbert space with reproducing kernel K.

(b) *Let $\mathcal{H} \subseteq \mathbb{C}^X$ be a vector subspace with the structure of a Hilbert space with continuous point evaluations. Then $\mathcal{H}_{K^{\mathcal{H}}} = \mathcal{H}$ as Hilbert spaces.*

(c) *Let $\mathcal{H}$ be a Hilbert space, let $\varphi\colon X \to \mathcal{H}$ be a map, and define a linear map $\Phi\colon \mathcal{H} \to \mathbb{C}^X$ by $\Phi(v)(x) := \langle v, \varphi(x)\rangle$. Then*

$$K\colon X \times X \longrightarrow \mathbb{C}, \qquad (y,x) \longmapsto \langle \varphi(x), \varphi(y)\rangle$$

is a positive definite kernel, we have $\ker(\Phi) = \varphi(X)^{\perp}$, and the restriction of Φ to $\ker(\Phi)^{\perp} = \overline{\operatorname{span}(\varphi(X))}$ is an isometry onto $\mathcal{H}_K$.

Proof. We will first construct a Hilbert space $\mathcal{H}_K$ as in (a) for a positive definite kernel K on X. Then we will prove assertion (c). This will easily imply assertion (b) and the uniqueness statement in (a).

Let K be a positive definite kernel on X. Define a positive semi-definite sesquilinear form on $\mathbb{C}^{(X)}$ by

$$\langle f, g\rangle_0 := \sum_{x,y \in X} f(x)\, K(y,x)\, \overline{g(y)} \qquad \left(f, g \in \mathbb{C}^{(X)}\right).$$

Let $\mathcal{N} := \left\{f \in \mathbb{C}^{(X)};\ \langle f, f\rangle_0 = 0\right\}$ be the radical of this form. Let $\mathcal{K}$ with scalar product $\langle \cdot, \cdot \rangle_1$ be the Hilbert space completion of the quotient space $\mathbb{C}^{(X)}/\mathcal{N}$. For $f \in \mathbb{C}^{(X)}$, set $[f] := f + \mathcal{N} \in \mathcal{K}$, and for $x \in X$, let $\delta_x \in \mathbb{C}^{(X)}$ be the characteristic function of $\{x\}$. Define a linear map $\Phi\colon \mathcal{K} \to \mathbb{C}^X$ into the vector space of all complex-valued functions on X by $\Phi(v)(x) := \langle v, [\delta_x]\rangle_1$. This map is injective because $\{[\delta_x];\ x \in X\}$ is a total subset of $\mathcal{K}$. Thus Φ

provides a realization of $\mathcal{K}$ as a linear subspace of $\mathbb{C}^X$. Set $\mathcal{H}_K := \Phi(\mathcal{K})$, and define a complex scalar product $\langle \cdot, \cdot \rangle_K$ on $\mathcal{H}_K$ by requiring Φ to be an isometry. Fix $x \in X$, and define $K_x := K(\cdot, x) \in \mathbb{C}^X$. If $y \in X$ then

$$\Phi([\delta_x])(y) = \langle [\delta_x], [\delta_y] \rangle_1 = \langle \delta_x, \delta_y \rangle_0 = K(y, x) = K_x(y),$$

so that $K_x = \Phi([\delta_x]) \in \mathcal{H}_K$. Let $f \in \mathcal{H}_K$. Then there is a unique $v \in \mathcal{K}$ such that $f = \Phi(v)$, and

$$f(x) = \Phi(v)(x) = \langle v, [\delta_x] \rangle_1 = \langle \Phi(v), \Phi([\delta_x]) \rangle_K = \langle f, K_x \rangle_K.$$

Hence $\mathcal{H}_K$ is a Hilbert space with continuous point evaluations, and $K_x^{\mathcal{H}_K} = K_x$. If $x, y \in X$ then

$$\begin{aligned} K^{\mathcal{H}_K}(y, x) &= \langle K_x^{\mathcal{H}_K}, K_y^{\mathcal{H}_K} \rangle_K = \langle K_x, K_y \rangle_K \\ &= \langle \Phi([\delta_x]), \Phi([\delta_y]) \rangle_K = \langle [\delta_x], [\delta_y] \rangle_1 = \langle \delta_x, \delta_y \rangle_0 = K(y, x). \end{aligned}$$

Thus we have proved the existence statement in (a).

Let $\varphi\colon X \to \mathcal{H}$ be a map into a Hilbert space, and define $\Phi\colon \mathcal{H} \to \mathbb{C}^X$ and $K\colon X \times X \to \mathbb{C}$ as in assertion (c). A calculation analogous to the proof of Lemma 1.1 shows that K is a positive definite kernel. Let $\mathcal{H}_K$ and $\{K_x;\, x \in X\} \subseteq \mathcal{H}_K$ be defined as above. The definition of Φ implies that $\ker(\Phi) = \varphi(X)^\perp$, so that $\ker(\Phi)^\perp = \overline{\operatorname{span}(\varphi(X))}$. Therefore, we may assume that $\varphi(X)$ is a total subset of $\mathcal{H}$. If $x, y \in X$ then

$$\Phi(\varphi(x))(y) = \langle \varphi(x), \varphi(y) \rangle = K(y, x) = K_x(y),$$

so that $\Phi(\varphi(x)) = K_x$. Hence $\Phi(\varphi(X)) \subseteq \mathcal{H}_K$. Since

$$\langle \Phi(\varphi(x)), \Phi(\varphi(y)) \rangle_K = \langle K_x, K_y \rangle_K = K(y, x) = \langle \varphi(x), \varphi(y) \rangle,$$

the restriction of Φ to the span of $\varphi(X)$ is an isometry into $\mathcal{H}_K$. This restriction extends to an isometry $\tilde{\Phi}\colon \mathcal{H} \to \mathcal{H}_K$ because $\{K_x;\, x \in X\}$ is a total subset of $\mathcal{H}_K$. If $v \in \mathcal{H}$ and $x \in X$ then

$$\tilde{\Phi}(v)(x) = \langle \tilde{\Phi}(v), K_x \rangle_K = \langle \tilde{\Phi}(v), \tilde{\Phi}(\varphi(x)) \rangle_K = \langle v, \varphi(x) \rangle = \Phi(v)(x).$$

We conclude that $\tilde{\Phi} = \Phi$. This completes the proof of (c).

Let $\mathcal{H} \subseteq \mathbb{C}^X$ be a Hilbert space with continuous point evaluations. Set $\varphi\colon X \to \mathcal{H},\ x \mapsto K_x^{\mathcal{H}}$, and define a positive definite kernel K on X as in (c). Since $\varphi(X)$ is a total subset of $\mathcal{H}$, we have just proved that φ gives rise to an isometric isomorphism $\Phi\colon \mathcal{H} \to \mathcal{H}_K$ which satisfies $\Phi(K_x^{\mathcal{H}}) = K_x$ for every $x \in X$. If $f \in \mathcal{H}$ and $x \in X$ then

$$\Phi(f)(x) = \langle \Phi(f), K_x \rangle_K = \langle \Phi(f), \Phi(K_x^{\mathcal{H}}) \rangle_K = \langle f, K_x^{\mathcal{H}} \rangle = f(x).$$

Thus $\Phi(f) = f$ and $\mathcal{H} = \mathcal{H}_K$, and we have proved (b).

Finally, let K be a positive definite kernel on X, and let $\mathcal{H} \subseteq \mathbb{C}^X$ be a vector subspace with the structure of a Hilbert space with continuous point evaluations such that $K^{\mathcal{H}} = K$. Then assertion (b) shows that $\mathcal{H} = \mathcal{H}_{K^{\mathcal{H}}} = \mathcal{H}_K$ as Hilbert spaces. This proves the uniqueness statement in (a). $\square$

1.3 Lemma (Sums of kernels). *Let K_1 and K_2 be positive definite kernels on a set X, and set $K := K_1 + K_2$. Then the linear map*

$$\Phi\colon \mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2} \longrightarrow \mathcal{H}_K, \quad (f_1, f_2) \longmapsto f_1 + f_2$$

is a surjective contraction. Its adjoint is an isometric embedding which maps K_x to $(K_{1,x}, K_{2,x})$ for every $x \in X$.

Proof. Define $\varphi\colon X \to \mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2}$, $x \mapsto (K_{1,x}, K_{2,x})$, and apply Theorem 1.2 (c). The map Φ defined there maps $(f_1, f_2) \in \mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2}$ to the map

$$x \longmapsto \langle (f_1, f_2), (K_{1,x}, K_{2,x}) \rangle = f_1(x) + f_2(x)\colon X \to \mathbb{C},$$

so that it coincides with the map Φ defined in the present lemma. Similarly, the positive definite kernel defined in Theorem 1.2 (c) maps $(y, x) \in X \times X$ to $\langle (K_{1,x}, K_{2,x}), (K_{1,y}, K_{2,y}) \rangle = K_1(y, x) + K_2(y, x)$, so that it coincides with the kernel $K = K_1 + K_2$. Hence Theorem 1.2 shows that Φ is a surjective contraction.

The adjoint Φ^* of Φ is an isometry of $\mathcal{H}_K$ onto the closed linear span of $\varphi(X)$ because Φ maps this space isometrically onto $\mathcal{H}_K$ and its orthogonal complement to 0. If $x \in X$ and $(f_1, f_2) \in \mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2}$ then

$$\begin{aligned}
&\langle (f_1, f_2), \Phi^*(K_x) \rangle_K = \langle \Phi(f_1, f_2), K_x \rangle_K = \langle f_1 + f_2, K_x \rangle_K \\
&= f_1(x) + f_2(x) = \langle f_1, K_{1,x} \rangle_{K_1} + \langle f_2, K_{2,x} \rangle_{K_2} = \langle (f_1, f_2), (K_{1,x}, K_{2,x}) \rangle.
\end{aligned}$$

Hence $\Phi^*(K_x) = (K_{1,x}, K_{2,x})$. □

2 The Gelfand–Naĭmark–Segal construction

A **-algebra* is a complex associative algebra A with a conjugate-linear anti-multiplicative involution $a \mapsto a^*\colon A \to A$. Its *unitization* $A^1 = A + \mathbb{C} \cdot 1$ is defined as A if A has a unit element, and as the direct sum $A \oplus \mathbb{C}$ of vector spaces with algebra multiplication $(a, \alpha) \cdot (b, \beta) := (ab + \alpha b + \beta a, \alpha\beta)$ and involution $(a, \alpha)^* := (a^*, \overline{\alpha})$ if A does not have a unit element. Even for a *-algebra A, we will write A^* for the space of linear functionals from A into $\mathbb{C}$. Let

$$\operatorname{Pos}(A, \mathbb{C}) := \{\omega \in A^*;\ \forall\, a \in A\colon \omega(a^* a) \geq 0\}$$

be the convex cone of *positive linear functionals.* An easy calculation shows that every element $\omega \in \operatorname{Pos}(A, \mathbb{C})$ gives rise to a positive definite kernel on A by

$$K_\omega\colon A \times A \longrightarrow \mathbb{C}, \quad (a, b) \longmapsto \omega(ab^*).$$

Let $\mathcal{H}_\omega := \mathcal{H}_{K_\omega} \subseteq \mathbb{C}^A$ be the Hilbert space with reproducing kernel K_ω. Define a linear map

$$p_\omega\colon A \longrightarrow \mathcal{H}_\omega, \quad a \longmapsto K_{\omega, a^*} = K_\omega(\cdot, a^*) = \omega(\cdot\ a).$$

We will often use the dense subspace $\mathcal{H}_\omega^0 := p_\omega(A)$ of $\mathcal{H}_\omega$. If $a, b \in A$ then

$$\langle p_\omega(a), p_\omega(b)\rangle = \langle K_{\omega,a^*}, K_{\omega,b^*}\rangle = K_\omega(b^*, a^*) = \omega(b^*a).$$

This implies the equation $\omega(b^*a) = \overline{\omega(a^*b)}$ and the Cauchy–Schwarz inequality

$$|\omega(a^*b)|^2 \leq \omega(a^*a)\;\omega(b^*b).$$

Note that $\mathcal{H}_\omega \subseteq A^*$. Indeed, if $f \in \mathcal{H}_\omega$, $a, b \in A$ and $\lambda \in \mathbb{C}$ then

$$\begin{aligned} f(a+\lambda b) &= \big\langle f, p_\omega\big((a+\lambda b)^*\big)\big\rangle \\ &= \big\langle f, p_\omega(a^*) + \overline{\lambda}\, p_\omega(b^*)\big\rangle \\ &= \big\langle f, p_\omega(a^*)\big\rangle + \lambda\big\langle f, p_\omega(b^*)\big\rangle = f(a) + \lambda f(b). \end{aligned}$$

2.1 Remark (Alternative construction). The point of view on the Gelfand–Naĭmark–Segal construction presented here is taken from Neeb [7]. In many other books, one defines $A_\omega := \{a \in A;\ \forall\, b \in A\colon \omega(ba) = 0\} = \ker(p_\omega)$ directly, proves that ω induces a scalar product on A/A_ω, and defines the Hilbert space $\mathcal{H}_\omega$ as the completion of A/A_ω. Then one can define a map $\varphi\colon A \to \mathcal{H}_\omega$, $A \mapsto a^* + A_\omega$ and use Theorem 1.2 (c) in order to identify $\mathcal{H}_\omega$ with the reproducing kernel Hilbert space defined above. Since A_ω is a left ideal of A, the construction yields a left A-module structure on A/A_ω, to which we will now turn our attention.

The vector space $\mathbb{C}^A$ is a left A-module[3] under the action of A by right multiplication in the argument, which means that $(a.f)(b) := f(ba)$ whenever $a, b \in A$ and $f \in \mathbb{C}^A$. Note that A^* is a submodule of $\mathbb{C}^A$, and that we can write $p_\omega(a) = a.\omega$ for $a \in A$. If $a, b \in A$ then

$$a.p_\omega(b) = a.(b.\omega) = ab.\omega = p_\omega(ab).$$

Thus p_ω is a homomorphism of A-modules, the pre-Hilbert space $\mathcal{H}_\omega^0$ is a submodule of $\mathbb{C}^A$, and we obtain a representation

$$\pi_\omega^0\colon A \longrightarrow \mathrm{End}(\mathcal{H}_\omega^0), \quad a \longmapsto (f \mapsto a.f).$$

We show that this is a *-representation. Let $a \in A$ and $f, g \in \mathcal{H}_\omega^0$, and choose $x, y \in A$ such that $f = p_\omega(x)$ and $g = p_\omega(y)$. Then

$$\begin{aligned} \langle \pi_\omega^0(a).f, g\rangle &= \langle a.p_\omega(x), p_\omega(y)\rangle = \langle p_\omega(ax), p_\omega(y)\rangle \\ &= \omega(y^*ax) = \omega\big((a^*y)^*x\big) \\ &= \langle p_\omega(x), p_\omega(a^*y)\rangle = \langle p_\omega(x), a^*.p_\omega(y)\rangle = \langle f, \pi_\omega^0(a^*).g\rangle. \end{aligned}$$

[3] Even if A is a non-unital algebra, the definitions include that all A-modules are vector spaces and all module homomorphisms are linear maps.

2.2 Proposition (Neeb [7, III.1.3]: invariance of $\mathcal{H}_\omega$). *Let ω be a positive linear functional on a *-algebra A, and choose $a \in A$. Then $a.\mathcal{H}_\omega \subseteq \mathcal{H}_\omega$ if and only if $\pi_\omega^0(a)$ is bounded. In this case, the unique continuous extension of $\pi_\omega^0(a)$ to $\mathcal{H}_\omega$ is given by the action of a.*

Proof. Assume that $a.\mathcal{H}_\omega \subseteq \mathcal{H}_\omega$. Let $f \in \mathcal{H}_\omega$ and $g \in \mathcal{H}_\omega^0$, and choose $b \in A$ such that $g = p_\omega(b)$. Then

$$\begin{aligned}\langle a.f, g\rangle = \langle a.f, p_\omega(b)\rangle &= (a.f)(b^*) = f(b^*a)\\ &= \langle f, p_\omega(a^*b)\rangle = \langle f, a^*.p_\omega(b)\rangle = \langle f, a^*.g\rangle.\end{aligned}$$

Choose a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{H}_\omega$ which converges to some element $f \in \mathcal{H}_\omega$ such that the sequence $(a.f_n)_{n\in\mathbb{N}}$ converges to some $h \in \mathcal{H}_\omega$. If $g \in \mathcal{H}_\omega^0$ then

$$\langle h, g\rangle = \lim_n \langle a.f_n, g\rangle = \lim_n \langle f_n, a^*.g\rangle = \langle f, a^*.g\rangle = \langle a.f, g\rangle.$$

Hence $h = a.f$, and the Closed Graph Theorem (see Rudin [14, 2.15]) shows that $f \mapsto a.f \colon \mathcal{H}_\omega \to \mathcal{H}_\omega$ is continuous. Therefore, the restriction $\pi_\omega^0(a)$ of this map to $\mathcal{H}_\omega^0$ is continuous.

Conversely, assume that $\pi_\omega^0(a)$ is continuous, so that it extends uniquely to a bounded operator $\pi_\omega(a) \in B(\mathcal{H}_\omega)$. If $f, g \in \mathcal{H}_\omega^0$ then

$$\langle \pi_\omega(a)^*.f, g\rangle = \langle f, \pi_\omega(a).g\rangle = \langle f, \pi_\omega^0(a).g\rangle = \langle \pi_\omega^0(a^*).f, g\rangle.$$

Hence $\pi_\omega(a)^*|_{\mathcal{H}_\omega^0} = \pi_\omega^0(a^*)$. Choose $f \in \mathcal{H}_\omega$. If $b \in A$ then

$$\begin{aligned}(\pi_\omega(a).f)(b) &= \langle \pi_\omega(a).f, p_\omega(b^*)\rangle = \langle f, \pi_\omega(a)^*.p_\omega(b^*)\rangle\\ &= \langle f, \pi_\omega^0(a^*).p_\omega(b^*)\rangle = \langle f, p_\omega(a^*b^*)\rangle = f(ba) = (a.f)(b).\end{aligned}$$

Hence $\pi_\omega(a).f = a.f$. In particular, this shows that $a.\mathcal{H}_\omega \subseteq \mathcal{H}_\omega$. ☐

For the following definition, recall that a C*-semi-norm on a *-algebra A is a semi-norm σ on A which satisfies $\sigma(a^*a) = \sigma(a)^2$ for all $a \in A$. Sebestyén's Theorem [16] (cf. Palmer [9, 9.5.14]) states that a C*-semi-norm σ is automatically sub-multiplicative, which means that the inequality $\sigma(ab) \le \sigma(a)\,\sigma(b)$ holds for all $a, b \in A$.

2.3 Definition. (a) A positive linear functional ω on a *-algebra A is called *exponentially bounded* or *admissible* if $\mathcal{H}_\omega$ is a submodule of $\mathbb{C}^A$ or, equivalently, if the endomorphism $\pi_\omega^0(a)$ of $\mathcal{H}_\omega(a)$ is bounded for every $a \in A$. (The term "exponentially bounded" is used by Neeb [7, III.1.9], the term "admissible" was introduced by Rickart [11, IV.5].) In this case, we obtain a representation

$$\pi_\omega \colon A \longrightarrow B(\mathcal{H}_\omega), \quad a \longmapsto (f \mapsto a.f).$$

This is a *-representation because for every $a \in A$, the operator $\pi_\omega(a)$ is the unique continuous extension of $\pi^0_\omega(a)$ to $\mathcal{H}_\omega$.

If $\omega \in \mathrm{Pos}(A, \mathbb{C})$ is exponentially bounded, the *C^*-semi-norm of* ω is defined as

$$\sigma_\omega \colon A \longrightarrow \mathbb{R}, \quad a \longmapsto \|\pi_\omega(a)\|.$$

(b) Let A be a *-algebra, and let $\varphi\colon A \to \mathbb{R}^+_0$ be an arbitrary function. Then an positive linear functional ω on A is called *bounded by* φ, or *φ-bounded* for short, if it is exponentially bounded with $\sigma_\omega \le \varphi$. This holds if and only if all $a, b \in A$ satisfy $\omega(b^*a^*ab) \le \varphi(a)^2\ \omega(b^*b)$. Indeed, the latter condition is equivalent to the inequality $\|\pi^0_\omega(a)\| \le \varphi(a)$.

Note that we can often assume φ to be a C*-semi-norm because an exponentially bounded positive linear functional ω on A is σ_ω-bounded.

If A is a Banach *-algebra, or more generally a (Mackey) complete continuous inverse *-algebra, then every positive linear functional on A is automatically bounded by the function $a \mapsto \sqrt{\rho(a^*a)}\colon A \to \mathbb{R}$, where ρ denotes the spectral radius (see [2] and Bonsall and Duncan [3, 37.6]).

Let $\omega \in \mathrm{Pos}(A, \mathbb{C})$. If ω is φ-bounded then all $a, b \in A$ satisfy

$$|\omega(b^*ab)| = |\langle \pi_\omega(a).p_\omega(b), p_\omega(b)\rangle| \le \|\pi_\omega(a)\| \cdot \|p_\omega(a)\|^2 \le \varphi(a)\ \omega(b^*b).$$

Conversely, if a function $\varphi\colon A \to \mathbb{R}^+_0$ satisfies $|\omega(b^*ab)| \le \varphi(a)\ \omega(b^*b)$ for all $a, b \in A$ then ω is bounded by the function $a \mapsto \sqrt{\varphi(a^*a)}\colon A \to \mathbb{R}$.

2.4 Lemma (Continuity implies boundedness). *Let σ be a sub-multiplicative semi-norm on a *-algebra A, and assume that $\omega \in \mathrm{Pos}(A, \mathbb{C})$ is continuous with respect to σ. Then ω is bounded by the function $a \mapsto \sqrt{\sigma(a^*a)}\colon A \to \mathbb{R}$, and also by the *-invariant sub-multiplicative semi-norm $a \mapsto \max\{\sigma(a), \sigma(a^*)\}$ on A.*

See Palmer [9, 9.4.12] for a more detailed result.

Proof. Continuity of ω with respect to σ means that there is a constant $C > 0$ such that $|\omega(a)| \le C\sigma(a)$ holds for all $a \in A$. Let $a, b \in A$. Using the Cauchy–Schwarz inequality, we inductively find that $\omega(b^*a^*ab)^{2^n} \le \omega\big(b^*(a^*a)^{2^n} b\big)\ \omega(b^*b)^{2^n-1}$ holds for all $n \in \mathbb{N}$. Sub-multiplicativity of σ yields

$$\omega(b^*a^*ab)^{2^n} \le C\ \sigma(b)\ \sigma(b^*)\ \sigma(a^*a)^{2^n}\ \omega(b^*b)^{2^n-1}.$$

Taking the 2^n-th root and letting n tend to infinity, we find that $\omega(b^*a^*ab) \le \sigma(a^*a)\ \omega(b^*b)$. Thus $\|\pi^0_\omega(a)\| \le \sqrt{\sigma(a^*a)} \le \max\{\sigma(a), \sigma(a^*)\}$. $\square$

2.5 Remark (Unital algebras). Let ω be an exponentially bounded positive linear functional on a *-algebra A, and assume that A has a unit element. The Cauchy–Schwarz inequality implies that every $a \in A$ satisfies

$|\omega(a)|^2 \leq \omega(1)\ \omega(a^*a) \leq \omega(1)^2\ \sigma_\omega(a)^2$ and hence $|\omega(a)| \leq \omega(1)\ \sigma_\omega(a)$. Thus ω is continuous with respect to σ_ω. We also infer the inclusions $\ker(\pi_\omega) \subseteq \ker(p_\omega) \subseteq \ker(\omega)$. Let $B \subseteq B(\mathcal{H}_\omega)$ be the closure of $\pi_\omega(A)$. Then ω induces a positive linear functional on the unital C*-algebra B. More precisely, there is a unique $\overline{\omega} \in \mathrm{Pos}(B, \mathbb{C})$ such that $\omega = \overline{\omega} \circ \pi_\omega$. If B is commutative then the Abstract Bochner Theorem as proved by Fell and Doran [6, 21.2] yields a Borel measure μ on the Gelfand spectrum $\hat{B}$, unique under certain regularity conditions, such that all $b \in B$ satisfy $\overline{\omega}(b) = \int_{\hat{B}} \hat{b}\, d\mu$. If B is not commutative, we can apply the elaborate theory of integral decompositions of positive linear functionals on C*-algebras (see Sakai [15, Chapter 3]), of which the Abstract Bochner Theorem can be seen as a special case.

This article is devoted to the corresponding results for non-unital algebras. In the commutative case, an Abstract Bochner Theorem still holds, and we obtain strong results on the uniqueness of the integral decomposition, which may be of interest for unital algebras as well. If A is not commutative but has countable dimension and ω is central, the non-degenerate part of ω corresponds to a trace on B which admits an integral decomposition.

3 Commutative *-algebras

3.1 Gelfand spectrum and Gelfand homomorphism

3.1 Definition. Let A be a commutative *-algebra. The *involutive Gelfand spectrum* of A is defined as $\hat{A} := \mathrm{Hom}^*(A, \mathbb{C}) \setminus \{0\}$, where $\mathrm{Hom}^*(A, \mathbb{C})$ denotes the space of *-algebra-homomorphisms from A into $\mathbb{C}$ with the topology of pointwise convergence on A. The elements of $\hat{A}$ are called the *involutive characters* of A.

For an arbitrary non-negative function $\varphi\colon A \to \mathbb{R}_0^+$, let the *$\varphi$-bounded involutive spectrum* of A be

$$\hat{A}(\varphi) := \left\{ \chi \in \hat{A};\ \ \forall\, a \in A\colon |\chi(a)| \leq \varphi(a) \right\}.$$

3.2 Remark. Let χ be an involutive character of a *-algebra A. Then χ is a positive linear functional on A. Since we have defined the notion of φ-boundedness for positive linear functionals as well, we have to check that it coincides with the new definition for involutive characters. The Gelfand–Naĭmark–Segal construction gives $\ker(p_\chi) = \{a \in A;\ \chi(a^*a) = 0\} = \ker(\chi)$ and $\mathcal{H}_\chi \cong \mathbb{C}$, and the representation π_χ is equivalent to χ, viewed as a representation of A on $\mathbb{C}$. Hence $\sigma_\chi(a) = |\chi(a)|$ holds for all $a \in A$. In particular, the two notions of φ-boundedness coincide for χ.

3.3 Remark (Topology of $\hat{A}$). (a) In the topology of pointwise convergence on A, the set $\hat{A}(\varphi) \cup \{0\}$ is closed in $\left\{ f \in \mathbb{C}^A;\ \ \forall\, a \in A\colon |f(a)| \leq \varphi(a) \right\}$, which is a compact space by Tychonov's Theorem. Hence $\hat{A}(\varphi)$ is a locally compact Hausdorff space. If the algebra A has a unit element 1 then all $\chi \in \hat{A}$ satisfy $\chi(1) = 1$, whence $\hat{A}(\varphi)$ is closed in $\hat{A}(\varphi) \cup \{0\}$ and therefore compact.

(b) Assume that for every $a \in A$, the spectrum

$$\mathrm{Sp}(a) := \left\{\lambda \in \mathbb{C};\ \lambda \cdot 1 - a \notin (A^1)^\times\right\}$$

is bounded. (This implies that every element has compact spectrum, see Palmer [8, 2.1.11].) The spectral radius of an element $a \in A$ is $\rho(a) := \sup\{|\lambda|;\ \lambda \in \mathrm{Sp}(a)\}$. If $a \in A$ and $\chi \in \hat{A}$ then

$$\mathrm{Sp}_A(a) \supseteq \mathrm{Sp}_{\mathbb{C}}(\chi(a)) = \{\chi(a)\},$$

so that $|\chi(a)| \leq \rho(a)$. We conclude that $\hat{A} = \hat{A}(\rho)$.

3.4 Definition. Let A be a commutative *-algebra. Each $a \in A$ gives rise to a continuous function

$$\hat{a}\colon \hat{A} \cup \{0\} \longrightarrow \mathbb{C}, \quad \chi \longmapsto \chi(a),$$

which is called the *Gelfand transform* of a. The restriction of $\hat{a}$ to a subset of $\hat{A} \cup \{0\}$ such as $\hat{A}$ or $\hat{A}(\varphi)$ will also be denoted by $\hat{a}$ if no confusion seems likely. Let $\varphi\colon A \to \mathbb{R}_0^+$ be a function. If $a \in A$ then the restriction of $\hat{a}$ to $\hat{A}(\varphi)$ belongs to the C*-algebra $C_0(\hat{A}(\varphi))$ of those continuous complex-valued functions on $\hat{A}(\varphi)$ which vanish at infinity. The *-algebra homomorphism $a \mapsto \hat{a}\colon A \to C_0(\hat{A}(\varphi))$ is called the *φ-bounded Gelfand homomorphism.* Its image is a subalgebra of $C_0(\hat{A}(\varphi))$ which is closed under pointwise conjugation, separates the points of $\hat{A}(\varphi)$, and does not vanish anywhere on $\hat{A}(\varphi)$. Hence this image is a uniformly dense subalgebra of $C_0(\hat{A}(\varphi))$ by the Stone–Weierstrass Theorem (cf. Fell and Doran [6, A8]).

3.5 Lemma (Compact subsets of $\hat{A} \cup \{0\}$). *Let A be a commutative *-algebra, and let $K \subseteq \hat{A}$ be a subset such that $K \cup \{0\}$ is compact. Define a C*-semi-norm on A by*

$$\sigma\colon A \longrightarrow \mathbb{R}, \quad a \longmapsto \left\|\hat{a}|_K\right\|_\infty = \sup\{|\chi(a)|;\ \chi \in K\}.$$

Then $K \subseteq \hat{A}(\sigma)$. If $K = \hat{A}(\varphi)$ for some function $\varphi\colon A \to \mathbb{R}_0^+$ then $K = \hat{A}(\sigma)$.

Proof. The inequality $|\chi(a)| \leq \sigma(a)$ holds for all $\chi \in K$ and all $a \in A$, so that $K \subseteq \hat{A}(\sigma)$. In the case that $K = \hat{A}(\varphi)$, the definition of σ shows that all $a \in A$ satisfy $\sigma(a) \leq \varphi(a)$. This implies the reverse inclusion $\hat{A}(\sigma) \subseteq K$. $\square$

3.6 Lemma ($\hat{A}(\sigma)$ for a C*-semi-norm σ). *Let σ be a C*-semi-norm on a commutative *-algebra A, and let $\pi_\sigma\colon A \to \overline{A}^\sigma$ be the natural map into the corresponding C*-algebra. Then*

$$\chi \longmapsto \chi \circ \pi_\sigma \colon \bigl(\overline{A}^\sigma\bigr)^\frown \longrightarrow \hat{A}(\sigma)$$

is a homeomorphism. In particular, all $a \in A$ satisfy $\sigma(a) = \bigl\|\hat{a}|_{\hat{A}(\sigma)}\bigr\|_\infty$.

Proof. The map π_σ is constructed as follows. Let $A_\sigma := \sigma^{-1}(\{0\})$ be the zero ideal of σ. Then σ induces a C*-norm on the quotient *-algebra A/A_σ. The completion of A/A_σ with respect to this norm is denoted by $\overline{A}^\sigma$, and we set $\pi_\sigma(a) := a + A_\sigma \in \overline{A}^\sigma$ for $a \in A$.

If $\chi \in \bigl(\overline{A}^\sigma\bigr)^\frown$ and $a \in A$ then $|\chi(\pi_\sigma(a))| \le \|\pi_\sigma(a)\| = \sigma(a)$, so that $\chi \circ \pi_\sigma \in \hat{A}(\sigma)$. Hence we have a map from $\bigl(\overline{A}^\sigma\bigr)^\frown \cup \{0\}$ into $\hat{A}(\sigma) \cup \{0\}$ which maps χ to $\chi \circ \pi_\sigma$. This map is continuous and injective, and it is closed because it is a map between compact Hausdorff spaces. Finally, it is surjective because every $\chi \in \hat{A}(\sigma)$ induces a continuous involutive character of A/A_σ, which extends to a character $\overline{\chi}$ of $\overline{A}^\sigma$ such that $\overline{\chi} \circ \pi_\sigma = \chi$. This proves the main assertion, which implies that every $a \in A$ satisfies

$$\begin{aligned}\sigma(a) = \|\pi_\sigma(a)\| &= \sup\bigl\{|\chi(\pi_\sigma(a))|;\ \chi \in \bigl(\overline{A}^\sigma\bigr)^\frown\bigr\} \\ &= \sup\Bigl\{|\chi(a)|;\ \chi \in \hat{A}(\sigma)\Bigr\} = \bigl\|\hat{a}|_{\hat{A}(\sigma)}\bigr\|_\infty,\end{aligned}$$

so that the second assertion follows immediately. □

3.7 Lemma. *Let A be a commutative *-algebra, and let $K \subseteq \hat{A}$ be compact. Then there exists $a \in A$ such that the Gelfand transform $\hat{a}$ does not have any zero on K.*

Proof. For each $\chi \in K$, choose an element $a_\chi \in A$ such that $\hat{a}_\chi(\chi) \neq 0$. Since K is compact, we can choose $\chi_1, \dots, \chi_n \in K$ such that

$$a := a_{\chi_1}^* a_{\chi_1} + \dots + a_{\chi_n}^* a_{\chi_n}$$

has the required property. □

3.8 Proposition (Extension of the Gelfand–Naĭmark–Segal representation). *Let σ be a C*-semi-norm on a commutative *-algebra A, and let $\omega \in \mathrm{Pos}(A, \mathbb{C})$ be σ-bounded. Then the Gelfand–Naĭmark–Segal representation $\pi_\omega\colon A \to B(\mathcal{H}_\omega)$ associated with ω factors through a unique *-representation $\overline{\pi}_\omega\colon C_0(\hat{A}(\sigma)) \to B(\mathcal{H}_\omega)$ in the sense that all $a \in A$ satisfy $\overline{\pi}_\omega(\hat{a}) = \pi_\omega(a)$.*

Note that every *-homomorphism between C*-algebras is a contraction (see Dixmier [5, 1.3.7]), so that $\|\overline{\pi}_\omega(\varphi)\| \le \|\varphi\|_\infty$ holds for every $\varphi \in C_0(\hat{A}(\sigma))$.

Proof. Let $B \subseteq B(\mathcal{H}_\omega)$ be the closure of the image $\pi_\omega(A)$. Then B is a commutative C*-algebra. If $\chi \in \hat{B}$ and $a \in A$ then

$$|\chi(\pi_\omega(a))| \leq \|\pi_\omega(a)\| = \sigma_\omega(a) \leq \sigma(a),$$

so that $\chi \circ \pi_\omega \in \hat{A}(\sigma)$. We obtain a map $\chi \mapsto \chi \circ \pi_\omega \colon \hat{B} \to \hat{A}(\sigma)$ which is continuous. This gives rise to a *-homomorphism

$$\varphi \longmapsto \big(\chi \mapsto \varphi(\chi \circ \pi_\omega)\big) \colon C_0(\hat{A}(\sigma)) \longrightarrow C_0(\hat{B}).$$

The composition of this homomorphism with the inverse of the Gelfand isomorphism $B \to C_0(\hat{B})$ and the inclusion $B \hookrightarrow B(\mathcal{H}_\omega)$ is the *-representation $\overline{\pi}_\omega \colon C_0(\hat{A}(\sigma)) \to B(\mathcal{H}_\omega)$. Thus for $\varphi \in C_0(\hat{A}(\sigma))$, the operator $\overline{\pi}_\omega(\varphi) \in B$ is characterized by the fact that its Gelfand transform maps $\chi \in \hat{B}$ to $\varphi(\chi \circ \pi_\omega)$, i.e. by the formula

$$\forall\, \varphi \in C_0(\hat{A}(\sigma)), \chi \in \hat{B} \colon \chi(\overline{\pi}_\omega(\varphi)) = \varphi(\chi \circ \pi_\omega).$$

In particular, if $a \in A$ then $\chi(\overline{\pi}_\omega(\hat{a})) = \hat{a}(\chi \circ \pi_\omega) = \chi(\pi_\omega(a))$ holds for all $\chi \in \hat{B}$, whence $\overline{\pi}_\omega(\hat{a}) = \pi_\omega(a)$. Since the image of the Gelfand homomorphism from A into $C_0(\hat{A}(\sigma))$ is uniformly dense, this property uniquely determines the *-representation $\overline{\pi}_\omega$. □

3.9 Remark. The relation $\chi(\overline{\pi}_\omega(\varphi)) = \varphi(\chi \circ \pi_\omega)$ for $\chi \in \hat{B}$ and $\varphi \in C_0(\hat{A}(\sigma))$ will be used again in the proof of Proposition 3.13.

3.2 Representing measures

Recall that the *Borel σ-algebra* $\mathfrak{B}(X)$ on a topological space X is the σ-algebra generated by the open sets. A *Borel measure* on X is a measure defined on $\mathfrak{B}(X)$.

3.10 Definition. Let ω be a positive linear functional on a commutative *-algebra A, and let σ be a C*-semi-norm on A. A Borel measure μ on $\hat{A}(\sigma)$ *represents* ω if $\left\{\hat{a}|_{\hat{A}(\sigma)};\ a \in A\right\} \subseteq L^3(\mu)$ and all $a, b, c \in A$ satisfy

$$\int_{\hat{A}(\sigma)} \hat{a}\hat{b}\hat{c}\; d\mu = \omega(abc).$$

Lemma 3.5 shows that we would not obtain a more general concept if we considered representing Borel measures on other compact subsets of $\hat{A} \cup \{0\}$.

3.11 Lemma (Representability implies exponential boundedness). *Let ω be a positive linear functional on a commutative *-algebra A, let σ be a C*-semi-norm on A, and let μ be a Borel measure on $\hat{A}(\sigma)$ which represents ω. Then μ takes finite values on all compact subsets of $\hat{A}(\sigma)$, and ω is σ-bounded. In particular, $\hat{A}(\sigma)$ contains $\hat{A}(\sigma_\omega)$.*

Proof. Let $K \subseteq \hat{A}(\sigma)$ be compact. Lemma 3.7 yields an element $a \in A$ such that $\varepsilon := \inf\{|\hat{a}(\chi)|;\ \chi \in K\}$ is strictly positive, so that

$$\mu(K) \leq \varepsilon^{-4} \int_{\hat{A}(\sigma)} |\hat{a}|^4 \, d\mu = \varepsilon^{-4}\, \omega\big((a^*a)^2\big).$$

Let $a, b \in A$. The Cauchy–Schwarz inequality and Lemma 3.6 show that

$$\begin{aligned} \omega(b^*a^*ab)^2 &\leq \omega(b^*b)\, \omega\big(b^*(a^*a)^2 b\big) = \omega(b^*b) \int_{\hat{A}(\sigma)} |\hat{a}|^4\, |\hat{b}|^2 \, d\mu \\ &\leq \omega(b^*b)\, \big\| \hat{a}|_{\hat{A}(\sigma)} \big\|_\infty^2 \int_{\hat{A}(\sigma)} |\hat{a}|^2\, |\hat{b}|^2 \, d\mu = \omega(b^*b)\, \sigma(a)^2\, \omega(b^*a^*ab). \end{aligned}$$

Hence $\omega(b^*a^*ab) \leq \sigma(a)^2\, \omega(b^*b)$, and we conclude that $\sigma_\omega(a) = \|\pi_\omega(a)\| \leq \sigma(a)$. This immediately implies that $\hat{A}(\sigma_\omega) \subseteq \hat{A}(\sigma)$. $\square$

The space of compactly supported complex-valued functions on a locally compact space X is denoted by $C_c(X)$.

3.12 Proposition (Characterization of representing measures). *Let ω be a positive linear functional on a commutative *-algebra A, and let σ be a C*-semi-norm on A. Then the following conditions are equivalent for a Borel measure μ on $\hat{A}(\sigma)$:*

(i) *the measure μ represents ω;*

(ii) $\forall\, \varphi \in C_c(\hat{A}(\sigma)),\ a \in A\colon \displaystyle\int_{\hat{A}(\sigma)} \varphi\, |\hat{a}|^2 \, d\mu = \langle \overline{\pi}_\omega(\varphi).p_\omega(a), p_\omega(a)\rangle;$

(iii) $\forall\, \varphi \in C_0(\hat{A}(\sigma)),\ a, b \in A\colon \displaystyle\int_{\hat{A}(\sigma)} \varphi\, \hat{a}\hat{b} \, d\mu = \langle \overline{\pi}_\omega(\varphi).p_\omega(a), p_\omega(b^*)\rangle$
(in particular, the integral exists).

If these conditions are satisfied then $\displaystyle\int_{\hat{A}(\sigma)} |\hat{a}|^2 \, d\mu \leq \omega(a^*a)$ *holds for every $a \in A$.*

Proof. If $a, b, c \in A$ then

$$\langle \overline{\pi}_\omega(\hat{c}).p_\omega(a), p_\omega(b^*)\rangle = \langle \pi_\omega(c).p_\omega(a), p_\omega(b^*)\rangle = \langle p_\omega(ca), p_\omega(b^*)\rangle = \omega(bca).$$

In particular, condition (iii) implies (i).

Assume that condition (i) holds. Choose $\varphi \in C_c(\hat{A}(\sigma))$ and $a \in A$. The calculation above shows that (ii) holds if φ is replaced with an element $c \in A$. Lemma 3.5 yields $b \in A$ such that $\hat{b}$ does not vanish anywhere on the support of φ. Extending the quotient $\varphi/|\hat{b}|^2$, which is defined on a neighbourhood of $\mathrm{supp}(\varphi)$, by zero, we view it as an element of $C_c(\hat{A}(\sigma))$. For an arbitrary element $c \in A$, we calculate

$$\begin{aligned}
&\left| \int_{\hat{A}(\sigma)} \varphi \, |\hat{a}|^2 \, d\mu - \langle \overline{\pi}_\omega(\varphi).p_\omega(a), p_\omega(a) \rangle \right| \\
&= \left| \int_{\hat{A}(\sigma)} \frac{\varphi}{|\hat{b}|^2} \, |\hat{a}|^2 \, |\hat{b}|^2 \, d\mu - \int_{\hat{A}(\sigma)} \hat{c} \, |\hat{a}|^2 \, |\hat{b}|^2 \, d\mu \right. \\
&\qquad \left. + \langle \overline{\pi}_\omega(\hat{c}).p_\omega(ab), p_\omega(ab) \rangle - \left\langle \overline{\pi}_\omega \left(\frac{\varphi}{|\hat{b}|^2} \, \hat{b} \right) .p_\omega(a), \overline{\pi}_\omega(\hat{b}).p_\omega(a) \right\rangle \right| \\
&\leq \left| \int_{\hat{A}(\sigma)} \left(\frac{\varphi}{|\hat{b}|^2} - \hat{c} \right) \, |\hat{a}|^2 \, |\hat{b}|^2 \, d\mu \right| + \left| \left\langle \overline{\pi}_\omega \left(\hat{c} - \frac{\varphi}{|\hat{b}|^2} \right) .p_\omega(ab), p_\omega(ab) \right\rangle \right| \\
&\leq \left\| \frac{\varphi}{|\hat{b}|^2} - \hat{c} \right\|_\infty \int_{\hat{A}(\sigma)} |\hat{a}|^2 \, |\hat{b}|^2 \, d\mu + \left\| \overline{\pi}_\omega \left(\hat{c} - \frac{\varphi}{|\hat{b}|^2} \right) \right\| \langle p_\omega(ab), p_\omega(ab) \rangle \\
&\leq \left\| \frac{\varphi}{|\hat{b}|^2} - \hat{c} \right\|_\infty \cdot 2 \, \omega(b^* a^* ab).
\end{aligned}$$

As the image of the Gelfand homomorphism is uniformly dense in $C_0(\hat{A}(\sigma))$, we can choose $c \in A$ such that the right-hand side is arbitrarily small. This proves condition (ii).

Assume that (ii) holds. Let $a \in A$. We will prove that $\int |\hat{a}|^2 \, d\mu \leq \omega(a^*a)$. For $n \in \mathbb{N}$, set $K_n := \{\chi \in \hat{A}(\sigma); |\hat{a}(\chi)| \geq \frac{1}{n}\}$. Since $\hat{a} \in C_0(\hat{A}(\sigma))$, each K_n is a compact subset of the interior of K_{n+1}. By Urysohn's Lemma, we find continuous functions $\varphi_n \colon \hat{A}(\sigma) \to [0,1]$ which are identically 1 on K_n and vanish outside K_{n+1}. The sequence $(\varphi_n)_{n\in\mathbb{N}}$ is increasing and converges pointwise to the characteristic function of the open set $\{\chi \in \hat{A}(\sigma); \hat{a}(\chi) \neq 0\}$. By Lebesgue's Monotone Convergence Theorem (cf. Rudin [13, 1.26]),

$$\begin{aligned}
\int_{\hat{A}(\sigma)} |\hat{a}|^2 \, d\mu = \lim_n \int_{\hat{A}(\sigma)} \varphi_n \, |\hat{a}|^2 \, d\mu &= \lim_n \langle \overline{\pi}_\omega(\varphi_n).p_\omega(a), p_\omega(a) \rangle \\
&\leq \langle p_\omega(a), p_\omega(a) \rangle = \omega(a^*a).
\end{aligned}$$

Since $C_c(\hat{A}(\sigma))$ is uniformly dense in $C_0(\hat{A}(\sigma))$, we conclude from (ii) that

$$\forall \, \varphi \in C_0(\hat{A}(\sigma)), \; a \in A \colon \int_{\hat{A}(\sigma)} \varphi \, |\hat{a}|^2 \, d\mu = \langle \overline{\pi}_\omega(\varphi).p_\omega(a), p_\omega(a) \rangle.$$

Condition (iii) follows because all $\varphi \in C_0(\hat{A}(\sigma))$ and $a, b \in A$ satisfy the polarization identities

$$\begin{aligned} \varphi\, \hat{a}\hat{b} &= \frac{1}{4}\, \varphi \sum_{k=0}^{3} i^k \left|(a+i^k b^*)\hat{}\right|^2 , \\ \left\langle \overline{\pi}_\omega(\varphi).p_\omega(a), p_\omega(b^*)\right\rangle &= \frac{1}{4}\sum_{k=0}^{3} i^k \left\langle \overline{\pi}_\omega(\varphi).p_\omega(a+i^k b^*), p_\omega(a+i^k b^*)\right\rangle . \end{aligned}$$

□

Let μ be a Borel measure on a topological space X. The *closed support* $\mathrm{supp}(\mu)$ of μ is the set of all $x \in X$ such that $\mu(U) > 0$ holds for every open neighbourhood $U \subseteq X$ of x. It is a closed subset of X. If every open subset $U \subseteq X$ satisfies the condition $\mu(U) = \sup\{\mu(K);\ K \subseteq U,\ K \text{ compact}\}$ (cf. Theorem 3.14) then any union of open subsets of measure 0 has measure 0, so that $\mathrm{supp}(\mu)$ is the complement in X of the largest open subset of measure 0.

Let μ be a Borel measure on a locally compact space X, and assume that $\varphi \in C_0(X)$ vanishes on $\mathrm{supp}(\mu)$. The set $\{x \in X;\ \varphi(x) \neq 0\}$ is the countable union of the sets $\left\{x \in X;\ |\varphi(x)| \geq \frac{1}{n}\right\}$ for $n \in \mathbb{N}$, and these sets are compact and disjoint from $\mathrm{supp}(\mu)$, so that they have measure 0. This implies that $\int_X \varphi\, d\mu = 0$.

3.13 Proposition (The support of a representing measure). *Let ω be a positive linear functional on a commutative *-algebra A, let σ be a C^*-semi-norm on A, and let μ be a Borel measure on $\hat{A}(\sigma)$ which represents ω. Then* $\mathrm{supp}(\mu) = \hat{A}(\sigma_\omega)$.

Proof. Lemma 3.11 shows that $\hat{A}(\sigma_\omega) \subseteq \hat{A}(\sigma)$.

Choose a non-negative function $\varphi \in C_0(\hat{A}(\sigma))$, and let $B \subseteq B(\mathcal{H}_\omega)$ be the closure of $\pi_\omega(A)$. Lemma 3.6 entails that

$$\hat{A}(\sigma_\omega) = \left\{\chi \circ \pi_\omega;\ \chi \in \hat{B}\right\},$$

and Remark 3.9 shows that all $\chi \in \hat{B}$ satisfy $\chi(\overline{\pi}_\omega(\varphi)) = \varphi(\chi \circ \pi_\omega)$. Hence φ vanishes on $\hat{A}(\sigma_\omega)$ if and only if $\overline{\pi}_\omega(\varphi) = 0$. Proposition 3.12 shows that $\overline{\pi}_\omega(\varphi) = 0$ if and only if all $a, b \in A$ satisfy $\int \varphi\ \hat{a}\hat{b}\ d\mu = 0$. This holds if φ vanishes on $\mathrm{supp}(\mu)$ by the remarks following the introduction of the closed support. Conversely, if $\varphi(\chi) \neq 0$ holds for some $\chi \in \mathrm{supp}(\mu)$ then every $a \in A$ with $\chi(a) \neq 0$ yields $\int \varphi\, |\hat{a}|^2\, d\mu > 0$. We conclude that a non-negative function $\varphi \in C_0(\hat{A}(\sigma))$ vanishes on $\hat{A}(\sigma_\omega)$ if and only if it vanishes on $\mathrm{supp}(\mu)$.

These two sets are closed in $\hat{A}(\sigma)$, and this is a locally compact Hausdorff space and hence completely regular. This implies that $\mathrm{supp}(\mu) = \hat{A}(\sigma_\omega)$. □

3.3 Existence and uniqueness of a representing measure

3.14 Theorem (Abstract Bochner Theorem). *Let A be a commutative *-algebra, and let ω be an exponentially bounded positive linear functional on A. Then there exists a unique Borel measure μ_ω on $\hat{A}(\sigma_\omega)$ which represents ω and satisfies the following conditions:*

(i) *all Borel subsets $E \subseteq \hat{A}(\sigma_\omega)$ satisfy*

$$\mu_\omega(E) = \inf\left\{\mu_\omega(U);\ E \subseteq U,\ U \subseteq \hat{A}(\sigma_\omega) \text{ open}\right\};$$

(ii) *if a Borel subset $E \subseteq \hat{A}(\sigma_\omega)$ is open or has finite measure then*

$$\mu_\omega(E) = \sup\left\{\mu_\omega(K);\ K \subseteq E,\ K \text{ compact}\right\}.$$

The measure μ_ω is called the Gelfand transform *of the positive linear functional ω.*

Proof. Existence of μ_ω will follow from the Riesz Representation Theorem for positive linear functionals on $C_c(\hat{A}(\sigma_\omega))$ (see Rudin [13, 2.14]). Let $\varphi \in C_c(\hat{A}(\sigma_\omega))$. Lemma 3.5 yields an element $a \in A$ such that $\hat{a}$ has no zero on the support of φ. We extend the quotient $\varphi/|\hat{a}|^2$ by zero to an element of $C_c(\hat{A}(\sigma_\omega))$ and set

$$\omega'(\varphi) := \left\langle \overline{\pi}_\omega\left(\frac{\varphi}{|\hat{a}|^2}\right).p_\omega(a), p_\omega(a)\right\rangle.$$

To see that this definition does not depend on the choice of a, let $b \in A$ be another element whose Gelfand transform has no zero on the support of φ. Then

$$\begin{aligned}
\left\langle \overline{\pi}_\omega\left(\frac{\varphi}{|\hat{a}|^2}\right).p_\omega(a), p_\omega(a)\right\rangle &= \left\langle \overline{\pi}_\omega\left(\frac{\varphi}{|\hat{a}|^2\,|\hat{b}|^2}\,\hat{b}\right).p_\omega(a), \overline{\pi}_\omega(\hat{b}).p_\omega(a)\right\rangle \\
&= \left\langle \overline{\pi}_\omega\left(\frac{\varphi}{|\hat{a}|^2\,|\hat{b}|^2}\right).p_\omega(ba), p_\omega(ba)\right\rangle \\
&= \left\langle \overline{\pi}_\omega\left(\frac{\varphi}{|\hat{a}|^2\,|\hat{b}|^2}\right).p_\omega(ab), p_\omega(ab)\right\rangle \\
&= \left\langle \overline{\pi}_\omega\left(\frac{\varphi}{|\hat{a}|^2\,|\hat{b}|^2}\,\hat{a}\right).p_\omega(b), \overline{\pi}_\omega(\hat{a}).p_\omega(b)\right\rangle \\
&= \left\langle \overline{\pi}_\omega\left(\frac{\varphi}{|\hat{b}|^2}\right).p_\omega(b), p_\omega(b)\right\rangle.
\end{aligned}$$

Thus ω' is a well-defined linear functional on $C_c(\hat{A}(\sigma_\omega))$, and using the square root of a compactly supported non-negative function, we find that ω

is positive. By the Riesz Representation Theorem [13, 2.14], there is a unique Borel measure μ_ω on $\hat{A}(\sigma_\omega)$ which satisfies conditions (i) and (ii) and

$$\forall \varphi \in C_c(\hat{A}(\sigma_\omega)) \colon \omega'(\varphi) = \int_{\hat{A}(\sigma_\omega)} \varphi \, d\mu_\omega \, .$$

Let $a \in A$ and $\varphi \in C_c(\hat{A}(\sigma_\omega))$ be arbitrary, and choose an element $c \in A$ whose Gelfand transform $\hat{c}$ does not vanish anywhere on the support of φ. Then

$$\begin{aligned}
\int_{\hat{A}(\sigma_\omega)} \varphi \, |\hat{a}|^2 \, d\mu_\omega = \omega'(\varphi \, |\hat{a}|^2) &= \left\langle \overline{\pi}_\omega \left(\frac{\varphi \, |\hat{a}|^2}{|\hat{c}|^2} \right) .p_\omega(c), p_\omega(c) \right\rangle \\
&= \left\langle \overline{\pi}_\omega \left(\frac{\varphi}{|\hat{c}|^2} \, \hat{a} \right) .p_\omega(c), \overline{\pi}_\omega(\hat{a}).p_\omega(c) \right\rangle \\
&= \left\langle \overline{\pi}_\omega \left(\frac{\varphi}{|\hat{c}|^2} \, \hat{c} \right) .p_\omega(a), \overline{\pi}_\omega(\hat{c}).p_\omega(a) \right\rangle \\
&= \langle \overline{\pi}_\omega(\varphi).p_\omega(a), p_\omega(a) \rangle.
\end{aligned}$$

Proposition 3.12 shows that the measure μ_ω represents ω.

Let μ be a Borel measure on $\hat{A}(\sigma_\omega)$ which represents ω and satisfies conditions (i) and (ii). Choose $\varphi \in C_c(\hat{A}(\sigma_\omega))$ and $a \in A$ such that $\hat{a}$ does not vanish anywhere on the support of φ. By Proposition 3.12,

$$\begin{aligned}
\int_{\hat{A}(\sigma_\omega)} \varphi \, d\mu &= \int_{\hat{A}(\sigma_\omega)} \frac{\varphi}{|\hat{a}|^2} \, |\hat{a}|^2 \, d\mu \\
&= \left\langle \overline{\pi}_\omega \left(\frac{\varphi}{|\hat{a}|^2} \right) .p_\omega(a), p_\omega(a) \right\rangle = \int_{\hat{A}(\sigma_\omega)} \varphi \, d\mu_\omega.
\end{aligned}$$

The uniqueness part of the Riesz Representation Theorem [13, 2.14] shows that $\mu = \mu_\omega$. □

3.15 Example. We show that the values of a general positive linear functional ω on single elements and on products of two elements are not represented by the Gelfand transform. In this sense, our notion of a representing measure is the best one can expect. The relevant additional conditions on ω will be studied in Proposition 3.16 and Corollary 3.23.

In the following two examples, consider $A := \mathbb{C}^2$ with the involution $(z, w)^* := (\overline{z}, \overline{w})$ and algebra multiplication to be defined. Set $\omega \colon A \to \mathbb{C}, (z, w) \mapsto \alpha z + \beta w$ with $\alpha, \beta \in \mathbb{C}$.

(a) Define a multiplication on A by $(z, w) \cdot (z', w') := (zz', 0)$. Considering the elements $(1, 0)$ and $(0, 1)$, we find that ω is multiplicative if and only if $\alpha \in \{0, 1\}$ and $\beta = 0$. Hence the full spectrum $\hat{A}$ consists of a single point, and the Gelfand homomorphism corresponds to the projection of A onto its first coordinate. The linear functional ω is positive if and only if

$\alpha \geq 0$. In this case, its Gelfand transform μ_ω satisfies $\int (z,w)\hat{}\, d\mu_\omega = \alpha z$. Hence there is no general relation between $\int \hat{a}\, d\mu_\omega$ and $\omega(a)$.

(b) Define a multiplication on A by $(z,w)\cdot(z',w') := (0, zz')$. Then ω is multiplicative if and only if $\alpha = \beta = 0$, whence the spectrum of A is empty. Moreover, ω is positive if and only if $\beta \geq 0$. If μ_ω is the Gelfand transform of ω then $\int |\hat{a}|^2\, d\mu_\omega$ vanishes for all $a \in A$, whereas $\omega(a^*a)$ may be strictly positive.

3.16 Proposition (Extension of ω to A^1). *Let ω be an exponentially bounded positive linear functional on a commutative *-algebra A with Gelfand transform μ_ω. Then ω extends to a positive linear functional on the unitization $A^1 = A + \mathbb{C}\cdot 1$ of A if and only if the measure μ_ω is finite and each $a \in A$ satisfies $\int \hat{a}\, d\mu_\omega = \omega(a)$.*

Proof. If A has a unit element 1 then $\mu_\omega(\hat{A}(\sigma_\omega)) = \omega(1^3) = \omega(1)$ is finite, and $\omega(a) = \omega(1^2\, a) = \int \hat{a}\, d\mu_\omega$ holds for each $a \in A$. Hence we will assume that A is non-unital.

Assume that μ_ω is finite and that each $a \in A$ satisfies $\omega(a) = \int \hat{a}\, d\mu_\omega$. Then $\omega(a^*) = \overline{\omega(a)}$ holds for each $a \in A$, and the Cauchy–Schwarz Inequality (cf. Rudin [13, 3.5]) shows that

$$\begin{aligned}|\omega(a)|^2 &= \left| \int_{\hat{A}(\sigma_\omega)} \hat{a}\, d\mu_\omega \right|^2 \\ &\leq \mu_\omega(\hat{A}(\sigma_\omega)) \int_{\hat{A}(\sigma_\omega)} |\hat{a}|^2\, d\mu_\omega = \mu_\omega(\hat{A}(\sigma_\omega))\, \omega(a^*a).\end{aligned}$$

Hence ω extends to a positive linear functional on A^1 (see Fell and Doran [6, VI.18.7] or Palmer [9, 9.4.7]).

Assume that there is a positive linear functional $\tilde{\omega}$ on $A^1 = A \oplus \mathbb{C}$ such that $\tilde{\omega}|_A = \omega$. We claim that $\tilde{\omega}$ is exponentially bounded. If $a \in A$ then

$$|\omega(a)| = |\tilde{\omega}(a)| \leq \sqrt{\tilde{\omega}(1)\, \tilde{\omega}(a^*a)} = \sqrt{\tilde{\omega}(1)\, \omega(a^*a)} = \sqrt{\tilde{\omega}(1)}\, \|p_\omega(a)\|.$$

Hence ω induces a linear functional on $\mathcal{H}_\omega^0$, which is continuous of norm at most $\sqrt{\tilde{\omega}(1)}$. Applying the Riesz Representation Theorem (see Rudin [14, 12.5]) to the continuous extension of that functional to $\mathcal{H}_\omega$, we obtain a unique vector $z_\omega \in \mathcal{H}_\omega$ such that all $a \in A$ satisfy $\omega(a) = \langle p_\omega(a), z_\omega\rangle$, and $\|z_\omega\| \leq \sqrt{\tilde{\omega}(1)}$. (Palmer [9, 9.4.5] calls z_ω the *canonical vector* of the extensible positive functional ω.) If $a, b \in A$ then

$$\langle p_\omega(b), \pi_\omega(a).z_\omega\rangle = \langle p_\omega(a^*b), z_\omega\rangle = \omega(a^*b) = \langle p_\omega(b), p_\omega(a)\rangle.$$

Therefore, all $a \in A$ satisfy $\pi_\omega(a).z_\omega = p_\omega(a)$, so that ω is represented by π_ω through the formula $\omega(a) = \langle \pi_\omega(a) z_\omega, z_\omega\rangle$. Set $r := \tilde{\omega}(1) - \|z_\omega\|^2 \geq 0$. Then every $(a,\alpha) \in A^1$ satisfies

$$\tilde{\omega}(a,\alpha) = \omega(a) + \alpha\, \tilde{\omega}(1) = \big\langle (\pi_\omega(a) + \alpha \cdot 1) z_\omega, z_\omega \big\rangle + \alpha r$$

and therefore $|\tilde{\omega}(a,\alpha)| \le \|z_\omega\|^2 \,\|\pi_\omega(a) + \alpha\cdot 1\| + r|\alpha|$. Hence $\tilde{\omega}$ is continuous with respect to the *-invariant sub-multiplicative semi-norm

$$(a,\alpha) \longmapsto \max\{\|\pi_\omega(a) + \alpha\cdot 1\|, |\alpha|\}\colon A^1 \longrightarrow \mathbb{R}.$$

Lemma 2.4 shows that $\tilde{\omega}$ is exponentially bounded with

$$\sigma_{\tilde{\omega}}(a,\alpha) \le \max\{\|\pi_\omega(a) + \alpha\cdot 1\|, |\alpha|\} \le \sigma_\omega(a) + |\alpha|.$$

Together with the formula $\sigma_{\tilde{\omega}}(a)^2 = \sup\big\{\tilde{\omega}(b^*a^*ab);\ b \in A^1, \tilde{\omega}(b^*b) \le 1\big\}$, this inequality implies that $\sigma_\omega(a) = \sigma_{\tilde{\omega}}(a,0)$ holds for all $a \in A$, and $|\alpha| = \sigma_{\tilde{\omega}}(0,\alpha)$ holds for all $\alpha \in \mathbb{C}$.

Every element of $\hat{A} \cup \{0\}$ has a unique extension to an element of $(A^1)^\frown$, and it is easy to see that this gives a homeomorphism $f\colon \hat{A} \to (A^1)^\frown \setminus \{\mathrm{pr}_2\}$, where $\mathrm{pr}_2\colon A^1 \to \mathbb{C}$, $(a,\alpha) \mapsto \alpha$. Let $\mu_{\tilde{\omega}}$ be the Gelfand transform of $\tilde{\omega}$. Since $f^*(\mu_{\tilde{\omega}})$ is a measure on $f^{-1}(\hat{A}(\sigma_{\tilde{\omega}}))$ which represents ω, Proposition 3.13 implies that $f(\hat{A}(\sigma_\omega)) = \hat{A}(\sigma_{\tilde{\omega}}) \setminus \{\mathrm{pr}_2\}$, and the uniqueness assertion of the Bochner Theorem 3.14 shows that the restriction of $f^*(\mu_{\tilde{\omega}})$ to $\hat{A}(\sigma_\omega)$ is equal to μ_ω. Hence

$$\mu_\omega(\hat{A}(\sigma_\omega)) = \mu_{\tilde{\omega}}\big(f(\hat{A}(\sigma_\omega))\big) = \tilde{\omega}(1) - \mu_{\tilde{\omega}}(\{\mathrm{pr}_2\})$$

is finite. If $a \in A$ then $(a,0)^\frown$ vanishes in $\mathrm{pr}_2 \in (A^1)^\frown$, so that

$$\int_{\hat{A}(\sigma_\omega)} \hat{a}\; d\mu_\omega = \int_{\hat{A}(\sigma_{\tilde{\omega}})} (a,0)^\frown\; d\mu_{\tilde{\omega}} = \tilde{\omega}(a,0) = \omega(a).$$

This completes the proof. □

3.17 Example. In these two examples, let X be a locally compact Hausdorff space, and let μ be a Borel measure on X which takes finite values on compact sets and satisfies conditions (i) and (ii) of the Bochner Theorem 3.14.

(a) Set $A := C_0(X) \cap L^1(\mu)$ with pointwise multiplication and involution. Define $\omega \in \mathrm{Pos}(A,\mathbb{C})$ by $\omega(\varphi) := \int \varphi\, d\mu$. Then ω is exponentially bounded; in fact, σ_ω is the supremum norm $\|\cdot\|_\infty$ on A. As the measure μ is finite on compact sets, the algebra A contains $C_c(X)$ and hence is uniformly dense in $C_0(X)$. Therefore, any $\|\cdot\|_\infty$-bounded involutive character of A has a unique extension to the C^*-algebra $C_0(X)$. Hence there is a natural homeomorphism from X onto the $\|\cdot\|_\infty$-bounded involutive spectrum of A. Under this homeomorphism, the Gelfand transform $\hat{\varphi}$ of $\varphi \in A$ corresponds to the function φ itself, and the Gelfand transform μ_ω of ω corresponds to the original measure μ. In particular, the equation $\omega(\varphi) = \int \hat{\varphi}\, d\mu_\omega$ holds for every element $\varphi \in A$. Nevertheless, if the measure μ is not finite then ω does not extend to a positive linear functional on the unitization A^1 of A.

(b) Set $A := C_0(X) \cap L^2(\mu)$ with pointwise multiplication and involution. Define $\omega(\varphi) := \int \varphi\, d\mu$ for $\varphi \in L^1(\mu)$, and extend ω arbitrarily to a linear functional on A. Then ω is an exponentially bounded positive linear functional with $\sigma_\omega = \|\cdot\|_\infty$. As above, the $\|\cdot\|_\infty$-bounded involutive spectrum of A is homeomorphic to X, the Gelfand transform $\hat{\varphi}$ of $\varphi \in A$ corresponds to the function φ itself, and the Gelfand transform μ_ω of ω corresponds to the original measure μ. This example shows that the Gelfand transforms of elements of A need not be μ_ω-integrable.

3.4 The Plancherel Theorem

3.18 Lemma. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and let $\Phi\colon \mathcal{H}_1 \to \mathcal{H}_2$ be a linear contraction. Assume that $V \leq \mathcal{H}_1$ is a closed linear subspace which is mapped isometrically onto $\mathcal{H}_2$. Then the kernel of Φ is the orthogonal complement of V.*

Proof. Choose $v \in V$, and write $\Phi^*\Phi(v) = v' + w$ with $v' \in V$ and $w \in V^\perp$. For an arbitrary element $v'' \in V$, we find that

$$\langle v, v'' \rangle = \langle \Phi(v), \Phi(v'') \rangle = \langle \Phi^*\Phi(v), v'' \rangle = \langle v' + w, v'' \rangle = \langle v', v'' \rangle.$$

Hence $v = v'$. As $\Phi^*\Phi$ is a contraction, this implies that $\Phi^*\Phi(v) = v$. Since $\Phi(V) = \mathcal{H}_2$, we conclude that the image of Φ^* is V. The kernel of Φ is the orthogonal complement of the image of its adjoint. $\square$

3.19 Definition. Let A be a *-algebra. Let $\mathcal{H}$ be a left Hilbert A-module, i.e. a Hilbert space equipped with a *-representation of A by bounded operators. Let $\mathcal{H}_1 \leq \mathcal{H}$ be the closure of the linear span of $A.\mathcal{H}$. The module $\mathcal{H}$ (or the corresponding *-representation of A) is called *non-degenerate* or *essential* if $\mathcal{H}_1 = \mathcal{H}$. The orthogonal complement of $\mathcal{H}_1$ in $\mathcal{H}$,

$$\begin{aligned}\mathcal{H}_0 := \mathcal{H}_1^\perp = (A.\mathcal{H})^\perp &= \{v \in \mathcal{H}; \langle v, A.\mathcal{H}\rangle = \{0\}\} \\ &= \{v \in \mathcal{H}; \langle A.v, \mathcal{H}\rangle = \{0\}\} = \{v \in \mathcal{H}; A.v = \{0\}\},\end{aligned}$$

is called the *totally degenerate* or *trivial part* of $\mathcal{H}$. The A-invariant orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ shows that the Hilbert A-module $\mathcal{H}_1$ is non-degenerate, whence it is called the *non-degenerate* or *essential part* of $\mathcal{H}$. Thus the Hilbert A-module $\mathcal{H}$ is the orthogonal direct sum of its totally degenerate part and its non-degenerate part. Note that $\mathcal{H}$ is non-degenerate if and only if $\mathcal{H}_0 = \{0\}$, i.e. if and only if $A.v = \{0\}$ implies $v = 0$ for $v \in \mathcal{H}$.

Assume that $\mathcal{H}$ is non-degenerate. Then every $v \in \mathcal{H}$ satisfies $v \in \overline{A.v}$. Indeed, the non-degenerate part of the Hilbert A-module $\mathbb{C}v + \overline{A.v}$ is $\overline{A.v}$. Hence if $v = v_0 + v_1$ with $v_0 \in (A.v)^\perp$ and $v_1 \in \overline{A.v}$ then v_0 belongs to the totally degenerate part of $\mathbb{C}v + \overline{A.v}$, which means that $A.v_0 = \{0\}$, and this implies that $v_0 = 0$ and $v = v_1 \in \overline{A.v}$.

For this and additional basic material on Hilbert modules see, for instance, Neeb [7, II.2.4].

The following result describes an isometric isomorphism from the non-degenerate part of the Gelfand–Naĭmark–Segal representation associated with a positive linear functional ω onto the L^2 space of the Gelfand transform of ω.

3.20 Theorem (Abstract Plancherel Theorem). *Let ω be an exponentially bounded positive linear functional on a commutative *-algebra A, and let μ_ω be the Gelfand transform of ω. Then there is a unique continuous linear map $\Phi_\omega \colon \mathcal{H}_\omega \to L^2(\mu_\omega)$ such that $\Phi_\omega(p_\omega(a)) = \hat{a}|_{\hat{A}(\sigma_\omega)}$ holds for every $a \in A$.*

This map is a homomorphism of A-modules and of $C_0(\hat{A}(\sigma_\omega))$-modules in the sense that

$$\forall\, \varphi \in C_0(\hat{A}(\sigma_\omega)),\, f \in \mathcal{H}_\omega \colon \Phi_\omega(\overline{\pi}_\omega(\varphi).f) = \varphi \cdot \Phi_\omega(f).$$

The kernel of Φ_ω is the totally degenerate part $\{f \in \mathcal{H}_\omega; \pi_\omega(A).f = \{0\}\}$ of $\mathcal{H}_\omega$, and Φ_ω maps the non-degenerate part $\overline{\operatorname{span}}(\pi_\omega(A).\mathcal{H}_\omega)$ of $\mathcal{H}_\omega$ isometrically onto $L^2(\mu_\omega)$.

Proof. If $a \in A$ belongs to $\ker(p_\omega) = \{x \in A;\, \omega(x^*x) = 0\}$ then $\hat{a}$ vanishes μ_ω-almost everywhere by Proposition 3.12. The linear map $p_\omega(a) \mapsto \hat{a} \colon \mathcal{H}^0_\omega \to L^2(\mu_\omega)$ is therefore a well-defined contraction, and so is its unique continuous extension $\Phi_\omega \colon \mathcal{H}_\omega \to L^2(\mu_\omega)$ to $\mathcal{H}_\omega$.

For $a, b \in A$ and $\varphi = \hat{a}$, $f = p_\omega(b)$, we have

$$\begin{aligned}\Phi_\omega(\overline{\pi}_\omega(\varphi).f) &= \Phi_\omega(\pi_\omega(a).p_\omega(b)) \\ &= \Phi_\omega(p_\omega(ab)) = \hat{a}\hat{b} = \hat{a} \cdot \Phi_\omega(p_\omega(b)) = \varphi \cdot \Phi_\omega(f).\end{aligned}$$

By continuity of $\overline{\pi}_\omega$ and Φ_ω, this equation extends to arbitrary pairs $(\varphi, f) \in C_0(\hat{A}(\sigma_\omega)) \times \mathcal{H}_\omega$.

For $a_1, a_2 \in A$, the Bochner Theorem 3.14 shows that $\|\Phi_\omega(p_\omega(a_1a_2))\|_2 = \|p_\omega(a_1a_2)\|$. Therefore, the restriction of Φ_ω to $V := \operatorname{span}(\pi_\omega(A).p_\omega(A))$ is an isometry, and so is the restriction of Φ_ω to $\overline{V}$, which is the non-degenerate part of $\mathcal{H}_\omega$. To see that Φ_ω maps $\overline{V}$ onto $L^2(\mu_\omega)$, it suffices to show that the closure of $\Phi_\omega(V)$ in $L^2(\mu_\omega)$ contains $C_c(\hat{A}(\sigma_\omega))$ (cf. Rudin [13, 3.14]). Choose an element $\varphi \in C_c(\hat{A}(\sigma_\omega))$. By Lemma 3.7, we may pick an element $a \in A$ whose Gelfand transform $\hat{a}$ does not vanish anywhere on the support of φ. The quotient $\varphi/\hat{a}$ may be extended by zero to an element of $C_c(\hat{A})$. For an arbitrary element $b \in A$, we calculate

$$\begin{aligned}\|\varphi - \hat{a}\hat{b}\|_2{}^2 &= \int_{\hat{A}(\sigma_\omega)} |\varphi - \hat{a}\hat{b}|^2 \, d\mu_\omega = \int_{\hat{A}(\sigma_\omega)} \left|\frac{\varphi}{\hat{a}} - \hat{b}\right|^2 |\hat{a}|^2 \, d\mu_\omega \\ &\le \left\|\frac{\varphi}{\hat{a}} - \hat{b}\right\|_\infty^2 \int_{\hat{A}(\sigma_\omega)} |\hat{a}|^2 \, d\mu_\omega \le \left\|\frac{\varphi}{\hat{a}} - \hat{b}\right\|_\infty^2 \omega(a^*a).\end{aligned}$$

Since the image of the Gelfand homomorphism is uniformly dense in the algebra $C_0(\hat{A}(\sigma_\omega))$, we conclude that the restriction of Φ_ω to the non-degenerate part $\overline{V}$ of $\mathcal{H}_\omega$ is an isometric isomorphism onto $L^2(\mu_\omega)$. Lemma 3.18 shows that the kernel of Φ_ω is the orthogonal complement $V^\perp$. The remarks preceding Theorem 3.20 contained the proof that $V^\perp$ is the degenerate part of $\mathcal{H}_\omega$. $\square$

3.21 Corollary. *Let ω be an exponentially bounded positive linear functional on a commutative *-algebra A with Gelfand transform μ_ω, and let* pr_1 *be the orthogonal projection of $\mathcal{H}_\omega$ onto its non-degenerate part. Then all $a, b \in A$ satisfy*

$$\int_{\hat{A}(\sigma_\omega)} \hat{a}\hat{b}\, d\mu_\omega = \big\langle \mathrm{pr}_1(p_\omega(a)), \mathrm{pr}_1(p_\omega(b^*))\big\rangle. \qquad \square$$

3.22 Corollary. *Let ω be an exponentially bounded positive linear functional on a commutative *-algebra A with Gelfand transform μ_ω. Then for each element $a \in A$, the following are equivalent:*

(i) *The equation $\int \hat{b}_1\hat{a}\, d\mu_\omega = \omega(b_1 a)$ holds for all $b_1 \in A$.*

(ii) *The equation $\int |\hat{a}|^2\, d\mu_\omega = \omega(a^*a)$ holds.*

(iii) *The vector $p_\omega(a) \in \mathcal{H}_\omega$ belongs to the non-degenerate part of $\mathcal{H}_\omega$.*

(iv) *For all $\varepsilon > 0$, there is a $b_2 \in A$ with $\omega\big((a - b_2 a)^*(a - b_2 a)\big) < \varepsilon$.*

(v) *For all $\varepsilon > 0$, there are $b_3, b_4 \in A$ with $\omega\big((a - b_3 b_4)^*(a - b_3 b_4)\big) < \varepsilon$.*

Proof. Let pr_1 be the projection of $\mathcal{H}_\omega$ onto its non-degenerate part, and set $f := p_\omega(a)$. Condition (i) trivially implies condition (ii). Condition (ii) means that $\|f\| = \|\mathrm{pr}_1(f)\|$, which implies (iii). If (iii) holds then all $b_1 \in A$ satisfy $\omega(b_1 a) = \langle f, \mathrm{pr}_1(b_1^*)\rangle = \int \hat{b}_1\hat{a}\, d\mu_\omega$.

We have seen in Definition 3.19 that (iii) implies that f belongs to the closure of $\pi_\omega(A).f$, which is condition (iv). This trivially implies (v), which is a reformulation of (iii). $\square$

3.23 Corollary. *Let ω be an exponentially bounded positive linear functional on a commutative *-algebra A with Gelfand transform μ_ω. Then π_ω is non-degenerate if and only if all $a, b \in A$ satisfy $\omega(ab) = \int \hat{a}\hat{b}\, d\mu_\omega$.* $\square$

3.24 Example. Set $A := \{\varphi \in C^2([0,1]); \varphi(0) = 0\}$ with pointwise multiplication and involution. Let $\omega_0 \colon A \to \mathbb{C},\ \varphi \mapsto \varphi''(0)$. Then $\omega_0(\varphi^*\varphi) = 2\,|\varphi'(0)|^2 \geq 0$, so that ω_0 is a positive linear functional. Since $\omega_0(\varphi_1\varphi_2\varphi_3) = 0$ for all $\varphi_1, \varphi_2, \varphi_3 \in A$, the positive functional ω_0 is bounded by every non-negative function on A, and its Gelfand transform is the zero measure.

Define

$$\omega_1 \colon A \longrightarrow \mathbb{C}, \quad \varphi \longmapsto \int_0^1 \varphi \, dx.$$

Then ω_1 is an exponentially bounded positive linear functional with $\sigma_{\omega_1} = \|\cdot\|_\infty$. Since any $\|\cdot\|_\infty$-bounded involutive character of A has a unique extension to the C*-algebra $C_0(]0,1])$, the $\|\cdot\|_\infty$-bounded involutive spectrum of A is naturally homeomorphic to the interval $]0,1]$. Under this homeomorphism, the Gelfand transform $\hat{\varphi}$ of $\varphi \in A$ corresponds to the function φ itself, and the Gelfand transform μ_{ω_1} of ω_1 corresponds to the Lebesgue measure on the Borel σ-algebra of $]0,1]$.

Set $\omega := \omega_0 + \omega_1$. Then the Gelfand transform μ_ω of ω equals μ_{ω_1}. An element $\varphi \in A$ satisfies $\omega(\varphi^*\varphi) = \int |\hat{\varphi}|^2 \, d\mu_\omega$ if and only if $\varphi'(0) = 0$. In particular, the Plancherel homomorphism $\Phi_\omega \colon \mathcal{H}_\omega \to L^2(\mu_\omega)$ is a proper contraction, whence the Gelfand–Naĭmark–Segal representation π_ω of A on $\mathcal{H}_\omega$ has a non-trivial degenerate part. However, the composition $\Phi_\omega \circ p_\omega \colon A \to L^2(\mu_\omega)$ is injective because it is just the Gelfand homomorphism. In other words, the dense subspace $\mathcal{H}_\omega^0$ of $\mathcal{H}_\omega$ has trivial intersection with the degenerate part of $\mathcal{H}_\omega$.

4 The non-degenerate part of a central positive linear functional

4.1 Proposition (Exponential boundedness of sums). *Let ω_1 and ω_2 be positive linear functionals on a *-algebra A. Then $\omega := \omega_1 + \omega_2$ is exponentially bounded if and only if both ω_1 and ω_2 are exponentially bounded. In this case, the associated C*-semi-norms satisfy $\sigma_\omega = \max\{\sigma_{\omega_1}, \sigma_{\omega_2}\}$.*

The following proof is a simplification of a similar result given by Neeb [7, II.4.21].

Proof. Assume that both ω_1 and ω_2 are exponentially bounded. If $a, b \in A$ then

$$\begin{aligned} \omega(b^*a^*ab) &= \omega_1(b^*a^*ab) + \omega_2(b^*a^*ab) \\ &\leq \sigma_{\omega_1}(a)^2 \, \omega_1(b^*b) + \sigma_{\omega_2}(a)^2 \, \omega_2(b^*b) \\ &\leq \big(\max\{\sigma_{\omega_1}(a), \sigma_{\omega_2}(a)\}\big)^2 \, \omega(b^*b). \end{aligned}$$

Hence ω is exponentially bounded with $\sigma_\omega \leq \max\{\sigma_{\omega_1}, \sigma_{\omega_2}\}$.

Conversely, assume that ω is exponentially bounded. Lemma 1.3 implies that $\mathcal{H}_{\omega_1} \subseteq \mathcal{H}_\omega$ and that the inclusion map $\Phi \colon \mathcal{H}_{\omega_1} \to \mathcal{H}_\omega$ is continuous. Denote the scalar product on $\mathcal{H}_{\omega_1}$ by $\langle \cdot, \cdot \rangle_1$ in order to distinguish it from the scalar product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}_\omega$. If $f \in \mathcal{H}_{\omega_1}$ and $a \in A$ then

$$\langle f, p_{\omega_1}(a) \rangle_1 = f(a^*) = \langle \Phi(f), p_\omega(a) \rangle = \langle f, \Phi^*(p_\omega(a)) \rangle_1 \, ,$$

which shows that $\Phi^*(p_\omega(a)) = p_{\omega_1}(a)$. Since Φ is a module homomorphism with respect to the A-module structure induced from $\mathbb{C}^A$, an easy calculation shows that Φ^* is a module homomorphism as well. Let $\Psi \in B(\mathcal{H}_\omega)$ be the positive square root of $\Phi\Phi^*$. If $a \in A$ then $\pi_\omega(a)$ commutes with $\Phi\Phi^*$ and hence with Ψ. For $a, b \in A$, we conclude that

$$\begin{aligned}
\|\pi_{\omega_1}(a).p_{\omega_1}(b)\|_1 &= \|p_{\omega_1}(ab)\|_1 = \|\Phi^*(p_\omega(ab))\|_1 \\
&= \sqrt{\langle \Phi^*(p_\omega(ab)), \Phi^*(p_\omega(ab))\rangle_1} = \|\Psi(p_\omega(ab))\| \\
&= \|\Psi(\pi_\omega(a).p_\omega(b))\| = \|\pi_\omega(a).\Psi(p_\omega(b))\| \\
&\leq \|\pi_\omega(a)\| \cdot \|\Psi(p_\omega(b))\| = \sigma_\omega(a) \cdot \|\Phi^*(p_\omega(b))\|_1 \\
&= \sigma_\omega(a) \cdot \|p_{\omega_1}(b)\|_1.
\end{aligned}$$

This shows that ω_1 is exponentially bounded with $\sigma_{\omega_1} \leq \sigma_\omega$. Analogously, the functional ω_2 is exponentially bounded with $\sigma_{\omega_2} \leq \sigma_\omega$. This implies that $\max\{\sigma_{\omega_1}, \sigma_{\omega_2}\} \leq \sigma_\omega$. $\square$

4.2 Definition. A positive linear functional ω on a *-algebra A is called *non-degenerate* if it is exponentially bounded and the *-representation π_ω of A on $\mathcal{H}_\omega$ is non-degenerate. Similarly, ω is called *totally degenerate* if it is exponentially bounded and the *-representation π_ω of A on $\mathcal{H}_\omega$ is totally degenerate.

Note that a positive linear functional on A is totally degenerate if and only if it vanishes on A^3. An exponentially bounded positive linear functional ω on A is non-degenerate if and only if for every $a \in A$ and every $\varepsilon > 0$, there are $x, y \in A$ such that $\omega\big((a-xy)^*(a-xy)\big) < \varepsilon$. Equivalently, for every $a \in A$ and every $\varepsilon > 0$, there is a $z \in A$ such that $\omega\big((a-za)^*(a-za)\big) < \varepsilon$.

4.3 Proposition (Uniqueness of the non-degenerate part of ω). *Let A be a *-algebra, let $\omega_0 \in \mathrm{Pos}(A, \mathbb{C})$ be totally degenerate, and let $\omega_1 \in \mathrm{Pos}(A, \mathbb{C})$ be non-degenerate. Set $\omega := \omega_0 + \omega_1$. Then the map $(f_0, f_1) \mapsto f_0 + f_1 \colon \mathcal{H}_{\omega_0} \oplus \mathcal{H}_{\omega_1} \to \mathcal{H}_\omega$ is an isometric isomorphism.*

Let $\mathrm{pr}_1 \colon \mathcal{H}_\omega \to \mathcal{H}_{\omega_1}$ denote the orthogonal projection. Then all $a, b \in A$ satisfy $\omega_1(b^*a) = \langle \mathrm{pr}_1(p_\omega(a)), \mathrm{pr}_1(p_\omega(b))\rangle$. In particular, the values $\omega_0(b^*a)$ and $\omega_1(b^*a)$ are uniquely determined by ω.

Proof. Since the A-module $\mathcal{H}_{\omega_0}$ is totally degenerate and the A-module $\mathcal{H}_{\omega_1}$ is non-degenerate, their intersection is $\{0\}$. Therefore, Lemma 1.3 shows that the map described in the statement is an isometric isomorphism. The same lemma shows that the inverse isomorphism maps $p_\omega(a)$ to the pair $(p_{\omega_0}(a), p_{\omega_1}(a))$ whenever $a \in A$, so that $p_{\omega_1}(a) = \mathrm{pr}_1(p_\omega(a))$. We infer that all $a, b \in A$ satisfy

$$\omega_1(b^*a) = \langle p_{\omega_1}(a), p_{\omega_1}(b)\rangle = \langle \mathrm{pr}_1(p_{\omega_1}(a)), \mathrm{pr}_1(p_{\omega_1}(b))\rangle.$$

Since pr_1 is the orthogonal projection of $\mathcal{H}_\omega$ onto its non-degenerate part, this map is uniquely determined by ω, whence so are the values $\omega_1(b^*a)$ and $\omega_0(b^*a) = \omega(b^*a) - \omega_1(b^*a)$. $\square$

4.4 Definition. A positive linear functional ω on a *-algebra A is called *central* if all $a, b \in A$ satisfy $\omega(ab) = \omega(ba)$. This condition is equivalent to $\omega(a^*a) = \omega(aa^*)$ for all $a \in A$ by the polarization identity

$$ab = \frac{1}{4}\sum_{k=0}^{3} i^k (i^k a^* + b)^* (i^k a^* + b) \qquad (a, b \in A).$$

4.5 Remark (Central positive linear functionals). (a) Let ω be a central positive linear functional on a *-algebra A. Then $\ker(p_\omega) \subseteq \ker(\pi_\omega) = \big\{a \in A;\ \omega\big((a^*a)^2\big) = 0\big\}$. The kernel of p_ω is a *-invariant ideal of A, so that a conjugate linear involution J_ω on $\mathcal{H}^0_\omega$ may be defined by $J_\omega(p_\omega(a)) := p_\omega(a^*)$. This involution is isometric and hence extends to an involution on $\mathcal{H}_\omega$, which will also be denoted by J_ω and which is also isometric. Since J_ω is conjugate linear, the latter condition means that $\langle J_\omega(f), J_\omega(g)\rangle = \langle g, f\rangle$ holds for all $f, g \in \mathcal{H}_\omega$. The decomposition of $\mathcal{H}_\omega$ as a direct sum of its totally degenerate and its non-degenerate part is invariant under J_ω because the non-degenerate part is the closed linear span of the J_ω-invariant set $p_\omega(AA)$.

(b) Assume, in addition, that ω is non-degenerate. Then

$$\begin{aligned}
\ker(p_\omega) &= \Big\{a \in A;\ p_\omega(a) \in \big(\pi_\omega(A).\mathcal{H}_\omega\big)^\perp\Big\} \\
&= \{a \in A;\ \ \forall\, b, c \in A\colon \omega(bca) = 0\} \\
&= \{a \in A;\ \ \forall\, b, c \in A\colon \omega(cab) = 0\} = \ker(\pi_\omega).
\end{aligned}$$

4.6 Definition. A *Hilbert algebra* is a *-algebra A with a (positive definite) scalar product such that the following axioms are satisfied:

(i) $\forall\, a, b \in A\colon \langle a, b\rangle = \langle b^*, a^*\rangle$;

(ii) $\forall\, a, b, c \in A\colon \langle ab, c\rangle = \langle b, a^*c\rangle$;

(iii) for every $a \in A$, the map $x \mapsto ax\colon A \to A$ is continuous;

(iv) $\mathrm{span}(AA)$ is dense in A.

4.7 Remark (Theory of Hilbert algebras). We collect some fundamental results from the theory of Hilbert algebras without giving proofs. A convenient reference for most of the material is Dixmier [5, Appendix A, 54–60]; see also Dixmier [4, I.5 and I.6], Palmer [9, 11.7], and Rieffel [12, § 1].

Let A be a Hilbert algebra. Then all $a, b, c \in A$ satisfy $\langle ab, c\rangle = \langle a, cb^*\rangle$, and for every $a \in A$, the map $x \mapsto xa\colon A \to A$ is continuous. Thus there is a perfect left-right symmetry, which our further exposition will suppress for the sake of brevity.

Let $\mathcal{H}_A$ be the Hilbert space completion of A, and let $J_A\colon \mathcal{H}_A \to \mathcal{H}_A$ be the continuous extension of the involution of A. For $a \in A$, let $U_a \in B(\mathcal{H}_A)$ be the continuous extension of the left translation map $x \mapsto ax\colon A \to A$ to $\mathcal{H}_A$. Then $a \mapsto U_a\colon A \to B(\mathcal{H}_A)$ is a non-degenerate *-representation. The weak closure of its image is denoted by $\mathcal{U}(A) \subseteq B(\mathcal{H}_A)$ and called the *left von Neumann algebra* of A.

An element $x \in \mathcal{H}_A$ is called *bounded* if the map $a \mapsto U_a.x\colon A \to \mathcal{H}_A$ is continuous. The continuous extension of this map to $\mathcal{H}_A$ is denoted by $V_x \in B(\mathcal{H}_A)$. If $x \in A$ then x is bounded, and $\|V_x\| = \|U_x\|$. The vector subspace $A' \subseteq \mathcal{H}_A$ of bounded elements of $\mathcal{H}_A$ is invariant under the involution J_A, and it becomes a Hilbert algebra if multiplication is defined by $xy := V_y.x$ for $x, y \in A'$. If $x, y \in A$ then $V_y.x = U_x.y = xy$, so that the multiplication on A' extends the multiplication on A. If $a \in A'$ then the map $x \mapsto ax\colon A' \to A'$ has a continuous extension to $\mathcal{H}_A$, which belongs to $\mathcal{U}(A)$ and is denoted by U_a.

Let $s \in \operatorname{Pos}(\mathcal{U}(A))$. If the positive square root of s in $\mathcal{U}(A)$ has the form U_a for an element $a \in A'$, set $\theta(s) := \langle a, a\rangle$. If there is no such $a \in A'$, set $\theta(s) := \infty$. Then $\theta\colon \operatorname{Pos}(\mathcal{U}(A)) \to [0,\infty]$ is a semi-finite faithful normal trace. It is called the *natural trace* defined by A. We have $\{t \in \mathcal{U}(A);\, \theta(t^*t) < \infty\} = \{U_a;\, a \in A'\} =: \mathfrak{n}_\theta$. There is a unique linear functional $\dot\theta$ on the ideal $\mathfrak{m}_\theta := \operatorname{span}(\mathfrak{n}_\theta \mathfrak{n}_\theta)$ of $\mathcal{U}(A)$ such that $\dot\theta(s) = \theta(s)$ holds for all $s \in \mathfrak{m}_\theta \cap \operatorname{Pos}(\mathcal{U}(A))$. All $a, b \in A'$ satisfy $\dot\theta(U_b^* U_a) = \langle a, b\rangle$.

Define the *Rieffel norm* on A' by $\|a\|' := \|a\|_{\mathcal{H}_A} + \|U_a\|_{B(\mathcal{H}_A)}$ for $a \in A'$. This norm is sub-multiplicative and *-invariant. It is also complete (Rieffel [12, 1.15]), and the Banach *-algebra $(A', \|\cdot\|')$ is hermitian (Palmer [9, 11.7.11]). This means that every element $a \in A'$ with $a^* = a$ satisfies $\operatorname{Sp}(a) \subseteq \mathbb{R}$. By the Shirali–Ford Theorem [17], a Banach *-algebra is hermitian if and only if it satisfies the apparently stronger condition that every element of the form $a_1^*a_1 + \cdots + a_n^*a_n$ has non-negative real spectrum (see also Bonsall and Duncan [3, 41.4, 41.5] and the exposition by Pták [10]).

4.8 Proposition (The natural positive linear functional on a Hilbert algebra). *Let A be a Hilbert algebra. Then there is a non-degenerate central positive linear functional ω_A on A such that all $a, b \in A$ satisfy $\omega_A(b^*a) = \langle a, b\rangle$. The map $p_{\omega_A}\colon A \to \mathcal{H}^0_{\omega_A}$ is an isometric isomorphism of A-modules, so that it extends to an isometric isomorphism of A-modules $\overline{p}_{\omega_A}\colon \mathcal{H}_A \to \mathcal{H}_{\omega_A}$. In particular, every $a \in A$ satisfies $\|U_a\| = \|\pi_{\omega_A}(a)\|$.*

These properties define ω_A uniquely on the linear span of AA only, and the extension to A is indeed arbitrary. Nevertheless, it seems justified to call ω_A "the natural" positive linear functional on A.

Proof. Let $\theta\colon \operatorname{Pos}(\mathcal{U}(A)) \to [0,\infty]$ be the natural trace defined by A, and let $\dot\theta$ be the associated linear functional on $\mathfrak{m}_\theta$. If $a \in \operatorname{span}(AA)$ then $U_a \in \mathfrak{m}_\theta$, and we set $\omega_A(a) := \dot\theta(U_a)$. This yields a linear functional on $\operatorname{span}(AA)$, which may be extended to a linear functional ω_A on A.

Let $a, b \in A$. We calculate $\omega_A(b^*a) = \dot\theta(U_{b^*a}) = \dot\theta(U_b^* U_a) = \langle a, b\rangle$. In particular, the linear functional ω_A is positive. It is also central because $\omega(ab) = \langle b, a^*\rangle = \langle a, b^*\rangle = \omega(ba)$. The assertions about p_{ω_A} follow from the calculations

$$\ker(p_{\omega_A}) = \{a \in A;\ \omega(a^*a) = 0\} = \{a \in A;\ \langle a, a\rangle = 0\} = \{0\}$$

and $\langle p_{\omega_A}(a), p_{\omega_A}(b)\rangle = \omega_A(b^*a) = \langle a, b\rangle$. The fact that AA is a total subset of A proves that ω is non-degenerate. □

4.9 Example (Commutative Hilbert algebras). Let A be a commutative Hilbert algebra, and let $\omega := \omega_A$ be the natural positive linear functional on A. Let μ_ω be the Gelfand transform of ω. The Plancherel Theorem 3.20 and Proposition 4.8 imply that the map $\Phi\colon A \to L^2(\mu_\omega),\ a \mapsto \hat a|_{\hat A(\sigma_\omega)}$ is an isometric homomorphism of A-modules which extends to an isometric isomorphism $\mathcal{H}_\omega \cong L^2(\mu_\omega)$.

Let X be a locally compact Hausdorff space, and let μ be a Borel measure on X which takes finite values on compact sets and satisfies conditions (i) and (ii) of the Bochner Theorem 3.14. Consider the *-algebra $L^2(\mu) \cap L^\infty(\mu)$ of essentially bounded square-integrable functions on X with pointwise multiplication and involution. Let $A \subseteq L^2(\mu) \cap L^\infty(\mu)$ be a subalgebra such that AA is a total subset of $L^2(\mu)$. (For example, this is the case if $C_c(X) \subseteq A$, see Rudin [13, 3.14].) Under the scalar product of $L^2(\mu)$, the algebra A is a commutative Hilbert algebra with $\mathcal{H}_A = L^2(\mu)$. The first paragraph of this remark shows that every commutative Hilbert algebra can be realized in this way, and if we wish, we can choose (X, μ) such that A is a dense subalgebra of $C_0(X)$. If $a \in A$ then U_a is multiplication by a. The Hilbert algebra of bounded elements of $L^2(\mu)$ is $L^2(\mu) \cap L^\infty(\mu)$, and if f is one of its elements then U_f is again multiplication by f. The Rieffel norm is given by $\|f\|' = \|f\|_2 + \|f\|_\infty$. If $f\colon X \to \mathbb{R}_0^+$ is bounded and integrable then the natural trace satisfies $\theta(U_f) = \int f\, d\mu$. In particular, if $a \in \operatorname{span}(AA)$ then the natural positive linear functional ω_A on A is given by $\omega_A(a) = \int a\, d\mu$.

Assume, in addition, that the measure μ is σ-finite, and that the uniform closure of A contains $C_0(X)$. We claim that the left von Neumann algebra of A satisfies $\mathcal{U}(A) = \{v \mapsto f \cdot v;\ f \in L^\infty(\mu)\}$. Dixmier [4, I.7.3] shows that the right-hand side is a von Neumann algebra in $L^2(\mu)$, so that it suffices to show that its subalgebra $\{U_a;\ a \in A\}$ is strongly dense. Choose $f \in L^\infty(\mu)$, $v \in L^2(\mu)$, and $\varepsilon > 0$. We have to find an element $a \in A$ such that $\|(f-a)v\|_2 < \varepsilon$. We may assume that $f \neq 0$. There is a measurable function $s\colon X \to \mathbb{C}$ with finite image such that $E := s^{-1}(\mathbb{C}^\times)$ has finite measure and

$\|v - s\|_2 < \varepsilon/(9\|f\|_\infty)$, see Rudin [13, 3.13]. Lusin's Theorem [13, 2.24] yields a function $g \in C_c(X)$ such that

$$\mu(\{x \in E;\ f(x) \neq g(x)\}) < \frac{\varepsilon^2}{36\|f\|_\infty{}^2\ \|s\|_\infty{}^2}$$

and $\|g\|_\infty \leq \|f\|_\infty$, so that

$$\|(f-g)s\|_2 = \left(\int_E |f-g|^2\ |s|^2\ d\mu\right)^{\frac{1}{2}} < \frac{\varepsilon}{3}.$$

By hypothesis, we find an element $a \in A$ such that

$$\|g - a\|_\infty < \max\left\{\|f\|_\infty,\ \frac{\varepsilon}{3\|s\|_2}\right\}.$$

These conditions imply that $\|(g-a)s\|_2 \leq \|g-a\|_\infty\ \|s\|_2 < \varepsilon/3$ and that $\|(f-a)(v-s)\|_2 \leq \|f-a\|_\infty\ \|v-s\|_2 < \varepsilon/3$. We conclude that

$$\|(f-a)v\|_2 = \|(f-a)(v-s) + (f-g)s + (g-a)s\|_2 < \varepsilon.$$

This proves our claim.

Specialize the above situation by choosing $X = \mathbb{R}$, and let μ be Lebesgue measure. For $n \in \mathbb{N}$, let $a_n\colon X \to \left[0, \frac{1}{n}\right]$ be continuous with $\operatorname{supp}(a_n) \subseteq [0, n^2]$ and $f([1, n^2-1]) = \left\{\frac{1}{n}\right\}$. Then $\|a_n\|' \leq 1 + \frac{1}{n}$ and $\omega_A(a_n) \geq n - \frac{2}{n}$. This proves that ω_A need not be continuous.

4.10 Theorem (The non-degenerate part of ω). *Let ω be an exponentially bounded central positive linear functional on a *-algebra A. Then there exist central positive linear functionals ω_0 and ω_1 on A such that ω_0 is totally degenerate, ω_1 is non-degenerate, and $\omega = \omega_0 + \omega_1$. On the *-algebra $\pi_\omega(A)$, a scalar product may be defined by $\langle \pi_\omega(x), \pi_\omega(y)\rangle := \omega_1(y^*x)$ for $x, y \in A$. This scalar product turns $\pi_\omega(A)$ into a Hilbert algebra.*

If $\pi\colon A \to B$ is a *-homomorphism from A onto a Hilbert algebra such that all $x, y \in A$ satisfy $\omega_1(y^*x) = \langle \pi(x), \pi(y)\rangle$ then there is an isometric isomorphism $\psi\colon \pi_\omega(A) \to B$ such that $\pi = \psi \circ \pi_\omega$.

Proposition 4.3 shows that ω_0 and ω_1 are essentially uniquely determined by ω.

Proof. The last assertion follows immediately from the observation that

$$\begin{aligned}\ker(\pi_\omega) &= \{a \in A;\ \langle \pi_\omega(a), \pi_\omega(a)\rangle = 0\}\\ &= \{a \in A;\ \langle \pi(a), \pi(a)\rangle = 0\} = \ker(\pi).\end{aligned}$$

We will first define the Hilbert algebra structure on $B := \pi_\omega(A)$. The decomposition of ω will be constructed from the natural positive linear functional on B. Let $\mathrm{pr}_1 \colon \mathcal{H}_\omega \to \mathcal{H}_1$ be the orthogonal projection of $\mathcal{H}_\omega$ onto its non-degenerate part, and set $p_1 := \mathrm{pr}_1 \circ p_\omega$. Note that the involution J_ω of $\mathcal{H}_\omega$ defined in Remark 4.5 satisfies $J_\omega \circ \mathrm{pr}_1 = \mathrm{pr}_1 \circ J_\omega$, so that $J_\omega(p_1(a)) = p_1(a^*)$ holds for all $a \in A$. Set $H := p_1(A) \subseteq \mathcal{H}_1$. Since

$$\begin{aligned}\ker(p_1) &= \Big\{a \in A;\ p_\omega(a) \in \mathcal{H}_1^\perp\Big\} \\ &= \{a \in A;\ \ \forall\, b, c \in A \colon \omega(bca) = 0\} \\ &= \{a \in A;\ \ \forall\, b, c \in A \colon \omega(cab) = 0\} = \ker(\pi_\omega),\end{aligned}$$

there is an isomorphism $\pi_\omega(a) \mapsto p_1(a) \colon B \to H$ of A-modules. In particular, we can define a scalar product on B by

$$\langle \pi_\omega(a), \pi_\omega(b)\rangle := \langle p_1(a), p_1(b)\rangle.$$

Let us check that B is a Hilbert algebra. Let $b_1, b_2, b_3 \in B$, and choose $a_1, a_2, a_3 \in A$ such that $b_j = \pi_\omega(a_j)$ holds for $j \in \{1, 2, 3\}$. For the first two axioms, we calculate

$$\begin{aligned}\langle b_2^*, b_1^*\rangle &= \langle p_1(a_2^*), p_1(a_1^*)\rangle \\ &= \langle J_\omega(p_1(a_2)), J_\omega(p_1(a_1))\rangle = \langle p_1(a_1), p_1(a_2)\rangle = \langle b_1, b_2\rangle\end{aligned}$$

and

$$\begin{aligned}\langle b_2, b_1^* b_3\rangle &= \big\langle p_1(b_2), p_1(b_1^* b_3)\big\rangle = \big\langle p_1(b_2), \mathrm{pr}_1\big(\pi_\omega(b_1^*).p_\omega(b_3)\big)\big\rangle \\ &= \big\langle p_1(b_2), \pi_\omega(b_1^*).p_1(b_3)\big\rangle \\ &= \big\langle \pi_\omega(b_1).p_1(b_2), p_1(b_3)\big\rangle = \big\langle p_1(b_1 b_2), p_1(b_3)\big\rangle = \langle b_1 b_2, b_3\rangle.\end{aligned}$$

Under the isomorphism of B onto H, the left multiplication map $x \mapsto b_1 x \colon B \to B$ corresponds to the restriction of $\pi_\omega(a_1)$ to H, which shows that it is continuous. Similarly, the linear span of BB corresponds to the linear span of $p_1(AA) = p_\omega(AA)$, which proves the fourth axiom. Thus $B = \pi_\omega(A)$ is a Hilbert algebra.

Let ω_B be the natural positive linear functional on B defined in Proposition 4.8, so that all $x, y \in B$ satisfy $\omega_B(y^* x) = \langle x, y\rangle$. This is pulled back to a central positive linear functional on A by $\omega_1 := \omega_B \circ \pi_\omega$, so that $\langle \pi_\omega(x), \pi_\omega(y)\rangle = \omega_1(y^* x)$ holds for all $x, y \in A$. If $f \in \mathcal{H}_1$ and $a \in A$ then $f(a) = \langle f, p_\omega(a^*)\rangle = \langle f, p_1(a^*)\rangle$. Therefore, the positive definite kernel $\mathcal{K}^{\mathcal{H}_1}$ on A associated with $\mathcal{H}_1$ by Lemma 1.1 satisfies

$$K^{\mathcal{H}_1}(y, x) = \langle p_1(x^*), p_1(y^*)\rangle = \langle \pi_\omega(x^*), \pi_\omega(y^*)\rangle = \omega_1(y x^*) = K_{\omega_1}(y, x)$$

for all $x, y \in A$. Theorem 1.2 shows that $\mathcal{H}_1 = \mathcal{H}_{\mathcal{K}^{\mathcal{H}_1}} = \mathcal{H}_{\omega_1}$. In particular, the positive linear functional ω_1 is non-degenerate. Define a linear functional on A by $\omega_0 := \omega - \omega_1$. Then ω_0 is positive because

$$\omega_0(a^* a) = \omega(a^* a) - \omega_1(a^* a) = \|p_\omega(a)\|^2 - \|p_1(a)\|^2 \geq 0$$

holds for all $a \in A$. Since all $a, b \in A$ satisfy $p_\omega(ab) = \pi_\omega(a).p_\omega(b) \in \mathcal{H}_1$ and therefore

$$\omega_0(b^*a^*ab) = \omega(b^*a^*ab) - \omega_1(b^*a^*ab) = \|p_\omega(ab)\|^2 - \|p_1(ab)\|^2 = 0,$$

the positive linear functional ω_0 is totally degenerate. $\square$

4.11 Remark (The commutative case). Let ω be an exponentially bounded central positive linear functional on a *-algebra A, and denote the orthogonal projection of $\mathcal{H}_\omega$ onto its non-degenerate part by $\mathrm{pr}_1 \colon \mathcal{H}_\omega \to \mathcal{H}_1$. In the proof of Theorem 4.10, the theory of Hilbert algebras was needed in order to construct a positive linear functional ω_1 on A such that all $a, b \in A$ satisfy $\langle \mathrm{pr}_1(p_\omega(a)), \mathrm{pr}_1(p_\omega(b))\rangle = \omega_1(b^*a)$.

Assume that A is commutative. Then the Plancherel Theorem 3.20 allows us to construct ω_1 without using Hilbert algebras. Indeed, let μ_ω be the Gelfand transform of ω, and define the Plancherel homomorphism $\Phi_\omega \colon \mathcal{H}_\omega \to L^2(\mu_\omega)$ as in Theorem 3.20. If $a \in A$ is such that $\hat{a} \in L^1(\mu_\omega)$, set $\omega_1(a) := \int \hat{a}\, d\mu_\omega$, and extend ω_1 arbitrarily to a linear functional on A. Then ω_1 is a positive linear functional with the property described in the first paragraph, and we obtain the decomposition $\omega = \omega_0 + \omega_1$ as in Theorem 4.10.

If σ is a a semi-norm on A such that $\left|\int \hat{a}\, d\mu_\omega\right| \leq \sigma(a)$ holds for all $a \in A$ with $\hat{a} \in L^1(\mu_\omega)$ then ω_1 can be chosen such that $\omega_1 \leq \sigma$, as follows from the Hahn–Banach Theorem on dominated extension (see Rudin [14, 3.3]).

4.12 Proposition (The hermitian Banach *-algebra of ω). *Let ω be a non-degenerate central positive linear functional on a *-algebra A. By Remark 4.5* (b)*, we may define a norm on the algebra $\pi_\omega(A)$ by the formula $\|\pi_\omega(a)\|' := \|p_\omega(a)\| + \|\pi_\omega(a)\|$ for $a \in A$. Let B be the completion of $\pi_\omega(A)$ with respect to this norm. Then B is a hermitian Banach *-algebra with isometric involution, and there is a non-degenerate central positive linear functional $\overline{\omega}$ on B such that $\omega(xy) = \overline{\omega}(\pi_\omega(xy))$ holds for all $x, y \in A$.*

Since the inclusion of $\pi_\omega(A)$ into $B(\mathcal{H}_\omega)$ is continuous with respect to the norm $\|\cdot\|'$, we may view B as a subalgebra of $B(\mathcal{H}_\omega)$. In particular, the norm of $B(\mathcal{H}_\omega)$ is a C*-norm on B.

Proof. Since ω is non-degenerate, Theorem 4.10 shows that the scalar product defined on the algebra $B_0 := \pi_\omega(A) \subseteq B(\mathcal{H}_\omega)$ by $\langle \pi_\omega(x), \pi_\omega(y)\rangle := \omega(y^*x) = \langle p_\omega(x), p_\omega(y)\rangle$ gives this algebra the structure of a Hilbert algebra. Let $\mathcal{H}$ be the Hilbert space completion of B_0, and choose $b \in B_0$. Since $\mathcal{H} \cong \mathcal{H}_\omega$ as B_0-modules, the continuous extension U_b of left multiplication by b to $\mathcal{H}$ satisfies $\|U_b\|_{B(\mathcal{H})} = \|b\|_{B(\mathcal{H}_\omega)}$.

As in Remark 4.7, we denote the Hilbert algebra of bounded elements of $\mathcal{H}$ by $B_0' \subseteq \mathcal{H}$. The algebra B_0' is a hermitian Banach *-algebra with respect to the *-invariant norm defined by $\|b\|' := \|b\|_{\mathcal{H}} + \|U_b\|_{B(\mathcal{H})}$. If

$a \in A$ then $\|\pi_\omega(a)\|' = \|p_\omega(a)\| + \|\pi_\omega(a)\|$. Hence we can identify B with the closure of B_0 in B_0'. A closed *-subalgebra of a hermitian Banach *-algebra is hermitian (see Palmer [9, 11.4.2]).

Let $\omega_{B_0'}$ be the natural positive linear functional on B_0', and set $\overline{\omega} := \omega_{B_0'}|_B$. Then $\overline{\omega}$ is a central positive linear functional on B, and it is non-degenerate because BB contains B_0B_0 and hence is total in $\mathcal{H}$. Finally, all $x, y \in A$ satisfy

$$\overline{\omega}(\pi_\omega(xy)) = \omega_{B_0'}(\pi_\omega(x^*)^*\pi_\omega(y)) = \langle \pi_\omega(y), \pi_\omega(x^*) \rangle = \omega(xy). \qquad \square$$

4.13 Remark. (a) The proof shows that $\overline{\omega}$ is most naturally defined on the Hilbert algebra of bounded elements defined by the Hilbert algebra $\pi_\omega(A)$, which is larger than the completion B. We restrict $\overline{\omega}$ to B just in order to obtain a more elementary statement.

(b) The final remarks of Example 4.9 show that $\overline{\omega}$ may not be continuous.

(c) We could now derive the Bochner Theorem 3.14 from a version for non-degenerate positive linear functionals on hermitian commutative Banach *-algebras such as it is given by Fell and Doran [6, 21.4]. However, this would be a logical detour. In fact, the direct proof of Theorem 3.14 which we have given above is quite similar to the proof of the more special result by Fell and Doran.

4.14 Remark (Integral decompositions for separable C*-algebras). The relation between central positive linear functionals and Hilbert algebras allows us to apply the theory of integral decompositions of traces on separable C*-algebras (Dixmier [5, 8.8.2]).

Let ω be a non-degenerate central positive linear functional on a *-algebra A. Assume that the dimension of A is at most countable. Then the closure $B \subseteq B(\mathcal{H}_\omega)$ of $\pi_\omega(A)$ is a separable C*-algebra. The quasi-equivalence classes [5, 5.3.2] of factorial representations [5, 5.2.6] of B form a measurable space [5, 7.2.2], which is called the quasi-spectrum $\mathrm{QSp}(B)$ of B. The restriction of the natural trace θ defined by the Hilbert algebra $\pi_\omega(A)$ to $\mathrm{Pos}(B)$ is a semi-finite lower semi-continuous trace [5, 6.1.5]. The decomposition theorem [5, 8.8.2] yields a standard measure μ on $\mathrm{QSp}(B)$ and a family $(\theta_\zeta)_{\zeta \in \mathrm{QSp}(B)}$ of pure traces on B with the following properties:

(a) the representation of B associated to θ_ζ belongs to the quasi-equivalence class $\zeta \in \mathrm{QSp}(B)$ almost everywhere;

(b) for every $b \in \mathrm{Pos}(B)$, the function $\zeta \mapsto \theta_\zeta(b)\colon \mathrm{QSp}(B) \to [0, \infty]$ is measurable;

(c) $\forall\, b \in \mathrm{Pos}(B)\colon \theta(b) = \displaystyle\int_{\mathrm{QSp}(B)} \theta_\zeta(b)\, d\mu(\zeta)$.

In particular, if $a \in A$ then

$$\omega(a^*a) = \theta\big(\pi_\omega(a^*a)\big) = \int_{\mathrm{QSp}(B)} \theta_\zeta\big(\pi_\omega(a^*a)\big)\, d\mu(\zeta).$$

Assume that the separable C*-algebra B is commutative. Then a representation of B is factorial if and only if it is one-dimensional, and quasi-equivalence of factorial representations is equality of the corresponding characters. Hence $\mathrm{QSp}(B) = \hat{B}$ as sets; in fact, the σ-algebra defined on $\mathrm{QSp}(B)$ equals the Borel σ-algebra of $\hat{B}$. (All this follows more or less immediately from the definitions [5].) The pure traces of B are exactly the positive scalar multiples of elements of $\hat{B}$ by [5, 6.7.8], and the representation associated with a pure trace is given by the corresponding character [5, 6.8.3].

Apply the above decomposition in the commutative situation. The measure μ is a Borel measure on $\hat{B}$. For every $\chi \in \hat{B}$ up to a set of measure 0, statement (a) implies that there is a positive number $t(\chi) \in \mathbb{R}^+$ such that $\theta_\chi = t(\chi) \cdot \chi$. By (b), the function $\hat{b} \cdot t\colon \hat{B} \to \mathbb{C}$ is measurable for all $b \in B$. We claim that $t\colon \hat{B} \to \mathbb{R}^+$ is measurable. Let $E \subseteq \mathbb{R}^+$ be measurable, and let $(b_n)_{n\in\mathbb{N}}$ be a total sequence in B. Set $U_n := \hat{b}_n^{-1}(\mathbb{C}^\times) \subseteq \hat{B}$ for $n \in \mathbb{N}$, so that $\hat{B} = \bigcup_n U_n$. The restriction of t to U_n is a measurable function because it is the quotient of $\hat{b}_n \cdot t$ by $\hat{b}_n$. We conclude that $t^{-1}(E) = \bigcup_n \{\chi \in U_n;\ t(\chi) \in E\}$ is indeed a measurable subset of $\hat{B}$.

Since t is a measurable function, a Borel measure on $\hat{B}$ is defined by $\nu(E) := \int_E t\, d\mu$ for $E \in \mathfrak{B}(\hat{B})$ (see Rudin [13, 1.29]). Statement (c) shows that $\theta(b) = \int_{\hat{B}} \hat{b} \cdot t\, d\mu = \int_{\hat{B}} \hat{b}\, d\nu$ holds for all $b \in \mathrm{Pos}(B)$. By polarization, this implies that all $a_1, a_2 \in A$ satisfy $\omega(a_1a_2) = \int_{\hat{B}} \pi_\omega(a_1a_2)^\frown\, d\nu$. As in the proof of Lemma 3.11, one shows that ν takes finite values on compact subsets of $\hat{B}$. Since B is separable and $\hat{B} \cup \{0\}$ is weak*-compact, this set is a compact metrizable space (Rudin [14, 3.16]), which implies that every open subset is σ-compact. We conclude that ν is a regular Borel measure [13, 2.18]. Transporting ν to $\hat{A}(\sigma_\omega)$ by means of the homeomorphism $\chi \mapsto \chi \circ \pi_\omega\colon \hat{B} \to \hat{A}(\sigma_\omega)$, we obtain a regular Borel measure ν' on $\hat{A}(\sigma_\omega)$ such that $\omega(a_1a_2) = \int \hat{a}_1\hat{a}_2\, d\nu'$ holds for all $a_1, a_2 \in A$. If $\omega' \in \mathrm{Pos}(A, \mathbb{C})$ is totally degenerate then ν' is a representing measure for the positive linear functional $\omega'' := \omega + \omega'$ because all $a_1, a_2, a_3 \in A$ satisfy

$$\omega''(a_1a_2a_3) = \omega(a_1a_2a_3) = \int_{\hat{A}(\sigma_\omega)} \hat{a}_1\hat{a}_2\hat{a}_3\, d\nu'.$$

For commutative algebras of countable dimension, the existence part of the Bochner Theorem 3.14 is thus a special case of a more general integral decomposition.

The same arguments apply if A is a separable topological *-algebra and ω is a non-degenerate central positive linear functional on A such that π_ω is continuous with respect to the norm of $B(\mathcal{H}_\omega)$. Continuity of π_ω is automatic if A is a Banach *-algebra (see Bonsall and Duncan [3, 37.3]) or a (Mackey) complete continuous inverse *-algebra with continuous involution [2].

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